Conditional stability for backward parabolic equations with Log $\mathrm{Lip}_t \times \mathrm{Lip}_x$-coefficients

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ABSTRACT

In this work we present an improvement of Del Santo and Prizzi (2009), where the authors proved a result concerning continuous dependence for backward-parabolic operators whose coefficients are Log-Lipschitz in t and C^2 in x. In that paper, the C^2 regularity with respect to x had to be assumed for technical reasons: here we remove this assumption, replacing it with Lipschitz-continuity. The main tools in the proof are Littlewood–Paley theory and Bony's paraproduct.

1. Introduction

In this paper, we study the continuous dependence of solutions to the Cauchy problem for a backward-parabolic operator, namely

$$Pu = \partial_t u + \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t,x)\partial_{x_k} u) = 0 \tag{1.1}$$

on the strip $[0,T] \times \mathbb{R}^n_x$ with data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}^n_x. \tag{1.2}$$

The coefficients are supposed to be real valued, measurable and bounded. The matrix $(a_{jk})_{j,k=1,\dots,n}$ is symmetric and positive definite, i.e. there exists a $\kappa > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j\xi_k \geq \kappa|\xi|^2, \quad \forall\,(t,x,\xi) \in [0,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi.$$

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It is well known that the Cauchy problem (1.1), (1.2) is not well-posed in the sense of Hadamard [10,11]. On the one side the smoothing effect of parabolic operators prevents existence results backward in time in any reasonable function space, and on the other side relatively elementary examples show that uniqueness is also not valid without additional assumptions on the solutions and on the operator (see [17]; for a more precise discussion on uniqueness of the solutions to the Cauchy problem for a backward-parabolic equation we quote the papers [5,6,8,14,16]).

In the celebrated paper [13], John introduced the notion of well-behaved problem in which also not well-posed problems can be included: roughly speaking a problem is well-behaved if its solutions in a space $\mathcal{H}$ depend continuously on the data belonging to a space $\mathcal{K}$, provided the solutions satisfy a prescribed bound in possibly another space $\mathcal{H}'$. This property goes also under the name of conditional stability.

The well-behavedness for (1.1), (1.2) in the space

$$\mathcal{H} = C^0([0,T], L^2(\mathbb{R}^n_x)) \cap C^0([0,T), H^1(\mathbb{R}^n_x)) \cap C^1([0,T), L^2(\mathbb{R}^n_x)) \tag{1.3}$$

with continuous dependence with respect to the data in $L^2(\mathbb{R}^n_x)$, can be deduced with the so called logarithmic convexity of the norm of the solutions to (1.1), as proved by Agmon and Nirenberg in [1]. A similar result was obtained by Glagoleva in [9] and in a more precise and general form by Hurd in [12]. Hurd's result can be summarized as follows:

suppose that the coefficients a_{jk} are Lipschitz-continuous; for every $T' \in (0,T)$ and $D > 0$, there exist $\rho > 0$, $\delta \in (0,1)$ and $M > 0$ such that if $u \in \mathcal{H}$ ($\mathcal{H}$ defined in (1.3)) is a solution of $Pu = 0$ on $[0,T] \times \mathbb{R}^n_x$, with $\|u(0,\cdot)\|_{L^2} \le \rho$ and $\|u(t,\cdot)\|_{L^2} \le D$ for all $t \in [0,T]$, then

$$\sup_{t\in[0,T']} \|u(t,\cdot)\|_{L^2} \le M \|u(0,\cdot)\|^{\delta}_{L^2}, \tag{1.4}$$

where the constants ρ, M and δ depend only on T', D, the ellipticity constant of P and the Lipschitz constant of the coefficients with respect to t.

Hurd's proof relies on some rather complicated weighted energy estimates and it turns out that the Lipschitz-continuity of the coefficients a_{jk} is an essential requirement.

In the present paper, we are interested in relaxing the regularity hypothesis on the coefficients a_{jk}. Our starting point are the results contained in [7]. In that paper an example showed that if the coefficients a_{jk} are not Lipschitz-continuous in time, then the estimate (1.4) does not hold in general, and if the coefficients are Log-Lipschitz-continuous in time, then an estimate weaker than (1.4) is valid. However, in order to obtain this weaker estimate, a technical difficulty imposed to assume C^2-regularity for the a_{jk} with respect to the space variables.

Here we overcome this point and we remove this supplementary and unnatural requirement. Our result is the following:

suppose that the coefficients a_{jk} are Lipschitz-continuous with respect to x and Log-Lipschitz-continuous with respect to t; for every $T' \in (0,T)$, $D > 0$ and $s \in (0,1)$, there exist $\rho > 0$, $\delta \in (0,1)$ and $M, N > 0$ such that if $u \in \mathcal{H}$ is a solution of $Pu = 0$ on $[0,T] \times \mathbb{R}^n_x$, with $\|u(0,\cdot)\|_{H^{-s}} \le \rho$ and $\|u(t,\cdot)\|_{L^2} \le D$ for all $t \in [0,T]$, then

$$\sup_{t\in[0,T']} \|u(t,\cdot)\|_{L^2} \le M \exp\left(-N |\log(\|u(0,\cdot)\|_{H^{-s}})|^{\delta}\right),$$

where the constants ρ, M, N and δ depend only on T', D, s, the ellipticity constant of P, the Lipschitz constant of the coefficients with respect to x and the Log-Lipschitz constant of the coefficients with respect to t.

The main tool in proving this statement is Bony's paraproduct (see [15]).

Outline of the content. In Section 2.2, we state our main theorems and make some remarks regarding the comparison with the results of [7].

In Section 3.1, we present elements of the Littlewood–Paley theory and we develop the necessary machinery of Bony's paraproduct for our proof. After that we prove auxiliary estimates that will be crucial for the proof of our weighted energy estimate in Sections 3.3 and 3.4. Some proofs are shifted to Appendix in order to make the main results easier to read.

In Section 4, we prove the weighted energy estimate for solutions of (1.1) from which the conditional stability result in Theorem 2.4 follows. The derivation of the conditional stability result from the weighted energy estimate is shown in Section 5.

2. Results

2.1. Notation

We consider the backward-parabolic equation

$$Pu = \partial_t u + \sum_{j,k=1}^{n} \partial_{x_j}(a_{jk}(t,x)\partial_{x_k} u) = 0 \tag{2.1}$$

on the strip $[0,T] \times \mathbb{R}^n_x$. We suppose that

- for all $(t,x) \in [0,T] \times \mathbb{R}^n_x$ and for all $j,k = 1,\dots,n$,

$$a_{jk}(t,x) = a_{kj}(t,x);$$

- there exists a $\kappa \in (0,1)$ such that for all $(t,x,\xi) \in [0,T] \times \mathbb{R}^n_x \times \mathbb{R}^n_\xi$,

$$\kappa|\xi|^2 \le \sum_{j,k=1}^{n} a_{jk}(t,x)\xi_j\xi_k \le \frac{1}{\kappa}|\xi|^2; \tag{2.2}$$

- for all $j,k = 1,\dots,n$, $a_{jk} \in \mathrm{LogLip}([0,T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0,T], \mathrm{Lip}(\mathbb{R}^n_x))$.

We set

$$A_{LL} := \sup\left\{ \frac{|a_{jk}(t,x) - a_{jk}(s,x)|}{|t-s|(1+|\log|t-s||)} : j,k = 1,\dots,n,\ (t,s,x) \in [0,T]^2 \times \mathbb{R}^n_x,\ 0 < |s-t| \le T \right\},$$
$$A := \sup\{\|\partial_x^\alpha a_{jk}(t,\cdot)\|_{L^\infty} : |\alpha| \le 1,\ t \in [0,T]\}.$$

Remark 2.1. If one would like to include lower order terms in (2.1), one has to suppose that those are L^∞ with respect to t and also Lip with respect to x. The constants will then additionally depend on constants B and C similarly defined to A.

Remark 2.2. We will often use a letter, say C, to denote a generic numerical constant; and different appearances of the letter C will not necessarily denote the same numerical constant, even in the same line of text. When a constant actually depends on one of the parameters of the problem, it shall be indicated by an index. Sometimes it might be necessary to differentiate between constants so that we will count them with an index.

2.2. Main results—conditional stability and weighted energy estimates

We denote by

$$\mathcal{H} := C^0([0,T], L^2(\mathbb{R}^n_x)) \cap C^0([0,T), H^1(\mathbb{R}^n_x)) \cap C^1([0,T), L^2(\mathbb{R}^n_x))$$

the space of solutions of (2.1) for which we prove the conditional stability result.

First we restate the precise local result of [7]; we also want to compare the two estimates in the sequel. Keep in mind that in this case the constant A also contains the L^∞ norms of the second spatial derivative of the principal part coefficients.

Theorem 2.3 (*Th. 1 in [7]*). *There exists a positive constant α_1 and, setting $\sigma := \min\{T, \frac{1}{\alpha_1}\}$, $\bar\sigma = \frac{\sigma}{8}$, there exist constants ρ, δ, M and N, such that, whenever $u \in \mathcal{H}$ is a solution of (2.1) with $\|u(0,\cdot)\|_{L^2} \le \rho$, the inequality*

$$\sup_{t\in[0,\bar\sigma]} \|u(t,\cdot)\|_{L^2} \le M(1 + \|u(\sigma,\cdot)\|_{L^2}) \exp(-N(|\log(\|u(0,\cdot)\|_{L^2})|^\delta))$$

holds true.

The constant α_1 depends only on A_{LL}, A, κ and n, while the constants ρ, δ, M and N depend on A_{LL}, A, κ, n and T.

Let us stress again that the constants α_1, ρ, δ, M, N depend also on constants B and C, similar to A, if one considers also lower order terms. See Remark 2.1.

The next results improves Theorem 2.3: now the principal part coefficients are only Lipschitz continuous with respect to x.

Theorem 2.4 (*Conditional Stability (Local)*). *Let $s \in (0,1)$. There exists a positive constant α_1 and, setting $\sigma := \min\{T, \frac{1-s}{\alpha_1}\}$, $\bar\sigma = \frac{\sigma}{8}$, there exist constants ρ, δ, M and N, such that, whenever $u \in \mathcal{H}$ is a solution of (2.1) with $\|u(0,\cdot)\|_{H^{-s}} \le \rho$, the inequality*

$$\sup_{t\in[0,\bar\sigma]} \|u(t,\cdot)\|_{L^2} \le M\left(1 + \frac{1}{\sigma} \sup_{t\in\left[\frac{5}{8}\sigma, \frac{7}{8}\sigma\right]} \|u(t,\cdot)\|_{L^2}\right) \exp(-N(|\log(\|u(0,\cdot)\|_{H^{-s}})|^\delta)) \tag{2.3}$$

holds true.

The constant α_1 depends only on A_{LL}, A, κ, s and n, while the constants ρ, δ, M and N depend on A_{LL}, A, κ, s, n and T.

Iterating the local result of Theorem 2.4 a finite number of times, one obtains the following global result.

Theorem 2.5 (*Conditional Stability (Global)*). *Let $s \in (0,1)$. Then, for $T' \in (0,T)$ and $D > 0$ there exist positive constants ρ', δ', M' and N', depending only on A_{LL}, A, κ, n, s, T' and D such that if $u \in \mathcal{H}$ is a solution of (2.1) satisfying $\|u(0,\cdot)\|_{H^{-s}} \le \rho'$ and $\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^2} \le D$, the inequality*

$$\sup_{t\in[0,T']} \|u(t,\cdot)\|_{L^2} \le M' \exp\left(-N'|\log(\|u(0,\cdot)\|_{H^{-s}})|^{\delta'}\right)$$

holds true.

Remark 2.6. Theorems 2.4 and 2.5 also hold if one considers Eq. (2.1) with lower order terms. As already mentioned, one has to assume Lipschitz-regularity in x and the additional dependence of the constants on the L^∞-norm and the Lip-norm of those coefficients.

Remark 2.7. Theorems 2.4 and 2.5 are stated in the case of principal part coefficients which are log-Lipschitz continuous with respect to t and Lipschitz continuous with respect to x. It is not excluded that similar results are valid for operators having regularity with respect to the x variables which go below the Lipschitz continuity. This should be the content of further studies. In the different context of Carleman estimates similar results have been proved in [5]. In that paper the uniqueness in backward parabolic operators is shown in particular in the case that the coefficients of the principal part are log-Lipschitz in time and $\log^{1/2}$-Lipschitz in space.

2.2.1. *Weighted energy estimates*

The proof of Theorem 2.4 relies on an appropriate weighted energy estimate. The choice of the weight function is connected with the modulus of continuity with respect to t as in [7]. A similar situation occurred in [6,8], where backward-uniqueness for parabolic operators by means of suitable Carleman estimates was obtained. In both cases, the weight function was deduced as a solution of a second order non-linear ordinary differential equation.

Let us now introduce the weight function that we are going to use here. For $s > 0$, let $\mu(s) = s(1 + |\log(s)|)$. For $\tau \geq 1$, we define

$$\theta(\tau) := \int_{\frac{1}{\tau}}^{1} \frac{1}{\mu(s)} ds = \log(1 + |\log(\tau)|).$$

The function $\theta : [1, +\infty) \to [0, +\infty)$ is bijective and strictly increasing. For $y \in (0, 1]$ and $\lambda > 1$, we set $\psi_\lambda(y) = \theta^{-1}(-\lambda \log(y)) = \exp(y^{-\lambda} - 1)$ and we define

$$\Phi_\lambda(y) := -\int_y^1 \psi_\lambda(z) dz.$$

The function $\Phi_\lambda : (0, 1] \to (-\infty, 0]$ is bijective and strictly increasing; moreover, it satisfies

$$y\Phi_\lambda''(y) = -\lambda(\Phi_\lambda'(y))^2 \mu\left(\frac{1}{\Phi_\lambda'(y)}\right) = -\lambda\Phi_\lambda'(y)\left(1 + \left|\log\left(\frac{1}{\Phi_\lambda'(y)}\right)\right|\right). \tag{2.4}$$

This is the second order non-linear differential equation we mentioned above. The reason for this choice is made clear in [7, Sec. 2]. The computations in [6,8] lead to a different differential equation and consequently to a different weight. In the next lemma, we collect some properties of the functions ψ_λ and Φ_λ. The proof is left to the reader.

