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Research Article

Differentiation Theory over Infinite-Dimensional Banach Spaces

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We study, for any positive integer k and for any subset I of $\mathbf{N}^*$, the Banach space E_I of the bounded real sequences $\{x_n\}_{n\in I}$ and a measure over $(\mathbf{R}^I, \mathscr{B}^{(I)})$ that generalizes the k-dimensional Lebesgue one. Moreover, we expose a differentiation theory for the functions defined over this space. The main result of our paper is a change of variables' formula for the integration of the measurable real functions on $(\mathbf{R}^I, \mathscr{B}^{(I)})$. This change of variables is defined by some infinite-dimensional functions with properties that generalize the analogous ones of the standard finite-dimensional diffeomorphisms.

1. Introduction

The aim of this paper is to generalize the results of article [1], where, for any positive integer k and for any subset I of $\mathbf{N}^*$, we study a particular infinite-dimensional measure $\lambda_{N,a,v}^{(k,I)}$ that, in the case $I = \{1, \ldots, k\}$, coincides with the k-dimensional Lebesgue one on $\mathbf{R}^k$. The measure $\lambda_{N,a,v}^{(k,I)}$ is a product indexed by I of σ-finite measures on the Borel σ-algebra $\mathscr{B}$ on $\mathbf{R}$ (by using a generalization of the Jessen theorem), and it is defined over the measurable space $(\mathbf{R}^I, \mathscr{B}^{(I)})$, and in particular over $(E_I, \mathscr{B}_I)$, where $\mathscr{B}^{(I)}$ is the product indexed by I of the same σ-algebra $\mathscr{B}$, $E_I \subset \mathbf{R}^I$ is the Banach space of the bounded real sequences $\{x_n\}_{n\in I}$, and $\mathscr{B}_I$ is the restriction to E_I of $\mathscr{B}^{(I)}$.

In the mathematical literature, some articles introduced infinite-dimensional measures analogue of the Lebesgue one (see, e.g., the paper of Léandre [2], in the context of the noncommutative geometry, that one of Tsilevich et al. [3], which studies a family of σ-finite measures on $\mathbf{R}^+$, and that one of Baker [4], which defines a measure on $\mathbf{R}^{\mathbf{N}^*}$ that is not σ-finite).

In paper [1], the main result is a change of variables' formula for the integration of the measurable real functions on the space $(E_I, \mathscr{B}_I)$. This change of variables is defined by a particular class of linear functions over E_I, called (m, σ)-standard. A related problem is studied in the paper of Accardi et al. [5], where the authors describe the transformations of generalized measures on locally convex spaces under smooth transformations of these spaces.

In this paper, we prove that the change of variables' formula given in [1] can be extended by defining some infinite-dimensional functions with properties that generalize the analogous ones of the standard finite-dimensional diffeomorphisms.

In Section 2, we construct the infinite-dimensional Banach space E_I, and we define the continuous functions and the homeomorphisms over the open subsets of this space. Moreover, we recall some results about the integration of the measurable real functions defined on a measurable product space. In Section 3, we expose a differentiation theory in the infinite-dimensional context, and in particular we define the functions C^1 and the diffeomorphisms. Moreover, we introduce a class of functions, called (m, σ)-standard, that generalizes the set of the linear (m, σ)-standard functions given in [1], and we expose some properties of these functions. In Section 4, we present the main result of our paper, that is, a change of variables' formula for the integration of the measurable real functions on $(\mathbf{R}^I, \mathscr{B}^{(I)})$; this change of variables is defined by the biunique, C^1 and (m, σ)-standard functions, with further properties (Theorem 47). This result agrees with the analogous finite-dimensional result. In Section 5, we expose some ideas for further study in the probability theory.

2. Construction of an Infinite-Dimensional Banach Space

Henceforth, we will indicate by $\mathbf{N}^*$ and $\mathbf{R}^*$ the sets $\mathbf{N} \setminus \{0\}$ and $\mathbf{R} \setminus \{0\}$, respectively. Let $I \neq \varnothing$ be a set and let $k \in \mathbf{N}^*$; indicate by τ, by $\tau^{(k)}$, by $\tau^{(I)}$, by $\mathscr{B}$, by $\mathscr{B}^{(k)}$, by $\mathscr{B}^{(I)}$, by Leb, and by $\text{Leb}^{(k)}$, respectively, the Euclidean topology on $\mathbf{R}$, the Euclidean topology on $\mathbf{R}^k$, the topology $\bigotimes_{i\in I}\tau$, the Borel σ-algebra on $\mathbf{R}$, the Borel σ-algebra on $\mathbf{R}^k$, the σ-algebra $\bigotimes_{i\in I}\mathscr{B}$, the Lebesgue measure on $\mathbf{R}$, and the Lebesgue measure on $\mathbf{R}^k$. Moreover, for any set $A \subset \mathbf{R}$, indicate by $\mathscr{B}(A)$ the σ-algebra induced by $\mathscr{B}$ on A, and by $\tau(A)$ the topology induced by τ on A; analogously, for any set $A \subset \mathbf{R}^I$, define the σ-algebra $\mathscr{B}^{(I)}(A)$ and the topology $\tau^{(I)}(A)$. Finally, if $S = \prod_{i\in I}S_i$ is a Cartesian product, for any $(x_i : i \in I) \in S$ and for any $\varnothing \neq H \subset I$, define $x_H = (x_i : i \in H) \in \prod_{i\in H}S_i$, and define the projection $\pi_{I,H}$ on $\prod_{i\in H}S_i$ as the function $\pi_{I,H} : S \to \prod_{i\in H}S_i$ given by $\pi_{I,H}(x_I) = x_H$.

Theorem 1. *Let $I \neq \varnothing$ be a set and, for any $i \in I$, let (S_i, Σ_i, μ_i) be a measure space such that μ_i is finite. Moreover, suppose that, for some countable set $J \subset I$, μ_i is a probability measure for any $i \in I \setminus J$ and $\prod_{j\in J}\mu_j(S_j) \in \mathbf{R}^+$. Then, over the measurable space $(\prod_{i\in I}S_i, \bigotimes_{i\in I}\Sigma_i)$, there is a unique finite measure μ, indicated by $\bigotimes_{i\in I}\mu_i$, such that, for any $H \subset I$ such that $|H| < +\infty$ and for any $A = \prod_{h\in H}A_h \times \prod_{i\in I\setminus H}S_i \in \bigotimes_{i\in I}\Sigma_i$, where $A_h \in \Sigma_h$, $\forall h \in H$, we have $\mu(A) = \prod_{h\in H}\mu_h(A_h)\prod_{j\in J\setminus H}\mu_j(S_j)$. In particular, if I is countable, then $\mu(A) = \prod_{i\in I}\mu_i(A_i)$ for any $A = \prod_{i\in I}A_i \in \bigotimes_{i\in I}\Sigma_i$.*

Proof. See the proof of Corollary 4 in Asci [1]. □

Henceforth, we will suppose that I, J are sets such that $\varnothing \neq I, J \subset \mathbf{N}^*$; moreover, for any $k \in \mathbf{N}^*$, we will indicate by $I_k = \{i_1, \ldots, i_k\}$ the set of the first k elements of I (with the natural order and with the convention $I_k = I$ if $|I| < k$) and analogously J_k; finally, for any $j = i_n \in I$, set $|j| = n$.

Theorem 2. *Let (S_i, Σ_i, μ_i) be a probability space, for any $i \in I$, let $(S, \Sigma, \mu) = (\prod_{i\in I}S_i, \bigotimes_{i\in I}\Sigma_i, \bigotimes_{i\in I}\mu_i)$, and let $f \in L^1(S, \Sigma, \mu)$. Moreover, for any $H \subset I$ such that $\varnothing \neq |H| < +\infty$, define the measurable function $f_{H^c} : (S, \Sigma) \to (\mathbf{R}, \mathscr{B})$ by*

$$f_{H^c}(x) = \int_{S_H} f(x_H, x_{H^c})\, d\mu_H(x_H), \tag{1}$$

where (S_H, Σ_H, μ_H) is the probability space $(\prod_{i\in H}S_i, \bigotimes_{i\in H}\Sigma_i, \bigotimes_{i\in H}\mu_i)$. Then, μ-a.e. one has $\lim_{n\to+\infty} f_{I_n^c} = \int_S f\, d\mu$.

Proof. See, for example, Theorem 3, page 349, in Rao [6]. □

Corollary 3. *Let (S_i, Σ_i, μ_i) be a measure space such that μ_i is finite, for any $i \in I$, and $\prod_{i\in I}\mu_i(S_i) \in [0, +\infty)$; moreover, let $(S, \Sigma, \mu) = (\prod_{i\in I}S_i, \bigotimes_{i\in I}\Sigma_i, \bigotimes_{i\in I}\mu_i)$, let $f \in L^1(S, \Sigma, \mu)$, and let the measurable function $f_{I_n^c} : (S, \Sigma) \to (\mathbf{R}, \mathscr{B})$ defined by (1), for any $n \in \mathbf{N}^*$. Then, μ-a.e. one has $\lim_{n\to+\infty} f_{I_n^c} = \int_S f\, d\mu$.*

Proof. Suppose that $\mu_i(S_i) \neq 0$, $\forall i \in I$; then, $\overline{\mu}_i = \mu_i/\mu_i(S_i)$ is a probability measure, and also $\overline{\mu} = \bigotimes_{i\in I}\overline{\mu}_i$, $\overline{\mu}_{I_n} = \bigotimes_{i\in I_n}\overline{\mu}_i$, and $\overline{\mu}_{I_n^c} = \bigotimes_{i\in I_n^c}\overline{\mu}_i$, $\forall n \in \mathbf{N}^*$; then, define the function $\overline{f}_{I_n^c} : (S, \Sigma) \to (\mathbf{R}, \mathscr{B})$ by

$$\overline{f}_{I_n^c}(x) = \int_{S_{I_n}} f\left(x_{I_n}, x_{I_n^c}\right) d\overline{\mu}_{I_n}\left(x_{I_n}\right). \tag{2}$$

From Theorem 2, $\overline{\mu}$-a.e., we have

$$\begin{aligned} \lim_{n\to+\infty} f_{I_n^c} &= \lim_{n\to+\infty} \left(\prod_{i\in I_n}\mu_i(S_i)\right)\overline{f}_{I_n^c} \\ &= \prod_{i\in I}\mu_i(S_i)\int_S f\, d\overline{\mu} = \int_S f\, d\mu. \end{aligned} \tag{3}$$

Conversely, if $\mu_i(S_i) = 0$, for some $i \in I$, then

$$\lim_{n\to+\infty} f_{I_n^c} = 0 = \int_S f\, d\mu. \tag{4}$$

Thus, since $\mu = (\prod_{i\in I}\mu_i(S_i))\overline{\mu}$, we obtain the statement. □

Definition 4. For any set $I \neq \varnothing$, define the function $\|\cdot\|_I : \mathbf{R}^I \to [0, +\infty]$ by

$$\|x\|_I = \sup_{i\in I}|x_i|, \quad \forall x = (x_i : i \in I) \in \mathbf{R}^I, \tag{5}$$

and define the vector space

$$E_I = \left\{x \in \mathbf{R}^I : \|x\|_I < +\infty\right\}. \tag{6}$$

Moreover, indicate by $\mathscr{B}_I$ the σ-algebra $\mathscr{B}^{(I)}(E_I)$, by τ_I the topology $\tau^{(I)}(E_I)$ induced on E_I by the product topology $\tau^{(I)}(\mathbf{R}^I)$, and by $\tau_{\|\cdot\|_I}$ the topology induced on E_I by the distance $d : E_I \times E_I \to [0, +\infty)$ defined by $d(x, y) = \|x - y\|_I$, $\forall x, y \in E_I$; furthermore, for any set $A \subset E_I$, indicate by $\tau_{\|\cdot\|_I}(A)$ the topology induced by $\tau_{\|\cdot\|_I}$ on A. Finally, for any $x_0 \in E_I$ and for any $\delta > 0$, indicate by $B_I(x_0, \delta)$ the set $\{x \in E_I : \|x - x_0\|_I < \delta\}$.

Observe that $\tau^{(I)}(A) \subset \tau_{\|\cdot\|_I}(A)$, $\forall A \subset E_I$; moreover, E_I is a Banach space, with the norm $\|\cdot\|_I$ (see, e.g., the proof of Remark 2 in [1]).

Proposition 5. *Let H and I be sets such that $\varnothing \neq H \subset I$; then, the function $\pi_{I,H} : (\mathbf{R}^I, \tau^{(I)}) \to (\mathbf{R}^H, \tau^{(H)})$ is continuous and open.*

Proof. See, for example, the theory of the product spaces in Weidmann's book [7]. □

Proposition 6. *Let H and I be sets such that $\varnothing \neq H \subset I$, and let $\overline{\pi}_{I,H} : E_I \to E_H$ be the function given by $\overline{\pi}_{I,H}(x) = \pi_{I,H}(x)$, for any $x \in E_I$; then,*

(1) *$\overline{\pi}_{I,H} : (E_I, \tau_I) \to (E_H, \tau_H)$ is continuous and open;*

(2) *$\overline{\pi}_{I,H} : (E_I, \tau_{\|\cdot\|_I}) \to (E_H, \tau_{\|\cdot\|_H})$ is continuous and open.*

Proof of (1). $\forall A \in \tau_H$, there exists $B \in \tau^{(H)}$ such that $A = B \cap E_H$, and so $\overline{\pi}_{I,H}^{-1}(A) = (\pi_{I,H}^{-1}(B \cap E_H)) \cap E_I = \pi_{I,H}^{-1}(B) \cap E_I$; then, since $\pi_{I,H}^{-1}(B) \in \tau^{(I)}$ by Proposition 5, we have $\overline{\pi}_{I,H}^{-1}(A) \in \tau_I$, and so $\overline{\pi}_{I,H}$ is a continuous function. Moreover, $\forall A \in \tau_I$, there exists $B \in \tau^{(I)}$ such that $A = B \cap E_I$, and so $\overline{\pi}_{I,H}(A) = \pi_{I,H}(B \cap E_I) = \pi_{I,H}(B) \cap E_H$; then, since $\pi_{I,H}(B) \in \tau^{(H)}$ by Proposition 5, we have $\overline{\pi}_{I,H}(A) \in \tau_H$, and so $\overline{\pi}_{I,H}$ is an open function. □

Proof of (2). $\forall A \in \tau_{\|\cdot\|_H}$ and $\forall x = (x_i : i \in I) \in \overline{\pi}_{I,H}^{-1}(A)$, we have $(x_i : i \in H) \in A$, and so there exists $a \in \mathbf{R}^+$ and $(y_i : i \in H) \in A$ such that $(x_i : i \in H) \in \prod_{i\in H}(y_i - a, y_i + a) \subset A$; let $z = (z_i : i \in I) \in \overline{\pi}_{I,H}^{-1}(A)$ such that $z_i = x_i, \forall i \in I \setminus H$, and $z_i = y_i, \forall i \in H$; then, $x \in \prod_{i\in I}(z_i - a, z_i + a) \subset \overline{\pi}_{I,H}^{-1}(A)$, and so $\overline{\pi}_{I,H}$ is a continuous function. Moreover, $\forall A \in \tau_{\|\cdot\|_I}$ and $\forall x = (x_i : i \in H) \in \overline{\pi}_{I,H}(A)$, let $y = (y_i : i \in I) \in A$ such that $y_i = x_i, \forall i \in H$; since $A \in \tau_{\|\cdot\|_I}$, there exists $a \in \mathbf{R}^+$ and $(z_i : i \in I) \in A$ such that $y \in \prod_{i\in I}(z_i - a, z_i + a) \subset A$, and so $x \in \prod_{i\in H}(z_i - a, z_i + a) \subset \overline{\pi}_{I,H}(A)$; then, $\overline{\pi}_{I,H}$ is an open function. □

Definition 7. Let $U \in \tau_{\|\cdot\|_J}$, let $x_0 \in U$, let $l \in E_I$, and let $\varphi : U \subset E_J \to E_I$ be a function; we say that $\lim_{x\to x_0}\varphi(x) = l$ if, for any neighbourhood $M \in \tau_{\|\cdot\|_I}$ of $\varphi(x_0)$, there exists a neighbourhood $N \in \tau_{\|\cdot\|_J}(U)$ of x_0 such that, for any $x \in N \setminus \{x_0\}$, we have $\varphi(x) \in M$.

