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Research Article

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The Dirichlet problem for gradient dependent prescribed mean curvature equations in the Lorentz–Minkowski space

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Abstract: We discuss existence, multiplicity, localisation and stability properties of solutions of the Dirichlet problem associated with the gradient dependent prescribed mean curvature equation in the Lorentz–Minkowski space

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The obtained results display various peculiarities, which are due to the special features of the involved differential operator and have no counterpart for elliptic problems driven by other quasilinear differential operators. This research is also motivated by some recent achievements in the study of prescribed mean curvature graphs in certain Friedmann–Lemaître–Robertson–Walker, as well as Schwarzschild–Reissner–Nordström, spacetimes.

Keywords: Mean curvature, Lorentz–Minkowski space, partial differential equation, quasilinear elliptic problem, Dirichlet condition, lower and upper solutions, existence, multiplicity, order stability, order instability

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Dedicated to Professor Ivan Tarielovich Kiguradze

1 Introduction

Let us consider the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with regular boundary $\partial\Omega$ and f satisfies the L^∞-Carathéodory conditions. Graphs of solutions of (1.1) are surfaces of prescribed mean curvature in the Lorentz–Minkowski

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space $\mathbb{L}^{N+1} = \{(x,t) : x \in \mathbb{R}^N,\ t \in \mathbb{R}\}$ with metric $\sum_{i=1}^N dx_i^2 - dt^2$. We will be concerned with strictly spacelike solutions of (1.1), that is, weak, or strong, solutions u of (1.1) satisfying $\|\nabla u\|_\infty < 1$; a non-exhaustive list of references about this problem includes [2, 3, 10, 22, 26, 28, 30] and the bibliographies therein.

A motivation for considering equations in (1.1), where the right-hand side f depends explicitly on the gradient of the solution, derives from the interest in various issues of differential geometry about the following class of anisotropic mean curvature equations,

$$-\frac{1}{N}\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \mathcal{H}(x, u, \mathcal{N}(u)),$$

in which the prescribed mean curvature $\mathcal{H}$ depends on the unit upward normal to the graph of u,

$$\mathcal{N}(u) = \frac{(\nabla u, 1)}{\sqrt{1-|\nabla u|^2}}.$$

These equations may also arise as Euler–Lagrange equations of some weighted area functionals (cf. [8, 9, 13, 26, 27, 29]), such as

$$\int_\Omega \mathcal{A}(x,u)\sqrt{1-|\nabla u|^2}\,dx + \int_\Omega \mathcal{B}(x,u)dx,$$

as well as they occur in the study of prescribed mean curvature graphs in certain Friedmann–Lemaître–Robertson–Walker, or Schwarzschild–Reissner–Nordström, spacetimes (cf. [4, 19, 20, 26]).

The aim of this paper is to work out a general lower and upper solution method for (1.1). Rather than the solvability of (1.1), which as we will see is always guaranteed without placing any additional qualitative or quantitative assumption on the right-hand side f, the interest of using lower and upper solutions in this context mainly relies on the localisation, the multiplicity and the stability information that they may provide. In this respect, due to the special features of the mean curvature operator in the Lorentz–Minkowski space, various peculiarities are displayed, which have no counterpart for elliptic problems driven by other quasilinear differential operators, such as the p-Laplace operator, or the mean curvature operator in the Euclidean space. In particular, the simple knowledge of just one lower solution α, or just one upper solution β, allows to localise solutions in terms of α, or β, whereas the existence of a couple of lower and upper solutions α, β with $\alpha \nleq \beta$ yields multiple solutions, whose stability or instability properties can be detected and specified. Here we use the notion of order stability: for a discussion of the relationships between this concept and other classical ones considered in the literature we refer to [18, 21, 24]. It is worthy to point out that our stability, or instability, conclusions will follow, as in [16–18], without assuming any additional regularity hypotheses on the function f besides the L^∞-Carathéodory conditions.

We finally recall that some preliminary results related to the topics of this paper, but confined to the simpler problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

were announced in [14]. We refer to that paper for some applications of the lower and upper solutions method to the existence of multiple positive solutions of (1.2) (see also [5–7, 11, 12, 15] for further results). It should be stressed that, if compared with (1.2), the study of (1.1) needs more care and requires the introduction of some new technical devices.

Notations. We list some notations that are used throughout this paper. For $s \in \mathbb{R}$, we set $s^+ = \max\{s, 0\}$ and $s^- = -\min\{s, 0\}$. We denote by $B_R(y) = \{x \in \mathbb{R}^N : |x-y| < R\}$ the open ball in $\mathbb{R}^N$ centered at y and having radius R; the subscript R indicating the radius, as well as the indication of the center y, may be omitted if irrelevant in the context. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^N$ with boundary $\partial\mathcal{O}$. For functions $u, v : \mathcal{O} \to \mathbb{R}$, we write $u \le v$ in $\mathcal{O}$ if $u(x) \le v(x)$ for a.e. $x \in \mathcal{O}$. Whenever $u, v : \overline{\mathcal{O}} \to \mathbb{R}$ are continuous, we also write: $u \le v$ on $\partial\mathcal{O}$ if $u(x) \le v(x)$ for all $x \in \partial\mathcal{O}$; $u \le v$ in $\overline{\mathcal{O}}$ if $u(x) \le v(x)$ for all $x \in \overline{\mathcal{O}}$; $u < v$ in $\overline{\mathcal{O}}$ if $u \le v$ in $\overline{\mathcal{O}}$ and $u(x_0) < v(x_0)$ for some $x_0 \in \mathcal{O}$; $u \ll v$ in $\overline{\mathcal{O}}$ if there is $\varepsilon > 0$ such that $u(x) + \varepsilon\,\operatorname{dist}(x, \partial\mathcal{O}) \le v(x)$ for all $x \in \overline{\mathcal{O}}$. We finally set $C_0^1(\overline{\mathcal{O}}) = \{u \in C^1(\overline{\mathcal{O}}) : u = 0 \text{ on } \partial\mathcal{O}\}$.

2 Preliminaries

Throughout this paper the following assumptions are considered:

(h1) Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^2,

(h2) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the L^∞-Carathéodory conditions, i.e.,
- for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $f(\cdot\,, s, \xi)$ is measurable in Ω,
- for a.e. $x \in \Omega$, $f(x, \cdot\,, \cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}^N$,
- for each $\rho > 0$,

$$\operatorname*{ess\,sup}_{\Omega\times[-\rho,\rho]\times[-\rho,\rho]^N} |f(x, s, \xi)| < +\infty.$$

The following notion of solution of problem (1.1) is adopted.

Definition 2.1. We say that a function $u : \overline{\Omega} \to \mathbb{R}$ is a solution of (1.1) if $u \in C^{0,1}(\overline{\Omega})$ and satisfies
- $\|\nabla u\|_\infty < 1$,
- for every $w \in W_0^{1,1}(\Omega)$,

$$\int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{1 - |\nabla u|^2}}\, dx = \int_\Omega f(x, u, \nabla u) w\, dx, \tag{2.1}$$

- $u = 0$ on $\partial\Omega$.

Remark 2.1. A direct consequence of this definition is that any solution u of (1.1) satisfies $\|u\|_\infty < \frac{1}{2}\operatorname{diam}(\Omega)$.

Next we state the following comparison result, which is a direct consequence of [3, Lemma 1.2].

Lemma 2.1. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\mathcal{O}$, and suppose that $v_1, v_2 \in L^\infty(\mathcal{O})$ satisfy $v_1 \le v_2$ in $\mathcal{O}$. Let, for $i = 1, 2$, $u_i \in C^{0,1}(\overline{\mathcal{O}})$ be such that $\|\nabla u_i\|_\infty < 1$ and*

$$\int_{\mathcal{O}} \frac{\nabla u_i \cdot \nabla w}{\sqrt{1 - |\nabla u_i|^2}}\, dx = \int_{\mathcal{O}} v_i w\, dx$$

for all $w \in W_0^{1,1}(\mathcal{O})$. Then we have

$$\min_{\partial\mathcal{O}}(u_2 - u_1) = \min_{\overline{\mathcal{O}}}(u_2 - u_1). \tag{2.2}$$

Proof. Fix $v \in L^\infty(\mathcal{O})$. Let $u \in C^{0,1}(\overline{\mathcal{O}})$ be such that $\|\nabla u\|_\infty < 1$ and

$$\int_{\mathcal{O}} \frac{\nabla u \cdot \nabla w}{\sqrt{1 - |\nabla u|^2}}\, dx = \int_{\mathcal{O}} v w\, dx \tag{2.3}$$

for all $w \in W_0^{1,1}(\mathcal{O})$. Set

$$\mathcal{C}_u = \{w \in C^{0,1}(\overline{\mathcal{O}}) : \|\nabla w\|_\infty \le 1 \text{ and } w = u \text{ on } \partial\mathcal{O}\}$$

and define the functional $\mathcal{J}_v : \mathcal{C}_u \to \mathbb{R}$ by

$$\mathcal{J}_v(w) = \int_{\mathcal{O}} \sqrt{1 - |\nabla w|^2}\, dx + \int_{\mathcal{O}} v w\, dx.$$

We claim that u maximises $\mathcal{J}_v$ in $\mathcal{C}_u$. Indeed, pick any $w \in \mathcal{C}_u$. Taking $u - w$ as test function in (2.3), we get

$$\int_{\mathcal{O}} \frac{\nabla u \cdot \nabla(u - w)}{\sqrt{1 - |\nabla u|^2}}\, dx = \int_{\mathcal{O}} v(u - w)\, dx. \tag{2.4}$$

Let $g : \overline{B_1(0)} \to \mathbb{R}$ be defined by $g(y) = \sqrt{1 - |y|^2}$. By the concavity and the differentiability of g in $B_1(0)$, we obtain

$$\int_{\mathcal{O}} \sqrt{1 - |\nabla w|^2}\, dx - \int_{\mathcal{O}} \sqrt{1 - |\nabla u|^2}\, dx \le \int_{\mathcal{O}} \frac{\nabla u \cdot \nabla(u - w)}{\sqrt{1 - |\nabla u|^2}}\, dx. \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\mathcal{J}_v(w) \le \mathcal{J}_v(u).$$

Accordingly, we have that u_1 and u_2 are global maximisers of $\mathcal{J}_{v_1}$ in $\mathcal{C}_{u_1}$ and of $\mathcal{J}_{v_2}$ in $\mathcal{C}_{u_2}$, respectively. Hence [3, Lemma 1.2] applies, implying that (2.2) holds. □

Now we prove an existence and regularity result for the problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = v & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}, \end{cases} \tag{2.6}$$

where $\mathcal{O}$ is an open bounded set in $\mathbb{R}^N$ with boundary of class C^2 and $v \in L^\infty(\mathcal{O})$. This result is based on the gradient estimates obtained in [3, Corollary 3.4, Theorem 3.5].

Lemma 2.2. *Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\mathcal{O}$ of class C^2, and suppose that $v \in L^\infty(\mathcal{O})$. Then problem (2.6) has a unique solution u with $u \in W^{2,r}(\mathcal{O})$, for all finite $r \geq 1$. Moreover, for any given $\Lambda > 0$ and $r \in]N, +\infty[$, there exist constants $\vartheta = \vartheta(\mathcal{O}, \Lambda) \in]0,1[$ and $c = c(\mathcal{O}, \Lambda, r) > 0$ such that, for every $v \in L^\infty(\mathcal{O})$ with $\|v\|_\infty \leq \Lambda$, the following estimates hold:*

$$\|\nabla u\|_\infty < 1 - \vartheta \tag{2.7}$$

and

$$\|u\|_{W^{2,r}} \leq c\|v\|_{L^r}. \tag{2.8}$$

Proof. *Uniqueness.* Uniqueness of solution of (2.6) follows immediately from Lemma 2.1.

Existence. Let $\Lambda > 0$ and $r \in]N, +\infty[$ be fixed. Take a function $v \in L^\infty(\mathcal{O})$ with $\|v\|_\infty \leq \Lambda$. We first assume that v further satisfies $v \in C^{0,1}(\overline{\mathcal{O}})$. Combining [3, Corollary 3.4] and [3, Theorem 3.5] provides the existence of a constant $\vartheta = \vartheta(\mathcal{O}, \Lambda) \in]0, 1[$ such that any solution $u \in C^2(\mathcal{O}) \cap C^1(\overline{\mathcal{O}})$ of (2.6) satisfies (2.7); according to Remark 2.1, u also satisfies $\|u\|_\infty < \frac{1}{2}\operatorname{diam}(\Omega)$.

