

Dynamically self-regular quantum harmonic black holes



Euro Spallucci*, Anais Smailagic

Dipartimento di Fisica Teorica, Università di Trieste and INFN, Sezione di Trieste, Italy

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ABSTRACT

The recently proposed UV self-complete quantum gravity program is a new and very interesting way to envision Planckian/trans-Planckian physics. In this new framework, high energy scattering is dominated by the creation of micro black holes, and it is experimentally impossible to probe distances shorter than the horizon radius. In this letter we present a model which realizes this idea through the creation of *self-regular* quantum black holes admitting a minimal size extremal configuration. Their radius provides a *dynamically generated minimal length* acting as a universal short-distance cutoff. We propose a quantization scheme for this new kind of microscopic objects based on a Bohr-like approach, which does not require a detailed knowledge of quantum gravity. The resulting black hole quantum picture resembles the energy spectrum of a quantum harmonic oscillator. The mass of the extremal configuration plays the role of zero-point energy. Large quantum number re-establishes the classical black hole description. Finally, we also formulate a “quantum hoop conjecture” which is satisfied by all the mass eigenstates and sustains the existence of quantum black holes sourced by Gaussian matter distributions.

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1. Introduction

The idea that non-perturbative quantum gravity can “cure” ultraviolet divergences, including its own, dates back to the seventies [1–3]; the idea was further developed by Sri Ram and Dass [4] and very recently embodied in the so-called “UV self-complete quantum gravity” by Dvali and collaborators [5,6]. The novelty of this approach consists in the assumption that Planckian energy scattering will be dominated by the production of micro black holes (BHs). So far, the paradigm of modern high-energy physics is that the energy of an accelerated particle allows to probe shorter and shorter distances without any lower bound. The present LHC peak energy, 14 TeV, sets the experimental limit up to 10^{-17} cm. Hypothetically, an ultra-Planckian particle accelerator would be even able to probe distances below 10^{-33} cm. Although there is no chance to build such a machine in a foreseeable future, the theoretical argument remains valid.

Nevertheless, if one considers the collision of two elementary particles with high enough center of mass energy and small impact parameter, a huge energy concentration would be reached requiring, according to UV self-complete quantum gravity hypothesis, a proper account of non-perturbative gravitational effects. Such a situation is expected to lead to the creation of a micro BH, as

a realization of the “quantum hoop conjecture” (QHC). QHC extends the classical statement that a macroscopic object of arbitrary shape, of mass M , passing through a ring of radius $R = 2MG$, will necessarily collapse into a BH [7]. In the quantum case,¹ the macroscopic object is replaced by the target-projectile pair and the condition for a BH creation is $2\sqrt{s}G \leq b$, where \sqrt{s} is the total center of mass energy of the colliding system, G the gravitational coupling constant and b is the impact parameter. Thus, whenever the effective Schwarzschild radius is lower or equal to the impact parameter, the BH production channel opens up. If Planckian scattering regime is BH creation dominated [9–17], the idea higher-energy/shorter-distance needs a substantial revision [18–24]. In other words, increasing \sqrt{s} instead of reaching lower and lower wavelengths, stops at the threshold of BH creation. Any further energy increase leads to growing BHs, thus shielding distances below their horizon from experimental reach.

This idea has been recently incorporated in the framework of large extra-dimension models, where quantum gravity effects are expected to be dominant around 10–100 TeV. In this scenario “TeV BHs” production is slightly above the LHC energy and hopefully reachable by the next generation particle colliders. Far beyond this energy scale, contrary to the standard expectation, gravitational dynamics becomes classical again (“large BHs”), thus

* Corresponding author.

E-mail addresses: spallucci@ts.infn.it (E. Spallucci), anaiss@ts.infn.it (A. Smailagic).

¹ So far, there is no unique formulation of QHC. For an alternative definition see [8].

ultra-Planckian regime is unexpectedly dominated by *classical* field configurations, i.e. “classicalons” [6,25–27].

If minimal size BHs are to be produced, their length scale will serve as a natural short-distance cut-off, i.e. a *minimal length*. The existence of a minimal length, l_0 , in the space-time fabric was also implied by different approaches to quantum gravity including string theory, loop quantum gravity, non-commutative geometry, etc. [28]. From a conventional point of view, l_0 is identified with the Planck length $l_P = \sqrt{G}$, but as described above could be lowered near *TeV* scale. Therefore, it can be expected that *TeV* BHs should be sensible to the presence of l_0 .

