IL NUOVO CIMENTO **38 C** (2015) 164
DOI 10.1393/ncc/i2015-15164-4

Colloquia: VILASIFEST

Fock modules and noncommutative line bundles

Giovanni Landi(*)

Università di Trieste and INFN, Sezione di Trieste - Trieste, Italy

received 11 January 2016

Dedicato a Gaetano Vilasi con amicizia e stima.

Summary. — To a line bundle over a noncommutative space there is naturally associated a Fock module. The algebra of corresponding creation and annihilation operators is the total space algebra of a principal $U(1)$-bundle over the noncommutative space. We describe the general construction and illustrate it with examples.

PACS `02.40.Gh` – Noncommutative geometry.
PACS `03.65.Fd` – Algebraic methods.

1. – Introduction

Algebraically, a vector bundle $M \to X$ over a (compact finite-dimensional) manifold M is completely characterized by its smooth sections $\Gamma(M, X)$. In this context, the space of sections is a (right) module over the algebra $C^\infty(X)$ of smooth functions over M. Indeed, by the Serre-Swan theorem (initially stated for continuos functions and sections [22], and extended to the smooth case [8]), finite-rank complex vector bundles over a compact Hausdorff space M correspond canonically to finite projective modules over the algebra $C^\infty(X)$. Indeed, by this theorem a $C^\infty(X)$-module $\mathcal{E}$ is isomorphic to a module $\Gamma(M, X)$ of smooth sections, if and only if it is finite projective.

For a Hermitian bundle there is extra structure: the Hermitian inner product $\langle \cdot, \cdot \rangle_x$ on each fiber M_x, $x \in X$, gives a $C^\infty(X)$-valued Hermitian map on the module $\Gamma(M, X)$,

$$(\mathbf{1}.1) \qquad \langle \cdot, \cdot \rangle : \Gamma(M, X) \times \Gamma(M, X) \to C^\infty(X), \qquad \langle \xi, \eta \rangle(x) =: \langle \xi(x), \eta(x) \rangle_x,$$

(*) E-mail: `landi@units.it`

for any $\xi, \eta \in \Gamma(M, X)$. By its definition, this map satisfy the properties:

$$\begin{aligned} &\langle \xi, \eta b \rangle = \langle \xi, \eta \rangle b, \qquad \langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle, \\ \text{and} \quad &\langle \eta, \eta \rangle \geq 0, \quad \langle \eta, \eta \rangle = 0 \Leftrightarrow \eta = 0, \qquad \text{for} \quad b \in C^\infty(X), \; \xi, \eta \in \Gamma(M, X). \end{aligned} \tag{1.2}$$

Next, let $\mathrm{End}(M) \to X$ be the endomorphism bundle with corresponding sections $\Gamma(\mathrm{End}(M), X)$. The latter is an algebra under composition and there is an identification

$$\Gamma(\mathrm{End}(M), X) \simeq \mathrm{End}_{C^\infty(X)}(\Gamma(M, X)),$$

with the algebra of $C^\infty(X)$-endomorphisms of the module $\Gamma(M, X)$. By its definition $\mathrm{End}_{C^\infty(X)}(\Gamma(M, X))$ acts on the left on the module $\Gamma(M, X)$. Moreover, in parallel with (**1**.1) there is a $\mathrm{End}_{C^\infty(X)}(\Gamma(M, X))$-valued Hermitian product on $\Gamma(M, X)$

$$|\cdot\rangle\langle\cdot| : \Gamma(M, X) \times \Gamma(M, X) \to \mathrm{End}_{C^\infty(X)}(\Gamma(M, X)), \tag{1.3}$$

where, for any $\xi, \eta \in \Gamma(M, X)$, the endomorphism $\xi\langle\eta|$ is defined by

$$|\xi\rangle\langle\eta|(\xi) := \xi\langle\eta, \xi\rangle \quad \text{for} \quad \xi \in \Gamma(M, X). \tag{1.4}$$

This Hermitian product has properties analogous to the one in (**1**.2), with linearity now in the first entry, that is, for $T \in \mathrm{End}_{C^\infty(X)}(\Gamma(M, X))$ and $\xi, \eta \in \Gamma(M, X)$:

$$\begin{aligned} &|T\xi\rangle\langle\eta| = T\langle\xi, \eta\rangle, \qquad (|\xi\rangle\langle\eta|)^* = |\eta\rangle\langle\xi|, \\ \text{and} \quad &|\eta\rangle\langle\eta| \geq 0, \quad |\eta\rangle\langle\eta| = 0 \;\Leftrightarrow\; \eta = 0. \end{aligned} \tag{1.5}$$

The fact that $\Gamma(M, X)$ is a $(\mathrm{End}_{C^\infty(X)}(\Gamma(M, X)), C^\infty(X))$-bimodule and is endowed with two Hermitian products which are compatible in a sense that generalises the relation (**1**.4), put it in the context of Morita equivalence that we shall describe in sect. **2**.

On the other hand, one sees that the vector bundle $M \to X$ is a line bundle if and only if $\mathrm{End}_{C^\infty(X)}(\Gamma(M, X)) \simeq C^\infty(X)$. This motivates calling noncommutative line bundle over the noncommutative algebra $\mathcal{B}$ (having the role of $C^\infty(X)$), a self-Morita equivalence bimodule for $\mathcal{B}$, that is a $\mathcal{B}$-bimodule $\mathcal{E}$ (having the role of $(\Gamma(M, X))$ with extra structures (roughly, two compatible $\mathcal{B}$-valued Hermitian products on $\mathcal{E}$).

In the present paper we illustrate how to naturally associate a Fock module over the (noncommutative) algebra $\mathcal{B}$ to any such a noncommutative line bundle over the algebra $\mathcal{B}$ of the base space. The algebra of corresponding creation and annihilation operators acting on a Hilbert module (or rigged Hilbert space) can then be realised as the total space algebra of a noncommutative principal $U(1)$-bundle over the algebra $\mathcal{B}$.

2. – Hilbert modules and Morita equivalence

Hilbert modules generalize Hilbert spaces, with the complex algebra of scalars $\mathbb{C}$ replaced by a complex $*$-algebra $\mathcal{B}$. Geometrically (and in the light of Gel'fand-Naimark and Serre-Swan theorems), as allude to in the previous section, a Hilbert module over $\mathcal{B}$ is the module of sections of a noncommutative Hermitian vector bundle over the noncommutative space "dual" to the algebra $\mathcal{B}$.

We recall here some of the definitions and results that we need later on in the paper. Our main references for this section are [14, 19]. Typically, we let $\mathcal{B}$ denote a pre-C^*-algebra, that is a normed $*$-algebra, whose closure is a C^*-algebra denoted B; the algebra $\mathcal{B}$ has the role of smooth functions while its closure B that of continuous functions.

2˙1. *Hilbert-modules*. – A right *pre-Hilbert-module* over $\mathcal{B}$ is a right $\mathcal{B}$-module $\mathcal{E}$ with a $\mathcal{B}$-valued Hermitian product $\langle\cdot,\cdot\rangle_{\mathcal{B}} : \mathcal{E}\times\mathcal{E}\to\mathcal{B}$ satisfying the conditions:

$$\langle\xi,\eta b\rangle_{\mathcal{B}} = \langle\xi,\eta\rangle_{\mathcal{B}} b, \qquad \langle\xi,\eta\rangle_{\mathcal{B}} = \left(\langle\eta,\xi\rangle_{\mathcal{B}}\right)^*,$$
$$\text{and} \qquad \langle\xi,\xi\rangle_{\mathcal{B}} \geq 0, \quad \langle\xi,\xi\rangle_{\mathcal{B}} = 0 \Leftrightarrow \xi = 0,$$

for all $\xi,\eta\in\mathcal{E}$ and for all $b\in\mathcal{B}$. In fact, the element $\langle\xi,\xi\rangle_{\mathcal{B}}$ is required to be positive in the completion B. An alternative name for $\mathcal{E}$ is a right $\mathcal{B}$-*rigged Hilbert space*.

By Lemma 2.16 of [19], a pre-Hilbert-module $\mathcal{E}$ over $\mathcal{B}$ can be completed to a Hilbert-module E over B, where E is the completion of $\mathcal{E}$ in the norm $\|\cdot\|_{\mathcal{E}}$ on $\mathcal{E}$ defined by

$$\|\xi\|_{\mathcal{E}} = \left(\|\langle\xi,\xi\rangle_{\mathcal{B}}\|_B\right)^{1/2}, \tag{2.1}$$

using the norm $\|\cdot\|_B$ on B. One says that the pre-Hilbert-module $\mathcal{E}$ (or better the Hilbert-module E) is *full* if the ideal $\langle\mathcal{E},\mathcal{E}\rangle_{\mathcal{B}} := \mathrm{Span}_{\mathcal{B}}\{\langle\xi,\eta\rangle_{\mathcal{B}}\,|\,\xi,\eta\in\mathcal{E}\}$ is dense in B. There are analogous definitions for *left* modules, with Hermitian product denoted ${}_{\mathcal{B}}\langle,\rangle$ and taken to be $\mathcal{B}$-linear in the first entry. Clearly, when $B=\mathbb{C}$ a Hilbert module is a usual Hilbert space. To lighten notations we write $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\mathcal{B}}$ whenever possible and use the name Hilbert $\mathcal{B}$-module to mean a (pre-)Hilbert module over the $*$-algebra $\mathcal{B}$.