Lemma 2.8. *Let $\zeta > 1$. Then, for $y \in (0, 1/\zeta]$,*

$$\psi_\lambda(\zeta y) = \exp(\zeta^{-\lambda} - 1)(\psi_\lambda(y))^{\zeta^{-\lambda}}.$$

Define $\Lambda_\lambda(y) := y\Phi_\lambda(1/y)$. Then the function $\Lambda_\lambda : [1, +\infty) \to (-\infty, 0]$ is bijective and

$$\lim_{z\to-\infty} -\frac{1}{z}\psi_\lambda\left(\frac{1}{\Lambda_\lambda^{-1}(z)}\right) = +\infty.$$

With these preparations, we are ready to state the weighted energy estimate which will be needed to prove Theorem 2.4.

Proposition 2.9 (*Weighted Energy Estimate*). *Let $s \in (0, 1)$. Then, there exist positive constants $\bar\lambda > 1$, $\bar\gamma$, α_1 and $M > 0$ such that, setting $\alpha := \max\{\alpha_1, T^{-1}\}$, $\sigma := \frac{1-s}{\alpha}$, $\tau := \frac{\sigma}{4}$, letting $\beta \geq \sigma + \tau$ be a free parameter, whenever $u \in \mathcal{H}$ is a solution of Eq. (2.1), one has*

$$\begin{aligned}
&\int_0^p e^{2\gamma t} e^{-2\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)} \|u(t,\cdot)\|^2_{H^{1-s-\alpha t}} dt \\
&\leq M\left((p+\tau)e^{2\gamma p} e^{-2\beta\Phi_\lambda\left(\frac{p+\tau}{\beta}\right)} \|u(p,\cdot)\|^2_{H^{1-s-\alpha p}} + \tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right) e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} \|u(0,\cdot)\|^2_{H^{-s}}\right)
\end{aligned} \tag{2.5}$$

for all $p \in [0, \frac{7}{8}\sigma]$, $\lambda \geq \bar\lambda$ and $\gamma \geq \bar\gamma$. The constant α_1 depends only on A_{LL}, A, κ, s and n, while the constants $\bar\lambda$, $\bar\gamma$ and M depend on A_{LL}, A, κ, s, n and T.

Remark 2.10. There are two aspects to be underlined in the estimate (2.5). On the one side we were able to perform our estimate only in negative Sobolev spaces, instead of the usual L^2 framework. We notice that there was the same difficulty also in [5,8]. On the other side the energy inequality (2.5) undergoes a loss of derivatives. This essentially means that the index of the Sobolev norm of the solutions depends on time and becomes smaller and smaller while the time increases, denoting a sort of degradation of the regularity of the solutions itself. This phenomenon also occurred in [3,4] in the context of hyperbolic equations with Log-Lipschitz coefficients.

3. Littlewood–Paley theory and Bony's paraproduct

In this section, we review some elements of the Littlewood–Paley decomposition which we shall use throughout this paper to define Bony's paraproduct. The proofs which are not contained in this section can be found in [7,8,15].

3.1. Littlewood–Paley decomposition

Let $\chi \in C_0^\infty(\mathbb{R})$ with $0 \le \chi(s) \le 1$ be an even function and such that $\chi(s) = 1$ for $|s| \le 11/10$ and $\chi(s) = 0$ for $|s| \ge 19/10$. We now define $\chi_k(\xi) = \chi(2^{-k}|\xi|)$ for $k \in \mathbb{Z}$ and $\xi \in \mathbb{R}^n_\xi$. Denoting by $\mathcal{F}$ the Fourier-transform $x \to \xi$ and by $\mathcal{F}^{-1}$ its inverse, we define the operators

$$S_{-1}u = 0 \quad \text{and} \quad S_k u = \chi_k(D_x)u = \mathcal{F}^{-1}(\chi_k(\cdot)\mathcal{F}(u)(\cdot)), \quad k \ge 0,$$
$$\Delta_0 u = S_0 u \quad \text{and} \quad \Delta_k u = S_k u - S_{k-1}u, \quad k \ge 1.$$

We define

$$\operatorname{spec}(u) := \operatorname{supp}(\mathcal{F}(u))$$

and we will use the abbreviation $\Delta_k u = u_k$. For $u \in \mathcal{S}'(\mathbb{R}^n_x)$, we have

$$u = \lim_{k\to+\infty} S_k u = \sum_{k\ge 0} \Delta_k u$$

in the sense of $\mathcal{S}'(\mathbb{R}^n_x)$.

We shall make use of the classical:

Proposition 3.1 (*Bernstein's Inequalities*)**.** *Let $u \in \mathcal{S}'(\mathbb{R}^n_x)$. Then, for $k \ge 1$,*

$$2^{k-1}\|u_k\|_{L^2} \le \|\nabla_x u_k\|_{L^2} \le 2^{k+1}\|u_k\|_{L^2}. \tag{3.1}$$

The right inequality of (3.1) holds also for $k = 0$.

In the following two propositions, we recall the characterization of the classical Sobolev spaces and Lipschitz-continuous functions via Littlewood–Paley decomposition.

Proposition 3.2. *Let $s \in \mathbb{R}$. Then, a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n_x)$ belongs to $H^s(\mathbb{R}^n_x)$ iff the following two conditions hold:*
(i) *for all $k \ge 0$, $\Delta_k u \in L^2(\mathbb{R}^n_x)$,*
(ii) *the sequence $\{\delta_k\}_{k\in\mathbb{N}}$, where $\delta_k := 2^{ks}\|\Delta_k u\|_{L^2}$, belongs to $l^2(\mathbb{N})$.*
Moreover, there exists $C_s \ge 1$ such that, for all $u \in H^s(\mathbb{R}^n_x)$, we have

$$\frac{1}{C_s}\|u\|_{H^s} \le \|\{\delta_k\}_k\|_{l^2} \le C_s\|u\|_{H^s}.$$

Proposition 3.3. *Let $s \in \mathbb{R}$ and $R > 2$. If a sequence $\{u_k\}_{k\in\mathbb{N}} \subseteq L^2(\mathbb{R}^n_x)$ satisfies*
(i) $\operatorname{spec}(u_0) \subseteq \{|\xi| \le R\}$ *and* $\operatorname{spec}(u_k) \subseteq \{R^{-1}2^k \le |\xi| \le R2^k\}$, *for all $k \ge 1$,*
(ii) *the sequence $\{\delta_k\}_{k\in\mathbb{N}}$, where $\delta_k := 2^{ks}\|u_k\|_{L^2}$, belongs to $l^2(\mathbb{N})$,*
then $u = \sum_{k\ge 0} u_k \in H^s(\mathbb{R}^n_x)$ and there exists $C_s \ge 1$ such that

$$\frac{1}{C_s}\|u\|_{H^s} \le \|\{\delta_k\}_k\|_{l^2} \le C_s\|u\|_{H^s}.$$

In the previous statement, if $s > 0$ then, instead of (i), *it is enough to assume that*
(i') $\operatorname{spec}(u_k) \subseteq \{|\xi| \le R2^k\}$, *for all $k \ge 0$.*

Proposition 3.4. *A function $a \in L^\infty(\mathbb{R}^n_x)$ belongs to $\operatorname{Lip}(\mathbb{R}^n_x)$ iff*

$$\sup_{k\in\mathbb{N}} \|\nabla_x(S_k a)\|_{L^\infty} < +\infty.$$

Moreover, if $a \in \operatorname{Lip}(\mathbb{R}^n_x)$, there exists a positive constant C such that

$$\|\Delta_k a\|_{L^\infty} \le C2^{-k}\|a\|_{\mathrm{Lip}}, \quad \text{and} \quad \|\nabla_x(S_k a)\|_{L^\infty} \le C\|a\|_{\mathrm{Lip}}.$$

3.2. Bony's (modified) paraproduct

Let $a \in L^\infty(\mathbb{R}^n_x)$. Bony's paraproduct of a with $u \in H^s(\mathbb{R}^n_x)$ is defined as

$$T_a u = \sum_{k\ge 3} S_{k-3}a\,\Delta_k u.$$

For the proof of our conditional stability result it is essential that T_a is a positive operator. Unfortunately, this is not implied by $a(x) \geq \kappa > 0$. Therefore, we have to modify the paraproduct a little bit. We introduce the operator

$$T_a^m u = S_{m-1} a S_{m+2} u + \sum_{k \geq m+3} S_{k-3} a \Delta_k u, \tag{3.2}$$

where $m \in \mathbb{N}$; note $T_a^0 = T_a$. As it shall be shown, the operator T_a^m is a positive operator for positive a, provided that m is sufficiently large. The proofs of the subsequent propositions can be found in [8].

Proposition 3.5. *Let $m \in \mathbb{N}$, $s \in \mathbb{R}$ and $a \in L^\infty(\mathbb{R}_x^n)$. Then, T_a^m maps $H^s(\mathbb{R}_x^n)$ continuously into $H^s(\mathbb{R}_x^n)$, i.e. there exists a constant $C_{m,s} > 0$ such that*

$$\|T_a^m u\|_{H^s} \leq C_{m,s} \|a\|_{L^\infty} \|u\|_{H^s}.$$

If $m \in \mathbb{N}_{\geq 3}$, $s \in (0, 1)$ and $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$, then $a - T_a^m$ maps $H^{-s}(\mathbb{R}_x^n)$ continuously into $H^{1-s}(\mathbb{R}_x^n)$, i.e. there exists a constant $C_{m,s} > 0$ such that

$$\|au - T_a^m u\|_{H^{1-s}} \leq C_{m,s} \|a\|_{\mathrm{Lip}} \|u\|_{H^{-s}}.$$

The constant $C_{m,s}$ is independent of s, if s is chosen in a compact subset of $(0, 1)$.

We state the previously recalled positivity result for T_a^m.

Proposition 3.6. *Let $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$ and suppose that $a(x) \geq \kappa > 0$ for all $x \in \mathbb{R}_x^n$. Then, there exists an integer $m_0 = m_0(\kappa, \|a\|_{\mathrm{Lip}})$ such that*

$$\mathrm{Re} \left\langle T_a^m u | u \right\rangle_{L^2} \geq \frac{\kappa}{2} \|u\|_{L^2},$$

for all $u \in L^2(\mathbb{R}_x^n)$ and $m \geq m_0$. A similar result is true for vector-valued functions, if a is replaced by a positive symmetric matrix.

The next proposition is needed since T_a^m is not self-adjoint. However, the operator $(T_a^m - (T_a^m)^*)\partial_{x_j}$ is of order 0 and maps, if a is Lipschitz, L^2 continuously into L^2.

Proposition 3.7. *Let $m \in \mathbb{N}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$ and $u \in L^2(\mathbb{R}_x^n)$. Then, there exists a constant $C_m > 0$ such that*

$$\|(T_a^m - (T_a^m)^*)\partial_{x_j} u\|_{L^2} \leq C_m \|a\|_{\mathrm{Lip}} \|u\|_{L^2}.$$

3.3. *Auxiliary estimates for $a - T_a^m$*

Let $m \geq 3$. We set

$$(a - T_a^m)w = \sum_{k \geq m} \Delta_k a S_{k-3} w + \sum_{k \geq m} \left(\sum_{|j-k| \leq 2} \Delta_k a \Delta_j w \right) := \Omega_1 w + \Omega_2 w. \tag{3.3}$$

For our proof of the weighted energy estimate, from which we derive the conditional stability result, we need some estimates for terms involving $\Delta_\nu((a - T_a^m)w)$. To handle these terms, we introduce a second Littlewood–Paley decomposition depending on a parameter μ and we look at $\sum_{\mu \geq 0} \Delta_\nu((a - T_a^m)w_\mu)$. To derive estimates for those terms, we need appropriate estimates for $\Delta_\nu \Omega_1 w_\mu$ and $\Delta_\nu \Omega_2 w_\mu$. Let us first analyze the spectra of $\Delta_\nu \Omega_1 w$ and $\Delta_\nu \Omega_2 w$. From the definition of S_k and Δ_k in Section 3.1 we see that

$$\mathrm{spec}(\Delta_k a S_{k-3} w) \subseteq \{2^{k-2} \leq |\xi| \leq 2^{k+2}\}$$

and, therefore,

$$\Delta_\nu \Omega_1 w = \sum_{\substack{k \geq m \\ |k-\nu| \leq 2}} \Delta_\nu(\Delta_k a S_{k-3} w)$$

since $\Delta_\nu(\Delta_k a S_{k-3} w) \equiv 0$ for $|\nu - k| \geq 3$. Replacing now w by w_μ, we get

$$\mathrm{spec}(S_{k-3} w_\mu) \subseteq \begin{cases} \emptyset & : \quad k \leq \mu + 1, \\ \{2^{\mu-1} \leq |\xi| \leq 2^{\mu+1}\} & : \quad k \geq \mu + 2, \end{cases}$$

and, from this,

$$\mathrm{spec}(\Delta_k a S_{k-3} w_\mu) \subseteq \begin{cases} \emptyset & : \quad k \leq \mu + 1, \\ \{|\xi| \leq 2^{k+2}\} & : \quad k = \mu + 2, \\ \{2^{k-2} \leq |\xi| \leq 2^{k+2}\} & : \quad k \geq \mu + 3. \end{cases}$$

With this we get

$$\Delta_\nu \Omega_1 w_\mu = \sum_{\substack{k\geq \max\{m,\,\mu+2\}\\ |\nu-k|\leq 2}} \Delta_\nu(\Delta_k a S_{k-3} w_\mu).$$

Further, we also get $\Delta_\nu \Omega_1 w_\mu \equiv 0$ for $\nu \leq \mu - 1$. Now we look at $\Delta_\nu \Omega_2 w_\mu$. We have

$$\begin{aligned}\Delta_\nu \Omega_2 w_\mu &= \Delta_\nu \left(\sum_{k\geq m} \left(\sum_{|j-k|\leq 2} \Delta_k a \Delta_j w_\mu \right)\right) \\ &= \Delta_\nu \left(\sum_{k\geq m} \left(\sum_{\substack{|\mu-j|\leq 1\\ |j-k|\leq 2}} \Delta_k a \Delta_j w_\mu \right)\right)\end{aligned}$$

since

$$\operatorname{spec}(\Delta_j(\Delta_\mu w)) \subseteq \begin{cases} \emptyset & : \quad |j-\mu| \geq 2, \\ \{2^{\mu-1} \leq |\xi| \leq 2^{\mu+1}\} & : \quad |j-\mu| \leq 1. \end{cases}$$

From that we get

$$\Delta_\nu \Omega_2 w_\mu = \Delta_\nu \left(\sum_{|\mu-j|\leq 1} \left(\sum_{\substack{k\geq m\\ |k-j|\leq 2}} \Delta_k a \Delta_j(\Delta_\mu w) \right)\right) \tag{3.4}$$

with

$$\operatorname{spec}(\Delta_\nu \Omega_2 w_\mu) \subseteq \{|\xi| \leq 2^{\mu+5}\}, \quad \nu \leq \mu + 5.$$

For all $\nu \geq \mu + 6$ we have $\Delta_\nu \Omega_2 w_\mu \equiv 0$.