Let us introduce a function $A : \mathbb{R}^N \to \mathbb{R}^N$ satisfying the structure conditions assumed in [25, Theorem 1] and

$$A(\xi) = \frac{\xi}{\sqrt{1-|\xi|^2}} \quad \text{for all } \xi \in \mathbb{R}^N \text{ with } |\xi| \leq \vartheta.$$

Then [25, Theorem 1] applies and yields the existence of constants $\alpha = \alpha(\mathcal{O}, \Lambda) \in]0, 1]$ and $c_1 = c_1(\mathcal{O}, \Lambda) > 0$ such that $u \in C^{1,\alpha}(\overline{\mathcal{O}})$ and

$$\|u\|_{C^{1,\alpha}} < c_1.$$

We can also suppose that α has been taken so small that $W^{2,r}(\mathcal{O})$ is compactly embedded into $C^{1,\alpha}(\overline{\mathcal{O}})$; as a consequence, α and c_1 now depend on $\mathcal{O}$, Λ and r too.

Let us define

$$\mathcal{D} = \{w \in C^{1,\alpha}(\overline{\mathcal{O}}) : \|\nabla w\|_\infty < 1 - \vartheta,\ \|w\|_{C^{1,\alpha}} < c_1\};$$

$\mathcal{D}$ is an open bounded subset of $C^{1,\alpha}(\overline{\mathcal{O}})$ with $0 \in \mathcal{D}$. Pick any $w \in \overline{\mathcal{D}}$ and set, for $i, j = 1, \dots, N$,

$$a_{ij}(w) = \delta_{ij} a(|\nabla w|^2) + 2a'(|\nabla w|^2)\partial_{x_i} w\, \partial_{x_j} w,$$

where δ_{ij} is the Kronecker delta and $a(s) = (1-s)^{-\frac{1}{2}}$. Consider the Dirichlet problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} a_{ij}(w)\partial_{x_i x_j} z = v & \text{in } \mathcal{O}, \\ z = 0 & \text{on } \partial\mathcal{O}. \end{cases} \tag{2.9}$$

Note that the coefficients $a_{ij}(w)$ belong to $C^{0,\alpha}(\overline{\mathcal{O}})$ and are uniformly bounded in $C^{0,\alpha}(\overline{\mathcal{O}})$ with bound independent of $w \in \overline{\mathcal{D}}$ and ultimately depending on $\mathcal{O}$, Λ and r only; moreover, the ellipticity constant can be taken equal to 1. According to the L^p-regularity theory [23, Theorem 9.15, Theorem 9.13], problem (2.9) has a unique solution $z \in W^{2,r}(\mathcal{O})$ (depending on v and w) and there exists a constant $c_2 = c_2(\mathcal{O}, \Lambda, r) > 0$ such that

$$\|z\|_{W^{2,r}} \leq c_2(\|z\|_{L^r} + \|v\|_{L^r}).$$

Since in particular $r \in]\frac{N}{2}, +\infty[$, $W^{2,r}(\mathcal{O})$ is embedded into $L^\infty(\mathcal{O})$, and z satisfies

$$\|z\|_\infty \leq c_3$$

for some $c_3 = c_3(\mathcal{O}, \Lambda, r) > 0$. Combining these two estimates yields

$$\|z\|_{W^{2,r}} \le c\|v\|_{L^r} \tag{2.10}$$

for some constant $c = c(\mathcal{O}, \Lambda, r) > 0$ (depending on the indicated quantities only). Moreover, as $z \in C^{1,\alpha}(\overline{\mathcal{O}})$, $v \in C^{0,1}(\overline{\mathcal{O}})$ and $a_{ij}(w) \in C^{0,\alpha}(\overline{\mathcal{O}})$, for $i, j = 1, \dots, N$, the Schauder regularity theory [23, Corollary 6.9] applies locally and allows us to conclude that $z \in C^{2,\alpha}(\mathcal{O})$; hence, in particular, $z \in W^{2,r}(\mathcal{O}) \cap C^2(\mathcal{O})$.

Let us denote by $\mathcal{L} : \overline{\mathcal{D}} \to C^{1,\alpha}(\overline{\mathcal{O}})$ the operator which sends each $w \in \overline{\mathcal{D}}$ onto the unique solution $z \in W^{2,r}(\mathcal{O})$ of (2.9). Let us verify that $\mathcal{L}$ is completely continuous. We first prove that $\mathcal{L}$ has a relatively compact range. Let $(w_n)_n$ be a sequence in $\overline{\mathcal{D}}$. By (2.10) the sequence $(\mathcal{L}(w_n))_n$ is bounded in $W^{2,r}(\mathcal{O})$. Hence there exists a subsequence $(\mathcal{L}(w_{n_k}))_k$ which converges weakly in $W^{2,r}(\mathcal{O})$ and strongly in $C^{1,\alpha}(\overline{\mathcal{O}})$ to some $z \in W^{2,r}(\mathcal{O})$. The continuity of $\mathcal{L}$ can be verified as follows. Let $(w_n)_n$ be a sequence in $\overline{\mathcal{D}}$ converging in $C^{1,\alpha}(\overline{\mathcal{O}})$ to some $w \in \overline{\mathcal{D}}$. We want to prove that $(\mathcal{L}(w_n))_n$ converges in $C^{1,\alpha}(\overline{\mathcal{O}})$ to $\mathcal{L}(w)$. Let us consider any subsequence $(\mathcal{L}(w_{n_k}))_k$ of $(\mathcal{L}(w_n))_n$ and verify that it has a subsequence converging to $\mathcal{L}(w)$. Arguing as above, there exists a subsequence $(\mathcal{L}(w_{n_{k_s}}))_s$ of $(\mathcal{L}(w_{n_k}))_k$ which converges weakly in $W^{2,r}(\mathcal{O})$ and strongly in $C^{1,\alpha}(\overline{\mathcal{O}})$ to some $z \in W^{2,r}(\mathcal{O})$. As each $z_{n_{k_s}} = \mathcal{L}(w_{n_{k_s}})$ satisfies the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} a_{ij}(w_{n_{k_s}})\partial_{x_i x_j} z_{n_{k_s}} = v & \text{in } \mathcal{O}, \\ z_{n_{k_s}} = 0 & \text{on } \partial\mathcal{O}, \end{cases}$$

we can pass to the limit, concluding that $z \in W^{2,r}(\mathcal{O})$ is a solution of (2.9) and hence, by uniqueness, $z = \mathcal{L}(w)$. We then deduce that the whole sequence $(\mathcal{L}(w_n))_n$ converges in $C^{1,\alpha}(\overline{\mathcal{O}})$ to $\mathcal{L}(w)$.

We further observe that, if $u \in \overline{\mathcal{D}}$ is a fixed point of $\mathcal{L}$, then u is a solution of (2.6) with $u \in W^{2,r}(\mathcal{O})$. In order to prove the existence of a fixed point of $\mathcal{L}$, we show that every solution $u \in \overline{\mathcal{D}}$ of

$$u = t\mathcal{L}(u), \tag{2.11}$$

for some $t \in [0, 1]$, belongs to $\mathcal{D}$. Note that (2.11) implies that $u \in W^{2,r}(\mathcal{O})$ is a solution of

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = tv & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

As $\|tv\|_\infty \le \Lambda$ and $v \in C^{0,1}(\overline{\mathcal{O}})$, by the previous argument we deduce that u satisfies $u \in W^{2,r}(\mathcal{O}) \cap C^2(\mathcal{O})$, $\|\nabla u\|_\infty < 1 - \vartheta$, $\|u\|_{C^{1,\alpha}} < c_1$, and hence $u \in \mathcal{D}$. Accordingly, the Leray–Schauder continuation theorem yields the existence of a fixed point $u \in \mathcal{D}$ of $\mathcal{L}$ and therefore of a solution in $W^{2,r}(\mathcal{O})$ of (2.6) which satisfies (2.7) and (2.8).

The general case of a function $v \in L^\infty(\mathcal{O})$ with $\|v\|_\infty \le \Lambda$, can be easily dealt with by approximation. Fix $r \in \,]N, +\infty[$ and let $(v_n)_n$ be a sequence in $C^{0,1}(\overline{\mathcal{O}})$ converging to v in $L^r(\mathcal{O})$ and satisfying $\|v_n\|_\infty \le \Lambda$ for all n. The corresponding solutions $(u_n)_n$ in $W^{2,r}(\mathcal{O})$ of (2.6) satisfy (2.7) and (2.8), where u is replaced by u_n, for all n. Arguing as above, we can extract a subsequence of $(u_n)_n$ which weakly converges in $W^{2,r}(\mathcal{O})$ to a solution u of (2.6). Clearly, estimate (2.7) is valid, possibly reducing ϑ. By the weak lower semi-continuity of the $W^{2,r}$-norm, (2.8) holds true as well. □

Remark 2.2. Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^N$ with boundary of class $C^{2,\alpha}$ and $v \in C^{0,\alpha}(\overline{\mathcal{O}})$ for some $\alpha \in \,]0, 1[$. Then the solution u of (2.6) belongs to $C^{2,\gamma}(\overline{\mathcal{O}})$ for some $\gamma \in \,]0, 1[$. Indeed, let us fix $r \in \,]N, +\infty[$. Lemma 2.2 implies that $u \in W^{2,r}(\mathcal{O})$ and hence $u \in C^{1,\beta}(\overline{\mathcal{O}})$ with $\beta = 1 - \frac{N}{r}$. For $i, j = 1, \dots, N$, let us define the functions $a_{ij} \in C^{0,\beta}(\overline{\mathcal{O}})$ by

$$a_{ij} = \delta_{ij} a(|\nabla u|^2) + 2a'(|\nabla u|^2)\partial_{x_i} u\, \partial_{x_j} u,$$

where δ_{ij} is the Kronecker delta and, as before, $a(s) = (1-s)^{-\frac{1}{2}}$. Then u is a solution of the linear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} a_{ij}\partial_{x_i x_j} z = v & \text{in } \mathcal{O}, \\ z = 0 & \text{on } \partial\mathcal{O}. \end{cases}$$

The Schauder regularity theory [23, Theorem 6.14] applies and allows us to conclude that $u \in C^{2,\gamma}(\overline{\mathcal{O}})$ for some $\gamma \in\,]0,1[$.

Remark 2.3. Lemma 2.2 guarantees in particular that, if $u \in C^{0,1}(\overline{\Omega})$ is a solution of (1.1), then $u \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, and hence it is a strong solution of (1.1). Further, if, for some $\alpha \in\,]0,1[$, Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $C^{2,\alpha}$ and $f \in C^{0,\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$, then Remark 2.2 implies that any solution u of (1.1) belongs to $C^{2,\gamma}(\overline{\Omega})$, for some $\gamma \in\,]0,1[$, and thus it is a classical solution of (1.1).

Let us denote by

$$\mathcal{B} = \{u \in C^1_0(\overline{\Omega}) : \|\nabla u\|_\infty < 1\}$$

the unit open ball in $C^1_0(\overline{\Omega})$ and by $\mathcal{I}$ the identity operator in $C^1_0(\overline{\Omega})$.

Lemma 2.3. *Assume* (h1) *and let* $\mathcal{N} : C^{0,1}(\overline{\Omega}) \to L^\infty(\Omega)$ *be an operator satisfying*

(h3) *for any sequence* $(v_n)_n$ *in* $C^{0,1}(\overline{\Omega})$ *converging in* $C^{0,1}(\overline{\Omega})$ *to some* $v \in C^{0,1}(\overline{\Omega})$, $\lim_{n\to+\infty} \mathcal{N}(v_n) = \mathcal{N}(v)$ *a.e. in* Ω,

(h4) *for any bounded sequence* $(v_n)_n$ *in* $C^{0,1}(\overline{\Omega})$, *there is a constant* $\Lambda > 0$ *such that* $\|\mathcal{N}(v_n)\|_\infty \leq \Lambda$ *for all* n.

Let $\mathcal{P} : C^{0,1}(\overline{\Omega}) \to C^1_0(\overline{\Omega})$ *be the operator which sends any function* $v \in C^{0,1}(\overline{\Omega})$ *onto the unique solution* u *of the problem*

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \mathcal{N}(v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Then $\mathcal{P}$ *is completely continuous and*

$$\deg(\mathcal{I} - \mathcal{P}, \mathcal{B}, 0) = 1. \tag{2.13}$$

Proof. We divide the proof into two steps.

