Maximal elements of quasi upper semicontinuous preorders on compact spaces

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Abstract We introduce the concept of quasi upper semicontinuity of a not necessarily total preorder on a topological space and we prove that there exists a maximal element for a preorder on a compact topological space provided that it is quasi upper semicontinuous. In this way, we generalize many classical and well known results in the literature. We compare the concept of quasi upper semicontinuity with the other semicontinuity concepts to arrive at the conclusion that our definition can be viewed as the most appropriate and natural when dealing with maximal elements of preorders on compact spaces.

Keywords Quasi upper semicontinuous preorder · Weak utility · Quasi utility

1 Introduction

The deep and rich literature concerning the existence of maximal elements for preference relations (typically *acyclic binary relations*) on a compact topological space from an economical viewpoint dates back essentially to the seminal papers of Bergstrom (1975), whose fundamental result states that an acyclic binary relation $\prec$ on a compact topological space (X, τ) has a maximal element provided that $\prec$ is *upper semicontin-uous* (i.e., $l(x) = \{z \in X : z \prec x\}$ is an open subset of X for every $x \in X$).

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Campbell and Walker (1990) introduced a semicontinuity concept concerning an acyclic binary relation called *weak lower continuity* (*weak upper continuity* in the present paper).

Several authors presented generalizations of the fundamental result of Bergstrom, To this aim, suitable notions of semicontinuity besides *weak lower continuity* were introduced. For example, Mehta (1989) and Subiza and Peris (1997) considered *transfer lower continuity*.

More recently, sufficient conditions for the existence of maximal elements for ordered preferences on a compact topological space were presented by Tian and Zhou (1995), Bosi and Zuanon (2013), Luc and Soubeyran (2013) and Nosratabadi (2014). In particular, the concept of *transfer weak upper continuity* introduced by Tian and Zhou (1995) appears by now as the most general and appropriate. Kukushkin (2008) was concerned with the existence of maximal elements for an *interval order* on a compact metric space.

In a slightly different context, conditions for the existence of *undominated maximal elements* for preferences with intransitive indifference on a compact topological space are found in Alcantud et al. (2010).

The problem concerning the existence of maximal elements over non-compact subsets of linear topological spaces was solved by Yannelis (1985), who used an extension of Browder's fixed point theorem.

In this paper, we concentrate our attention specifically on the existence of maximal elements for preorders on a compact topological space. We introduce the concept of a *quasi upper semicontinuous* preorder on a topological space and we show that such a concept is the most suitable to be considered in connection with the aforementioned problem of determining the maximal elements. Such a notion generalizes the already encountered concepts of *upper semicontinuity* of a preorder on a topological space (which were studied both in connection with the existence of maximal elements and of an upper semicontinuous order-preserving function). In particular, we show that while there are quasi upper semicontinuous preorders on compact spaces which are not transfer weakly upper continuous, quasi upper semicontinuity implies transfer weak upper continuity under some conditions. We also discuss the representability of quasi upper semicontinuous preorders and in particular we show that such an assumption is necessary and sufficient for the existence of an upper semicontinuous *weak utility* for the strict part of the preorder in case that the topological space is *second countable*.

2 Notation and preliminaries

Let $\precsim$ be a preorder (i.e., a *reflexive* and *transitive* binary relation) on a set X. As usual, the *strict part* of $\precsim$ will be denoted by $\prec$ (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \precsim y)$ and *not* $(y \precsim x)$). For the sake of convenience, we shall occasionally write $(x, y) \in \prec$ instead of $x \prec y$. For any subset A of X and any element $x \in X$, the scripture $A \precsim x$ means "$z \precsim x$ for all $z \in A$".

Denote by $\bowtie$ the *incomparability relation* associated to a preorder $\precsim$ on a set X (i.e., for all $x, y \in X$, $x \bowtie y$ if and only if *not* $(x \precsim y)$ *and not* $(y \precsim x)$). For any element $x \in X$, $I(x)$ stands for the set of all the elements that are incomparable with x, i.e., $I(x) = \{z \in x : z \bowtie x\}$.

The *indifference relation* $\sim$ associated to a preorder $\precsim$ on X is denoted by $\sim$ (i.e., for all $x, y \in X$, $x \sim y$ if and only if $(x \precsim y)$ and $(y \precsim x)$). It is an equivalence, and the associated *quotient space* is denoted by $X_{|\sim}$. Further, $\precsim_{|\sim}$ is the *quotient order* on the quotient space $X_{|\sim}$ (i.e., for any two elements $x, y \in X$, $[x] \precsim_{|\sim} [y]$ if and only if $x \precsim y$). A *jump* j *in* $(X_{|\sim}, \precsim_{|\sim})$ is a pair $j = ([x], [y])$ such that for no $z \in X$ it occurs that $[x] \prec_{|\sim} [z] \prec_{|\sim} [y]$. The set of all the jumps in $(X_{|\sim}, \precsim_{|\sim})$ will be denoted by J.