In a series of papers we have given BH solutions naturally incorporating $l_0 (= \sqrt{\theta})$, where θ is a parameter measuring the amount of coordinate non-commutativity at short distance. In other papers, l_0 was engraved in the space-time fabric through a $*$ -product embedded into the very definition of the metric tensor $g_{\mu\nu}$, in terms of the vierbein field e^a_μ :

$$\hat{g}_{\mu\nu} \equiv \eta_{ab} e^a_\mu * e^b_\nu \quad (1)$$

The latter approach faces the basic difficulty that any attempt to solve the Einstein equations requires a truncated perturbative expansion in l_0 , leading to an effective field theory with derivative couplings of arbitrary order. The resulting Feynman expansion still contains planar graphs which are divergent one by one, in spite of the presence of l_0 [29,30]. This is the consequence of the (truncated) perturbative treatment which changes the original meaning of l_0 from a natural UV cut-off into the (dimensional) strength of non-renormalizable derivative interactions. The difficulty with the perturbative treatment of the $*$ -product can be summarized as follows: in spite of the presence of l_0 in the theory, some of the resulting Feynman diagrams remain divergent. To have a genuine non-perturbative approach we argued that the effects of l_0 can be implemented correctly in General Relativity by keeping the standard form of the Einstein tensor in the l.h.s. of the field equations and introduced an energy-momentum tensor with a modified source [31–36]. The resulting solution for *neutral, non-rotating*, BH exhibits:

- “regularity”, i.e. absence of curvature singularities;
- extremal configuration corresponding to a minimal size near l_0 .

Regularity is an immediate consequence of the presence of l_0 in the space-time geometry, while the existence of a minimal mass, extremal configuration, is a surprising property, at least from the point of view of the BH textbook solutions.

In the first part of this letter we present the regular Schwarzschild solution that exhibits extremal configuration with radius $r_0 = l_0$. This is what one expects in a theory where distances below l_0 have no physical meaning.

All up to date experiments indicate that $l_0 < 10^{-17}$ cm, which means that minimal BHs created in a Planckian collision, will be certainly quantum objects. Thus, neither classical nor semi-classical description is satisfactory and one should quantize BHs themselves.

In the absence of a proper quantum mechanical description of BHs, we propose a quantization scheme based on the analogy with the quantum harmonic oscillator. This quantization scheme is discussed in Section 3, where we also provide a new formulation of QHC. Finally, in Section 4 we summarize the main results obtained.

2. Self-regular Schwarzschild solution

In this section we construct regular Schwarzschild solution of the Einstein equations, where the minimal length is dynamically induced, in a self-consistent way.

We are looking for a static, spherically symmetric, asymptotically flat metric of the form

$$ds^2 = - \left(1 - \frac{2m(r)}{r} \right) dt^2 + \left(1 - \frac{2m(r)}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (2)$$

where (r, θ, ϕ) are standard polar coordinates and t is the time measured by an asymptotic Minkowskian observer; $m(r)$ is an unknown function determined by the Einstein equations once the source is given. An energy-momentum tensor compatible with the symmetry of the problem is the one of an anisotropic fluid:

$$T^\nu_\mu = p_\theta \delta^\nu_\mu + (\rho + p_\theta) (u_\mu u^\nu - l_\mu l^\nu), \quad (3)$$

$$p_r + \rho = 0, \quad (4)$$

$$T^\nu_{\mu; \nu} = 0 \quad (5)$$

In the chosen coordinate system $u^\mu = \delta^\mu_0$, $l^\mu = \sqrt{g_{rr}} \delta^\mu_r$; ρ is the energy density, p_r is the radial pressure and p_θ is the tangential pressure determined in terms of ρ by the covariant divergence-free condition (5).