Step 1. The operator $\mathcal{P}$ *is completely continuous.* We first prove the continuity of $\mathcal{P}$. Fix $r \in\,]N, +\infty[$. Let $(v_n)_n$ be a sequence in $C^{0,1}(\overline{\Omega})$ converging to some $v \in C^{0,1}(\overline{\Omega})$. By assumption the sequence $(\mathcal{N}(v_n))_n$ converges to $\mathcal{N}(v)$ a.e. in Ω. Set, for each n, $u_n = \mathcal{P}(v_n)$ and $u = \mathcal{P}(v)$. We aim to prove that $\lim_{n\to+\infty} u_n = u$ in $C^1(\overline{\Omega})$. Let $(u_{n_k})_k$ be a subsequence of $(u_n)_n$. From (h1), (h4) and Lemma 2.2 we infer that $(u_{n_k})_k$ is bounded in $W^{2,r}(\Omega)$. Therefore, there exists a subsequence $(u_{n_{k_j}})_j$ of $(u_{n_k})_k$ which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to some $z \in W^{2,r}(\Omega)$; moreover there exists $\vartheta = \vartheta(\Omega, \Lambda) \in\,]0,1[$ such that

$$\|\nabla u_{n_{k_j}}\|_\infty < 1 - \vartheta$$

for all j. In particular, we have $z \in C^1_0(\overline{\Omega})$ and $\|\nabla z\|_\infty \leq 1 - \vartheta$. Furthermore, as, for each j, $u_{n_{k_j}}$ solves (2.12), it satisfies

$$\int_\Omega \frac{\nabla u_{n_{k_j}} \cdot \nabla w}{\sqrt{1 - |\nabla u_{n_{k_j}}|^2}}\, dx = \int_\Omega \mathcal{N}(v_{n_{k_j}}) w\, dx \tag{2.14}$$

for all $w \in W^{1,1}_0(\Omega)$. Letting $j \to +\infty$ in (2.14), we get by the dominated convergence theorem

$$\int_\Omega \frac{\nabla z \cdot \nabla w}{\sqrt{1 - |\nabla z|^2}}\, dx = \int_\Omega \mathcal{N}(v) w\, dx$$

for all $w \in W^{1,1}_0(\Omega)$. Thus we conclude that $z \in W^{2,r}(\Omega)$ is a solution of problem (2.12). By uniqueness of the solution, we conclude that $z = \mathcal{P}(v) = u$. Therefore it follows that $\lim_{n\to+\infty} u_n = u$ in $C^1(\overline{\Omega})$.

Next we show that $\mathcal{P}$ sends bounded subsets of $C^{0,1}(\overline{\Omega})$ into relatively compact subsets of $C^1_0(\overline{\Omega})$. Let $(v_n)_n$ be a bounded sequence in $C^{0,1}(\overline{\Omega})$. Then, by condition (h4), there exists a constant $\Lambda > 0$ such that $\|\mathcal{N}(v_n)\|_\infty \leq \Lambda$ for all n. Set $u_n = \mathcal{P}(v_n)$ for all n. Arguing as above, we deduce the existence of a subsequence $(u_{n_k})_k$ of $(u_n)_n$ which strongly converges in $C^1(\overline{\Omega})$. We conclude that the operator $\mathcal{P}$ is completely continuous.

Step 2. $\deg(\mathcal{I} - \mathcal{P}, \mathcal{B}, 0) = 1$. According to assumption (h4), there exists $\Lambda_1 > 0$ such that $\|\mathcal{N}(v)\|_\infty \leq \Lambda_1$ for all $v \in \overline{\mathcal{B}}$. Using Lemma 2.2, we find a constant $\eta \in\,]0,1[$ such that any solution $u = \mathcal{P}(v)$ of (2.12) satisfies $\|\nabla u\|_\infty \leq \eta$. Hence $\mathcal{P}$ maps $\overline{\mathcal{B}}$ into $\mathcal{B}$ and, a fortiori, for each $t \in [0,1[$, also $t\mathcal{P}$ maps $\overline{\mathcal{B}}$ into $\mathcal{B}$. The invariance under homotopy of the topological degree yields $\deg(\mathcal{I} - \mathcal{P}, \mathcal{B}, 0) = \deg(\mathcal{I}, \mathcal{B}, 0) = 1$. □

Remark 2.4. Under the assumptions of Lemma 2.3, we see in particular that there exists a solution $u \in W^{2,r}(\Omega)$, for every finite $r \geq 1$, of the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = \mathcal{N}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 2.5. Assume (h1) and (h2). Then we can define the operator $\mathcal{T} : C^{0,1}(\overline{\Omega}) \to C^1_0(\overline{\Omega})$, which sends any function $v \in C^{0,1}(\overline{\Omega})$ onto the unique solution u of the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = f(x, v, \nabla v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Clearly, $u \in C^{0,1}(\overline{\Omega})$ is a solution of (1.1) if and only if u is a fixed point of $\mathcal{T}$. Let $\mathcal{N} : C^{0,1}(\overline{\Omega}) \to L^\infty(\Omega)$ be the superposition operator defined by $\mathcal{N}(v) = f(\cdot, v, \nabla v)$. Observe that, by (h2), $\mathcal{N}$ satisfies (h3) and (h4). Applying Lemma 2.3 to $\mathcal{N}$ we see that the operator $\mathcal{T}$ is completely continuous and $\deg(\mathcal{I} - \mathcal{T}, \mathcal{B}, 0) = 1$.

3 A lower and upper solution method

The following notion of lower and upper solutions of problem (1.1) is adopted.

Definition 3.1. We say that a function $\alpha : \overline{\Omega} \to \mathbb{R}$ is a lower solution of (1.1) if $\alpha \in C^{0,1}(\overline{\Omega})$ and satisfies

- $\|\nabla \alpha\|_\infty < 1$,
- for every $w \in W^{1,1}_0(\Omega)$ with $w \geq 0$ in Ω,

$$\int_\Omega \frac{\nabla \alpha \cdot \nabla w}{\sqrt{1-|\nabla \alpha|^2}}\, dx \leq \int_\Omega f(x, \alpha, \nabla \alpha) w \, dx, \tag{3.1}$$

- $\alpha \leq 0$ on $\partial\Omega$.

We say that a lower solution α of (1.1) is proper if it is not a solution. Further, we say that a lower solution α of (1.1) is strict if every solution u of (1.1) with $u \geq \alpha$ in $\overline{\Omega}$ satisfies $u \gg \alpha$ in $\overline{\Omega}$.

Similarly, we say that a function $\beta : \overline{\Omega} \to \mathbb{R}$ is an upper solution of (1.1) if $\beta \in C^{0,1}(\overline{\Omega})$ and satisfies

- $\|\nabla \beta\|_\infty < 1$,
- for every $w \in W^{1,1}_0(\Omega)$ with $w \geq 0$ in Ω,

$$\int_\Omega \frac{\nabla \beta \cdot \nabla w}{\sqrt{1-|\nabla \beta|^2}}\, dx \geq \int_\Omega f(x, \beta, \nabla \beta) w \, dx,$$

- $\beta \geq 0$ on $\partial\Omega$.

We say that an upper solution β of (1.1) is proper if it is not a solution. Further, we say that an upper solution β of (1.1) is strict if every solution u of (1.1) with $u \leq \beta$ in $\overline{\Omega}$ satisfies $u \ll \beta$ in $\overline{\Omega}$.

Remark 3.1. Note that u is a solution of (1.1) if and only if it is simultaneously a lower solution and an upper solution of (1.1).

The following result holds in the presence of a couple of ordered lower and upper solutions.

Proposition 3.1. *Assume* (h1) *and* (h2). *Suppose that there exist a lower solution α and an upper solution β of* (1.1) *with $\alpha \leq \beta$ in $\overline{\Omega}$. Then problem* (1.1) *has solutions v, w with $\alpha \leq v \leq w \leq \beta$ in $\overline{\Omega}$ such that every solution u of* (1.1) *with $\alpha \leq u \leq \beta$ in $\overline{\Omega}$ satisfies $v \leq u \leq w$ in $\overline{\Omega}$. Further, if α and β are strict, then*

$$\deg(I - \mathcal{T}, \mathcal{U}, 0) = 1, \tag{3.2}$$

where $\mathcal{T}$ is defined by (2.15) *and*

$$\mathcal{U} = \{z \in C^1_0(\overline{\Omega}) : \alpha \ll z \ll \beta \text{ in } \overline{\Omega} \text{ and } \|\nabla z\|_\infty < 1\}. \tag{3.3}$$

Proof. The proof is divided into three parts.

Part 1. Existence of a solution u of (1.1) *with $\alpha \le u \le \beta$ in $\overline{\Omega}$.*

Step 1. Construction of a modified problem. We define a function $\gamma : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by setting, for all $x \in \overline{\Omega}$,

$$\gamma(x,s) = \begin{cases} \alpha(x) & \text{if } s < \alpha(x), \\ s & \text{if } \alpha(x) \le s < \beta(x), \\ \beta(x) & \text{if } \beta(x) \le s, \end{cases}$$

and an operator $\mathcal{F} : C^{0,1}(\overline{\Omega}) \to L^\infty(\Omega)$ by setting

$$\mathcal{F}(u) = f(\cdot\,, \gamma(\cdot\,, u), \nabla(\gamma(\cdot\,, u))).$$

Note that, for each $u \in C^{0,1}(\overline{\Omega})$, we have, for a.e. $x \in \Omega$,

$$\mathcal{F}(u)(x) = f(x, \alpha(x), \nabla\alpha(x)) \quad \text{if } u(x) \le \alpha(x),$$

and

$$\mathcal{F}(u)(x) = f(x, \beta(x), \nabla\beta(x)) \quad \text{if } u(x) \ge \beta(x).$$

Then we consider the modified problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \mathcal{F}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

Step 2. Every solution u of (3.4) *satisfies $\alpha \le u \le \beta$ in $\overline{\Omega}$.* Let u be a solution of (3.4). In order to prove that $u \ge \alpha$ in $\overline{\Omega}$, we set $w = (u-\alpha)^- \in W_0^{1,1}(\Omega)$. Taking w as a test function both in

$$\int_\Omega \frac{\nabla u \cdot \nabla w}{\sqrt{1-|\nabla u|^2}}\,dx = \int_\Omega \mathcal{F}(u) w\,dx$$

and in (3.1), we get

$$\int_{\{u<\alpha\}} \frac{\nabla u \cdot \nabla(u-\alpha)}{\sqrt{1-|\nabla u|^2}}\,dx = -\int_\Omega \frac{\nabla u \cdot \nabla(u-\alpha)^-}{\sqrt{1-|\nabla u|^2}}\,dx = -\int_\Omega \mathcal{F}(u)\,(u-\alpha)^-\,dx = \int_{\{u<\alpha\}} \mathcal{F}(u)\,(u-\alpha)\,dx$$

and

$$-\int_{\{u<\alpha\}} \frac{\nabla \alpha \cdot \nabla(u-\alpha)}{\sqrt{1-|\nabla \alpha|^2}}\,dx = \int_\Omega \frac{\nabla \alpha \cdot \nabla(u-\alpha)^-}{\sqrt{1-|\nabla \alpha|^2}}\,dx \le \int_\Omega f(x,\alpha,\nabla\alpha)\,(u-\alpha)^-\,dx = -\int_{\{u<\alpha\}} f(x,\alpha,\nabla\alpha)\,(u-\alpha)\,dx,$$

respectively. Summing up we obtain

$$\int_{\{u<\alpha\}} \Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} - \frac{\nabla \alpha}{\sqrt{1-|\nabla \alpha|^2}}\Big)\cdot(\nabla u - \nabla\alpha)\,dx \le \int_{\{u<\alpha\}} (\mathcal{F}(u) - f(x,\alpha,\nabla\alpha))\,(u-\alpha)\,dx = 0. \tag{3.5}$$

Define $\psi : B_1(0) \to \mathbb{R}^N$ by

$$\psi(y) = \frac{y}{\sqrt{1-|y|^2}}.$$

As a consequence of the strict monotonicity of ψ, from (3.5) we deduce that

$$\int_{\{u<\alpha\}} \Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}} - \frac{\nabla \alpha}{\sqrt{1-|\nabla \alpha|^2}}\Big)\cdot(\nabla u - \nabla\alpha)\,dx = 0;$$

then either the N-dimensional measure of the set $\{u < \alpha\}$ is equal to 0 or $\nabla(u-\alpha) = 0$ in $\{u < \alpha\}$. In both cases we get $(u-\alpha)^- = 0$ and hence $u \ge \alpha$, in $\overline{\Omega}$. In a completely similar way we prove that $u \le \beta$ in $\overline{\Omega}$.

Step 3. Problem (1.1) *has at least one solution u with $\alpha \le u \le \beta$ in $\overline{\Omega}$.* Observe that the operator $\mathcal{N} = \mathcal{F}$ satisfies (h3) and (h4). By Remark 2.4 there exists a solution u of problem (3.4) which, by the result of Step 2, satisfies $\alpha \le u \le \beta$ in $\overline{\Omega}$ and, in particular, is a solution of (1.1) as well.

Part 2. Existence of extremal solutions. We know that the solutions of (1.1) are precisely the fixed points of the operator $\mathcal{T}$. By the complete continuity of $\mathcal{T}$ proved in Remark 2.5, the closed bounded subset of $C^1_0(\overline{\Omega})$,

$$\mathcal{S} = \{u \in C^1_0(\overline{\Omega}) : u = \mathcal{T}(u) \text{ and } \alpha \le u \le \beta \text{ in } \overline{\Omega}\},$$

is compact. In Part 1 we have seen that $\mathcal{S}$ is not empty.

