

BIFURCATION OF POSITIVE SOLUTIONS FOR A ONE-DIMENSIONAL INDEFINITE QUASILINEAR NEUMANN PROBLEM

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ABSTRACT. We study the structure of the set of the positive regular solutions of the one-dimensional quasilinear Neumann problem involving the curvature operator

$$-\left(u'/\sqrt{1+(u')^2}\right)' = \lambda a(x)f(u), \quad u'(0) = 0, \quad u'(1) = 0.$$

Here $\lambda \in \mathbb{R}$ is a parameter, $a \in L^1(0, 1)$ changes sign, and $f \in C(\mathbb{R})$. We focus on the case where the slope of f at 0, $f'(0)$, is finite and non-zero, and the potential of f is superlinear at infinity, but also the two limiting cases where $f'(0) = 0$, or $f'(0) = +\infty$, are discussed. We investigate, in some special configurations, the possible development of singularities and the corresponding appearance in this problem of bounded variation solutions.

1. INTRODUCTION

The main goal of this paper is analyzing the positive regular solutions of the quasilinear Neumann problem

$$\begin{cases} -\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x)f(u), & 0 < x < 1, \\ u'(0) = u'(1) = 0, \end{cases} \quad (1.1)$$

where $\lambda \in \mathbb{R}$ is a parameter, $a \in L^1(0, 1)$, and $f \in C(\mathbb{R})$. By a regular solution we mean a function $u \in W^{2,1}(0, 1)$ which satisfies the equation a.e. in $(0, 1)$ and the Neumann conditions $u'(0) = u'(1) = 0$. We also assume that the weight a changes sign, f vanishes at 0 and is strictly increasing, and the potential of f ,

$$F(u) := \int_0^u f(s) ds, \quad (1.2)$$

is superlinear at infinity. As we shall see, under these assumptions on f , the existence of a positive solution of (1.1) entails that the sign of a must change. This research is also motivated by the large amount of studies devoted to the existence of positive solutions for semilinear elliptic problems with indefinite nonlinearities, that started nearly three decades ago with [6, 1, 2, 8, 7, 3] and since then have had a tremendous development in several different directions.

Date: September 27, 2016.

2010 *Mathematics Subject Classification.* Primary: 35J93, 34B18; Secondary: 35B32, 34A47, 34B15, 35B36.

Key words and phrases. quasilinear equation, prescribed curvature equation, Neumann boundary condition, indefinite weight, positive solution, regular or singular solution, bounded variation solution, local and global bifurcation, topological degree, critical point theory.

The first and the second named authors have been supported by “The Spanish Ministry of Economy and Competitiveness of Spain under Research Grants MTM2012-30669 and MTM2015-65899-P”, and by the IMI of Complutense University. The second named author has been also supported by “Università degli Studi di Trieste – Finanziamento di Ateneo per Progetti di Ricerca Scientifica – FRA 2015”. This paper was written under the auspices of INdAM-GNAMPA.

This problem is a special one-dimensional counterpart of the elliptic problem

$$\begin{cases} -\operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = h(x, u), & \text{in } \Omega, \\ -\frac{\nabla u \cdot \nu}{\sqrt{1 + |\nabla u|^2}} = \sigma, & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

which plays a relevant role in the mathematical analysis of various physical or geometrical issues, such as when describing capillarity phenomena for incompressible fluids, or modeling reaction-diffusion processes where the flux response to an increase of gradients slows down and ultimately approaches saturation at large gradients, or studying prescribed mean curvature problems for cartesian graphs in the Euclidean space; significant references related to these topics include [34, 51, 9, 16, 25, 21, 29, 27, 24, 30, 31, 28, 33, 13].

It is a well established fact that introducing the mean curvature operator

$$-\frac{1}{N} \operatorname{div} \left(\nabla u / \sqrt{1 + |\nabla u|^2} \right)$$

determines a deep impact on the morphology of the solution patterns of elliptic problems, the most notable of which is the possibility of discontinuous equilibrium states [33, 10, 11, 45, 13, 47, 18, 17]. Accordingly, the space of bounded variation solutions is usually considered as an appropriate framework where settling problem (1.3), and hence a suitable notion of solution, involving a variational inequality, has been introduced and systematically used in, e.g., [44, 35, 45, 46, 47, 48, 49, 18, 41]. It was also noticed in [46] that, by virtue of the results in [5], such definition, when referred to (1.1), can be reformulated as follows: a function $u \in BV(0, 1)$ is a bounded variation (BV, for short) solution of (1.1) if

$$\int_0^1 \frac{(Du)^a (D\phi)^a}{\sqrt{1 + |(Du)^a|^2}} dx + \int_0^1 \operatorname{sgn} \left(\frac{Du}{|Du|} \right) \frac{D\phi}{|D\phi|} |D\phi|^s = \int_0^1 a f(u) \phi dx,$$

for all $\phi \in BV(0, 1)$ such that $|D\phi|^s$ is absolutely continuous with respect to $|Du|^s$. Here, for any given $v \in BV(0, 1)$,

$$Dv = (Dv)^a dx + (Dv)^s$$

is the Lebesgue-Nikodym decomposition of the measure Dv , the distributional derivative of v , in its absolutely continuous part $(Dv)^a dx$, with density function $(Dv)^a$, and its singular part, $(Dv)^s$, with respect to the Lebesgue measure in \mathbb{R} . If $|Dv|$ denotes the absolute variation of Dv ,

$$|Dv| = |Dv|^a dx + |Dv|^s$$

is the Lebesgue-Nikodym decomposition of $|Dv|$; in addition, $\frac{Dv}{|Dv|}$ stands for the density function of Dv with respect to its absolute variation $|Dv|$. We refer to [4] for additional information about bounded variation functions.

In strong contrast with the semilinear case, no result can be found in the available literature concerning the existence of positive solutions of (1.3) in the presence of indefinite superlinear nonlinearities. Therefore, in our recent paper [41] we began this study, starting from the simplest prototype problem (1.1), and providing several existence and multiplicity results in the frame of bounded variation solutions, under various configurations at 0 and at infinity of the potential F of f . By using variational methods, in [41] we proved among others the following theorem.

Theorem 1.1. *Assume that*

- $a \in L^1(0, 1)$ is such that $\int_0^1 a dx < 0$ and $a(x) > 0$ a.e. on an interval $K \subset [0, 1]$,
- the function a changes sign finitely many times in $(0, 1)$, in the sense that there is a decomposition

$$[0, 1] = \bigcup_{i=1}^k [\alpha_i, \beta_i], \quad \text{with } \alpha_i < \beta_i = \alpha_{i+1} < \beta_{i+1}, \quad \text{for } i = 1, \dots, k-1,$$

such that

$$(-1)^i a(x) \geq 0 \quad \text{a.e. in } (\alpha_i, \beta_i), \quad \text{for } i = 1, \dots, k,$$

or

$$(-1)^i a(x) \leq 0 \quad \text{a.e. in } (\alpha_i, \beta_i), \quad \text{for } i = 1, \dots, k.$$

- $f \in C^1(\mathbb{R})$ is such that $f(0) = 0$ and $f'(u) > 0$ for $u \geq 0$,
- there exist $q > 1$ and $h > 0$ such that

$$\lim_{u \rightarrow +\infty} \frac{F(u)}{u^q} = h,$$

- there exists $\vartheta > 1$ such that

$$\lim_{u \rightarrow +\infty} \frac{\vartheta F(u) - f(u)u}{u} = 0,$$

- there exists

$$\lim_{u \rightarrow 0} \frac{F(u)}{u^2} = 1,$$

with F defined in (1.2).

Then, there is $\lambda^* > 0$ such that, for all $\lambda \in (0, \lambda^*)$, there exists a bounded variation solution u of (1.1), with $\text{ess inf } u > 0$. This function u is such that

$$u \in W_{\text{loc}}^{2,1}(\alpha, \beta) \cap W^{1,1}(\alpha, \beta)$$

for each interval $(\alpha, \beta) \subset (0, 1)$ where the function a has a constant sign. Moreover, $u \in W_{\text{loc}}^{2,1}[0, \beta)$, with $u'(0) = 0$, if $\alpha = 0$, while $u \in W_{\text{loc}}^{2,1}(\alpha, 1]$, with $u'(1) = 0$, if $\beta = 1$. In addition, for every pair of adjacent intervals, $(\alpha, \beta), (\beta, \gamma) \subset (0, 1)$ with $a(x) \geq 0$ a.e. in (α, β) and $a(x) \leq 0$ a.e. in (β, γ) (respectively, $a(x) \leq 0$ a.e. in (α, β) and $a(x) \geq 0$ a.e. in (β, γ)), either

$$u \in W_{\text{loc}}^{2,1}(\alpha, \gamma),$$

or

$$u(\beta^-) \geq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = -\infty = u'(\beta^+)$$

$$(\text{respectively, } u(\beta^-) \leq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = +\infty = u'(\beta^+)),$$

where $u'(\beta^-), u'(\beta^+)$ are, respectively, the left and the right Dini derivatives of u at β . Finally, u satisfies the equation in (1.1) a.e. in $[0, 1]$.

This kind of bounded variation solutions, which are piecewise regular, but possibly discontinuous, will be in the sequel referred to as *singular* solutions of (1.1). However Theorem 1.1, yielding only the existence of positive singular solutions, which in some cases are the only solutions one may expect, leaves completely open the question of ascertaining the existence of positive regular solutions. In this paper we mainly address this issue: in order to fill this gap, we provide several information about the structure of the set of the positive regular solutions of (1.1) and we investigate the development of singularities and the inherent formation of bounded variation solutions, so establishing a direct connection between the existence results obtained in this work and those in [41], in particular with Theorem 1.1.

Let us now introduce the precise assumptions that will be used throughout most of this paper. In many circumstances, we will suppose that

$$(Ha) \quad a \in L^\infty(0, 1) \text{ changes sign in } [0, 1],$$

and sometimes we will also assume that

$$(Hf) \quad f \in C^1(\mathbb{R}) \text{ satisfies } f(0) = 0, f'(0) = 1 > 0, f'(u) > 0 \text{ for all } u > 0.$$

The condition (Hf) can be relaxed up to assume $f'(0) > 0$ by inter-exchanging λ by $\mu := \lambda f'(0)$ in (1.1). According to it, the associated potential F satisfies

$$\lim_{u \rightarrow 0} \frac{F(u)}{u^2} = \lim_{u \rightarrow 0} \frac{F'(u)}{2u} = \lim_{u \rightarrow 0} \frac{f(u) - f(0)}{2u} = \frac{1}{2}.$$

Hence, F is quadratic at 0. Most of the results of this paper under condition (Hf) will be obtained when, in addition, there exist $q > 1$ and $h > 0$ such that

$$\lim_{u \rightarrow +\infty} \frac{f(u)}{u^{q-1}} = qh. \quad (1.4)$$

This implies that F is superlinear at infinity, i.e., there exists $q > 1$ and $h > 0$ such that

$$\lim_{u \rightarrow +\infty} \frac{F(u)}{u^q} = h, \quad (1.5)$$

as

$$\lim_{u \rightarrow +\infty} \frac{F(u)}{u^q} = \lim_{u \rightarrow +\infty} \frac{f(u)}{qu^{q-1}} = h.$$

The most paradigmatic example satisfying (Hf) and (1.4) is obviously $f(u) = u$.

Subsequently, we will describe some of the main findings of this paper concerning the existence of positive regular solutions for problem (1.1). According to Lemma 2.1, when a and f satisfy (Ha) and (Hf), necessarily $\int_0^1 a \, dx < 0$ if (1.1) possesses a positive regular solution for some $\lambda > 0$. Actually, by Theorem 3.1, under these conditions, there exist $\lambda_0 > 0$, to be characterized in Section 2, and an unbounded component, $\mathfrak{C}_{\lambda_0}^+$, of the set of positive regular solutions of (1.1) in $[0, +\infty) \times \mathcal{C}^1[0, 1]$ such that $(\lambda_0, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+$. If, in addition, f satisfies (1.4) and there exist $r, s \in (0, 1)$ such that $\text{ess inf}_{[r,s]} a = \omega > 0$, then, by Theorem 6.1, (1.1) cannot admit a positive regular solution for sufficiently large $\lambda \geq \lambda_0$. Thus, under these circumstances, the λ -projection of the component $\mathfrak{C}_{\lambda_0}^+$ is a bounded interval, J . An extremely challenging problem that has been partially solved in this paper is to ascertain whether or not $0 \in \bar{J}$. So far, the main available result is Theorem 7.1 in Section 7, where it has been established that $\lambda^* = \inf J > 0$ under the additional hypothesis that $a^{-1}(0) \cap (0, 1) = \{z\}$ and that there exist $q \in (1, 2]$ and $h > 0$ such that

$$\lim_{u \rightarrow +\infty} \frac{f'(u)}{u^{q-2}} = q(q-1)h. \quad (1.6)$$

Note that (1.6) implies (1.4) with $q \in (1, 2]$. Actually, according to Theorem 7.1, the problem (1.1) cannot admit a positive regular solution under these conditions for sufficiently small $\lambda > 0$. In particular, this occurs for the special — but extremely important — case when $f(u) = u$. As a byproduct, (1.1) possesses at least two components of positive regular solutions: $\mathfrak{C}_{\lambda_0}^+$ and $\{(0, \kappa) : \kappa > 0\}$. Finally, at least for the choice $f(u) = u$, where standard phase plane techniques can be applied, our results in Section 8 establish that $\mathfrak{C}_{\lambda_0}^+$ must accumulate at some continuous singular solution of (1.1) whenever the component becomes unbounded in $(\lambda, u) \in \mathbb{R} \times \mathcal{C}^1[0, 1]$. As according to Theorem 1.1, the problem (1.1) admits a bounded variation solution for sufficiently small $\lambda > 0$, we conjecture the validity of the bifurcation diagram sketched in Figure 1 under the previous general assumptions.

Figure 1 represents positive solutions, (λ, u) , both regular and singular, plotting the norm $\|u\|_{\mathcal{C}[0,1]}$ in ordinates versus the value of λ in abscissas. The set of positive solutions consists, at least, of a curve of positive regular solutions emanating from $u = 0$ at $\lambda = \lambda_0$ globally defined for all value of the parameter λ in the interval (λ^*, λ_0) , which have been plotted by using a continuous line; each point of this line representing a solution of (1.1), (λ, u) . Actually, this curve represents the component $\mathfrak{C}_{\lambda_0}^+$. Much like in the case when $f(u) = u$, we conjecture that, under the previous general assumptions, there is a value of λ , $\lambda^* > 0$, where the regular solutions of $\mathfrak{C}_{\lambda_0}^+$ loose their a priori bounds in $\mathbb{R} \times \mathcal{C}^1[0, 1]$ accumulating to some positive continuous singular solution, (λ^*, u^*) . For smaller values of λ , the problem (1.1) possesses a further component of singular bounded variation solutions, $\mathfrak{C}_{\text{BV}}^+$, whose λ -projection should contain the entire interval $(0, \lambda^*)$, in complete agreement with Theorem 1.1, and such that $(\lambda^*, u^*) \in \mathfrak{C}_{\text{BV}}^+$. This component of singular solutions has been represented with a dashed line in Figure 1.

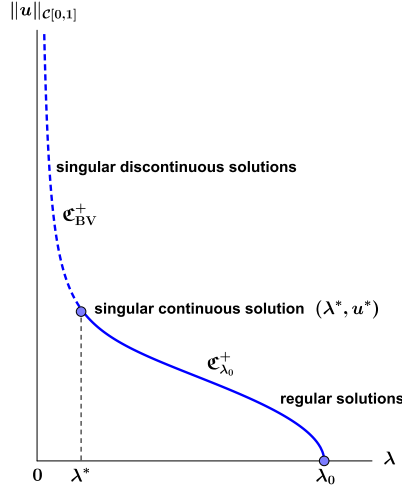


FIGURE 1. Bifurcation diagram when F is quadratic at zero and superlinear at infinity.

More generally, we conjecture that, as soon as the weight a satisfies (Ha), with $\int_0^1 a \, dx < 0$, and the function f satisfies (Hf) and (1.6), there is a component, \mathfrak{C}^+ , of the set of positive bounded variation solutions of (1.1) such that $(\lambda_0, 0) \in \mathfrak{C}^+$ and $(0, +\infty) \in \mathfrak{C}^+$. Moreover,

$$\mathfrak{C}^+ = \mathfrak{C}_{\lambda_0}^+ \cup \mathfrak{C}_{\text{BV}}^+.$$

Furthermore, $(\lambda, u) \in \mathfrak{C}_{\lambda_0}^+$ if $(\lambda, u) \in \mathfrak{C}^+$ with λ sufficiently close to λ_0 , whereas $(\lambda, u) \in \mathfrak{C}_{\text{BV}}^+$ for sufficiently small $\lambda > 0$.

Naturally, much like to semilinear elliptic problems of superlinear indefinite type, (1.1) might admit more than two components within the appropriate ranges of values of the parameters involved in the setting of this problem (see, e.g., [42], [43] and [39]). But this analysis remained outside the general scope of this paper.

Lastly, this paper analyzes how changes the global bifurcation diagram sketched in Figure 1 when, instead of being quadratic at zero, the associated potential, F , is assumed to be either subquadratic, or superquadratic. Our analysis strongly suggests that, under condition (1.5), if there exists $p \in (1, 2)$ and $L > 0$ such that

$$\lim_{u \rightarrow 0^+} \frac{F(u)}{u^p} = L, \quad (1.7)$$

then the component \mathfrak{C}^+ of Figure 1 together with the bounded segment

$$\{(\lambda, 0) : \lambda \in (0, \lambda_0]\}$$

perturb, as $p < 2$ separates away from 2, into the component sketched in the first plot of Figure 2, while if (1.7) holds with $p > 2$, then the component \mathfrak{C}^+ together with the unbounded segment

$$\{(\lambda, 0) : \lambda \geq \lambda_0\}$$

perturb, as $p > 2$ separates away from 2, into the component sketched in the second plot of Figure 2. As an immediate consequence, when F is subquadratic at zero (1.1) possesses two regular solutions for $\lambda < \lambda^*$ sufficiently large, while it admits a regular solution and a discontinuous singular solution for sufficiently small $\lambda > 0$. When, instead, F is assumed to be superquadratic at zero, then (1.1) admits a positive solution for each $\lambda > 0$, which is regular if $\lambda > \lambda_*$ and discontinuous if $\lambda < \lambda_*$.

The distribution of this paper is the following. Section 2 analyzes the linearized stability of $u = 0$ as a steady state solution of the parabolic counterpart of (1.1) and shows that $\int_0^1 a \, dx < 0$ is necessary

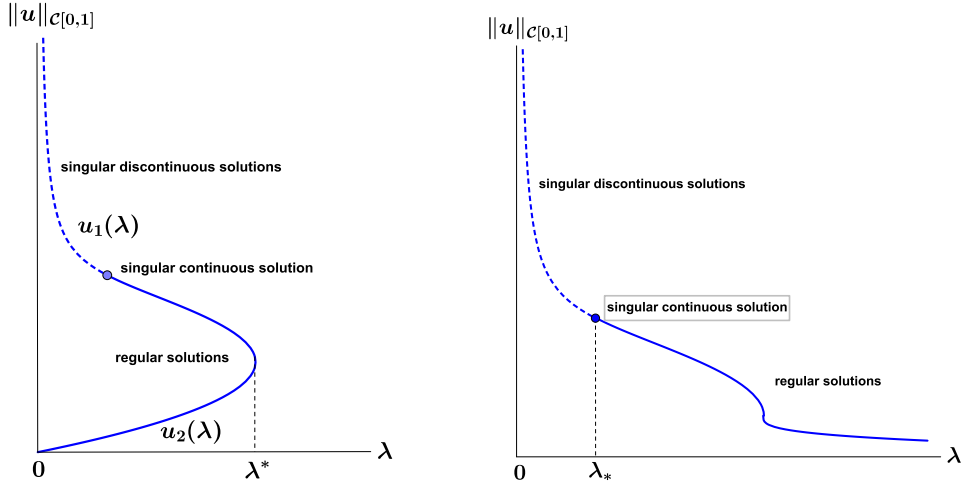


FIGURE 2. Bifurcation diagrams when F is superlinear at infinity. The left picture represents the diagram when the potential F is subquadratic at zero, while the right one plots an admissible global bifurcation diagram when F is superquadratic at zero.

for the existence of a positive regular solution of (1.1) if $\lambda > 0$. Although in [41] it was already established that $\int_0^1 a \, dx < 0$ is necessary for the existence of a bounded variation solution, and regular solutions are bounded variation solutions, the proof of this result for regular solutions is free of the technical difficulties we had to overcome in [41]. So, we shall give it. The main result of Section 2 establishes that under condition $\int_0^1 a \, dx < 0$ there exists $\lambda_0 > 0$ such that $u = 0$ is linearly stable if $\lambda \in [0, \lambda_0)$ and linearly unstable if $\lambda > \lambda_0$.