We prove now some technical lemmas which we will use later on.

Lemma 3.8. *Let $s' \in (0, 1)$, $m \in \mathbb{N}$, $a \in L^\infty(\mathbb{R}^n_x) \cap \operatorname{Lip}(\mathbb{R}^n_x)$ and $w \in L^2(\mathbb{R}^n_x)$. Then, there exist a constant $C > 0$ and a sequence $\{c_\nu^{(\mu)}\}_{\nu\in\mathbb{N}} \in l^2(\mathbb{N})$, depending on $\Delta_\mu w$, with $\|\{c_\nu^{(\mu)}\}_\nu\|_{l^2} \leq 1$ for all $\mu \geq 0$, such that*

$$\|\Delta_\nu \Omega_1 w_\mu\|_{L^2} \leq C \|a\|_{\operatorname{Lip}} 2^{-\nu(1-s')} c_\nu^{(\mu)} \|w_\mu\|_{H^{-s'}}. \tag{3.5}$$

Proof. From our considerations above we have that

$$\Delta_\nu \Omega_1 w_\mu = \sum_{|k-\nu|\leq 2} \Delta_\nu(\Delta_k a S_{k-3} w_\mu)$$

and, therefore,

$$\begin{aligned}\|\Delta_\nu(\Omega_1 w_\mu)\|_{L^2} &\leq \sum_{|k-\nu|\leq 2} \|\Delta_k a S_{k-3} w_\mu\|_{L^2} \\ &\leq \sum_{|k-\nu|\leq 2} \|\Delta_k a\|_{L^\infty} \|S_{k-3} w_\mu\|_{L^2} \\ &\leq C \sum_{|k-\nu|\leq 2} \|a\|_{\operatorname{Lip}} 2^{-k} \sum_{j\leq k} \|\Delta_j w_\mu\|_{L^2} \\ &= C\|a\|_{\operatorname{Lip}} \sum_{|k-\nu|\leq 2} 2^{-k} \sum_{j\leq k} 2^{ks'} 2^{-ks'} 2^{js'} \underbrace{2^{-js'} \|\Delta_j w_\mu\|_{L^2}}_{:=\varepsilon_j^{(\mu)}} \\ &\leq C\|a\|_{\operatorname{Lip}} \sum_{|k-\nu|\leq 2} 2^{-(1-s')k} \underbrace{\sum_{j\leq k} 2^{-(k-j)s'} \varepsilon_j^{(\mu)}}_{:=f_k^{(\mu)}} \\ &= C\|a\|_{\operatorname{Lip}} \sum_{|k-\nu|\leq 2} 2^{-(1-s')k} f_k^{(\mu)} \\ &\leq C\|a\|_{\operatorname{Lip}} 2^{-(1-s')\nu} \sum_{|k-\nu|\leq 2} f_k^{(\mu)},\end{aligned} \tag{3.6}$$

where $\{\varepsilon_j^{(\mu)}\}_{j\in\mathbb{N}} \in l^2(\mathbb{N})$ with $\|\{\varepsilon_j^{(\mu)}\}\|_{l^2} \approx \|w_\mu\|_{H^{-s'}}$; see Proposition 3.2. The sequence $\{f_k^{(\mu)}\}_{k\in\mathbb{N}}$ is a convolution of the sequences $\{\varepsilon_j^{(\mu)}\}_{j\in\mathbb{N}}$ and $d_k := 2^{-ks'}$. Using Young's inequality, we obtain

$$\|\{f_k^{(\mu)}\}_k\|_{l^2} = \|\{\{\varepsilon_j^{(\mu)}\} *_{(j)} \{d_k\}\}_k\|_{l^2} \le \|\{d_k\}_k\|_{l^1} \|\{\varepsilon_j^{(\mu)}\}_j\|_{l^2}.$$

From the formula of the geometric series and the integral criterion, we obtain

$$\|\{d_k\}_k\|_{l^1} \le \frac{1}{1-2^{-s'}} \le \frac{C}{s'}$$

and, hence,

$$\|\{f_k^{(\mu)}\}_k\|_{l^2} \le \frac{C}{s'}\|w_\mu\|_{H^{-s'}}.$$

We define

$$c_\nu := \frac{f_{\nu-2}^{(\mu)} + f_{\nu-1}^{(\mu)} + f_\nu^{(\mu)} + f_{\nu+1}^{(\mu)} + f_{\nu+2}^{(\mu)}}{C_{s'}\|w_\mu\|_{H^{-s'}}},$$

where $C_{s'}$ can be chosen such that $\sum_{\nu\ge 0}(c_\nu^{(\mu)})^2 \le 1$. With this, we get from (3.6)

$$\|\Delta_\nu \Omega_1 w_\mu\|_{L^2} \le C\|a\|_{\mathrm{Lip}} 2^{-(1-s')\nu} c_\nu^{(\mu)} \|w_\mu\|_{H^{-s'}}. \quad \square$$

The next lemma deals with the estimate of $\Delta_\nu \Omega_2 w$.

Lemma 3.9. *Let $m \in \mathbb{N}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$ and $w \in L^2(\mathbb{R}_x^n)$. Then, there exist a constant $C > 0$ and a sequence $\{\tilde{c}_\nu^{(\mu)}\}_{\nu\in\mathbb{N}} \in l^2(\mathbb{N})$, depending on $\Delta_\mu w$, with $\|\{\tilde{c}_\nu^{(\mu)}\}_\nu\|_{l^2} \le 1$ for all $\mu \ge 1$, such that*

$$\|\Delta_\nu \Omega_2 w_\mu\|_{L^2} \le C\|a\|_{\mathrm{Lip}} \tilde{c}_\nu^{(\mu)} 2^{-\mu}\|w_\mu\|_{L^2}.$$

Proof. Straightforward computations on (3.4) show that $\Omega_2 w \in L^2(\mathbb{R}_x^n)$ if $w \in L^2(\mathbb{R}_x^n)$. Hence, there exists a sequence $\{c_\nu^{(\mu)}\}_{\nu\in\mathbb{N}}$, depending on w_μ, with $\|\{c_\nu^{(\mu)}\}_\nu\|_{l^2} \approx \|\Omega_2 w_\mu\|_{L^2}$. From (3.4), we obtain

$$\begin{aligned}
\|\Delta_\nu \Omega_2 w_\mu\|_{L^2} &\le \tilde{c}_\nu^{(\mu)} \|\Omega_2 w_\mu\|_{L^2} \\
&\le \tilde{c}_\nu^{(\mu)} \sum_{|\mu-j|\le 1} \sum_{\substack{k\ge m \\ |k-j|\le 2}} \|\Delta_k a \Delta_j(\Delta_\mu w)\|_{L^2} \\
&\le \tilde{c}_\nu^{(\mu)} \sum_{|j-\mu|\le 1} \sum_{\substack{k\ge m \\ |j-k|\le 2}} 2^{-k}\|a\|_{\mathrm{Lip}}\|w_\mu\|_{L^2} \\
&\le \|a\|_{\mathrm{Lip}} \tilde{c}_\nu^{(\mu)} 2^{-\mu}\|w_\mu\|_{L^2},
\end{aligned}$$

where $\tilde{c}_\nu^{(\mu)} = c_\nu^{(\mu)}/\|\Omega_2 w_\mu\|_{L^2}$. By construction we have $\sum_{\nu\ge 0}(\tilde{c}_\nu^{(\mu)})^2 \le 1$ for all $\mu \ge 0$. This concludes the proof. $\square$

The next proposition is an essential tool in our proof and contains information about the behavior of the Littlewood–Paley pieces of $(a - T_a)w$.

Proposition 3.10. *Let $s \in (0, 1)$, $m \in \mathbb{N}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$, $\alpha > 0$ and $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$. Then there exists a constant $C > 0$ such that, for all $w \in \mathcal{H}$, we have*

$$\sum_{\nu\ge 0} 2^{-(s+\alpha t)\nu} \left\langle \partial_{x_i}\partial_t v_\nu(t,\cdot) | \Delta_\nu((a - T_a^m)\partial_{x_j} w(t,\cdot))\right\rangle_{L^2} \le \frac{1}{N}\sum_{\nu\ge 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + CN\|a\|_{\mathrm{Lip}}^2 \sum_{\nu\ge 0} 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2$$

for every $N > 0$ and with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

The proof of this proposition can be found in the Appendix. Following the same ideas one can also prove

Proposition 3.11. *Let $s \in (0, 1)$, $m \in \mathbb{N}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$, $\alpha > 0$ and $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$. Then there exists a constant $C > 0$ such that, for all $w \in \mathcal{H}$, we have*

$$\sum_{\nu\ge 0} 2^{-(s+\alpha t)\nu}\nu \left\langle \partial_{x_i} v_\nu(t,\cdot) | \Delta_\nu((a - T_a^m)\partial_{x_j} w(t,\cdot))\right\rangle_{L^2} \le C\|a\|_{\mathrm{Lip}} \sum_{\nu\ge 0} 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2$$

with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

3.4. Auxiliary estimates for $[\Delta_\nu, T_a^m]$

The next result about commutation will also be crucial in our proof of the weighted energy estimate (2.5). Results on commutation play an essential role also in the proof of Carleman estimates for (1.1) with low-regular coefficients in [8] and in the proof of well-posedness for hyperbolic equations with low-regular coefficients in [3].

Proposition 3.12. *Let $m \in \mathbb{N}_{\geq 3}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$ and $s \in (0,1)$. Then, for $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$ there exists a constant $C_m > 0$ such that, for all $w \in \mathcal{H}$,*

$$\sum_{\nu\geq 0} 2^{-(s+\alpha t)\nu} \left\langle \partial_t \partial_{x_j} v_\nu(t,\cdot) | [\Delta_\nu, T_a^m]\partial_{x_h} w(t,\cdot)\right\rangle_{L^2} \leq \frac{1}{N}\sum_{\nu\geq 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + \frac{C_m}{1-s}\|a\|_{\mathrm{Lip}}^2 N \sum_{\nu\geq 0} 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2,$$

for every $N > 0$ and with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

This follows from the following lemma whose proof can be found in the Appendix.

Lemma 3.13. *Let $m \in \mathbb{N}_{\geq 3}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$. Then there exists a constant $C_m > 0$ such that, for all $w \in \mathcal{H}$,*

$$\sum_{\nu\geq 0} 2^{-2(s+\alpha t)\nu} \left\| \partial_{x_j}[\Delta_\nu, T_a^m]\partial_{x_h} w(t,\cdot)\right\|_{L^2}^2 \leq \frac{C_m}{1-s}\|a\|_{\mathrm{Lip}}^2 \sum_{\nu\geq 0} 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2,$$

with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

Also the next proposition follows immediately from this lemma.

Proposition 3.14. *Let $m \in \mathbb{N}_{\geq 3}$, $a \in L^\infty(\mathbb{R}_x^n) \cap \mathrm{Lip}(\mathbb{R}_x^n)$ and $s \in (0,1)$. Then, for $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$ there exists a constant $C_m > 0$ such that, for all $w \in H^{1-s-\alpha t}(\mathbb{R}_x^n)$,*

$$\sum_{\nu\geq 0} 2^{-(s+\alpha t)\nu} \nu \left\langle \partial_{x_j} v_\nu(t,\cdot) | [\Delta_\nu, T_a^m]\partial_{x_h} w(t,\cdot)\right\rangle_{L^2} \leq \frac{C_m}{1-s}\|a\|_{\mathrm{Lip}} \sum_{\nu\geq 0} 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2,$$

with $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$.

4. Proof of Proposition 2.9

In order to simplify the presentation, we shall write the proof only for $n = 1$. As already mentioned, one may also include lower-order terms with the appropriate regularity in x; see Section 2.2. The latter can be handled with the techniques of the present work following the scheme of [7].

To make the proof more readable, we divide it into several steps. First the operator will be transformed by a change of variables involving the weight function, and then we shall introduce the paraproduct and microlocalize the operator. After that, we shall use the estimates of Section 3.2 and conclude the proof for $\nu = 0$ and $\nu \geq 1$ separately. After that, in Section 5, we shall show how the stability estimate follows from the energy estimate.