The simplest example is the algebra $\mathcal{B}$ itself with respect to the Hermitian product $\langle a,b\rangle = a^*b$. The Hilbert module $\mathcal{B}^n$ consists of n-tuples of elements of $\mathcal{B}$, with component-wise operations, and with Hermitian product defined by

$$\langle(a_1,\ldots,a_n),(b_1,\ldots b_n)\rangle = \sum\nolimits_{j=1}^{n}\langle a_j,b_j\rangle.$$

A Hilbert $\mathcal{B}$-module $\mathcal{E}$ is (*algebraically*) *finitely generated* if there exists a finite collection $\{\eta_i\}_{j=1}^n$ of elements of $\mathcal{E}$ such that every $\xi\in\mathcal{E}$ is of the form $\xi=\sum_{j=1}^n \eta_j b_j$ for some b_j's in $\mathcal{B}$. A Hilbert $\mathcal{B}$-module $\mathcal{E}$ is *projective* if it is a direct summand in the free module $\mathcal{B}^m$ for some positive integer m. By 15.4.8 of [24], every algebraically finitely generated Hilbert-module over a unital algebra is projective.

If $\mathcal{E},\mathcal{F}$ are two Hilbert $\mathcal{B}$-modules over the same algebra $\mathcal{B}$, one says that an operator $T:\mathcal{E}\to\mathcal{F}$ is *adjointable* if there exists an operator $T^*:\mathcal{F}\to\mathcal{E}$ such that

$$\langle T\xi,\eta\rangle = \langle\xi,T^*\eta\rangle, \qquad \text{for all} \quad \xi\in\mathcal{E},\eta\in\mathcal{F}.$$

An adjointable operator is automatically $\mathcal{B}$-linear and bounded. The collection of adjointable operators from $\mathcal{E}$ to $\mathcal{F}$ is denoted $\mathcal{L}_{\mathcal{B}}(\mathcal{E},\mathcal{F})$. A bounded $\mathcal{B}$-linear operator need not be adjointable, thus the need for the definition. Clearly, if $T\in\mathcal{L}_{\mathcal{B}}(\mathcal{E},\mathcal{F})$, then $T^*\in\mathcal{L}_{\mathcal{B}}(\mathcal{F},\mathcal{E})$. In particular, $\mathcal{L}_{\mathcal{B}}(\mathcal{E}) := \mathcal{L}_{\mathcal{B}}(\mathcal{E},\mathcal{E})$ is a $*$-algebra.

There is an important class of operators which is built from "finite rank" operators. For any $\xi\in\mathcal{F}$ and $\eta\in\mathcal{E}$ one defines the operator $\theta_{\xi,\eta}:\mathcal{E}\to\mathcal{F}$ as

$$\theta_{\xi,\eta}(\zeta) = \xi\langle\eta,\zeta\rangle, \qquad \text{for all} \quad \zeta\in\mathcal{E}. \tag{2.2}$$

In a Dirac ket-bra notation this could be denoted $\theta_{\xi,\eta}(\zeta) = |\xi\rangle\langle\eta|$. Every such $\theta_{\xi,\eta}$ is adjointable, with adjoint $\theta^*_{\xi,\eta} := \theta_{\eta,\xi} : \mathcal{F} \to \mathcal{E}$. The closed linear subspace of $\mathcal{L}_\mathcal{B}(\mathcal{E}, \mathcal{F})$ spanned by $\{\theta_{\xi,\eta} \,|\, \xi, \eta \in \mathcal{E}\}$ is denoted by $\mathcal{K}_B(\mathcal{E}, \mathcal{F})$. In particular $\mathcal{K}_\mathcal{B}(\mathcal{E}) := \mathcal{K}_\mathcal{B}(\mathcal{E}, \mathcal{E}) \subseteq \mathcal{L}_\mathcal{B}(\mathcal{E})$; this is a closed ideal, whose elements are referred to as *compact endomorphisms*. One should remark that both $\mathcal{L}_\mathcal{B}(\mathcal{E})$ and $\mathcal{K}_\mathcal{B}(\mathcal{E})$ are C^*-algebras for the operator norm. When possible we write $\mathcal{L}(\mathcal{E}) = \mathcal{L}_\mathcal{B}(\mathcal{E})$ and $\mathcal{K}(\mathcal{E}) = \mathcal{K}_\mathcal{B}(\mathcal{E})$.

The dual of $\mathcal{E}$, denoted by $\mathcal{E}^*$, is defined as the space

$$\mathcal{E}^* := \{\phi \in \mathrm{Hom}_\mathcal{B}(\mathcal{E}, \mathcal{B}) | \exists \xi \in \mathcal{E} \text{ such that } \phi(\eta) = \langle \xi, \eta \rangle \ \forall \eta \in \mathcal{E}\}, \tag{2.3}$$

that is $\mathcal{E}^* := \mathcal{K}_\mathcal{B}(\mathcal{E}, \mathcal{B})$. Thus, with $\xi \in \mathcal{E}$, and $\lambda_\xi : \mathcal{E} \to \mathcal{B}$ the operator defined by $\lambda_\xi(\eta) = \langle \xi, \eta \rangle$, for all $\eta \in \mathcal{E}$, every element of $\mathcal{E}^*$ is of the form λ_ξ for some $\xi \in \mathcal{E}$. The module $\mathcal{E}$ is called *self-dual* if all elements of $\mathcal{L}_\mathcal{B}(\mathcal{E}, \mathcal{B})$ are of this form, *i.e.* if the module map $E \ni \xi \mapsto \lambda_\xi \in \mathcal{L}_\mathcal{B}(\mathcal{E}, \mathcal{B})$, is surjective, so that $\mathcal{E}^*$ coincides with the whole of $\mathcal{L}_\mathcal{B}(\mathcal{E}, \mathcal{B})$. If $\mathcal{B}$ is unital, then $\mathcal{B}^n$ is self-dual for any $n \geq 1$. As a consequence, every finitely generated projective Hilbert module over a unital algebra is also self-dual.

2˙2. *Morita equivalence.* – Given a right Hilbert $\mathcal{B}$-module $\mathcal{E}$, by construction, compact endomorphisms act from the left on $\mathcal{E}$. Then, by defining:

$${}_{\mathcal{K}(\mathcal{E})}\langle \xi, \eta \rangle := \theta_{\xi,\eta},$$

we obtain a natural $\mathcal{K}(\mathcal{E})$-valued Hermitian product on $\mathcal{E}$. Note the it is *left* linear over $\mathcal{K}(\mathcal{E})$, that is ${}_{\mathcal{K}(\mathcal{E})}\langle T \cdot \xi, \eta \rangle = T \cdot ({}_{\mathcal{K}(\mathcal{E})}\langle \xi, \eta \rangle)$ for $T \in \mathcal{K}(\mathcal{E})$. With the notation $\theta_{\xi,\eta}(\zeta) = |\xi\rangle\langle\eta|$ this Hermitian product reads just like the one in (**1**.3) and (**1**.4). Thus $\mathcal{E}$ is a left Hilbert $\mathcal{K}(\mathcal{E})$-module and by the very definition of $\mathcal{K}(\mathcal{E})$, $\mathcal{E}$ is full over $\mathcal{K}(\mathcal{E})$. One readily checks the compatibility condition

$${}_{\mathcal{K}(\mathcal{E})}\langle \xi, \eta \rangle \zeta = \xi \langle \eta, \zeta \rangle_\mathcal{B}, \qquad \text{for all} \quad \xi, \eta, \zeta \in \mathcal{E}. \tag{2.4}$$

By its definition, $\mathcal{K}(\mathcal{E})$ acts by adjointable operators on the right $\mathcal{B}$-module $\mathcal{E}$. On the other hand, with $b \in \mathcal{B}$ and $\xi, \eta, \zeta \in \mathcal{E}$, one computes

$${}_{\mathcal{K}(\mathcal{E})}\langle \xi b, \eta \rangle \zeta = (\xi b) \langle \eta, \zeta \rangle_\mathcal{B} = \xi \langle \eta b^*, \zeta \rangle_\mathcal{B} = {}_{\mathcal{K}(\mathcal{E})}\langle \xi, \eta b^* \rangle \zeta,$$

that is, $\mathcal{B}$ acts by adjointable operators on the left $\mathcal{K}(\mathcal{E})$-module $\mathcal{E}$.

This example motivates the following definition:

Let $\mathcal{A}$ and $\mathcal{B}$, be pre-C^*- algebras with C^*- algebra closures A and B. An $(\mathcal{A}, \mathcal{B})$-*equivalence bimodule* $\mathcal{E}$ is a right pre-Hilbert $\mathcal{B}$-module with $\mathcal{B}$-valued Hermitian product $\langle\ ,\ \rangle_\mathcal{B}$, which is also a left pre-Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued Hermitian product ${}_\mathcal{A}\langle\ ,\ \rangle$ ($\langle\ ,\ \rangle_\mathcal{B}$ is right B-linear, while ${}_A\langle\ ,\ \rangle$ is left A-linear) with the additional properties that:

1. the Hermitian products are compatible, that is

$$\xi \langle \eta, \zeta \rangle_\mathcal{B} =_\mathcal{A} \langle \xi, \eta \rangle \zeta, \qquad \text{for all} \quad \xi, \eta, \zeta \in \mathcal{E}.$$

2. ${}_{\mathcal{A}}\langle \mathcal{E}, \mathcal{E} \rangle$ and $\langle \mathcal{E}, \mathcal{E} \rangle_{\mathcal{B}}$ span dense ideals of the completions A and B respectively.