From the Einstein equations one finds

$$\begin{aligned} m(r) &= -4\pi \int_0^r dr' r'^2 T^\nu_{\nu} \\ &= 4\pi \int_0^r dr' r'^2 \rho(r') \end{aligned} \quad (6)$$

The textbook Schwarzschild solution for a BH of mass M is obtained by the choice

$$\rho = M \frac{\delta(r)}{4\pi r^2} \quad (7)$$

which describes a *point-like* source. In our case we choose a smeared matter distribution given by a Gaussian as:

$$\rho(r) \equiv M \sigma(r) = \left(\frac{3}{l_0} \right)^3 \frac{M}{(4\pi)^{3/2}} \exp \left(-\frac{9r^2}{4l_0^2} \right) \quad (8)$$

where M is the total mass-energy of the system as measured by an asymptotic Minkowskian observer:

$$M \equiv \lim_{r \rightarrow \infty} m(r) = 4\pi \int_0^\infty dr' r'^2 \rho(r') \quad (9)$$

This choice draws its motivation from the fact that in ordinary Quantum Mechanics the minimal uncertainty states, i.e. the closest states to a point-like object, are given by Gaussian wave-packets. In the limit $l_0 \rightarrow 0$ the function (8) goes into the singular density (7).

By inserting (8) in (6) we obtain

$$\begin{aligned} ds^2 &= - \left(1 - \frac{4M}{\sqrt{\pi} r} \gamma \left(3/2, 9r^2/4l_0^2 \right) \right) dt^2 \\ &\quad + \left(1 - \frac{4M}{\sqrt{\pi} r} \gamma \left(3/2, 9r^2/4l_0^2 \right) \right)^{-1} dr^2 + r^2 d\Omega^2, \end{aligned} \quad (10)$$

where the incomplete gamma function γ is defined as

$$\gamma(a/b; x) \equiv \int_0^x \frac{du}{u} u^{a/b} e^{-u} \quad (11)$$

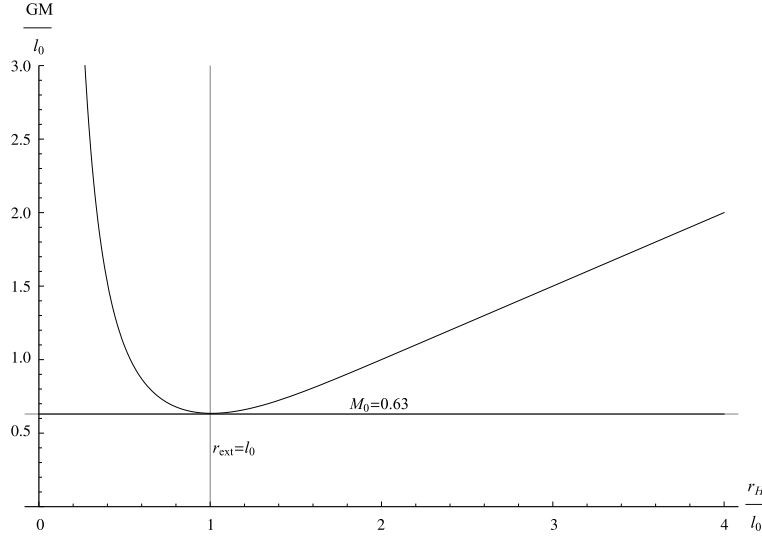


Fig. 1. Plot of the mass M as a function of the radius of the horizon.

As a consistency check, we showed the relation $M = M_{ADM}$, where M_{ADM} is the Arnowitt, Deser, Misner mass [37] derived from the metric (10). The calculation is straightforward and will be reproduced here.

Horizons correspond to the solutions of the equation

$$g_{rr}^{-1} = 0 \longrightarrow M = \frac{\sqrt{\pi} r_h}{4\gamma (3/2, 9r_h^2/4l_0^2)} \quad (12)$$

Eq. (12) cannot be solved analytically as in the standard Schwarzschild case, but by plotting the function $M = M(r)$ (see Fig. 1) one sees the existence of a pair of horizons, merging into a single, degenerate horizon at the minimum with estimated radius

$$r_{min.} = l_0 + 0.01 \times l_0 \quad (13)$$

Neglecting the one per cent corrections, the minimum mass results to be

$$M_0 = \frac{l_0 \sqrt{\pi}}{4\gamma (3/2, 9/4)} \quad (14)$$

Thus, for any $M > M_0$ the solution describes a *non-extremal* BH of radius $r_+ > l_0$. For $M = M_0$ we have a *minimal-size, extremal* BH of radius l_0 which gives a physical meaning to the, up to now arbitrarily introduced, cut-off l_0 . In other words, the existence of a minimal length is a strict consequence of the existence of minimal size BH of the same radius. This goes under the name of *self-regular* BH meaning that the non-perturbative dynamics of gravity determines a natural cut-off, thus realizing the UV self-completeness hypothesis in this model.