Section 3 shows the existence of a component, $\mathfrak{C}_{\lambda_0}^+$, of the set of positive regular solutions of (1.1) bifurcating from $u = 0$ at $\lambda = \lambda_0$, the value of the parameter where the stability of $u = 0$ is lost. Moreover, it establishes that $\mathfrak{C}_{\lambda_0}^+$ is unbounded in $[0, +\infty) \times \mathcal{C}^1[0, 1]$. As we are not imposing f to be of class \mathcal{C}^2 in this section, the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [19] cannot be applied to get the local result. As the global unilateral theorems of P. H. Rabinowitz [50] are wrong as stated (see E. N. Dancer [20]), we must invoke to the unilateral theorem [36, Theorem 6.4.3] to get the main theorem given here. As we are imposing Neumann boundary conditions, (1.1) possesses another component of (positive) regular solutions, \mathfrak{C}_0^+ ; the one containing all constant solutions, $(\lambda, u) = (0, \kappa)$ with $\kappa > 0$. One of the main goals of this paper consists in establishing that \mathfrak{C}_0^+ and $\mathfrak{C}_{\lambda_0}^+$ are separated away from each other, as illustrated in Figure 1.

Section 4 uses the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [19] to show that in a neighborhood of $(\lambda_0, 0)$ the component $\mathfrak{C}_{\lambda_0}^+$ is a smooth curve if $f \in \mathcal{C}^2(\mathbb{R})$. Moreover, it ascertains the nature of the local bifurcation to positive regular solutions at λ_0 according to the sign of $f''(0)$ establishing that it is transcritical if $f''(0) \neq 0$, namely, supercritical if $f''(0) < 0$ and subcritical if $f''(0) > 0$, and a subcritical pitchfork bifurcation if $f''(0) = 0$, as it occurs in the most paradigmatic case when $f(u) = u$. This suggests the existence of at least two positive regular solutions when $f''(0) > 0$ and $\lambda > \lambda_0$.

Section 5 discusses very shortly the formation of singularities along the curves of positive regular solutions of problem (1.1). Essentially, it shows how, as soon as the solutions remain bounded, the singularities do arise through a blowing-up phenomenon of u' at some of the nodes of the weight function a .

Section 6 shows that, under condition (1.4) with $q \in (1, 2]$, i.e., for superlinear and subquadratic potentials at infinity, (1.1) cannot admit a positive regular solution for sufficiently large $\lambda > 0$ provided

$a(x) > 0$ a.e. on an interval $K \subset [0, 1]$. Assuming, in addition, that $f \in \mathcal{C}^2(\mathbb{R})$ satisfies (1.6) with $q \in (1, 2]$ and there exists $z > 0$ such that either $a(x) > 0$ for all $x \in (0, z)$ and $a(x) < 0$ for all $x \in (z, 1)$, or $a(x) < 0$ for all $x \in (0, z)$ and $a(x) > 0$ for all $x \in (z, 1)$, the main result of Section 7 establishes that (1.1) cannot admit a positive regular solution for sufficiently small $\lambda > 0$. Therefore, under the assumptions of Theorem 1.1, the existing bounded variation solution guaranteed by that theorem must be singular for sufficiently small $\lambda > 0$, which seems to be a very sharp result.

Section 8 focuses attention into the most paradigmatic case when $f(u) = u$ for a special choice of the function a , step-wise constant, satisfying the assumptions of the main theorem of Section 7. By using some elementary phase plane techniques the existence of regular and continuous singular solutions can be established showing in addition that there exists a $\lambda^* > 0$ where the regular solutions must become singular at z , the point where the function a changes sign. When, $\lambda < \lambda^*$ the problem possesses at least one bounded variation solution, (λ, u) , such that

$$u(z-) > u(z+), \quad u'(z-) = u'(z+) = -\infty.$$

All these features suggest the validity of the global bifurcation diagram sketched in Figure 1, at least for superlinear and either quadratic, or subquadratic, potentials at infinity.

Finally, Sections 9 and 10 analyze the case when, instead of being quadratic at zero, as in the previous sections, the underlying potential, F , is subquadratic and superquadratic, respectively. The main result of Section 9, Theorem 9.1, establishes that if F is subquadratic at zero, then (1.1) possesses at least one regular solution for sufficiently small positive λ , as illustrated in the left picture of Figure 2. The proof here is variational and relies on the introduction of an auxiliary truncated problem in which the degenerate part of the curvature operator is replaced by a uniformly elliptic operator. Suitable estimates imply that the minimizers of the modified problem are in fact solutions of the original one, provided that $\lambda > 0$ is taken small enough. The main result of Section 10, Theorem 10.1, shows instead that if F is superquadratic at zero, then (1.1) possesses at least one regular solution for sufficiently large λ , as illustrated in the right picture of Figure 2. Here the proof is topological: by a natural change of variables, the given problem can be interpreted, for λ large enough, as a small perturbation of another, simpler, problem, where the curvature operator is again replaced by a uniformly elliptic operator. Since the coincidence degree of the modified operator equation can be computed explicitly and is non-zero, the Rouché property of the degree yields the solvability of the original problem. Theorems 9.1 and 10.1, establishing the existence of regular solutions, complement in various directions the results obtained in [41]. We finally notice that the solutions obtained here should perturb from $u = 0$ as $p \rightarrow 2$, however, we will establish this structural property elsewhere.

Throughout this paper it should be noted that, due to the notion of regular solution adopted here, (1.1) can be equivalently expressed in the form

$$\begin{cases} -u'' = \lambda a(x)f(u)g(u'), & 0 < x < 1, \\ u'(0) = u'(1) = 0, \end{cases} \quad (1.8)$$

(see, e.g., [14, 15]), where

$$g(\xi) := (1 + \xi^2)^{\frac{3}{2}}, \quad \xi \in \mathbb{R}. \quad (1.9)$$

Obviously,

$$g(0) = 1, \quad g'(0) = 0, \quad g'(\xi) > 0 \text{ for all } \xi > 0 \text{ and } g(1) = 2\sqrt{2}.$$

Moreover, for every $r < s$ and $V \in L^\infty(r, s)$, we denote by

$$\sigma[-D^2 + V(x); \mathcal{N}, (r, s)], \quad \text{where } D^2 := \frac{d^2}{dx^2},$$

the lowest eigenvalue of the linear Neumann eigenvalue problem

$$\begin{cases} -u'' + V(x)u = \tau u, & r < x < s, \\ u'(r) = u'(s) = 0. \end{cases}$$

Similarly, $\sigma[-D^2 + V(x); \mathcal{D}, (r, s)]$ stands for the lowest eigenvalue of the Dirichlet eigenvalue problem

$$\begin{cases} -u'' + V(x)u = \tau u, & r < x < s, \\ u(r) = u(s) = 0. \end{cases}$$

If u is a regular positive solution of (1.1), i.e., (1.8) holds, then u must be the principal eigenfunction associated with

$$\sigma[-D^2 - \lambda a(x) \frac{f(u)}{u} g(u'); \mathcal{N}, (0, 1)] = 0$$

and therefore,

$$u(x) > 0 \quad \text{for all } x \in [0, 1]. \quad (1.10)$$

Lastly, note that

$$Q(u) := \frac{f(u)}{u}, \quad u > 0,$$

admits a continuous extension to $u = 0$ by setting $Q(0) := 1$.

Lastly, given two arbitrary Banach spaces, U , V , and a linear continuous operator, $T : U \rightarrow V$, we denote by $N[T]$ the null space or kernel of T , $\ker T$, and by $R[T]$ the range or image of T , $\text{im} T$.

2. LINEARIZED STABILITY OF $u = 0$. A SIMPLE NECESSARY CONDITION FOR THE EXISTENCE OF POSITIVE SOLUTIONS

This section analyzes the linearized stability of $u = 0$ as a steady-state of the parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \left(\frac{u'}{\sqrt{1+(u')^2}} \right)' = \lambda a(x) f(u), & 0 < x < 1, \quad t > 0, \\ u'(t, 0) = u'(t, 1) = 0, & t > 0, \\ u(\cdot, 0) = u_0 & \text{in } (0, 1). \end{cases} \quad (2.1)$$

Since $f'(0) = 1$, and $u \sim 0$ in $\mathcal{C}^1[0, 1]$ if and only if $u \sim 0$ and $u' \sim 0$ in $\mathcal{C}[0, 1]$, it is evident that

$$\frac{u'}{\sqrt{1+(u')^2}} \sim u' \quad \text{in } \mathcal{C}^1[0, 1] \quad \text{for } u \sim 0$$

and therefore, the linearized stability of $u = 0$ is determined by the sign of the principal eigenvalue

$$\Sigma(\lambda) := \sigma[-D^2 - \lambda a(x); \mathcal{N}, (0, 1)], \quad \lambda \in \mathbb{R}, \quad (2.2)$$

much like the linearized stability of $u = 0$ as a steady-state of the parabolic problem

$$\begin{cases} u_t - u_{xx} = \lambda a(x) f(u) g(u'), & 0 < x < 1, \quad t > 0, \\ u'(t, 0) = u'(t, 1) = 0, & t > 0, \\ u(\cdot, 0) = u_0 & \text{in } (0, 1), \end{cases} \quad (2.3)$$

because $f(0) = 0$ and $f'(0) = g(0) = 1$.

Precisely, $(\lambda, u) = (\lambda, 0)$ is linearly stable if and only if $\Sigma(\lambda) > 0$, while it is linearly unstable if $\Sigma(\lambda) < 0$. In any circumstances, $\Sigma(0) = 0$. The next result provides us with the structure of $\Sigma(\lambda)$ according to the sign of $\int_0^1 a \, dx$. It is a refinement of a classical result of K. J. Brown and S. S. Lin [12]. As the version given here sharpens the former ones, a short self-contained proof is given.

Theorem 2.1. *For every $a \in L^\infty(0, 1)$, $\Sigma(\lambda)$ is a real analytic function of $\lambda \in \mathbb{R}$ which is concave, in the sense that $\Sigma''(\lambda) \leq 0$ for all $\lambda \in \mathbb{R}$. Moreover, if a changes sign in $(0, 1)$, then*

$$\lim_{\lambda \rightarrow \pm\infty} \Sigma(\lambda) = -\infty. \quad (2.4)$$

Actually, in such case, there exists a unique $\lambda_m \in \mathbb{R}$ such that

$$\text{sign } \Sigma'(\lambda) = \text{sign } (\lambda_m - \lambda) \quad \text{for all } \lambda \in \mathbb{R}. \quad (2.5)$$

Furthermore, we have

$$\text{sign } \lambda_m = -\text{sign} \int_0^1 a \, dx.$$

Therefore, whenever $\int_0^1 a \, dx \neq 0$, there exists a (unique) $\lambda_0 \in \mathbb{R} \setminus \{0\}$ such that $\Sigma(\lambda_0) = 0$ and

$$\lambda_0 \int_0^1 a \, dx < 0.$$

Proof. The first part of the result is a direct consequence of Theorems 9.1 and 9.4 of [37]. The second one is a direct consequence of the identity

$$\dot{\Sigma}(0) = - \int_0^1 a \, dx, \quad (2.6)$$

where $\dot{\cdot}$ stands for $\frac{d}{d\lambda}$. Indeed, according to a classical perturbation result of T. Kato [32] (see Lemma 2.1.1 of [36]), $\Sigma(\lambda)$ admits a principal eigenfunction $\varphi_\lambda > 0$ such that $\varphi_0 := \varphi(0) = 1$. Note that 1 is a principal eigenfunction of $\Sigma(0)$. In particular, we have

$$-\varphi_\lambda'' - \lambda a(x)\varphi_\lambda = \Sigma(\lambda)\varphi_\lambda \quad \text{for all } \lambda \in \mathbb{R},$$

and differentiating with respect to λ yields

$$-\dot{\varphi}_\lambda'' - a(x)\varphi_\lambda - \lambda a(x)\dot{\varphi}_\lambda = \dot{\Sigma}(\lambda)\varphi_\lambda + \Sigma(\lambda)\dot{\varphi}_\lambda \quad \text{for all } \lambda \in \mathbb{R}.$$

Thus, particularizing at $\lambda = 0$, we find that

$$-\dot{\varphi}_0'' - a(x)\varphi_0 = \dot{\Sigma}(0)\varphi_0$$

and hence

$$\dot{\Sigma}(0) = -\dot{\varphi}_0'' - a,$$

because $\varphi_0 = 1$. Therefore, integrating in $(0, 1)$ yields

$$\dot{\Sigma}(0) = \int_0^1 \dot{\varphi}_0'' \, dx - \int_0^1 a \, dx = - \int_0^1 a \, dx,$$

because

$$\dot{\varphi}_0'(0) = \dot{\varphi}_0'(1) = 0.$$

The remaining assertions of the statement are easy consequences of the identity (2.6). \square

Figure 3 shows the three possibilities according to the sign of $\int_0^1 a \, dx$. It helps to visualize the proof of the last assertions of the theorem.

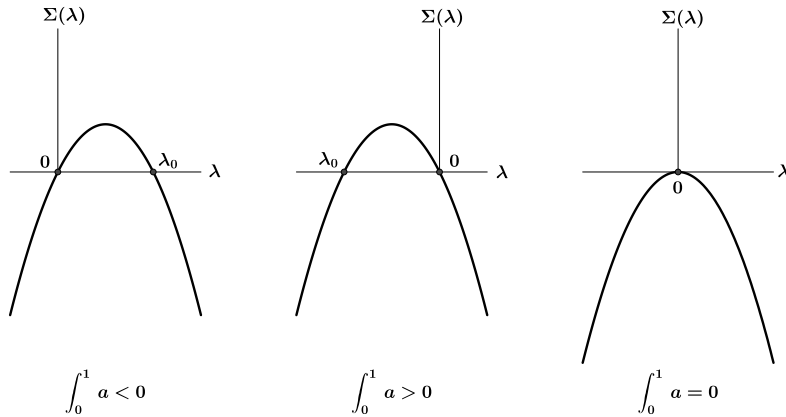


FIGURE 3. All admissible graphs of $\Sigma(\lambda)$ according to the sign of $\int_0^1 a \, dx$.

According to the linearized stability principle, in case $\int_0^1 a \, dx < 0$, $(\lambda, 0)$ is linearly stable if and only if $\lambda \in (0, \lambda_0)$, while in case $\int_0^1 a \, dx > 0$ this occurs if and only if $\lambda \in (\lambda_0, 0)$; at least when it is regarded as a steady state solution of the semilinear parabolic problem (2.3).

Throughout the rest of this paper, without lost of generality, we will assume that

$$\int_0^1 a \, dx < 0. \quad (2.7)$$

The next result provides us with a simple necessary condition for the existence of positive solutions of (1.1) under condition (2.7). According to it, throughout the remaining we will assume that $\lambda \geq 0$.

Lemma 2.1. *Suppose f and a satisfy (Hf), (Ha) and (2.7). Then, $\lambda \geq 0$ if (1.1) admits a positive solution. Moreover, the solution must be constant in $[0, 1]$ if $\lambda = 0$.*

Proof. Suppose (λ, u) is a positive solution of (1.1), then $u(x) > 0$ for all $x \in [0, 1]$ and hence

$$\begin{aligned} \lambda a(x) &= - \left(\frac{u'(x)}{\sqrt{1 + (u'(x))^2}} \right)' \frac{1}{f(u(x))} \\ &= - \left(\frac{1}{f(u(x))} \frac{u'(x)}{\sqrt{1 + (u'(x))^2}} \right)' + \left(\frac{1}{f(u(x))} \right)' \frac{u'(x)}{\sqrt{1 + (u'(x))^2}}. \end{aligned} \quad (2.8)$$

Thus, since $u'(0) = u'(1) = 0$ and f satisfies (Hf), we get

$$\begin{aligned} \lambda \int_0^1 a \, dx &= \int_0^1 \left(\frac{1}{f(u(x))} \right)' \frac{u'(x)}{\sqrt{1 + (u'(x))^2}} \, dx \\ &= - \int_0^1 \frac{f'(u(x))}{f^2(u(x))} \frac{(u'(x))^2}{\sqrt{1 + (u'(x))^2}} \, dx \leq 0. \end{aligned}$$

Therefore, by (2.7), we necessarily have $\lambda \geq 0$.

Suppose $\lambda = 0$. Then, since $u(x) > 0$ for all $x \in [0, 1]$ and hence $f'(u(x)) > 0$, we find from

$$\int_0^1 \frac{f'(u(x))}{f^2(u(x))} \frac{(u'(x))^2}{\sqrt{1 + (u'(x))^2}} \, dx = 0$$

that $u'(x) = 0$ for all $x \in [0, 1]$. Therefore, we conclude $u(x) = u(0)$ for all $x \in [0, 1]$, which ends the proof. \square

3. GLOBAL BIFURCATION OF POSITIVE SOLUTIONS FROM $u = 0$

Throughout this section, for any positive integer k , $\mathcal{C}_{\mathcal{N}}^k[0, 1]$ stands for the Banach space of the functions $u : [0, 1] \rightarrow \mathbb{R}$ of class \mathcal{C}^k such that $u'(0) = u'(1) = 0$ equipped with the norm

$$\|u\|_{\mathcal{C}^k[0,1]} := \sum_{j=0}^k \|D^j u\|_{\infty}, \quad D^j = \frac{d^j}{dx^j}, \quad 0 \leq j \leq k,$$

and P stands for the cone of non-negative functions of $\mathcal{C}_{\mathcal{N}}^1[0, 1]$, i.e., P is the positive cone of $\mathcal{C}_{\mathcal{N}}^1[0, 1]$ regarded as an ordered Banach space with respect to the usual ordering. The interior of the cone P , $\text{int } P$, consists of the functions $u \in \mathcal{C}_{\mathcal{N}}^1[0, 1]$ such that $u(x) > 0$ for all $x \in [0, 1]$. According to (1.8), any positive regular solution, (λ, u) , of (1.1) satisfies $u \in \text{int } P$.

The following result establishes the existence of a component of regular positive solutions of (1.8) bifurcating from the line $(\lambda, u) = (\lambda, 0)$ at $\lambda = \lambda_0$. By a component it is meant a closed and connected subset that it is maximal for the inclusion.

Theorem 3.1. *Suppose $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f'(0) = 1$, $a \in L^\infty(0, 1)$ changes sign in $(0, 1)$ and $\int_0^1 a < 0$. Let $\lambda_0 > 0$ be the unique real number such that $\Sigma(\lambda_0) = 0$, whose existence and uniqueness was established by Theorem 2.1. Then, there exists an unbounded component*

$$\mathfrak{C}_{\lambda_0}^+ \subset [0, +\infty) \times \text{int } P$$

of the set of positive regular solutions of (1.1) such that $(\lambda_0, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+$ and

$$(\lambda, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+ \quad \text{with} \quad \lambda \neq 0 \quad \implies \quad \lambda = \lambda_0. \quad (3.1)$$

Proof. Note that (λ, u) is a positive solution of (1.1), or (1.8), if and only if

$$u = (-D^2 + 1)^{-1}[u + \lambda af(u)g(u')]$$

where

$$(-D^2 + 1)^{-1} : L^\infty(0, 1) \rightarrow W_{\mathcal{N}}^{2,\infty}(0, 1)$$

stands for the resolvent operator of $-D^2 + 1$ in $(0, 1)$ under Neumann boundary conditions and

$$W_{\mathcal{N}}^{2,\infty}(0, 1) := \{u \in W^{2,\infty}(0, 1) : u'(0) = u'(1) = 0\}.$$

As the imbedding

$$W_{\mathcal{N}}^{2,\infty}(0, 1) \hookrightarrow \mathcal{C}_{\mathcal{N}}^1[0, 1] := \{u \in C^1[0, 1] : u'(0) = u'(1) = 0\}.$$

is compact, introducing the compact operator $\mathcal{K}(\lambda, \cdot) : \mathcal{C}_{\mathcal{N}}^1[0, 1] \rightarrow \mathcal{C}_{\mathcal{N}}^1[0, 1]$ defined by

$$\mathcal{K}(\lambda, u) := (-D^2 + 1)^{-1}[u + \lambda af(u)g(u')], \quad \lambda \in \mathbb{R}, \quad u \in \mathcal{C}_{\mathcal{N}}^1[0, 1],$$

the problem (1.8) can be expressed as the fixed point equation

$$u = \mathcal{K}(\lambda, u), \quad \lambda \in \mathbb{R}, \quad u \in \mathcal{C}_{\mathcal{N}}^1[0, 1]. \quad (3.2)$$

Equivalently, setting

$$\mathfrak{F}(\lambda, u) := u - \mathcal{K}(\lambda, u), \quad (\lambda, u) \in \mathbb{R} \times \mathcal{C}_{\mathcal{N}}^1[0, 1],$$

the solutions of (1.8) can be regarded as the zeroes of the operator \mathfrak{F} , i.e., the solutions of the equation

$$\mathfrak{F}(\lambda, u) = 0.$$

The operator $\mathfrak{F}(\lambda, u)$ is of class \mathcal{C}^1 and it is a compact perturbation of the identity map. Thus, it is Fredholm of index zero. Moreover, $\mathfrak{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$ and its linearization at $(\lambda, 0)$ is given by

$$\mathfrak{L}(\lambda)u := D_u \mathfrak{F}(\lambda, 0)u = u - (-D^2 + 1)^{-1}[(1 + \lambda a)u], \quad u \in \mathcal{C}_{\mathcal{N}}^1[0, 1],$$

for all $\lambda \in \mathbb{R}$, because $f(0) = g'(0) = 0$. Let $\varphi > 0$ denote any solution of

$$\begin{cases} -\varphi'' = \lambda_0 a(x)\varphi & \text{in } (0, 1), \\ \varphi'(0) = \varphi'(1) = 0. \end{cases} \quad (3.3)$$

By the maximum principle, $\varphi(x) > 0$ for all $x \in [0, 1]$. Moreover, φ is unique up to a multiplicative constant and

$$N[\mathfrak{L}(\lambda_0)] = \text{span}[\varphi].$$

We claim that

$$\mathfrak{L}'(\lambda_0)\varphi = -(-D^2 + 1)^{-1}(a\varphi) \notin R[\mathfrak{L}(\lambda_0)]. \quad (3.4)$$

On the contrary, suppose there is $u \in \mathcal{C}_{\mathcal{N}}^1[0, 1]$ such that

$$\mathfrak{L}(\lambda_0)u = -(-D^2 + 1)^{-1}(a\varphi).$$

Then, $u \in W_{\mathcal{N}}^{2,\infty}(0, 1)$, since $a \in L^\infty(0, 1)$, and hence

$$-u'' - \lambda_0 au = -a\varphi \quad \text{in } (0, 1), \quad u'(0) = u'(1) = 0. \quad (3.5)$$

Multiplying (3.5) by φ and integrating by parts in $(0, 1)$, (3.3) implies

$$\int_0^1 a\varphi^2 = 0. \quad (3.6)$$

On the other hand, multiplying (3.3) by φ and integrating in $[0, 1]$ yields

$$\lambda_0 \int_0^1 a\varphi^2 = - \int_0^1 \varphi\varphi'' = \int_0^1 (\varphi')^2 > 0 \quad (3.7)$$

and therefore, since $\lambda_0 > 0$, we find that $\int_0^1 a\varphi^2 > 0$, which contradicts (3.6) and shows (3.4). Note that φ cannot be constant because $\lambda_0 > 0$ and $a \neq 0$. Therefore, the transversality condition of M. G. Crandall and P. H. Rabinowitz [19] holds, i.e.,

$$\mathfrak{L}'(\lambda_0)(N[\mathfrak{L}(\lambda_0)] \oplus R[\mathfrak{L}(\lambda_0)]) = \mathcal{C}_{\mathcal{N}}^1[0, 1], \quad (3.8)$$

though the main bifurcation theorem of [19] cannot be applied unless $f \in \mathcal{C}^2(\mathbb{R})$. But this is far from important here. Indeed, thanks to (3.8), the algebraic multiplicity of J. Esquinas and J. López-Gómez [22] satisfies

$$\chi[\mathfrak{L}(\lambda); \lambda_0] = 1$$

(see Chapter 4 of [40] if necessary). Thus, owing to [36, Theorem 5.6.2], or [40, Prop. 12.3.1], the local index of $u = 0$ as a fixed point of $\mathcal{K}(\lambda, \cdot)$ changes as λ crosses λ_0 . Consequently, according to the unilateral bifurcation theorem of J. López-Gómez [36, Theorem 6.4.3], it becomes apparent that either $\mathfrak{C}_{\lambda_0}^+$ is unbounded in $[0, +\infty) \times \mathcal{C}_N^1[0, 1]$, or there is $\lambda_1 \in [0, +\infty) \setminus \{\lambda_0\}$ such that $(\lambda_1, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+$. Note that the unilateral theorems of P. H. Rabinowitz [50] cannot be used to get this global result because they were wrong as stated (see the counterexample constructed by E. N. Dancer [20]).