4.1. Preliminaries—transformation, microlocalization, approximation

Let $u \in \mathcal{H}$ be a solution of the equation

$$Pu = \partial_t u + \partial_x(a(t,x)\partial_x u) = 0$$

on the strip $[0,T] \times \mathbb{R}_x$. In what follows, $\alpha_1 > 0$, $\bar\lambda > 1$ and $\bar\gamma > 0$ are constants to be determined later. Set $\alpha := \max\{\alpha_1, T^{-1}\}$, take $s \in (0,1)$, and set $\sigma := \frac{1-s}{\alpha}$, $\tau := \frac{\sigma}{4}$. For $\gamma \geq \bar\gamma$, $\lambda \geq \bar\lambda$ and $\beta \geq \sigma + \tau$, define $w(t,x) = e^{\gamma t} e^{-\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)} u(t,x)$. Then w satisfies the following equation:

$$w_t - \gamma w + \Phi_\lambda'\left(\frac{t+\tau}{\beta}\right) w + \partial_x(a(t,x)\partial_x w) = 0.$$

Now we add and subtract $\partial_x T_a^m \partial_x w$, with T_a^m as defined in (3.2), and obtain

$$w_t - \gamma w + \Phi_\lambda'\left(\frac{t+\tau}{\beta}\right) w + \partial_x(T_a^m\partial_x w) + \partial_x((a - T_a^m)\partial_x w) = 0. \tag{4.1}$$

We set $u_\nu = \Delta_\nu u$, $w_\nu = \Delta_\nu w$ and $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$. The function v_ν satisfies

$$\begin{aligned}\partial_t v_\nu &= \gamma v_\nu - \Phi_\lambda'\left(\frac{t+\tau}{\beta}\right) v_\nu - \partial_x(T_a^m \partial_x v_\nu) - \alpha\log(2)\nu v_\nu \\ &\quad - 2^{-(s+\alpha t)\nu}\partial_x([\Delta_\nu, T_a^m]\partial_x w) - 2^{-(s+\alpha t)\nu}\Delta_\nu\partial_x((a - T_a^m)\partial_x w).\end{aligned} \tag{4.2}$$

Next, we form the scalar product of (4.2) with $(t+\tau)\partial_t v_\nu$ and obtain

$$
\begin{aligned}
(t+\tau)\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 =\ & \gamma(t+\tau)\,\langle v_\nu|\partial_t v_\nu(t,\cdot)\rangle_{L^2} - (t+\tau)\left\langle \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& -(t+\tau)\left\langle \partial_x(T_a^m\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} - \alpha\log(2)(t+\tau)\nu\,\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} \\
& -(t+\tau)2^{-(s+\alpha t)\nu}\left\langle \partial_x([\Delta_\nu, T_a^m]\partial_x w(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& -(t+\tau)2^{-(s+\alpha t)\nu}\left\langle \Delta_\nu\partial_x((a-T_a^m)\partial_x w(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2}. \qquad (4.3)
\end{aligned}
$$

To proceed, we shall regularize the coefficient $a(t,x)$ with respect to t. Therefore, we pick an even, non-negative $\rho\in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}(\rho)\subseteq[-\frac{1}{2},\frac{1}{2}]$ and $\int_{\mathbb{R}}\rho(s)ds=1$. For $\varepsilon\in(0,1]$, we set

$$
a_\varepsilon(t,x) = \frac{1}{\varepsilon}\int_{\mathbb{R}} a(s,x)\rho\left(\frac{t-s}{\varepsilon}\right)ds.
$$

A straightforward computation shows that for all $\varepsilon\in(0,1]$, we have

$$
a_\varepsilon(t,x)\ge a_0>0, \qquad (4.4)
$$

$$
|a_\varepsilon(t,x)-a(t,x)|\le A_{LL}\varepsilon(|\log(\varepsilon)|+1), \qquad (4.5)
$$

as well as

$$
|\partial_t a_\varepsilon(t,x)|\le A_{LL}\|\rho'\|_{L^1}(|\log(\varepsilon)|+1),
$$

for all $(t,x)\in[0,T]\times\mathbb{R}_x$. From these properties of $a_\varepsilon(t,x)$, the fact that $T_{a+b}=T_a+T_b$ and Proposition 3.5, we immediately get:

Lemma 4.1. *Let $m\in\mathbb{N}$ and $u\in L^2(\mathbb{R}_x^n)$. Then*

$$
\|(T_a^m-T_{a_\varepsilon}^m)u\|_{L^2}\le C_m A_{LL}\varepsilon(|\log(\varepsilon)|+1)\|u\|_{L^2}
$$

and

$$
\|T_{\partial_t a_\varepsilon}^m u\|_{L^2}\le C_m A_{LL}\|\rho'\|_{L^1}(|\log(\varepsilon)|+1)\|u\|_{L^2}
$$

hold.

We introduce

$$
a_\nu(t,x) := a_\varepsilon(t,x), \quad \text{with } \varepsilon = 2^{-2\nu}.
$$

We replace T_a^m by $T_{a_\nu}^m + T_a^m - T_{a_\nu}^m$ in the third term of the right hand side of (4.3) and we obtain

$$
\begin{aligned}
(t+\tau)\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 =\ & \gamma(t+\tau)\,\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} - (t+\tau)\left\langle \Phi'_\lambda\left(\frac{t+\tau}{\beta}\right) v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& -(t+\tau)\left\langle \partial_x(T_{a_\nu}^m\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& -(t+\tau)\left\langle \partial_x((T_a^m-T_{a_\nu}^m)\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& -\alpha\log(2)(t+\tau)\nu\,\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} \\
& -(t+\tau)2^{-(s+\alpha t)\nu}\left\langle \partial_x([\Delta_\nu, T_a^m]\partial_x w(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& -(t+\tau)2^{-(s+\alpha t)\nu}\left\langle \Delta_\nu\partial_x((a-T_a^m)\partial_x w(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2}. \qquad (4.6)
\end{aligned}
$$

Now we replace $\partial_t v_\nu(t,\cdot)$ in

$$
-\alpha\log(2)(t+\tau)\nu\,\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2}
$$

by the expression on the right hand side of (4.2) and we obtain

$$
\begin{aligned}
-\alpha\log(2)(t+\tau)\nu\,\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} =\ & -\alpha\gamma\log(2)\nu(t+\tau)\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& +\alpha\log(2)(t+\tau)\Phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\nu\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& +\alpha\log(2)(t+\tau)\nu\left\langle v_\nu(t,\cdot)|\partial_x T_a^m\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \\
& +\alpha^2(\log(2))^2(t+\tau)\nu^2\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& +\alpha\log(2)\nu 2^{-(s+\alpha t)\nu}(t+\tau)\left\langle v_\nu(t,\cdot)|\partial_x\big([\Delta_\nu, T_a^m]\partial_x w(t,\cdot)\big)\right\rangle_{L^2} \\
& +\alpha\log(2)\nu 2^{-(s+\alpha t)\nu}(t+\tau)\left\langle v_\nu(t,\cdot)|\Delta_\nu\partial_x\big((a-T_a^m)\partial_x w(t,\cdot)\big)\right\rangle_{L^2}. \qquad (4.7)
\end{aligned}
$$

Taking into account (4.6) and (4.7), it follows

$$
\begin{aligned}
(t+\tau)\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 =\ & \gamma(t+\tau)\,\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} - (t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} \\
& - (t+\tau)\left\langle \partial_x(T^m_{a_\nu}\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} - (t+\tau)\left\langle \partial_x((T^m_a - T^m_{a_\nu})\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& + \alpha\log(2)(t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\nu\|v_\nu(t,\cdot)\|_{L^2}^2 + \alpha\log(2)(t+\tau)\nu\left\langle v_\nu(t,\cdot)|\partial_x T^m_a\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \\
& + \alpha^2(\log(2))^2(t+\tau)\nu^2\|v_\nu(t,\cdot)\|_{L^2}^2 - \alpha\gamma\log(2)(t+\tau)\nu\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& + \alpha\log(2)(t+\tau)\nu 2^{-(s+\alpha t)\nu}\left\langle v_\nu(t,\cdot)|\partial_x\big([\Delta_\nu, T^m_a]\partial_x w(t,\cdot)\big)\right\rangle_{L^2} \\
& + \alpha\log(2)(t+\tau)\nu 2^{-(s+\alpha t)\nu}\left\langle v_\nu(t,\cdot)|\Delta_\nu\partial_x\big((a-T^m_a)\partial_x w(t,\cdot)\big)\right\rangle_{L^2} \\
& - (t+\tau)2^{-(s+\alpha t)\nu}\left\langle \partial_x([\Delta_\nu, T^m_a]\partial_x w(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& - (t+\tau)2^{-(s+\alpha t)\nu}\left\langle \Delta_\nu\partial_x((a-T^m_a)\partial_x w(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2}.
\end{aligned}
$$

Integration by parts with respect to t yields

$$
\gamma(t+\tau)\,\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} = \frac{\gamma}{2}\frac{d}{dt}\Big((t+\tau)\|v_\nu(t,\cdot)\|_{L^2}^2\Big) - \frac{\gamma}{2}\|v_\nu(t,\cdot)\|_{L^2}^2
$$

and

$$
\begin{aligned}
& -(t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\langle v_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2} \\
& \quad = -\frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2\right) + \frac{1}{2}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2(\mathbb{R}^n)}^2.
\end{aligned}
$$

Next, we investigate the term $-(t+\tau)\left\langle \partial_x(T^m_{a_\nu}\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2}$. From (3.2) it can be seen that $\partial_t T^m_{a_\nu} = T^m_{\partial_t a_\nu} + T^m_a\partial_t$. A straightforward computation shows that

$$
\begin{aligned}
-(t+\tau)\left\langle \partial_x(T^m_{a_\nu}\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} =\ & \frac{1}{2}\frac{d}{dt}\Big((t+\tau)\left\langle T^m_{a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2}\Big) \\
& - \frac{1}{2}\left\langle T^m_{a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \\
& - \frac{1}{2}(t+\tau)\left\langle T^m_{\partial_t a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \\
& - \frac{1}{2}(t+\tau)\left\langle \partial_t\partial_x v_\nu(t,\cdot)|((T^m_{a_\nu})^* - T^m_{a_\nu})\partial_x v_\nu(t,\cdot)\right\rangle_{L^2}.
\end{aligned}
$$

Therefore, we have the following equality:

$$
\begin{aligned}
(t+\tau)\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 =\ & \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_\nu(t,\cdot)\|_{L^2}^2\right) - \frac{\gamma}{2}\|v_\nu(t,\cdot)\|_{L^2}^2 - \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2\right) \\
& + \frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& - (t+\tau)\left\langle \partial_x((T^m_a - T^m_{a_\nu})\partial_x v_\nu(t,\cdot))|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} \\
& + \frac{1}{2}\frac{d}{dt}\left((t+\tau)\left\langle T^m_{a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2}\right) - \frac{1}{2}\left\langle T^m_{a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \\
& - \frac{1}{2}(t+\tau)\left\langle T^m_{\partial_t a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \\
& - \frac{1}{2}(t+\tau)\left\langle \partial_t\partial_x v_\nu(t,\cdot)|((T^m_{a_\nu})^* - T^m_{a_\nu})\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \\
& - \alpha\gamma\log(2)(t+\tau)\nu\|v_\nu(t,\cdot)\|_{L^2}^2 + \alpha\log(2)(t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\nu\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& - \alpha\log(2)(t+\tau)\nu\left\langle \partial_x v_\nu(t,\cdot)|T^m_a\partial_x v_\nu(t,\cdot)\right\rangle_{L^2} + \alpha^2(\log(2))^2(t+\tau)\nu^2\|v_\nu(t,\cdot)\|_{L^2}^2 \\
& + \alpha\log(2)\nu 2^{-(s+\alpha t)\nu}(t+\tau)\,\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\rangle_{L^2} \\
& - (t+\tau)2^{-(s+\alpha t)\nu}\,\langle \mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2},
\end{aligned}
\tag{4.8}
$$

where we have set

$$
\mathcal{X}_\nu(t,\cdot) := \partial_x([\Delta_\nu, T^m_a]\partial_x w(t,\cdot)) + \Delta_\nu(\partial_x((a-T^m_a)\partial_x w(t,\cdot))).
$$

4.2. Estimates for $\nu = 0$

Setting $\nu = 0$, we get from (4.8)

$$
\begin{aligned}
(t+\tau)\|\partial_t v_0(t,\cdot)\|_{L^2}^2 &= \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_0(t,\cdot)\|_{L^2}^2\right) - \frac{\gamma}{2}\|v_0(t,\cdot)\|_{L^2}^2 - \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2\right) \\
&+ \frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2 \\
&- (t+\tau)\left\langle \partial_x((T_a^m - T_{a_0}^m)\partial_x v_0(t,\cdot)) | \partial_t v_0(t,\cdot)\right\rangle_{L^2} \\
&+ \frac{1}{2}\frac{d}{dt}\left((t+\tau)\left\langle T_{a_0}^m \partial_x v_0(t,\cdot)|\partial_x v_0(t,\cdot)\right\rangle_{L^2}\right) \\
&- \frac{1}{2}\left\langle T_{a_0}^m \partial_x v_0(t,\cdot)|\partial_x v_0(t,\cdot)\right\rangle_{L^2} - \frac{1}{2}(t+\tau)\left\langle \partial_x v_0(t,\cdot)|T_{\partial_t a_0}^m \partial_x v_0(t,\cdot)\right\rangle_{L^2} \\
&- \frac{1}{2}(t+\tau)\left\langle \partial_t\partial_x v_0(t,\cdot)|((T_{a_0}^m)^* - T_{a_0}^m)\partial_x v_0(t,\cdot)\right\rangle_{L^2} - (t+\tau)\left\langle \mathcal{X}_0(t,\cdot)|\partial_t v_0(t,\cdot)\right\rangle_{L^2}.
\end{aligned}
$$

Using Propositions 3.1, 3.5 and Lemma 4.1, for $N_1, N_2 > 0$, we get

$$
\begin{aligned}
&\left|\left\langle \partial_x v_0(t,\cdot)|T_{\partial_t a_0}^m \partial_x v_0(t,\cdot)\right\rangle_{L^2}\right| \le C_{a,m}^{(1)}\|v_0\|_{L^2}^2, \\
&\left|\left\langle T_{a-a_0}^m \partial_x v_0(t,\cdot)|\partial_x\partial_t v_0(t,\cdot)\right\rangle_{L^2}\right| \le C_{a,m}^{(2)} N_1\|v_0(t,\cdot)\|_{L^2}^2 + \frac{1}{N_1}\|\partial_t v_0(t,\cdot)\|_{L^2}^2, \\
&\left|\left\langle ((T_{a_0}^m)^* - T_{a_0}^m)\partial_x v_0(t,\cdot)|\partial_t\partial_x v_0(t,\cdot)\right\rangle_{L^2}\right| \le C_{a,m}^{(3)} N_2\|v_0(t,\cdot)\|_{L^2}^2 + \frac{1}{N_2}\|\partial_t v_0(t,\cdot)\|_{L^2}^2.
\end{aligned}
$$