3. Bohr quantization of micro BHs

In this section we present a Bohr-like quantization of BHs. For the sake of simplicity we limit ourselves to neutral objects only.

From the discussion regarding neutral BHs in Section 2 it results that quantum effects are dominant in the near extremal region where the behavior of the function $M(r_+)$ significantly differs from the usual Schwarzschild case.

We shall follow a Bohr-like quantization scheme which does not require the knowledge of a full quantum gravity theory. The idea

comes from the form of (8) which is reminiscent of the ground-state for an isotropic, 3D, harmonic oscillator²

$$\sigma(r) \longleftrightarrow |\psi_{000}(r)|^2 \quad (15)$$

where

$$\psi_{000}(r) \propto e^{-mr^2\omega/2} \quad (16)$$

is the ground state wave function [49]. To relate the two different systems, i.e. our BH and the quantum harmonic oscillator, we establish a formal correspondence between the mass m and the angular frequency ω with the corresponding quantities in (8)

$$m\omega = \frac{9}{2l_0^2} \quad (17)$$

In this identification the mass of the extremal BH represents the equivalent of the ground-state energy of the harmonic oscillator, i.e. M_0 is the zero-point energy

$$\frac{3}{2}\omega = M_0 \quad (18)$$

By solving the two equations (17), (18) we find

$$m = \frac{27}{4} \frac{1}{l_0^2 M_0}, \quad \omega = \frac{2}{3} M_0 \quad (19)$$

The need of BH mass quantization has been also recently stressed in [38]. Thus, we describe non-extremal BHs as “excited” energy states labeled by an integer *principal quantum number* n as

$$M_n = \frac{2}{3} M_0 \left(n + \frac{3}{2} \right), \quad n = 0, 2, 4, \dots \quad (20)$$

Due to the spherical symmetry only even oscillator states are allowed for the 3D, isotropic, harmonic oscillator. In this quantization scheme the extremal BH configuration with $r_0 = l_0$ represents the *zero-point energy* of the gravitational system. The result can be interpreted as the realization of earlier attempts to dynamically generate a *zero-point length* [39–42] thus eliminating ultraviolet divergences through quantum fluctuations of gravity itself.

² A similar idea has been recently proposed in [43] in the framework of Bose–Einstein condensates model of quantum BHs [44–48].

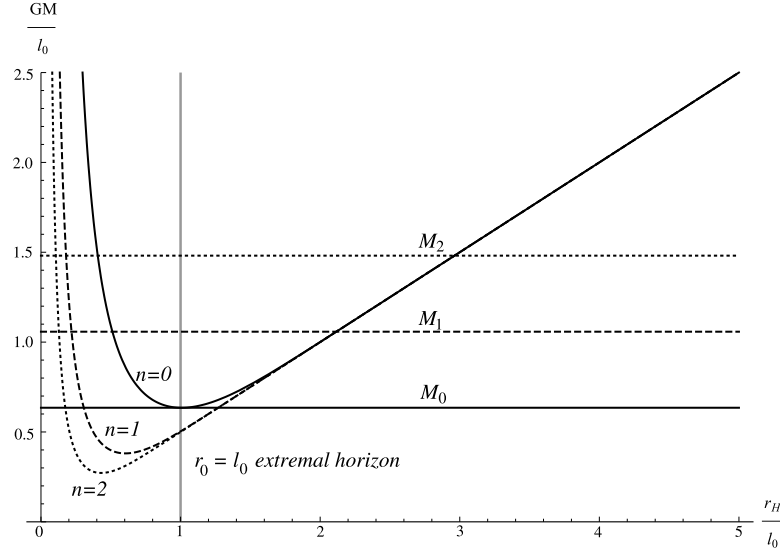


Fig. 2. Plot of the function $\mathcal{M}_n(r)$ for $n = 0, 2, 4$. Horizontal lines correspond to quantized mass levels.