To establish (3.1), let (λ_n, u_n) , $n \geq 1$, be a sequence of positive solutions of (1.8) such that

$$\lim_{n \rightarrow +\infty} \lambda_n = \lambda \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|u_n\|_{\mathcal{C}^1[0,1]} = 0 \quad \text{for some } \lambda \neq 0. \quad (3.9)$$

Then, setting

$$v_n := \frac{u_n}{\|u_n\|_\infty}, \quad n \geq 1,$$

we have that

$$v_n = (-D^2 + 1)^{-1} \left[v_n + \lambda a \frac{f(u_n)}{\|u_n\|_\infty} g(u'_n) \right] + (\lambda_n - \lambda) (-D^2 + 1)^{-1} \left[a \frac{f(u_n)}{\|u_n\|_\infty} g(u'_n) \right] \quad (3.10)$$

for all $n \geq 1$. According to (3.9), we get

$$\lim_{n \rightarrow +\infty} g(u'_n) = 1, \quad \lim_{n \rightarrow +\infty} \frac{f(u_n)}{\|u_n\|_\infty} = f'(0) = 1$$

and hence

$$\lim_{n \rightarrow +\infty} \left((\lambda_n - \lambda) (-D^2 + 1)^{-1} \left[a \frac{f(u_n)}{\|u_n\|_\infty} g(u'_n) \right] \right) = 0.$$

Moreover, $\|v_n\|_\infty = 1$ for all $n \geq 1$. Thus, since

$$(-D^2 + 1)^{-1} : L^\infty(0, 1) \rightarrow \mathcal{C}_N^1[0, 1]$$

is compact and the sequence

$$v_n + \lambda a \frac{f(u_n)}{\|u_n\|_\infty} g(u'_n), \quad n \geq 1,$$

is bounded, there exists a subsequence of v_n , relabeled by n , such that $\lim_{n \rightarrow +\infty} v_n = \psi$ for some $\psi \geq 0$, $\|\psi\|_\infty = 1$. Thus, letting $n \rightarrow +\infty$ in (3.10) yields

$$\psi = (-D^2 + 1)[\psi + \lambda a \psi],$$

or, equivalently,

$$-\psi'' = \lambda a \psi \quad \text{in } (0, 1), \quad \psi'(0) = \psi'(1) = 0.$$

Therefore, since $\lambda \neq 0$, from Theorem 2.1 we may conclude that $\lambda = \lambda_0$.

It remains to prove that $\mathfrak{C}_{\lambda_0}^+$ is unbounded. This is a direct consequence from [36, Theorem 6.4.3] and (3.1) if $(0, 0) \notin \bar{\mathfrak{C}}_{\lambda_0}^+$. If $(0, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+$, then $(\lambda, u) = (0, \kappa) \in \mathfrak{C}_{\lambda_0}^+$ for all constant $\kappa > 0$ and therefore, $\mathfrak{C}_{\lambda_0}^+$ is unbounded. This ends the proof. \square

Similarly, the next result establishes the existence of a component of positive solutions of (3.1), \mathfrak{C}_0^+ , with $(0, 0) \in \bar{\mathfrak{C}}_0^+$.

Theorem 3.2. *Suppose $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f'(0) = 1$, $a \in L^\infty(0, 1)$ changes sign in $(0, 1)$ and $\int_0^1 a \, dx < 0$. Then, there exists an unbounded component of the set of positive solutions of (1.8),*

$$\mathfrak{C}_0^+ \subset [0, +\infty) \times \text{int } P,$$

such that

$$\{(\lambda, u) = (0, \kappa) : \kappa > 0\} \subset \mathfrak{C}_0^+.$$

Moreover,

$$(\lambda, 0) \in \bar{\mathfrak{C}}_0^+ \quad \text{with } \lambda \neq 0 \implies \lambda = \lambda_0 \quad (3.11)$$

and consequently, should this occur, $\mathfrak{C}_0^+ = \mathfrak{C}_{\lambda_0}^+$. Otherwise, $\mathfrak{C}_0^+ \cap \mathfrak{C}_{\lambda_0}^+ = \emptyset$.

Proof. We will maintain the notations introduced in the proof of Theorem 3.1. Now,

$$N[\mathfrak{L}(0)] = \text{span}[1]$$

and

$$\mathfrak{L}'(0)1 = -(-D^2 + 1)^{-1}a \notin R[\mathfrak{L}(0)]. \quad (3.12)$$

On the contrary, suppose

$$\mathfrak{L}(0)u = -(-D^2 + 1)^{-1}a$$

for some $u \in \mathcal{C}_{\mathcal{N}}^1[0, 1]$. Then, $u \in W^{2,p}(0, 1)$ for all $p > 1$, $u'(0) = u'(1) = 0$, and $u'' = a$ in $(0, 1)$. Thus, integrating in $(0, 1)$ yields $\int_0^1 a \, dx = 0$, which contradicts $\int_0^1 a \, dx < 0$. Consequently, also in this case the transversality condition of M. G. Crandall and P. H. Rabinowitz [19] holds. Thus, arguing as in the proof of Theorem 3.1, the existence of a (global) component, \mathfrak{C}_0^+ , of the set of positive solutions of (1.1) in $[0, +\infty) \times \text{int } P$ with $(0, 0) \in \mathfrak{C}_0^+$ holds. Since all positive constants $\kappa > 0$ provide us with solutions of (1.8) at $\lambda = 0$, necessarily $(\lambda, u) = (0, \kappa) \in \mathfrak{C}_0^+$ for all $\kappa > 0$. In particular, \mathfrak{C}_0^+ is unbounded. As the proof of (3.11) can be easily adapted from the proof of (3.1), we omit the technical details here. Finally, if $(\lambda_0, 0) \in \bar{\mathfrak{C}}_0^+$, then $(\lambda_0, 0) \in \bar{\mathfrak{C}}_0^+ \cap \bar{\mathfrak{C}}_{\lambda_0}^+$ and therefore, $\mathfrak{C}_0^+ = \mathfrak{C}_{\lambda_0}^+$. This ends the proof. \square

Hereafter, we will denote by \mathcal{P}_λ the λ -projection operator defined by

$$\mathcal{P}_\lambda(\lambda, u) := \lambda.$$

Figure 4 shows three admissible bifurcation diagrams of positive solutions of (1.1) when $\int_0^1 a \, dx < 0$. In the first one,

$$\mathcal{P}_\lambda(\mathfrak{C}_{\lambda_0}^+) = (0, \lambda_0) \quad \text{and} \quad \mathfrak{C}_0^+ \cap \mathfrak{C}_{\lambda_0}^+ = \emptyset.$$

Consequently, the solutions along $\mathfrak{C}_{\lambda_0}^+$ must become unbounded as $\lambda \rightarrow 0^+$. In the second one,

$$\mathcal{P}_\lambda(\mathfrak{C}_{\lambda_0}^+) = (\lambda^*, \lambda_0)$$

for some $\lambda^* \in (0, \lambda_0)$. The third picture represents a case where $\mathfrak{C}_{\lambda_0}^+ = \mathfrak{C}_0^+$. Much of this paper is devoted to ascertain whether or not each of these situations can occur. Actually, one of the main open problems addressed in this paper is getting the intervals $\mathcal{P}_\lambda(\mathfrak{C}_0^+)$ and $\mathcal{P}_\lambda(\mathfrak{C}_{\lambda_0}^+)$. Based on the results in the forthcoming sections, we conjecture that the second diagram occurs if the associated potential, F , is superlinear at infinity.

4. LOCAL SOLUTION CURVES AT $(0, 0)$ AND $(\lambda_0, 0)$ WHEN $f \in \mathcal{C}^2(\mathbb{R})$

When $f \in \mathcal{C}^2(\mathbb{R})$, thanks to the uniqueness obtained from the local bifurcation theorem of M. G. Crandall and P. H. Rabinowitz [19], one can complement Theorems 3.1 and 3.2 with the next result of a local nature, which will be extremely useful later.

Theorem 4.1. *Suppose $f \in \mathcal{C}^r(\mathbb{R})$, $r \geq 2$, satisfies $f'(0) = 1$ and $a \in L^\infty(\Omega)$ changes sign and $\int_0^1 a < 0$. Then, in a neighborhood of $(\lambda, u) = (0, 0)$, \mathfrak{C}_0^+ consists of the curve $(0, \kappa)$, $\kappa > 0$.*

Similarly, setting

$$V := \left\{ v \in \mathcal{C}_{\mathcal{N}}^1[0, 1] : \int_0^1 v(x)\varphi(x) \, dx = 0 \right\},$$

where $\varphi > 0$ is any principal eigenfunction associated with (3.3), there exist $\varepsilon > 0$ and two maps of class \mathcal{C}^{r-1} ,

$$\lambda : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}, \quad v : (-\varepsilon, \varepsilon) \rightarrow V,$$

such that

- (a) $\lambda(0) = \lambda_0$ and $v(0) = 0$;
- (b) $(\lambda(s), s(\varphi + v(s)))$ solves (1.1) for all $s \in (-\varepsilon, \varepsilon)$;
- (c) besides $(\lambda, 0)$, $(\lambda(s), s(\varphi + v(s)))$ are the unique solutions of (1.1) in a neighborhood of $(\lambda_0, 0)$.

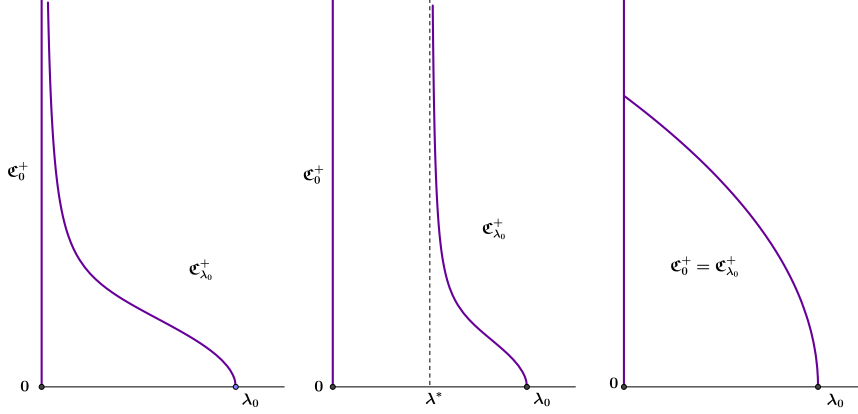


FIGURE 4. Three possible bifurcation diagrams for regular solutions in case $\int_0^1 a dx < 0$. We are plotting the parameter λ , in abscissas, versus $\|u\|_{\mathcal{C}^1[0,1]}$, in ordinates.

Note that V satisfies

$$\text{span}[\varphi] \oplus V = \mathcal{C}_{\mathcal{N}}^1[0, 1]$$

and that $\varepsilon > 0$ can be shortened, if necessary, so that

$$(\lambda(s), s(\varphi + v(s))) \in \mathfrak{E}_{\lambda_0}^+ \quad \text{for all } s \in (0, \varepsilon).$$

The next result provides us with the bifurcation direction of the bifurcated curve $(\lambda(s), s(\varphi + v(s)))$, $s \sim 0$, in all possible cases. In the most classical case when $f(u) = u$ it establishes that the local bifurcation from $u = 0$ at $\lambda = \lambda_0$ is always subcritical, independently of the nature of the weight function a . Subsequently, when these derivatives do make sense, we will set

$$\lambda_1 := \lambda'(0), \quad v_1 := v'(0), \quad \lambda_2 := \frac{\lambda''(0)}{2}, \quad v_2 := \frac{v''(0)}{2}.$$

Then,

$$\lambda(s) = \lambda_0 + s\lambda_1 + s^2\lambda_2 + o(s^2), \quad v(s) = sv_1 + s^2v_2 + o(s^2), \quad \text{as } s \rightarrow 0,$$

and

$$[1 + s^2(\varphi' + sv_1' + o(s))^2]^{\frac{3}{2}} = 1 + \frac{3}{2}(\varphi')^2s^2 + o(s^2) \quad \text{as } s \rightarrow 0.$$

Theorem 4.2. *Under the general assumptions of Theorem 4.1, we have that*

$$\lambda_1 = -\frac{1}{2}f''(0)\lambda_0 \frac{\int_0^1 a\varphi^3 dx}{\int_0^1 a\varphi^2 dx}. \quad (4.1)$$

Thus, the bifurcation at λ_0 is transcritical if $f''(0) \neq 0$. In particular, the bifurcation to positive solutions is supercritical, $\lambda_1 > 0$, if $f''(0) < 0$ and subcritical, $\lambda_1 < 0$, if $f''(0) > 0$.

Suppose, in addition, that $r \geq 3$ and $f''(0) = 0$. Then, $\lambda_1 = 0$ and

$$\lambda_2 = -\frac{3}{2}\lambda_0 \frac{\int_0^1 a\varphi^2(\varphi')^2 dx}{\int_0^1 a\varphi^2 dx} < 0. \quad (4.2)$$

Therefore, the bifurcation at λ_0 is a genuine subcritical pitchfork bifurcation if $f''(0) = 0$.

Proof. Throughout this proof, the notations introduced in the proof of Theorem 4.1 are kept and we set

$$u(s) = s(\varphi + v(s)) \quad \text{for all } s \in (-\varepsilon, \varepsilon).$$

The component $\mathfrak{C}_{\lambda_0}^+$ is the maximal closed and connected subset of $[0, +\infty) \times \text{int } P$ containing the arc of curve $(\lambda(s), u(s))$, $0 \leq s < \varepsilon$.

Substituting (λ, u) by $(\lambda(s), u(s))$ in (1.8) and dividing by s , we find that

$$\begin{aligned} -(\varphi + sv_1 + o(s))'' &= (\lambda_0 + s\lambda_1 + o(s))a(x)(\varphi + sv_1 + o(s)) \\ &\cdot \left[1 + \frac{f''(0)}{2}s(\varphi + sv_1 + o(s)) + o(s)\right] \left[1 + \frac{3}{2}(\varphi')^2s^2 + o(s^2)\right] \end{aligned}$$

for sufficiently small s . Particularizing at $s = 0$, it becomes apparent that

$$-\varphi'' = \lambda_0 a \varphi, \tag{4.3}$$

which is true by the definition of λ_0 and φ . Identifying terms of order s yields

$$-v_1'' = \lambda_0 a v_1 + \lambda_0 \frac{f''(0)}{2} a \varphi^2 + \lambda_1 a \varphi.$$

Multiplying this equation by φ and integrating by parts in $(0, 1)$, we find from (4.3) that

$$\frac{1}{2} \lambda_0 f''(0) \int_0^1 a \varphi^3 dx + \lambda_1 \int_0^1 a \varphi^2 dx = 0. \tag{4.4}$$

On the other hand, multiplying (4.3) by φ and integrating in $(0, 1)$, we find that

$$\lambda_0 \int_0^1 a \varphi^2 dx = - \int_0^1 \varphi'' \varphi dx = \int_0^1 (\varphi')^2 dx > 0$$

and hence, (4.1) holds by eliminating λ_1 in (4.4). Note that multiplying (4.3) by φ^2 and integrating in $(0, 1)$ yields

$$\lambda_0 \int_0^1 a \varphi^3 dx = - \int_0^1 \varphi'' \varphi^2 dx = \int_0^1 \varphi' (\varphi^2)' dx = 2 \int_0^1 \varphi (\varphi')^2 dx > 0.$$

Therefore, since $\lambda_0 > 0$, it follows from (4.1) that

$$\text{sign } \lambda_1 = -\text{sign } f''(0).$$

Subsequently, we suppose $r \geq 3$ and $f''(0) = 0$. Then, $\lambda_1 = 0$ and hence,

$$-v_1'' = \lambda_0 a v_1.$$

Thus, there exists $\alpha \in \mathbb{R}$ such that $v_1 = \alpha \varphi$. Therefore, since $v_1 \in V$, we find that $\alpha = 0$, which implies $v_1 = 0$. Consequently, substituting $(\lambda(s), u(s))$ in (1.8) and dividing by s yields

$$\begin{aligned} -(\varphi + s^2 v_2 + o(s))'' &= (\lambda_0 + s^2 \lambda_2 + o(s^2))a(x)(\varphi + s^2 v_2 + o(s^2)) \\ &\cdot \left[1 + \frac{f''(0)}{2}s(\varphi + s^2 v_2 + o(s^2)) + o(s)\right] \left[1 + \frac{3}{2}(\varphi')^2s^2 + o(s^2)\right]. \end{aligned}$$

Consequently, identifying terms of order s^2 , we obtain that

$$-v_2'' = \lambda_0 a v_2 + \frac{3}{2} \lambda_0 a \varphi (\varphi')^2 + \lambda_2 a \varphi. \tag{4.5}$$

Thus, multiplying (4.5) by φ and integrating by parts in $(0, 1)$ gives

$$\frac{3}{2} \lambda_0 \int_0^1 a \varphi^2 (\varphi')^2 dx + \lambda_2 \int_0^1 a \varphi^2 dx = 0$$

and hence, since $\int_0^1 a \varphi^2 > 0$, the identity (4.2) holds.