Now, we choose N_1 and N_2 so large that

$$
\frac{1}{N_1} + \frac{1}{N_2} - \frac{1}{2} < 0
$$

and $\bar{\gamma}$ so large that

$$
-\frac{\gamma}{4} + \left(C_{a,m}^{(1)} + C_{a,m}^{(2)}N_1 + C_{a,m}^{(3)}N_2\right)\left(\frac{7}{8}\sigma + \tau\right) < 0
$$

for $\gamma \ge \bar{\gamma}$. Hence, the term

$$
C_{a,m}^{(1)}(t+\tau)\|v_0(t,\cdot)\|_{L^2}^2 + C_{a,m}^{(2)}N_1(t+\tau)\|v_0(t,\cdot)\|_{L^2}^2 + C_{a,m}^{(3)}N_2(t+\tau)\|v_0(t,\cdot)\|_{L^2}^2
$$

is absorbed by $-\frac{\gamma}{4}\|v_\nu(t,\cdot)\|_{L^2}^2$ and the term

$$
\frac{1}{N_1}(t+\tau)\|\partial_t v_0(t,\cdot)\|_{L^2}^2 + \frac{1}{N_2}(t+\tau)\|\partial_t v_0(t,\cdot)\|_{L^2}^2
$$

is absorbed by $-\frac{1}{2}(t+\tau)\|\partial_t v_0(t,\cdot)\|_{L^2}^2$. Hence, we get

$$
\begin{aligned}
\frac{1}{2}(t+\tau)\|\partial_t v_0(t,\cdot)\|_{L^2}^2 &\le \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_0(t,\cdot)\|_{L^2}^2\right) - \frac{\gamma}{4}\|v_0(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2 \\
&- \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2\right) + \frac{1}{2}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2 \\
&+ \frac{1}{2}\frac{d}{dt}\left((t+\tau)\left\langle T_{a_0}^m \partial_x v_0(t,\cdot)|\partial_x v_0(t,\cdot)\right\rangle_{L^2}\right) - (t+\tau)\left\langle \mathcal{X}_0|\partial_t v_0(t,\cdot)\right\rangle_{L^2}.
\end{aligned}
$$

Further, we recall that Φ fulfills Eq. (2.4), i.e.

$$
y\Phi_\lambda''(y) = -\lambda(\Phi_\lambda'(y))^2\mu\left(\frac{1}{\Phi_\lambda'(y)}\right) = -\lambda\Phi_\lambda'(y)\left(1 + \left|\log\left(\frac{1}{\Phi_\lambda'(y)}\right)\right|\right)
$$

for $\lambda > 1$. From this, we see that

$$
\frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2 < 0
$$

holds, and thus, we get

$$\frac{\gamma}{8}\|v_0(t,\cdot)\|_{L^2}^2 \le -\frac{1}{2}(t+\tau)\|\partial_t v_0(t,\cdot)\|_{L^2}^2 + \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_0(t,\cdot)\|_{L^2}^2\right) - \frac{\gamma}{8}\|v_0(t,\cdot)\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\left((t+\tau)\left\langle T_{a_0}^m \partial_x v_0(t,\cdot)|\partial_x v_0\right\rangle_{L^2}\right) - (t+\tau)\left\langle \mathcal{X}_0|\partial_t v_0(t,\cdot)\right\rangle_{L^2} - \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_0(t,\cdot)\|_{L^2}^2\right).$$

Using Propositions 3.6 and 3.5 as well as integrating in t over $[0,p]\subseteq[0,\frac{7}{8}\sigma]$, we obtain

$$\frac{\gamma}{8}\int_0^p \|v_0(t,\cdot)\|_{L^2}^2 dt \le \left(\frac{\gamma}{2}+C_{m,a}^{(4)}\right)(p+\tau)\|v_0(p,\cdot)\|_{L^2}^2 + \frac{1}{2}\tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)\|v_0(0,\cdot)\|_{L^2}^2 - \frac{\gamma}{8}\int_0^p \|v_0(t,\cdot)\|_{L^2}^2 dt - \frac{1}{2}\int_0^p (t+\tau)\|\partial_t v_0(t,\cdot)\|_{L^2}^2 dt - \int_0^p (t+\tau)\left\langle \mathcal{X}_0(t,\cdot)|\partial_t v_0(t,\cdot)\right\rangle_{L^2} dt,$$

where we have used

$$\left|\left\langle \partial_x v_0(p,\cdot)|T_{a_0}^m \partial_x v_0(p,\cdot)\right\rangle_{L^2}\right| \le C_{m,a}^{(4)}\|v_0(p,\cdot)\|_{L^2}^2$$

and, applying Proposition 3.6,

$$\left\langle \partial_x v_0(t,\cdot)|T_{a_0}^m \partial_x v_0(t,\cdot)\right\rangle_{L^2} \ge \frac{\kappa}{2}\|\partial_x v_0(t,\cdot)\|_{L^2}^2$$

choosing m large enough.

4.3. *Estimates for* $\nu \ge 1$

Now, we consider (4.8) for $\nu \ge 1$. From Lemma 4.1, for N_3 and $N_4 > 0$, we obtain

$$\left|\left\langle (T_a^m - T_{a_\nu}^m)\partial_x v_\nu(t,\cdot)|\partial_x\partial_t v_\nu(t,\cdot)\right\rangle_{L^2}\right| \le C_{a,m}^{(5)} N_3 \nu^2 \|v_\nu(t,\cdot)\|_{L^2}^2 + \frac{1}{N_3}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 \le C_{a,m}^{(5)} N_3 \nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2 + \frac{1}{N_3}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 \tag{4.9}$$

and

$$\left|\left\langle \partial_x v_\nu(t,\cdot)|T_{\partial_t a_\nu}^m \partial_x v_\nu(t,\cdot)\right\rangle_{L^2}\right| \le C_{a,m}^{(6)}\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2, \tag{4.10}$$

as well as

$$\left|\left\langle ((T_{a_\nu}^m)^* - T_{a_\nu}^m)\partial_x v_\nu(t,\cdot)|\partial_t\partial_x v_\nu(t,\cdot)\right\rangle_{L^2}\right| \le C_{a,m}^{(7)} N_4 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2 + \frac{1}{N_4}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 \tag{4.11}$$

which follows from Proposition 3.7. Using again the positivity estimate in Proposition 3.6, we obtain

$$-\alpha\log(2)(t+\tau)\nu\left\langle \partial_x v_\nu(t,\cdot)|T_{a_\nu}^m \partial_x v_\nu(t,\cdot)\right\rangle_{L^2} \le -\alpha C_{a,m}^{(8)}(t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2. \tag{4.12}$$

Now, we choose N_3 and N_4 so large that

$$\frac{1}{N_3}+\frac{1}{N_4}-\frac{1}{2}<0,$$

and α_1 large enough such that

$$-\frac{\alpha_1}{2}C_{a,m}^{(8)} + N_3 C_{a,m}^{(5)} + C_{a,m}^{(6)} + C_{a,m}^{(7)} N_4 < 0,$$

and we set $\alpha := \max\{T^{-1},\alpha_1\}$. With this choice, we get

$$\frac{\gamma}{4}\|v_\nu(t,\cdot)\|_{L^2}^2 + \frac{1}{2}(t+\tau)\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 \le \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_\nu(t,\cdot)\|_{L^2}^2\right) - \frac{\gamma}{4}\|v_\nu(t,\cdot)\|_{L^2}^2 - \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2+\frac{1}{2}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2\\
&+\frac{1}{2}\frac{d}{dt}\left((t+\tau)\left\langle T^m_{a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2}\right)\\
&-\alpha\gamma\log(2)(t+\tau)\nu\|v_\nu(t,\cdot)\|_{L^2}^2-\frac{1}{2}\left\langle T^m_{a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2}\\
&+\alpha\log(2)(t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\nu\|v_\nu(t,\cdot)\|_{L^2}^2\\
&+\alpha^2(\log(2))^2\nu^2(t+\tau)\|v_\nu(t,\cdot)\|_{L^2}-\frac{\alpha}{2}C^{(8)}_{a,m}(t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2\\
&+\alpha\log(2)\nu 2^{-(s+\alpha t)\nu}(t+\tau)\left\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\right\rangle_{L^2}\\
&-(t+\tau)2^{-(s+\alpha t)\nu}\left\langle \mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2}.
\end{aligned}\tag{4.13}
$$

Since $y\Phi_\lambda''(y)=-\lambda\Phi_\lambda'(y)(1+|\log(\Phi_\lambda'(y))|)$, if we take $\lambda\geq\bar\lambda>2$, we have

$$
\frac{1}{4}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\leq-\frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right),
$$

and hence, the term $\frac{1}{2}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2$ in (4.13) is absorbed by the term $\frac{1}{4}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2$. Now we need to absorb

$$
\alpha\log(2)(t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\nu\|v_\nu(t,\cdot)\|_{L^2}^2.\tag{4.14}
$$

There are two terms in (4.13) that will help to achieve this. One is

$$
-\frac{\alpha}{4}C^{(8)}_{a,m}(t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2\tag{4.15}
$$

and the other one is

$$
\frac{1}{4}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|_{L^2}^2.\tag{4.16}
$$

Let $\tilde C^{(8)}_{a,m}=\min\{4\log(2),C^{(8)}_{a,m}\}$. If $\nu\geq\frac{1}{2\log 2}\log\left(\frac{4\log(2)}{\tilde C^{(8)}_{a,m}}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\right)$, then

$$
-\frac{C^{(8)}_{a,m}}{4}\alpha\nu 2^{2\nu}\leq-\alpha\log(2)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\nu.
$$

On the contrary, if $\nu<\frac{1}{2\log 2}\log\left(\frac{4\log(2)}{\tilde C^{(8)}_{a,m}}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\right)$, then $\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)>2^\nu$ and, hence, by (2.4), we obtain

$$
\begin{aligned}
\frac{1}{4}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right)&=-\frac{1}{4}\lambda\left(\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\right)^2\mu\left(\frac{1}{\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)}\right)\\
&\leq-\frac{1}{4}\lambda\left(\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\right)^2\mu\left(\frac{1}{\frac{4\log(2)}{\tilde C^{(8)}_{a,m}}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)}\right)\\
&\leq-\frac{1}{4}\lambda\frac{\tilde C^{(8)}_{a,m}}{4\log(2)}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\left(1+\left|\log\left(\frac{4\log(2)}{\tilde C^{(8)}_{a,m}}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\right)\right|\right)\\
&\leq-\frac{1}{4}\lambda\frac{\tilde C^{(8)}_{a,m}}{4\log(2)}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)(1+\nu\log(2))\\
&\leq-\lambda C^{(9)}_{a,m}\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\nu,
\end{aligned}
$$

where we have used the fact that the function $\varepsilon\mapsto\varepsilon(|\log\varepsilon|+1)$ is increasing. Consequently, if we choose $\lambda\geq\bar\lambda$ with

$$
\bar\lambda\geq\frac{\alpha\log(2)\left(\frac{7}{8}\sigma+\tau\right)}{C^{(9)}_{a,m}},
$$

we have

$$\frac{1}{4}\frac{t+\tau}{\beta}\Phi_\lambda''\left(\frac{t+\tau}{\beta}\right) \le -\alpha\log(2)(t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\nu$$

and hence, the term (4.14) is compensated by (4.15) and (4.16). Now we consider the term

$$(t+\tau)\alpha^2\log^2(2)\nu^2\|v_\nu(t,\cdot)\|_{L^2}. \tag{4.17}$$

If $\nu \ge (\log(2))^{-1}\log\left(\frac{\alpha(2\log(2))^2}{C^{(8)}_{a,m}}\right) =: \bar{\nu}_1$, then

$$-\frac{C^{(8)}_{a,m}}{4}\alpha\nu 2^{2\nu} + \alpha^2\log^2(2)\nu^2 \le 0.$$

If $\nu \le \bar{\nu}_1$, then we eventually choose a possibly larger $\bar{\gamma}$ such that

$$\frac{\gamma}{4} \ge \alpha^2\log^2(2)\bar{\nu}_1^2\left(\frac{7}{8}\sigma+\tau\right)$$

for all $\gamma \ge \bar{\gamma}$. We obtain

$$-\frac{\gamma}{4} + \alpha^2\log^2(2)\nu \le 0,$$

and, consequently, (4.17) is absorbed by

$$-\frac{\alpha}{4}C^{(8)}_{a,m}(t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|^2_{L^2} - \frac{\gamma}{4}\|v_\nu(t,\cdot)\|^2_{L^2}.$$

The term $-\alpha\gamma\log(2)(t+\tau)\nu\|v_\nu(t,\cdot)\|^2_{L^2}$ can be neglected since it is negative. However, we stress here that it is a crucial term in order to achieve our energy estimate for an equation including also lower order terms. Eventually, recalling also Propositions 3.1 and 3.6, we obtain

$$\begin{aligned}
\frac{1}{2}(t+\tau)\|\partial_t v_\nu(t,\cdot)\|^2_{L^2} + \frac{\gamma}{8}\|v_\nu(t,\cdot)\|^2_{L^2} \le\ & \frac{\gamma}{2}\frac{d}{dt}\left((t+\tau)\|v_\nu(t,\cdot)\|^2_{L^2}\right) - \frac{1}{2}\frac{d}{dt}\left((t+\tau)\Phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\|v_\nu(t,\cdot)\|^2_{L^2}\right)\\
&+\frac{1}{2}\frac{d}{dt}\left((t+\tau)\left\langle T^m_{a_\nu}\partial_x v_\nu(t,\cdot)|\partial_x v_\nu(t,\cdot)\right\rangle_{L^2}\right) - \frac{\kappa}{8}2^{2\nu}\|v_\nu(t,\cdot)\|^2_{L^2}\\
&-\frac{\alpha}{2}\log(2)C^{(8)}_{a,m}(t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|^2_{L^2}\\
&+\alpha\log(2)\nu 2^{-(s+\alpha t)\nu}(t+\tau)\left\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\right\rangle_{L^2}\\
&-(t+\tau)2^{-(s+\alpha t)\nu}\left\langle \mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2} - \frac{\gamma}{8}\|v_\nu(t,\cdot)\|^2_{L^2}.
\end{aligned}$$