Definition 8. Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function; we say that φ is continuous in $x_0 \in U$ if $\lim_{x\to x_0}\varphi(x) = \varphi(x_0)$, and we say that φ is continuous in U if φ is continuous in x_0 for any $x_0 \in U$.

Remark 9. Let $U \in \tau_{\|\cdot\|_J}$, let $V \in \tau_{\|\cdot\|_I}$, and let $\varphi : U \subset E_J \to V \subset E_I$ be a function; then, $\varphi : (U, \tau_{\|\cdot\|_J}(U)) \to (V, \tau_{\|\cdot\|_I}(V))$ is continuous if and only if φ is continuous in U.

Remark 10. Let H, I, and J be sets such that $\varnothing \neq H \subset I$, let $U \in \tau_{\|\cdot\|_J}$, and let $\varphi : U \subset E_J \to E_I$ be a function continuous in U; then, the function $(\pi_{I,H} \circ \varphi) : U \to E_H$ is continuous in U.

Proof. The statement follows from Proposition 6. □

Definition 11. Let $U \in \tau_{\|\cdot\|_J}$ and let $V \in \tau_{\|\cdot\|_I}$; a function $\varphi : U \subset E_J \to V \subset E_I$ is called homeomorphism if φ is biunique and the functions $\varphi : (U, \tau_{\|\cdot\|_J}(U)) \to (V, \tau_{\|\cdot\|_I}(V))$ and $\varphi^{-1} : (V, \tau_{\|\cdot\|_I}(V)) \to (U, \tau_{\|\cdot\|_J}(U))$ are continuous.

3. Differentiation Theory in the Infinite-Dimensional Context

The following concept generalizes Definition 6 in [1] (see also the theory in the Lang's book [8]).

Definition 12. Let $A = (a_{ij})_{i\in I, j\in J}$ be a real matrix $I \times J$ (eventually infinite); then, define the linear function $A = (a_{ij})_{i\in I, j\in J} : E_J \to \mathbf{R}^I$, and write $x \to Ax$, in the following manner:

$$(Ax)_i = \sum_{j\in J} a_{ij}x_j, \quad \forall x \in E_J, \ \forall i \in I, \tag{7}$$

on condition that, for any $i \in I$, the sum in (7) converges to a real number.

Definition 13. Let $U \in \tau_{\|\cdot\|_J}$; a function $\varphi : U \subset E_J \to E_I$ is called differentiable in $x_0 \in U$ if there exists a linear and continuous function $A : E_J \to E_I$ defined by a real matrix $A = (a_{ij})_{i\in I, j\in J}$, and one has

$$\lim_{h\to 0} \frac{\|\varphi(x_0 + h) - \varphi(x_0) - Ah\|_I}{\|h\|_J} = 0. \tag{8}$$

If φ is differentiable in x_0 for any $x_0 \in U$, φ is called differentiable in U. The function A is called differential of the function φ in x_0, and it is indicated by the symbol $d\varphi(x_0)$.

Remark 14. Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi, \psi : U \subset E_J \to E_I$ be differentiable functions in $x_0 \in U$; then, for any $\alpha, \beta \in \mathbf{R}$, the function $\alpha\varphi + \beta\psi$ is differentiable in x_0, and $d(\alpha\varphi + \beta\psi)(x_0) = \alpha d\varphi(x_0) + \beta d\psi(x_0)$.

Proof. Observe that

$$\begin{aligned}
&\lim_{h\to 0} \frac{\|(\alpha\varphi + \beta\psi)(x_0 + h) - (\alpha\varphi + \beta\psi)(x_0) - (\alpha d\varphi(x_0) + \beta d\psi(x_0))h\|_I}{\|h\|_J} \\
&\quad = \lim_{h\to 0} \frac{\|\alpha\varphi(x_0 + h) - \alpha\varphi(x_0) - \alpha d\varphi(x_0)h + \beta\psi(x_0 + h) - \beta\psi(x_0) - \beta d\psi(x_0)h\|_I}{\|h\|_J} \\
&\quad \leq \lim_{h\to 0} \frac{\|\alpha\varphi(x_0 + h) - \alpha\varphi(x_0) - \alpha d\varphi(x_0)h\|_I}{\|h\|_J} + \lim_{h\to 0} \frac{\|\beta\psi(x_0 + h) - \beta\psi(x_0) - \beta d\psi(x_0)h\|_I}{\|h\|_J} = 0 \Longrightarrow \\
&d(\alpha\varphi + \beta\psi)(x_0) = \alpha d\varphi(x_0) + \beta d\psi(x_0).
\end{aligned} \tag{9}$$

□

Remark 15. A linear and continuous function $A = (a_{ij})_{i\in I, j\in J} : E_J \to E_I$, defined by

$$(Ax)_i = \sum_{j\in J} a_{ij} x_j, \quad \forall x \in E_J,\ \forall i \in I, \tag{10}$$

is differentiable and $d\varphi(x_0) = A$, for any $x_0 \in E_J$.

Remark 16. Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$; then, for any $i \in I$, the component $\varphi_i : U \to \mathbf{R}$ is differentiable in x_0, and $d\varphi_i(x_0)$ is the matrix A_i given by the ith row of $A = d\varphi(x_0)$. Moreover, if $|I| < +\infty$ and $\varphi_i : U \subset E_J \to \mathbf{R}$ is differentiable in x_0, for any $i \in I$, then $\varphi : U \subset E_J \to E_I$ is differentiable in x_0.

Remark 17. Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$; then, φ is continuous in x_0.

Proof. $\forall x \in U$, set

$$\sigma(x, x_0) = f(x) - f(x_0) - df(x_0)(x - x_0). \tag{11}$$

From (8), we have $\lim_{x\to x_0} \|\sigma(x, x_0)\|_I = 0$; moreover,

$$\begin{aligned} \|f(x) - f(x_0)\|_I &= \|df(x_0)(x - x_0) + \sigma(x, x_0)\|_I \\ &\le \|df(x_0)\| \, \|x - x_0\|_J \\ &\quad + \|\sigma(x, x_0)\|_I, \end{aligned} \tag{12}$$

from which $\lim_{x\to x_0} f(x) = f(x_0)$. □

Definition 18. Let $U \in \tau_{\|\cdot\|_J}$ and let $v \in E_J$ such that $\|v\|_J = 1$; a function $\varphi : U \subset E_J \to \mathbf{R}$ is called derivable in $x_0 \in U$ in the direction v if there exists the limit

$$\lim_{t\to 0} \frac{\varphi(x_0 + tv) - \varphi(x_0)}{t}. \tag{13}$$

This limit is indicated by $(\partial\varphi/\partial v)(x_0)$, and it is called derivative of φ in x_0 in the direction v. If for some $j \in J$ one has $v = e_j$, where $(e_j)_k = \delta_{jk}$, for any $k \in J$, indicate $(\partial\varphi/\partial v)(x_0)$ by $(\partial\varphi/\partial x_j)(x_0)$, and call it partial derivative of φ in x_0, with respect to x_j. Moreover, if there exists the linear function defined by the matrix $J_\varphi(x_0)$, where $(J_\varphi(x_0))_{ij} = (\partial\varphi_i/\partial x_j)(x_0)$, for any $i \in I$ and $j \in J$, then $J_\varphi(x_0)$ is called Jacobian matrix of the function φ in x_0.

Remark 19. Let $U \in \tau_{\|\cdot\|_J}$ and suppose that a function $\varphi : U \subset E_J \to E_I$ is differentiable in $x_0 \in U$; then, for any $v \in E_J$ such that $\|v\|_J = 1$ and for any $i \in I$, the function $\varphi_i : U \subset E_J \to \mathbf{R}$ is derivable in x_0 in the direction v, and one has

$$\frac{\partial\varphi_i}{\partial v}(x_0) = d\varphi_i(x_0)\, v. \tag{14}$$

Proof. $\forall h \in E_J$, by setting $h = tv$, for some matrix $A = (a_{ij})_{i\in I, j\in J}$ we have

$$\begin{aligned} &\lim_{t\to 0} \frac{|\varphi_i(x_0 + tv) - \varphi_i(x_0) - A_i(tv)|}{|t|} \\ &\quad = \lim_{h\to 0} \frac{|\varphi_i(x_0 + h) - \varphi_i(x_0) - A_i h|}{\|h\|_J} = 0. \end{aligned} \tag{15}$$

Then, since $A_i(tv) = tA_i v$, we have

$$\lim_{t\to 0} \frac{\varphi_i(x_0 + tv) - \varphi_i(x_0)}{t} = A_i v, \tag{16}$$

from which

$$\frac{\partial\varphi_i}{\partial v}(x_0) = A_i v = d\varphi_i(x_0)\, v. \tag{17}$$

□

Corollary 20. *Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$; then, the linear function $J_\varphi(x_0) : E_J \to E_I$ is defined and continuous; moreover, for any $h \in E_J$, one has $d\varphi(x_0)(h) = J_\varphi(x_0)h$.*

Proof. From Remark 19, $\forall i \in I$ and $\forall j \in J$, we have $(d\varphi(x_0))_{ij} = d\varphi_i(x_0)(e_j) = (\partial\varphi_i/\partial x_j)(x_0) = (J_\varphi(x_0))_{ij}$. □

Theorem 21. *Let $U \in \tau_{\|\cdot\|_J}$, let $\varphi : U \subset E_J \to E_I$ be a function differentiable in $x_0 \in U$, let $V \in \tau_{\|\cdot\|_I}$ such that $V \supset \varphi(U)$, and let $\psi : V \subset E_I \to E_H$ be a function differentiable in $y_0 = \varphi(x_0)$. Then, the function $\psi \circ \varphi$ is differentiable in x_0, and one has $d(\psi \circ \varphi)(x_0) = d\psi(y_0) \circ d\varphi(x_0)$.*

Proof. The proof is analogous to that one true in the particular case $|H| < +\infty$, $|I| < +\infty$, $|J| < +\infty$ (see, e.g., the Lang's book [9]). □

Definition 22. Let $U \in \tau_{\|\cdot\|_J}$, let $i, j \in J$, and let $\varphi : U \subset E_J \to \mathbf{R}$ be a function derivable in $x_0 \in U$ with respect to x_i, such that the function $\partial\varphi/\partial x_i$ is derivable in x_0 with respect to x_j. Indicate $(\partial/\partial x_j)(\partial\varphi/\partial x_i)(x_0)$ by $(\partial^2\varphi/\partial x_j \partial x_i)(x_0)$ and call it second partial derivative of φ in x_0 with respect to x_i and x_j. If $i = j$, it is indicated by $(\partial^2\varphi/\partial x_i^2)(x_0)$. Analogously, for any $k \in \mathbf{N}^*$ and for any $j_1, \ldots, j_k \in J$, define $(\partial^k\varphi/(\partial x_{j_k} \cdots \partial x_{j_1}))(x_0)$ and call it kth partial derivative of φ in x_0 with respect to $x_{j_1}, \ldots, x_{j_k}$.

Definition 23. Let $U \in \tau_{\|\cdot\|_J}$ and let $k \in \mathbf{N}^*$; a function $\varphi : U \subset E_J \to E_I$ is called C^k in $x_0 \in U$ if, in a neighbourhood $V \in \tau_{\|\cdot\|_J}(U)$ of x_0, for any $i \in I$ and for any $j_1, \ldots, j_k \in J$, there exists the function defined by $x \to (\partial^k\varphi_i/(\partial x_{j_k} \cdots \partial x_{j_1}))(x)$, and this function is continuous in x_0; φ is called C^k in U if, for any $x_0 \in U$, φ is C^k in x_0. Moreover, φ is called strongly C^1 in $x_0 \in U$ if, in a neighbourhood $V \in \tau_{\|\cdot\|_J}(U)$ of x_0, there exists the function defined by $x \to J_\varphi(x)$, and this function is continuous in x_0, with $\|J_\varphi(x_0)\| < +\infty$. Finally, φ is called strongly C^1 in U if, for any $x_0 \in U$, φ is strongly C^1 in x_0.

Definition 24. Let $U \in \tau_{\|\cdot\|_J}$ and let $V \in \tau_{\|\cdot\|_I}$; a function $\varphi : U \subset E_J \to V \subset E_I$ is called diffeomorphism if φ is biunique and C^1 in U, and the function $\varphi^{-1} : V \subset E_I \to U \subset E_J$ is C^1 in V.

Remark 25. Let $U \in \tau_{\|\cdot\|_J}$ and let $\varphi : U \subset E_J \to E_I$ be a function C^1 in $x_0 \in U$, where $|I| < +\infty$, $|J| < +\infty$; then, φ is strongly C^1 in x_0.

Theorem 26. *Let $U \in \tau_{\|\cdot\|_J}$, let $\varphi : U \subset E_J \to \mathbf{R}$ be a function C^k in $x_0 \in U$, let $i_1, \ldots, i_k \in J$, and let $j_1, \ldots, j_k \in J$ be a permutation of $i_1, \ldots, i_k$. Then, one has*

$$\frac{\partial^k \varphi}{\partial x_{i_1} \cdots \partial x_{i_k}}(x_0) = \frac{\partial^k \varphi}{\partial x_{j_1} \cdots \partial x_{j_k}}(x_0). \tag{18}$$

Proof. The proof is analogous to that one true in the particular case $|J| < +\infty$ (see, e.g., the Lang's book [9]). □

Proposition 27. *Let $U = (\prod_{j\in J} U_j) \cap E_J \in \tau_{\|\cdot\|_J}$, where $U_j \in \tau$, for any $j \in J$, and let $\varphi : U \subset E_J \to E_I$ be a function C^1 in $x_0 \in U$, such that*

$$\varphi_i(x) = \sum_{j\in J} \varphi_{ij}(x_j), \quad \forall x = (x_j : j \in J) \in U, \ \forall i \in I, \tag{19}$$

where $\varphi_{ij} : U_j \to \mathbf{R}$, $\forall i \in I$ and $\forall j \in J$; moreover, suppose that there exists a neighbourhood $V \in \tau_{\|\cdot\|_J}(U)$ of x_0 such that $\sup_{x\in V} \|J_\varphi(x)\| < +\infty$. Then, φ is continuous in x_0; in particular, if φ is strongly C^1 in x_0 and $|I| < +\infty$, φ is differentiable in x_0.