It remains to show that $\lambda_2 < 0$. Indeed, integrating in $(0, 1)$ the identity

$$(\varphi^2)' [(\varphi')^2]' = [(\varphi^2)' (\varphi')^2]' - (\varphi^2)'' (\varphi')^2$$

it becomes apparent that

$$\begin{aligned}
\int_0^1 (\varphi^2)'[(\varphi')^2]' dx &= - \int_0^1 (\varphi^2)''(\varphi')^2 dx \\
&= - \int_0^1 (\varphi''\varphi + 2\varphi'\varphi' + \varphi\varphi'')(\varphi')^2 dx \\
&= -2 \int_0^1 (\varphi''\varphi + (\varphi')^2)(\varphi')^2 dx \\
&= -2 \int_0^1 \varphi\varphi''(\varphi')^2 dx - 2 \int_0^1 (\varphi')^4 dx
\end{aligned}$$

and hence, since $-\varphi'' = \lambda_0 a\varphi$, we obtain that

$$\int_0^1 (\varphi^2)'[(\varphi')^2]' dx = 2\lambda_0 \int_0^1 a\varphi^2(\varphi')^2 dx - 2 \int_0^1 (\varphi')^4 dx. \quad (4.6)$$

On the other hand,

$$(\varphi^2)'[(\varphi')^2]' = 4\varphi\varphi'\varphi'\varphi'' = -4\lambda_0 a\varphi^2(\varphi')^2$$

and so, substituting in (4.6), yields

$$6\lambda_0 \int_0^1 a\varphi^2(\varphi')^2 dx = 2 \int_0^1 (\varphi')^4 dx.$$

As $\lambda_0 > 0$ and $a \neq 0$, we already know that φ cannot be constant and hence,

$$\lambda_0 \int_0^1 a\varphi^2(\varphi')^2 dx > 0.$$

Thus, $\int_0^1 a\varphi^2(\varphi')^2 dx > 0$. Moreover, due to (3.7), $\int_0^1 a\varphi^2 dx > 0$. Consequently, $\lambda_2 < 0$. This ends the proof. \square

5. ON THE DEVELOPMENT OF SINGULARITIES

Throughout this section we will assume that

(HC) $a \in \mathcal{C}[0, 1]$, $a^{-1}(0) \cap (0, 1) = \{z_j : 1 \leq j \leq N\}$ for some integer $N \geq 1$ and

$$a(x) \begin{cases} > 0 & \text{if } x \in (0, z_1) \cup \bigcup_{3 \leq 2j+1 \leq N} (z_{2j}, z_{2j+1}), \\ < 0 & \text{if } x \in \bigcup_{0 \leq 2j \leq N-2} (z_{2j+1}, z_{2(j+1)}). \end{cases}$$

Although these requirements can be relaxed substantially, they are sufficiently general for our purposes in this section.

Suppose $\int_0^1 a dx < 0$ and let (λ, u) be a positive regular solution of (1.1) with $\lambda > 0$. In each component, I_+ , of $a^{-1}((0, +\infty))$ we have that

$$u''(x) - \lambda a(x)f(u(x))g(u'(x)) < 0$$

for all $x \in I_+$ and hence, u is strictly concave. Similarly, in each component, I_- , of $a^{-1}((-\infty, 0))$ we have that $u''(x) > 0$ for all $x \in I_-$ and hence, u is strictly convex. In particular, the convexity properties of the solutions change at the nodes, z_j , $1 \leq j \leq N$, of a . This entails that u possesses a unique critical point in each nodal interval (z_j, z_{j+1}) ; naturally, $x = 0$ is the unique critical point in $[0, z_1)$ and $x = 1$ the unique one in $(z_N, 1]$. Consequently, the shape of u is strongly reminiscent of the one of a .

Now, let $0 < \alpha < \beta$ be such that (1.1) possesses a positive solution, (λ, u_λ) , for all $\lambda \in (\alpha, \beta]$, and consider, for every $\lambda \in (\alpha, \beta]$, the function

$$\varphi_\lambda(x) := \frac{-u'_\lambda(x)}{\sqrt{1 + (u'_\lambda(x))^2}}, \quad x \in [0, 1]. \quad (5.1)$$

By construction,

$$\varphi_\lambda(0) = \varphi_\lambda(1) = 0 \quad \text{and} \quad \varphi_\lambda(x) \in (-1, 1) \quad \text{for all } x \in (0, 1).$$

Moreover, according to (1.1),

$$\varphi'_\lambda(x) = \lambda a(x) f(u_\lambda(x)) \begin{cases} > 0 & \text{if } a(x) > 0, \\ < 0 & \text{if } a(x) < 0. \end{cases}$$

Thus, the interior critical points of φ_λ are the nodes, z_j , $1 \leq j \leq N$, of the weight function a . Precisely, under assumption (HC), z_1 is a local maximum of φ_λ , while z_2 is a local minimum, and so on.

Suppose there exists $\varepsilon > 0$ such that

$$|\varphi_\lambda(x)| \leq 1 - \varepsilon \quad \text{for all } (x, \lambda) \in [0, 1] \times (\alpha, \beta]. \quad (5.2)$$

Then, there exists a constant $C > 0$ such that

$$|u'_\lambda(x)| \leq C \quad \text{for all } (x, \lambda) \in [0, 1] \times (\alpha, \beta]. \quad (5.3)$$

Thus, by a standard compactness argument, it is easily seen that

$$u_\alpha := \lim_{\lambda \rightarrow \alpha^+} u_\lambda \quad (5.4)$$

provides us with a non-negative regular solution of (1.1) for $\lambda = \alpha$ if $\{u_\lambda\}_{\lambda \in (\alpha, \beta]}$ is bounded. Moreover, by Theorem 3.1, $u_\alpha > 0$ if $\alpha \neq \lambda_0$. On the contrary, when (5.2) fails, necessarily

$$\limsup_{\lambda \rightarrow \alpha^+} \|\varphi_\lambda\|_{C[0,1]} = 1$$

and hence,

$$\limsup_{\lambda \rightarrow \alpha^+} \|u'_\lambda(x)\|_{C[0,1]} = +\infty. \quad (5.5)$$

Since the critical points of φ_λ are the nodes of a for all $\lambda \in (\alpha, \beta]$, it becomes apparent that there exists $j \in \{1, \dots, N\}$ such that

$$\limsup_{\lambda \rightarrow \alpha^+} |\varphi_\lambda(z_j)| = 1.$$

Equivalently,

$$\limsup_{\lambda \rightarrow \alpha^+} |u'_\lambda(z_j)| = +\infty.$$

Therefore, along any solution curve of (1.1), the singularities can only arise at the nodes of a , by a blow-up of the derivative, u'_λ , as λ approaches α , provided such an α exists.

Although the example of Section 8 strongly suggests that the curves of regular solutions of (1.1) can actually be continued by paths consisting of bounded variation solutions of (1.1), as discussed by the authors in [41], it remains an open problem to ascertain whether or not this is a general phenomenology for the class of problems dealt with in this paper.

6. NONEXISTENCE FOR F SUPERLINEAR AT $+\infty$ AND λ LARGE

The main result of this section can be stated as follows. Note that all the assumptions hold for the most classical case when $f(u) = u$.

Theorem 6.1. *Suppose f and a satisfy (Hf), (Ha), $\int_0^1 a \, dx < 0$, and there exist $r, s \in (0, 1)$ such that*

$$\operatorname{ess\,inf}_{[r,s]} a = \omega > 0. \quad (6.1)$$

Assume, in addition, that (1.4) holds for some $q \in (1, 2]$ and $h > 0$. Then, (1.1) cannot admit a positive regular solution for sufficiently large $\lambda > 0$.

Proof. We will argue by contradiction. Suppose there is a sequence, (λ_n, u_n) , $n \geq 1$, of positive solutions of (1.1) such that

$$\lim_{n \rightarrow +\infty} \lambda_n = +\infty.$$

Then, integrating the differential equation in $[r, s]$ yields

$$\lambda_n \int_r^s a(x) f(u_n(x)) dx = \frac{u'_n(r)}{\sqrt{1 + (u'_n(r))^2}} - \frac{u'_n(s)}{\sqrt{1 + (u'_n(s))^2}} \leq 1 + 1 = 2.$$

Thus,

$$0 \leq \omega \int_r^s f(u_n(x)) dx \leq \frac{2}{\lambda_n}$$

for all $n \geq 1$ and hence, letting $n \rightarrow +\infty$ yields

$$\lim_{n \rightarrow +\infty} \|f(u_n)\|_{L^1(r,s)} = 0.$$

In particular, there exists a subsequence of $\{f(u_n)\}_{n \geq 1}$, still denoted by $\{f(u_n)\}_{n \geq 1}$, such that

$$\lim_{n \rightarrow +\infty} f(u_n(x)) = 0 \quad \text{a.e. in } [r, s].$$

Therefore, since $f^{-1}(0) \cap [0, +\infty] = \{0\}$, we find that

$$\lim_{n \rightarrow +\infty} u_n(x) = 0 \quad \text{a.e. in } [r, s]. \quad (6.2)$$

Subsequently, we consider the auxiliary function Q defined by

$$Q(u) := \begin{cases} \frac{f(u)}{u} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases} \quad (6.3)$$

$Q \in \mathcal{C}(\mathbb{R})$ because $f \in \mathcal{C}^1(\mathbb{R})$. Moreover, by (1.4), there exist $u_0 > 0$ and a constant $C > 0$ such that

$$\frac{f(u)}{u^{q-1}} \leq C \quad \text{for all } u \geq u_0. \quad (6.4)$$

As we are assuming that $0 < q - 1 \leq 1$, for every $u \geq u_0$ we have that

$$Q(u) = \frac{f(u)}{u} = \frac{f(u)}{u^{q-1}} \frac{1}{u^{2-q}} \leq \frac{C}{u_0^{2-q}}.$$

Hence, Q is globally bounded in $[0, +\infty)$. Consequently, $\{Q(u_n)\}_{n \geq 1}$ is dominated in $L^1(r, s)$ by large constants.

By definition, for every $n \geq 1$ and a.e. $x \in [r, s]$,

$$\begin{aligned} -u_n''(x) &= \lambda_n a(x) f(u_n(x)) [1 + (u_n'(x))^2]^{\frac{3}{2}} \\ &> \lambda_n a(x) f(u_n(x)) = \lambda_n a(x) Q(u_n(x)) u_n(x). \end{aligned} \quad (6.5)$$

Thus, since $u_n(r) > 0$ and $u_n(s) > 0$, u_n provides us with a strict positive supersolution in $[r, s]$ of the second order operator

$$\mathfrak{L}_n := -D^2 - \lambda_n a(x) Q(u_n(x))$$

subject to homogeneous Dirichlet boundary conditions, \mathfrak{D} . Consequently, thanks to [38, Theorem 2.1] (see [37, Theorem 7.10]), it is apparent that it satisfies the strong maximum principle, or, equivalently, its principal eigenvalue must be positive. Thus, we get

$$\Sigma_n := \sigma[-D^2 - \lambda_n a(x) Q(u_n(x)), \mathfrak{D}, (r, s)] > 0 \quad \text{for all } n \geq 1. \quad (6.6)$$

By the variational characterization of Σ_n , we also have that

$$\Sigma_n = \inf_{\substack{\psi \in H_0^1(r,s) \\ \psi > 0, \int_r^s \psi^2 = 1}} \left\{ \int_r^s (\psi')^2 dx - \lambda_n \int_r^s a(x) Q(u_n(x)) \psi(x) dx \right\} > 0 \quad (6.7)$$

for all $n \geq 1$. Actually, the infimum is reached at any principal eigenfunction, $\varphi_n > 0$, associated to Σ_n . By (6.2) and (6.3), we obtain

$$\lim_{n \rightarrow +\infty} Q(u_n(x)) = 1 \quad \text{a.e. in } [r, s].$$

Therefore, since $Q(u_n)$ is dominated in $L^1(r, s)$ by the large positive constants, the theorem of Lebesgue establishes that

$$\lim_{n \rightarrow +\infty} \int_r^s a(x)Q(u_n(x))\psi(x) dx = \int_r^s a(x)\psi(x) dx > 0 \quad (6.8)$$

for all $\psi \in H_0^1(r, s)$ such that $\psi > 0$ in (r, s) and $\int_r^s \psi^2 dx = 1$. Consequently, since $\lim_{n \rightarrow +\infty} \lambda_n = +\infty$,

$$\lim_{n \rightarrow +\infty} \left(\int_r^s (\psi')^2 dx - \lambda_n \int_r^s a(x)Q(u_n(x))\psi(x) dx \right) = -\infty,$$

which contradicts (6.7), and ends the proof. \square

Remark 6.1. The result stated in Theorem 6.1 is still valid for the singular (bounded variation) solutions whose existence is guaranteed by Theorem 1.1, provided that condition (6.1) holds. This follows from an inspection of the above proof, observing that, from (6.5) on, the interval $[r, s]$ should be replaced by a subinterval $[r', s']$ with $r < r' < s' < s$, in order to guarantee that $u_n \in W^{2,1}(r', s')$ for all $n \geq 1$. Then the remainder of the proof, until the conclusion, should be modified accordingly, replacing $[r, s]$ with $[r', s']$ everywhere.

Remark 6.2. When $q > 2$ the argument of the previous proof fails, because $2 - q < 0$ and, in such case, $Q(u)$ is not bounded above. However, if we would be able to establish

$$\lim_{n \rightarrow +\infty} u_n(x) = 0 \quad \text{uniformly in } [r, s], \quad (6.9)$$

then we would get the same result with a rather direct argument. Indeed, in such case,

$$\lim_{n \rightarrow +\infty} \frac{f(u_n)}{u_n} = 1 \quad \text{uniformly in } [r, s],$$

and hence, for sufficiently large n , say $n \geq n_0$,

$$-u_n''(x) > \lambda_n a(x) f(u_n(x)) > \frac{\lambda_n}{2} a(x), \quad x \in [r, s],$$

which implies

$$\sigma[-D^2 - \frac{\lambda_n}{2} a(x), \mathfrak{D}, (r, s)] > 0, \quad n \geq n_0,$$

though,

$$\lim_{n \rightarrow +\infty} \sigma[-D^2 - \frac{\lambda_n}{2} a(x), \mathfrak{D}, (r, s)] = -\infty,$$

which concludes the proof.

7. NONEXISTENCE FOR F SUPERLINEAR AT $+\infty$ AND $\lambda > 0$ SMALL

The main theorem of this section establishes the non-existence of a regular positive solutions of (1.1) for sufficiently small $\lambda > 0$ under the assumptions of Theorem 6.1 for a special, but significant, class of weight functions a . Note that, thanks to Theorem 1.1, the problem possesses a singular (bounded variation) solution for such range of λ 's.

Theorem 7.1. *Suppose $f \in \mathcal{C}^2(\mathbb{R})$ satisfies (Hf), $a \in L^\infty(0, 1)$, $\int_0^1 a dx < 0$, and there exists $z \in (0, 1)$ such that $a(x) > 0$ for all $x \in (0, z)$ and $a(x) < 0$ for all $x \in (z, 1)$. Assume, in addition, that there exist $q \in (1, 2]$ and $h > 0$ such that (1.6) holds. Then, (1.1) cannot admit a positive regular solution for sufficiently small $\lambda > 0$.*

Naturally, a result similar to Theorem 7.1 holds if $a(x) < 0$ in $(0, z)$ and $a(x) > 0$ in $(z, 1)$.

Theorem 7.1 measures how sharp are Theorems 1.1 and 3.1, establishing that the solution provided by Theorem 1.1 must be singular for sufficiently small $\lambda > 0$. Note that (1.6) implies

$$\lim_{u \rightarrow +\infty} \frac{f(u)}{u^{q-1}} = qh \quad \text{and} \quad \lim_{u \rightarrow +\infty} \frac{F(u)}{u^q} = h.$$

Hence f and F satisfy (1.4) and (1.5), respectively, with $1 < q \leq 2$. Moreover, since $q \in (1, 2]$, there exists a constant $C > 0$ such that

$$|f'(u)| \leq C \quad \text{for all } u \in [0, +\infty). \quad (7.1)$$

The proof of Theorem 7.1 follows after a series of lemmas of technical nature. Under the assumptions imposed to a , we have that

$$u''(x) = -\lambda a(x)f(u(x))g(u'(x)) \begin{cases} < 0 & \text{if } x \in (0, z), \\ > 0 & \text{if } x \in (z, 1). \end{cases}$$

Thus, u' is decreasing in $(0, z)$ and increasing in $(z, 1)$. In particular, since $u'(0) = u'(1) = 0$, we find that $u'(x) < 0$ for all $x \in (0, 1)$. Therefore,

$$u \text{ is strictly decreasing in } (0, 1). \quad (7.2)$$

Actually, this is why we have chosen a with this so special nodal configuration. The proof of Theorem 7.1 will proceed by contradiction assuming that

(H) problem (1.1) possesses a sequence of positive regular solutions, $\{(\lambda_n, u_n)\}_{n \geq 1}$, such that

$$\lim_{n \rightarrow +\infty} \lambda_n = 0.$$

Lemma 7.1. *Suppose (H). Then, under the same assumptions of Theorem 7.1,*

$$\lim_{n \rightarrow +\infty} u_n(0) = +\infty. \quad (7.3)$$

Proof. On the contrary, suppose that there exists a constant $C > 0$ such that, along some subsequence, relabeled by n ,

$$0 < u_n(0) \leq C, \quad n \geq 1. \quad (7.4)$$

Then, integrating (1.1) in $(0, z)$, we find from (Hf), (7.2) and (7.4) that

$$\frac{-u'_n(z)}{\sqrt{1 + (u'_n(z))^2}} = \lambda_n \int_0^z a(x)f(u_n(x)) dx \leq \lambda_n f(C) \int_0^z a(x) dx.$$

On the other hand, by (H), there exists an integer, $n_0 \in \mathbb{N}$, such that

$$\lambda_n < \frac{1}{2f(C) \int_0^z a(x) dx}, \quad n \geq n_0.$$

Hence,

$$\frac{-u'_n(z)}{\sqrt{1 + (u'_n(z))^2}} < \frac{1}{2} \quad \text{for all } n \geq n_0. \quad (7.5)$$

This estimate entails the existence of a constant $C_1 > 0$ such that

$$\|u'_n\|_{L^\infty(0,1)} \leq C_1 \quad \text{for all } n \geq n_0. \quad (7.6)$$

Therefore, $\{u_n : n \geq n_0\}$ is bounded and equicontinuous in $\mathcal{C}[0, 1]$ and hence, by the Ascoli–Arzelá theorem, along some subsequence, again labeled by n , we find that

$$\lim_{n \rightarrow +\infty} u_n = u_\omega \quad \text{in } \mathcal{C}[0, 1] \quad (7.7)$$

for some $u_\omega \in \mathcal{C}[0, 1]$.

On the other hand, it follows from the definition of (λ_n, u_n) that

$$-u''_n = \lambda_n a f(u_n) g(u'_n), \quad n \geq 1,$$

and, since

$$\|a f(u_n) g(u'_n)\|_{L^\infty(0,1)} \leq \|a\|_{L^\infty(0,1)} f(C) g(C_1),$$

the function

$$V_n(x) := \lambda_n a(x) f(u_n(x)) g(u_n'(x)), \quad n \geq 1, \quad x \in [0, 1],$$

satisfies

$$-u_n''(x) = V_n(x) \quad \text{for all } x \in [0, 1] \quad \text{and} \quad \lim_{n \rightarrow +\infty} \|V_n\|_{L^\infty(0,1)} = 0.$$

Consequently,

$$u_n(x) = u_n(0) - \int_0^x \int_0^s V_n(\sigma) d\sigma ds, \quad n \geq 1. \quad (7.8)$$

Since $V_n \rightarrow 0$ uniformly in $[0, 1]$, letting $n \rightarrow +\infty$ in (7.8) yields

$$\lim_{n \rightarrow +\infty} u_n(x) = u_\omega(0) \quad \text{for all } x \in [0, 1].$$

In particular, u_ω must be constant.