3. for all $\xi, \eta \in \mathcal{E}$ and $a \in \mathcal{A}, b \in \mathcal{B}$, it holds that

$$\langle a\xi, \eta \rangle_{\mathcal{B}} = \langle \xi, a^* \eta \rangle_{\mathcal{B}} \qquad \text{and} \qquad {}_{A}\langle \xi b, \eta \rangle = {}_{A}\langle \xi, \eta b^* \rangle.$$

The compatibility of the two Hermitian products yields that the corresponding $\mathcal{B}$-valued and $\mathcal{K}(\mathcal{E})$-valued norms as in (**2**.1) coincide (see [19], Lemma 2.30). Then by Prop. 3.12 of [19], the $(\mathcal{A}, \mathcal{B})$-equivalence bimodule $\mathcal{E}$ can be completed to a (A, B)-equivalence bimodule $\mathcal{E}$ where E is the completion of $\mathcal{E}$ in this norm.

If there exist an $(\mathcal{A}, \mathcal{B})$-equivalence bimodule one says that the two pre-C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are *Morita equivalent* (to be precise rather the C^*-algebras A and B). From the considerations above on the algebra $\mathcal{K}(\mathcal{E})$ of compact endomorphisms and in particular the compatibility condition (**2**.4), it is not surprising that the algebra $\mathcal{K}(\mathcal{E})$ has a central role for Morita equivalence. Indeed, one shows (see [19], Prop. 3.8) that every full Hilbert $\mathcal{B}$-module $\mathcal{E}$ is a $(\mathcal{K}(\mathcal{E}), \mathcal{B})$-equivalence bimodule with $\mathcal{K}(\mathcal{E})$-valued Hermitian product given by ${}_{\mathcal{K}(\mathcal{E})}\langle \xi, \eta \rangle = \theta_{\xi,\eta}$. Conversely, if $\mathcal{E}$ is an $(\mathcal{A}, \mathcal{B})$ equivalence bimodule, then there exists an isomorphism $\phi : \mathcal{A} \to \mathcal{K}(\mathcal{E})$ such that

$$\phi\left({}_{\mathcal{A}}\langle \xi, \eta \rangle\right) =_{\mathcal{K}(\mathcal{E})} \langle \xi, \eta \rangle, \qquad \text{for all} \quad \xi, \eta \in \mathcal{E}.$$

Thus, two pre-C^*- algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent if and only if $\mathcal{A} \simeq \mathcal{K}_{\mathcal{B}}(\mathcal{E}_{\mathcal{B}})$ for a full right pre-Hilbert $\mathcal{B}$-module $E_{\mathcal{B}}$, or equivalently, if and only if $\mathcal{B} \simeq {}_{\mathcal{A}}\mathcal{K}({}_{\mathcal{A}}\mathcal{E})$ for a full left pre-Hilbert $\mathcal{A}$-module ${}_{\mathcal{A}}\mathcal{E}$. In fact, Morita equivalence is an equivalence relation, with transitivity obtained by taking the interior tensor product of bimodules. We do not dwell upon the details of the construction here while referring to §3.2 of [19], for instance.

For $B = \mathbb{C}$ so that $\mathcal{E} = \mathcal{H}$ is a Hilbert space, the algebra $\mathcal{K}(\mathcal{E})$ is the algebra of compact operators $\mathcal{K}(\mathcal{H})$ and ${}_{\mathcal{K}(\mathcal{H})}\langle \xi, \eta \rangle = |\xi\rangle\langle\eta| = \xi^* \otimes \eta$. The Hilbert space $\mathcal{H}$ is a Morita equivalence bimodule between $\mathcal{K}(\mathcal{H})$, acting on the left, and $\mathbb{C}$, acting on the left.

2˙3. *Frames*. – Suppose $\mathcal{A}, \mathcal{B}$ are two unital pre-C^*-algebras, and let $\mathcal{E}$ be a finitely generated $(\mathcal{A}, \mathcal{B})$-equivalence bimodule. Since $\mathcal{A} \simeq \mathcal{K}(\mathcal{E})$, there exists a finite collection of elements $\eta_1, \dots, \eta_n \in E$ with the property that

$$\sum\nolimits_j {}_{\mathcal{A}}\langle \eta_j, \eta_j \rangle = 1_{\mathcal{A}}.$$

As a consequence, one can reconstruct any element $\xi \in \mathcal{E}$ as

$$\xi = \sum\nolimits_j {}_{\mathcal{A}}\langle \eta_j, \eta_j \rangle \xi = \sum\nolimits_j \eta_j \langle \eta_j, \xi \rangle_{\mathcal{B}}, \tag{2.5}$$

using the compatibility condition of point 1. before. This motivates the following [20]:

With the algebra $\mathcal{B}$ unital, a *finite standard module frame* for the right Hilbert $\mathcal{B}$-module $\mathcal{E}$ is a finite family of elements $\{\eta_j\}_{j=1}^n$ of $\mathcal{E}$ so that, for all $\xi \in \mathcal{E}$, the reconstruction formula (**2**.5) holds true. The existence of a finite frame is a geometrical condition. Whenever one has a right Hilbert $\mathcal{B}$-module $\mathcal{E}$ with a finite standard module frame, $\mathcal{E}$ is algebraically finitely generated and projective as a right module, with the frame explicitly providing a projection for $\mathcal{E}$: the matrix $p = (p_{jk})$ with entries $p_{jk} = \langle \eta_j, \eta_k \rangle_{\mathcal{B}}$ is a projection in the matrix algebra $\mathrm{M}_n(\mathcal{B})$. By construction $(p_{jk})^* = p_{kj}$ and, using (**2**.5),

$$(p^2)_{jl} = \sum\nolimits_k \langle \eta_j, \eta_k \rangle_{\mathcal{B}} \langle \eta_k, \eta_l \rangle_{\mathcal{B}} = \sum\nolimits_k \langle \eta_j, \eta_k \langle \eta_k, \eta_l \rangle_{\mathcal{B}} \rangle_{\mathcal{B}} = \langle \eta_j, \eta_l \rangle_{\mathcal{B}} = p_{jl}.$$

This establishes the finite right $\mathcal{B}$-module projectivity of $\mathcal{E}$ and identifies $\mathcal{E} \simeq p\mathcal{B}^n$. Furthermore, $\mathcal{E}$ is self-dual for its Hermitian product.

More generally, the module $\mathcal{E}$ is finitely generated projective whenever there exist two finite sets $\{\eta_j\}_{j=1}^n$ and $\{\zeta_j\}_{j=1}^n$ of elements of $\mathcal{E}$ with the property that

$$\sum\nolimits_j {}_{\mathcal{K}(\mathcal{E})}\langle \eta_j, \zeta_j \rangle = 1_{\mathcal{K}(\mathcal{E})}. \tag{2.6}$$

Then, any element $\xi \in \mathcal{E}$ is reconstructed as $\xi = \sum_j \eta_j \langle \zeta_j, \xi \rangle_{\mathcal{B}}$, and one gets an idempotent matrix $e_{jk} = \langle \zeta_j, \eta_k \rangle_{\mathcal{B}}$ in $\mathrm{M}_n(\mathcal{B})$, $(e^2)_{jk} = e_{jk}$, and $\mathcal{E} \simeq e\mathcal{B}^n$ as a right $\mathcal{B}$-module.

3. – Noncommutative line bundles

From sect. **1** we know that the module of sections $\Gamma(M, X)$ of a vector bundle $M \to X$ is a Morita equivalence between the algebra $C^\infty(X)$ acting on the right, and the endomorphism algebra $\mathrm{End}_{C^\infty(X)}(\Gamma(M, X))$ acting on the left. As already mentioned, the vector bundle is in fact a line bundle if and only if $\mathrm{End}_{C^\infty(X)}(\Gamma(M, X)) \simeq C^\infty(X)$.

Based on this, one could define a *noncommutative line bundles* over the noncommutative algebra $\mathcal{B}$ to be the same as a *self-Morita equivalence bimodule* for $\mathcal{B}$, that is a pair $(\mathcal{E}, \phi)$ with $\mathcal{E}$ a full right (pre-)Hilbert $\mathcal{B}$-module $\mathcal{E}$ and $\phi : \mathcal{B} \to \mathcal{K}(\mathcal{E})$ an isomorphism.

If $(\mathcal{E}, \phi)$ is a noncommutative line bundle over $\mathcal{B}$, the dual $\mathcal{E}^*$ as defined in (**2**.3), is made into a noncommutative line bundle over $\mathcal{B}$ as well. Firstly, $\mathcal{E}^*$ is given the structure of a (right) Hilbert module over $\mathcal{B}$ via ϕ. Recall that elements of $\mathcal{E}^*$ are of the form λ_ξ for some $\xi \in \mathcal{E}$, with $\lambda_\xi(\eta) = \langle \xi, \eta \rangle$, for all $\eta \in \mathcal{E}$. The right action of $\mathcal{B}$ on $\mathcal{E}^*$ is then

$$\lambda_\xi \, b := \lambda_\xi \circ \phi(b) = \lambda_{\phi(b)^* \xi}.$$

The $\mathcal{B}$-valued Hermitian product on $\mathcal{E}^*$ uses the $\mathcal{K}(\mathcal{E})$-valued Hermitian product on $\mathcal{E}$:

$$\langle \lambda_\xi, \lambda_\eta \rangle := \phi^{-1}(\theta_{\xi,\eta}),$$

and $\mathcal{E}^*$ is full as well. Next, define a $*$-homomorphism $\phi^* : \mathcal{B} \to \mathcal{L}(\mathcal{E}^*)$ by

$$\phi^*(b)(\lambda_\xi) := \lambda_{\xi \cdot b^*},$$

which is in fact an isomorphism $\phi^* : \mathcal{B} \to \mathcal{K}(\mathcal{E}^*)$. Thus, the pair $(\mathcal{E}^*, \phi^*)$ is a noncommutative line bundle over B as well.