Following further analogy with Bohr quantization, where the quantum/classical transition is achieved for large n , we redefine a “quantum” mass/energy distribution which for large n approaches a Dirac delta sourcing a standard Schwarzschild metric [31,32,34]. In other words, we are adopting a kind of Correspondence Principle, *a la* Bohr, applied to the matter energy density:

$$\rho_n(r) \equiv M_n \sigma_n(r) = \left(\frac{n+3/2}{l_0} \right)^3 \frac{M_n}{\pi^{3/2}} \exp \left[-\frac{r^2}{l_0^2} (n+3/2)^2 \right] \quad (21)$$

Notice that

$$\rho_n(r) \longrightarrow M_n \frac{\delta(r)}{4\pi r^2}, \quad n \gg 1 \quad (22)$$

leading to a standard Schwarzschild geometry. One may wonder why we do not push the analogy to an extreme and use the known excited state wave-functions of the harmonic oscillator. We considered this approach but did not pursue it for the following reasons:

- We called our BH quantization conjecture “Bohr-like”, in the same spirit in which old quantum mechanics was formulated before Schrödinger, since there is no wave-equation for quantum BHs as there is for quantum harmonic oscillator.
- Even if one ignores the previous comment, the use of excited harmonic oscillator wave-functions leads to multi-horizon geometries with an increasing number of different extremal configurations. The resulting excited BHs have a geometrical structure completely different from the minimal size, self-regular, solution we started from.

Therefore, the “quantized” version of (10) reads (see Fig. 2)

$$ds^2 = -g_{00} dt^2 + g_{00}^{-1} dr^2 + r^2 d\Omega^2 \quad (23)$$

$$g_{00} = 1 - \frac{2\mathcal{M}_n(r)}{r}, \quad (24)$$

$$\mathcal{M}_n(r) = M_n \gamma \left[3/2, \frac{r^2}{l_0^2} (n+3/2)^2 \right] / \Gamma(3/2) \quad (25)$$

Several comments are in order.

First, due to the identification of the extremal BH with the $n = 0$ state, we obtain the *zero-point metric*, with quantization picking up only the ground-state mass $M = M_0$.

Second, the metric (25), for “large- n ” reduces to the ordinary Schwarzschild line element while still keeping a quantized mass spectrum.

$$ds^2 = - \left(1 - \frac{2M_n}{r} \right) dt^2 + \left(1 - \frac{2M_n}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 \quad (26)$$

where $M_n \approx 2nM_0/3$. This limit is due to the fact $\gamma(nx) \rightarrow \Gamma(n)$ for n large enough. As a matter of fact, already for $n > 2$ the metric (26) is a very good approximation of the exact quantum metric (25). The relative energy difference between nearby levels $\Delta M_n/M \sim 1/n$ for $n \gg 1$ and the mass spectrum becomes *effectively* continuous.

Two limiting cases are of particular interest: ground state $n = 0$, and the *classical (large BH)* limit $n \gg 1$.

One expects that the ground state of the system is only “vacuum energy”, i.e. that the extremal BH configuration is only a vacuum-fluctuation. From the effective geometry (25) one obtains the “semi-classical” horizon equation

$$r_+ = 2M_n \gamma \left(3/2, \frac{r_+^2}{l_0^2} (n+3/2)^2 \right) / \Gamma(3/2) \quad (27)$$

which we translate into a quantum framework as the equation for the average values of horizon radius in a given quantum state. We identify the trivial solution $r_+ = 0$ as the vacuum average value:

$$\langle 0 | r_+ | 0 \rangle = 0 \Leftrightarrow r_+ = 0 \quad (28)$$

However, this vanishing mean value has an “uncertainty”:

$$\Delta r_+ = \sqrt{\langle 0 | r_+^2 | 0 \rangle} \quad (29)$$

By squaring (27) we get the equation for the vacuum average of r_+^2 :

$$\langle 0 | r_+^2 | 0 \rangle = \left[2M_0 \gamma \left(3/2, \frac{9 \langle 0 | r_+^2 | 0 \rangle}{4l_0^2} \right) / \Gamma(3/2) \right]^2 \quad (30)$$

By taking into account the definition (14) of M_0 it is immediate to check that Eq. (30) is solved by

$$\langle 0 | r_+^2 | 0 \rangle = l_0^2 \quad (31)$$

and $\Delta r_+ = l_0$.