On the other hand, integrating the differential equation of (1.1) in the interval $(0, 1)$ provides us with the identity

$$\int_0^1 a f(u_n) dx = 0, \quad n \geq 1. \quad (7.9)$$

Thus, letting $n \rightarrow +\infty$ in (7.9) yields

$$0 = \int_0^1 a f(u_\omega) dx = f(u_\omega(0)) \int_0^1 a dx,$$

which implies $f(u_\omega(0)) = 0$. Therefore, by (Hf), $u_\omega = 0$. Consequently, $(\lambda, u) = (0, 0)$ is a bifurcation point to positive solutions of the form (λ_n, u_n) , $n \geq 1$, with $\lambda_n > 0$. This contradicts Theorem 4.1 and ends the proof. \square

Lemma 7.2. *Suppose (H). Then, under the same assumptions of Theorem 7.1,*

$$\lim_{n \rightarrow +\infty} u_n(z) = +\infty. \quad (7.10)$$

Consequently, thanks to (7.2),

$$\lim_{n \rightarrow +\infty} u_n = +\infty \quad \text{uniformly in } [0, z]. \quad (7.11)$$

Proof. On the contrary, suppose that there exists a constant $C > 0$ such that, along some subsequence relabeled by n ,

$$0 < u_n(z) \leq C, \quad n \geq 1. \quad (7.12)$$

Then, by (7.2) and (7.9),

$$\begin{aligned} \int_0^z a(x) f(u_n(x)) dx &= - \int_z^1 a(x) f(u_n(x)) dx \\ &\leq -f(u_n(z)) \int_z^1 a(x) dx \leq -f(C) \int_z^1 a(x) dx. \end{aligned}$$

Thus, there exists a constant $\tilde{C} > 0$ such that

$$\int_0^z a(x) f(u_n(x)) dx \leq \tilde{C}, \quad n \geq 1,$$

and hence,

$$\frac{-u_n'(z)}{\sqrt{1 + (u_n'(z))^2}} = \lambda_n \int_0^z a(x) f(u_n(x)) dx \leq \tilde{C} \lambda_n, \quad n \geq 1.$$

Therefore, we have

$$|u_n'(z)| \leq \frac{\lambda_n \tilde{C}}{\sqrt{1 - \lambda_n^2 \tilde{C}^2}}$$

for sufficiently large $n \geq 1$, and, consequently,

$$\lim_{n \rightarrow +\infty} u_n'(z) = 0,$$

which entails

$$\lim_{n \rightarrow +\infty} \|u'_n\|_{L^\infty(0,1)} = 0, \quad (7.13)$$

because the minimum of u'_n is reached at z (see Section 5, if necessary). Since

$$u_n(0) = u_n(z) - \int_0^z u'_n(x) dx,$$

(7.12) and (7.13) imply that $\{u_n(0)\}_{n \geq 1}$ is bounded, which is impossible by Lemma 7.1. The proof is complete. \square

Lemma 7.3. *Suppose (H). Then, under the same assumptions of Theorem 7.1,*

$$\lim_{n \rightarrow +\infty} u'_n(z) = -\infty. \quad (7.14)$$

Proof. On the contrary, suppose that along some subsequence, relabeled by n ,

$$|u'_n(z)| \leq C, \quad n \geq 1. \quad (7.15)$$

Then, since the minimum of u'_n is reached at z , we have that

$$-u'_n(z) = \|u'_n\|_{L^\infty(0,1)} \leq C, \quad n \geq 1. \quad (7.16)$$

In particular, the sequence

$$v_n := \frac{u_n}{\|u_n\|_{L^\infty(0,1)}}, \quad n \geq 1,$$

satisfies $\|v_n\|_{L^\infty(0,1)} = 1$ for all $n \geq 1$ and, owing to (7.15) and (7.16),

$$\|v'_n\|_{L^\infty(0,1)} = \frac{\|u'_n\|_{L^\infty(0,1)}}{\|u_n\|_{L^\infty(0,1)}} = \frac{-u'_n(z)}{u_n(0)} \leq \frac{C}{u_n(0)}$$

for all $n \geq 1$. Thus, thanks to Lemma 7.1,

$$\lim_{n \rightarrow +\infty} \|v'_n\|_{L^\infty(0,1)} = 0.$$

Thus, thanks to the Ascoli–Arzelá theorem, we can assume, without loss of generality, that there exists $v_\omega \in \mathcal{C}[0,1]$ such that

$$\lim_{n \rightarrow +\infty} v_n = v_\omega \quad \text{in } \mathcal{C}[0,1].$$

Letting $n \rightarrow +\infty$ in the relation

$$v_n(x) - v_n(0) = \int_0^x v'_n(t) dt, \quad x \in [0,1],$$

we conclude that v_ω is constant and, as $v_n(0) = 1$ for all $n \geq 1$,

$$v_\omega = 1 \quad \text{in } [0,1].$$

Consequently, we find that

$$\lim_{n \rightarrow +\infty} v_n = \lim_{n \rightarrow +\infty} \frac{u_n}{\|u_n\|_{L^\infty(0,1)}} = \lim_{n \rightarrow +\infty} \frac{u_n}{u_n(0)} = 1 \quad \text{uniformly in } [0,1]. \quad (7.17)$$

Finally, from (7.9) it is easily seen that

$$\int_0^z a(x) f(u_n(x)) dx = - \int_z^1 a(x) f(u_n(x)) dx, \quad n \geq 1,$$

and hence,

$$\int_0^z a(x) \frac{f(u_n(x))}{u_n^{q-1}(0)} dx = - \int_z^1 a(x) \frac{f(u_n(x))}{u_n^{q-1}(0)} dx, \quad n \geq 1. \quad (7.18)$$

Consequently, letting $n \rightarrow +\infty$ in (7.18), from (1.4) —which follows from (1.6)— and (7.17), it becomes apparent that

$$qh \int_0^z a dx = -qh \int_z^1 a dx,$$

which contradicts $\int_0^1 a < 0$ and ends the proof. \square

Lemma 7.4. *Suppose (H). Then, under the same assumptions of Theorem 7.1,*

$$\begin{aligned} f(u_n(1)) &< \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \frac{1}{\lambda_n \int_z^1 (-a) dx} \\ &< f(u_n(z)) < \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \frac{1}{\lambda_n \int_0^z a dx} < f(u_n(0)) \end{aligned} \quad (7.19)$$

for all $n \geq 1$.

Proof. Arguing as in the proof of Lemma 2.1, it becomes apparent that

$$\lambda_n \int_0^z a dx = \frac{1}{f(u_n(z))} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} + \int_0^z \left(\frac{1}{f(u_n(x))} \right)' \frac{u'_n(x)}{\sqrt{1+(u'_n(x))^2}} dx.$$

Moreover,

$$\frac{u'_n(x)}{\sqrt{1+(u'_n(x))^2}} > \frac{u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \quad \text{for all } x \in [0, z),$$

because

$$\left(\frac{u'_n(x)}{\sqrt{1+(u'_n(x))^2}} \right)' = -\lambda_n a(x) f(u_n(x)) \begin{cases} < 0 & \text{if } x \in (0, z), \\ > 0 & \text{if } x \in (z, 1). \end{cases}$$

Thus, since $x \mapsto f(u_n(x))$ is decreasing for all $n \geq 1$, we find that

$$\begin{aligned} \lambda_n \int_0^z a dx &> \frac{1}{f(u_n(z))} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} + \frac{u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \int_0^z \left(\frac{1}{f(u_n(x))} \right)' dx \\ &= \frac{u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \left[-\frac{1}{f(u_n(z))} + \frac{1}{f(u_n(z))} - \frac{1}{f(u_n(0))} \right] \\ &= \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \frac{1}{f(u_n(0))}, \end{aligned}$$

which provides us with the last estimate of (7.19). Moreover,

$$\lambda_n \int_0^z a dx < \frac{1}{f(u_n(z))} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}},$$

which provides us with the third one.

Similarly, integrating in the interval $(z, 1)$ yields

$$\lambda_n \int_z^1 a dx = \frac{1}{f(u_n(z))} \frac{u'_n(z)}{\sqrt{1+(u'_n(z))^2}} + \int_z^1 \left(\frac{1}{f(u_n(x))} \right)' \frac{u'_n(x)}{\sqrt{1+(u'_n(x))^2}} dx.$$

Hence, we have

$$\begin{aligned} \lambda_n \int_z^1 (-a) dx &= \frac{1}{f(u_n(z))} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} + \int_z^1 \left(\frac{1}{f(u_n(x))} \right)' \frac{-u'_n(x)}{\sqrt{1+(u'_n(x))^2}} dx \\ &> \frac{1}{f(u_n(z))} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}}, \end{aligned}$$

which establishes the second estimate of (7.19). Finally, we get

$$\begin{aligned}
\lambda_n \int_z^1 (-a) dx &= \frac{1}{f(u_n(z))} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} + \int_z^1 \left(\frac{1}{f(u_n(x))} \right)' \frac{-u'_n(x)}{\sqrt{1+(u'_n(x))^2}} dx \\
&< \frac{1}{f(u_n(z))} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} + \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \int_z^1 \left(\frac{1}{f(u_n(x))} \right)' dx \\
&= \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \left[\frac{1}{f(u_n(z))} + \frac{1}{f(u_n(1))} - \frac{1}{f(u_n(z))} \right] \\
&= \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \frac{1}{f(u_n(1))},
\end{aligned}$$

which provides us with the first estimate of (7.19). The proof is complete. \square

Corollary 7.1. *Suppose (H). Then, under the same assumptions of Theorem 7.1,*

$$\begin{aligned}
\limsup_{n \rightarrow +\infty} [\lambda_n f(u_n(1))] &\leq \left(- \int_z^1 a dx \right)^{-1} \\
&\leq \liminf_{n \rightarrow +\infty} [\lambda_n f(u_n(z))] \leq \limsup_{n \rightarrow +\infty} [\lambda_n f(u_n(z))] \\
&\leq \left(\int_0^z a dx \right)^{-1} \leq \liminf_{n \rightarrow +\infty} [\lambda_n f(u_n(0))].
\end{aligned} \tag{7.20}$$

Proof. It follows easily from Lemma 7.3 multiplying (7.19) by λ_n and letting $n \rightarrow +\infty$ in the resulting inequalities. \square

As a direct consequence from (7.19), the following estimates hold

$$\begin{aligned}
f(u_n(0)) - f(u_n(1)) &> \frac{1}{\lambda_n} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \left(\int_0^z a dx - \int_z^1 (-a) dx \right) \\
&= \frac{1}{\lambda_n} \frac{-u'_n(z)}{\sqrt{1+(u'_n(z))^2}} \frac{- \int_0^z a dx}{\int_0^z a dx \int_z^1 (-a) dx}
\end{aligned}$$

and, letting $n \rightarrow +\infty$, Lemma 7.3 implies that

$$\lim_{n \rightarrow +\infty} [f(u_n(0)) - f(u_n(1))] = +\infty.$$

Consequently, by (1.6), the maximal oscillation of u_n in the interval $[0, 1]$ blows-up as $n \rightarrow +\infty$.

Proof of Theorem 7.1. Suppose (H) and consider the sequence

$$v_n(x) = \lambda_n f(u_n(x)), \quad x \in [0, 1], \quad n \geq 1. \tag{7.21}$$

Then, according to Corollary 7.1,

$$\left(- \int_z^1 a dx \right)^{-1} \leq \liminf_{n \rightarrow +\infty} v_n(z) \leq \limsup_{n \rightarrow +\infty} v_n(z) \leq \left(\int_0^z a dx \right)^{-1}. \tag{7.22}$$

In particular, the sequence $\{v_n\}_{n \geq 1}$ is uniformly bounded in the interval $[z, 1]$.

Moreover, differentiating with respect to x ,

$$v'_n(x) = \lambda_n f'(u_n(x)) u'_n(x) \leq 0, \quad x \in [0, 1], \quad n \geq 1,$$

and hence, for every $x \in [0, 1]$ and $n \geq 1$,

$$- \left(\frac{\frac{v'_n(x)}{\lambda_n f'(u_n(x))}}{\sqrt{1 + \left(\frac{v'_n(x)}{\lambda_n f'(u_n(x))} \right)^2}} \right)' = a(x) v_n(x).$$

Equivalently,

$$-\left(\frac{v'_n(x)}{\sqrt{(\lambda_n f'(u_n(x)))^2 + (v'_n(x))^2}}\right)' = a(x)v_n(x) \quad (7.23)$$

for all $x \in [0, 1]$ and $n \geq 1$.

Let $y \in [z, 1]$ be. Integrating in $[y, 1]$ the identity (7.23) yields

$$\frac{v'_n(y)}{\sqrt{(\lambda_n f'(u_n(y)))^2 + (v'_n(y))^2}} = \int_y^1 a(\sigma)v_n(\sigma) d\sigma. \quad (7.24)$$

Therefore, thanks to (7.1), letting $n \rightarrow +\infty$ in the identity (7.24) and taking into account that $\lim_{n \rightarrow +\infty} \lambda_n = 0$ reveal that

$$\lim_{n \rightarrow +\infty} \int_y^1 a(\sigma)v_n(\sigma) d\sigma = -1 \quad \text{for all } y \in [z, 1], \quad (7.25)$$

because $v'_n(y) < 0$ for all $y \in [z, 1]$ and $n \geq 1$.

As the sequence $\{v_n\}_{n \geq 1}$ is bounded in $L^\infty(z, 1)$, there exists a subsequence, still labeled as $\{v_n\}_{n \geq 1}$, which converges in the weak* topology to some $\bar{v} \in L^\infty(z, 1)$. Hence, for any given $y \in [z, 1]$,

$$\lim_{n \rightarrow +\infty} \int_y^1 a(\sigma)v_n(\sigma) d\sigma = \int_y^1 a(\sigma)\bar{v}(\sigma) d\sigma.$$

This implies that

$$\int_y^1 a(\sigma)\bar{v}(\sigma) d\sigma = -1 \quad \text{for all } y \in [z, 1]. \quad (7.26)$$

Therefore, differentiating (7.26), we get $\bar{v} = 0$ a.e. in $[z, 1]$, which contradicts (7.22) and ends the proof. \square

8. A PARADIGMATIC EXAMPLE

Throughout this section we will assume that the function a is given by

$$a(x) := \begin{cases} A & \text{if } 0 \leq x < z, \\ -B & \text{if } z < x \leq 1, \end{cases} \quad (8.1)$$

for some $z \in (0, 1)$ and some constants, $A > 0$ and $B > 0$ such that

$$\int_0^1 a dx = zA - (1-z)B < 0, \quad (8.2)$$

and that $f(u) = u$ for all $u \geq 0$. Thus, we are dealing with the problem

$$\begin{cases} -u'' = \lambda a(x)u[1 + (u')^2]^{\frac{3}{2}}, & 0 < x < 1, \\ u'(0) = u'(1) = 0, \end{cases} \quad (8.3)$$

Multiplying the differential equation of (8.3) by u' yields

$$-u'u''[1 + (u')^2]^{-\frac{3}{2}} = \lambda a u u',$$

which can be written down as

$$\left(\frac{1}{\sqrt{1+v^2}}\right)' = \lambda a u v, \quad v \equiv u'. \quad (8.4)$$

According to Theorem 3.1, the set of positive solutions of (8.3) possesses a component, $\mathfrak{C}_{\lambda_0}^+$, such that $(\lambda_0, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+$ and

$$(\lambda, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+ \quad \text{with } \lambda \neq 0 \implies \lambda = \lambda_0.$$

Moreover, by Theorem 7.1, the problem (8.3) cannot admit a positive regular solution, (λ, u) , for sufficiently small $\lambda > 0$, and, thanks to Theorem 6.1, it cannot admit a positive solution for sufficiently large $\lambda > \lambda_0$ neither. Therefore,

$$\mathcal{P}_\lambda(\mathfrak{C}_{\lambda_0}^+) \text{ is a compact subinterval of } (0, +\infty).$$

Our main goal in this section is describing analytically all the positive solutions of (8.3) with $\lambda > 0$ to show that the regular solutions of this problem do indeed develop singularities at z as the parameter λ reaches some critical value, $\lambda^* > 0$.

Suppose $\lambda > 0$ and (λ, u) is a positive solution of (8.3). As we are working under the general assumptions of Theorem 7.1, $u(x)$ is strictly decreasing in $(0, 1)$. In particular,

$$u_0 := u(0) > u_z := u(z) > u_1 := u(1).$$

Moreover, $v(z) = u'(z) \in (-\infty, 0)$. Thus, setting

$$w \equiv w(z) := 1 - \frac{1}{\sqrt{1 + v^2(z)}}, \quad (8.5)$$

it is apparent that $v(z)$ runs in $(-\infty, 0)$ if and only if $w(z) \in (0, 1)$. Moreover, $v(z) = 0$ if $w(z) = 0$, while $v(z) = -\infty$ if $w(z) = 1$. Throughout most of this section we will consider $v(z)$, or equivalently $w(z)$, as a parameter ranging in $(-\infty, 0)$. In such case, $w(z)$ ranges in $(0, 1)$. Singular bounded variation solutions, as discussed by the authors in [41] will arise when $v(z) = -\infty$, i.e., $w(z) = 1$.

8.1. Analysis in the interval $(0, z)$. In the interval $(0, z)$, $a \equiv A$ and (8.4) becomes

$$\left(\frac{1}{\sqrt{1 + v^2}} - \frac{\lambda A}{2} u^2 \right)' = 0.$$

Thus, since $v(0) = u'(0) = 0$,

$$\frac{1}{\sqrt{1 + v^2(x)}} - \frac{\lambda A}{2} u^2(x) = 1 - \frac{\lambda A}{2} u_0^2 = \frac{1}{\sqrt{1 + v^2(z)}} - \frac{\lambda A}{2} u_z^2$$

for all $x \in [0, z]$. Hence,

$$\frac{1}{\sqrt{1 + v^2(x)}} = 1 - \frac{\lambda A}{2} (u_0^2 - u^2(x)) \quad \text{for all } x \in [0, z].$$

Moreover, using (8.5),

$$\frac{\lambda A}{2} u_z^2 = \frac{\lambda A}{2} u_0^2 - w. \quad (8.6)$$

Consequently, since $v = u' < 0$,

$$v(x) = -\sqrt{\left[1 - \frac{\lambda A}{2} (u_0^2 - u^2(x))\right]^{-2} - 1}, \quad x \in [0, z], \quad (8.7)$$

and therefore, performing the changes of variables $u = u(x)$ and $u = u_0\theta$ yields

$$z = -\int_{u_0}^{u_z} \frac{du}{\sqrt{\left[1 - \frac{\lambda A}{2} (u_0^2 - u^2)\right]^{-2} - 1}} = \int_{\frac{u_z}{u_0}}^1 \frac{u_0 d\theta}{\sqrt{\left[1 - \frac{\lambda A}{2} u_0^2 (1 - \theta^2)\right]^{-2} - 1}}. \quad (8.8)$$

Subsequently, we will set

$$\gamma := \frac{\lambda A}{2} u_0^2. \quad (8.9)$$

Owing to (8.6), we have that

$$\frac{u_z^2}{u_0^2} = 1 - \frac{w}{\gamma} > 0.$$

So, $\gamma > w$ and (8.8) can be expressed as

$$z = u_0 I(\gamma, w), \quad (8.10)$$

where

$$I(\gamma, w) := \int_{\sqrt{1-\frac{w}{\gamma}}}^1 \frac{d\theta}{\sqrt{[1-\gamma(1-\theta^2)]^{-2}-1}}, \quad \gamma > w, \quad 0 \leq w \leq 1. \quad (8.11)$$

Note that

$$C := \lim_{(\gamma, w) \rightarrow (1, 1)} I(\gamma, w) = I(1, 1) = \int_0^1 \frac{d\theta}{\sqrt{\theta^{-4}-1}} = 0.59907 \dots < 1$$

is the constant defined in [14, Eq. (1.7)]. By using the change of variable

$$t = \gamma(1 - \theta^2), \quad \sqrt{1 - \frac{w}{\gamma}} \leq \theta \leq 1,$$

$I(\gamma, w)$ can be written as

$$I(\gamma, w) = \frac{1}{2\gamma} \int_0^w \frac{1-t}{\sqrt{2-t}\sqrt{t}\sqrt{1-\frac{t}{\gamma}}} dt = \frac{1}{2\sqrt{\gamma}} \int_0^w \frac{1-t}{\sqrt{2-t}\sqrt{t}\sqrt{\gamma-t}} dt. \quad (8.12)$$

Obviously, $I(\gamma, w)$ is a decreasing function of γ such that

$$\lim_{\gamma \rightarrow +\infty} (\gamma I(\gamma, w)) = \frac{1}{2} \int_0^w \frac{1-t}{\sqrt{2-t}\sqrt{t}} dt = \frac{1}{2} \sqrt{2w-w^2}, \quad (8.13)$$

because

$$\int_0^w \frac{1-t}{\sqrt{2-t}\sqrt{t}} dt = \frac{1}{2} \int_0^w (2-2t)(2t-t^2)^{-\frac{1}{2}} dt = \left[\sqrt{2t-t^2} \right]_{t=0}^{t=w} = \sqrt{2w-w^2}.$$

8.2. Analysis in the interval $(z, 1)$. In the interval $(z, 1)$, $a \equiv -B$ and (8.4) becomes

$$\left(\frac{1}{\sqrt{1+v^2}} + \frac{\lambda B}{2} u^2 \right)' = 0.$$

Thus, since $v(1) = u'(1) = 0$,

$$\frac{1}{\sqrt{1+v^2(x)}} + \frac{\lambda B}{2} u^2(x) = \frac{1}{\sqrt{1+v^2(z)}} + \frac{\lambda B}{2} u_z^2 = 1 + \frac{\lambda B}{2} u_1^2 \quad (8.14)$$

for all $x \in [z, 1]$. Hence, we get

$$\frac{1}{\sqrt{1+v^2(x)}} = 1 - \frac{\lambda B}{2} (u^2(x) - u_1^2) \quad \text{for all } x \in [z, 1],$$

and using (8.5) yields

$$\frac{\lambda B}{2} u_z^2 = \frac{\lambda B}{2} u_1^2 + w. \quad (8.15)$$

Consequently, since $v = u' < 0$,

$$v(x) = -\sqrt{\left[1 - \frac{\lambda B}{2} (u^2(x) - u_1^2)\right]^{-2} - 1}, \quad x \in [z, 1], \quad (8.16)$$

and therefore, performing the changes of variables $u = u(x)$ and $u = u_1\theta$ shows that

$$1 - z = - \int_{u_z}^{u_1} \frac{du}{\sqrt{\left[1 - \frac{\lambda B}{2} (u^2 - u_1^2)\right]^{-2} - 1}} = \int_1^{\frac{u_z}{u_1}} \frac{u_1 d\theta}{\sqrt{\left[1 - \frac{\lambda B}{2} u_1^2 (\theta^2 - 1)\right]^{-2} - 1}}. \quad (8.17)$$

Subsequently, we set

$$\eta := \frac{\lambda B}{2} u_1^2 > 0. \quad (8.18)$$

Then, it follows from (8.15) that

$$\frac{u_z^2}{u_1^2} = 1 + \frac{w}{\eta}$$

and hence, (8.17) can be expressed as

$$1 - z = u_1 J(\eta, w), \quad (8.19)$$

where

$$J(\eta, w) := \int_1^{\sqrt{1+\frac{w}{\eta}}} \frac{d\theta}{\sqrt{[1-\eta(\theta^2-1)]^{-2}-1}}. \quad (8.20)$$

Note that, performing the change of variable

$$t = \eta(\theta^2 - 1), \quad 1 \leq \theta \leq \sqrt{1 + \frac{w}{\eta}},$$

it becomes apparent that

$$J(\eta, w) = \frac{1}{2\eta} \int_0^w \frac{1-t}{\sqrt{2-t}\sqrt{t}\sqrt{1+\frac{t}{\eta}}} dt = \frac{1}{2\sqrt{\eta}} \int_0^w \frac{1-t}{\sqrt{2-t}\sqrt{t}\sqrt{\eta+t}} dt. \quad (8.21)$$

Obviously, $J(\eta, w)$ is decreasing in η . Moreover, since

$$\int_0^w \frac{1-t}{t\sqrt{2-t}} dt = +\infty, \quad w \in (0, 1],$$

we find that

$$\lim_{\eta \rightarrow 0^+} (\sqrt{\eta} J(\eta, w)) = +\infty, \quad w \in (0, 1]. \quad (8.22)$$

Lastly,

$$\lim_{\eta \rightarrow +\infty} (\eta J(\eta, w)) = \frac{1}{2} \int_0^w \frac{1-t}{\sqrt{2-t}\sqrt{t}} dt = \frac{1}{2} \sqrt{2w-w^2}. \quad (8.23)$$

8.3. Existence of regular and singular positive solutions in $(0, 1)$. As a by-product of the previous analysis, by well known properties of the integral curves of (8.3), the next result holds.