Integrating over $[0,p] \subseteq [0,\frac{7}{8}\sigma]$, we get

$$\begin{aligned}
&\frac{\kappa}{8}\int_0^p 2^{2\nu}\|v_\nu(t,\cdot)\|^2_{L^2}dt + \frac{\gamma}{8}\int_0^p \|v_\nu(t,\cdot)\|^2_{L^2}dt\\
&\quad\le \tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)\|v_\nu(0,\cdot)\|^2_{L^2} + \left(\frac{\gamma}{2}+C^{(10)}_{a,m}2^{2\nu}\right)(p+\tau)\|v_\nu(p,\cdot)\|^2_{L^2}\\
&\qquad-\frac{\alpha}{2}\log(2)C^{(8)}_{a,m}\int_0^p(t+\tau)\nu 2^{2\nu}\|v_\nu(t,\cdot)\|^2_{L^2}dt - \frac{\gamma}{8}\int_0^p\|v_\nu(t,\cdot)\|^2_{L^2}dt\\
&\qquad-\frac{1}{2}\int_0^p(t+\tau)\|\partial_t v_\nu(t,\cdot)\|^2_{L^2}dt - \int_0^p(t+\tau)2^{-(s+\alpha t)\nu}\left\langle\mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\right\rangle_{L^2}dt\\
&\qquad+\alpha\log(2)\int_0^p\nu 2^{-(s+\alpha t)\nu}(t+\tau)\left\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\right\rangle_{L^2}dt.
\end{aligned}$$

Now we sum over ν and we obtain

$$\begin{aligned}
&\frac{\kappa}{8}\int_0^p\sum_{\nu\ge 0}2^{2\nu}\|v_\nu(t,\cdot)\|^2_{L^2}dt + \frac{\gamma}{8}\int_0^p\sum_{\nu\ge 0}\|v_\nu(t,\cdot)\|^2_{L^2}dt\\
&\quad\le \tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)\sum_{\nu\ge 0}\|v_\nu(0,\cdot)\|^2_{L^2} - \frac{\gamma}{8}\int_0^p\sum_{\nu\ge 0}\|v_\nu(t,\cdot)\|^2_{L^2}dt - \frac{1}{2}\int_0^p(t+\tau)\sum_{\nu\ge 0}\|\partial_t v_\nu(t,\cdot)\|^2_{L^2}dt
\end{aligned}$$

$$+\frac{\gamma}{2}(p+\tau)\sum_{\nu\geq 0}\|v_\nu(p,\cdot)\|_{L^2}^2+C_{a,m}^{(10)}(p+\tau)\sum_{\nu\geq 0}2^{2\nu}\|v_\nu(p,\cdot)\|_{L^2}^2$$
$$-\frac{\alpha}{2}\log(2)C_{a,m}^{(8)}\int_0^p(t+\tau)\sum_{\nu\geq 0}\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt-\int_0^p(t+\tau)\sum_{\nu\geq 0}2^{-(s+\alpha t)\nu}\langle\mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2}\,dt$$
$$+\alpha\log(2)\int_0^p(t+\tau)\sum_{\nu\geq 0}\nu 2^{-(s+\alpha t)\nu}\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\rangle_{L^2}\,dt.$$

Using the results from Sections 3.3 and 3.4, we have the estimates

$$-\int_0^p(t+\tau)\sum_{\nu\geq 0}2^{-(s+\alpha t)\nu}\langle\mathcal{X}_\nu(t,\cdot)|\partial_t v_\nu(t,\cdot)\rangle_{L^2}\,dt$$
$$\leq\eta\int_0^p(t+\tau)\sum_{\nu\geq 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2dt+\frac{C_{a,m,s}^{(11)}}{\eta}\int_0^p(t+\tau)\sum_{\nu\geq 0}2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt$$

and

$$\alpha\log(2)\int_0^p(t+\tau)\sum_{\nu\geq 0}\nu 2^{-(s+\alpha t)\nu}\langle v_\nu(t,\cdot)|\mathcal{X}_\nu(t,\cdot)\rangle_{L^2}\,dt\leq\alpha\log(2)C_{a,m,s}^{(12)}\int_0^p(t+\tau)\sum_{\nu\geq 0}2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt.$$

4.4. *End of the proof*

So far we have obtained

$$\frac{\kappa}{8}\int_0^p\sum_{\nu\geq 0}2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt+\frac{\gamma}{8}\int_0^p\sum_{\nu\geq 0}\|v_\nu(t,\cdot)\|_{L^2}^2dt$$
$$\leq\tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)\sum_{\nu\geq 0}\|v_\nu(0,\cdot)\|_{L^2}^2-\frac{\gamma}{8}\int_0^p\sum_{\nu\geq 0}\|v_\nu(t,\cdot)\|_{L^2}^2dt$$
$$+\frac{\gamma}{2}(p+\tau)\sum_{\nu\geq 0}\|v_\nu(p,\cdot)\|_{L^2}^2+C_{a,m}^{(10)}(p+\tau)\sum_{\nu\geq 0}2^{2\nu}\|v_\nu(p,\cdot)\|_{L^2}^2$$
$$-\frac{\alpha}{2}\log(2)C_{a,m}^{(8)}\int_0^p(t+\tau)\sum_{\nu\geq 0}\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt$$
$$-\frac{1}{2}\int_0^p(t+\tau)\sum_{\nu\geq 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2dt+\eta\int_0^p(t+\tau)\sum_{\nu\geq 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2dt$$
$$+\left(\alpha\log(2)C_{a,m,s}^{(12)}+\frac{C_{a,m,s}^{(11)}}{\eta}\right)\int_0^p(t+\tau)\sum_{\nu\geq 0}2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt.$$

Now we take $\eta<\frac{1}{2}$ and choose $\bar{\nu}_2:=\left\lceil\left(\alpha\log(2)C_{a,m,s}^{(12)}+\frac{C_{a,m,s}^{(11)}}{\eta}\right)\frac{2}{\alpha\log(2)C_{a,m}^{(8)}}\right\rceil$. With this, we have

$$-\frac{\alpha}{2}\log(2)C_{a,m}^{(8)}\int_0^p(t+\tau)\sum_{\nu\geq\bar{\nu}_2}\nu 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt$$
$$+\left(\alpha\log(2)C_{a,m,s}^{(12)}+\frac{C_{a,m,s}^{(11)}}{\eta}\right)\int_0^p(t+\tau)\sum_{\nu\geq\bar{\nu}_2}2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt\leq 0.$$

To absorb the remaining parts of the sum, we choose $\bar{\gamma}$ larger (if necessary) such that

$$-\frac{\gamma}{8}+\left(\frac{7}{8}\sigma+\tau\right)\left(\alpha\log(2)C_{a,m,s}^{(12)}+\frac{C_{a,m,s}^{(11)}}{\eta}\right)2^{2\bar{\nu}_2}<0$$

for all $\gamma\geq\bar{\gamma}$. This leads to

$$-\frac{\gamma}{8}\int_0^p\sum_{\nu<\bar{\nu}_2}\|v_\nu(t,\cdot)\|_{L^2}^2dt+\left(\alpha\log(2)C_{a,m,s}^{(12)}+\frac{C_{a,m,s}^{(11)}}{\eta}\right)\int_0^p(t+\tau)\sum_{\nu<\bar{\nu}_2}2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2dt\leq 0.$$

All in all, we finally obtain

$$\frac{\kappa}{8}\int_0^p \sum_{\nu\geq 0} 2^{2\nu}\|v_\nu(t,\cdot)\|_{L^2}^2 dt + \frac{\gamma}{8}\int_0^p \sum_{\nu\geq 0}\|v_\nu(t,\cdot)\|_{L^2}^2 dt$$
$$\leq \tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)\sum_{\nu\geq 0}\|v_\nu(0,\cdot)\|_{L^2}^2 + \frac{\gamma}{2}(p+\tau)\sum_{\nu\geq 0}\|v_\nu(p,\cdot)\|_{L^2}^2 + C_{a,m}^{(10)}(p+\tau)\sum_{\nu\geq 0}2^{2\nu}\|v_\nu(p,\cdot)\|_{L^2}^2.$$

From this, going back to u_ν, we have, for $p\in[0,\frac{7}{8}\sigma]$, $\sigma=\frac{1-s}{\alpha}$, $\tau=\frac{\sigma}{4}$ and $\beta\geq\sigma+\tau$,

$$\frac{\kappa}{8}\int_0^p e^{2\gamma t}e^{-2\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\sum_{\nu\geq 0}2^{2(1-s-\alpha t)\nu}\|u_\nu(t,\cdot)\|_{L^2}^2 dt + \frac{\gamma}{8}\int_0^p e^{2\gamma t}e^{-2\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\sum_{\nu\geq 0}2^{-2(s+\alpha t)\nu}\|u_\nu(t,\cdot)\|_{L^2}^2 dt$$
$$\leq C_{a,m}^{(10)}(p+\tau)e^{2\gamma p}e^{-2\beta\Phi_\lambda\left(\frac{p+\tau}{\beta}\right)}\sum_{\nu\geq 0}2^{2(1-s-\alpha p)\nu}\|u_\nu(p,\cdot)\|_{L^2}^2$$
$$+\frac{\gamma}{2}(p+\tau)e^{2\gamma p}e^{-2\beta\Phi_\lambda\left(\frac{p+\tau}{\beta}\right)}\sum_{\nu\geq 0}2^{-(s+\alpha p)\nu}\|u_\nu(p,\cdot)\|_{L^2}^2 + \tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\sum_{\nu\geq 0}2^{-2s\nu}\|u_\nu(0,\cdot)\|_{L^2}^2.$$

Using Proposition 3.2, the weighted energy estimate (2.5) follows. □

5. Proof of Theorem 2.4

In this section, we show how the conditional stability estimate in Theorem 2.4 follows from the weighted energy estimate in Proposition 2.9. To this end, we need two lemmas whose proof is left to the reader.

Lemma 5.1. *There exists $\gamma_0>0$ such that if $\gamma\geq\gamma_0$, whenever $u\in\mathcal{H}$ is a solution of (2.1), then the function $E(t)=e^{2\gamma t}\|u(t,\cdot)\|_{L^2}^2$ is not decreasing in $[0,T]$.*

The next lemma contains an estimate of the H^1-norm of a solution of (2.1) by its L^2-norm. This estimate is crucial in gaining (2.3) from (2.5).

Lemma 5.2. *There exists a positive constant C such that, whenever $u\in\mathcal{H}$ is a solution of (2.1) in $[0,T]$, then*

$$\inf_{t\in\left[\frac{5}{7}\sigma,\frac{7}{8}\sigma\right]}\|u(t,\cdot)\|_{H^1}^2\leq\frac{C}{\sigma}\sup_{t\in\left[\frac{5}{7}\sigma,\frac{7}{8}\sigma\right]}\|u(t,\cdot)\|_{L^2}^2.$$

The constant C depends only on κ in (2.2).

We start from the inequality (2.5), that is

$$\int_0^p e^{2\gamma t}e^{-2\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\|u(t,\cdot)\|_{H^{1-s-\alpha t}}^2 dt$$
$$\leq M\left[(p+\tau)e^{2\gamma p}e^{-2\beta\Phi_\lambda\left(\frac{p+\tau}{\beta}\right)}\|u(p,\cdot)\|_{H^{1-s-\alpha p}}^2+\tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\right]$$

which is valid for $p\in[0,\frac{7}{8}\sigma]$, $\sigma:=\frac{1-s}{\alpha}$. For every $\sigma^*\in(\frac{5}{8}\sigma,\frac{7}{8}\sigma)$, we have

$$\int_0^{\sigma^*} e^{2\gamma t}e^{-2\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\|u(t,\cdot)\|_{H^{1-s-\alpha t}}^2 dt$$
$$\leq M\left[(\sigma^*+\tau)e^{2\gamma\sigma^*}e^{-2\beta\Phi_\lambda\left(\frac{\sigma^*+\tau}{\beta}\right)}\|u(\sigma^*,\cdot)\|_{H^{1-s-\alpha\sigma^*}}^2+\tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\right],$$

where $\beta\geq\sigma+\tau$. Now we take $p\in[0,\bar\sigma]$ with $\bar\sigma=\frac{1}{2}\left(\frac{\sigma}{2}-\tau\right)=\frac{\sigma}{8}$, so $2p+\tau\leq 2\bar\sigma+\tau=\frac{\sigma}{2}<\frac{5}{8}\sigma<\sigma^*$. Hence,

$$\int_p^{2p+\tau} e^{2\gamma t}e^{-2\beta\Phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\|u(t,\cdot)\|_{H^{1-s-\alpha t}}^2 dt$$
$$\leq M\left[(\sigma^*+\tau)e^{2\gamma\sigma^*}e^{-2\beta\Phi_\lambda\left(\frac{\sigma^*+\tau}{\beta}\right)}\|u(\sigma^*,\cdot)\|_{H^{1-s-\alpha\sigma^*}}^2+\tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\right].$$