In the large- n limit Eq. (27) reduces to the definition of the classical Schwarzschild radius

$$\langle n | r_+ | n \rangle = 2M_n, \quad n \gg 1 \quad (32)$$

and

$$\langle n | r_+^2 | n \rangle = 4M_n^2 \longrightarrow \Delta r_+ \rightarrow 0 \quad (33)$$

As it was expected, the extremal BH, corresponding to the zero-point energy of the system, is a pure quantum fluctuation, while highly excited states behave as “classical” objects ($\Delta r_+ = 0$) described by an effective Schwarzschild metric. Thanks to the properties of the γ -function, a good approximation of the full spectrum is

$$\langle n | r_+ | n \rangle \simeq 2M_n \gamma\left(\frac{3}{2}, n^2\right) / \Gamma(3/2) \quad (34)$$

3.1. Quantum hoop conjecture

Strictly speaking, the density (21) is non-vanishing everywhere, even if it quickly drops to zero already at distances of few l_0 . Nevertheless, the skeptics may rise the question whether a BH can be formed at all by such a smeared distribution. In order to remove these doubts, we evoke the classical *hoop conjecture* [7] and adapt it to the present situation.³ Firstly, we define a *mean radius* of the mass distribution and the mean value of the square radius, as

$$\langle n | r | n \rangle \equiv 4\pi \int_0^\infty dr r^3 \sigma_n(r) = \frac{l_0}{(n+3/2)} \frac{1}{\Gamma(3/2)}, \quad (35)$$

$$\langle n | r^2 | n \rangle \equiv 4\pi \int_0^\infty dr r^4 \sigma_n(r) = \frac{l_0^2}{(n+3/2)^2} \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma(3/2)} \quad (36)$$

Secondly, we evaluate the mean square deviation as

$$\begin{aligned} \Delta r_n^2 &\equiv \langle n | r^2 | n \rangle - \langle n | r | n \rangle^2 \\ &= \frac{l_0^2}{(n+3/2)^2} \left[\frac{3\pi}{8} - 1 \right] \end{aligned} \quad (37)$$

which vanishes for large n : $\Delta r_n \rightarrow 0$.

Finally, we define the quantum hoop conjecture as the condition that *whenever the mean radius of the mass distribution is smaller than the mean value of the horizon radius, then a quantum BH forms*:

$$\langle n | r | n \rangle \leq \langle n | r_+ | n \rangle \quad (38)$$

which leads to the relation

$$\frac{l_0}{n+3/2} \leq 2M_n \gamma\left[\frac{3}{2}; n^2\right] \quad (39)$$

It is sufficient to verify (39) in the “worst case scenario” $n = 1$, which turns out to be satisfied. For larger n the width of the Gaussian distribution shrinks while the radius of the horizon increases, maintaining the QHC.

³ A quantum formulation of the hoop conjecture has been recently proposed in [8].

4. Conclusions and discussion

In the first part of this letter we described a regular Schwarzschild geometry, incorporating a “minimal” length l_0 which, due to gravitational quantum dynamics, turns out to be the radius of the minimal size extremal configuration. This solution is a first realization, within a specific model, of the UV self-complete quantum gravity program. In other words, quantum gravity effects dynamically generate a short distance cut-off shielding the Planck scale physics from experimental probe. In the last part of this work, we have proposed a Bohr-like BH quantization scheme. For the sake of computational simplicity, we have described neutral quantum BHs. The generalization to charged BHs will follow the same pattern in a bit more technically involved manner.

The main outcome of the proposed quantization scheme can be listed as:

- Any phenomenon occurring at distances smaller than l_0 is not experimentally measurable.
- The mass of the extremal configuration is equivalent to the zero-point energy of the quantum harmonic oscillator.
- The non-extremal configurations correspond to harmonic oscillators excited states, with a discrete mass spectrum of equally spaced levels.
- The model satisfies a Bohr-like Correspondence Principle for large n , where it reproduces standard Schwarzschild BH.
- We also formulated a “quantum hoop conjecture”, which supports the existence of quantum BHs whenever the condition $\langle n | r | n \rangle \leq \langle n | r_+ | n \rangle$ is met.
- One may wonder what is the thermodynamics of quantum BHs. We believe that concepts like the Hawking temperature and the Bekenstein entropy refer to semi-classical BHs where mass and size are continuous variables. Intrinsically quantum BHs do not radiate thermally being stationary state configurations. Non-thermal quantum BHs have been recently discussed in [50]. In the quantum phase absorption and emission proceed through discrete quantum jumps between different energy states, instead of emitting a continuous thermal spectrum. Thus, there is no Hawking radiation at the quantum level and BHs are just another kind of “particle” in the quantum zoo.

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