More generally, one considers tensor products of noncommutative line bundles starting from the interior tensor product $\mathcal{E} \widehat{\otimes}_\phi \mathcal{E}$ of $\mathcal{E}$ with itself over $\mathcal{B}$; this is naturally a right $\mathcal{B}$-module and can be made into a noncommutative line bundle over $\mathcal{B}$. The construction can be iterated yielding for $n > 0$, the n-fold interior tensor power of $\mathcal{E}$ over $\mathcal{B}$,

$$\mathcal{E}^{\widehat{\otimes}_\phi n} := \mathcal{E} \widehat{\otimes}_\phi \mathcal{E} \widehat{\otimes}_\phi \cdots \widehat{\otimes}_\phi \mathcal{E}, \qquad n\text{-factors};$$

again a noncommutative line bundle over B. Details are in ref [14], chapt. 4 and in ref [14], sect. 2.3.

The collection of (isomorphic classes of) noncommutative line bundles over $\mathcal{B}$ has a natural group structure with respect to the interior tensor product. The inverse of the noncommutative line bundle $(\mathcal{E}, \phi)$ is the dual noncommutative line bundle $(\mathcal{E}^*, \phi^*)$. This group is the *Picard group* of $\mathcal{B}$, denoted $\mathrm{Pic}(\mathcal{B})$ in analogy with the classical Picard group of a space, — the group of isomorphism classes of line bundles with group operation just tensor product. For a commutative unital C^*-algebra $B = C(X)$ the Picard group is the semidirect product of the classical Picard group of X with the group of automorphisms of the algebra B (which is the same as the group of homeomorphisms of X) [2].

4. – Fock modules and Pimsner algebras

With a noncommutative line bundle as a self-Morita equivalence bimodule, one constructs an algebra of (creation and annihilation) operators acting on a "Fock module". It has a natural role as the total space algebra of a noncommutative $U(1)$-bundle. To avoid entangling in technical details we give it at the continuous level that is with C^*-algebras.

4˙1. *The Pimsner algebra.* – To every pair (E, ϕ), where E is a right Hilbert B-module, for a C^*-algebra B, and $\phi : B \to \mathcal{L}_B(E)$ is an isometric $*$-homomorphism, Pimsner associates in [18] a very natural and universal C^*-algebra. This important work has attracted a lot of attention and has been meanwhile generalized in several directions. We shall not work in full generality here, but rather under the assumption that the pair (E, ϕ) is a noncommutative line bundle for B, that is ϕ is an isomorphism.

Given a noncommutative line bundle (E, ϕ) for the C^*-algebra B, in sect. **3** we described the interior tensor product $E \widehat{\otimes}_\phi E$, itself a noncommutative line bundle and, more generally, the tensor product $E^{\widehat{\otimes}_\phi n}$, for $n > 0$. To lighten notation, we denote

$$E^{(n)} := \begin{cases} E^{\widehat{\otimes}_\phi n}, & n > 0, \\ B, & n = 0, \\ (E^*)^{\widehat{\otimes}_{\phi^*}(-n)}, & n < 0. \end{cases}$$

Clearly, $E^{(1)} = E$ and $E^{(-1)} = E^*$ and from the definition of these Hilbert B-modules, one has isomorphisms $\mathcal{K}(E^{(n)}, E^{(m)}) \simeq E^{(m-n)}$. Out of them, one constructs a *two-sided Fock module*, as the Hilbert B-module E_∞ given by a direct sum

$$E_\infty := \bigoplus\nolimits_{n\in\mathbb{Z}} E^{(n)}.$$

As on usual Fock spaces, one defines creation and annihilation operators. Firstly, for each $\xi \in E$ one has a bounded adjointable operator (a creation operator) $S_\xi : E_\infty \to E_\infty$, shifting the degree by $+1$, defined on simple tensors by

$$\begin{aligned} S_\xi(b) &:= \xi \cdot b, & b \in B, \\ S_\xi(\xi_1 \otimes \cdots \otimes \xi_n) &:= \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n, & n > 0, \\ S_\xi(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-n}}) &:= \lambda_{\xi_2 \cdot \phi^{-1}(\theta_{\xi_1,\xi})} \otimes \lambda_{\xi_3} \otimes \cdots \otimes \lambda_{\xi_{-n}}, & n < 0. \end{aligned}$$

The adjoint of S_ξ (an annihilation operator) is found to be $S_{\lambda_\xi} := S^*_\xi : E_\infty \to E_\infty$ as

$$\begin{aligned} S_{\lambda_\xi}(b) &:= \lambda_\xi \cdot b, && b \in B, \\ S_{\lambda_\xi}(\xi_1 \otimes \cdots \otimes \xi_n) &:= \phi(\langle \xi, \xi_1 \rangle)\xi_2 \otimes \xi_3 \otimes \cdots \otimes \xi_n, && n > 0, \\ S_{\lambda_\xi}(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-n}}) &:= \lambda_\xi \otimes \lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_{-n}}, && n < 0. \end{aligned}$$

In particular, $S_\xi(\lambda_{\xi_1}) = \phi^{-1}(\theta_{\xi,\xi_1}) \in B$ and $S_{\lambda_\xi}(\xi_1) = \langle \xi, \xi_1 \rangle \in B$.

The *Pimsner algebra* $\mathcal{O}_E$ associated with the pair (E, ϕ) is the smallest C^*-subalgebra of $\mathcal{L}_B(E_\infty)$ containing the creation operators S_ξ for all $\xi \in E$.

There is an injective $*$-homomorphism $i : B \to \mathcal{O}_E$. This is induced by the injective $*$-homomorphism $\phi : B \to \mathcal{L}_B(E_\infty)$ defined by

$$\begin{aligned} \phi(b)(b') &:= b \cdot b', \\ \phi(b)(\xi_1 \otimes \cdots \otimes \xi_n) &:= \phi(b)(\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n, \\ \phi(b)(\lambda_{\xi_1} \otimes \cdots \otimes \lambda_{\xi_n}) &:= \phi^*(b)(\lambda_{\xi_1}) \otimes \lambda_{\xi_2} \otimes \cdots \otimes \lambda_{\xi_n} = \lambda_{\xi_1 \cdot b^*} \otimes \lambda_{\xi_2} \otimes \cdots \otimes \lambda_{\xi_n}, \end{aligned}$$

whose image is in $\mathcal{O}_E$. In particular, for all $\xi, \eta \in E$ it holds that $S_\xi S^*_\eta = i(\phi^{-1}(\theta_{\xi,\eta}))$, that is the operator $S_\xi S^*_\eta$ on E_∞ is right-multiplication by the element $\phi^{-1}(\theta_{\xi,\eta}) \in B$.

A Pimsner algebra is universal in the following sense (see ref. [18], Thm. 3.12): Let C be a C^*-algebra and $\sigma : B \to C$ a $*$-homomorphism. Suppose there exist elements $s_\xi \in C$ such that, for all $\xi, \eta \in E$, $b \in B$ and $\alpha, \beta \in \mathbb{C}$ it holds that:

$$\begin{aligned} \alpha s_\xi + \beta s_\eta &= s_{\alpha\xi + \beta\eta}, \\ s_\xi \sigma(b) &= s_{\xi b} \qquad \text{and} \qquad \sigma(b) s_\xi = s_{\phi(b)(\xi)}, \\ s^*_\xi s_\eta &= \sigma(\langle \xi, \eta \rangle) \qquad \text{and} \qquad s_\xi s^*_\eta = \sigma\left(\phi^{-1}(\theta_{\xi,\eta})\right). \end{aligned}$$

Then, there is a unique $*$-homomorphism $\widetilde{\sigma} : \mathcal{O}_E \to C$ so that $\widetilde{\sigma}(S_\xi) = s_\xi$ for all $\xi \in E$.

4˙1.1. Examples. The first example was already in [18]. Consider a C^*-algebra B with an automorphism $\alpha : B \to B$. Then (B, α) is naturally a self-Morita equivalence for B. The right Hilbert B-module structure is the standard one, with right B valued Hermitian product $\langle a, b \rangle_B = a^* b$. The automorphism α is used to define the left action as $a \cdot b = \alpha(a) b$ and the left B-valued Hermitian product by ${}_B\langle a, b \rangle = \alpha(a^* b)$. For all $n \in \mathbb{Z}$, the module $E^{(n)}$ is isomorphic to B as a vector space,

$$E^{(n)} \ni a \cdot (x_1 \otimes \cdots \otimes x_n) \longmapsto \alpha^n(a)\alpha^{n-1}(x_1) \cdots \alpha(x_{n-1}) x_n \in B,$$

and the module E_∞ is isomorphic to a direct sum of copies of B. The corresponding Pimsner algebra $\mathcal{O}_E$ coincides with the crossed product algebra $B \rtimes_\alpha \mathbb{Z}$.