Lemma 8.1. *The problem (8.3) admits a positive regular solution (λ, u) with $\lambda > 0$ if, and only if, there exists $w \in (0, 1)$ such that*

$$\frac{z}{u_0} = I(\gamma, w), \quad \gamma = \frac{\lambda A}{2} u_0^2, \quad \frac{1-z}{u_1} = J(\eta, w), \quad \eta = \frac{\lambda B}{2} u_1^2, \quad (8.24)$$

where $u_0 = u(0)$ and $u_1 = u(1)$. Moreover, in such case,

$$v(z) = u'(z) = -\sqrt{(1-w)^{-2}-1} \quad (8.25)$$

and

$$\frac{u_z^2}{u_0^2} = 1 - \frac{w}{\gamma}, \quad \frac{u_z^2}{u_1^2} = 1 + \frac{w}{\eta}, \quad (8.26)$$

where $u_z = u'(z)$. Equivalently, (8.3) admits a positive regular solution (λ, u) with $\lambda > 0$ and $u'(z) = v(z) \in (-\infty, 0)$ if, and only if, (8.24) holds for some $u_0 > 0$, $u_1 > 0$, and w given by

$$w = 1 - \frac{1}{\sqrt{1+(v(z))^2}}, \quad (8.27)$$

Should this be the case, then $u(0) = u_0$, $u(1) = u_1$ and (8.26) holds with $u_z = u'(z)$.

The previous analysis can be repeated almost *mutatis mutandis* up to characterize the existence of continuous singular solutions of (8.3), i.e., solutions (λ, u) such that $u \in \mathcal{C}^1[0, z) \cap \mathcal{C}^1(z, 1] \cap \mathcal{C}[0, 1]$ and $v(z) = u'(z) = -\infty$. Naturally, in such case $w = 1$ and hence, the next result holds.

Lemma 8.2. *The problem (8.3) admits a positive continuous singular solution (λ, u) with $\lambda > 0$ if, and only if,*

$$\frac{z}{u_0} = I(\gamma, 1), \quad \gamma = \frac{\lambda A}{2} u_0^2, \quad \frac{1-z}{u_1} = J(\eta, 1), \quad \eta = \frac{\lambda B}{2} u_1^2, \quad (8.28)$$

where $u_0 = u(0)$ and $u_1 = u(1)$. Moreover, in such case,

$$\frac{u_z^2}{u_0^2} = 1 - \frac{1}{\gamma}, \quad \frac{u_z^2}{u_1^2} = 1 + \frac{1}{\eta}, \quad (8.29)$$

where $u_z = u'(z)$.

Suppose (8.3) possesses a positive regular solution, (λ, u) with $\lambda > 0$. Then, according to Lemma 8.1, it follows from (8.24) that

$$u_0 = \sqrt{\frac{2\gamma}{\lambda A}}, \quad u_1 = \sqrt{\frac{2\eta}{\lambda B}}. \quad (8.30)$$

Thus, substituting (8.30) into the first and third identities of (8.24) yields

$$\frac{z\sqrt{\lambda A}}{\sqrt{2\gamma}} = I(\gamma, w), \quad \frac{(1-z)\sqrt{\lambda B}}{\sqrt{2\eta}} = J(\eta, w), \quad (8.31)$$

where

$$w = 1 - \frac{1}{\sqrt{1 + (u'(z))^2}} \in (0, 1).$$

Moreover, (8.26) holds. So,

$$u_0^2 \left(1 - \frac{w}{\gamma}\right) = u_1^2 \left(1 + \frac{w}{\eta}\right)$$

and, thanks to (8.24),

$$\frac{2\gamma}{\lambda A} \left(1 - \frac{w}{\gamma}\right) = \frac{2\eta}{\lambda B} \left(1 + \frac{w}{\eta}\right).$$

Thus, simplifying and reordering this identity yields

$$\gamma = \frac{A}{B}(\eta + w) + w. \quad (8.32)$$

Now, eliminating λ in each of the identities (8.31), it becomes apparent that

$$\frac{\sqrt{2\eta}J(\eta, w)}{(1-z)\sqrt{B}} = \sqrt{\lambda} = \frac{\sqrt{2\gamma}I(\gamma, w)}{z\sqrt{A}}. \quad (8.33)$$

In particular, invoking to (8.32), we find that

$$\frac{\sqrt{\eta}J(\eta, w)}{(1-z)\sqrt{B}} = \frac{\sqrt{\frac{A}{B}(\eta + w) + w} \cdot I\left(\frac{A}{B}(\eta + w) + w, w\right)}{z\sqrt{A}}. \quad (8.34)$$

Therefore, setting

$$G(\eta, w) := \frac{\sqrt{\eta}J(\eta, w)}{(1-z)\sqrt{B}} - \frac{\sqrt{\frac{A}{B}(\eta + w) + w} \cdot I\left(\frac{A}{B}(\eta + w) + w, w\right)}{z\sqrt{A}} \quad (8.35)$$

for every $\eta > 0$ and $w \in (0, 1)$, the following consequence from Lemma 8.1 holds

Corollary 8.1. *The problem (8.3) admits a positive regular solution, (λ, u) with $\lambda > 0$, if, and only if, there exist $\eta > 0$ and $w \in (0, 1)$ such that $G(\eta, w) = 0$. Moreover, in such case, λ satisfies (8.33), u_0 and u_1 are given by (8.30), with γ given by (8.32), $u_z = u'(z)$ satisfies (8.29), and $u'(z)$ is given by (8.25).*

Actually, the positive solutions of (8.33), with $v(z) = u'(z)$ fixed, are in one-to-one correspondence with the zeroes of the map $G(\cdot, w)$, where

$$w = 1 - \frac{1}{\sqrt{1 + (u'(z))^2}} \in (0, 1).$$

Similarly, as a direct consequence from Lemma 8.2 the next result holds:

Corollary 8.2. *The problem (8.3) admits a positive continuous singular solution, (λ, u) , with $\lambda > 0$, if, and only if, there exists $\eta > 0$ such that $G(\eta, 1) = 0$. Moreover, in such case,*

$$\lambda = \left(\frac{\sqrt{2\eta}J(\eta, 1)}{(1-z)\sqrt{B}} \right)^2,$$

u_0 and u_1 are given by (8.30), with $\gamma = \frac{A}{B}(\eta + 1) + 1$, and u_z is given by (8.29).

The next result establishes the existence of (positive) regular and singular solutions of (8.3) within the appropriate ranges of values of the parameters involved in its formulation.

Proposition 8.1. *For every $v \in [-\infty, 0)$, the problem (8.3) admits, at least, one positive solution, (λ, u) , with $\lambda > 0$ and such that $u'(z) = v$. Moreover, these solutions are in one-to-one correspondence with the zeroes of the map $G(\cdot, w)$, where*

$$w = 1 - \frac{1}{\sqrt{1+v^2}} \in (0, 1].$$

Proof. Let $v \in [-\infty, 0)$ be and consider $w = 1 - \frac{1}{\sqrt{1+v^2}}$. Since

$$\lim_{\eta \rightarrow 0} \frac{\sqrt{\frac{A}{B}(\eta+w)+w} \cdot I(\frac{A}{B}(\eta+w)+w, w)}{z\sqrt{A}} = \frac{\sqrt{(\frac{A}{B}+1)w} \cdot I((\frac{A}{B}+1)w, w)}{z\sqrt{A}} < +\infty,$$

letting $\eta \rightarrow 0$ in (8.35) it follows from (8.22) that

$$\lim_{\eta \rightarrow 0} G(\eta, w) = +\infty. \quad (8.36)$$

Moreover, multiplying (8.35) by $2\sqrt{\eta}$ yields

$$2\sqrt{\eta}G(\eta, w) := \frac{2\eta J(\eta, w)}{(1-z)\sqrt{B}} - \frac{2\sqrt{\eta}\sqrt{\frac{A}{B}(\eta+w)+w} \cdot I(\frac{A}{B}(\eta+w)+w, w)}{z\sqrt{A}} \quad (8.37)$$

for all $\eta > 0$. Thus, thanks to (8.23), we have that

$$\lim_{\eta \rightarrow +\infty} (2\eta J(\eta, w)) = \sqrt{2w - w^2}.$$

Similarly, thanks to (8.23), it becomes apparent that

$$\begin{aligned} & \lim_{\eta \rightarrow +\infty} \left(2\sqrt{\eta}\sqrt{\frac{A}{B}(\eta+w)+w} \cdot I(\frac{A}{B}(\eta+w)+w, w) \right) \\ &= \lim_{\eta \rightarrow +\infty} \frac{\sqrt{\frac{A}{B}(\eta^2 + \eta w) + \eta w}}{\frac{A}{B}(\eta+w)+w} \lim_{\eta \rightarrow +\infty} \left(2 \left(\frac{A}{B}(\eta+w)+w \right) I(\frac{A}{B}(\eta+w)+w, w) \right) \\ &= \sqrt{\frac{B}{A}} \sqrt{2w - w^2}. \end{aligned}$$

Consequently, letting $\eta \rightarrow +\infty$ in (8.37) yields

$$\begin{aligned} \lim_{\eta \rightarrow +\infty} (2\sqrt{\eta}G(\eta, w)) &= \left(\frac{1}{(1-z)\sqrt{B}} - \frac{\sqrt{B}}{zA} \right) \sqrt{2w - w^2} \\ &= \frac{zA - B(1-z)}{A\sqrt{B}z(1-z)} \sqrt{2w - w^2} = \frac{\int_0^1 a \, dx}{A\sqrt{B}z(1-z)} \sqrt{2w - w^2} < 0, \end{aligned} \quad (8.38)$$

because of (8.2). Thus, $G(\eta, w) < 0$ for sufficiently large $\eta > 0$ and therefore, by (8.36), there exists $\eta > 0$ such that $G(\eta, w) = 0$. The remaining assertions of the proposition are a by-product of Corollaries 8.1 and 8.2. \square

Remark 8.1. For every $w \in (0, 1]$, the map $G(\cdot, w)$ is real analytic in $\eta > 0$. Therefore, since $G(\eta, w) > 0$ for sufficiently small $\eta > 0$ and $G(\eta, w) < 0$ for sufficiently large $\eta > 0$, the number of zeros of $G(\cdot, w)$, counting them according to their orders, must be odd. Therefore, generically, for every $v \in [-\infty, 0)$, the problem (8.3) possesses an odd number of positive solutions. Actually, they must arise, or shrink to disappear, by pairs.

According to Proposition 8.1, (8.3) possesses a positive continuous singular solution, (λ, u) , with $\lambda > 0$, i.e., a solution with $u'(z) = -\infty$. The next result complements the analysis already done in Section 5 by establishing that any unbounded sequence of positive regular solutions must approach some positive singular solution.

Theorem 8.1. *Let $\{(\lambda_n, u_n)\}_{n \geq 1}$ be a sequence of positive regular solutions of (8.3), with $\lambda_n > 0$, such that*

$$\lim_{n \rightarrow +\infty} \|u_n\|_{C^1[0,1]} = +\infty. \quad (8.39)$$

Then, one can extract a subsequence, $\{(\lambda_{n_m}, u_{n_m})\}_{m \geq 1}$, and a positive continuous singular solution of (8.3), (λ^, u^*) , with $\lambda^* > 0$, such that*

$$\lim_{m \rightarrow +\infty} (\lambda_{n_m}, u_{n_m}) = (\lambda^*, u^*) \quad \text{in } \mathbb{R} \times \mathcal{C}[0, 1]. \quad (8.40)$$

Remark 8.2. As the component $\mathfrak{C}_{\lambda_0}^+$ is unbounded in $\mathbb{R} \times \mathcal{C}^1[0, 1]$ and, thanks to Theorems 6.1 and 7.1, there exists $0 < \alpha \leq \beta$ such that $\mathcal{P}_\lambda(\mathfrak{C}_{\lambda_0}^+) = [\alpha, \beta]$, according to Theorem 8.1, the problem (8.3) possesses a positive singular solution, (λ^*, u^*) , with $\lambda^* \in [\alpha, \beta]$, such that (λ^*, u^*) is a limit point of $\mathfrak{C}_{\lambda_0}^+$ in $\mathbb{R} \times \mathcal{C}[0, 1]$.

Proof. For every $n \geq 1$, we have that

$$\|u_n\|_{C^1[0,1]} = \|u_n\|_{C[0,1]} + \|u_n'\|_{C[0,1]} = u_n(0) - u_n'(z).$$

Thanks to Theorem 7.1, there exists $\alpha > 0$ such that $\lambda_n \geq \alpha$ for all $n \geq 1$. Suppose that, along some subsequence, $\{u_{n_m}\}_{m \geq 1}$,

$$\lim_{m \rightarrow +\infty} u_{n_m}(0) = +\infty. \quad (8.41)$$

Then,

$$\gamma_{n_m} := \frac{\lambda_{n_m} A}{2} u_{n_m}^2(0) \geq \frac{\alpha A}{2} u_{n_m}^2(0) \rightarrow +\infty \quad \text{as } m \rightarrow +\infty.$$

Note that, since

$$w_n := 1 - \frac{1}{\sqrt{1 + (u_n'(z))^2}} \in (0, 1], \quad n \geq 1,$$

setting

$$\eta_n := \frac{B}{A} (\gamma_n - w_n) - w_n, \quad n \geq 1,$$

we also have that

$$\lim_{m \rightarrow +\infty} \eta_{n_m} = +\infty.$$

Moreover, by compactness, along some subsequence, relabeled by n_m ,

$$\lim_{m \rightarrow +\infty} w_{n_m} = w^* \in [0, 1].$$

By Corollary 8.1, $G(\eta_{n_m}, w_{n_m}) = 0$ for all $n \geq 1$ and hence, based on (8.12), (8.21) and (8.38), it becomes apparent that

$$0 = \lim_{m \rightarrow +\infty} (2\sqrt{\eta_{n_m}} G(\eta_{n_m}, w_{n_m})) = \frac{\int_0^1 a \, dx}{A\sqrt{Bz}(1-z)} \sqrt{2w^* - (w^*)^2}.$$

Thus, $w^* = 0$ and consequently,

$$\lim_{m \rightarrow +\infty} u_{n_m}'(z) = 0.$$

So, since

$$-u_n'(z) = \|u_n'\|_{C[0,1]}, \quad n \geq 1,$$

we find that

$$\lim_{m \rightarrow +\infty} \|u'_{n_m}\|_{C[0,1]} = 0. \quad (8.42)$$

Hence, for every $\varepsilon > 0$, there exists $m_0 = m_0(\varepsilon) \in \mathbb{N}$ such that

$$\|u'_{n_m}\|_{C[0,1]} \leq \varepsilon \quad \text{for all } m \geq m_0.$$

Consequently, for every $x \in [0, 1]$ and $m \geq m_0$,

$$|u_{n_m}(x) - u_{n_m}(0)| \leq \varepsilon|x| \leq \varepsilon$$

and therefore, (8.41) yields

$$\lim_{m \rightarrow +\infty} u_{n_m} = +\infty \quad \text{uniformly in } [0, 1]. \quad (8.43)$$

Actually, thanks to Theorem 6.1 and 7.1, without loss of generality, we may assume that

$$\lim_{m \rightarrow +\infty} \lambda_{n_m} = \lambda^* > 0. \quad (8.44)$$

Setting

$$v_m := \frac{u_{n_m}}{\|u_{n_m}\|_{C[0,1]}}, \quad m \geq 1,$$

and dividing by $\|u_{n_m}\|_{C[0,1]}$ the differential equation satisfied by (λ_{n_m}, u_{n_m}) , $m \geq 1$, it is apparent that

$$-v_m'' = \lambda_{n_m} a v_m [1 + (v_m')^2]^{\frac{3}{2}}, \quad m \geq 1. \quad (8.45)$$

Combining (8.42) and (8.44) with a standard compactness argument and letting $m \rightarrow +\infty$ in (8.45) shows that there exist a subsequence of $\{v_m\}_{m \geq 1}$, labeled again by m , and a function $\varphi \in C^1[0, 1] \cap C^2[0, z] \cap C^2(z, 1]$ such that

$$\lim_{m \rightarrow +\infty} v_m = \varphi \quad \text{in } C^1[0, 1] \quad (8.46)$$

and

$$-\varphi'' = \lambda^* a \varphi, \quad \varphi'(0) = \varphi'(1) = 0, \quad \varphi \geq 0, \quad \|\varphi\|_{C[0,1]} = 1.$$

By Theorem 2.1, $\lambda^* = \lambda_0$ and φ must be a principal eigenfunction associated with $\Sigma(\lambda_0)$. Moreover, according to (8.46),

$$\lim_{m \rightarrow +\infty} \frac{u'_{n_m}}{\|u_{n_m}\|_{C[0,1]}} = \varphi'.$$

Thus, (8.42) and (8.43) imply that $\varphi' = 0$ and hence, $\varphi = 1$ in $[0, 1]$, which entails $\lambda_0 a = 0$, a contradiction. Therefore, there exists a constant $C > 0$ such that

$$\|u_n\|_{C[0,1]} = u_n(0) \leq C \quad \text{for all } n \geq 1, \quad (8.47)$$

whereas

$$\lim_{n \rightarrow +\infty} \|u'_n\|_{C[0,1]} = - \lim_{n \rightarrow +\infty} u'_n(z) = +\infty. \quad (8.48)$$

According to Theorems 6.1 and 7.1, we can extract a subsequence, relabeled by n , such that

$$\lim_{n \rightarrow +\infty} \lambda_n = \lambda^* \quad \text{and} \quad \lim_{n \rightarrow +\infty} u_n(0) = u_0^* \quad (8.49)$$

for some $\lambda^* > 0$ and $u_0^* \geq 0$. Should u_0^* be equal zero, the sequence (λ_n, u_n) , $n \geq 1$, would bifurcate from zero and, according to Theorem 4.1, we should also have

$$\lim_{n \rightarrow +\infty} \|u'_n\|_{C[0,1]} = 0,$$

which contradicts (8.48). Thus, $u_0^* > 0$. Moreover, by (8.48),

$$\lim_{n \rightarrow +\infty} w_n = \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{\sqrt{1 + (u'_n(z))^2}} \right) = 1.$$

Similarly,

$$\lim_{n \rightarrow +\infty} \gamma_n = \lim_{n \rightarrow +\infty} \frac{\lambda_n}{2} A(u_n(0))^2 = \frac{\lambda^*}{2} A(u_0^*)^2 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \eta_n = \eta^* := \frac{B}{A}(\gamma^* - 1) - 1.$$

Finally, letting $n \rightarrow +\infty$ in $G(\eta_n, w_n) = 0$, $n \geq 1$, yields $G(\eta^*, 1) = 0$ and therefore, Corollary 8.2 ends the proof. It should be noted that, along some subsequence, we also have that

$$u_z^* := \lim_{n \rightarrow +\infty} u_n(z)$$

is well defined. \square

8.4. Existence of singular discontinuous solutions. As a byproduct of the analysis already done in the previous subsections, the following result holds.