Step 1. There exists $\min \mathcal{S}$. For each $u \in \mathcal{S}$, define the closed subset of $\mathcal{S}$

$$\mathcal{K}_u = \{z \in \mathcal{S} : z \le u \text{ in } \overline{\Omega}\}.$$

The family $\{\mathcal{K}_u : u \in \mathcal{S}\}$ has the finite intersection property. Indeed, if $u_1, \dots, u_k \in \mathcal{S}$, let $u_0 = \min\{u_1, \dots, u_k\}$: it satisfies $\alpha \le u_0 \le \beta$ in $\overline{\Omega}$. We prove the existence of a solution u of (1.1) with $\alpha \le u \le u_0$ in $\overline{\Omega}$. For all $i = 0, 1, \dots, k$, define the function $\gamma_i : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma_i(x, s) = \begin{cases} \alpha(x) & \text{if } s < \alpha(x), \\ s & \text{if } \alpha(x) \le s < u_i(x), \\ u_i(x) & \text{if } u_i(x) \le s, \end{cases}$$

for all $x \in \overline{\Omega}$, and the operator $\mathcal{F}_i : C^{0,1}(\overline{\Omega}) \to L^\infty(\Omega)$ by

$$\mathcal{F}_i(u) = f(\cdot, \gamma_i(\cdot, u), \nabla(\gamma_i(\cdot, u))).$$

Next, we set $\underline{\mathcal{F}} = \mathcal{F}_0 - \sum_{i=1}^k |\mathcal{F}_0 - \mathcal{F}_i|$ and observe that the operator $\mathcal{N} = \underline{\mathcal{F}}$ satisfies (h3) and (h4). By Remark 2.4 there exists a solution u of the problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1 - |\nabla u|^2}}\Big) = \underline{\mathcal{F}}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

We prove now that any solution z of (3.6) satisfies $\alpha \le z \le u_0$ in $\overline{\Omega}$. We first notice that, for all $i = 0, 1, \dots, k$ and for a.e. $x \in \Omega$, we have

$$\mathcal{F}_i(z)(x) = f(x, \alpha(x), \nabla\alpha(x)) \quad \text{if } z(x) < \alpha(x),$$

and, hence,

$$\underline{\mathcal{F}}(z)(x) = f(x, \alpha(x), \nabla\alpha(x)) \quad \text{if } z(x) < \alpha(x); \tag{3.7}$$

on the other hand, for all $i = 0, 1, \dots, k$, $u_i \ge u_0$ in $\overline{\Omega}$, then we get, for a.e. $x \in \Omega$,

$$\mathcal{F}_i(z)(x) = \mathcal{F}_i(u_i)(x) = f(x, u_i(x), \nabla u_i(x)) \quad \text{if } z(x) > u_i(x). \tag{3.8}$$

Similarly to Step 2 in Part 1, testing now (3.1) and

$$\int_\Omega \frac{\nabla z \cdot \nabla w}{\sqrt{1 - |\nabla z|^2}}\, dx = \int_\Omega \underline{\mathcal{F}}(z) w \, dx \tag{3.9}$$

against $w = (z - \alpha)^- \in W^{1,1}_0(\Omega)$ and taking advantage of (3.7), we get

$$\begin{aligned} 0 &\le \int_{\{z<\alpha\}} \Big(\frac{\nabla z}{\sqrt{1 - |\nabla z|^2}} - \frac{\nabla \alpha}{\sqrt{1 - |\nabla \alpha|^2}}\Big) \cdot (\nabla z - \nabla \alpha)\, dx \\ &\le \int_{\{z<\alpha\}} (\underline{\mathcal{F}}(z) - f(x, \alpha, \nabla\alpha))(z - \alpha)\, dx = 0. \end{aligned}$$

We then deduce that $z \geq \alpha$ in $\overline{\Omega}$. For any given $j = 1, \dots, k$, we will prove that $z \leq u_j$ in $\overline{\Omega}$. Testing (2.1), where u is replaced by u_j, and (3.9) against $w = (z - u_j)^+ \in W_0^{1,1}(\Omega)$, and using (3.8) yield

$$\begin{aligned}
0 &\leq \int_{\{z>u_j\}} \Big(\frac{\nabla z}{\sqrt{1-|\nabla z|^2}} - \frac{\nabla u_j}{\sqrt{1-|\nabla u_j|^2}} \Big) \cdot (\nabla z - \nabla u_j)\, dx \\
&= \int_{\{z>u_j\}} (\underline{\mathcal{F}}(z) - f(x, u_j, \nabla u_j))(z - u_j)\, dx \\
&= \int_{\{z>u_j\}} \Big(\mathcal{F}_0(z) - f(x, u_j, \nabla u_j) - \sum_{i=1}^{k} |\mathcal{F}_0(z) - \mathcal{F}_i(z)| \Big)(z - u_j)\, dx \\
&= \int_{\{z>u_j\}} \Big(\mathcal{F}_0(z) - f(x, u_j, \nabla u_j) - |\mathcal{F}_0(z) - \mathcal{F}_j(z)| - \sum_{\substack{i=1\\ i\neq j}}^{k} |\mathcal{F}_0(z) - \mathcal{F}_i(z)| \Big)(z - u_j)\, dx \\
&\leq \int_{\{z>u_j\}} (\mathcal{F}_0(z) - f(x, u_j, \nabla u_j) - |\mathcal{F}_0(z) - \mathcal{F}_j(z)|)(z - u_j)\, dx \\
&= \int_{\{z>u_j\}} (\mathcal{F}_0(z) - \mathcal{F}_j(z) - |\mathcal{F}_0(z) - \mathcal{F}_j(z)|)(z - u_j)\, dx \\
&\leq 0.
\end{aligned}$$

We then obtain $z \leq u_j$ in $\overline{\Omega}$. Hence we conclude that $z \leq u_0$ in $\overline{\Omega}$.

The estimates above prove that the solution u of (3.6) satisfies $\alpha \leq u \leq u_0 \leq \beta$ in $\overline{\Omega}$, therefore u is also a solution of (1.1). In particular, we have $u \in \bigcap_{i=1}^{k} \mathcal{K}_{u_i}$, which entails the validity of the finite intersection property for the family $\{\mathcal{K}_u : u \in \mathcal{S}\}$. By the compactness of $\mathcal{S}$, there exists $v \in \bigcap_{u \in \mathcal{S}} \mathcal{K}_u$. Clearly, $v = \min \mathcal{S}$, that is v is the minimum solution of (1.1) lying between α and β.

Step 2. There exists $\max \mathcal{S}$. The procedure is similar to the one developed in the previous step.

Part 3. Degree computation. Let $\mathcal{P}$ be the operator defined by (2.12), where $\mathcal{N} = \mathcal{F}$. Let us assume that α and β are, respectively, a strict lower and a strict upper solution of (1.1). Since there exists a solution u of (1.1) with $\alpha \leq u \leq \beta$ in $\overline{\Omega}$, and such a solution satisfies $\alpha \ll u \ll \beta$ in $\overline{\Omega}$, it follows that $\alpha \ll \beta$ in $\overline{\Omega}$. Hence the set $\mathcal{U}$ defined in (3.3) is a non-empty open bounded subset of $C_0^1(\overline{\Omega})$ such that there is no fixed point either of $\mathcal{T}$ or of $\mathcal{P}$ on its boundary $\partial\mathcal{U}$. Moreover, as $\mathcal{T}$ and $\mathcal{P}$ coincide in $\mathcal{U}$, we have

$$\deg(I - \mathcal{T}, \mathcal{U}, 0) = \deg(I - \mathcal{P}, \mathcal{U}, 0).$$

Since $\mathcal{P}$ is fixed point free in $\overline{\mathcal{B}} \setminus \mathcal{U}$, the excision property of the degree and (2.13) imply that

$$\deg(I - \mathcal{P}, \mathcal{U}, 0) = \deg(I - \mathcal{P}, \mathcal{B}, 0) = 1.$$

Thus we conclude that (3.2) holds. □

The counterpart result to Proposition 3.1, in the presence of a couple of non-ordered strict lower and strict upper solutions of (1.1), is formulated below.

Proposition 3.2. *Assume* (h1) *and* (h2). *Suppose that there exist a strict lower solution α and a strict upper solution β of* (1.1) *with $\alpha \not\leq \beta$ in $\overline{\Omega}$. Then problem* (1.1) *has at least three solutions u_1, u_2, u_3 with*

$$u_1 < u_2 < u_3, \quad u_1 \ll \beta, \quad u_2 \not\geq \alpha, \quad u_2 \not\leq \beta, \quad u_3 \gg \alpha \quad \text{in } \overline{\Omega}. \tag{3.10}$$

Proof. The proof is divided into three steps.

Step 1. Construction of a modified problem. Set

$$M = \max\{\|\alpha\|_\infty, \|\beta\|_\infty, \tfrac{1}{2}\operatorname{diam}(\Omega)\}, \tag{3.11}$$

and define $f_M : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ by

$$f_M(x, s, \xi) = \eta(s, \xi) f(x, s, \xi),$$

where $\eta : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a continuous function such that $\operatorname{supp} \eta \subset [-(M+1), M+1] \times B_2(0)$ and $\eta(s, \xi) = 1$ in $[-M, M] \times B_1(0)$. Note that f_M satisfies the L^∞-Carathéodory conditions. We consider the modified problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f_M(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.12}$$

Due to the choice of M, Remark 2.1 implies that any solution of (3.12) is a solution of (1.1), α and β are strict lower and upper solutions of (3.12), and the constants $\bar{\alpha} = -(M+1)$ and $\bar{\beta} = M+1$ are strict lower and upper solutions of (3.12), respectively.

Step 2. Degree computation. Let us define the following open bounded subsets of $C_0^1(\overline{\Omega})$:

$$\mathcal{U}_{\bar{\alpha}}^{\beta} = \{u \in C_0^1(\overline{\Omega}) : \bar{\alpha} \ll u \ll \beta \text{ in } \overline{\Omega} \text{ and } \|\nabla u\|_\infty < 1\},$$
$$\mathcal{U}_{\alpha}^{\bar{\beta}} = \{u \in C_0^1(\overline{\Omega}) : \alpha \ll u \ll \bar{\beta} \text{ in } \overline{\Omega} \text{ and } \|\nabla u\|_\infty < 1\},$$
$$\mathcal{U}_{\bar{\alpha}}^{\bar{\beta}} = \{u \in C_0^1(\overline{\Omega}) : \bar{\alpha} \ll u \ll \bar{\beta} \text{ in } \overline{\Omega} \text{ and } \|\nabla u\|_\infty < 1\}.$$

Notice that $\mathcal{U}_{\bar{\alpha}}^{\beta} \subset \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}$, $\mathcal{U}_{\alpha}^{\bar{\beta}} \subset \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}$, and, as $\alpha \not\leq \beta$ in $\overline{\Omega}$, $\mathcal{U}_{\bar{\alpha}}^{\beta} \cap \mathcal{U}_{\alpha}^{\bar{\beta}} = \emptyset$. Moreover, since both α and $\bar{\alpha}$ are strict lower solutions of (3.12), and β and $\bar{\beta}$ are strict upper solutions of (3.12), we have

$$0 \notin (\mathcal{I} - \mathcal{T}_M)(\partial\mathcal{U}_{\alpha}^{\bar{\beta}} \cup \partial\mathcal{U}_{\bar{\alpha}}^{\beta} \cup \partial\mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}), \tag{3.13}$$

where $\mathcal{T}_M : C^{0,1}(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is the operator which sends any function $v \in C^{0,1}(\overline{\Omega})$ onto the unique solution $u \in C_0^1(\overline{\Omega})$ of

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f_M(x, v, \nabla v) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Define now the following open bounded subset of $C_0^1(\overline{\Omega})$:

$$\mathcal{V} = \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}} \setminus \overline{(\mathcal{U}_{\alpha}^{\bar{\beta}} \cup \mathcal{U}_{\bar{\alpha}}^{\beta})}.$$

By (3.13), using the excision property of the degree, we get

$$\deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}, 0) = \deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}} \setminus (\partial\mathcal{U}_{\alpha}^{\bar{\beta}} \cup \partial\mathcal{U}_{\bar{\alpha}}^{\beta}), 0)$$

and hence the additivity property of the degree implies

$$\deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}, 0) = \deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\bar{\alpha}}^{\beta}, 0) + \deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\alpha}^{\bar{\beta}}, 0) + \deg(\mathcal{I} - \mathcal{T}_M, \mathcal{V}, 0).$$