Proof. Since φ is C^1 in x_0, from (19) there exists a neighbourhood of $x_0 = (x_{0j} : j \in J)$ that we can suppose to be V, such that $V = \prod_{j\in J} V_j \subset U$, where $V_j = (x_{0j} - \gamma_j, x_{0j} + \delta_j)$, $\forall j \in J$, and such that, $\forall i \in I$ and $\forall j \in J$, φ'_{ij} exists in V_j. Let $x = (x_j : j \in J) \in V$; from the Lagrange theorem, $\forall i \in I$ and $\forall j \in J$, there exists $\xi_{ij} \in (\min\{x_{0j}, x_j\}, \max\{x_{0j}, x_j\}) \subset V_j$ such that $\varphi_{ij}(x_j) - \varphi_{ij}(x_{0j}) = \varphi'_{ij}(\xi_{ij})(x_j - x_{0j})$. Then,

$$\begin{gathered} \varphi_i(x) - \varphi_i(x_0) = \sum_{j\in J} \varphi'_{ij}(\xi_{ij})(x_j - x_{0j}) \Longrightarrow \\ \varphi(x) - \varphi(x_0) = A_\varphi(x, x_0)(x - x_0), \end{gathered} \tag{20}$$

where $A_\varphi(x, x_0) = (\varphi'_{ij}(\xi_{ij}))_{i\in I, j\in J}$. Thus, if $\sup_{x\in V} \|J_\varphi(x)\| = c < +\infty$, $\forall x \in V$, we have

$$\begin{aligned} \|A_\varphi(x, x_0)\| &= \sup_{i\in I} \left\|(A_\varphi(x, x_0))_i\right\| \\ &\le \sup_{i\in I} \left\|J_\varphi(\xi_{ij} : j \in J)\right\| \le c, \end{aligned} \tag{21}$$

from which

$$\|\varphi(x) - \varphi(x_0)\|_I \le c\,\|x - x_0\|_J; \tag{22}$$

then, φ is continuous in x_0. Moreover, we have

$$\begin{aligned} &\varphi(x) - \varphi(x_0) - J_\varphi(x_0)(x - x_0) \\ &\quad = (A_\varphi(x, x_0) - J_\varphi(x_0))(x - x_0) \Longrightarrow \\ &\frac{\|\varphi(x) - \varphi(x_0) - J_\varphi(x_0)(x - x_0)\|_I}{\|x - x_0\|_J} \\ &\quad \le \|A_\varphi(x, x_0) - J_\varphi(x_0)\| \\ &\quad \le \sup_{i\in I} \|J_\varphi(\xi_{ij} : j \in J) - J_\varphi(x_0)\|. \end{aligned} \tag{23}$$

Then, if φ is strongly C^1 in x_0 and $|I| < +\infty$, we obtain

$$\lim_{x\to x_0} \frac{\|\varphi(x) - \varphi(x_0) - J_\varphi(x_0)(x - x_0)\|_I}{\|x - x_0\|_J} = 0, \tag{24}$$

and so φ is differentiable in x_0, with $d\varphi(x_0) = J_\varphi(x_0)$. □

Definition 28. Let $m \in \mathbf{N}^*$, let $U = (U^{(m)} \times \prod_{j\in I\setminus I_m} A_j) \cap E_I \in \tau_{\|\cdot\|_I}$, where $U^{(m)} \in \tau^{(m)}$, $A_j \subset \mathbf{R}$ is an open interval, for any $j \in I \setminus I_m$, and let $\sigma : I \setminus I_m \to I \setminus I_m$ be an increasing function. A function $\varphi : U \subset E_I \to E_I$ is called (m, σ)-standard if

(1) $\forall i \in I$, there exist some functions $\varphi_i^{(m,m)} : U^{(m)} \to \mathbf{R}$ and $\{\varphi_{ij}\}_{j\in I\setminus I_m} : A_j \to \mathbf{R}$ such that, $\forall x \in U$, one has $\varphi_i(x) = \varphi_i^{(m,m)}(x_{i_1}, \ldots, x_{i_m}) + \sum_{j\in I\setminus I_m} \varphi_{ij}(x_j)$;

(2) $\forall i \in I \setminus I_m$ and $\forall x \in U$, one has $\varphi_i^{(m,m)}(x_{i_1}, \ldots, x_{i_m}) = 0$ and $\varphi_{ij}(x_j) = 0$, $\forall j \ne \sigma(i)$;

(3) $\forall i \in I \setminus I_m$, the function $g_i = \varphi_{i,\sigma(i)}$ is constant or injective derivable; moreover, $\forall x = (x_j : j \in I \setminus I_m) \in \prod_{j\in I\setminus I_m} A_j$, there exists $\prod_{i\in \mathcal{I}_\varphi} (-1)^{|i|+|\sigma(i)|} g'_i(x_{\sigma(i)}) \in \mathbf{R}^*$, where $\mathcal{I}_\varphi = \{i \in I \setminus I_m : g_i \text{ is injective derivable}\}$.

If $\sigma(i) = i$, $\forall i \in I \setminus I_m$, φ is called m-standard; moreover, if the sequence $\{(-1)^{|i|+|\sigma(i)|} g'_i(x_{\sigma(i)})\}_{i\in\mathcal{I}_\varphi}$ converges uniformly to 1, with respect to $x = (x_j : j \in I \setminus I_m) \in \prod_{j\in I\setminus I_m} A_j$, then φ is called strongly (m, σ)-standard.

Furthermore, $\forall n \ge m$, define the function $\varphi^{(n,n)} : \pi_{I,I_n}(U) \to \mathbf{R}^n$ by $\varphi^{(n,n)}(\mathbf{x}) = (\varphi_{i_1}^{(n,n)}(\mathbf{x}), \ldots, \varphi_{i_n}^{(n,n)}(\mathbf{x}))$, where

$$\begin{gathered} \varphi_i^{(n,n)}(\mathbf{x}) = \varphi_i^{(m,m)}(x_{i_1}, \ldots, x_{i_m}) + \sum_{j\in I_n\setminus I_m} \varphi_{ij}(x_j), \\ \forall \mathbf{x} = (x_j : j \in I_n) \in \pi_{I,I_n}(U), \ \forall i \in I_n. \end{gathered} \tag{25}$$

Finally, define the (m, σ)-standard function $\varphi^{(n)} : U \subset E_I \to E_I$ in the following manner:

$$\begin{aligned} &\varphi_i^{(n)}(x) = \varphi_i^{(n,n)}(x_{i_1}, \ldots, x_{i_n}), \\ &\qquad\qquad \forall x = (x_j : j \in I) \in U, \ \forall i \in I_m, \\ &\varphi_i^{(n)}(x) = \varphi_i(x), \\ &\qquad\qquad \forall x = (x_j : j \in I) \in U, \ \forall i \in I \setminus I_m, \end{aligned} \tag{26}$$

and indicate $\varphi^{(m)}$ by $\overline{\varphi}$.

Remark 29. Let $\varphi : U \subset E_I \to E_I$ be a (m, σ)-standard function. Then,

(1) if σ is injective, for any $n \in \mathbf{N}$, $n \ge m$, φ is (n, σ)-standard;

(2) σ is biunique if and only if $\sigma(i) = i$, $\forall i \in I \setminus I_m$;

(3) if $\prod_{j\in I\setminus I_m} A_j \subset E_{I\setminus I_m}$, there exist $a \in \mathbf{R}^+$ and $m_0 \in \mathbf{N}$, $m_0 \ge m$, such that, for any $j \in I \setminus I_{m_0}$, one has $A_j \subset (-a, a)$.

Proof. Points (1) and (2) follow from the fact that σ is increasing. Moreover, the proof of the point (3) is trivial. □

Remark 30. Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function such that $\prod_{j\in I\setminus I_m} A_j \subset E_{I\setminus I_m}$, and σ is injective; then, there exists $m_1 \in \mathbf{N}$, $m_1 \geq m$, such that, for any $i \in I \setminus I_{m_1}$, g_i is bounded. In particular, if $|I| = +\infty$, φ is not surjective.

Proof. Since $\prod_{j\in I\setminus I_m} A_j \subset E_{I\setminus I_m}$, from Remark 29, there exists $m_0 \in \mathbf{N}, m_0 \geq m$, such that, $\forall j \in I\setminus I_{m_0}$, the set A_j is bounded; then, let $\mathscr{H} = \{i \in I \setminus I_{m_0} : g_i \text{ is not bounded}\} \subset \mathscr{I}_\varphi$; since σ is injective and increasing, we have $\sigma(\mathscr{H}) \subset I \setminus I_{m_0}$. Moreover, we have $|\mathscr{H}| < +\infty$; indeed, $\forall i \in \mathscr{H}$, the set $A_{\sigma(i)}$ is bounded, and so there exists $t_i \in A_{\sigma(i)}$ such that $|g_i'(t_i)| > 2$; by supposing by contradiction $|\mathscr{H}| = +\infty$, $\forall x = (x_j : j \in I \setminus I_m) \in \prod_{j\in I\setminus I_m} A_j$ such that $x_{\sigma(i)} = t_i$, $\forall i \in \mathscr{H}$, we would obtain $\prod_{i\in\mathscr{I}_\varphi}|g_i'(x_{\sigma(i)})| = \prod_{i\in\mathscr{I}_\varphi\setminus\mathscr{H}}|g_i'(x_{\sigma(i)})|\prod_{i\in\mathscr{H}}|g_i'(t_i)| = +\infty$ (a contradiction). Then, there exists $m_1 \in \mathbf{N}$, $m_1 \geq m$, such that, $\forall i \in I\setminus I_{m_1}$, g_i is bounded. In particular, $\forall i \in I\setminus I_{m_1}$, the function g_i is not surjective; then, if $|I| = +\infty$, φ is not surjective. □

Proposition 31. *Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function; then,*

(1) *suppose that φ is injective, $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, the function $\varphi_i : U \to \mathbf{R}$ is C^1, for any $i \in I_m$, and $\det J_{\varphi^{(m,m)}}(\mathbf{x}) \neq 0$, $\forall \mathbf{x} \in U^{(m)}$; then, the functions g_i, for any $i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique.*

(2) *if φ is biunique, then the functions g_i, for any $i \in I\setminus I_m$, $\varphi^{(m,m)}$, and σ are biunique.*

Proof of (1). Suppose that φ is injective, $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, and let $i \in I \setminus I_m$; we have $i \in \mathscr{I}_\varphi$, since otherwise we would obtain $g_i(x) = c$, $\forall x \in A_{\sigma(i)}$, for some $c \in \mathbf{R}$, from which $\pi_{\{i\}}(\varphi(U)) = \{c\} \notin \tau = \tau^{(\{i\})}$, and this should contradict the assumption; then, g_i is injective. Moreover, σ must be injective; in fact, by supposing by contradiction that $\sigma(i_1) = \sigma(i_2)$, for some $m < i_1 < i_2$, then

$$\begin{aligned}\pi_{I,\{i_1,i_2\}}(\varphi(U)) &= \{(y_1,y_2) \in \mathbf{R}^2 : y_1 = g_{i_1}(x),\ y_2 \\ &= g_{i_2}(x),\ \text{for some } x \in A_{\sigma(i_1)}\} = \{(y_1,y_2) \\ &\in \mathbf{R}^2 : y_1 \in g_{i_1}(A_{\sigma(i_1)}),\ y_2 = g_{i_2}(g_{i_1}^{-1}(y_1))\} \\ &\notin \tau^{(2)} = \tau^{(\{i_1,i_2\})}\end{aligned} \tag{27}$$

(a contradiction). Moreover, σ is surjective; in fact, suppose by contradiction that there exists $n \in (I\setminus I_m)\setminus\sigma(I\setminus I_m)$; since φ is injective, $\forall y \in \varphi(U)$, there is a unique $x = (x_j : j \in I) \in U$ such that $\varphi(x) = y$, and so

$$\begin{gathered}y_i = \varphi_i(x) = \varphi_i^{(m,m)}(x_{i_1},\ldots,x_{i_m}) + \varphi_{in}(x_n) \\ + \sum_{j\in I\setminus(I_m\cup\{n\})}\varphi_{ij}(x_j), \quad \forall i \in I_m \Longrightarrow \\ \varphi_i^{(m,m)}(x_{i_1},\ldots,x_{i_m}) + \varphi_{in}(x_n) - y_i \\ + \sum_{j\in I\setminus(I_m\cup\{n\})}\varphi_{ij}(x_j) = 0, \quad \forall i \in I_m.\end{gathered} \tag{28}$$

Then, consider the function $F : U^{(m)} \times A_n \to \mathbf{R}^m$ defined by

$$\begin{gathered}F_i(\mathbf{z},t) = \varphi_i^{(m,m)}(\mathbf{z}) + \varphi_{in}(t) - y_i \\ + \sum_{j\in I\setminus(I_m\cup\{n\})}\varphi_{ij}(x_j), \\ \forall \mathbf{z} \in U^{(m)},\ \forall t \in A_n,\ \forall i \in I_m.\end{gathered} \tag{29}$$

From (28) and by assumption, we have

$$\begin{gathered}F(x_{i_1},\ldots,x_{i_m},x_n) = \mathbf{0}, \\ \det\frac{\partial F}{\partial \mathbf{z}}(x_{i_1},\ldots,x_{i_m},x_n) \neq 0;\end{gathered} \tag{30}$$

then, there exist a neighbourhood $V_1 \subset A_n$ of x_n and a neighbourhood $V_2 \subset U^{(m)}$ of $(x_{i_1},\ldots,x_{i_m})$ such that, $\forall t \in V_1$, there exists a unique $\mathbf{z} = f(t) \in V_2$ such that $F(\mathbf{z},t) = \mathbf{0}$ (a contradiction with the uniqueness of $x = (x_j : j \in I) \in U$ such that $\varphi(x) = y$). Finally, let $\mathbf{x},\mathbf{y} \in U^{(m)}$ be such that $\varphi^{(m,m)}(\mathbf{x}) = \varphi^{(m,m)}(\mathbf{y})$, and let $\overline{x},\overline{y} \in U$ be such that $\overline{x}_i = x_i$ and $\overline{y}_i = y_i$, $\forall i \in I_m$, $\overline{x}_i = \overline{y}_i$, $\forall i \in I \setminus I_m$. We have

$$\begin{gathered}\varphi_i(\overline{x}) = \varphi_i^{(m,m)}(\mathbf{x}) + \sum_{j\in I\setminus I_m}\varphi_{ij}(\overline{x}_j) \\ = \varphi_i^{(m,m)}(\mathbf{y}) + \sum_{j\in I\setminus I_m}\varphi_{ij}(\overline{y}_j) = \varphi_i(\overline{y}), \\ \forall i \in I_m;\end{gathered} \tag{31}$$

$$\varphi_i(\overline{x}) = g_i(\overline{x}_{\sigma(i)}) = g_i(\overline{y}_{\sigma(i)}) = \varphi_i(\overline{y}), \quad \forall i \in I \setminus I_m,$$

from which $\varphi(\overline{x}) = \varphi(\overline{y})$; then, since φ is injective, we have $\overline{x} = \overline{y}$, and so $\mathbf{x} = \mathbf{y}$. Then, the function $\varphi^{(m,m)}$ is injective. □