Since $\frac{1}{8}(1-s)\leq 1-s-\alpha t\leq 1-s$, we have

$$\|u(t,\cdot)\|_{H^{1-s-\alpha t}}\geq\|u(t,\cdot)\|_{L^2}.$$

Hence, applying Lemma 5.1, we have

$$e^{2\gamma p}(p+\tau)\|u(p,\cdot)\|_{L^2}^2 e^{-2\beta\Phi_\lambda\left(\frac{2p+2\tau}{\beta}\right)} \leq M\left[(\sigma^*+\tau)e^{2\gamma\sigma^*}e^{-2\beta\Phi_\lambda\left(\frac{\sigma^*+\tau}{\beta}\right)}\|u(\sigma^*,\cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 + \tau\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\right].$$

Since $\Phi_\lambda' \geq 1$, we have

$$\begin{aligned}\|u(p,\cdot)\|_{L^2}^2 &\leq M\frac{\sigma^*+\tau}{\tau}e^{2\gamma\sigma^*}\Phi_\lambda'\left(\frac{\tau}{\beta}\right)\Big[e^{2\beta\Phi_\lambda\left(\frac{\sigma/2+\tau}{\beta}\right)-2\beta\Phi_\lambda\left(\frac{\sigma^*+\tau}{\beta}\right)}\|u(\sigma^*,\cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 \\ &\quad + e^{2\beta\Phi_\lambda\left(\frac{\sigma/2+\tau}{\beta}\right)-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\Big] \\ &\leq \tilde{M}\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{2\beta\Phi_\lambda\left(\frac{\sigma/2+\tau}{\beta}\right)-2\beta\Phi_\lambda'\left(\frac{\sigma^*+\tau}{\beta}\right)}\left[\|u(\sigma^*,\cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\right].\end{aligned}$$

Now we use $\frac{\sigma^*+\tau}{\beta} \geq \frac{\frac{5}{8}\sigma+\tau}{\beta}$, which implies

$$\Phi_\lambda\left(\frac{\sigma^*+\tau}{\beta}\right) \geq \Phi_\lambda\left(\frac{\frac{5}{8}\sigma+\tau}{\beta}\right)$$

and, hence,

$$\|u(p,\cdot)\|_{L^2}^2 \leq \tilde{M}\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{2\beta\Phi_\lambda\left(\frac{\sigma/2+\tau}{\beta}\right)-2\beta\Phi_\lambda\left(\frac{5\sigma/8+\tau}{\beta}\right)}\left[\|u(\sigma^*,\cdot)\|_{H^{1-s-\alpha\alpha\sigma^*}}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\right].$$

By the concavity of Φ_λ, we have

$$\begin{aligned}&2\beta\Phi_\lambda\left(\frac{\sigma/2+\tau}{\beta}\right) - 2\beta\Phi_\lambda\left(\frac{5\sigma/8+\tau}{\beta}\right) \\ &\quad\leq 2\beta\Phi_\lambda'\left(\frac{5\sigma/8+\tau}{\beta}\right)\left(\frac{\sigma/2+\tau}{\beta} - \frac{5\sigma/8+\tau}{\beta}\right) = -2\beta\Phi_\lambda'\left(\frac{5\sigma/8+\tau}{\beta}\right)\frac{\sigma}{8\beta}.\end{aligned}$$

This implies

$$\|u(p,\cdot)\|_{L^2}^2 \leq \tilde{M}\Phi_\lambda'\left(\frac{\tau}{\beta}\right)e^{-\frac{\sigma}{4}\Phi_\lambda'\left(\frac{5\sigma/8+\tau}{\beta}\right)}\left(\|u(\sigma^*,\cdot)\|_{H^{1-s-\alpha\sigma^*}}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}\right).$$

By Lemma 2.8, we have

$$\begin{aligned}\Phi_\lambda'\left(\frac{5\sigma/8+\tau}{\beta}\right) &= \psi_\lambda\left(\frac{5\sigma/8+\tau}{\tau}\frac{\tau}{\beta}\right) \\ &= \exp\left(\left(\frac{5\sigma/8+\tau}{\tau}\right)^{-\lambda}-1\right)\psi_\lambda\left(\frac{\tau}{\beta}\right)^{\left(\frac{5\sigma/8+\tau}{\tau}\right)^{-\lambda}}.\end{aligned}$$

Setting $\tilde{\delta} := \left(\frac{5\sigma/8+\tau}{\tau}\right)^{-\lambda}$, we have

$$\|u(p,\cdot)\|_{L^2}^2 \leq \tilde{M}\left(\psi_\lambda\left(\frac{\tau}{\beta}\right)\right)^{\tilde{\delta}}e^{-\tilde{N}\psi_\lambda\left(\frac{\tau}{\beta}\right)^{\tilde{\delta}}}\left(\|u(\sigma^*,\cdot)\|_{H^1}^2 + e^{-2\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|_{H^{-s}}^2\right).$$

Now we choose β such that

$$e^{-\beta\Phi_\lambda\left(\frac{\tau}{\beta}\right)} = \|u(0,\cdot)\|_{H^{-s}}^{-1},$$

that is

$$\beta = \tau\Lambda^{-1}\left(\frac{1}{\tau}\log\|u(0,\cdot)\|_{H^{-s}}\right).$$

Then there exists $\bar{\rho} > 0$ such that, if $\|u(0,\cdot)\|_{L^2} \leq \bar{\rho}$, then $\beta \geq \sigma+\tau$. With this choice and thanks to Lemma 2.8, we get

$$\|u(p,\cdot)\|_{L^2}^2 \leq \tilde{\tilde{M}}\exp\left(-\tilde{\tilde{N}}\left[\frac{1}{\tau}|\log(\|u(0,\cdot)\|_{H^{-s}})|\right]^{\tilde{\delta}}\right)\left(\|u(\sigma^*,\cdot)\|_{H^1}^2+1\right)$$

for all $\sigma^* \in [\frac{5}{8}\sigma, \frac{7}{8}\sigma]$ and for all $p \in [0, \frac{\sigma}{8}]$. By Lemma 5.2, we finally get

$$\|u(p,\cdot)\|_{L^2}^2 \le Ce^{-\tilde{\tilde{N}}\left[\frac{1}{\tau}\left|\log\left(\|u(0,\cdot)\|_{H^{-s}}\right)\right|\right]^{\tilde{\delta}}} \left(\max_{t\in\left[\frac{5}{8}\sigma,\frac{7}{8}\sigma\right]} \|u(t,\cdot)\|_{L^2}^2 + 1 \right).$$

This completes the proof of Theorem 2.4. □

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Appendix

Proof of Proposition 3.10. To estimate

$$\sum_{\nu\ge 0} 2^{-(s+\alpha t)\nu} \left\langle \partial_{x_i}\partial_t v_\nu(t,\cdot) | \Delta_\nu((a-T_a^m)\partial_{x_j} w(t,\cdot))\right\rangle_{L^2},$$

we introduce a second Littlewood–Paley decomposition: setting $w(t,\cdot) = \sum_{\mu\ge 0} w_\mu(t,\cdot)$ and $w_\mu(t,\cdot) = 2^{(s+\alpha t)\mu} v_\mu(t,\cdot)$ (see Section 4.1) we obtain, using Proposition 3.1, that

$$\begin{aligned}
&\sum_{\nu\ge 0} 2^{-(s+\alpha t)\nu} \left\langle \partial_{x_i}\partial_t v_\nu(t,\cdot) | \Delta_\nu((a-T_a^m)\partial_{x_j} w(t,\cdot))\right\rangle_{L^2} \\
&\quad = \sum_{\nu\ge 0}\sum_{\mu\ge 0} 2^{-(s+\alpha t)\nu} \left\langle \partial_{x_i}\partial_t v_\nu(t,\cdot) | \Delta_\nu((a-T_a^m)\partial_{x_j} w_\mu(t,\cdot))\right\rangle_{L^2} \\
&\quad = \sum_{\nu\ge 0}\sum_{\mu\ge 0} 2^{-(s+\alpha t)(\nu-\mu)} \left\langle \partial_{x_i}\partial_t v_\nu(t,\cdot) | \Delta_\nu((a-T_a^m)\partial_{x_j} v_\mu(t,\cdot))\right\rangle_{L^2} \\
&\quad \le \sum_{\nu\ge 0}\sum_{\mu\ge 0} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t,\cdot)\|_{L^2} \|\Delta_\nu((a-T_a^m)\partial_{x_j} v_\mu(t,\cdot))\|_{L^2} \\
&\quad \le \sum_{\nu\ge 0}\sum_{\mu\le \nu} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t,\cdot)\|_{L^2} \|\Delta_\nu \Omega_1 \partial_{x_j} v_\mu(t,\cdot)\|_{L^2} \\
&\qquad + \sum_{\nu\ge 0}\sum_{\mu\ge \nu-5} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t,\cdot)\|_{L^2} \|\Delta_\nu \Omega_2 \partial_{x_j} v_\mu(t,\cdot)\|_{L^2}.
\end{aligned}$$

Since $w(t,\cdot) \in H^1(\mathbb{R}^n_x)$, we have $\partial_x v_\mu(t,\cdot) \in H^{-s}(\mathbb{R}^n_x)$ and, taking an $s' \in (0,s)$, also $\partial_x v_\mu(t,\cdot) \in H^{-s'}(\mathbb{R}^n_x)$. By Lemma 3.8, it follows

$$\|\Delta_\nu \Omega_1 \partial_{x_j} v_\mu(t,\cdot)\|_{L^2} \le C\|a\|_{\mathrm{Lip}} c_\nu^{(\mu)} 2^{-(1-s')\nu} 2^\mu \|v_\mu(t,\cdot)\|_{H^{-s}}$$

and therefore,

$$\begin{aligned}
&\sum_{\nu\ge 0}\sum_{\mu\le \nu} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t,\cdot)\|_{L^2} \|\Delta_\nu \Omega_1 \partial_{x_j} v_\mu(t,\cdot)\|_{L^2} \\
&\quad \le C\|a\|_{\mathrm{Lip}} \sum_{\nu\ge 0}\sum_{\mu\le \nu} 2^{-(s+\alpha t)(\nu-\mu)} 2^\nu \|\partial_t v_\nu(t,\cdot)\|_{L^2} c_\nu^{(\mu)} 2^{-(1-s')\nu} 2^\mu 2^{-s'\mu} \|v_\mu(t,\cdot)\|_{L^2} \\
&\quad \le C\|a\|_{\mathrm{Lip}} \sum_{\nu\ge 0}\sum_{\mu\le \nu} 2^{-s\alpha t\nu} 2^{s\alpha t\mu} \left(2^{(s'-s)(\nu-\mu)} \|\partial_t v_\nu(t,\cdot)\|_{L^2}\right) \left(c_\nu^{(\mu)} 2^\mu \|v_\mu(t,\cdot)\|_{L^2}\right) \\
&\quad \le \frac{1}{N} \sum_{\nu\ge 0}\sum_{\mu\le \nu} 2^{-2(s-s')(\nu-\mu)} \|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N \sum_{\nu\ge 0}\sum_{\mu\le \nu} (c_\nu^{(\mu)})^2 2^{2\mu} \|v_\mu(t,\cdot)\|_{L^2}^2 \\
&\quad \le \frac{1}{N} \sum_{\nu\ge 0} \left(\sum_{\mu\le\nu} 2^{2(s-s')\mu}\right) 2^{-2(s-s')\nu} \|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N \sum_{\mu\ge 0} \left(\sum_{\nu\ge 0} (c_\nu^{(\mu)})^2\right) 2^{2\mu} \|v_\mu(t,\cdot)\|_{L^2}^2 \\
&\quad = \frac{1}{N} \sum_{\nu\ge 0} \frac{2^{2(s-s')(\nu+1)}-1}{2^{2(s-s')}-1} 2^{-2(s-s')\nu} \|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N \sum_{\mu\ge 0} 2^{2\mu} \|v_\mu(t,\cdot)\|_{L^2}^2 \\
&\quad \le \frac{1}{N} \frac{2^{2(s-s')}}{2^{2(s-s')}-1} \sum_{\nu\ge 0} \|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N \sum_{\mu\ge 0} 2^{2\mu} \|v_\mu(t,\cdot)\|_{L^2}^2.
\end{aligned}$$

By the summation formula of the geometric sum and the integral criterion, we obtain

$$\frac{2^{2(s-s')}}{2^{2(s-s')}-1} \le \frac{2^{2(s-s')}}{2^{2(s-s')}(1-2^{-2(s-s')})} \le \frac{C}{s-s'}$$

and, hence,

$$\sum_{\nu\ge 0}\sum_{\mu\ge 0} 2^{-(s+\alpha t)(\nu-\mu)}2^{\nu}\|\partial_t v_\nu(t,\cdot)\|_{L^2}\|\Delta_\nu \Omega_1\partial_x v_\mu(t,\cdot)\|_{L^2}$$
$$\le \frac{1}{N}\frac{C}{s-s'}\sum_{\nu\ge 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N\sum_{\mu\ge 0}2^{2\mu}\|v_\mu(t,\cdot)\|_{L^2}^2.$$

On the other hand, we have from Lemma 3.9 that

$$\|\Delta_\nu \Omega_2\partial_x v_\mu(t,\cdot)\|_{L^2} \le C\|a\|_{\mathrm{Lip}}\tilde{c}_\nu^{(\mu)}\|v_\mu(t,\cdot)\|_{L^2},$$

and therefore, we get

$$\sum_{\nu\ge 0}\sum_{\mu\ge \nu-5} 2^{-(s+\alpha t)(\nu-\mu)}2^{\nu}\|\partial_t v_\nu(t,\cdot)\|_{L^2}\|\Delta_\nu \Omega_2\partial_x v_\mu(t,\cdot)\|_{L^2}$$
$$\le C\|a\|_{\mathrm{Lip}}\sum_{\nu\ge 0}\sum_{\mu\ge \nu-5} 2^{-(s+\alpha t)(\nu-\mu)}2^{\nu}\|\partial_t v_\nu(t,\cdot)\|_{L^2}\tilde{c}_\nu^{(\mu)}2^{-\mu}2^{\mu}\|v_\mu(t,\cdot)\|_{L^2}$$
$$\le C\|a\|_{\mathrm{Lip}}\sum_{\nu\ge 0}\sum_{\mu\ge \nu-5} 2^{(1-s-\alpha t)\nu}2^{-(1-s-\alpha t)\mu}\tilde{c}_\nu^{(\mu)}\|\partial_t v_\nu(t,\cdot)\|_{L^2}2^{\mu}\|v_\mu(t,\cdot)\|_{L^2}$$
$$\le \frac{1}{N}\sum_{\nu\ge 0}\sum_{\mu\ge \nu-5} 2^{2(1-s-\alpha t)\nu}2^{-2(1-s-\alpha t)\mu}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N\sum_{\nu\ge 0}\sum_{\mu\ge \nu-5}(c_\nu^{(\mu)})^2 2^{2\mu}\|v_\mu(t,\cdot)\|_{L^2}^2$$
$$\le \frac{1}{N}\frac{2^{10(1-s-\alpha t)}}{1-2^{-2(1-s-\alpha t)}}\sum_{\nu\ge 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N\sum_{\mu\ge 0}\left(\sum_{\nu\le\mu+5}(c_\nu^{(\mu)})^2\right)2^{2\mu}\|v_\mu(t,\cdot)\|_{L^2}^2.$$