Since, by Proposition 3.1, we have

$$\deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\bar{\alpha}}^{\bar{\beta}}, 0) = \deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\bar{\alpha}}^{\beta}, 0) = \deg(\mathcal{I} - \mathcal{T}_M, \mathcal{U}_{\alpha}^{\bar{\beta}}, 0) = 1,$$

we finally get

$$\deg(\mathcal{I} - \mathcal{T}_M, \mathcal{V}, 0) = -1.$$

Step 3. Existence of solutions. Since $\mathcal{U}_{\bar{\alpha}}^{\beta}$, $\mathcal{U}_{\alpha}^{\bar{\beta}}$, $\mathcal{V}$ are pairwise disjoint, the previous degree calculations imply that there are three distinct fixed points u_1, u_2, u_3 of the operator $\mathcal{T}_M$ with

$$u_1 \in \mathcal{U}_{\bar{\alpha}}^{\beta}, \quad u_2 \in \mathcal{V}, \quad u_3 \in \mathcal{U}_{\alpha}^{\bar{\beta}}.$$

This means that

$$u_1 \ll \beta, \quad u_2 \not\geq \alpha, \quad u_2 \not\leq \beta, \quad u_3 \gg \alpha \quad \text{in } \overline{\Omega}.$$

Let v and w be, respectively, the minimum and the maximum solution of (3.12) lying between $\bar{\alpha}$ and $\bar{\beta}$. Then, possibly replacing u_1 with v and u_3 with w, we immediately conclude that (3.12) and, hence, (1.1) have three distinct solutions for which (3.10) holds. □

Remark 3.2. By combining Step 1 of Part 2 in the proof of Proposition 3.1 with the truncation argument of the proof of Proposition 3.2, we infer that if $\alpha_1, \dots, \alpha_n$ are lower solutions of (1.1) (respectively, $\beta_1, \dots, \beta_n$ are upper solutions of (1.1)), then there exists a solution u of (1.1) with $u \geq \max\{\alpha_1, \dots, \alpha_n\}$ in $\overline{\Omega}$ (respectively, there exists a solution u of (1.1) with $u \leq \min\{\beta_1, \dots, \beta_n\}$ in $\overline{\Omega}$).

4 Existence, multiplicity and localisation results

In this section we formulate some existence, multiplicity and localisation results for problem (1.1), which are consequence of the conclusions achieved in the previous section.

Theorem 4.1. *Assume* (h1) *and* (h2).

(i) *Suppose that there exists a lower solution α of* (1.1). *Then problem* (1.1) *has at least one solution $u \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, with*
$$u \geq \alpha \quad \text{in } \overline{\Omega}.$$

(ii) *Suppose that there exists an upper solution β of* (1.1). *Then problem* (1.1) *has at least one solution $u \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, with*
$$u \leq \beta \quad \text{in } \overline{\Omega}.$$

(iii) *Suppose that there exist a lower solution α and an upper solution β of* (1.1) *with $\alpha \leq \beta$ in $\overline{\Omega}$. Then problem* (1.1) *has at least one solution $u \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, with*
$$\alpha \leq u \leq \beta \quad \text{in } \overline{\Omega}.$$

(iv) *Suppose that there exist a lower solution α and an upper solution β of* (1.1) *with $\alpha \nleq \beta$ in $\overline{\Omega}$. Then problem* (1.1) *has at least two solutions $u_1, u_2 \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, with*
$$u_1 < u_2, \quad u_1 \leq \beta, \quad u_2 \geq \alpha \quad \text{in } \overline{\Omega}. \tag{4.1}$$

(v) *Suppose that there exist a strict lower solution α and a strict upper solution β of* (1.1) *with $\alpha \nleq \beta$ in $\overline{\Omega}$. Then problem* (1.1) *has at least three solutions $u_1, u_2, u_3 \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, with*
$$u_1 < u_2 < u_3, \quad u_1 \ll \beta, \quad u_2 \ngeq \alpha, \quad u_2 \nleq \beta, \quad u_3 \gg \alpha \quad \text{in } \overline{\Omega}. \tag{4.2}$$

(vi) *Suppose that there exist lower solutions $\alpha, \bar{\alpha}$ and upper solutions $\beta, \bar{\beta}$ of* (1.1) *such that α and β are strict, $\bar{\alpha} \leq \min\{\alpha, \beta\} \leq \max\{\alpha, \beta\} \leq \bar{\beta}$ and $\alpha \nleq \beta$ in $\overline{\Omega}$. Then problem* (1.1) *has at least three solutions $u_1, u_2, u_3 \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, with*
$$\bar{\alpha} \leq u_1 < u_2 < u_3 \leq \bar{\beta}, \quad u_1 \ll \beta, \quad u_2 \ngeq \alpha, \quad u_2 \nleq \beta, \quad u_3 \gg \alpha \quad \text{in } \overline{\Omega}. \tag{4.3}$$

Proof. In order to prove (i), we consider the modified problem (3.12) constructed in Step 1 of the proof of Proposition 3.2, with the choice
$$M_1 = \max\{\|\alpha\|_\infty, \tfrac{1}{2}\operatorname{diam}(\Omega)\}.$$
By Remark 2.1, we see that any solution of the modified problem (3.12) with $M = M_1$ is a solution of the original one (1.1). Let us set $\bar{\beta} = M_1 + 1$. We have that α is a lower solution and $\bar{\beta}$ is an upper solution of (3.12) with $\alpha \leq \bar{\beta}$ in $\overline{\Omega}$. By Proposition 3.1 there exists at least one solution u of (3.12) with $\alpha \leq u \leq \bar{\beta}$ in $\overline{\Omega}$, and hence of (1.1).

A similar argument implies the validity of (ii).

The statement in (iii) follows from Proposition 3.1.

Let us prove (iv). Let α be a lower solution and β an upper solution of (1.1) with $\alpha \nleq \beta$ in $\overline{\Omega}$. Let M be the positive constant defined in (3.11) and set $\bar{\alpha} = -(M + 1)$ and $\bar{\beta} = M + 1$. Consider the modified problem (3.12). Observe that $\alpha, \bar{\alpha}$ are lower solutions and $\beta, \bar{\beta}$ are upper solutions of (3.12), which satisfy $\bar{\alpha} \leq \beta$ and $\alpha \leq \bar{\beta}$ in $\overline{\Omega}$. According to (iii) and to Remark 2.1 applied to the modified problem (3.12), there exist two solutions u_1, u_2 of (3.12) which satisfy
$$\bar{\alpha} \leq u_1 \leq \beta, \quad \alpha \leq u_2 \leq \bar{\beta} \quad \text{in } \overline{\Omega}$$

and $\|u_i\|_\infty < M$. Therefore u_1 and u_2 are solutions of (1.1). Proposition 3.1 provides a minimum solution v and a maximum solution w of (1.1) lying between $\bar{\alpha}$ and $\bar{\beta}$. Possibly replacing u_1 with v and u_2 with w, from the assumption $\alpha \nleq \beta$ in $\overline{\Omega}$, we have $u_1 < u_2$ in $\overline{\Omega}$, thus (4.1) holds.

The statement in (v) is precisely the one of Proposition 3.2.

We finally prove (vi). We define the function $\gamma : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma(x,s) = \begin{cases} \bar{\alpha}(x) & \text{if } s < \bar{\alpha}(x), \\ s & \text{if } \bar{\alpha}(x) \le s < \bar{\beta}(x), \\ \bar{\beta}(x) & \text{if } \bar{\beta}(x) \le s, \end{cases}$$

for all $x \in \overline{\Omega}$ and the operator $\mathcal{F} : C^{0,1}(\overline{\Omega}) \to L^\infty(\Omega)$ by

$$\mathcal{F}(u) = f(\,\cdot\,, \gamma(\,\cdot\,, u), \nabla(\gamma(\,\cdot\,, u))). \tag{4.4}$$

We consider problem (3.4), where the operator on the right-hand side of the equation is given by (4.4). From the proof of Proposition 3.1 we infer that any solution u of (3.4) satisfies $\bar{\alpha} \le u \le \bar{\beta}$ in $\overline{\Omega}$. We notice that α and β are still a strict lower solution and a strict upper solution of (3.4), respectively. Then, applying statement (v) to problem (3.4), we deduce the existence of three solutions u_1, u_2, u_3 of (3.4) which satisfy (4.2). As $\bar{\alpha} \le u_1 < u_2 < u_3 \le \bar{\beta}$ in $\overline{\Omega}$, we conclude that u_1, u_2, u_3 are solutions of (1.1), satisfying (4.3). □

We conclude with a kind of "universal" existence result. We notice that the solvability of (1.1), where the right-hand side explicitly depends on the gradient, has been raised in [28] as an open question.

Theorem 4.2. *Assume* (h1) *and* (h2). *Then problem* (1.1) *has at least one solution* $u \in W^{2,r}(\Omega)$, *for all finite* $r \ge 1$.

Proof. Set $M = \frac{1}{2}\operatorname{diam}(\Omega)$ and consider the modified problem (3.12). Take the constant functions in Ω given by $\bar{\alpha} = -(M+1)$ and $\bar{\beta} = M+1$. Then $\bar{\alpha}$ is a lower solution and $\bar{\beta}$ is an upper solution of (3.12), which satisfies $\bar{\alpha} < \bar{\beta}$ in $\overline{\Omega}$. According to Proposition 3.1 and to Remark 2.1 applied to the modified problem (3.12), there exists a solution u of (3.12) which satisfies $\bar{\alpha} \le u \le \bar{\beta}$ in $\overline{\Omega}$, and $\|u\|_\infty < M$. Therefore u is a solution of problem (1.1). □

5 Stability analysis

In this section we show how certain stability properties of the solutions of problem (1.1) can be detected by the use of lower and upper solutions. We introduce a concept of order stability and order instability, adapted to the present setting from [24, Chapter I]. Our analysis follows patterns developed in [16–18].

Definition 5.1. We say that a solution u of problem (1.1) is order stable (respectively, properly order stable) from below if there exists a sequence $(\alpha_n)_n$ of lower solutions (respectively, proper lower solutions) such that, for each n, $\alpha_n < \alpha_{n+1}$ in $\overline{\Omega}$ and $\lim_{n\to+\infty} \alpha_n = u$ in $C^{0,1}(\overline{\Omega})$.

We say that a solution u of problem (1.1) is order stable (respectively, properly order stable) from above if there exists a sequence $(\beta_n)_n$ of upper solutions (respectively, proper upper solutions) such that, for each n, $\beta_n > \beta_{n+1}$ in $\overline{\Omega}$ and $\lim_{n\to+\infty} \beta_n = u$ in $C^{0,1}(\overline{\Omega})$.

We say that a solution u of problem (1.1) is order stable (respectively, properly order stable) if u is order stable (respectively, properly order stable) both from below and from above.

Definition 5.2. We say that a solution u of problem (1.1) is order unstable (respectively, properly order unstable) from below if there exists a sequence $(\beta_n)_n$ of upper solutions (respectively, proper upper solutions) such that, for each n, $\beta_n < \beta_{n+1}$ in $\overline{\Omega}$ and $\lim_{n\to+\infty} \beta_n = u$ in $C^{0,1}(\overline{\Omega})$.

We say that a solution u of problem (1.1) is order unstable (respectively, properly order unstable) from above if there exists a sequence $(\alpha_n)_n$ of lower solutions (respectively, proper lower solutions) such that, for each n, $\alpha_n > \alpha_{n+1}$ in $\overline{\Omega}$ and $\lim_{n\to+\infty} \alpha_n = u$ in $C^{0,1}(\overline{\Omega})$.

We begin by stating some preliminary results.

Lemma 5.1. *Assume* (h1) *and* (h2). *Let* $\mathcal{S}$ *be a non-empty set of solutions of* (1.1). *Then there exist a minimal solution* v *of* (1.1) *and a maximal solution* w *of* (1.1) *in* $\overline{\mathcal{S}}$, *where* $\overline{\mathcal{S}}$ *is the closure in* $C^1(\overline{\Omega})$ *of* $\mathcal{S}$.