Proof of (2). Suppose that g_i is injective, $\forall i \in I \setminus I_m$, and let $x, y \in U$ be such that $\varphi(x) = \varphi(y)$; then, $\forall i \in I \setminus I_m$ we have $g_i(x_{\sigma(i)}) = \varphi_i(x) = \varphi_i(y) = g_i(y_{\sigma(i)})$, from which $x_{\sigma(i)} = y_{\sigma(i)}$; then, if σ is biunique, from Remark 29, we have $\sigma(i) = i$, and so $(x_i : i \in I \setminus I_m) = (y_i : i \in I \setminus I_m)$. This implies that

$$\begin{gathered}\varphi_i^{(m,m)}(x_{i_1},\ldots,x_{i_m}) = \varphi_i(x) - \sum_{j\in I\setminus I_m}\varphi_{ij}(x_j) \\ = \varphi_i(y) - \sum_{j\in I\setminus I_m}\varphi_{ij}(y_j) \\ = \varphi_i^{(m,m)}(y_{i_1},\ldots,y_{i_m}), \\ \forall i \in I_m \Longrightarrow\end{gathered} \tag{32}$$

$$\varphi^{(m,m)}(x_{i_1},\ldots,x_{i_m}) = \varphi^{(m,m)}(y_{i_1},\ldots,y_{i_m}),$$

and so, if $\varphi^{(m,m)}$ is injective, we have $(x_{i_1},\dots,x_{i_m}) = (y_{i_1},\dots,y_{i_m})$; then, $x = y$; that is, φ is injective. □

Proposition 32. *Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function; then:*

(1) *Suppose that φ is biunique, the function $\varphi_i : U \to \mathbf{R}$ is C^1, for any $i \in I_m$, and $\det J_{\varphi^{(m,m)}}(\mathbf{x}) \neq 0$, for any $\mathbf{x} \in U^{(m)}$. Then, the functions g_i, for any $i \in I \setminus I_m$, σ and $\varphi^{(m,m)}$ are biunique.*

(2) *If the functions g_i, for any $i \in I \setminus I_m$, σ and $\varphi^{(m,m)}$ are biunique, then φ is biunique.*

Proof of (1). If φ is biunique, $\forall H \subset I\setminus I_m$ such that $0 < |H| \le 2$; we have $\pi_{I,H}(\varphi(U)) = \mathbf{R}^H \in \tau^{(H)}$; then, from Proposition 31, the functions g_i, $\forall i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique; thus, $\forall i \in I \setminus I_m$, we have $\sigma(i) = i$. Moreover, $\forall i \in I \setminus I_m$, g_i is surjective, since φ is surjective. Furthermore, $\forall \mathbf{y} = (y_{i_1},\dots,y_{i_m}) \in \mathbf{R}^m$, let $(x_j : j \in I \setminus I_m) \in (\prod_{j\in I\setminus I_m} A_j) \cap E_{I\setminus I_m}$ and let $\overline{y} \in E_I$ be such that

$$\begin{aligned} \overline{y}_i &= y_i + \sum_{j\in I\setminus I_m} \varphi_{ij}\left(x_j\right), \quad \forall i \in I_m, \\ \overline{y}_i &= g_i\left(x_i\right), \quad \forall i \in I \setminus I_m; \end{aligned} \tag{33}$$

moreover, let $\overline{x} = \varphi^{-1}(\overline{y}) \in U$, from which $\overline{y}_i = g_i(\overline{x}_i)$, $\forall i \in I \setminus I_m$. Since g_i is injective, we have $\overline{x}_i = x_i$, $\forall i \in I \setminus I_m$; then, $\forall i \in I_m$, we have

$$\begin{aligned} \varphi_i^{(m,m)}\left(\overline{x}_{i_1},\dots,\overline{x}_{i_m}\right) &+ \sum_{j\in I\setminus I_m} \varphi_{ij}\left(x_j\right) = \overline{y}_i \\ &= y_i + \sum_{j\in I\setminus I_m} \varphi_{ij}\left(x_j\right) \Longrightarrow \\ y_i &= \varphi_i^{(m,m)}\left(\overline{x}_{i_1},\dots,\overline{x}_{i_m}\right) \Longrightarrow \\ \mathbf{y} &= \varphi^{(m,m)}\left(\mathbf{x}\right), \end{aligned} \tag{34}$$

where $\mathbf{x} = (\overline{x}_{i_1},\dots,\overline{x}_{i_m}) \in U^{(m)}$; then, $\varphi^{(m,m)}$ is surjective. □

Proof of (2). If the functions g_i, $\forall i \in I \setminus I_m$, σ and $\varphi^{(m,m)}$ are biunique, from Proposition 31, we obtain that φ is injective. Moreover, $\forall y \in E_I$, define $x \in U^{(m)} \times \prod_{j\in I\setminus I_m} A_j$ in the following manner:

$$\begin{aligned} x_i &= g_i^{-1}\left(y_i\right) \in A_i, \quad \forall i \in I \setminus I_m, \\ \left(x_{i_1},\dots,x_{i_m}\right) &= \left(\varphi^{(m,m)}\right)^{-1}\left(z_{i_1},\dots,z_{i_m}\right) \in U^{(m)}, \end{aligned} \tag{35}$$

where

$$z_i = y_i - \sum_{j\in I\setminus I_m} \varphi_{ij}\left(x_j\right), \quad \forall i \in I_m. \tag{36}$$

Let $x_0 = (x_{0,i} : i \in I) \in U$; $\forall i \in I \setminus I_m$, we have

$$\begin{aligned} \left|x_i\right| &= \left|g_i^{-1}\left(y_i\right) - x_{0,i} + x_{0,i}\right| \\ &\le \left|g_i^{-1}\left(y_i\right) - g_i^{-1}\left(g_i\left(x_{0,i}\right)\right)\right| + \left|x_{0,i}\right|; \end{aligned} \tag{37}$$

moreover, the function $g_i^{-1} : \mathbf{R} \to A_i$ is derivable, and

$$\left(g_i^{-1}\right)'(t) = \frac{1}{g_i'\left(g_i^{-1}(t)\right)} \in \mathbf{R}^*, \quad \forall i \in I \setminus I_m,\ \forall t \in \mathbf{R}; \tag{38}$$

then, the Lagrange theorem implies that, for some $\xi_i \in (\min\{y_i, g_i(x_{0,i})\}, \max\{y_i, g_i(x_{0,i})\})$, we have

$$\begin{aligned} &\left|g_i^{-1}\left(y_i\right) - g_i^{-1}\left(g_i\left(x_{0,i}\right)\right)\right| \\ &\quad = \left|\left(g_i^{-1}\right)'\left(\xi_i\right)\right| \left|y_i - g_i\left(x_{0,i}\right)\right|; \end{aligned} \tag{39}$$

thus, from (37) and (38), we obtain

$$\left|x_i\right| \le \frac{\left|y_i - g_i\left(x_{0,i}\right)\right|}{\left|g_i'\left(g_i^{-1}\left(\xi_i\right)\right)\right|} + \left|x_{0,i}\right|. \tag{40}$$

Moreover, we have $\prod_{i\in I\setminus I_m} |g_i'(g_i^{-1}(\xi_i))| = \prod_{i\in \mathcal{I}_\varphi} |g_i'(g_i^{-1}(\xi_i))| \in \mathbf{R}^+$, from which there exists $i_0 \in I \setminus I_m$ such that $\forall i \in I \setminus I_m$, $i > i_0$, we have $|g_i'(g_i^{-1}(\xi_i))| > 1/2$; then, there exists $c \in \mathbf{R}^+$ such that $\sup_{i\in I\setminus I_m} |g_i'(g_i^{-1}(\xi_i))|^{-1} \le c$, and so formula (40) implies

$$\sup_{i\in I\setminus I_m} \left|x_i\right| \le c\left(\|y\|_I + \left\|\varphi\left(x_0\right)\right\|_I\right) + \left\|x_0\right\|_I < +\infty; \tag{41}$$

then, we have $x \in E_I$, from which $x \in U$. Finally, it is easy to prove that $\varphi(x) = y$, and so φ is surjective. □

Remark 33. Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function such that $\varphi_{ij}(x_j) = 0$, for any $x_j \in A_j$, for any $i \in I_m$, and for any $j \in I \setminus I_m$; then,

(1) if φ is injective, and $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \le 2$, then the functions g_i, for any $i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique.

(2) if φ is biunique, then the functions g_i, for any $i \in I\setminus I_m$, $\varphi^{(m,m)}$ and σ are biunique.

Proof of (1). Suppose that φ is injective; by proceeding as in the proof of Proposition 31, we have that the functions g_i, $\forall i \in I \setminus I_m$, $\varphi^{(m,m)}$, and σ are injective. Moreover, σ is surjective; in fact, suppose by contradiction that there exists $n \in (I \setminus I_m) \setminus \sigma(I \setminus I_m)$, and let $x = (x_j : j \in I)$; moreover, $\forall t \in A_n$, $t \neq x_n$, let $\overline{x} = (\overline{x}_j : j \in I) \in U$ be the sequence defined by $\overline{x}_j = x_j$, $\forall j \neq n$, and $\overline{x}_n = t$; we have

$$\begin{aligned} \varphi_i\left(x\right) &= \varphi_i^{(m,m)}\left(x_{i_1},\dots,x_{i_m}\right) = \varphi_i^{(m,m)}\left(\overline{x}_{i_1},\dots,\overline{x}_{i_m}\right) \\ &= \varphi_i\left(\overline{x}\right), \quad \forall i \in I_m, \\ \varphi_i\left(x\right) &= g_i\left(x_{\sigma(i)}\right) = g_i\left(\overline{x}_{\sigma(i)}\right) = \varphi_i\left(\overline{x}\right), \quad \forall i \in I \setminus I_m; \end{aligned} \tag{42}$$

thus, we have $\varphi(x) = \varphi(\overline{x})$, and so φ is not injective (a contradiction). □

Proof of (2). Since φ is surjective, the functions g_i, $\forall i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are surjective; moreover, $\forall H \subset I \setminus I_m$ such that $0 < |H| \le 2$, we have $\pi_{I,H}(\varphi(U)) = \mathbf{R}^H \in \tau^{(H)}$; then, since φ is injective, from point 1, the functions g_i, $\forall i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique. □

Corollary 34. *Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function such that $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, and $\overline{\varphi}$ is injective; then, the functions φ, $\varphi^{(n)}$, and $\varphi^{(n,n)}$, for any $n \in \mathbf{N}$, $n \geq m$, are injective.*

Proof. Observe that $\overline{\varphi}$ is (m,σ)-standard, and $\pi_{I,H}(\overline{\varphi}(U)) = \pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, $\forall H \subset I \setminus I_m$ such that $0 < |H| \leq 2$; then, from Remark 33, we have that the functions g_i, $\forall i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique; then, from Proposition 31, φ is injective; analogously, since $\forall n \in \mathbf{N}$, $n \geq m$, the function $\varphi^{(n)}$ is (n,σ)-standard, from Proposition 31 $\varphi^{(n)}$ is injective, Moreover, we have $\pi_{I,H}(\varphi^{(n)}(U)) = \pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, $\forall H \subset I \setminus I_n$ such that $0 < |H| \leq 2$; then, from Remark 33, $\varphi^{(n,n)} = (\varphi^{(n)})^{(n,n)}$ is injective. □

Proposition 35. *Let $\varphi : U \subset E_I \to E_I$ be a function C^1 in $x_0 = (x_{0,j} : j \in I) \in U$ and (m,σ)-standard. Then, for any $n \geq m$, the function $\varphi^{(n,n)} : \pi_{I,I_n}(U) \to \mathbf{R}^n$ is C^1 in $(x_{0,j} : j \in I_n)$, the function $\varphi^{(n)} : U \subset E_I \to E_I$ is C^1 in x_0, there exists the function $J_{\varphi^{(n)}}(x_0) : E_I \to E_I$, and it is continuous. Moreover, if φ is C^1 in x_0 and strongly (m,σ)-standard, then $\varphi^{(n)}$ is differentiable in x_0. Finally, if φ is strongly C^1 in x_0 and strongly (m,σ)-standard, then φ is differentiable in x_0.*

Proof. By assumption, there exists a neighbourhood $V = \prod_{j\in I} V_j \in \tau_{\|\cdot\|_I}(U)$ of x_0 such that, $\forall i, j \in I$, there exists the function $x \to \partial\varphi_i(x)/\partial x_j$ on V, and this function is continuous in x_0; then, $\forall \mathbf{x} = (x_j : j \in I_n) \in \prod_{j\in I_n} V_j$, let $x \in V$ such that $(x_j : j \in I_n) = \mathbf{x}$; since φ is a (m,σ)-standard function, $\forall i, j \in I_n$, we have

$$\frac{\partial\varphi_i^{(n,n)}(\mathbf{x})}{\partial x_j} = \frac{\partial\varphi_i(x)}{\partial x_j}, \tag{43}$$

from which $\varphi^{(n,n)}$ is C^1 in $(x_{0,j} : j \in I_n)$. Moreover, $\forall x \in V$, we have

$$\frac{\partial\varphi_i^{(n)}(x)}{\partial x_j} = \begin{cases} \dfrac{\partial\varphi_i(x)}{\partial x_j} & \text{if } (i,j) \in (I \times I_m) \cup ((I \setminus I_m) \times (I \setminus I_m)) \\ 0 & \text{if } (i,j) \in I_m \times (I \setminus I_m), \end{cases} \tag{44}$$

and so $\varphi^{(n)}$ is C^1 in x_0. Furthermore, $\forall i \in I$, the function $\varphi_i^{(n)} : U \subset E_I \to \mathbf{R}$ is differentiable in x_0, since $\varphi_i^{(n)}$ depends only on a finite number of variables, and so there exists the function $J_{\varphi_i^{(n)}}(x_0) : E_I \to \mathbf{R}$; moreover, since $\forall i \in I \setminus I_m$ we have $\|J_{\varphi_i^{(n)}}(x_0)\| = |g_i'(x_{0,\sigma(i)})|$ and since the sequence $\{|g_i'(x_{0,\sigma(i)})|\}_{i\in\mathcal{F}_\varphi}$ is convergent, this implies that $\sup_{i\in I\setminus I_m}|g_i'(x_{0,\sigma(i)})| < +\infty$, and so $\sup_{i\in I}\|J_{\varphi_i^{(n)}}(x_0)\| < +\infty$; then, there exists the function $J_{\varphi^{(n)}}(x_0) : E_I \to E_I$, and it is continuous.