As a second example, take the B-module E to be finitely generated and projective, so that it admits a finite frame $\{\eta_j\}_{j=1}^n$. Then, the reconstruction formula (**2**.5) yields

$$\phi(b)\eta_j = \sum\nolimits_k \eta_k \langle \eta_k, \phi(b)\eta_j \rangle_{\mathcal{B}}, \qquad \text{for any} \quad b \in B.$$

The C^*-algebra $\mathcal{O}_E$ can be realised in terms of generators and relations [13]. It is indeed the universal C^*-algebra generated by B together with n operators $S_1, \dots, S_n$, satisfying

$$(\mathbf{4}.1) \qquad S^*_k S_j = \langle \eta_k, \eta_j \rangle_{\mathcal{B}}, \quad \sum\nolimits_j S_j S^*_j = 1, \quad \text{and} \quad b\, S_j = \sum\nolimits_k S_k \langle \eta_k, \phi(b)\eta_j \rangle_{\mathcal{B}},$$

for $b \in B$, and $j = 1, \ldots, n$. The generators S_j are partial isometries if and only if the frame satisfies $\langle \eta_k, \eta_j \rangle = 0$ for $k \neq j$. For $B = \mathbb{C}$ and E a Hilbert space of dimension n, one recovers the original Cuntz algebra $\mathcal{O}_n$ of [7].

4·2. *Generalized crossed products.* – In the present paper we consider algebras endowed with a $U(1)$-action and their relation with Pimsner algebras. For this a somewhat better framework is that of generalized crossed products. These were introduced in [1] and are naturally associated with Hilbert bimodules via the notion of a covariant representation.

Let E be a Hilbert (B, B)-bimodule (not necessarily full). A covariant representation of E on a C^*-algebra C is a pair $(\pi, \mathcal{T})$ where

1. $\pi : B \to C$ is a $*$-homomorphism of algebras,

2. $\mathcal{T} : E \to C$ satisfies, or all $b \in B$ and $\xi, \eta \in E$,

$$\mathcal{T}(\xi)\pi(b) = \mathcal{T}(\xi b), \qquad \pi(b)\mathcal{T}(\xi) = \mathcal{T}(b\xi)$$
$$\text{and} \qquad \mathcal{T}(\xi)^*\mathcal{T}(\eta) = \pi(\langle \xi, \eta \rangle_{\mathcal{B}}), \quad \mathcal{T}(\xi)\mathcal{T}(\eta)^* = \pi({}_{\mathcal{B}}\langle \xi, \eta \rangle).$$

If E is a Hilbert (B, B)-bimodule, the *generalized crossed product* $B \rtimes_E \mathbb{Z}$ of B by E is the universal C^*-algebra generated by the covariant representations of E.

A generalized crossed product need not be a Pimsner algebra in general, since the representation of B giving the left action need not be injective. However, by the universality of a Pimsner algebra, one shows that for a self-Morita equivalence bimodule the two constructions yield the same algebra $\mathcal{O}_E = B \rtimes_E \mathbb{Z}$. The advantage of using generalized crossed products is that a C^*-algebra carrying a $U(1)$-action that satisfies a suitable completeness condition, can be re-obtained as a generalized crossed product.

5. – Algebras and U(1)-actions

A Pimsner algebra $\mathcal{O}_E$ carries a natural $U(1)$-action. The map

$$S_\xi \to \alpha_z(S_\xi) := z^* S_\xi, \qquad \text{for } z \in U(1),$$

extends to an automorphism of $\mathcal{O}_E$ by universality (with $C = \mathcal{O}_E$, $\sigma = i$ the injection of B into $\mathcal{O}_E$, and $s_\xi := z^* S_\xi$). The degree n part of $\mathcal{O}_E$ for this action is defined as usual, as the weight space $(\mathcal{O}_E)_n := \{x \in \mathcal{O}_E : \alpha_z(x) = z^{-n}x\}$.

In general, let A be a C^*-algebra and $\{\sigma_z\}_{z \in \mathrm{U}(1)}$ be a strongly continuous action of the group $U(1)$ on A. For each $n \in \mathbb{Z}$, one defines the spectral subspaces

$$A_n := \Big\{ x \in A \mid \sigma_z(x) = z^{-n}\, x \quad \text{for all } z \in U(1) \Big\}.$$

Clearly, the invariant subspace $A_0 \subseteq A$ is a C^*- subalgebra of A, with unit whenever A is unital; this is the *fixed-point subalgebra*. Moreover, the subspace $A_n A_m$, meant as the *closed* linear span of the set of products xy with $x \in A_n$ and $y \in A_m$, is contained in A_{n+m}. Thus, the algebra A is $\mathbb{Z}$-graded and the grading is compatible with the involution, that is $A_n^* = A_{-n}$ for all $n \in \mathbb{Z}$.

In particular, for any $n \in \mathbb{Z}$ the space $A_n^* A_n$ is a closed two-sided ideal in A_0. Thus, each spectral subspace A_n has a natural structure of Hilbert A_0-bimodule (not necessarily full) with left and right Hermitian products:

$$ {}_{A_0}\langle x, y\rangle = xy^*, \qquad \langle x, y\rangle_{A_0} = x^* y, \quad \text{for all } x, y \in A_n. \tag{5.1}$$

It was shown in ref. [1], Thm. 3.1, that a C^*-algebra A with a strongly continuous $U(1)$-action is isomorphic to the generalized crossed product $A_0 \rtimes_{A_1} \mathbb{Z}$ if and only if A is generated, as a C^*-algebra, by the fixed point algebra A_0 and the first spectral subspace A_1 of the $U(1)$-action. When this is the case, one says that the action is *semi-saturated* [10]. This condition is fulfilled in a large class of examples, like crossed product by the integers, and noncommutative principal circle bundles, as we shall see in sect. **6** below.

We see that the module A_1 has a central role. If it is a full bimodule, that is if

$$ A_1^* A_1 = A_0 = A_1 A_1^*, \tag{5.2}$$

the action σ is said to have *large spectral subspaces* (*cf.* [17], §2), a slightly stronger condition than semi-saturatedness (*cf.* [5], Prop. 3.4).

Now, the condition (**5**.2) is equivalent to the condition that all bimodules A_n are full, that is $A_n^* A_n = A_0 = A_n A_n^*$ for all $n \in \mathbb{Z}$. When this happens, all bimodules A_n are noncommutative line bundles for A_0, with isomorphism $\phi : A_0 \to \mathcal{K}_{A_0}(A_n)$ given by

$$ \phi(a)(\xi) := a\,\xi, \qquad \text{for all } a \in A_0,\ \xi \in A_n. \tag{5.3}$$

Combining the result above with the fact that for a self-Morita equivalence the generalized crossed product is the same as a Pimsner algebra, one has that (see ref. [5], Thm. 3.5):

Let A be a C^*-algebra with a strongly continuous action of the circle. Suppose that the first spectral subspace A_1 is a full and countably generated Hilbert bimodule over A_0. Then the Pimsner algebra $\mathcal{O}_{A_1}$ of the self-Morita equivalence (A_1, ϕ), with ϕ as in is (**5**.3), is isomorphic to A. The isomorphism is given by $S_\xi \mapsto \xi$ for all $\xi \in A_1$.

6. – Principal bundles and Pimsner algebras

At an algebraic level noncommutative (or quantum) principal circle bundles are intimately related to $\mathbb{Z}$-graded $*$-algebras. When completing with natural norms one gets continuous $U(1)$-actions on a C^*-algebra with $\mathbb{Z}$-grading given by spectral subspaces, that is the framework described in sect. **5**, and the total space algebras are indeed examples of Pimsner algebras. For commutative algebras this was already in Prop. 5.8 of ref. [11]:

Let A be a unital, commutative C^*-algebra carrying a $U(1)$-action. Suppose that the first spectral subspace $E = A_1$ is finitely generated and projective over $B = A_0$. Suppose furthermore that E generates A as a C^*-algebra. Then: 1) $B = C(X)$ for some compact space X; 2) E is the module of sections of some line bundle $L \to X$; 3) $A = C(P)$, where $P \to X$ is the principal $U(1)$-bundle over X associated with the line bundle L, and the $U(1)$-action on A comes from the principal $U(1)$-action on P.

6˙1. *Principal circle bundles and line bundles*. – We aim at exploring the connections between (noncommutative) principal $U(1)$-bundles, frames for modules as described in sect. **2**˙3, and $\mathbb{Z}$-graded algebras. The $U(1)$-action is dualized in a coaction of the dual group Hopf algebra. Thus, we need to consider the unital complex algebra

$$\mathcal{O}(\mathrm{U}(1)) := \mathbb{C}[z, z^{-1}]/\langle 1 - zz^{-1}\rangle,$$

where $\langle 1 - zz^{-1}\rangle$ is the ideal generated by $1 - zz^{-1}$ in the polynomial algebra $\mathbb{C}[z, z^{-1}]$ on two variables. The algebra $\mathcal{O}(\mathrm{U}(1))$ is a Hopf algebra by defining, for any $n \in \mathbb{Z}$, the coproduct $\Delta : z^n \mapsto z^n \otimes z^n$, the antipode $S : z^n \mapsto z^{-n}$ and the counit $\epsilon : z^n \mapsto 1$.

Let $\mathcal{A}$ be a complex unital algebra and suppose in addition it is a right comodule algebra over $\mathcal{O}(\mathrm{U}(1))$, that is $\mathcal{A}$ carries a coaction of $\mathcal{O}(\mathrm{U}(1))$,

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(\mathrm{U}(1)),$$

— a homomorphism of unital algebras. Let $\mathcal{B} := \{x \in \mathcal{A} \mid \Delta_R(x) = x \otimes 1\}$ be the unital subalgebra of $\mathcal{A}$ made of coinvariant elements for Δ_R.