Let us consider, for every point $x \in X$, the following subsets of X:

$$l(x) = \{y \in X \mid y \prec x\}, \quad r(x) = \{y \in X \mid x \prec y\}$$
$$d(x) = \{z \in X \mid z \precsim x\}, \quad i(x) = \{z \in X \mid x \precsim z\}.$$

A point $x_0 \in X$ is said to be *maximal* with respect to $\precsim$ if for no $x \in X$ it occurs that $x_0 \prec x$ (i.e., $r(x_0) = \emptyset$).

A subset D of a *related set* (X, R) is said to be R-*decreasing* if $\{y \in X \mid yRx\} \subset D$ for all $x \in D$.

If $(X, \precsim)$ is a preordered set, then a function $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ is said to be

(i) *increasing* if, for every $x, y \in X$, $[x \precsim y \Rightarrow u(x) \leq u(y)]$,
(ii) *order-preserving* if it is increasing, and for every $x, y \in X$, $[x \prec y \Rightarrow u(x) < u(y)]$.

In the economic literature, an order-preserving function is often referred to as a *Richter–Peleg utility function* (see e.g., Richter 1966). Denote by τ_{nat} the *natural topology* on the real line $\mathbb{R}$.

A real-valued function u on X is said to be a *weak utility* for the strict part $\prec$ of a preorder $\precsim$ on a set X (see Peleg 1970) if, for all $x, y \in X$,

$$x \prec y \Rightarrow u(x) < u(y).$$

We say that a real-valued function u on X is a *quasi utility* for the strict part $\prec$ of a preorder $\precsim$ on a set X if, for all $x, y \in X$,

$$x \prec y \Rightarrow u(x) \leq u(y).$$

It is clear that an increasing function u for a preorder $\precsim$ on a set X is in particular a *quasi utility* for its strict part $\prec$, while the converse is not true since a quasi utility for $\prec$ does not necessarily preserve the associated indifference relation $\sim$.

Definition 2.1 A preorder $\precsim$ on a topological space (X, τ) is said to be

(i) *upper semicontinuous of type 1*, if $l(x) = \{z \in X \mid z \prec x\}$ is an open subset of X, for every $x \in X$;
(ii) *upper semicontinuous of type 2*, if $i(x) = \{z \in X \mid x \precsim z\}$ is a closed subset of X, for every $x \in X$;

(iii) *weakly upper semicontinuous*, if for every pair $(x, y) \in \prec$ there exists some open $\precsim$-decreasing subset $O_{x,y}$ of X such that $x \in O_{x,y}$ and $y \in X \setminus O_{x,y}$;

(iv) *weakly upper continuous*, if for every pair $(x, y) \in \prec$ there exists some open set $V(x)$ containing x such that $V(x) \precsim y$;

(v) *transfer weakly upper continuous*, if for every pair $(x, y) \in \prec$ there exist some point $y' \in X$ and an open set $V(x)$ containing x such that $V(x) \precsim y'$;

(vi) *partially upper continuous*, if the set J of jumps in $X_{|\sim}$ is countable and for every pair $(x, y) \in \prec \setminus \{(t, v) : t \in [t'], v \in [v'], ([t'], [v']) \in J\}$ there exists an open set $V(x)$ containing x such that $V(x) \precsim y$.

Remark 2.2 It is known from the literature that a preorder $\precsim$ on a compact topological space (X, τ) admits a maximal element provided that it satisfies any of the semicontinuity conditions of Definition 2.1 In particular, the sufficiency of condition (i) was proved by Bergstrom (1975), of condition (ii) by Evren and Ok (2011), of condition (iii) by Bosi and Zuanon (2013, Theorem 3.3), of conditions (iv) and (v) by Tian and Zhou (1995, Proposition 2) and of condition (vi) by Nosratabadi (2014, Theorem 2). Further, Nosratabadi (2014) showed in the proof of Theorem 2 that a partially upper continuous preorder on a compact space is actually transfer weakly upper continuous.

Let us introduce the main definition of this paper.

Definition 2.3 A preorder $\precsim$ on a topological space (X, τ) is said to be *quasi upper semicontinuous* if there exists an upper semicontinuous of type 2 preorder $\lesssim$ on (X, τ) such that $\prec \subset <$.

The following proposition presents a characterization of a quasi upper semicontinuous preorder.