Proof. We only prove the existence of a maximal solution w; the proof of the existence of a minimal solution v being similar. Let us fix $r \in\,]N, +\infty[$. We first notice that, as any solution $u \in \overline{\mathcal{S}}$ satisfies $\|u\|_\infty < \frac{1}{2}\operatorname{diam}(\Omega)$ and $\|\nabla u\|_\infty < 1$, assumption (h2) and Lemma 2.2 imply that

$$\sup_{u\in\overline{\mathcal{S}}} \|u\|_{C^1} < +\infty \tag{5.1}$$

and

$$\sup_{u\in\overline{\mathcal{S}}} \|u\|_{W^{2,r}} < +\infty. \tag{5.2}$$

Next we show that $(\overline{\mathcal{S}}, \leq)$ is inductively ordered. Let $\mathcal{C} = \{u_i : i \in I\}$ be a totally ordered subset of $\overline{\mathcal{S}}$ and let us prove that $\mathcal{C}$ has an upper bound in $\overline{\mathcal{S}}$. Set, for each $x \in \overline{\Omega}$,

$$u(x) = \sup_{i\in I} u_i(x).$$

Let $\mathcal{D} = \{x_m : m \in \mathbb{N}\}$ be a countable dense subset of Ω and define a sequence in $\mathcal{C}$ as follows: for $n = 1$, take $u_1 \in \mathcal{C}$ such that $u_1(x_1) \geq u(x_1) - 1$, for $n = 2$, take $u_2 \in \mathcal{C}$ with $u_2 \geq u_1$ in $\overline{\Omega}$ such that $u_2(x_2) \geq u(x_2) - \frac{1}{2}$, $u_2(x_1) \geq u(x_1) - \frac{1}{2}$, and so on. In this way, we construct a sequence $(u_n)_n$ in $\mathcal{C}$ with

$$u_1 \leq u_2 \leq \cdots \leq u_n \leq u_{n+1} \leq \cdots \quad \text{in } \overline{\Omega},$$

such that $u_n(x_k) \geq u(x_k) - \frac{1}{n}$ for $k = 1, \ldots, n$. It is clear that $(u_n)_n$ converges to u pointwise on $\mathcal{D}$. On the other hand, as $(u_n)_n$ satisfies (5.1) and (5.2), we conclude that any subsequence of $(u_n)_n$ has a further subsequence which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to some function $\hat{u} \in W^{2,r}(\Omega)$. Actually, by monotonicity, the whole sequence $(u_n)_n$ converges pointwise in $\overline{\Omega}$ to $\hat{u}$, which is therefore independent of the chosen subsequence. Hence we infer that $(u_n)_n$ converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to $\hat{u}$, which is a solution of (1.1). Moreover, we have $\hat{u} = u$ on $\mathcal{D}$ and $\hat{u} \leq u$ in $\overline{\Omega}$. Let us show that $\hat{u} = u$ in $\overline{\Omega}$. Indeed, otherwise, one can find a point $x_0 \in \Omega$ and a function $u_0 \in \mathcal{C}$ such that $\hat{u}(x_0) < u_0(x_0) \leq u(x_0)$. The continuity of both $\hat{u}$ and u_0 and the density of $\mathcal{D}$ in Ω yield a contradiction. This proves that $u \in \overline{\mathcal{S}}$ is an upper bound of $\mathcal{C}$.

Finally, since $(\overline{\mathcal{S}}, \leq)$ is inductively ordered, Zorn lemma guarantees the existence of a maximal element $w \in \overline{\mathcal{S}}$. □

The following elementary result is immediately deduced from [17, Lemma 2.1] and [18, Proposition 1.10]: it will be crucial in the sequel in order to supply some monotonicity to problem (1.1) or to variations thereof.

Lemma 5.2. *Assume that* $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *satisfies the* L^∞*-Carathéodory conditions. Then, for each* $\rho > 0$, *there exists a* L^∞*-Carathéodory function* $h : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *such that*
(i) *for a.e.* $x \in \Omega$ *and every* $(r, \xi) \in [-\rho, \rho] \times \mathbb{R}^N$, $h(x, \cdot, r, \xi)|_{[-\rho,\rho]}$ *is strictly increasing,*
(ii) *for a.e.* $x \in \Omega$ *and every* $(s, \xi) \in [-\rho, \rho] \times \mathbb{R}^N$, $h(x, s, \cdot, \xi)|_{[-\rho,\rho]}$ *is strictly decreasing,*
(iii) *for a.e.* $x \in \Omega$ *and every* $(r, s, \xi) \in [-\rho, \rho] \times [-\rho, \rho] \times \mathbb{R}^N$, $h(x, s, r, \xi) = -h(x, r, s, \xi)$,
(iv) *for a.e.* $x \in \Omega$ *and every* $(r, s, \xi) \in [-\rho, \rho] \times [-\rho, \rho] \times \mathbb{R}^N$ *with* $r < s$,

$$|f(x, s, \xi) - f(x, r, \xi)| < h(x, s, r, \xi).$$

We first prove the following technical conclusion.

Lemma 5.3. *Assume* (h1) *and* (h2). *Let* z *be a solution of* (1.1).
(i) *If* α *is a proper lower solution of* (1.1) *such that* $\alpha < z$ *in* $\overline{\Omega}$, *then there exists a proper lower solution* $\bar{\alpha}$ *of* (1.1), *satisfying* $\bar{\alpha} \in W^{2,r}(\Omega)$, *for all finite* $r \geq 1$, $\bar{\alpha} = 0$ *on* $\partial\Omega$, *and* $\alpha < \bar{\alpha} < z$ *in* $\overline{\Omega}$.
(ii) *If* β *is a proper upper solution of* (1.1) *such that* $\beta > z$ *in* $\overline{\Omega}$, *then there exists a proper upper solution* $\bar{\beta}$ *of* (1.1), *satisfying* $\bar{\beta} \in W^{2,r}(\Omega)$, *for all finite* $r \geq 1$, $\bar{\beta} = 0$ *on* $\partial\Omega$, *and* $z < \bar{\beta} < \beta$ *in* $\overline{\Omega}$.

Proof. We only prove the former statement; the proof of the latter being similar. Let h be the function associated with f by Lemma 5.2 and corresponding to $\rho = \max\{\|\alpha\|_\infty, \|z\|_\infty\}$. Consider the problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f(x,\alpha,\nabla u) - h(x,u,\alpha,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

The right-hand side of the equation satisfies the L^∞-Carathéodory conditions. Moreover, as α is a proper lower solution and z is a proper upper solution of (5.3) with $\alpha < z$ in $\overline{\Omega}$, Proposition 3.1 implies that (5.3) has a solution $\bar{\alpha}$, satisfying $\alpha < \bar{\alpha} < z$ in $\overline{\Omega}$. The properties of h imply that $\bar{\alpha}$ is a proper lower solution of problem (1.1). □

Now we state an order stability result. We point out that our conclusions are obtained without assuming any additional regularity condition on f, like, e.g., Lipschitz continuity, as it is generally required to associate an order preserving operator with the considered problem (see, e.g., [1, 24]).

Proposition 5.4. *Assume* (h1) *and* (h2). *Let z be a solution of* (1.1).
(i) *Suppose that there exists a proper lower solution α of* (1.1) *such that $z > \alpha$ in $\overline{\Omega}$ and there is no solution u of* (1.1) *satisfying $\alpha < u < z$ in $\overline{\Omega}$. Then z is properly order stable from below.*
(ii) *Suppose that there exists a proper upper solution β of* (1.1) *such that $z < \beta$ in $\overline{\Omega}$ and there is no solution u of* (1.1) *satisfying $z < u < \beta$ in $\overline{\Omega}$. Then z is properly order stable from above.*

Proof. We only prove the former statement; the proof of the latter being similar. Repeating recursively the argument in the proof of Lemma 5.3, we get a sequence of proper lower solutions $(\alpha_n)_n$ such that $\alpha_0 = \alpha$ and, for each $n \geq 1$, $\alpha_n \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, $\alpha < \alpha_{n-1} < \alpha_n < z$ in $\overline{\Omega}$, and α_n is a solution of

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f(x,\alpha_{n-1},\nabla u) - h(x,u,\alpha_{n-1},\nabla u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where h is defined as in Lemma 5.3. Arguing as in the proof of Lemma 5.1, we see that the sequence $(\alpha_n)_n$ converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to a solution u of (1.1), which satisfies $\alpha < u \leq z$ in $\overline{\Omega}$ and therefore must be z. □

Proposition 5.4 immediately yields the proper order stability from below of the minimum solution and the proper order stability from above of the maximum solution of (1.1), lying between a couple of proper lower and upper solutions α and β with $\alpha \leq \beta$.

Theorem 5.5. *Assume* (h1) *and* (h2). *Suppose that α and β are, respectively, a proper lower solution and a proper upper solution of* (1.1) *with $\alpha < \beta$ in $\overline{\Omega}$. Then the minimum solution v and the maximum solution w of* (1.1), *lying between α and β, are, respectively, properly order stable from below and properly order stable from above.*

We now provide the basic tool for carrying out our analysis further.

Lemma 5.6. *Assume* (h1) *and* (h2). *Suppose that u_1, u_2 are solutions of* (1.1) *such that $u_1 < u_2$ in $\overline{\Omega}$ and there is no solution u of* (1.1) *with $u_1 < u < u_2$ in $\overline{\Omega}$. Then one of the following statements holds.*
- *There exists a sequence $(\alpha_n)_n$ of proper lower solutions of* (1.1) *such that, for each n, $\alpha_n \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, $\alpha_n = 0$ on $\partial\Omega$, and $u_1 < \alpha_n < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_1.*
- *There exists a sequence $(\beta_n)_n$ of proper upper solutions of* (1.1) *such that, for each n, $\beta_n \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, $\beta_n = 0$ on $\partial\Omega$, and $u_1 < \beta_n < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_2.*

Proof. The proof is inspired by [17, Lemma 2.8] (see also [18, Lemma III-3.1], [16, Proposition 2.18]). As in Proposition 3.1 we define a function

$$\gamma : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$$

by setting, for all $x \in \overline{\Omega}$,

$$\gamma(x,s) = \begin{cases} u_1(x) & \text{if } s < u_1(x), \\ s & \text{if } u_1(x) \le s < u_2(x), \\ u_2(x) & \text{if } u_2(x) \le s. \end{cases}$$

Clearly, γ is continuous and, for each $x \in \overline{\Omega}$, $\gamma(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is increasing. For $i = 1, 2$, let us set, for a.e. $x \in \Omega$ and every $\varepsilon > 0$,

$$\omega_i(x,\varepsilon) = \max_{|\xi|\le\varepsilon} |f(x, u_i(x), \nabla u_i(x) + \xi) - f(x, u_i(x), \nabla u_i(x))|,$$

and, for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$,

$$\omega(x,s) = \begin{cases} \omega_1(x, u_1(x) - s) & \text{if } s < u_1(x), \\ 0 & \text{if } u_1(x) \le s < u_2(x), \\ -\omega_2(x, s - u_2(x)) & \text{if } u_2(x) \le s. \end{cases}$$

Let h be the function introduced in Lemma 5.2, associated with f and $\rho = \max\{\|u_1\|_\infty, \|u_2\|_\infty\}$, and consider the following problems:

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = f(x,\gamma(x,u),\nabla u) + \omega(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.4}$$

and, for $\mu \in [0,1]$,

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \mu(f(x,\gamma(x,u),\nabla u) + \omega(x,u)) & \\ \qquad + (1-\mu)(f(x,u_1,\nabla u) + h(x,u_1,\gamma(x,u),\nabla u) + \omega(x,u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

and

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) = \mu(f(x,\gamma(x,u),\nabla u) + \omega(x,u)) & \\ \qquad + (1-\mu)(f(x,u_2,\nabla u) + h(x,u_2,\gamma(x,u),\nabla u) + \omega(x,u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

Clearly, the right-hand sides of the equations in (5.5) and (5.6) satisfy the L^∞-Carathéodory conditions. Notice that, if u is a solution of (5.4) satisfying $u_1 \le u \le u_2$ in $\overline{\Omega}$, then u is a solution of (1.1). Moreover, the choice $\mu = 1$ reduces both problems (5.5) and (5.6) to (5.4).