One says that the datum $(\mathcal{A}, \mathcal{O}(\mathrm{U}(1)), \mathcal{B})$ is a *noncommutative (or quantum) principal $U(1)$-bundle* when the *canonical map*,

$$\mathrm{can} : \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(\mathrm{U}(1)), \quad x \otimes y \mapsto x\, \Delta_R(y),$$

is an isomorphism. In fact, this is the statement that the right comodule algebra $\mathcal{A}$ is a $\mathcal{O}(\mathrm{U}(1))$ Hopf-Galois extension of $\mathcal{B}$, and this is equivalent (in the present context) by Prop. 1.6 of ref. [12], to the bundle being a noncommutative principal bundle for the universal differential calculus in the sense of [6].

Next, if $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ is a $\mathbb{Z}$-graded unital algebra, the unital algebra homomorphism,

$$\Delta_R : \mathcal{A} \to \mathcal{A} \otimes \mathcal{O}(\mathrm{U}(1)), \quad x \mapsto x \otimes z^{-n}, \quad \text{for} \quad x \in \mathcal{A}_n.$$

turns $\mathcal{A}$ into a right comodule algebra over $\mathcal{O}(\mathrm{U}(1))$. The unital subalgebra of coinvariant elements coincides with $\mathcal{A}_0$. A necessary and sufficient condition for the corresponding canonical map to be bijective is given in Thm. 4.3 of ref. [5] (*cf.* also [21], Lem. 5.1). It is more manageable in general, and in particular it can be applied for examples like the quantum lens spaces as principal circle bundles over quantum weighted projective lines [5, 9]:

One shows that the triple $(\mathcal{A}, \mathcal{O}(\mathrm{U}(1)), \mathcal{A}_0)$ is a noncommutative principal $U(1)$-bundle if and only if there exist finite sequences

$$\{\xi_j\}_{j=1}^N, \ \{\beta_i\}_{i=1}^M \text{ in } \mathcal{A}_1 \quad \text{and} \quad \{\eta_j\}_{j=1}^N, \ \{\alpha_i\}_{i=1}^M \text{ in } \mathcal{A}_{-1} \qquad (\mathbf{6}.1)$$

such that one has identities:

$$\sum\nolimits_{j=1}^N \eta_j \xi_j = 1_{\mathcal{A}} = \sum\nolimits_{i=1}^M \alpha_i \beta_i. \qquad (\mathbf{6}.2)$$

Out of the proof in Thm. 4.3 of ref. [15], we just report the explicit form of the inverse map $\mathrm{can}^{-1} : \mathcal{A} \otimes \mathcal{O}(\mathrm{U}(1)) \to \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}$, given by the formula

$$\text{(6.3)} \qquad \mathrm{can}^{-1} : x \otimes z^n \mapsto \begin{cases} \sum_{j_k=1}^{N} x\, \xi_{j_1} \cdot \ldots \cdot \xi_{j_n} \otimes \eta_{j_n} \cdot \ldots \cdot \eta_{j_1}, & n > 0, \\ x \otimes 1, & n = 0, \\ \sum_{i_k=1}^{M} x\, \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_{-n}} \otimes \beta_{i_{-n}} \cdot \ldots \cdot \beta_{i_1}, & n < 0. \end{cases}$$

Now, conditions (**6**.2) are exactly the frame relations (**2**.6) for $\mathcal{A}_1$ and $\mathcal{A}_{-1}$, which imply that they are finitely generated and projective over $\mathcal{A}_0$ (see ref. [5], Cor. 4.5).

Explicitly, with the ξ's and the η's as above, one defines the module homomorphisms

$$\begin{aligned} & \Phi_{(1)} : \mathcal{A}_1 \to (\mathcal{A}_0)^N, \quad \Phi_{(1)}(\zeta) = (\eta_1 \zeta, \eta_2\, \zeta, \cdots, \eta_N\, \zeta)^{tr} \\ \text{and} \quad & \Psi_{(1)} : (\mathcal{A}_0)^N \to \mathcal{A}_1, \quad \Psi_{(1)}(x_1, x_2, \cdots, x_N)^{tr} = \xi_1\, x_1 + \xi_2\, x_2 + \cdots + \xi_N\, x_N. \end{aligned}$$

It then follows that $\Psi_{(1)}\Phi_{(1)} = \mathrm{id}_{\mathcal{A}_1}$. Thus $\mathbf{e}_{(1)} := \Phi_{(1)}\Psi_{(1)}$ is an idempotent in $M_N(\mathcal{A}_0)$, and $\mathcal{A}_1 \simeq \mathbf{e}_{(1)}(\mathcal{A}_0)^N$. Similarly, with the α's and the β's as above, one defines

$$\begin{aligned} & \Phi_{(-1)} : \mathcal{A}_1 \to (\mathcal{A}_0)^M, \quad \Phi_{(-1)}(\zeta) = (\beta_1 \zeta, \beta_2\, \zeta, \cdots, \beta_M\, \zeta)^{tr} \\ \text{and} \quad & \Psi_{(-1)} : (\mathcal{A}_0)^M \to \mathcal{A}_1, \quad \Psi_{(-1)}(x_1, x_2, \cdots, x_M)^{tr} = \alpha_1\, x_1 + \alpha_2\, x_2 + \cdots + \alpha_M\, x_M, \end{aligned}$$

again module homomorphisms, with $\Psi_{(-1)}\Phi_{(-1)} = \mathrm{id}_{\mathcal{A}_{-1}}$. Then $\mathbf{e}_{(-1)} := \Phi_{(-1)}\Psi_{(-1)}$ is an idempotent in $M_M(\mathcal{A}_0)$ and $\mathcal{A}_{-1} \simeq \mathbf{e}_{(-1)}(\mathcal{A}_0)^M$.

We see that the modules $\mathcal{A}_1$ and $\mathcal{A}_{-1}$ emerge as *line bundles* over the noncommutative space dual to the algebra $\mathcal{A}_0$. In the same vein all modules $\mathcal{A}_n$ for $n \in \mathbb{Z}$ are line bundles as well. Firstly, given any natural number d consider the $\mathbb{Z}$-graded unital algebra

$$\text{(6.4)} \qquad \mathcal{A}^{\mathbb{Z}_d} := \oplus_{n\in\mathbb{Z}} \mathcal{A}_{dn},$$

which can be seen as a fixed point algebra for an action of $\mathbb{Z}_d := \mathbb{Z}/d\mathbb{Z}$ on the starting algebra $\mathcal{A}$. Suppose $(\mathcal{A}, \mathcal{O}(\mathrm{U}(1)), \mathcal{A}_0)$ is a noncommutative principal $U(1)$-bundle. Then, for all $d \in \mathbb{N}$, the datum $\big(\mathcal{A}^{\mathbb{Z}_d}, \mathcal{O}(\mathrm{U}(1)), \mathcal{A}_0\big)$ is a noncommutative principal $U(1)$-bundle.

The proof of this result goes by showing that the right modules $\mathcal{A}_d$ and $\mathcal{A}_{-d}$ are finitely generated projective over $\mathcal{A}_0$ for all $d \in \mathbb{N}$. For this, let the finite sequences $\{\xi_j\}_{j=1}^N$, $\{\beta_i\}_{i=1}^M$ in $\mathcal{A}_1$ and $\{\eta_j\}_{j=1}^N$, $\{\alpha_i\}_{i=1}^M$ in $\mathcal{A}_{-1}$ be as in (**6**.1) and (**6**.2). Then, for each multi-index $J \in \{1, \ldots, N\}^d$ and each multi-index $I \in \{1, \ldots, M\}^d$ the elements

$$\begin{aligned} & \xi_J := \xi_{j_1} \cdot \ldots \cdot \xi_{j_d}, \quad \beta_I := \beta_{i_d} \cdot \ldots \cdot \beta_{i_1} \in \mathcal{A}_d \\ \text{and} \quad & \eta_J := \eta_{j_d} \cdot \ldots \cdot \eta_{j_1}, \quad \alpha_I := \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_d} \in \mathcal{A}_{-d}, \end{aligned}$$

are clearly such that $\sum_{J\in\{1,\ldots,N\}^d} \xi_J\, \eta_J = 1_{\mathcal{A}^{\mathbb{Z}_d}} = \sum_{I\in\{1,\ldots,M\}^d} \alpha_I\, \beta_I$. These allow one, as before, to show principality and to construct idempotents $\mathbf{e}_{(-d)}$ and $\mathbf{e}_{(d)}$, thus showing the finite projectivity of the right modules $\mathcal{A}_d$ and $\mathcal{A}_{-d}$ for all $d \in \mathbb{N}$.

6·2. *Strongly graded algebras.* – The relevance of graded algebras for noncommutative principal bundles was already shown in [23]. If G is any (multiplicative) group with unit e, an algebra $\mathcal{A}$ is a *G-graded algebra* if its admits a direct sum decomposition labelled by elements of G, that is $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$, with the property that $\mathcal{A}_g\mathcal{A}_h \subseteq \mathcal{A}_{gh}$, for all $g, h \in G$. If $\mathcal{H} := \mathbb{C}G$ denotes the group algebra, it is well know that $\mathcal{A}$ is G-graded if and only if $\mathcal{A}$ is a right $\mathcal{H}$-comodule algebra for the coaction $\delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}$ defined on homogeneous elements $a_g \in \mathcal{A}_g$ by $\delta(a_g) = a_g \otimes g$. Clearly, the coinvariants are given by $\mathcal{A}^{co\mathcal{H}} = \mathcal{A}_e$, the identity components. One has then the following result (*cf.* [15], 8.1.7): The datum $(\mathcal{A}, \mathcal{H}, \mathcal{A}_e)$ is a noncommutative principal $\mathcal{H}$-bundle for the canonical map

$$\mathrm{can} : \mathcal{A} \otimes_{\mathcal{A}_e} \mathcal{A} \to \mathcal{A} \otimes \mathcal{H}, \quad a \otimes b \mapsto \sum\nolimits_g ab_g \otimes g,$$

if and only if $\mathcal{A}$ is *strongly graded*, that is $\mathcal{A}_g\mathcal{A}_h = \mathcal{A}_{gh}$, for all $g, h \in G$.