Proposition 2.4 *Let $\precsim$ be a preorder on a topological space (X, τ). Then the following conditions are equivalent:*

(i) *$\precsim$ is quasi upper semicontinuous.*

(ii) *For every pair $(x, y) \in \prec$ there exists some open $\prec$-decreasing subset $O_{x,y}$ of X such that $x \in O_{x,y}$ and $y \in X \setminus O_{x,y}$.*

(iii) *For every $x \in X$ that is not a minimal element with respect to $\precsim$ there exists a uniquely determined open $\prec$-decreasing subset $l^0(x)$ of X such that $x \notin l^0(x)$ and $l(x) \subset l^0(x)$.*

(iv) *For every pair $(x, y) \in \prec$ there exists an upper semicontinuous function $u_{x,y} : (X, \precsim, \tau) \longmapsto (\mathbb{R}, \leq, \tau_{\text{nat}})$ that is a quasi utility for $\prec$ such that $u_{x,y}(x) < u_{x,y}(y)$.*

Proof (i) $\Rightarrow$ (ii). Let $\precsim$ be a quasi upper semicontinuous preorder and let $\lesssim$ be an upper semicontinuous of type 2 preorder on (X, τ) such that $\prec \subset <$. For any pair $(x, y) \in \prec$, define $O_{x,y} = X \setminus i_{\lesssim}(y) = X \setminus \{z \in X : y \lesssim z\}$ to immediately verify that $O_{x,y}$ is an open $\prec$-decreasing subset of X such that $x \in O_{x,y}$ and $y \in X \setminus O_{x,y}$.
(ii) $\Rightarrow$ (iii). Define, for every $x \in X$ that is not a minimal element relative to $\prec$,

$$l^0(x) = \bigcup_{\{z\in X : z\prec x\}} \bigcup \{O_{z,x} : O_{z,x} \text{is an open} \prec -\text{decreasing subset of } X \text{ such that } z \in O_{z,x} \text{ and } x \in X \backslash O_{z,x}\Big\}.$$

(iii) $\Rightarrow$ (iv). Consider a pair $(x, y) \in \prec$, and assume that property (iii) is verified. Then one may define the desired upper semicontinuous quasi utility $u_{x,y}$ for $\prec$ such that $u_{x,y}(x) < u_{x,y}(y)$ by setting for all $z \in X$:

$$u_{x,y}(z) := \begin{cases} 0 & \text{if} \quad z \in l^0(y) \\ 1 & \text{if} \quad z \in X \backslash l^0(y) \end{cases}.$$

(iv) $\Rightarrow$ (i). Let $\precsim$ be a preorder on (X, τ). Then we denote by $U(\precsim)$ the set of all upper semicontinuous quasi utility functions u for the strict part $\prec$ of $\precsim$ such that there exists some pair $(x, y) \in\prec$ with $u(x) < u(y)$ and set

$$\lesssim := \{(x, y) \in X \times X \mid \forall u \in U(\precsim) \quad (u(x) \leq u(y))\}.$$

Clearly, $\lesssim$ is preorder on (X, τ) such that $\prec\subset<$. Hence, it suffices to verify that $\lesssim$ is upper semicontinuous of type 2. Let, therefore, (x, y) be a pair of $X \times X$ such that *not*$(y \lesssim x)$. Then there exists some function $u \in U(\precsim)$ such that $u(x) < u(y)$. The upper semicontinuity of u guarantees that $u^{-1}(] - \infty, u(y)[)$ is an open subset of X such that $x \in u^{-1}(] - \infty, u(y)[)$, $y \notin u^{-1}(] - \infty, u(y)[)$. Finally, it is almost immediate to check that $u^{-1}(] - \infty, u(y)[)$ is $\lesssim$-decreasing, which obviously implies that $u^{-1}(] - \infty, u(y)[) \cap i(y) = \emptyset$. Therefore, the preorder $\precsim$ on (X, τ) is quasi upper semicontinuous. □

It is clear that the notion of a quasi upper semicontinuous preorder generalizes that the concept of a weakly upper semicontinuous preorder.

Remark 2.5 It is easily seen that a preorder $\precsim$ on (X, τ) is quasi upper semicontinuous provided that it is either upper semicontinuous of type 1 or upper semicontinuous of type 2 (see Herden and Levin 2012) or else there exists an upper semicontinuous weak utility u for the strict part $\prec$ of $\precsim$ (in particular, $\precsim$ admits an upper semicontinuous order-preserving function).

From a result in Nosratabadi (2014), we immediately deduce the following proposition.

Proposition 2.6 *Let $\precsim$ be a total preorder on a second countable topological space (X, τ). If $\precsim$ is partially upper continuous, then it is quasi upper semicontinuous.*

Proof Since $\precsim$ is total, and (X, τ) is second countable, from Nosratabadi (2014, Theorem 1) we have that there exists an upper semicontinuous order-preserving function for $\precsim$, This fact implies that $\precsim$ is quasi upper semicontinuous (see Remark 2.5). □

The following example shows that there exist quasi upper semicontinuous total preorders which are not partially upper continuous.