Moreover, suppose that φ is C^1 in x_0 and strongly (m,σ)-standard; then, the sequence $\{(-1)^{|i|+|\sigma(i)|} g_i'(x_{\sigma(i)})\}_{i\in\mathcal{F}_\varphi}$ converges uniformly to 1, with respect to $x = (x_j : j \in I \setminus I_m) \in \prod_{j\in I\setminus I_m} A_j$; thus, $\forall\varepsilon > 0$, there exists $\bar{i} \in I \setminus I_m$ such that, $\forall i \in \mathcal{F}_\varphi$, $i > \bar{i}$, and $\forall x \in U$, we have

$$\left|(-1)^{|i|+|\sigma(i)|} g_i'(x_{\sigma(i)}) - 1\right| < \frac{\varepsilon}{2}. \tag{45}$$

Observe that, since $\forall i \leq \bar{i}$ the function $\varphi_i^{(n)}$ is differentiable in x_0, there exists a neighbourhood $N \in \tau_{\|\cdot\|_I}(U)$ of x_0 such that, $\forall x \in N \setminus \{x_0\}$, we have

$$\sup_{i\in I: i\leq\bar{i}} \frac{\left|\varphi_i^{(n)}(x) - \varphi_i^{(n)}(x_0) - J_{\varphi_i^{(n)}}(x_0)(x - x_0)\right|}{\|x - x_0\|_J} < \varepsilon. \tag{46}$$

Moreover, $\forall i \in I$ such that $i > \bar{i}$, g_i is derivable in $A_{\sigma(i)}$ and so, from the Lagrange theorem, there exists $\xi_i \in (\min\{x_{0,\sigma(i)}, x_{\sigma(i)}\}, \max\{x_{0,\sigma(i)}, x_{\sigma(i)}\})$ such that

$$g_i(x_{\sigma(i)}) - g_i(x_{0,\sigma(i)}) = g_i'(\xi_i)(x_{\sigma(i)} - x_{0,\sigma(i)}), \tag{47}$$

from which

$$\begin{aligned} &\frac{\left|\varphi_i^{(n)}(x) - \varphi_i^{(n)}(x_0) - J_{\varphi_i^{(n)}}(x_0)(x - x_0)\right|}{\|x - x_0\|_J} \\ &= \frac{\left|g_i(x_{\sigma(i)}) - g_i(x_{0,\sigma(i)}) - g_i'(x_{0,\sigma(i)})(x_{\sigma(i)} - x_{0,\sigma(i)})\right|}{\|x - x_0\|_J} \\ &= \frac{\left|(-1)^{|i|+|\sigma(i)|} g_i'(\xi_i) - (-1)^{|i|+|\sigma(i)|} g_i'(x_{0,\sigma(i)})\right| \left|x_{\sigma(i)} - x_{0,\sigma(i)}\right|}{\|x - x_0\|_J} \\ &\leq \left(\left|(-1)^{|i|+|\sigma(i)|} g_i'(\xi_i) - 1\right| + \left|(-1)^{|i|+|\sigma(i)|} g_i'(x_{0,\sigma(i)}) - 1\right|\right) \\ &\cdot 1_{\mathcal{F}_\varphi}(i) < \varepsilon. \end{aligned} \tag{48}$$

Then, from (46) and (48), we have

$$\frac{\left\|\varphi^{(n)}(x) - \varphi^{(n)}(x_0) - J_{\varphi^{(n)}}(x_0)(x - x_0)\right\|_I}{\|x - x_0\|_J} < \varepsilon, \tag{49}$$

and so $\varphi^{(n)}$ is differentiable in x_0.

Finally, if φ is strongly C^1 in x_0 and strongly (m,σ)-standard, the function $\psi = \varphi - \overline{\varphi} : U \subset E_I \to E_I$ given by

$$\psi_i(x) = \begin{cases} \displaystyle\sum_{j\in I\setminus I_m} \varphi_{ij}(x_j) & \forall i \in I_m,\ \forall x \in U \\ 0 & \forall i \in I \setminus I_m,\ \forall x \in U \end{cases} \tag{50}$$

is strongly C^1 in x_0, and so it is differentiable in x_0 from Proposition 27; then, since $\overline{\varphi}$ is differentiable in x_0, this is true for $\varphi = \psi + \overline{\varphi}$ too, from Remark 14. □

Proposition 36. *Let $\varphi : U \subset E_I \to E_I$ be a function C^1 and (m,σ)-standard; then, $\varphi : U \to \mathbf{R}^I$ is $(\mathscr{B}^{(I)}(U), \mathscr{B}^{(I)})$-measurable.*

Proof. Let $i \in I$, and let $n \in \mathbf{N}$, $n \geq \max\{|i|, m\}$; from Proposition 35, the function $\varphi_i^{(n,n)} : \pi_{I,I_n}(U) \to \mathbf{R}$ is C^1.

Then, $\forall B \in \tau$, we have $(\varphi_i^{(n,n)})^{-1}(B) \in \tau^{(n)} \subset \mathscr{B}^{(n)}$; moreover, if we consider $\varphi_i^{(n,n)}$ as a function from U to $\mathbf{R}$, we have $\pi_{I,I_n}((\varphi_i^{(n,n)})^{-1}(B)) = \mathbf{R}^{I\setminus I_n}$, and so $(\varphi_i^{(n,n)})^{-1}(B) \in \mathscr{B}^{(I)}$; then, since $\sigma(\tau) = \mathscr{B}$, $\forall B \in \mathscr{B}$, we have $(\varphi_i^{(n,n)})^{-1}(B) \in \mathscr{B}^{(I)}$, and so $(\varphi_i^{(n,n)})^{-1}(B) \in \mathscr{B}^{(I)}(U)$. Moreover, since $\lim_{n\to+\infty}\varphi_i^{(n,n)} = \varphi_i$, the function φ_i is $(\mathscr{B}^{(I)}(U), \mathscr{B})$-measurable. Let

$$\Sigma(I) = \left\{ B = \prod_{i\in I} B_i : B_i \in \mathscr{B},\ \forall i \in I \right\}; \tag{51}$$

$\forall B \in \Sigma(I)$, we have

$$\varphi^{-1}(B) = \bigcap_{i\in I} (\varphi_i)^{-1}(B_i) \in \mathscr{B}^{(I)}(U). \tag{52}$$

Finally, since $\sigma(\Sigma(I)) = \mathscr{B}^{(I)}$, $\forall B \in \mathscr{B}^{(I)}$, we have $\varphi^{-1}(B) \in \mathscr{B}^{(I)}(U)$. □

Proposition 37. *Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function, such that $\overline{\varphi} : U \to \overline{\varphi}(U)$ is a homeomorphism. Then, the functions $\varphi^{(m,m)} : U^{(m)} \to \varphi^{(m,m)}(U^{(m)})$ and $g_i : A_i \to g_i(A_i)$, $\forall i \in I \setminus I_m$, are homeomorphisms, and σ is biunique.*

Proof. Since $\overline{\varphi}$ is a homeomorphism, we have $\overline{\varphi}(U) \in \tau_{\|\cdot\|_I}$; then, from Proposition 6, $\forall H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, we have $\pi_{I,H}(\overline{\varphi}(U)) \in \tau_{\|\cdot\|_H} = \tau^{(H)}$; thus, since $\overline{\varphi}$ is injective, from Remark 33, the functions g_i, $\forall i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique; then, the functions g_i and g_i^{-1}, $\forall i \in I \setminus I_m$, are derivable, and so they are continuous. Moreover, we have $\varphi^{(m,m)}(U^{(m)}) = \pi_{I,I_m}(\overline{\varphi}(U)) \in \tau^{(m)}$, $g_i(A_i) = \pi_{I,\{i\}}(\overline{\varphi}(U)) \in \tau$, $\forall i \in I \setminus I_m$. Finally, from Remark 10, the function $(\pi_{I,I_m} \circ \varphi) : (U, \tau_{\|\cdot\|_I}(U)) \to (\mathbf{R}^m, \tau^{(m)})$ is continuous; then, $\forall B \in \tau^{(m)}$, we have $(\varphi^{(m,m)})^{-1}(B) \times \prod_{j\in I\setminus I_m} A_j = (\pi_{I,I_m} \circ \varphi)^{-1}(B) \in \tau_{\|\cdot\|_I}(U)$, from which $(\varphi^{(m,m)})^{-1}(B) \in \tau^{(m)}(U^{(m)})$; then, $\varphi^{(m,m)}$ is continuous; analogously, we can prove that the function $(\varphi^{(m,m)})^{-1}$ is continuous. □

Proposition 38. *Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function such that $\pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$. Then, $\overline{\varphi} : U \to \overline{\varphi}(U)$ is a diffeomorphism if and only if the functions $\varphi^{(m,m)} : U^{(m)} \to \varphi^{(m,m)}(U^{(m)})$ and $g_i : A_i \to g_i(A_i)$, $\forall i \in I \setminus I_m$, are diffeomorphisms, and σ is biunique.*

Proof. We have $\pi_{I,H}(\overline{\varphi}(U)) = \pi_{I,H}(\varphi(U)) \in \tau^{(H)}$, for any $H \subset I \setminus I_m$ such that $0 < |H| \leq 2$. If $\overline{\varphi}$ is a diffeomorphism, then $\overline{\varphi}$ is injective, and so, from Remark 33, the functions g_i, $\forall i \in I \setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique. Moreover, $\overline{\varphi}$ is C^1 in U, and so, from Proposition 35, $\varphi^{(m,m)} = \overline{\varphi}^{(m,m)}$ is C^1 in $U^{(m)} = \pi_{I,I_m}(U)$; analogously, since $(\overline{\varphi})^{-1} : \overline{\varphi}(U) \to U$ is C^1 in $\overline{\varphi}(U)$, then $(\varphi^{(m,m)})^{-1} = ((\overline{\varphi})^{-1})^{(m,m)}$ is C^1 in $\varphi^{(m,m)}(U^{(m)}) = \pi_{I,I_m}(\overline{\varphi}(U)) \in \tau^{(m)}$; then, $\varphi^{(m,m)}$ is a diffeomorphism. Moreover, $\forall i \in I \setminus I_m$, $\forall x \in A_i$, let $\overline{x} = (\overline{x}_j : j \in I) \in U$ such that $\overline{x}_i = x$; we have $g_i'(x) = (\partial\overline{\varphi}_i/\partial x_i)(\overline{x})$, and so g_i is C^1 in A_i; analogously, $\forall i \in I \setminus I_m$, $\forall y \in g_i(A_i)$, let $\overline{y} = (\overline{y}_j : j \in I) \in \overline{\varphi}(U)$ such that $\overline{y}_i = y$; we have $(g_i^{-1})'(y) = (\partial(\overline{\varphi}^{-1})_i/\partial y_i)(\overline{y})$, and so g_i^{-1} is C^1 in $g_i(A_i) = \pi_{I,\{i\}}(\overline{\varphi}(U)) \in \tau$; then, g_i is a diffeomorphism.

Conversely, if the functions $\varphi^{(m,m)}$ and g_i, $\forall i \in I \setminus I_m$, are diffeomorphisms and σ is biunique, then $\overline{\varphi}$ is injective from Remark 33; moreover, $\forall x = (x_j : j \in I) \in U$, set $\mathbf{x} = (x_j : j \in I_m) \in U^{(m)}$; we have

$$\frac{\partial\overline{\varphi}_i(x)}{\partial x_j} = \begin{cases} \dfrac{\partial\varphi_i^{(m,m)}(\mathbf{x})}{\partial x_j} & \text{if } (i,j) \in (I_m \times I_m) \\ g_i'(x_i) & \text{if } i \in I \setminus I_m,\ j = i \\ 0 & \text{otherwise,} \end{cases} \tag{53}$$

and so $\overline{\varphi}$ is C^1 in U; analogously, $\forall y = (y_j : j \in I) \in \overline{\varphi}(U)$, set $\mathbf{y} = (y_j : j \in I_m) \in \pi_{I,I_m}(\overline{\varphi}(U))$; we have

$$\frac{\partial\left(\overline{\varphi}^{-1}\right)_i(y)}{\partial y_j} = \begin{cases} \dfrac{\partial\left(\varphi^{-1}\right)_i^{(m,m)}(\mathbf{y})}{\partial y_j} & \text{if } (i,j) \in (I_m \times I_m) \\ \left(g_i^{-1}\right)'(y_i) & \text{if } i \in I \setminus I_m,\ j = i \\ 0 & \text{otherwise,} \end{cases} \tag{54}$$

and so $\overline{\varphi}^{-1}$ is C^1 in $\overline{\varphi}(U)$; then, $\overline{\varphi}$ is a diffeomorphism. □

Proposition 39. *Let $\varphi : U \subset E_I \to E_I$ be a (m,σ)-standard function such that $\varphi_i : U \to \mathbf{R}$ is C^1, for any $i \in I_m$; moreover, suppose that $\overline{\varphi}$ is injective and $\overline{\varphi}(U) \in \tau_{\|\cdot\|_I}$; then,*

(1) *if $\varphi^{(m,m)} : (U^{(m)}, \tau^{(m)}(U^{(m)})) \to (\varphi^{(m,m)}(U^{(m)}), \tau^{(m)}(\varphi^{(m,m)}(U^{(m)})))$ is an open function, then, for any $n \in \mathbf{N}$, $n \geq m$, the function $\varphi^{(n,n)} : \pi_{I,I_n}(U) \to \varphi^{(n,n)}(\pi_{I,I_n}(U))$ is a homeomorphism, and one has $\varphi^{(n)}(U) \in \tau_{\|\cdot\|_I}$.*

(2) *if $\overline{\varphi} : U \to \overline{\varphi}(U)$ is a diffeomorphism, then, for any $n \in \mathbf{N}$, $n \geq m$, the functions $\varphi^{(n,n)} : \pi_{I,I_n}(U) \to \varphi^{(n,n)}(\pi_{I,I_n}(U))$ and $\varphi^{(n)} : U \to \varphi^{(n)}(U)$ are diffeomorphisms.*

Proof of (1). Since $\overline{\varphi}(U) \in \tau_{\|\cdot\|_I}$, $\forall H \subset I \setminus I_m$ such that $0 < |H| \leq 2$, we have $\pi_{I,H}(\varphi(U)) = \pi_{I,H}(\overline{\varphi}(U)) \in \tau_{\|\cdot\|_H} = \tau^{(H)}$; then, from Corollary 34, $\forall n \in \mathbf{N}$, $n \geq m$, the function $\varphi^{(n,n)}$ is injective. Moreover, from Remark 33, the functions g_i, $\forall i \in I\setminus I_m$, and $\varphi^{(m,m)}$ are injective, and σ is biunique. Furthermore, from Proposition 35 and the continuity of the functions g_i, $\forall i \in I_n \setminus I_m$, we have that $\varphi^{(n,n)}$ is continuous in $\pi_{I,I_n}(U)$. Finally, $\forall y \in \varphi^{(n,n)}(\pi_{I,I_n}(U))$, let $x = (\varphi^{(n,n)})^{-1}(y) \in \pi_{I,I_n}(U)$; $\forall i \in I_n \setminus I_m$, we have $x_i = g_i^{-1}(y_i)$; then, $\forall i \in I_m$,

$$y_i = \varphi_i^{(m,m)}\left(x_{i_1}, \ldots, x_{i_m}\right) + \sum_{j\in I_n\setminus I_m} \varphi_{ij}\left(g_j^{-1}\left(y_j\right)\right), \tag{55}$$

and so

$$\left(x_{i_1},\dots,x_{i_m}\right)=\left(\varphi^{(m,m)}\right)^{-1}\left(z_{i_1},\dots,z_{i_m}\right),$$
$$x_i=g_i^{-1}\left(y_i\right),\quad \forall i\in I_n\setminus I_m, \tag{56}$$

where

$$z_i=y_i-\sum_{j\in I_n\setminus I_m}\varphi_{ij}\left(g_j^{-1}\left(y_j\right)\right),\quad \forall i\in I_m. \tag{57}$$