For the proof, one first notes that $\mathcal{A}$ is strongly graded if and only if $\mathcal{A}_g\mathcal{A}_{g^{-1}} = \mathcal{A}_e$, for all $g \in G$. Then one constructs an inverse of the canonical map pretty much as in (**6**.3). Since, for each $g \in G$, the unit $1_\mathcal{A} \in \mathcal{A}_e = \mathcal{A}_{g^{-1}}\mathcal{A}_g$, there exists $\xi_{g^{-1},j}$ in $\mathcal{A}_g$ and $\eta_{g,j} \in \mathcal{A}_{g^{-1}}$, such that $\sum_j \eta_{g,j}\xi_{g^{-1},j} = 1_\mathcal{A}$. Then, $\mathrm{can}^{-1} : \mathcal{A} \otimes \mathcal{H} \to \mathcal{A} \otimes_{\mathcal{A}_e} \mathcal{A}$, is given by

$$\mathrm{can}^{-1} : a \otimes g \mapsto \sum\nolimits_j a\, \xi_{g^{-1},j} \otimes \eta_{g,j}.$$

For the particular case of $G = \mathbb{Z} = \widehat{\mathrm{U}(1)}$, so that $\mathbb{C}G = \mathcal{O}(U(1))$, this translates the result above on the principality of the bundle into the following:

The datum $(\mathcal{A}, \mathcal{O}(\mathrm{U}(1)), \mathcal{A}_0)$ is a noncommutative principal $U(1)$-bundle if and only if the algebra $\mathcal{A}$ is strongly graded over $\mathbb{Z}$, that is $\mathcal{A}_n\mathcal{A}_m = \mathcal{A}_{n+m}$, for all $n, m \in \mathbb{Z}$.

In the context of strongly graded algebras, the fact that all right modules $\mathcal{A}_n$ for all $n \in \mathbb{Z}$ are finite projective is a consequence of (see ref. [16], Cor. I.3.3).

6·3. *Pimsner algebras from principal circle bundles.* – From the considerations above — and in particular, if one compares (**5**.2) and the strongly graded condition —, it is clear that a C^*-algebra A is strongly $\mathbb{Z}$-graded if and only if it carries a $U(1)$-action with large spectral subspaces. One is then led to consider Pimsner algebras coming from principal circle bundles. As mentioned, for commutative algebras this is (see ref. [11], Prop. 5.8).

More generally, let us start with $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ a graded $*$-algebra. Denote by σ the $U(1)$-action coming from the grading. In addition, suppose there is a C^*- norm on $\mathcal{A}$, and that σ is isometric with respect to this norm:

$$\|\sigma_z(a)\| = \|a\|, \qquad \text{for all } z \in \mathrm{U}(1),\ a \in \mathcal{A}. \tag{6.5}$$

Denoting by A the completion of $\mathcal{A}$, one has that the action $\{\sigma_z\}_{z\in\mathrm{U}(1)}$ extends by continuity to a strongly continuous action of U(1) on A §3.6 of ref [15]. Furthermore, each spectral subspace A_n for the extended action agrees with the closure of $\mathcal{A}_n \subseteq A$.

The left and right Hermitian product as in (**5**.1) will make each spectral subspace A_n a (not necessarily full) Hilbert module over A_0. These become full exactly when $\mathcal{A}$ is strongly graded. The result at the end of sect. **5** leads then to:

Let $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ be a strongly graded $*$-algebra. Then, its C^*-closure A is generated, as a C^*-algebra, by A_1, and A is isomorphic to the Pimsner algebra $\mathcal{O}_{A_1}$ over A_0.

7. – Examples

In this section, we describe some examples. We shall give them only at the algebraic level while referring to [4] for more details, in particular on how to extend them to the continuous category thus getting Pimsner algebras. Additional examples are quantum lens spaces over quantum weighted projective spaces of [3] and [5]. They are obtained by twisting the product of the algebras of a given principal bundle by an automorphism.

7˙1. *Twisting of graded algebras.* – Let $\mathcal{A} = \oplus_{n\in\mathbb{Z}}\mathcal{A}_n$ be a $\mathbb{Z}$-graded unital $*$-algebra. And let γ be a graded unital $*$-automorphism of $\mathcal{A}$. A new unital graded $*$-algebra $(\mathcal{A}, \star_\gamma) =: \mathcal{B} = \bigoplus_{n\in\mathbb{Z}} \mathcal{B}_n$ is defined as follows: as a vector space $\mathcal{B}_n = \mathcal{A}_n$, the involution is unchanged, and the product is

$$a \star_\gamma b = \gamma^n(a)\gamma^{-k}(b), \qquad \text{for all } a \in \mathcal{B}_k,\ b \in \mathcal{B}_n, \tag{7.1}$$

where the product on the right hand side is the one in $\mathcal{A}$.

It is indeed straightforward to check that the new product satisfies

i) associativity: for all $a \in \mathcal{A}_k$, $b \in \mathcal{A}_m$, $c \in \mathcal{A}_n$ it holds that
$$(a \star_\gamma b) \star_\gamma c = a \star_\gamma (b \star_\gamma c) = \gamma^{m+n}(a)\gamma^{n-k}(b)\gamma^{-k-m}(c),$$
ii) $(a \star_\gamma b)^* = b^* \star_\gamma a^*$, for all a, b.

Furthermore, the unit is preserved, that is: $1 \star_\gamma a = a \star_\gamma 1 = a$ for all a and the degree zero subalgebra has undeformed product: $\mathcal{B}_0 = \mathcal{A}_0$. Finally,

$$a \star_\gamma \xi = \gamma^n(a)\xi, \qquad \xi \star_\gamma a = \xi\gamma^{-n}(a)\ , \qquad \text{for all } a \in \mathcal{B}_0,\ \xi \in \mathcal{B}_n.$$

Thus, the left $\mathcal{B}_0$-module structure of $\mathcal{B}_n$ is the one of $\mathcal{A}_n$ twisted with γ^n, and the right $\mathcal{B}_0$-module structure is the one of $\mathcal{A}_n$ twisted with γ^{-n}. We write this as $\mathcal{B}_n =_{\gamma^n} (\mathcal{A}_n)_{\gamma^{-n}}$.

When $\mathcal{A}$ is commutative, from the deformed product (**7**.1) one has commutation rules:

$$a \star_\gamma b = \gamma^{-2k}(b) \star_\gamma \gamma^{2n}(a), \qquad \text{for all } a \in \mathcal{B}_k,\ b \in \mathcal{B}_n. \tag{7.2}$$

Assume the datum $\big(\mathcal{A}, \mathcal{O}(\mathrm{U}(1)), \mathcal{A}_0\big)$ is a noncommutative principal $U(1)$-bundle. Then, the datum $\big(\mathcal{B}, \mathcal{O}(\mathrm{U}(1)), \mathcal{A}_0\big)$ is a noncommutative principal $U(1)$-bundle as well.

With the notation of (**6**.1), denoting $\alpha_i^\gamma = \gamma^{-1}(\alpha_i)$, $\beta_i^\gamma = \gamma^{-1}(\beta_i)$, $\xi_i^\gamma = \gamma(\xi_i)$ and $\eta_i^\gamma = \gamma(\eta_i)$, the collections $\{\xi_i^\gamma\}_{i=1}^N, \{\beta_i^\gamma\}_{i=1}^M \subset \mathcal{B}_1$ and $\{\eta_i^\gamma\}_{i=1}^N, \{\alpha_i^\gamma\}_{i=1}^M \subset \mathcal{B}_{-1}$ obey:

$$\sum\nolimits_{i=1}^N \xi_i^\gamma \star_\gamma \eta_i^\gamma = \sum\nolimits_{i=1}^N \xi_i\eta_i = 1, \quad \sum\nolimits_{i=1}^M \alpha_i^\gamma \star_\gamma \beta_i^\gamma = \sum\nolimits_{i=1}^M \alpha_i\beta_i = 1,$$

which is what is needed for principality.

An isomorphism of bimodules ${}_{\gamma^n}(\mathcal{A}_n)_{\gamma^{-n}} \simeq {}_{\gamma^{2n}}(\mathcal{A}_n)_{\mathrm{id}}$, is implemented by the map $a \mapsto \gamma^n(a)$, for $a \in \mathcal{A}_n$. This map intertwines the deformed product $\star_\gamma$ with the product

$$a \star'_\gamma b = \gamma^{2n}(a)b, \qquad \text{for all } a \in \mathcal{B}_k,\ b \in \mathcal{B}_n,$$

and the undeformed involution with a new involution,

$$a^\dagger = \gamma^{-2n}(a^*), \quad \text{for all } a \in \mathcal{B}_n.$$

By construction $(\mathcal{A},\star_\gamma)$ is isomorphic to $(\mathcal{A},\star'_\gamma)$ with deformed involution.

If $\mathcal{A}$ is dense in a graded C^*-algebra A and γ extends to a C^*-automorphism, the completion E_n of ${}_{\gamma^{2n}}(\mathcal{A}_n)_{\mathrm{id}}$ is a self-Morita equivalence A_0-bimodule (with $\phi=\gamma^{2n}$) and, by sect. **6**˙3, the completion of $\mathcal{B}$ is the Pimsner algebra over A_0 for $E_1={}_{\gamma^2}(\mathcal{A}_1)_{\mathrm{id}}$.