Corollary 8.3. *The pair (λ, u) , with $\lambda > 0$, solves the singular boundary value problem*

$$\begin{cases} -u'' = \lambda au[1 + (u')^2]^{\frac{3}{2}} & \text{in } (0, z), \\ u'(0) = 0, \quad u(0) = u_0 > 0, \\ u'(z) = -\infty, \quad u(z) = u_{z,-} > 0, \end{cases} \quad (8.50)$$

if, and only if,

$$z = u_0 I(\gamma, 1), \quad \gamma = \frac{\lambda A}{2} u_0^2, \quad u_{z,-} = u_0 \sqrt{1 - \gamma^{-1}}. \quad (8.51)$$

Similarly, (λ, u) , with $\lambda > 0$, solves the singular problem

$$\begin{cases} -u'' = \lambda au[1 + (u')^2]^{\frac{3}{2}} & \text{in } (z, 1), \\ u'(z) = -\infty, \quad u(z) = u_{z,+} > 0, \\ u(1) = u_1 > 0, \quad u'(1) = 0, \end{cases} \quad (8.52)$$

if, and only if,

$$1 - z = u_1 J(\eta, 1), \quad \eta = \frac{\lambda B}{2} u_1^2, \quad u_{z,+} = u_1 \sqrt{1 + \eta^{-1}}. \quad (8.53)$$

According to the analysis already done in Section 8.1, we have that $\gamma \mapsto I(\gamma, 1)$ is decreasing and that

$$C := I(1, 1) = \int_0^1 \frac{d\theta}{\sqrt{\theta^{-4} - 1}} = 0.59907\dots, \quad \lim_{\gamma \rightarrow +\infty} (\gamma I(\gamma, 1)) = \frac{1}{2}. \quad (8.54)$$

Thus, the problem (8.50) possesses a solution if, and only if, $u_0 > \frac{z}{C}$. Note that if $u_0 = \frac{z}{C}$, then (8.51) implies $I(\gamma, 1) = C$ and hence,

$$\gamma = 1, \quad \lambda = \frac{2\gamma}{A u_0^2} = \frac{2C^2}{A z^2}, \quad u_{z,-} = 0.$$

Thus, (8.50) admits a singular solution with $u(z) = 0$ if $u_0 = \frac{z}{C}$. Moreover, by the monotonicity of $I(\gamma, 1)$, the singular solution is unique for every $u_0 \geq \frac{z}{C}$ and

$$\gamma = \gamma(u_0) := I^{-1}\left(\frac{z}{u_0}, 1\right). \quad (8.55)$$

Hence,

$$\lambda = \lambda(u_0) := \frac{2\gamma(u_0)}{A u_0^2}. \quad (8.56)$$

In addition, by the properties of $I(\gamma, 1)$, we have that

$$\lim_{u_0 \rightarrow +\infty} \gamma(u_0) = +\infty.$$

Thus, it follows from (8.54) that

$$\lim_{u_0 \rightarrow +\infty} (\gamma(u_0) I(\gamma(u_0), 1)) = \frac{1}{2}.$$

Equivalently,

$$\lim_{u_0 \rightarrow +\infty} \frac{\gamma(u_0) z}{u_0} = \frac{1}{2}. \quad (8.57)$$

Therefore, from (8.56) and (8.57) it becomes apparent that

$$\lim_{u_0 \rightarrow +\infty} (u_0 \lambda(u_0)) = \lim_{u_0 \rightarrow +\infty} \frac{2\gamma(u_0)}{A u_0} = \frac{1}{zA}.$$

In other words,

$$\lambda(u_0) \sim \frac{1}{zAu_0} \quad \text{as } u_0 \rightarrow +\infty. \quad (8.58)$$

In particular,

$$\lim_{u_0 \rightarrow +\infty} \lambda(u_0) = 0$$

and so, (8.50) possesses a positive solution for all

$$\lambda \in \left(0, \frac{2C^2}{z^2A}\right).$$

If we allow $u_{z,-} = 0$, then (8.50) also has a positive solution, with $u(z) = 0$, for $\lambda = \frac{2C^2}{z^2A}$. As a byproduct, thanks to Theorem 3.2,

$$\lambda_0 \leq \frac{2C^2}{z^2A}.$$

Naturally, according to Theorem 7.1, the problem (8.52) cannot admit a solution with $u_{z,+} = u_{z,-}$ for sufficiently small $\lambda > 0$.

Similarly, by the results of Section 8.2, the map $\eta \mapsto J(\eta, 1)$ is decreasing and, thanks to (8.22) and (8.23), we have that

$$\lim_{\eta \rightarrow 0^+} J(\eta, 1) = +\infty, \quad \lim_{\eta \rightarrow +\infty} (\eta J(\eta, 1)) = \frac{1}{2}. \quad (8.59)$$

Thus, the singular problem (8.52) admits a unique solution for every $u_1 > 0$. Moreover,

$$\eta = \eta(u_1) := J^{-1}\left(\frac{1-z}{u_1}, 1\right) \quad (8.60)$$

and hence, by (8.53),

$$\lambda = \lambda(u_1) := \frac{2\eta(u_1)}{Bu_1^2}. \quad (8.61)$$

By the properties of $J(\eta, 1)$, it follows from (8.60) that

$$\lim_{u_1 \rightarrow +\infty} \eta(u_1) = +\infty.$$

Consequently, by (8.59),

$$\lim_{u_1 \rightarrow +\infty} (\eta(u_1)J(\eta(u_1), 1)) = \frac{1}{2}.$$

Equivalently,

$$\lim_{u_1 \rightarrow +\infty} \frac{\eta(u_1)(1-z)}{u_1} = \frac{1}{2}. \quad (8.62)$$

Therefore, it follows from (8.61) and (8.62) that

$$\lim_{u_1 \rightarrow +\infty} (u_1 \lambda(u_1)) = \lim_{u_1 \rightarrow +\infty} \frac{2\eta(u_1)}{Bu_1} = \frac{1}{(1-z)B}.$$

In other words,

$$\lambda(u_1) \sim \frac{1}{(1-z)Bu_1} \quad \text{as } u_1 \rightarrow +\infty. \quad (8.63)$$

As we are assuming that

$$\int_0^1 a \, dx = zA - (1-z)B < 0,$$

we have that

$$\frac{1}{(1-z)B} < \frac{1}{zA}.$$

On the other hand, by (8.58) and (8.63), we may infer that

$$\frac{1}{zAu_0} \sim \lambda(u_0) = \lambda(u_1) \sim \frac{1}{(1-z)Bu_1}$$

for sufficiently large u_0 and u_1 . Therefore, if (8.50) and (8.52) possesses a positive solution for sufficiently small $\lambda > 0$, then $u_0 > u_1$. Moreover, since

$$\frac{u_{z,-}^2(u_0)}{u_0^2} = 1 - \gamma^{-1}(u_0), \quad \frac{u_{z,+}^2(u_1)}{u_1^2} = 1 - \eta^{-1}(u_1),$$

it becomes apparent that

$$\lim_{u_0 \rightarrow +\infty} \frac{u_{z,-}^2(u_0)}{u_0^2} = 1, \quad \lim_{u_1 \rightarrow +\infty} \frac{u_{z,+}^2(u_1)}{u_1^2} = 1.$$

Consequently, since $u_0 > u_1$, any solution of (8.50) and (8.52) must satisfy

$$u_{z,-}(u_0) > u_{z,+}(u_1) \quad (8.64)$$

for sufficiently small $\lambda > 0$. Actually, this analysis establishes the non-existence of a positive regular solution of (8.3) for the choice (8.1) for sufficiently small $\lambda > 0$, as well as the existence of a singular discontinuous solution satisfying (8.64). Naturally, these positive singular solutions are bounded variation solutions as discussed in Section 1: a proof of this simple fact is given below.

Lemma 8.3. *Let $u : [0, 1] \rightarrow \mathbb{R}$ solve the singular problems (8.50) and (8.51). Then, u is a bounded variation solution of (1.1).*

Proof. Note that $u \in W^{1,1}(0, z) \cap W_{\text{loc}}^{2,1}[0, z]$, $u \in W^{1,1}(z, 1) \cap W_{\text{loc}}^{2,1}(z, 1]$, and, in particular, $u \in BV(0, 1)$. Pick a test function $\phi \in BV(0, 1)$ such that $|D\phi|^s \ll |Du|^s$. Clearly, $\phi \in W^{1,1}(0, z)$ and $\phi \in W^{1,1}(z, 1)$. Rewrite the equations in (8.50) and (8.51), respectively, in the form

$$-\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x)u, \quad 0 < x < z, \quad -\left(\frac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x)u, \quad z < x < 1,$$

Multiply by ϕ both equations and integrate by parts on $(0, z)$ and $(z, 1)$, respectively. Using the conditions satisfied by u' at the points $0, z, 1$, namely $u'(0) = u'(1) = 0$ and $u'(z^-) = u'(z^+) = -\infty$, we find

$$\begin{aligned} \int_0^1 a u \phi dx &= \int_0^z \frac{u' \phi'}{\sqrt{1+(u')^2}} dx + \int_z^1 \frac{u' \phi'}{\sqrt{1+(u')^2}} dx - \left[\frac{u' \phi}{\sqrt{1+(u')^2}} \right]_0^z - \left[\frac{u' \phi}{\sqrt{1+(u')^2}} \right]_z^1 \\ &= \int_0^1 \frac{(Du)^a (D\phi)^a}{\sqrt{1+|(Du)^a|^2}} dx + \phi(z^-) - \phi(z^+) \\ &= \int_0^1 \frac{(Du)^a (D\phi)^a}{\sqrt{1+|(Du)^a|^2}} dx + \text{sgn}(u(z^-) - u(z^+))(\phi(z^-) - \phi(z^+)) \\ &= \int_0^1 \frac{(Du)^a (D\phi)^a}{\sqrt{1+|(Du)^a|^2}} dx + \int_0^1 \text{sgn}\left(\frac{Du}{|Du|}\right) \frac{D\phi}{|D\phi|} |D\phi|^s, \end{aligned}$$

according to the notations introduced in Section 1. \square

We conjecture that this behavior of the bounded variation solutions of problem (1.1) is not specific of the choice (8.1), but also holds for every $a \in L^\infty(0, 1)$ changing sign in $(0, 1)$ such that $\int_0^1 a dx < 0$ and any function f satisfying (Hf). This is suggested by the results in [41] too.

9. THE CASE WHERE F IS SUBQUADRATIC AT ZERO

The next result establishes the existence of a positive regular solution of (1.1), for sufficiently small $\lambda > 0$, if F is sub-quadratic at zero.

Theorem 9.1. *Assume that*

- $a \in L^1(0, 1)$ is such that $\int_0^1 a dx < 0$ and $a(x) > 0$ a.e. on an interval $K \subset [0, 1]$,
- $f \in C^0[0, +\infty)$ is such that $f(u) \geq 0$ for $u \geq 0$,

- there exist $p \in (1, 2)$ and $L > 0$ such that

$$\lim_{u \rightarrow 0^+} \frac{F(u)}{u^p} = L, \quad (9.1)$$

with $F(u) = \int_0^u f(s) ds$. Then, there exists $\lambda^* > 0$ such that, for every $\lambda \in (0, \lambda^*)$, (1.1) has a positive solution $u \in W^{2,1}(0, 1)$. Moreover, these solutions, u_λ , can be chosen to satisfy

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{C^1[0,1]} = 0. \quad (9.2)$$

Proof. Without loss of generality, we can suppose that there exists $\delta \in (0, 1)$ such that $f(s) > 0$ for all $s \in (0, \delta)$, because otherwise (1.1) admits, for every $\lambda > 0$, a sequence of constants positive solutions $\{u_n\}_{n \geq 1}$, with $\lim_{n \rightarrow +\infty} u_n = 0$. Hence, in this case, $\lambda^* = +\infty$. Choosing, for each $\lambda > 0$, a solution u_n such that $0 < u_n < \lambda$, we conclude that (9.2) holds as well. Note also that $f(0) = 0$, because of (9.1). The rest of the proof is divided into three steps.

Step 1. A modified problem. Let us define two functions $\varphi, h : \mathbb{R} \rightarrow \mathbb{R}$, by setting

$$\varphi(s) := \begin{cases} \frac{s}{\sqrt{1+s^2}} & \text{if } |s| \leq 1, \\ \varphi(-1) + \varphi'(-1)(s+1) & \text{if } s < -1, \\ \varphi(1) + \varphi'(1)(s-1) & \text{if } s > 1, \end{cases} \quad h(s) := \begin{cases} 0 & \text{if } s < 0 \text{ or } s > 1, \\ f(s) & \text{if } 0 \leq s \leq \delta, \\ f(\delta) \frac{1-s}{1-\delta} & \text{if } \delta < s \leq 1, \end{cases}$$

and consider, for each $\lambda > 0$, the modified problem

$$\begin{cases} -(\varphi(u'))' = \lambda a(x)h(u) & \text{in } (0, 1), \\ u'(0) = 0, u'(1) = 0. \end{cases} \quad (9.3)$$

Subsequently, we will find a positive solution $u = u_\lambda$ of (9.3) as a global minimizer of the functional $\mathcal{I}_\lambda : H^1(0, 1) \rightarrow \mathbb{R}$ defined by

$$\mathcal{I}_\lambda(v) = \int_0^1 \Phi(v') dx - \lambda \int_0^1 a H(v) dx,$$

where

$$\Phi(s) := \int_0^s \varphi(t) dt, \quad H(s) := \int_0^s h(t) dt.$$

Note that Φ is even, convex and has quadratic growth at infinity, in the sense that

$$\lim_{|s| \rightarrow +\infty} \frac{\Phi(s)}{s^2} > 0,$$

H is bounded, and both functions are increasing for $s > 0$.

Step 2. Solving the modified problem. Let $\lambda > 0$ be given. The functional \mathcal{I}_λ is bounded from below in $H^1(0, 1)$. Indeed, by the properties of the functions Φ, h, H , we have that

$$\mathcal{I}_\lambda(v) = \int_0^1 \Phi(v') dx - \lambda \int_0^1 a H(v) dx \geq -\lambda \|a\|_{L^1(0,1)} H(1).$$

Moreover, for every $u \in H^1(0, 1)$, the function

$$v = \min\{\max\{u, 0\}, 1\} \in H^1(0, 1)$$

reduces \mathcal{I}_λ , in the sense that $\mathcal{I}_\lambda(v) \leq \mathcal{I}_\lambda(u)$. Indeed, using again the properties of Φ, h, H , we find

$$\begin{aligned} \mathcal{I}_\lambda(u) &= \int_0^1 \Phi(u') dx - \lambda \int_0^1 a H(u) dx \\ &= \int_{\{u < 0\}} \Phi(u') dx + \int_{\{0 \leq u \leq 1\}} \Phi(u') dx + \int_{\{u > 1\}} \Phi(u') dx \\ &\quad - \lambda \int_{\{u < 0\}} a H(u) dx - \lambda \int_{\{0 \leq u \leq 1\}} a H(u) dx - \lambda \int_{\{u > 1\}} a H(u) dx \\ &\geq \int_{\{u < 0\}} \Phi(0) dx + \int_{\{0 \leq u \leq 1\}} \Phi(v') dx + \int_{\{u > 1\}} \Phi(0) dx \\ &\quad - \lambda \int_{\{u < 0\}} a H(0) dx - \lambda \int_{\{0 \leq u \leq 1\}} a H(v) dx - \lambda \int_{\{u > 1\}} a H(1) dx \\ &= \int_0^1 \Phi(v') dx - \lambda \int_0^1 a H(v) dx = \mathcal{I}_\lambda(v). \end{aligned}$$

Therefore, if $\{u_n\}_{n \geq 1}$ is a minimizing sequence of \mathcal{I}_λ and we set

$$v_n = \min\{\max\{u_n, 0\}, 1\}, \quad n \geq 1,$$

then the sequence $\{v_n\}_{n \geq 1}$ is a minimizing sequence that is bounded in $L^\infty(0, 1)$. Further, as

$$\sup_{n \geq 1} |\mathcal{I}_\lambda(v_n)| < +\infty$$

and H is bounded, we infer that

$$\sup_{n \geq 1} \int_0^1 \Phi(v'_n) dx < +\infty,$$

and, as Φ is asymptotically quadratic,

$$\sup_{n \geq 1} \|v'_n\|_{L^2(0,1)} < +\infty.$$

Therefore, we conclude that

$$\sup_{n \geq 1} \|v_n\|_{H^1(0,1)} < +\infty.$$

Accordingly, there exists a subsequence of $\{v_n\}_{n \geq 1}$, which converges, weakly in $H^1(0, 1)$ and strongly in $L^\infty(0, 1)$, to some function $u = u_\lambda \in H^1(0, 1)$, satisfying

$$0 \leq u_\lambda(x) \leq 1 \quad \text{in } (0, 1).$$

Since \mathcal{I}_λ is weakly lower semicontinuous in $H^1(0, 1)$, u_λ is a global minimizer of \mathcal{I}_λ in $H^1(0, 1)$. As \mathcal{I}_λ is of class C^1 , u_λ is a critical point of \mathcal{I}_λ and hence a solution of (9.3).

Finally, we prove that u_λ is non-trivial. It suffices to show that $\mathcal{I}_\lambda(u_\lambda) < 0$. As there is an interval $K \subset (0, 1)$ such that $a(x) > 0$ a.e. in K , we can pick a function $z \in C^1[0, 1]$, with $\text{supp } z \subset K$, such that $z(x) = 1$ in an interval $K_0 \subset K$. Since $H(s) \geq 0$ for all $s \in \mathbb{R}$ and $H(0) = 0$, we infer from the properties of Φ, H and (9.1) that, for sufficiently small $t > 0$,

$$\begin{aligned} \mathcal{I}_\lambda(tz) &= \int_0^1 \Phi(tz') dx - \int_{K_0} a F(t) dx - \int_{K \setminus K_0} a F(tz) dx \\ &\leq \int_0^1 \frac{t^2 (z')^2}{1 + \sqrt{1 + t^2 (z')^2}} dx + \frac{1}{2n} \int_0^1 t^2 (z')^2 dx - F(t) \int_{K_0} a dx \\ &\leq t^p \left(t^{2-p} \int_0^1 (z')^2 dx - \frac{F(t)}{t^p} \int_{K_0} a dx \right) < 0, \end{aligned}$$

because $2 - p > 0$. This implies that

$$\mathcal{I}_\lambda(u_\lambda) = \min_{v \in H^1(0,1)} \mathcal{I}_\lambda(v) < 0.$$

Therefore, u_λ is a positive solution of (9.3).

Step 3. Existence of classical positive solutions of (1.1) for small λ . For each $\lambda > 0$, u_λ satisfies the equation

$$-u_\lambda'' = \lambda \frac{a(x)h(u_\lambda)}{\varphi'(u_\lambda')} \quad \text{a.e. in } (0, 1).$$

Since h is bounded and φ' is bounded away from 0, there exists a constant $C > 0$ such that

$$\|u_\lambda''\|_{L^1} \leq \lambda C$$

and hence,

$$\|u_\lambda'\|_\infty \leq \lambda C \quad \text{and} \quad \|w_\lambda\|_\infty \leq \lambda C, \quad (9.4)$$

where the splitting $u_\lambda = w_\lambda + r_\lambda$, with $r_\lambda = \int_0^1 u_\lambda dx \in [0, 1]$, is used.

Let us prove that

$$\lim_{\lambda \rightarrow 0^+} r_\lambda = 0. \quad (9.5)$$

To this end we argue by contradiction. Suppose there is a sequence, $\{\lambda_n\}_{n \geq 1}$, with $\lim_{n \rightarrow +\infty} \lambda_n = 0$ such that, setting

$$r_n := r_{\lambda_n}, \quad u_n := u_{\lambda_n}, \quad w_n := w_{\lambda_n}, \quad n \geq 1,$$

we have that

$$\lim_{n \rightarrow +\infty} r_n = \bar{r} > 0.$$

As $\lim_{n \rightarrow +\infty} \lambda_n = 0$, (9.4) implies that

$$\lim_{n \rightarrow +\infty} \|u_n'\|_\infty = 0, \quad \lim_{n \rightarrow +\infty} \|w_n\|_\infty = 0.$$

Thus,

$$\lim_{n \rightarrow +\infty} h(r_n + w_n) = h(\bar{r}) \quad \text{and} \quad \lim_{n \rightarrow +\infty} H(r_n + w_n) = H(\bar{r}) \quad \text{uniformly in } [0, 1].$$

Therefore, letting $n \rightarrow +\infty$ in the relation

$$\int_0^1 a h(u_n) dx = 0, \quad n \geq 1,$$

yields $\int_0^1 a h(\bar{r}) dx = 0$ and, since $\int_0^1 a dx < 0$, we find that $h(\bar{r}) = 0$, which entails $\bar{r} = 1$. On the other hand, we already know that

$$0 > \mathcal{I}_{\lambda_n}(u_n) \geq -\lambda_n \int_0^1 a H(r_n + w_n) dx, \quad n \geq 1,$$

and hence

$$\int_0^1 a H(r_n + w_n) dx > 0, \quad n \geq 1.$$

Letting $n \rightarrow +\infty$ finally shows that

$$\int_0^1 a H(1) dx \geq 0,$$

which is impossible, because $\int_0^1 a dx < 0$ and $H(1) > 0$. Therefore, (9.5) holds and consequently

$$\lim_{\lambda \rightarrow 0^+} \|u_\lambda\|_{C^1[0,1]} = 0.$$

Accordingly, there exists $\lambda^* > 0$ such that u_λ is a solution of (1.1) for all $\lambda \in (0, \lambda^*)$, because any solution, u , of (9.3) satisfying $0 \leq u \leq 1$ solves (1.1). \square

Theorem 9.1 complements Theorems 1.2 and 1.6 of [41] from two different perspectives. First, it establishes the existence of a positive regular solution of (1.1), for sufficiently small $\lambda > 0$, provided the potential F is sub-quadratic at zero, and this independently of its growth at infinity. Secondly, when, in addition, F is superlinear at infinity, in the sense that (1.5) holds, and there further exists $\vartheta > 1$ such that

$$\lim_{u \rightarrow +\infty} \frac{\vartheta F(u) - f(u)u}{u} = 0,$$

then it establishes that the second positive bounded variation solution, u_2 , constructed in Theorem 1.6 of [41] can be taken to be regular. As in this case, rather naturally, the global bifurcation diagram of positive bounded variation solutions of (1.1) should be a perturbation of the global bifurcation diagram sketched in Figure 1, at least for $p = 2 - \varepsilon$ with sufficiently small $\varepsilon > 0$, we conjecture that the global bifurcation diagram of the bounded variation solutions of (1.1) in the setting of Theorem 1.6 of [41] looks like shows the first plot of Figure 2. The branch of minimal solutions, $\{u_2(\lambda) : \lambda \in (0, \lambda^*)\}$, consisting of regular solutions, should perturb from $u = 0$ as ε perturbs from 0. Consequently,

$$\lim_{\varepsilon \rightarrow 0^+} \lambda^*(\varepsilon) = \lambda_0.$$

The upper branch of the bifurcation diagram, $\{u_1(\lambda) : \lambda \in (0, \lambda^*)\}$, should approximate the component \mathfrak{C}^+ of bounded variation solutions of Figure 1 as $\varepsilon \rightarrow 0^+$, so that there is a continuous transition between both global bifurcation diagrams as $\varepsilon \rightarrow 0^+$.