Then, since the functions $(\varphi^{(m,m)})^{-1}$, φ_{ij}, and g_j^{-1} are continuous, the function $(\varphi^{(n,n)})^{-1}$ is continuous too, and so $\varphi^{(n,n)}(\pi_{I,I_n}(U))\in\tau^{(n)}$; then, $\varphi^{(n,n)}:\pi_{I,I_n}(U)\to\varphi^{(n,n)}(\pi_{I,I_n}(U))$ is a homeomorphism. Furthermore, since $\overline{\varphi}(U)\in\tau_{\|\cdot\|_I}$, we have $\pi_{I,I\setminus I_n}(\overline{\varphi}(U))\in\tau_{\|\cdot\|_{I\setminus I_n}}$, and so

$$\varphi^{(n)}(U)=\varphi^{(n,n)}\left(\pi_{I,I_n}(U)\right)\times\pi_{I,I\setminus I_n}\left(\overline{\varphi}(U)\right)\in\tau_{\|\cdot\|_I}. \tag{58}$$

□

Proof of (2). If $\overline{\varphi}:U\to\overline{\varphi}(U)$ is a diffeomorphism, from Proposition 38, the functions $\varphi^{(m,m)}:U^{(m)}\to\varphi^{(m,m)}(U^{(m)})$ and $g_i:A_i\to g_i(A_i)$, $\forall i\in I\setminus I_m$, are diffeomorphisms; in particular, the functions $(\varphi^{(m,m)})^{-1}$ and g_i^{-1}, $\forall i\in I_n\setminus I_m$, are C^1; then, since $\forall i\in I_m$, $\forall j\in I_n\setminus I_m$, the function φ_{ij} is C^1, from (56) and (57), we obtain that the function $(\varphi^{(n,n)})^{-1}$ is C^1 in $\varphi^{(n,n)}(\pi_{I,I_n}(U))$. Moreover, from Proposition 35 and since, $\forall i\in I_n\setminus I_m$, the functions g_i are C^1, $\varphi^{(n,n)}$ is C^1 in $\pi_{I,I_n}(U)$, and so $\varphi^{(n,n)}$ is a diffeomorphism. Finally, since $\overline{\varphi}(U)\in\tau_{\|\cdot\|_I}$, $\forall H\subset I\setminus I_n$ such that $0<|H|\le 2$, we have $\pi_{I,H}(\varphi^{(n)}(U))=\pi_{I,H}(\overline{\varphi}(U))\in\tau_{\|\cdot\|_H}=\tau^{(H)}$; then, from Proposition 38, $\varphi^{(n)}:U\to\varphi^{(n)}(U)$ is a diffeomorphism. □

Remark 40. A linear function $A=(a_{ij})_{i,j\in I}:E_I\to E_I$ is (m,σ)-standard, where $m\in\mathbf{N}^*$ and $\sigma:I\setminus I_m\to I\setminus I_m$ is an increasing function, if

(1) $a_{ij}=0$, $\forall(i,j)\notin(I_m\times I)\cup\bigcup_{n\in I\setminus I_m}\{(n,\sigma(n))\}$;

(2) there exists $\prod_{i\in I\setminus I_m:\lambda_i\neq0}\lambda_i\in\mathbf{R}^*$, where $\lambda_i=(-1)^{|i|+|\sigma(i)|}a_{i,\sigma(i)}$, $\forall i\in I\setminus I_m$.

Recall the following concept, defined in [1].

Definition 41. Let $A=(a_{ij})_{i,j\in I}:E_I\to E_I$ be a linear (m,σ)-standard function; define the determinant of A, and call it $\det_{(m,\sigma)}A$, or $\det A$, the real number

$$\det{}_{(m,\sigma)}A=\begin{cases}\det A^{(m,m)}\prod_{i\in I\setminus I_m}\lambda_i & \text{if }\sigma\text{ is biunique}\\ 0 & \text{if }\sigma\text{ is not biunique,}\end{cases} \tag{59}$$

where $A^{(m,m)}$ is the $m\times m$ real matrix defined by $A^{(m,m)}(i,j)=a_{ij}$, $\forall i,j\in I_m$.

Proposition 42. *Let $A=(a_{ij})_{i,j\in I}:E_I\to E_I$ be a linear and injective (m,σ)-standard function; then, A is biunique if and only if A is $(\tau_{\|\cdot\|_I},\tau_{\|\cdot\|_I})$-open.*

Proof. From Remark 9 in [1], A is continuous; then, if A is biunique, from the Banach theorem of the open function, A is $(\tau_{\|\cdot\|_I},\tau_{\|\cdot\|_I})$-open. Conversely, if A is $(\tau_{\|\cdot\|_I},\tau_{\|\cdot\|_I})$-open, $\forall H\subset I\setminus I_m$ such that $0<|H|\le 2$, we have $\pi_{I,H}(A(E_I))\in\tau_{\|\cdot\|_H}=\tau^{(H)}$; moreover, since A is linear, the function $A_i:E_I\to\mathbf{R}$ is C^1, $\forall i\in I_m$; furthermore, by proceeding as in the proof of Proposition 31, we obtain that $A^{(m,m)}$ is injective, and so $\det J_{A^{(m,m)}}(\mathbf{x})\neq0$, $\forall\mathbf{x}\in E_I$; then, from Proposition 31, the functions g_i, $\forall i\in I\setminus I_m$, are injective, and σ is biunique. Thus, since the functions $A^{(m,m)}$ and g_i, $\forall i\in I\setminus I_m$, are linear, they are biunique, and so A is biunique from Proposition 32. □

4. Change of Variables' Formula

Henceforth, we will suppose $|I|=+\infty$.

Definition 43. Let $k\in\mathbf{N}^*$, let $M,N\in\mathbf{R}^+$, let $a=(a_i:i\in I)\in[0,+\infty)^I$ such that $\prod_{i\in I:a_i\neq0}a_i\in\mathbf{R}^+$, and let $v=(v_i:i\in I)\in E_I$; define the following sets in $\mathscr{B}_I$:

$$E_{N,a,v}^{(k,I)}=\mathbf{R}^k\times\prod_{i\in I\setminus I_k}\left(v_i-Na_i,v_i+Na_i\right);$$
$$E_{M,N,a,v}^{(k,I)}=(-M,M)^k\times\prod_{i\in I\setminus I_k}\left(v_i-Na_i,v_i+Na_i\right). \tag{60}$$

Moreover, define the σ-finite measure $\lambda_{N,a,v}^{(k,I)}$ over $(\mathbf{R}^I,\mathscr{B}^{(I)})$ in the following manner:

$$\lambda_{N,a,v}^{(k,I)}=\left(\bigotimes_{i\in I_k}\frac{1}{2N}\mathrm{Leb}\right)\otimes\left(\bigotimes_{i\in I\setminus I_k}\frac{1}{2N}\mathrm{Leb}\left(\cdot\cap\left(v_i-Na_i,v_i+Na_i\right)\right)\right). \tag{61}$$

Proposition 44. *Let $\varphi:U\subset E_I\to E_I$ be a (m,σ)-standard function such that $\overline{\varphi}$ is biunique; moreover, let $N\in\mathbf{R}^+$, let $a=(a_i:i\in I)\in[0,+\infty)^I$ such that $\prod_{i\in I:a_i\neq0}a_i\in\mathbf{R}^+$, and let $v\in E_I$; then,*

(1) *there exist $b=(b_i:i\in I)\in[0,+\infty)^I$ and $z\in E_I$ such that $\prod_{i\in I:b_i\neq0}b_i\in\mathbf{R}^+$ and such that, for any $n,k\in\mathbf{N}$, $n\ge m$, $k\ge m$, one has*

$$\varphi^{-1}\left(E_{N,a,v}^{(k,I)}\right)=\left(\varphi^{(n)}\right)^{-1}\left(E_{N,a,v}^{(k,I)}\right)=E_{N,b,z}^{(k,I)}; \tag{62}$$

(2) *suppose that $\varphi_i:U\to\mathbf{R}$ is C^1, for any $i\in I_m$, and the function $\varphi^{(m,m)}:(U^{(m)},\tau^{(m)}(U^{(m)}))\to(\mathbf{R}^m,\tau^{(m)}$*

is open; then, for any $M_1 \in \mathbf{R}^+$, there exists $M_2 \in \mathbf{R}^+$ such that, for any $n, k \in \mathbf{N}$, $n \geq m$, $k \geq m$, one has

$$\varphi^{-1}\left(E^{(k,I)}_{M_1,N,a,v}\right) \subset E^{(k,I)}_{M_2,N,b,z},$$
$$\left(\varphi^{(n)}\right)^{-1}\left(E^{(k,I)}_{M_1,N,a,v}\right) \subset E^{(k,I)}_{M_2,N,b,z}. \tag{63}$$

Proof of (1). Since $\overline{\varphi}$ is biunique, from Remark 33, the functions g_i, $\forall i \in I \setminus I_m$, and σ are biunique. Set $b = (b_i : i \in I) \in [0,+\infty)^I$, $z = (z_i : i \in I) \in E_I$ such that $b_i = z_i = 1$, $\forall i \in I_m$; moreover, define

$$b_i = \frac{\left|g_i^{-1}(v_i + Na_i) - g_i^{-1}(v_i - Na_i)\right|}{2N}, \quad \forall i \in I \setminus I_m,$$
$$z_i = \frac{g_i^{-1}(v_i + Na_i) + g_i^{-1}(v_i - Na_i)}{2}, \quad \forall i \in I \setminus I_m. \tag{64}$$

Observe that, $\forall i \in I \setminus I_m$, we have $b_i \neq 0$ if and only if $a_i \neq 0$; then, by definition of $b = (b_i : i \in I)$, we have

$$\prod_{i\in I:b_i\neq 0} b_i = \prod_{i\in I:a_i\neq 0} b_i = \left(\prod_{i\in I\setminus I_m:a_i\neq 0}\frac{\left|g_i^{-1}(v_i + Na_i) - g_i^{-1}(v_i - Na_i)\right|}{2Na_i}\right) \cdot \left(\prod_{i\in I\setminus I_m:a_i\neq 0} a_i\right). \tag{65}$$

Moreover, since $\forall i \in I \setminus I_m$ the function g_i^{-1} is derivable on $\mathbf{R}$, if $a_i \neq 0$, the Lagrange theorem implies that, for some $\xi_i \in (v_i - Na_i, v_i + Na_i)$, we have

$$\frac{g_i^{-1}(v_i + Na_i) - g_i^{-1}(v_i - Na_i)}{2Na_i} = \left(g_i^{-1}\right)'(\xi_i) = \frac{1}{g_i'\left(g_i^{-1}(\xi_i)\right)}, \tag{66}$$

from which $\prod_{i\in I:b_i\neq 0} b_i \in \mathbf{R}^+$. Furthermore, $\forall i \in I \setminus I_m$, we have

$$g_i^{-1}\left((v_i - Na_i, v_i + Na_i)\right) = \begin{cases}\left(g_i^{-1}(v_i - Na_i), g_i^{-1}(v_i + Na_i)\right) & \text{if } g_i \text{ is increasing}\\ \left(g_i^{-1}(v_i + Na_i), g_i^{-1}(v_i - Na_i)\right) & \text{if } g_i \text{ is decreasing}\end{cases} = (z_i - Nb_i, z_i + Nb_i), \tag{67}$$

from which

$$g_i\left((z_i - Nb_i, z_i + Nb_i)\right) = (v_i - Na_i, v_i + Na_i). \tag{68}$$

Then, from (67) and (68), $\forall k \in \mathbf{N}$, $k \geq m$, we obtain

$$\varphi^{-1}\left(E^{(k,I)}_{N,a,v}\right) \subset E^{(k,I)}_{N,b,z},$$
$$\varphi\left(E^{(k,I)}_{N,b,z}\right) \subset E^{(k,I)}_{N,a,v} \Longrightarrow$$
$$E^{(k,I)}_{N,b,z} \subset \varphi^{-1}\left(E^{(k,I)}_{N,a,v}\right) \Longrightarrow$$
$$\varphi^{-1}\left(E^{(k,I)}_{N,a,v}\right) = E^{(k,I)}_{N,b,z}. \tag{69}$$

Analogously, $\forall n \in \mathbf{N}$, $n \geq m$, from (67) and (68), we have $(\varphi^{(n)})^{-1}(E^{(k,I)}_{N,a,v}) = E^{(k,I)}_{N,b,z}$. □

Proof of (2). Suppose that $\varphi_i : U \to \mathbf{R}$ is C^1, for any $i \in I_m$, and $\varphi^{(m,m)} : (U^{(m)}, \tau^{(m)}(U^{(m)})) \to (\mathbf{R}^m, \tau^{(m)})$ is an open function; thus, $\forall k \in \mathbf{N}$, $k \geq m$, and $\forall x = (x_i : i \in I) \in \varphi^{-1}(E^{(k,I)}_{M_1,N,a,v})$, let $y = \varphi(x) \in (E^{(k,I)}_{M_1,N,a,v})$; $\forall i \in I \setminus I_m$, we have $x_i = g_i^{-1}(y_i)$; then, $\forall i \in I_m$,

$$y_i = \varphi_i^{(m,m)}\left(x_{i_1}, \ldots, x_{i_m}\right) + \sum_{j\in I\setminus I_m}\varphi_{ij}\left(g_j^{-1}(y_j)\right), \tag{70}$$

and so

$$\left(x_{i_1}, \ldots, x_{i_m}\right) = \left(\varphi^{(m,m)}\right)^{-1}\left(z_{i_1}, \ldots, z_{i_m}\right),$$
$$x_i = g_i^{-1}(y_i), \quad \forall i \in I \setminus I_m, \tag{71}$$

where

$$z_i = y_i - \sum_{j\in I\setminus I_m}\varphi_{ij}\left(g_j^{-1}(y_j)\right), \quad \forall i \in I_m. \tag{72}$$

Moreover, since $\forall i \in I_m$, $\forall j \in I \setminus I_m$, the functions φ_{ij} and g_j^{-1} are continuous and $|y_j| \leq \max\{M_1, \|v\|_{I\setminus I_m} + N\|a\|_{I\setminus I_m}\}$, there exists $\overline{M} \in \mathbf{R}^+$ such that

$$\sup_{i\in I_m}\sum_{j\in I\setminus I_m}\left|\varphi_{ij}\left(g_j^{-1}(y_j)\right)\right| \leq \overline{M}, \tag{73}$$

and so $\|(z_{i_1}, \ldots, z_{i_m})\|_{I_m} \leq M_1 + \overline{M}$; then, since the function $(\varphi^{(m,m)})^{-1}$ is continuous, there exists $\widetilde{M} \in \mathbf{R}^+$ such that $\|(x_{i_1}, \ldots, x_{i_m})\|_{I_m} < \widetilde{M}$; finally, if b, z are the sequences defined by the point (1), by setting $M_2 = \max\{\widetilde{M}, \|z\|_{I\setminus I_m} + N\|b\|_{I\setminus I_m}\}$, from (67), we have $\varphi^{-1}(E^{(k,I)}_{M_1,N,a,v}) \subset E^{(k,I)}_{M_2,N,b,z}$. Analogously, $\forall n \in \mathbf{N}$, $n \geq m$, we have

$$\sup_{i\in I_m}\sum_{j\in I_n\setminus I_m}\left|\varphi_{ij}\left(g_j^{-1}(y_j)\right)\right| \leq \overline{M}, \tag{74}$$

from which $(\varphi^{(n)})^{-1}(E^{(k,I)}_{M_1,N,a,v}) \subset E^{(k,I)}_{M_2,N,b,z}$. □