Claim 1. *For any $\mu \in [0,1]$, every solution u of (5.5), or (5.6), satisfies $u_1 \le u \le u_2$ in $\overline{\Omega}$. In particular, u_1 and u_2 are the only solutions of problem (5.4).*

Let u be a solution of (5.5). We prove that $u \ge u_1$ in $\overline{\Omega}$. Set $v = u - u_1$ and assume that $\min_{\overline{\Omega}} v < 0$. Let x_0 be such that $v(x_0) = \min_{\overline{\Omega}} v < 0$ and let Ω_0 be the maximum open connected subset of Ω such that $x_0 \in \Omega_0$ and $v(x) < 0$ for all $x \in \Omega_0$. Define $K = \{y \in \Omega_0 : v(y) = \min_{\overline{\Omega}} v\}$. For each $y \in K$ pick an open ball $B(y)$ centered at y with $\overline{B(y)} \subset \Omega_0$ and such that $|\nabla v| \le |v|$ in $B(y)$. As K is compact, there exists a finite open covering $\mathcal{O} = \bigcup_{i=1}^n B(y_i)$ of K. Let $\mathcal{O}_0$ be a connected component of $\mathcal{O}$. Clearly, $\mathcal{O}_0$ is a bounded domain with Lipschitz boundary $\partial\mathcal{O}_0$. Then we have

$$\begin{aligned} -\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) &= \mu(f(x,\gamma(x,u),\nabla u) + \omega(x,u)) + (1-\mu)(f(x,u_1,\nabla u) + h(x,u_1,\gamma(x,u),\nabla u) + \omega(x,u)) \\ &= f(x,u_1,\nabla u) + \omega_1(x,|v|) \\ &\ge f(x,u_1,\nabla u) - f(x,u_1,\nabla u_1 + \nabla v) + f(x,u_1,\nabla u_1) \\ &= -\operatorname{div}\Big(\frac{\nabla u_1}{\sqrt{1-|\nabla u_1|^2}}\Big) \end{aligned}$$

in $\mathcal{O}_0$. Lemma 2.1 applies and yields

$$\min_{\partial\mathcal{O}_0} v \le \min_{\overline{\mathcal{O}_0}} v,$$

which is a contradiction, as $\partial\mathcal{O}_0$ does not contain any minimum point of v in Ω_0. To prove that $u \le u_2$ in $\overline{\Omega}$ we argue similarly: set $v = u_2 - u$, define K and $\mathcal{O}_0$ as above and observe that, by the properties of h, we have

$$\begin{aligned} -\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) &= \mu f(x,u_2,\nabla u) + (1-\mu)(f(x,u_1,\nabla u) + h(x,u_1,u_2,\nabla u)) + \omega(x,u) \\ &\le \mu f(x,u_2,\nabla u) + (1-\mu) f(x,u_2,\nabla u) + \omega(x,u) \\ &\le f(x,u_2,\nabla u) - \omega_2(x,|v|) \\ &\le -\operatorname{div}\Big(\frac{\nabla u_2}{\sqrt{1-|\nabla u_2|^2}}\Big) \end{aligned}$$

in $\mathcal{O}_0$. The conclusions for (5.6) follow in a symmetric way.

Claim 2. *For every $\mu \in [0,1]$, any solution of (5.5) is a lower solution of (1.1) and any solution of (5.6) is an upper solution of (1.1).*

Fix $\mu \in [0,1]$ and let u be a solution of (5.5). By Claim 1, we have $u_1 \le u \le u_2$ in $\overline{\Omega}$ and hence, in particular,

$$|f(x,u_1,\nabla u) - f(x,u,\nabla u)| \le -h(x,u_1,u,\nabla u)$$

in Ω. Therefore we obtain

$$\begin{aligned} -\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\Big) &= \mu f(x,u,\nabla u) + (1-\mu)(f(x,u_1,\nabla u) + h(x,u_1,u,\nabla u)) \\ &\le \mu f(x,u,\nabla u) + (1-\mu) f(x,u,\nabla u) = f(x,u,\nabla u) \end{aligned}$$

in Ω. Similarly we prove the result for a solution u of (5.6).

Claim 3. *For every $\delta > 0$, $u_1 - \delta$ is a strict lower solution of (5.5) with $\mu = 0$, and $u_1 + \delta$ is an upper solution of (5.5) with $\mu = 0$. For every $\delta > 0$, $u_2 - \delta$ is a lower solution of (5.6) with $\mu = 0$, and $u_2 + \delta$ is a strict upper solution of (5.6) with $\mu = 0$.*

Observe that $\omega(\cdot, u_1 - \delta) \ge 0$ in $\overline{\Omega}$. Hence we compute

$$\begin{aligned} -\operatorname{div}\Big(\frac{\nabla(u_1-\delta)}{\sqrt{1-|\nabla(u_1-\delta)|^2}}\Big) &= f(x,u_1,\nabla u_1) \\ &\le f(x,u_1,\nabla(u_1-\delta)) + h(x,u_1,\gamma(x,u_1-\delta),\nabla(u_1-\delta)) + \omega(x,u_1-\delta) \end{aligned}$$

in Ω. This means that $u_1 - \delta$ is a lower solution of (5.5) with $\mu = 0$. Note that $u_1 - \delta$ is strict; indeed, if u is a solution of (5.5) satisfying $u \ge u_1 - \delta$ in $\overline{\Omega}$, then $u \ge u_1 \gg u_1 - \delta$ in $\overline{\Omega}$.

Consider now $u_1 + \delta$. Observe that $\omega(\cdot, u_1 + \delta) \le 0$ and $h(\cdot, u_1, \gamma(\cdot, u_1+\delta), \nabla(u_1+\delta)) \le 0$ in $\overline{\Omega}$. Hence we compute

$$\begin{aligned} -\operatorname{div}\Big(\frac{\nabla(u_1+\delta)}{\sqrt{1-|\nabla(u_1+\delta)|^2}}\Big) &= f(x,u_1,\nabla u_1) \\ &\ge f(x,u_1,\nabla(u_1+\delta)) + h(x,u_1,\gamma(x,u_1+\delta),\nabla(u_1+\delta)) + \omega(x,u_1+\delta) \end{aligned}$$

in Ω. This means that $u_1 + \delta$ is an upper solution of (5.5) with $\mu = 0$.

The proof for $u_2 - \delta$ and $u_2 + \delta$ is symmetric.

Claim 4. *Suppose that, for all $\delta_1 > 0$, there exists $\delta \in]0, \delta_1[$ such that $u_1 + \delta$ is an upper solution of (5.5) with $\mu = 0$ which is not strict. Then there is a sequence $(\alpha_n)_n$ of proper lower solutions of (1.1) such that, for each n, $\alpha_n \in W^{2,r}(\Omega)$, for all finite $r \ge 1$, $\alpha_n = 0$ on $\partial\Omega$, and $u_1 < \alpha_n < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_1.*

By assumption we can find a decreasing sequence of numbers $(\delta_n)_n$, satisfying $\lim_{n\to+\infty}\delta_n = 0$, and, for each n, a solution u_{δ_n} of (5.5) with $\mu = 0$ satisfying $u_{\delta_n} \le \min\{u_1 + \delta_n, u_2\}$ in $\overline{\Omega}$, and some $x_{\delta_n} \in \Omega$ with $u_{\delta_n}(x_{\delta_n}) = u_1(x_{\delta_n}) + \delta_n$; in particular, $\|u_1 - u_{\delta_n}\|_\infty = \delta_n$. Observe that u_{δ_n} is a proper lower solution of (1.1).

Moreover, by Lemma 2.2, there is a constant $c > 0$ such that $\|u_{\delta_n}\|_{W^{2,r}} \le c$ for all δ_n. Therefore we can easily construct a sequence $(\alpha_n)_n$ of proper lower solutions of (1.1) such that, for each n, $\alpha_n \in W^{2,r}(\Omega)$, for all finite $r \ge 1$, $\alpha_n = 0$ on $\partial\Omega$, and $u_1 < \alpha_n < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_1.

Claim 5. *Suppose that, for all $\delta_1 > 0$, there exists $\delta \in \,]0, \delta_1[$ such that $u_2 - \delta$ is a lower solution of* (5.6) *with $\mu = 0$ which is not strict. Then, there is a sequence $(\beta_n)_n$ of proper upper solutions of* (1.1) *such that, for each n, $\beta_n \in W^{2,r}(\Omega)$, for all finite $r \ge 1$, $\beta_n = 0$ on $\partial\Omega$, and $u_2 > \beta_n > u_1$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_2.*

The proof is similar to that one of Claim 4.

Conclusion of the proof. By Claim 4 and Claim 5 we may suppose that there exists $\delta_1 > 0$ such that, for all $\delta \in \,]0, \delta_1[$, $u_1 + \delta$ is a strict upper solution of (5.5) with $\mu = 0$, and $u_2 - \delta$ is a strict lower solution of (5.6) with $\mu = 0$. Assume, for convenience, that $\delta_1 < \frac{1}{2}\|u_1 - u_2\|_\infty$. For all $\delta \in \,]0, \delta_1[$ we set

$$\mathcal{U}_1^\delta = \{u \in C_0^1(\overline{\Omega}) : u_1 - \delta \ll u \ll u_1 + \delta \text{ in } \overline{\Omega} \text{ and } \|\nabla u\|_\infty < 1\}$$

and

$$\mathcal{U}_2^\delta = \{u \in C_0^1(\overline{\Omega}) : u_2 - \delta \ll u \ll u_2 + \delta \text{ in } \overline{\Omega} \text{ and } \|\nabla u\|_\infty < 1\}.$$

Moreover, for all $\mu \in [0, 1]$, we consider the solution operators $\mathcal{T}_{1,\mu}, \mathcal{T}_{2,\mu} : C^{0,1}(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ associated with problems (5.5) and (5.6), respectively. Since $u_1 - \delta$ and $u_1 + \delta$ are strict, Proposition 3.1 yields

$$\deg(\mathcal{I} - \mathcal{T}_{1,0}, \mathcal{U}_1^\delta, 0) = 1. \tag{5.7}$$

Similarly we have

$$\deg(\mathcal{I} - \mathcal{T}_{2,0}, \mathcal{U}_2^\delta, 0) = 1. \tag{5.8}$$

We also set

$$\mathcal{U} = \{u \in C_0^1(\overline{\Omega}) : u_1 - 1 \ll u \ll u_2 + 1 \text{ in } \overline{\Omega} \text{ and } \|\nabla u\|_\infty < 1\},$$

and we consider the solution operator $\mathcal{T} : C^{0,1}(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ associated with problem (5.4). Note that

$$\mathcal{T}_{1,1} = \mathcal{T}_{2,1} = \mathcal{T}.$$

Observe that $u_1 - 1$ and $u_2 + 1$ are, respectively, a strict lower solution and a strict upper solution of (5.4). Therefore Proposition 3.1 yields

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}, 0) = 1.$$

Using the fact that u_1 and u_2 are the only fixed points of $\mathcal{T}$, we conclude, by the additivity and the excision properties of the degree, that

$$1 = \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}, 0) = \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_1^\delta \cup \mathcal{U}_2^\delta, 0) = \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_1^\delta, 0) + \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_2^\delta, 0). \tag{5.9}$$

Now, let us assume that, for every $\delta_0 > 0$, there exists $\delta \in \,]0, \delta_0[$ such that, for every $\mu \in [0, 1]$, problem (5.5) has no solution on $\partial\mathcal{U}_1^\delta$ and problem (5.6) has no solution on $\partial\mathcal{U}_2^\delta$. The homotopy property of the degree then implies, by (5.7) and (5.8),

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_1^\delta, 0) = \deg(\mathcal{I} - \mathcal{T}_{1,1}, \mathcal{U}_1^\delta, 0) = \deg(\mathcal{I} - \mathcal{T}_{1,0}, \mathcal{U}_1^\delta, 0) = 1$$

and

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_2^\delta, 0) = \deg(\mathcal{I} - \mathcal{T}_{2,1}, \mathcal{U}_2^\delta, 0) = \deg(\mathcal{I} - \mathcal{T}_{2,0}, \mathcal{U}_2^\delta, 0) = 1,$$

thus contradicting (5.9).

Therefore, we conclude that there is $\delta_0 > 0$ such that, for all $\delta \in \,]0, \delta_0[$, either there is a solution α_δ of problem (5.5), for some $\mu \in [0, 1]$, such that $\alpha_\delta \in \partial\mathcal{U}_1^\delta$, or there is a solution β_δ of problem (5.6), for some $\mu \in [0, 1]$, such that $\beta_\delta \in \partial\mathcal{U}_2^\delta$. In the former case, the condition $\alpha_\delta \in \partial\mathcal{U}_1^\delta$, together with Claim 1, implies that $u_1 \le \alpha_\delta \le \min\{u_2, u_1 + \delta\}$ in $\overline{\Omega}$ and $\|u_1 - \alpha_\delta\|_\infty = \delta$. By Claim 2, α_δ is a lower solution of (1.1). Moreover, by

Lemma 2.2, there is a constant C such that $\|\alpha_\delta\|_{W^{2,r}} \le C$ for all δ. Therefore we can easily construct a sequence $(\alpha_n)_n$ of proper lower solutions of (1.1) such that, for each n, $\alpha_n \in W^{2,r}(\Omega)$, for all finite $r \ge 1$, $\alpha_n = 0$ on $\partial\Omega$, and $u_1 < \alpha_n < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_1. In the latter case, arguing in the same way, we can construct a sequence $(\beta_n)_n$ of proper upper solutions of (1.1) such that, for each n, $\beta_n \in W^{2,r}(\Omega)$, for all finite $r \ge 1$, $\beta_n = 0$ on $\partial\Omega$, and $u_1 < \beta_n < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_2. □

Lemma 5.6 yields in particular the existence of sequences of lower or upper solutions connecting a couple of consecutive solutions of (1.1).