A simple condition on f guaranteeing (9.1) is

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^{p-1}} = pL. \quad (9.6)$$

In such case, since $p \in (1, 2)$, we have that

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u} = +\infty.$$

Thus, the quotient function, $Q(u)$, defined by (6.3) is unbounded at $u = 0$ and the proof of Theorem 6.1 cannot be adapted to treat subquadratic potentials at zero. Consequently, it remains an open question to ascertain whether or not (1.1) admits a positive solution, either regular, or singular, for sufficiently large $\lambda > 0$.

10. THE CASE WHERE F IS SUPERQUADRATIC AT ZERO

The next result basically establishes the existence of a positive regular solution of (1.1), for sufficiently large $\lambda > 0$, if F is superquadratic at zero.

Theorem 10.1. *Assume that*

- $a \in L^1(0, 1)$ satisfies $\int_0^1 a \, dx < 0$ and it admits a decomposition

$$[0, 1] = \bigcup_{i=1}^k [\alpha_i, \beta_i], \quad \text{with } \alpha_i < \beta_i = \alpha_{i+1} < \beta_{i+1}, \quad \text{for } i = 1, \dots, k-1,$$

such that

$$(-1)^i a(x) \geq 0 \quad \text{a.e. in } (\alpha_i, \beta_i), \quad a \not\equiv 0 \quad \text{in } (\alpha_i, \beta_i), \quad \text{for } i = 1, \dots, k,$$

or

$$(-1)^i a(x) \leq 0 \quad \text{a.e. in } (\alpha_i, \beta_i), \quad a \not\equiv 0 \quad \text{in } (\alpha_i, \beta_i), \quad \text{for } i = 1, \dots, k,$$

- $f \in C^0[0, +\infty)$ satisfies $f(u) \geq 0$ for $u \geq 0$,
- there exist $p > 1$ and $L > 0$ such that

$$\lim_{u \rightarrow 0^+} \frac{f(u)}{u^p} = L. \quad (10.1)$$

Then, there exists $\lambda_* > 0$ such that, for every $\lambda > \lambda_*$, the problem (1.1) admits at least one strictly positive solution $u \in W^{2,1}(0, 1)$. Moreover, these solutions, u_λ , can be chosen to satisfy

$$\lim_{\lambda \rightarrow +\infty} \|u_\lambda\|_{C^1[0,1]} = 0. \quad (10.2)$$

Proof. The proof, which is divided into two steps, has a topological nature and combines a clever result recently obtained in [23], with the invariance property of the coincidence degree under small perturbations, that is, the Rouché theorem. For a thorough treatment of the coincidence degree theory we refer to [26].

Step 1. An auxiliary problem. Let us consider the semilinear problem

$$\begin{cases} -v'' = L a(x)v^p & \text{in } (0, 1), \\ v'(0) = 0, \quad v'(1) = 0, \end{cases} \quad (10.3)$$

where $p > 1$ and $L > 0$ come from (10.1). By our assumptions on the weight a , all hypotheses of [23, Theorem 3.1] are met. Let us set, for a.e. $x \in [0, 1]$ and every $s \geq 0$,

$$f(x, s) := L a(x)s^p.$$

As in the proof of [23, Theorem 3.1], there are constants, r, R , with $0 < r < R$, for which conditions (H_r) and (H_R) of [23, Theorem 2.1] hold. Subsequently, we consider the open bounded subset of $\mathcal{C}^1[0, 1]$

$$\Omega := \{v \in \mathcal{C}^1[0, 1] : r < \|v\|_\infty < R, \|v'\|_\infty < S\},$$

where $S > \|a\|_{L^1} R^p$. We also define the function

$$\tilde{f}(x, s) := \begin{cases} f(x, s) & \text{if } s \geq 0, \\ -s & \text{if } s < 0, \end{cases}$$

for a.e. $x \in [0, 1]$ and every $s \in \mathbb{R}$, and the operators

$$\begin{aligned} \mathcal{L} : \text{dom } \mathcal{L} := W_{\mathcal{N}}^{2,1}(0, 1) &\rightarrow L^1(0, 1), & \mathcal{L}v &:= -v'', \\ \mathcal{N} : \mathcal{C}^1[0, 1] &\rightarrow L^1(0, 1), & \mathcal{N}(v) &:= \tilde{f}(\cdot, v), \\ \Pi : L^1(0, 1) &\rightarrow \mathbb{R}, & \Pi w &:= \int_0^1 w(x) dx. \end{aligned}$$

Clearly, \mathcal{L} is a Fredholm operator of index zero such that

$$N[\mathcal{L}] = R[\Pi] = \mathbb{R}, \quad R[\mathcal{L}] = N[\Pi].$$

Moreover, the operator $\mathcal{K} : L^1(0, 1) \rightarrow \mathcal{C}^1[0, 1]$, which sends any function $w \in L^1(0, 1)$ onto the unique solution $v \in \text{dom } \mathcal{L} \cap N[\Pi]$ of the equation

$$\mathcal{L}v = w - \Pi w$$

is compact. Furthermore, according to the terminology in [26], the operator \mathcal{N} is \mathcal{L} -compact in $\bar{\Omega}$. Therefore, as in the proof of Theorem 2.1 in [23], we infer that

$$\mathcal{L}v \neq \mathcal{N}(v) \quad \text{for all } v \in \text{dom } \mathcal{L} \cap \partial\Omega \quad (10.4)$$

and

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega) = -1, \quad (10.5)$$

where $D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega)$ denotes the coincidence degree of \mathcal{L} and \mathcal{N} in Ω . By (10.4), since \mathcal{K} is continuous, it follows from [26, Lemma III.5] that

$$\inf_{v \in \text{dom } \mathcal{L} \cap \partial\Omega} \|\mathcal{L}v - \mathcal{N}(v)\|_{L^1} > 0. \quad (10.6)$$

Step 2. Existence of classical solutions of problem (1.1) for large λ . Recall that a function $u \in W^{2,1}(0, 1)$ is a solution of (1.1) if and only if u satisfies (1.8). Moreover, introducing the change of variable $v := \lambda^{\frac{1}{p-1}} u$, with $\lambda > 0$, in (1.8) it turns out that (1.1) can be equivalently written as

$$\begin{cases} -v'' = L a(x)v^p + a(x)g_\lambda(v, v') & \text{in } (0, 1), \\ v'(0) = 0, \quad v'(1) = 0, \end{cases} \quad (10.7)$$

where

$$g_\lambda(s, \xi) := \left[\lambda^{\frac{p}{p-1}} f(\lambda^{-\frac{1}{p-1}} s) - Ls^p \right] (1 + \lambda^{-\frac{2}{p-1}} \xi^2)^{3/2} + Ls^p \left[(1 + \lambda^{-\frac{2}{p-1}} \xi^2)^{3/2} - 1 \right],$$

for every $s \geq 0$, $\xi \in \mathbb{R}$ and $\lambda > 0$. Since

$$g_\lambda(s, \xi) = s^p \left[\frac{f(\lambda^{-\frac{1}{p-1}} s)}{(\lambda^{-\frac{1}{p-1}} s)^p} - L \right] (1 + \lambda^{-\frac{2}{p-1}} \xi^2)^{3/2} + Ls^p \left[(1 + \lambda^{-\frac{2}{p-1}} \xi^2)^{3/2} - 1 \right],$$

it follows from (10.1) that

$$\lim_{\lambda \rightarrow +\infty} g_\lambda(s, \xi) = 0$$

uniformly on all compact subsets of $(s, \xi) \in \mathbb{R}^+ \times \mathbb{R}$. Thus, introducing the function

$$\tilde{h}_\lambda(x, s, \xi) := \begin{cases} La(x)s^p + a(x)g_\lambda(s, \xi) & \text{if } s \geq 0, \\ -s & \text{if } s < 0, \end{cases}$$

for a.e. $x \in [0, 1]$ and every $s, \xi \in \mathbb{R}$, $\lambda > 0$, the corresponding superposition operator

$$\mathcal{N}_\lambda : \mathcal{C}^1[0, 1] \rightarrow L^1(0, 1), \quad \mathcal{N}_\lambda(v) = \tilde{h}_\lambda(\cdot, v, v'),$$

is \mathcal{L} -compact in $\bar{\Omega}$. Hence, the properties of g_λ and the definition of \tilde{h}_λ imply that

$$\lim_{\lambda \rightarrow +\infty} \sup_{v \in \bar{\Omega}} \|\mathcal{N}_\lambda(v) - \mathcal{N}(v)\|_{L^1} = 0. \quad (10.8)$$

Since (10.6), (10.8) and (10.5) hold and the operator \mathcal{K} is continuous, [26, Lemma III.5] yields the existence of $\lambda_* > 0$ such that

$$D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}_\lambda, \Omega) = D_{\mathcal{L}}(\mathcal{L} - \mathcal{N}, \Omega) = -1$$

for all $\lambda > \lambda_*$. Therefore, by [26, Theorem III.3], the problem

$$\begin{cases} -v'' = \tilde{h}_\lambda(x, v, v') & \text{in } (0, 1), \\ v'(0) = 0, v'(1) = 0, \end{cases}$$

has at least one solution $v_\lambda \in \text{dom } \mathcal{L} \cap \Omega$. As $v_\lambda \neq 0$, the definition of \tilde{h}_λ and the strong maximum principle, e.g., in the form of [23, Lemma 6.1], imply that $\min v_\lambda > 0$. Therefore v_λ satisfies (10.7). Finally, setting $u_\lambda = \lambda^{-\frac{1}{p-1}} v_\lambda$, for every $\lambda > \lambda_*$, we conclude that u_λ is a solution of (1.1) such that $u_\lambda \in W^{2,1}(0, 1)$, $\min u_\lambda > 0$, and (10.2) hold. \square

Since (10.1) implies that

$$\lim_{u \rightarrow 0^+} \frac{F(u)}{u^{p+1}} = \frac{L}{p+1} \quad (10.9)$$

with $p+1 > 2$, Theorem 10.1 complements Theorem 1.1 of [41] by providing some sufficient conditions so that the bounded variation solutions constructed therein can indeed be regular solutions for sufficiently large $\lambda > 0$. When, in addition, (1.5) holds, i.e., F is superlinear at infinity, then, thanks to Theorem 7.1, one can give some general sufficient conditions so that (1.1) cannot admit a regular solution for sufficiently small $\lambda > 0$. From this perspective, Theorem 10.1 is optimal. Rather naturally, at least for $p+1 = 2 + \varepsilon$ with sufficiently small $\varepsilon > 0$, the global bifurcation diagram of the bounded variation solutions of (1.1) should be a perturbation of the global bifurcation diagram sketched in Figure 1. Based on this, we conjecture that the global bifurcation diagram of the bounded variation solutions of (1.1) in the setting of Theorem 1.1 of [41] looks like the second plot of Figure 2 shows.

11. CONCLUSIONS, CONJECTURES AND OPEN QUESTIONS

In this paper we have studied regular and singular (bounded variation) solutions of the quasilinear Neumann problem (1.1), where $a \in L^\infty(0,1)$ changes sign in $(0,1)$ and $f(u)$ is an increasing function of class C^1 such that the associated potential, $F(u) := \int_0^u f ds$, is superlinear at infinity, though many of the results remain valid for $a \in L^1(0,1)$. Most of the attention in this paper has been focused on the special case where the potential $F(u)$ is quadratic at zero; however Sections 9 and 10 also cover the cases where $F(u)$ is either subquadratic, or superquadratic, at zero, respectively.

According to one of our previous results in [41], reported here as Theorem 1.1, when $a \in L^1(0,1)$ satisfies $\int_0^1 a dx < 0$ and F is quadratic at zero and superlinear at infinity, there exists a value of the parameter λ , $\lambda^* > 0$, such that, for every $\lambda \in (0, \lambda^*)$, the problem (1.1) possesses a bounded variation solution, u , such that on each subinterval, (α, γ) , where the weight function a changes its sign exactly once, at some $\beta \in (\alpha, \gamma)$, either u is regular in the entire interval (α, γ) , or it develops a singularity at β , in the sense that, either

$$u(\beta^-) \geq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = -\infty = u'(\beta^+) \quad (11.1)$$

(if $a(x) \geq 0$ in (α, β) and $a(x) \leq 0$ in (β, γ)), or

$$u(\beta^-) \leq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = +\infty = u'(\beta^+) \quad (11.2)$$

(if $a(x) \leq 0$ in (α, β) and $a(x) \geq 0$ in (β, γ)). However, this result does not ascertain whether or not this property should hold for any other bounded variation solution of (1.1), or whether $u(\beta^-) > u(\beta^-)$ should occur in (11.1), or $u(\beta^-) < u(\beta^+)$ in (11.2), neither whether or not (1.1) should possess some regular solution.

Being the resolution of these problems important challenges also from the point of view of their applications in fluid dynamics and reaction diffusion processes, in this paper we have combined a variety of technical tools from calculus of variations, global bifurcation theory, topological degree and the theory of autonomous planar dynamical systems, in order to study the existence and the hidden structure of the regular solutions of (1.1), as well as their relationships with the bounded variation solutions provided by Theorem 1.1; very specially, in trying to realize the mechanisms for the generation of jump singularities from the λ -paths of regular solutions of (1.1).

According to Lemma 2.1, the condition $\int_0^1 a dx < 0$ has been shown to be necessary for the existence of a regular positive solution of (1.1) for some $\lambda > 0$. Thus, throughout this paper we have always assumed that $\int_0^1 a dx < 0$.

Thanks to Theorem 3.1, there are $\lambda_0 > 0$ and a non-trivial (connected) component, $\mathfrak{C}_{\lambda_0}^+$, of the set of regular positive solutions of (1.1) such that $(\lambda_0, 0) \in \bar{\mathfrak{C}}_{\lambda_0}^+$. Since we are working under non-flux boundary conditions, for every $\kappa > 0$, $(\lambda, u) = (0, \kappa)$ is a (trivial) regular positive solution of (1.1); the trivial solutions might belong, or not, to the component $\mathfrak{C}_{\lambda_0}^+$. By Theorem 4.2, $(\lambda_0, 0)$ is a subcritical pitchfork bifurcation point if $f''(0) = 0$, and a transcritical bifurcation point if $f''(0) \neq 0$: supercritical if $f''(0) < 0$ and subcritical if $f''(0) > 0$.

If, in addition, $F(u)$ is quadratic or subquadratic at infinity and there are $r, s \in (0,1)$ such that $\inf_{[r,s]} a > 0$, then, by Theorem 6.1, problem (1.1) cannot admit a regular positive solution, nor actually a bounded variation solution, for sufficiently large $\lambda > 0$. More precisely, Theorem 6.1 establishes that if

$$\lim_{u \rightarrow \infty} \frac{f(u)}{u^{q-1}} = qh \quad (11.3)$$

for some $q \in (1, 2]$ and $h > 0$, then

$$\Lambda(q) \equiv \sup \{ \lambda > 0 : (1.1) \text{ has a bounded variation solution} \} < +\infty. \quad (11.4)$$

Although the assumption that $\inf_{[r,s]} a > 0$ on some $[r, s]$ holds for any continuous function a , and so it does not look so much restrictive, it remains an open problem to ascertain whether or not $\Lambda(q) < +\infty$ when $q > 2$. Nevertheless, under the assumptions of Theorem 6.1, the λ -projection of the component $\mathfrak{C}_{\lambda_0}^+$ must be a compact interval $[\alpha(q), \omega(q)]$, for some $0 \leq \alpha(q) \leq \lambda_0$ and $\lambda_0 \leq \omega(q) \leq \Lambda(q)$. It is

still open to know whether or not (1.1) can admit a singular bounded variation solution for some $\lambda \in [\alpha(q), \omega(q)]$, or whether or not (1.1) can possess another non-trivial component of regular positive solutions, besides $\mathfrak{C}_{\lambda_0}^+$.

If, in addition, $f \in \mathcal{C}^2(\mathbb{R})$ and

$$\lim_{u \rightarrow \infty} \frac{f'(u)}{u^{q-2}} = q(q-1)h, \quad (11.5)$$

for some $q \in (1, 2]$ and $h > 0$, and there exists $z \in (0, 1)$ such that $a(x) > 0$ for all $x \in (0, z)$ and $a(x) < 0$ for all $x \in (z, 1)$, then, by Theorem 7.1, problem (1.1) cannot admit a regular positive solution for sufficiently small $\lambda > 0$. As a by-product, we find that $\alpha(q) > 0$ and

$$\{(0, \kappa) : \kappa > 0\} \cap \mathfrak{C}_{\lambda_0}^+ = \emptyset.$$

Therefore, one can extract the following important consequences.

- The problem (1.1) possesses at least two components of regular positive solutions. Namely, $\mathfrak{C}_{\lambda_0}^+$ and $\mathfrak{C}_0^+ = \{(0, \kappa) : \kappa > 0\}$.
- For each $\lambda \in (0, \alpha(q))$, every bounded variation solution of (1.1) must possess some singularity. Moreover, owing to Theorem 1.1, there exists at least one bounded variation solution, u , such that

$$u(z^-) \geq u(z^+) \quad \text{and} \quad u'(z^-) = -\infty = u'(z^+). \quad (11.6)$$

Note that (11.5) implies (11.3). Moreover, the nodal assumptions on the weight a in the statement of Theorem 7.1 imply $\min_{[\varepsilon, z-\varepsilon]} a > 0$ for sufficiently small $\varepsilon > 0$. So, the hypotheses of Theorem 6.1 hold true under the assumptions of Theorem 7.1. We conjecture that, in such case, (11.6) holds for every bounded variation solution, u . Actually, $u(z^-) > u(z^+)$ should occur if $\lambda < \alpha(q)$, while for $\lambda = \lambda^* := \alpha(q)$ the problem (1.1) should admit a bounded variation solution, $u^* \in \mathcal{C}[0, 1]$, such that

$$(u^*)'(z^-) = -\infty = (u^*)'(z^+) \quad \text{and} \quad (u^*, \lambda^*) \in \bar{\mathfrak{C}}_{\lambda_0}^+,$$

as illustrated by Figure 1.

These conjectures are strongly supported by the analysis of Section 8, where the special, but pivotal case, when $f(u) = u$ (for which $q = 2$) has been exhaustively studied for a particular choice of the weight a that enabled us to use planar phase techniques. In this particular example the global bifurcation diagram of bounded variation solutions looks like shows Figure 1. The component of regular positive solutions $\mathfrak{C}_{\lambda_0}^+$, which bifurcates from $(\lambda, 0)$ at $\lambda = \lambda_0$, loses the a priori bounds in $\mathcal{C}^1[0, 1]$ at $\lambda^* = \alpha(2) \in (0, \lambda_0)$, where it links a curve of singular bounded variation solutions, $\mathfrak{C}_{\text{BV}}^+$, bifurcating from infinity at $\lambda = 0$. We conjecture that, actually, this is always the global bifurcation diagram under, at least, the assumptions of Theorem 7.1. Naturally, if a would have a more intricate nodal behavior, then the regular positive solutions will not be decreasing in $(0, 1)$ and hence, the proof of Theorem 7.1, based on such monotonicity, might be substantially harder. In spite of these technical troubles, we still conjecture the validity of the theorem. However, it remains an open question to ascertain whether or not the assumption that $q \in (1, 2]$ is relevant for the validity of these results.

More generally, the analysis of this paper is suggesting the existence of a (rather abstract) unilateral global bifurcation theorem in the context of bounded variation solutions, where regular solutions developing singularities, but having a regular epigraph, should not be considered as singular anymore, but actually should be regarded as regular as classical solutions are. Moreover, the solutions on the component of bounded variation solutions should be regular when their L^∞ -norms are sufficiently small, while, according to the analysis of Section 5, they should develop jump singularities, on some or several of the internal nodes of the weight a , if their L^∞ -norms are sufficiently large. The results of this paper suggest this occurs at least if $q \in (1, 2]$ but it seems a real challenge to ascertain if this phenomenology also occurs for $q \geq 2$.

As it happens with the simplest algebraic examples exhibiting some imperfect bifurcation phenomenology, like

$$x^2 - y^2 = \varepsilon,$$

where the bifurcation from $(x, y) = (0, 0)$ is broken down as ε perturbs away from $\varepsilon = 0$, when, instead of quadratic, the growth of $F(u)$ at zero becomes subquadratic or superquadratic, the bifurcation of

$\mathfrak{C}_{\lambda_0}^+$ from $(\lambda, u) = (\lambda_0, 0)$ is lost and, according to the nature of the growth of F at zero, either $(\lambda, u) = (\lambda, 0)$, $0 < \lambda < \lambda_0$, or $(\lambda, u) = (\lambda, 0)$, $\lambda > \lambda_0$, perturbs into a curve of the global bifurcation diagram of regular positive solutions of (1.1) (see Figure 2). As those solutions have small L^∞ -norm, they should perturb into regular positive solutions. Although these imperfect bifurcations are indeed confirmed by the (global) Theorems 9.1 and 10.1, the technical details of this perturbation analysis will appear elsewhere, as it escapes from the general scope imposed to this paper.

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