Proposition 45. *Let (S, Σ) be a measurable space, let $\mathcal{I}$ be a π-system on S, and let μ_1 and μ_2 be two measures on (S, Σ), σ-finite on $\mathcal{I}$; if $\sigma(\mathcal{I}) = \Sigma$ and μ_1 and μ_2 coincide on $\mathcal{I}$, then μ_1 and μ_2 coincide on Σ.*

Proof. See, for example, Theorem 10.3 in Billingsley [10]. □

Lemma 46. *Let $k \in \mathbf{N}^*$, let $N \in \mathbf{R}^+$, let $a = (a_i : i \in I) \in [0,+\infty)^I$ such that $\prod_{i\in I:a_i\neq 0} a_i \in \mathbf{R}^+$, and let $v = (v_i : i \in I) \in E_I$; then, for any measurable function $f : (\mathbf{R}^I, \mathcal{B}^{(I)}) \to (\mathbf{R}, \mathcal{B})$ such that f^+ (or f^-) is $\lambda^{(k,I)}_{N,a,v}$-integrable, one has*

$$\int_{\mathbf{R}^I} f\, d\lambda^{(k,I)}_{N,a,v} = \int_{E^{(k,I)}_{N,a,v}} f\, d\lambda^{(k,I)}_{N,a,v}. \tag{75}$$

Proof. Let $B = \prod_{i\in I} B_i \in \mathscr{B}^{(I)}$; by definition of $\lambda^{(k,I)}_{N,a,v}$, we have

$$\int_{\mathbf{R}^I} 1_B d\lambda^{(k,I)}_{N,a,v} = \int_{E^{(k,I)}_{N,a,v}} 1_B d\lambda^{(k,I)}_{N,a,v}; \tag{76}$$

then, consider the measures $\mu_1 = \lambda^{(k,I)}_{N,a,v}$ and $\mu_2 = \lambda^{(k,I)}_{N,a,v}(\cdot \cap E^{(k,I)}_{N,a,v})$ on $(\mathbf{R}^I, \mathscr{B}^{(I)})$; from (76), μ_1 and μ_2 coincide on the set $\mathscr{I} = \{B \in \mathscr{B}^{(I)} : B = \prod_{i\in I} B_i\}$; moreover, we have $\mathbf{R}^I = \bigcup_{n\in\mathbf{N}}([-n,n]^k \times \mathbf{R}^{I\setminus I_k})$, where $[-n,n]^k \times \mathbf{R}^{I\setminus I_k} \in \mathscr{I}$, $\mu_1([-n,n]^k \times \mathbf{R}^{I\setminus I_k}) = \mu_2([-n,n]^k \times \mathbf{R}^{I\setminus I_k}) < +\infty$, $\forall n \in \mathbf{N}$, and so μ_1 and μ_2 are σ-finite on $\mathscr{I}$. Then, since $\mathscr{I}$ is a π-system on $\mathbf{R}^I$ such that $\sigma(\mathscr{I}) = \mathscr{B}^{(I)}$, from Proposition 45, formula (76) is true $\forall B \in \mathscr{B}^{(I)}$. This implies that, if $\psi : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to ([0,+\infty), \mathscr{B}([0,+\infty)))$ is a simple function, we have

$$\int_{\mathbf{R}^I} \psi\, d\lambda^{(k,I)}_{N,a,v} = \int_{E^{(k,I)}_{N,a,v}} \psi\, d\lambda^{(k,I)}_{N,a,v}. \tag{77}$$

Then, if $f : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to ([0,+\infty), \mathscr{B}([0,+\infty)))$ is a measurable function, and $\{\psi_n\}_{n\in\mathbf{N}}$ is a sequence of increasing positive simple functions over $(\mathbf{R}^I, \mathscr{B}^{(I)})$ such that $\lim_{n\to+\infty}\psi_n = f$, from Beppo Levi theorem we have

$$\begin{aligned}\int_{\mathbf{R}^I} f\, d\lambda^{(k,I)}_{N,a,v} &= \lim_{n\to+\infty} \int_{\mathbf{R}^I} \psi_n d\lambda^{(k,I)}_{N,a,v} \\ &= \lim_{n\to+\infty} \int_{E^{(k,I)}_{N,a,v}} \psi_n d\lambda^{(k,I)}_{N,a,v} \\ &= \int_{E^{(k,I)}_{N,a,v}} f\, d\lambda^{(k,I)}_{N,a,v}.\end{aligned} \tag{78}$$

Then, for any measurable function $f : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to (\mathbf{R}, \mathscr{B})$ such that f^+ (or f^-) is $\lambda^{(k,I)}_{N,a,v}$-integrable,

$$\begin{aligned}\int_{\mathbf{R}^I} f\, d\lambda^{(k,I)}_{N,a,v} &= \int_{\mathbf{R}^I} f^+ d\lambda^{(k,I)}_{N,a,v} - \int_{\mathbf{R}^I} f^- d\lambda^{(k,I)}_{N,a,v} \\ &= \int_{E^{(k,I)}_{N,a,v}} f^+ d\lambda^{(k,I)}_{N,a,v} - \int_{E^{(k,I)}_{N,a,v}} f^- d\lambda^{(k,I)}_{N,a,v} \\ &= \int_{E^{(k,I)}_{N,a,v}} f\, d\lambda^{(k,I)}_{N,a,v}.\end{aligned} \tag{79}$$

□

Now, we can prove the main result of our paper, that improves Theorem 29 in [1] and generalizes the change of variables' formula for the integration of a measurable function on $\mathbf{R}^m$ with values in $\mathbf{R}$ (see, e.g., the Lang's book [9]).

Theorem 47 (change of variables' formula). *Let $\varphi : U \subset E_I \to E_I$ be a C^1 and (m,σ)-standard function, such that the function $\overline{\varphi} : U \to E_I$ is a diffeomorphism; moreover, let $N \in \mathbf{R}^+$, let $a = (a_i : i \in I) \in [0,+\infty)^I$ such that $\prod_{i\in I: a_i\neq 0} a_i \in \mathbf{R}^+$, let $v \in E_I$, and let $b \in [0,+\infty)^I$ and $z \in E_I$ defined by Proposition 44. Then, for any $k \in \mathbf{N}$, $k \geq m$, for any $B \in \mathscr{B}^{(I)}$, and for any measurable function $f : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to (\mathbf{R}, \mathscr{B})$ such that f^+ (or f^-) is $\lambda^{(k,I)}_{N,a,v}$-integrable, one has*

$$\int_B f\, d\lambda^{(k,I)}_{N,a,v} = \int_{\varphi^{-1}(B)} f(\varphi) \left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}. \tag{80}$$

Proof. Let $n, k \in \mathbf{N}$, $n \geq k \geq m$; moreover, since $\overline{\varphi}$ is biunique, $\forall M_1, N \in \mathbf{R}^+$, $\forall a = (a_i : i \in I) \in [0,+\infty)^I$ such that $\prod_{i\in I: a_i\neq 0} a_i \in \mathbf{R}^+$, and $\forall v \in E_I$, let $M_2 \in \mathbf{R}^+$ and let b and z be the sequences defined by Proposition 44. Then, $\forall B = \prod_{i\in I} B_i \in \mathscr{B}^{(I)}(E^{(k,I)}_{M_1,N,a,v})$, we have $B_i \in \mathscr{B}((v_i - Na_i, v_i + Na_i))$ and $g_i^{-1}(B_i) \in \mathscr{B}((z_i - Nb_i, z_i + Nb_i))$, $\forall i \in I \setminus I_k$, and so

$$\begin{aligned}\int_B d\lambda^{(k,I)}_{N,a,v} &= \int_{B_{i_1}\times\cdots\times B_{i_k}\times\prod_{q\in I\setminus I_k} B_q} d\Bigg(\Bigg(\bigotimes_{p=1}^{k} \frac{1}{2N}\mathrm{Leb}\Bigg) \\ &\quad \otimes \Bigg(\bigotimes_{q\in I\setminus I_k} \frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((v_q - Na_q, v_q + Na_q))}\Bigg)\Bigg) \\ &= \int_{B_{i_1}\times\cdots\times B_{i_n}\times\prod_{q\in I\setminus I_n} B_q} d\Bigg(\Bigg(\bigotimes_{p=1}^{n} \frac{1}{2N}\mathrm{Leb}\Bigg) \\ &\quad \otimes \Bigg(\bigotimes_{q\in I\setminus I_n} \frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((v_q - Na_q, v_q + Na_q))}\Bigg)\Bigg) \\ &= \int_{B_{i_1}\times\cdots\times B_{i_n}} d\Bigg(\bigotimes_{p=1}^{n} \frac{1}{2N}\mathrm{Leb}\Bigg) \\ &\quad \cdot \int_{\prod_{q\in I\setminus I_n} B_q} d\Bigg(\bigotimes_{q\in I\setminus I_n} \frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((v_q - Na_q, v_q + Na_q))}\Bigg).\end{aligned} \tag{81}$$

Moreover, we have

$$\begin{aligned}&\int_{\prod_{q\in I\setminus I_n} B_q} d\Bigg(\bigotimes_{q\in I\setminus I_n} \frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((v_q - Na_q, v_q + Na_q))}\Bigg) \\ &\quad = \int_{\prod_{q\in I\setminus I_n} B_q} d\Bigg(\bigotimes_{q\in I\setminus I_n} \frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}(B_q)}\Bigg) \\ &\quad = \lim_{p\to+\infty} \int_{\prod_{q\in I_p\setminus I_n} B_q} d\Bigg(\bigotimes_{q\in I_p\setminus I_n} \frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}(B_q)}\Bigg) \\ &\qquad \text{(by Theorem 1)} \\ &\quad = \lim_{p\to+\infty} \int_{\prod_{q\in I_p\setminus I_n} g_q^{-1}(B_q)} \prod_{q\in I_p\setminus I_n} \left|\det J_{g_q}(x_q)\right| \\ &\qquad \cdot d\Bigg(\bigotimes_{q\in I_p\setminus I_n} \frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}(g_q^{-1}(B_q))}\Bigg)(x_{I_p\setminus I_n}) \\ &\qquad \text{(by Proposition 38)}\end{aligned}$$

$$= \lim_{p\to+\infty} \frac{1}{\prod_{q\in I\setminus I_p}\left|\det J_{g_q}\left(x_q\right)\right|} \cdot \int_{\prod_{q\in I_p\setminus I_n} g_q^{-1}(B_q)} \prod_{q\in I\setminus I_n}\left|\det J_{g_q}\left(x_q\right)\right| \cdot d\left(\bigotimes_{q\in I_p\setminus I_n}\frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}(g_q^{-1}(B_q))}\right)\left(x_{I_p\setminus I_n}\right)$$
$$= \int_{\prod_{q\in I\setminus I_n} g_q^{-1}(B_q)} \prod_{q\in I\setminus I_n}\left|\det J_{g_q}\left(x_q\right)\right| \cdot d\left(\bigotimes_{q\in I\setminus I_n}\frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}(g_q^{-1}(B_q))}\right)\left(x_{I\setminus I_n}\right)$$
(by Corollary 3)
$$= \int_{\prod_{q\in I\setminus I_n} g_q^{-1}(B_q)} \prod_{q\in I\setminus I_n}\left|\det J_{g_q}\left(x_q\right)\right| \cdot d\left(\bigotimes_{q\in I\setminus I_n}\frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((z_q-Nb_q,z_q+Nb_q))}\right)\left(x_{I\setminus I_n}\right). \tag{82}$$

Then, from Proposition 39, formula (81) implies

$$\begin{aligned}\int_B d\lambda_{N,a,v}^{(k,I)} &= \int_{(\varphi^{(n,n)})^{-1}(B_{i_1}\times\cdots\times B_{i_n})}\left|\det J_{\varphi^{(n,n)}}\right| d\left(\bigotimes_{p=1}^{n}\frac{1}{2N}\mathrm{Leb}\right)\cdot\int_{\prod_{q\in I\setminus I_n} g_q^{-1}(B_q)}\prod_{q\in I\setminus I_n}\left|\det J_{g_q}\left(x_q\right)\right| \\ &\quad\cdot d\left(\bigotimes_{q\in I\setminus I_n}\frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((z_q-Nb_q,z_q+Nb_q))}\right)\left(x_{I\setminus I_n}\right) \\ &= \int_{(\varphi^{(n)})^{-1}(B)}\left|\det J_{\varphi^{(n)}}\right| d\left(\left(\bigotimes_{p=1}^{n}\frac{1}{2N}\mathrm{Leb}\right)\otimes\left(\bigotimes_{q\in I\setminus I_n}\frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((z_q-Nb_q,z_q+Nb_q))}\right)\right) \\ &= \int_{(\varphi^{(n)})^{-1}(B)}\left|\det J_{\varphi^{(n)}}\right| d\left(\left(\bigotimes_{p=1}^{k}\frac{1}{2N}\mathrm{Leb}\right)\otimes\left(\bigotimes_{q\in I\setminus I_k}\frac{1}{2N}\mathrm{Leb}\Big|_{\mathscr{B}((z_q-Nb_q,z_q+Nb_q))}\right)\right) \\ &\qquad\left(\text{since }\left(\varphi^{(n)}\right)^{-1}(B)\subset E_{N,b,z}^{(k,I)}\right) \\ &= \int_{(\varphi^{(n)})^{-1}(B)}\left|\det J_{\overline{\varphi}}\right| d\lambda_{N,b,z}^{(k,I)}.\end{aligned} \tag{83}$$

Consider the measures μ_1 and μ_2 on $\Sigma \equiv \mathscr{B}^{(I)}(E_{M_1,N,a,v}^{(k,I)})$ defined by

$$\mu_1(B) = \int_B d\lambda_{N,a,v}^{(k,I)}, \qquad \mu_2(B) = \int_{(\varphi^{(n)})^{-1}(B)}\left|\det J_{\overline{\varphi}}\right| d\lambda_{N,b,z}^{(k,I)}; \tag{84}$$

from (83), μ_1 and μ_2 coincide on the set

$$\mathscr{I} = \left\{B\in\Sigma : B = \prod_{i\in I} B_i\right\}; \tag{85}$$

moreover, we have $\mu_1(E_{M_1,N,a,v}^{(k,I)}) = \mu_2(E_{M_1,N,a,v}^{(k,I)}) < +\infty$, $E_{M_1,N,a,v}^{(k,I)} \in \mathscr{I}$, and so μ_1 and μ_2 are σ-finite on $\mathscr{I}$. Then, since $\mathscr{I}$ is a π-system on $E_{M_1,N,a,v}^{(k,I)}$ such that $\sigma(\mathscr{I}) = \Sigma$, from Proposition 45, $\forall B \in \mathscr{B}^{(I)}(E_{M_1,N,a,v}^{(k,I)})$, we have

$$\int_B d\lambda_{N,a,v}^{(k,I)} = \int_{(\varphi^{(n)})^{-1}(B)}\left|\det J_{\overline{\varphi}}\right| d\lambda_{N,b,z}^{(k,I)}; \tag{86}$$

moreover, since $(\varphi^{(n)})^{-1}(B) \subset E_{M_2,N,b,z}^{(k,I)}$ from Proposition 44, we have

$$\int_{E_{M_1,N,a,v}^{(k,I)}} 1_B d\lambda_{N,a,v}^{(k,I)} = \int_{E_{M_2,N,b,z}^{(k,I)}} 1_B\left(\varphi^{(n)}\right)\left|\det J_{\overline{\varphi}}\right| d\lambda_{N,b,z}^{(k,I)}. \tag{87}$$