Corollary 5.7. *Assume* (h1) *and* (h2). *Suppose that u_1, u_2 are solutions of* (1.1) *such that $u_1 < u_2$ in $\overline{\Omega}$ and there is no solution u of* (1.1) *with $u_1 < u < u_2$ in $\overline{\Omega}$. Then one of the following statements holds.*
- *There exists a double sequence $(\alpha_m)_{m\in\mathbb{Z}}$ of proper lower solutions of problem* (1.1) *such that, for each m, $\alpha_m \in W^{2,r}(\Omega)$, for all finite $r \ge 1$, $\alpha_m = 0$ on $\partial\Omega$, and $u_1 < \alpha_m < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_1 as $m \to -\infty$ and to u_2 as $m \to +\infty$.*
- *There exists a double sequence $(\beta_m)_{m\in\mathbb{Z}}$ of proper upper solutions of problem* (1.1) *such that, for each m, $\beta_m \in W^{2,r}(\Omega)$, for all finite $r \ge 1$, $\beta_m = 0$ on $\partial\Omega$, and $u_1 < \beta_m < u_2$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to u_2 as $m \to -\infty$ and to u_1 as $m \to +\infty$.*

Proof. The conclusion follows just combining Lemma 5.6 with Proposition 5.4. □

We now prove a result which provides the existence of order stable solutions of (1.1) in the presence of lower and upper solutions α, β with $\alpha \le \beta$ in $\overline{\Omega}$. It also yields information about the topological structure of the set of the order stable solutions lying between α, β.

Theorem 5.8. *Assume* (h1) *and* (h2). *Suppose that α is a proper lower solution and β is a proper upper solution of* (1.1) *satisfying*

$$\alpha \le \beta \quad \text{in } \overline{\Omega}.$$

Let v and w be, respectively, the minimum solution and the maximum solution of (1.1), *lying between α and β. Then there exists a non-empty totally ordered compact and connected set $\mathcal{K}$ in $C^1(\overline{\Omega})$ such that every $u \in \mathcal{K}$ is an order stable solution of* (1.1) *satisfying $v \le u \le w$ in $\overline{\Omega}$; moreover, $u_1 = \min \mathcal{K}$ is properly order stable from below and $u_2 = \max \mathcal{K}$ is properly order stable from above.*

Proof. Let us denote by $\mathcal{S}_1$ the set of all solutions u of (1.1) with $\alpha \le u \le \beta$ in $\overline{\Omega}$ which are properly order stable from below. Since the minimum solution v is properly order stable from below, $\mathcal{S}_1$ is not empty. By Lemma 5.1 there exists a maximal solution $u_1 \in \overline{\mathcal{S}_1}$, which, by a diagonal argument, is easily proved to be properly order stable from below and, hence, $u_1 \in \mathcal{S}_1$.

Let us denote by $\mathcal{S}_2$ the set of all solutions u of (1.1) with $u_1 \le u \le \beta$ in $\overline{\Omega}$ which are properly order stable from above. Since the maximum solution w is properly order stable from above, $\mathcal{S}_2$ is not empty. Arguing as above, we prove that there exists at least one minimal element $u_2 \in \mathcal{S}_2$ with $u_1 \le u_2$ in $\overline{\Omega}$.

If $u_1 = u_2$, the conclusion is achieved. Therefore, let us suppose that $u_1 < u_2$ in $\overline{\Omega}$ and let us denote by $\mathcal{S}_3$ the set of all solutions u of (1.1) with $u_1 \le u \le u_2$ in $\overline{\Omega}$.

Let us observe that there is no proper lower solution and no proper upper solution of (1.1) between u_1 and u_2. Indeed, if we assume that there exists, for instance, a proper lower solution α^* with $u_1 < \alpha^* < u_2$ in $\overline{\Omega}$, and we denote by z the minimum solution of (1.1) with $\alpha^* < z \le u_2$ in $\overline{\Omega}$, Proposition 5.4 implies that z is properly order stable from below, thus contradicting the maximality of u_1.

Next, we prove that if $z_1, z_2 \in \mathcal{S}_3$ with $z_1 < z_2$ in $\overline{\Omega}$, then there exists a solution z_3 of (1.1) such that $z_1 < z_3 < z_2$ in $\overline{\Omega}$. Indeed, if we assume that there is no solution z of (1.1) with $z_1 < z < z_2$ in $\overline{\Omega}$, then Lemma 5.6 guarantees either the existence of a proper lower solution α^* with $z_1 < \alpha^* < z_2$ in $\overline{\Omega}$, or the existence of a proper upper solution β^* with $z_1 < \beta^* < z_2$ in $\overline{\Omega}$, thus contradicting our preceding conclusion.

Now, let us fix a solution $u_0 \in \mathcal{S}_3$ and denote by $\mathcal{S}(u_0)$ a maximal totally ordered subset of $\mathcal{S}_3$ with $u_0 \in \mathcal{S}(u_0)$, which exists by Zorn lemma. Note that $u_1, u_2 \in \mathcal{S}(u_0)$ and for every $z_1, z_2 \in \mathcal{S}(u_0)$ with $z_1 < z_2$ in $\overline{\Omega}$, there is $z_3 \in \mathcal{S}(u_0)$ such that $z_1 < z_3 < z_2$ in $\overline{\Omega}$.

Since $\mathcal{S}(u_0)$ is bounded in $C^1(\overline{\Omega})$, arguing as in the proof of Lemma 5.1, we conclude that it is bounded in $W^{2,r}(\Omega)$, for any fixed $r \in]N, +\infty[$, and therefore it is relatively compact in $C^1(\overline{\Omega})$. In order to prove that $\mathcal{S}(u_0)$ is compact, let us show that it is closed in $C^1(\overline{\Omega})$. Let $(z_n)_n$ be a sequence in $\mathcal{S}(u_0)$ converging in $C^1(\overline{\Omega})$ to some function $z \in C^1(\overline{\Omega})$. It is clear that $u_1 \leq z \leq u_2$ in $\overline{\Omega}$. As in the proof of Lemma 5.1, we also see that $z \in W^{2,r}(\Omega)$ and it is a solution of (1.1). Let us show that $z \in \mathcal{S}(u_0)$, that is, for each $u \in \mathcal{S}(u_0)$, either $u \leq z$ or $u \geq z$ in $\overline{\Omega}$. Assume by contradiction that there exists $u \in \mathcal{S}(u_0)$ such that $u \nleq z$ and $u \ngeq z$ in $\overline{\Omega}$, i.e.,

$$\min\{\|(u-z)^+\|_\infty, \|(u-z)^-\|_\infty\} > 0.$$

Take n such that

$$\|z_n - z\|_\infty < \min\{\|(u-z)^+\|_\infty, \|(u-z)^-\|_\infty\}$$

and suppose, for instance, that $z_n \geq u$ in $\overline{\Omega}$. We have $(z_n - z)^+ \geq (u-z)^+$ in $\bar{\Omega}$ and hence

$$\|(u-z)^+\|_\infty \leq \|(z_n - z)^+\|_\infty \leq \|z_n - z\|_\infty < \|(u-z)^+\|_\infty,$$

which is a contradiction. Thus we conclude that $z \in \mathcal{S}(u_0)$ and hence $\mathcal{S}(u_0)$ is compact.

Now, take a continuous linear functional $\ell : C^1(\overline{\Omega}) \to \mathbb{R}$ such that $\ell(u) > 0$ if $u > 0$ in $\overline{\Omega}$. Since $\ell(\mathcal{S}(u_0)) \subset \mathbb{R}$ is compact and $\ell|_{\mathcal{S}(u_0)}$ is strictly increasing, $\ell|_{\mathcal{S}(u_0)}$ is a homeomorphism between $\mathcal{S}(u_0)$ and $\ell(\mathcal{S}(u_0))$. Since $\ell(\mathcal{S}(u_0))$ is also dense into itself, with respect to the ordering of $\mathbb{R}$, $\ell(\mathcal{S}(u_0))$ is an interval. Accordingly, $\mathcal{S}(u_0)$ is connected.

Finally, it is clear that every $u \in \mathcal{S}(u_0)$ is order stable and $u_1 = \min \mathcal{S}(u_0)$ and $u_2 = \max \mathcal{S}(u_0)$ are, respectively, properly order stable from below and properly order stable from above. The conclusion then follows setting $\mathcal{K} = \mathcal{S}(u_0)$. □

The following result is a counterpart, concerning instability, of Proposition 5.4.

Proposition 5.9. *Assume* (h1) *and* (h2). *Let z be a solution of* (1.1).
(i) *Suppose that there exists a strict lower solution α of* (1.1) *such that $\alpha \nleq z$ in $\overline{\Omega}$ and there is no solution u of* (1.1) *satisfying $u > z$ and $u \ngeq \alpha$ in $\overline{\Omega}$. Then z is properly order unstable from above.*
(ii) *Suppose that there exists a strict upper solution β of* (1.1) *such that $\beta \ngeq z$ in $\overline{\Omega}$ and there is no solution u of* (1.1) *satisfying $u < z$ and $u \nleq \beta$ in $\overline{\Omega}$. Then z is properly order unstable from below.*

Proof. We prove only the former statement; the proof of the latter being similar. Define

$$\mathcal{S} = \{u : u \text{ is a solution of (1.1) with } u \geq \max\{\alpha, z\} \text{ in } \overline{\Omega}\}.$$

Remark 3.2 implies that $\mathcal{S} \neq \emptyset$. Hence, by Lemma 5.1, there exists a minimal solution $v \in \mathcal{S}$. Since α is a strict lower solution, we have $v \gg \alpha$ and hence $v > \max\{\alpha, z\}$ in $\overline{\Omega}$. Let us observe that there is no solution u of (1.1) such that $z < u < v$ in $\overline{\Omega}$. Indeed, if u were such a solution, by the minimality of v, it should satisfy $u \ngeq \max\{\alpha, z\}$ in $\overline{\Omega}$ and hence $u \ngeq \alpha$ in $\overline{\Omega}$. This contradicts the assumptions on z.

Then Lemma 5.6 implies that either there exists a sequence $(\alpha_n)_n$ of proper lower solutions of (1.1) such that, for each n, $\alpha_n \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, $\alpha_n = 0$ on $\partial\Omega$, and $z < \alpha_n < v$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to z, or there exists a sequence $(\beta_n)_n$ of proper upper solutions of (1.1) such that, for each n, $\beta_n \in W^{2,r}(\Omega)$, for all finite $r \geq 1$, $\beta_n = 0$ on $\partial\Omega$, and $z < \beta_n < v$ in $\overline{\Omega}$, which converges weakly in $W^{2,r}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to v.

Let us show that the latter alternative cannot occur. Indeed, otherwise, as $v \gg \alpha$ in $\overline{\Omega}$, we could find an upper solution $\hat{\beta}$ of (1.1) with $\max\{\alpha, z\} \leq \hat{\beta} < v$ in $\overline{\Omega}$. Hence there should exist a solution u of (1.1) with $\max\{\alpha, z\} \leq u \leq \hat{\beta}$ in $\overline{\Omega}$ and therefore $z < u < v$ in $\overline{\Omega}$, as $z \ngeq \alpha$ in $\overline{\Omega}$. This yields a contradiction with a preceding conclusion. Therefore, the former alternative necessarily occurs, that is, z is properly order unstable from above. □

An immediate consequence of these statements is the following instability result, in the presence of a lower solution α and an upper solution β satisfying the condition $\alpha \nleq \beta$ in $\overline{\Omega}$. Let us set

$$\mathcal{V} = \{u \in C^1(\overline{\Omega}) : u \ngeq \alpha \text{ and } u \nleq \beta \text{ in } \overline{\Omega}\}.$$

Theorem 5.10. *Assume* (h1) *and* (h2). *Suppose that α is a strict lower solution and β is a strict upper solution of* (1.1) *satisfying*

$$\alpha \nleq \beta \quad \text{in } \overline{\Omega}.$$

Then any minimal solution v of (1.1) *in $\overline{\mathcal{V}}$ is order unstable from below and any maximal solution w of* (1.1) *in $\overline{\mathcal{V}}$ is order unstable from above.*

Remark 5.1. Proposition 3.2 and Lemma 5.1 guarantee the existence of minimal and maximal solutions of (1.1) in $\overline{\mathcal{V}}$.

We conclude with a kind of "universal" result concerning the existence of order stable solutions.

Theorem 5.11. *Assume* (h1) *and* (h2). *Then there exists a non-empty totally ordered compact and connected set $\mathcal{K}$ in $C^1(\overline{\Omega})$ such that every $u \in \mathcal{K}$ is an order stable solution of* (1.1). *Moreover, any minimal solution of* (1.1) *is properly order stable from below and any maximal solution of* (1.1) *is properly order stable from above.*

Proof. We argue as in Theorem 4.2 to construct a constant lower solution $\bar{\alpha}$ and a constant upper solution $\bar{\beta}$. The conclusions are then achieved by applying Theorem 5.8 and Proposition 5.4 to the modified problem (3.12) and by observing that the solutions of (3.12) are precisely the solutions of (1.1). □

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