Since $t\in[0,\frac{7}{8}\sigma]$, where $\sigma := \frac{1-s}{\alpha}$, we have $\frac{1}{8}(1-s)\le 1-s-\alpha t\le 1-s$ and hence

$$\frac{2^{10(1-s-\alpha t)}}{1-2^{-2(1-s-\alpha t)}} \le \frac{C}{1-s-\alpha t} \le \frac{C}{1-s}.$$

From that, we get

$$\sum_{\nu\ge 0}\sum_{\mu\ge \nu-5} 2^{-(s+\alpha t)(\nu-\mu)}2^{\nu}\|\partial_t v_\nu(t,\cdot)\|_{L^2}\|\Delta_\nu \Omega_2\partial_x v_\mu(t,\cdot)\|_{L^2}$$
$$\le \frac{1}{N}\frac{C}{1-s}\sum_{\nu\ge 0}\|\partial_t v_\nu(t,\cdot)\|_{L^2}^2 + C\|a\|_{\mathrm{Lip}}^2 N\sum_{\mu\ge 0}2^{2\mu}\|v_\mu(t,\cdot)\|_{L^2}.$$

This concludes the proof of the proposition. □

Proof of Lemma 3.13. The proof is very similar to that of [8, Prop. 3.7]. We detail it for the reader's convenience. We have

$$[\Delta_\nu, T_a^m]w = [\Delta_\nu, S_{m-1}a]S_{m+1}w + \sum_{k\ge m+2}[\Delta_\nu, S_{k-3}a]\Delta_k w$$

and get

$$\partial_{x_j}[\Delta_\nu, T_a^m]\partial_{x_h}w = \partial_{x_j}([\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w)) + \partial_{x_j}\left(\sum_{k\ge m+2}[\Delta_\nu, S_{k-3}a]\Delta_k(\partial_{x_h}w)\right).$$

Since Δ_ν and Δ_k commute, we have

$$\begin{aligned}[\Delta_\nu, S_{m-1}a]S_{m+1}w &= \Delta_\nu(S_{m-1}aS_{m+1}w) - S_{m-1}aS_{m+1}(\Delta_\nu w)\\ &= \Delta_\nu(S_{m-1}aS_{m+1}w) - S_{m-1}a\Delta_\nu(S_{m+1}w).\end{aligned}$$

This holds analogously for $[\Delta_\nu, S_{k-3}a]\Delta_k w$. Let us consider

$$\partial_{x_j}([\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w)) = \partial_{x_j}([\Delta_\nu, S_{m-1}a]\partial_{x_h}(S_{m+1}w)).$$

Looking at the spectrum of this term, we see that the term equals 0 if $\nu \geq m+4$. Moreover, the spectrum is contained in $\{|\xi| \leq 2^{m+3}\}$. From Bernstein's inequality, we have that

$$\|\partial_{x_j}([\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w))\|_{L^2} \leq 2^{m+3}\|[\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w)\|_{L^2}.$$

From the well known result of Coifman and Meyer [2, Th. 35], which essentially says that

$$\|[\Delta_\nu, b]\partial_x w\|_{L^2} \leq C\|\nabla_x b\|_{L^\infty}\|w\|_{L^2}, \tag{A.1}$$

where $b \in \mathrm{Lip}(\mathbb{R}^n_x)$ and $w \in H^1(\mathbb{R}^n_x)$, we get

$$\|[\Delta_\nu, S_{m-1}a]\partial_{x_h}(S_{m+1}w)\|_{L^2} \leq C\|a\|_{\mathrm{Lip}}\|S_{m+1}w\|_{L^2}.$$

Further, we have

$$\|S_{m+1}w\|_{L^2} \leq \sum_{k\leq m+1}\|\Delta_k w\|_{L^2} \leq C\sum_{k\leq m+1}2^{-(1-s-\alpha t)}\varepsilon_k,$$

where $\{\varepsilon_k\}_{k\in\mathbb{N}} \in l^2(\mathbb{N})$ with $\|\{\varepsilon_k\}_k\|_{l^2} \approx \|w\|_{H^{1-s-\alpha t}}$. Using now Hölder's inequality, we obtain

$$\|S_{m+1}w\|_{L^2} \leq C\left(\sum_{k\geq 0}2^{-2(1-s-\alpha t)}\right)^{\frac{1}{2}}\|w\|_{H^{1-s-\alpha t}} \leq \frac{C}{1-s}\|w\|_{H^{1-s-\alpha t}},$$

where we used the summation formula for the geometric sum as well as the assumption that $t \in [0, \frac{7}{8}\sigma]$, $\sigma := \frac{1-s}{\alpha}$. Consequently,

$$\|\partial_{x_j}([\Delta_\nu, S_{m-1}a]S_{m+1}(\partial_{x_h}w))\|_{L^2} \leq \frac{C}{1-s}\|a\|_{\mathrm{Lip}}\|w\|_{H^{1-s-\alpha t}},$$

and

$$\begin{aligned}&\sum_{\nu\geq 0}2^{-2(s+\alpha t)\nu}\left\|\partial_{x_j}[\Delta_\nu, S_{k-3}a]\Delta_k(\partial_{x_h}w)\right\|^2_{L^2}\\ &\quad= \sum_{0\leq\nu\leq m+3}2^{-2(s+\alpha t)\nu}\left\|\partial_{x_j}[\Delta_\nu, S_{k-3}a]\Delta_k(\partial_{x_h}w)\right\|^2_{L^2} \leq \frac{C_m}{(1-s)^2}\|a\|^2_{\mathrm{Lip}}\|w\|^2_{H^{1-s-\alpha t}}.\end{aligned} \tag{A.2}$$

It is worthy to remark that (A.2) can be obtained without using (A.1), since we can allow the constant C to depend on m.

Now, we consider

$$\partial_{x_j}\left(\sum_{k\geq m+2}[\Delta_\nu, S_{k-3}a]\Delta_k(\partial_{x_h}w)\right) = \partial_{x_j}\left(\sum_{k\geq m+2}[\Delta_\nu, S_{k-3}a]\partial_{x_h}(\Delta_k w)\right).$$

Looking at the spectrum of $([\Delta_\nu, S_{k-3}a]\Delta_k(\partial_{x_h}w))$, we see that $[\Delta_\nu, S_{k-3}a]\Delta_k(\partial_{x_h}w)$ is identically 0 if $|k-\nu| \geq 4$. This means that the sum runs over at most seven terms: from $\partial_{x_j}[\Delta_\nu, S_{\nu-6}a]\partial_{x_h}(\Delta_{\nu-3}w)$ up to $\partial_{x_j}[\Delta_\nu, S_\nu a]\partial_{x_h}(\Delta_{\nu+3}w)$, where each of them has a spectrum contained in a ball $\{|\xi| \leq C2^\nu\}$. We consider only one of these terms, e.g. $\partial_{x_j}[\Delta_\nu, S_{\nu-6}a]\partial_{x_h}(\Delta_{\nu-3}w)$ since the estimates for the others follow analogously. From Bernstein's inequality we get

$$\|\partial_{x_j}[\Delta_\nu, S_{\nu-6}a]\partial_{x_h}(\Delta_{\nu-3}w)\|_{L^2} \leq C2^\nu\|[\Delta_\nu, S_{\nu-6}a]\partial_{x_h}(\Delta_{\nu-3}w)\|_{L^2}$$

and, using again (A.1), we obtain

$$\|[\Delta_\nu, S_{\nu-6}a]\partial_{x_h}(\Delta_{\nu-3}w)\|_{L^2} \leq C\|a\|_{\mathrm{Lip}}\|\Delta_\nu w\|_{L^2},$$

where C does not depend on ν and in order to obtain this the use of Coifman and Meyer's result is essential. Hence, we have

$$\|\partial_{x_j}[\Delta_\nu, S_{\nu-6}a]\partial_{x_h}(\Delta_{\nu-3}w)\|_{L^2} \leq C2^\nu\|a\|_{\mathrm{Lip}}\|\Delta_\nu w\|_{L^2}.$$

Thus, squaring, multiplying by $2^{-2(s+\alpha t)\nu}$ and summing over ν, we get

$$\sum_{\nu\geq 0}2^{-2(s+\alpha t)\nu}\|\partial_{x_j}[\Delta_\nu, S_{\nu-6}]\partial_{x_h}(\Delta_{\nu-3}u)\|^2_{L^2} \leq C\|a\|^2_{\mathrm{Lip}}\sum_{\nu\geq 0}2^{2(1-s-\alpha t)\nu}\|\Delta_\nu w\|^2_{L^2}.$$

With $w \in H^{1-s-\alpha t}(\mathbb{R}^n_x)$ and Proposition 3.3, we finally get

$$\sum_{\nu\geq 0}2^{-2(s+\alpha t)\nu}\|\partial_{x_j}[\Delta_\nu, S_{\nu-6}]\partial_{x_h}(\Delta_{\nu-3}u)\|^2_{L^2} \leq C\|a\|^2_{\mathrm{Lip}}\|w\|^2_{H^{1-s-\alpha t}}.$$

As already mentioned, the other terms can be treated in the same way. We finally get

$$\sum_{\nu\geq 0} 2^{-2(s+\alpha t)\nu} \left\| \partial_{x_j} \left(\sum_{k\geq m+2} [\Delta_\nu, S_{k-3}] \partial_{x_h} (\Delta_k u) \right) \right\|_{L^2}^2 \leq C \|a\|_{\mathrm{Lip}}^2 \|w\|_{H^{1-s-\alpha t}}^2 . \tag{A.3}$$

Putting (A.2) and (A.3) together, and using the notation $v_\nu = 2^{-(s+\alpha t)\nu} w_\nu$, concludes the proof of the proposition. □

References

[1] S. Agmon, L. Nirenberg, Properties of solutions of ordinary differential equations in Banach spaces, Comm. Pure Appl. Math. 16 (1963) 121–239. http://dx.doi.org/10.1002/cpa.3160160204.
[2] R. Coifman, Y. Meyer, Au-Delà des Opérateurs Pseudo-Différentiel, in: Astérisque, vol. 57, Société Mathématique de France, Paris, 1978.
[3] F. Colombini, N. Lerner, Hyperbolic operators having non-Lipschitz coefficients, Duke Math. J. 77 (3) (1995) 657–698.
[4] F. Colombini, G. Métivier, The Cauchy problem for wave equations with non-Lipschitz coefficients; application to unique continuation of solutions of some nonlinear wave equations, Ann. Sci. Éc. Norm. Supér. (4) 41 (2) (2008) 177–220.
[5] D. Del Santo, Ch.P. Jäh, M. Paicu, Backward-uniqueness for parabolic operators with non-Lipschitz coefficients, Osaka J. Math. (2014) in press arXiv:1404.7405.
[6] D. Del Santo, M. Prizzi, Backward uniqueness for parabolic operators whose coefficients are non-Lipschitz continuous in time, J. Math. Pures Appl. (9) 84 (4) (2005) 471–491. http://dx.doi.org/10.1016/j.matpur.2004.09.004.
[7] D. Del Santo, M. Prizzi, Continuous dependence for backward parabolic operators with Log-Lipschitz coefficients, Math. Ann. 345 (1) (2009) 213–243. http://dx.doi.org/10.1007/s00208-009-0353-5.
[8] D. Del Santo, M. Prizzi, A new result on backward uniqueness for parabolic operators, Ann. Mat. Pura Appl. (4) (2014). http://dx.doi.org/10.1007/s10231-013-0381-3.
[9] R.Ja. Glagoleva, Continuous dependence on initial data of the solution to the first boundary value problem for parabolic equations with negative time, Sov. Math. Dokl. 4 (1963) 13–17. English. Russian original, translation from Dokl. Akad. Nauk SSSR 148 (1963), 20–23.
[10] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Dover Publications, New York, 1953.
[11] J. Hadamard, La Théorie des Équations aux Dérivées Partielles, Éditions Scientifiques, Gauthier-Villars Éditeur, Peking, Paris, 1964.
[12] A.E. Hurd, Backward continuous dependence for mixed parabolic problems, Duke Math. J. 34 (1967) 493–500.
[13] F. John, Continuous dependence on data for solutions of partial differential equations with a prescribed bound, Comm. Pure Appl. Math. 13 (1960) 551–585.
[14] N. Mandache, On a counterexample concerning unique continuation for elliptic equations in divergence form with Hölder continuous coefficients, Math. Phys. Anal. Geom. 1 (3) (1998) 273–292.
[15] G. Métivier, Para-differential Calculus and Applications to the Cauchy Problem for Nonlinear Systems, in: Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, vol. 5, Edizioni della Normale, Pisa, 2008.
[16] K. Miller, Nonunique continuation for uniformly parabolic and elliptic equations in self-adjoint divergence form with Hölder continuous coefficients, Arch. Ration. Mech. Anal. 54 (1974) 105–117.
[17] A. Tychonoff, Théorèmes d'unicité pour l'équation de la chaleur, Mat. Sb. 42 (2) (1935) 199–215.