This implies that, if $\psi : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to ([0,+\infty), \mathscr{B}([0,+\infty)))$ is a simple function such that $\psi(x) = 0$, $\forall x \notin E_{M_1,N,a,v}^{(k,I)}$, we have

$$\int_{E_{M_1,N,a,v}^{(k,I)}} \psi\, d\lambda_{N,a,v}^{(k,I)} = \int_{E_{M_2,N,b,z}^{(k,I)}} \psi\left(\varphi^{(n)}\right)\left|\det J_{\overline{\varphi}}\right| d\lambda_{N,b,z}^{(k,I)}. \tag{88}$$

Then, if $l : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to ([0,+\infty), \mathscr{B}([0,+\infty)))$ is a measurable function such that $l(x) = 0$, $\forall x \notin E_{M_1,N,a,v}^{(k,I)}$, and $\{\psi_i\}_{i\in\mathbf{N}}$ is a sequence of increasing positive simple functions over $(\mathbf{R}^I, \mathscr{B}^{(I)})$ such that $\lim_{i\to+\infty}\psi_i = l$, $\psi_i(x) = 0$, $\forall x \notin E^{(k,I)}_{M_1,N,a,v}$, $\forall i \in \mathbf{N}$, from Beppo Levi theorem we have

$$\begin{aligned}\int_{E^{(k,I)}_{M_1,N,a,v}} l\,d\lambda^{(k,I)}_{N,a,v} &= \lim_{i\to+\infty}\int_{E^{(k,I)}_{M_1,N,a,v}} \psi_i d\lambda^{(k,I)}_{N,a,v} \\ &= \lim_{i\to+\infty}\int_{E^{(k,I)}_{M_2,N,b,z}} \psi_i\left(\varphi^{(n)}\right)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z} \\ &= \int_{E^{(k,I)}_{M_2,N,b,z}} l\left(\varphi^{(n)}\right)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z} \\ &= \lim_{n\to+\infty}\int_{E^{(k,I)}_{M_2,N,b,z}} l\left(\varphi^{(n)}\right)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}.\end{aligned} \tag{89}$$

In particular, formula (89) is true if l is a function of the form

$$l(x) = l^{(m)}\left(x_{i_1},\dots,x_{i_k}\right)\prod_{j\in I\setminus I_k} 1_{B_j}\left(x_j\right), \quad \forall x = \left(x_j : j\in I\right)\in\mathbf{R}^I, \tag{90}$$

where $l^{(k)} : \mathbf{R}^k \to [0,1]$ is a continuous function such that $l^{(k)}(\mathbf{x}) = 0$, $\forall\mathbf{x}\notin(-M_1,M_1)^k$, and $B_j \in \mathscr{B}((v_j - Na_j, v_j + Na_j))$, $\forall j\in I\setminus I_k$. In this case, let $\{f_n\}_{n\ge k}$ be the sequence of the measurable functions $f_n : (E^{(k,I)}_{M_2,N,b,z}, \mathscr{B}^{(I)}(E^{(k,I)}_{M_2,N,b,z})) \to ([0,+\infty), \mathscr{B}([0,+\infty)))$ given by

$$f_n(x) = l\left(\varphi^{(n)}(x)\right)\left|\det J_{\overline{\varphi}}(x)\right|, \quad \forall x\in E^{(k,I)}_{M_2,N,b,z},\ \forall n\ge k; \tag{91}$$

$\forall n \ge k$, we have $|f_n| \le g$, where $g : (E^{(k,I)}_{M_2,N,b,z}, \mathscr{B}^{(I)}(E^{(k,I)}_{M_2,N,b,z})) \to ([0,+\infty), \mathscr{B}([0,+\infty)))$ is the measurable function defined by

$$g(x) = \left|\det J_{\overline{\varphi}}(x)\right|, \quad \forall x\in E^{(k,I)}_{M_2,N,b,z}. \tag{92}$$

Moreover, we have

$$g(x) = \left|\det J_{\varphi^{(m,m)}}\left(x_{i_1},\dots,x_{i_k}\right)\right|\prod_{j\in I\setminus I_k}\left|g'_j\left(x_j\right)\right|, \quad \forall x\in E^{(k,I)}_{M_2,N,b,z}; \tag{93}$$

then, by definition of $E^{(k,I)}_{M_2,N,b,z}$ and since the functions $J_{\varphi^{(m,m)}}$ and g'_j, $\forall j\in I\setminus I_m$, are continuous, there exists $\beta\in\mathbf{R}^+$ such that $g(x)\le\beta$, $\forall x\in E^{(k,I)}_{M_2,N,b,z}$, and so

$$\begin{aligned}\int_{E^{(k,I)}_{M_2,N,b,z}} g\,d\lambda^{(k,I)}_{N,b,z} &\le \beta\lambda^{(k,I)}_{N,b,z}\left(E^{(k,I)}_{M_2,N,b,z}\right) = \beta\prod_{p\in I_k}\frac{1}{2N} \\ &\cdot \operatorname{Leb}\left((-M_2,M_2)\right) \\ &\cdot \prod_{q\in I\setminus I_k}\frac{1}{2N}\operatorname{Leb}\left(\left(z_q - Nb_q, z_q + Nb_q\right)\right) = \frac{\beta M_2^k}{N^k} \\ &\cdot \prod_{q\in I\setminus I_k} b_q < +\infty.\end{aligned} \tag{94}$$

Moreover, we have $\lim_{n\to+\infty}\varphi^{(n)} = \varphi$, and so $\lim_{n\to+\infty}f_n(x) = l(\varphi(x))|\det J_{\overline{\varphi}}(x)|$, $\forall x\in E^{(k,I)}_{M_2,N,b,z}$; then, from the dominated convergence theorem:

$$\begin{aligned}\lim_{n\to+\infty}\int_{E^{(k,I)}_{M_2,N,b,z}} l\left(\varphi^{(n)}\right)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z} \\ = \int_{E^{(k,I)}_{M_2,N,b,z}} l(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z};\end{aligned} \tag{95}$$

consequently, from (89), we have

$$\int_{E^{(k,I)}_{M_1,N,a,v}} l\,d\lambda^{(k,I)}_{N,a,v} = \int_{E^{(k,I)}_{M_2,N,b,z}} l(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}. \tag{96}$$

Let $B = \prod_{j\in I} B_j \in \mathscr{B}^{(I)}(E^{(k,I)}_{M_1,N,a,v})$, where $B_j = (a_j, b_j)$, $\forall j\in I_k$; moreover, $\forall n\in\mathbf{N}^*$, consider the function $l_n : \mathbf{R}^I \to [0,1]$ defined by

$$l_n(x) = l_n^{(k)}\left(x_j : j\in I_k\right)\prod_{j\in I\setminus I_k} 1_{B_j}\left(x_j\right), \quad \forall x = \left(x_j : j\in I\right)\in\mathbf{R}^I, \tag{97}$$

where

$$l_n^{(k)}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x}\in\prod_{j\in I_k}\left(a_j + \dfrac{\delta_j}{n}, b_j - \dfrac{\delta_j}{n}\right) \\ \dfrac{\|\mathbf{x}-\mathbf{x}_2\|_{I_m}}{\|\mathbf{x}_1-\mathbf{x}_2\|_{I_m}} & \text{if } \mathbf{x}\in\left(\prod_{j\in I_k} B_j\right)\setminus\prod_{j\in I_k}\left(a_j + \dfrac{\delta_j}{n}, b_j - \dfrac{\delta_j}{n}\right) \\ 0 & \text{if } \mathbf{x}\notin\prod_{j\in I_k} B_j, \end{cases} \tag{98}$$

$\delta_j = (b_j - a_j)/2$, $\forall j\in I_k$, $\mathbf{x} = (x_j : j\in I_k)$, $\mathbf{x}_1 = r\cap\partial(\prod_{j\in I_k}(a_j + \delta_j/n, b_j - \delta_j/n))$, $\mathbf{x}_2 = r\cap\partial(\prod_{j\in I_k} B_j)$, where r is the half-line with initial point $\prod_{j\in I_k}((a_j + b_j)/2)$ and containing $\mathbf{x}$. Since $\{l_n\}_{n\in\mathbf{N}^*}$ is an increasing positive sequence such that $\lim_{n\to+\infty} l_n = 1_B$, and since the function $l_n^{(k)} : \mathbf{R}^m \to [0,1]$ is continuous, from Beppo Levi theorem and (96), we have

$$\begin{aligned}\int_{E^{(k,I)}_{M_1,N,a,v}} 1_B d\lambda^{(k,I)}_{N,a,v} &= \lim_{n\to+\infty}\int_{E^{(k,I)}_{M_1,N,a,v}} l_n d\lambda^{(k,I)}_{N,a,v} \\ &= \lim_{n\to+\infty}\int_{E^{(k,I)}_{M_2,N,b,z}} l_n(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z} \\ &= \int_{E^{(k,I)}_{M_2,N,b,z}} 1_B(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}.\end{aligned} \tag{99}$$

Moreover, Proposition 45 implies that formula (99) is true $\forall B\in\mathscr{B}^{(I)}(E^{(k,I)}_{M_1,N,a,v})$. Consider the measures μ and υ on $(E^{(k,I)}_{N,a,v}, \mathscr{B}^{(I)}(E^{(k,I)}_{N,a,v}))$ defined by

$$\begin{aligned}\mu(B) &= \int_B d\lambda^{(k,I)}_{N,a,v}, \\ \upsilon(B) &= \int_{\varphi^{-1}(B)}\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z},\end{aligned} \tag{100}$$

and set $B_n = B \cap E^{(k,I)}_{n,N,a,v}$, $\forall n \in \mathbf{N}^*$ and $\forall B \in \mathscr{B}^{(I)}(E^{(k,I)}_{N,a,v})$. Since $B_n \subset B_{n+1}$, $A^{-1}(B_n) \subset A^{-1}(B_{n+1})$, $\bigcup_{n\in\mathbf{N}^*} B_n = B$ and $\bigcup_{n\in\mathbf{N}^*} A^{-1}(B_n) = A^{-1}(B)$, from the continuity property of μ and v and (99), we have

$$\begin{aligned}\int_B d\lambda^{(k,I)}_{N,a,v} &= \lim_{n\to+\infty}\int_{B_n} d\lambda^{(k,I)}_{N,a,v}\\ &= \lim_{n\to+\infty}\int_{\varphi^{-1}(B_n)} \left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}\\ &= \int_{\varphi^{-1}(B)} \left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}.\end{aligned} \tag{101}$$

Then, suppose that $B \in \mathscr{B}^{(I)}$; from Lemma 46, formula (101), and Proposition 44, we have

$$\begin{aligned}\int_{\mathbf{R}^I} 1_B d\lambda^{(k,I)}_{N,a,v} &= \int_{B\cap E^{(k,I)}_{N,a,v}} d\lambda^{(k,I)}_{N,a,v}\\ &= \int_{\varphi^{-1}(B\cap E^{(k,I)}_{N,a,v})} \left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}\\ &= \int_{\varphi^{-1}(B)\cap E^{(k,I)}_{N,b,z}} \left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}\\ &= \int_{\mathbf{R}^I} 1_B(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z};\end{aligned} \tag{102}$$

thus, by proceeding as in the proof of formula (89), for any measurable function $f : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to ([0,+\infty), \mathscr{B}([0,+\infty)))$, we obtain

$$\int_{\mathbf{R}^I} f\, d\lambda^{(k,I)}_{N,a,v} = \int_{\mathbf{R}^I} f(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}. \tag{103}$$

Then, for any measurable function $f : (\mathbf{R}^I, \mathscr{B}^{(I)}) \to (\mathbf{R}, \mathscr{B})$ such that f^+ (or f^-) is $\lambda^{(k,I)}_{N,a,v}$-integrable and for any $B \in \mathscr{B}^{(I)}$, we have

$$\begin{aligned}\int_B f\, d\lambda^{(k,I)}_{N,a,v} &= \int_{\mathbf{R}^I} f^+ 1_B d\lambda^{(k,I)}_{N,a,v} - \int_{\mathbf{R}^I} f^- 1_B d\lambda^{(k,I)}_{N,a,v}\\ &= \int_{\mathbf{R}^I} (f^+ 1_B)(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}\\ &\quad - \int_{\mathbf{R}^I} (f^- 1_B)(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}\\ &= \int_{\mathbf{R}^I} (f 1_B)(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}\\ &= \int_{\varphi^{-1}(B)} f(\varphi)\left|\det J_{\overline{\varphi}}\right| d\lambda^{(k,I)}_{N,b,z}.\end{aligned} \tag{104}$$

□

5. Problems for Further Study

A natural extension of this paper is the generalization of Theorem 47, by substituting the (m,σ)-standard functions for more general functions $\varphi : U \subset E_I \to E_I$ such that, for any $i \in I \setminus I_m$, the function $\varphi_i : U \subset E_I \to \mathbf{R}$ depends on a finite number of variables.

Moreover, a natural application of this paper, in the probabilistic framework, is the development of the theory of the infinite-dimensional continuous random elements, defined in the paper [1]. In particular, we can prove the formula of the density of such random elements composed with the (m,σ)-standard functions given in the change of variables' formula in Theorem 47. Consequently, it is possible to introduce many random elements that generalize the well-known continuous random vectors in $\mathbf{R}^m$ (e.g., the Gaussian random elements in E_I defined by the (m,σ)-standard matrices) and to develop some theoretical results and some applications in the statistical inference. It is possible also to define a convolution between the laws of two independent and infinite-dimensional continuous random elements, as in the finite case.

Furthermore, we can generalize paper [11] by considering the recursion $\{X_n\}_{n\in\mathbf{N}}$ on $[0,p)^{\mathbf{N}^*}$ defined by

$$X_{n+1} = AX_n + B_n \pmod p, \tag{105}$$

where $X_0 = x_0 \in E_I$, A is a biunique, linear, integer and (m,σ)-standard function, $p \in \mathbf{R}^+$, and $\{B_n\}_{n\in\mathbf{N}}$ is a sequence of independent and identically distributed random elements on E_I. Our target is to prove that, with some assumptions on the law of B_n, the sequence $\{X_n\}_{n\in\mathbf{N}}$ converges with geometric rate to a random element with law $\bigotimes_{i\in\mathbf{N}^*}((1/p)\mathrm{Leb}|_{\mathscr{B}([0,p))})$. Moreover, we wish to quantify the rate of convergence in terms of A, p, m, and the law of B_n, and to prove that, if A has an eigenvalue that is a root of 1, then $O(p^2)$ steps are necessary to achieve randomness.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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