Examples of the above construction are noncommutative tori and related θ-deformed spheres and lens spaces, which we recall next.

7˙2. *The noncommutative torus.* – Being a crossed product, the noncommutative torus $C(\mathbb{T}^2_\theta)\simeq C(\mathbb{S}^1)\rtimes_\alpha\mathbb{Z}$ can be naturally seen as a Pimsner algebra over $C(\mathbb{S}^1)$. The automorphism α of $C(\mathbb{S}^1)$ is the one induced by the $\mathbb{Z}$-action generated by a rotation by $2\pi\mathrm{i}\theta$ on $\mathbb{S}^1$. As a preparation for the examples of next section, let us see how it emerges from the deformed construction considered in the previous section.

Let $\mathcal{A}=\mathcal{A}(\mathbb{T}^2)$ be the commutative unital $*$-algebra generated by two unitary elements u and v. This algebra is graded by assigning to u,v degree $+1$ and to their adjoints degree -1. The degree zero part is $\mathcal{A}_0\simeq\mathcal{A}(\mathbb{S}^1)$, generated by the unitary u^*v. Let $\theta\in\mathbb{R}$ and γ be the graded $*$-automorphism given by

$$\gamma_\theta(u)=e^{2\pi\mathrm{i}\theta}u,\qquad \gamma_\theta(v)=v.$$

From (**7**.2) we get

$$u\star_{\gamma_\theta}v=e^{2\pi\mathrm{i}\theta}v\star_{\gamma_\theta}u,$$

together with the relations $u\star_{\gamma_\theta}u^*=u^*\star_{\gamma_\theta}u=1$ and $v\star_{\gamma_\theta}v^*=v^*\star_{\gamma_\theta}v=1$. Thus the deformed algebra $\mathcal{B}:=(\mathcal{A},\star_{\gamma_\theta})=\mathcal{A}(\mathbb{T}^2_\theta)$ is the noncommutative torus algebra.

7˙3. *θ-deformed spheres and lens spaces.* – Let $\mathcal{A}=\mathcal{A}(\mathbb{S}^{2n+1})$ be the commutative unital $*$-algebra generated by elements $z_0,\dots,z_n$ and their adjoints, with relation $\sum_{i=0}^n z_i^*z_i=1$. This is graded by assigning to $z_0,\dots,z_n$ degree $+1$ and to their adjoints degree -1. For this grading the degree zero part is $\mathcal{A}_0\simeq\mathcal{A}(\mathbb{CP}^n)$.

Any matrix $(u_{ij})\in\mathrm{U}(n+1)$ defines a graded $*$-automorphism γ by

$$\gamma_u(z_i)=\sum\nolimits_{j=0}^n u_{ij}z_j,\qquad i=0,\dots,n.$$

Since a unitary matrix can be diagonalized by a unitary transformation, one can assume that (u_{ij}) is diagonal. Denote $\lambda_{ij}=u_{ii}^2\bar{u}_{jj}^2e^{2\pi\mathrm{i}\theta_{ij}}$; the matrix $\Theta=(\theta_{ij})$ is real (since $\lambda_{ij}\bar\lambda_{ij}=1$), and antisymmetric (since $\bar\lambda_{ij}=\lambda_{ji}$). From (**7**.2) one gets

$$z_i\star_{\gamma_u}z_j=\lambda_{ij}\,z_j\star_{\gamma_u}z_i,\qquad z_i\star_{\gamma_u}z_j^*=\bar\lambda_{ij}\,z_j^*\star_{\gamma_u}z_i,\quad\text{for all } i,j,$$

together with the conjugated relations, (and each z_i is normal for the deformed product, since $\lambda_{ii}=1$), and a sphere relation $\sum_{i=0}^n z_i^*\star_\gamma z_i=1$. As it is customary, we denote by $\mathcal{A}(\mathbb{S}^{2n+1}_\Theta)$ the algebra $\mathcal{A}(\mathbb{S}^{2n+1})$ with deformed product $\star_\gamma$.

With the same notation as in (**6**.4), for any natural number d, consider the algebra

$$\mathcal{A}(L^{2n+1}_\theta(d;\underline{1})):=\mathcal{A}(\mathbb{S}^{2n+1}_\Theta)^{\mathbb{Z}_d}=\oplus_{n\in\mathbb{Z}}\big(\mathcal{A}(\mathbb{S}^{2n+1}_\Theta)\big)_{dn},\tag{7.3}$$

which we think of as the coordinate algebra of the Θ-deformed lens spaces. From the general construction, it follows that the datum $\big(\mathcal{A}(L^{2n+1}_\theta(d;\underline{1})),\mathcal{O}(\mathrm{U}(1)),\mathcal{A}(\mathbb{CP}^n)\big)$ is a noncommutative principal $U(1)$-bundle. Clearly, for $d=1$ we get back the algebra $\mathcal{A}(\mathbb{S}^{2n+1}_\Theta)$ and the noncommutative principal $U(1)$-bundle $\big(\mathcal{A}(\mathbb{S}^{2n+1}_\Theta),\mathcal{O}(\mathrm{U}(1)),\mathcal{A}(\mathbb{CP}^n)\big)$.

Finally, let $C(\mathbb{CP}^n)$, $C(\mathbb{S}_\theta^{2n+1})$ and $C(L_\theta^{2n+1}(d;\underline{1}))$ denote the universal enveloping C^*-algebras for the coordinate algebras and let E_1 be the completion of the spectral subspace $\mathcal{B}_1$. Since the $U(1)$-action extended to $C(\mathbb{S}_\theta^{2n+1})$ has large spectral subspaces —being the one in (**7**.3) a strong grading— the d-th spectral subspace E_d agrees with $(E_1)^{\otimes d}$. With $*$-homomorphism $\phi : A_0 \to \mathcal{K}(E)$ the left multiplication one has:

For all integers $d \geq 1$, the C^*-algebra $C(L_\theta^{2n+1}(d;\underline{1}))$ is a Pimsner algebra over $C(\mathbb{CP}^n)$ for the Hilbert bimodule E_d.

* * *

This review is based on refs. [3-5]. I am grateful to my coworkers of those papers for our great collaborations. The author acknowledges the Italian Project "Prin 2010-11 – Operator Algebras, Noncommutative Geometry and Applications" for support.

REFERENCES

[1] Abadie B., Eilers S. and Exel R., *Trans. Amer. Math. Soc.*, **350** (1998) 3043.
[2] Abadie B. and Exel R., *Rev. Math. Phys.*, **9** (1997) 411.
[3] Arici F., Brain S. and Landi G., *J. Noncommut. Geom.*, **9** (2015) 1077.
[4] Arici F., D'Andrea F. and Landi G., *Pimsner algebras and noncommutative circle bundles*, arXiv:1506.03109.
[5] Arici F., Kaad J. and Landi G., *Pimsner algebras and Gysin sequences from principal $U(1)$-actions*, arXiv:1409.5335, to be published in *J. Noncomm. Geom.*
[6] Brzeziński T. and Majid S., *Commun. Math. Phys.*, **157** (1993) 591; *Erratum*, **167** (1995) 235.
[7] Cuntz J., *Commun. Math. Phys.*, **57** (1977) 173.
[8] Connes A., *Non-commutative Geometry and Physics*. In: *Gravitation and Quantizations*, Les Houches, Session LVII, 1992 (Elsevier) 1995.
[9] D'Andrea F. and Landi G., *Commun. Math. Phys.*, **340** (2015) 325.
[10] Exel R., *J. Funct. Anal.*, **122** (1994) 361.
[11] Gabriel O. and Grensing M., *Spectral triples and generalized crossed products*, arXiv:1310.5993.
[12] Hajac P. M., *Commun. Math. Phys.*, **182** (1996) 579.
[13] Kajiwara T., Pinzari C. and Watatani Y., *J. Funct. Anal.*, **159** (1998) 295.
[14] Lance E. C., *Hilbert -modules: a toolkit for operator algebraists, LMS Lect. Notes Ser.* Vol. **210** (Cambridge University Press) 1995.
[15] Montgomery S., *Hopf algebras and their actions on rings*, RCSM no. 82 (American Mathematical Society, Providence, R.I.) 1993.
[16] Nastasescu C. and Van Oystaeyen F., *Graded ring theory* (Elsevier) 1982.
[17] Paschke W. L., *J. Oper. Theory*, **6** (1981) 125.
[18] Pimsner M. V., *A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by* Z, Free probability theory (Waterloo, ON, 1995), in *Fields Institute Communications*, Vol. **12** (American Mathematical Society, Providence, R.I.) 1997, pp. 189–212.
[19] Raeburn I. and Williams D. P., *Morita equivalence and continuous-trace C^*-algebras*, MSM, Vol. **60** (American Mathematical Society, Providence, R.I.) 1998.
[20] Rieffel M. A., *J. K-theory*, **5** (2010) 39.
[21] Sitarz A. and Venselaar J. J., *Math. Phys. Anal. Geom.*, **18** (2015) 1.
[22] Swan R. G., *Trans. Am. Math. Soc.*, **105** (1962) 264.
[23] Ulbrich K. H., *Abh. Math. Sem. Univ. Hamburg*, **51** (1981) 136.
[24] Wegge-Olsen N. E., *K-Theory and C^*-algebras - a friendly approach* (Oxford University Press) 1993.