Example 2.7 Let X be the set $[1, 3] \cup [9, 10]$ endowed with the natural (induced) topology τ_{nat} and consider the total preorder $\precsim$ on X defined as follows for all $x, y \in X$:

$$x \precsim y \Leftrightarrow y \leq z^2 \Rightarrow x \leq z^2 \quad \text{for all } z \in X.$$

Then the strict part $\prec$ of $\precsim$ is defined as follows:

$$x \prec y \Leftrightarrow x \leq z^2 < y \quad \text{for some } z \in X.$$

The economic motivation of the present example is related to the consideration of preferences with nontransitive indifference. Indeed, it can be noted (see Bosi and Zuanon 2014, Example 2.1) that $\precsim$ is one of the *traces* associated to the *interval order* $\precsim'$ on X defined as follows for all $x, y \in X$:

$$x \precsim' y \Leftrightarrow x \leq y^2.$$

Since we have that $l(10) = [1, 3] \cup \{9\}$ is not an open set, and therefore, $i(10) =]9, 10]$ is not a closed set, we have that $\precsim$ is neither upper semicontinuous of type 1 nor upper semicontinuous of type 2. Further, since (X, τ_{nat}) is second countable and $\precsim$ is a total preorder, we have that $\precsim$ cannot be partially upper continuous since otherwise by Nosratabadi (2014, Theorem 1) there would exist an upper semicontinuous utility function u, contradicting the fact that the preorder is not upper semicontinuous of type 1. Nevertheless, $\precsim$ is quasi upper semicontinuous since the identity function i_X is an (upper semi)continuous weak utility for $\prec$. It is easily seen that every element in $]9, 10]$ is a maximal element for $\precsim$.

We now furnish an example of a quasi upper semicontinuous preorder which fails to be transfer weakly upper continuous.

Example 2.8 Let X be the set $[1, 3] \cup [9, 10]$ endowed with the natural (induced) topology τ_{nat} and consider the non-total preorder $\precsim$ on X defined as follows for all $x, y \in X$:

$$x \precsim y \Leftrightarrow \begin{cases} x \leq y \ (x, y \in [1, 3] \cup \{9\} \cup \{10\}) \\ x \leq y \ (x, y \in [9, 10] \cap \mathbb{Q}) \end{cases},$$

As in the previous example, $\precsim$ is quasi upper semicontinuous since the identity function is an (upper semi)continuous weak utility for $\prec$. On the other hand, we have that $9 \prec 10$ but there is no open subset $V(9)$ of X containing 9 such that for some element $y \in X$ it happens that $V(9) \precsim y$. Indeed, every open subset of X containing 9 contains elements $z \in]9, 10[\setminus\mathbb{Q}$ which are incomparable with any other element of X.

The following example of the *lexicographic order* is very popular in the literature. We present it as an illustration of the simplicity of quasi upper semicontinuity compared to the other existing concepts of semicontinuity of a preorder.

Example 2.9 Let $\precsim_L$ be the lexicographic order on $X = [0, 1] \times [0, 1] \subset \mathbb{R}^2$. It is known that $\precsim_L$ is transfer weakly upper continuous, not partially upper continuous (see Nosratabadi 2014, Example 3). It is simple to check that $\precsim_L$ is quasi upper semicontinuous. Consider two elements $x, y \in X$ such that $x = (x_1, x_2) \prec_L (y_1, y_2) = y$. If $x_1 < y_1$, then, for example, $[0, \frac{x_1+y_1}{2}[\times[0, 1[$ is an open $\prec_L$-decreasing subset of X containing x and excluding y. Analogously we can proceed when $x_1 = y_1$ and $x_2 < y_2$.

Recall that a topology τ on a set X is said to be T_1 if all its singleton sets $\{x\}$ are closed (in other words, if "points are closed"). Such a condition, which is obviously satisfied by every metric space, was already used by Campbell and Walker (1990).

We present a condition under which every quasi upper semicontinuous preorder is weakly upper continuous (therefore transfer weakly upper continuous).

Proposition 2.10 *Let $\precsim$ be a quasi upper semicontinuous preorder on a T_1 topological space (X, τ). If $I(x) = \{z \in X : z \bowtie x\}$ is a finite set for all $x \in X$, then $\precsim$ is weakly upper continuous.*

Proof Let $\precsim$ be a quasi upper semicontinuous preorder on (X, τ) and consider any two elements $x, y \in X$ such that $x \prec y$. Then, from condition (ii) of Proposition 2.4, there exists some open $\prec$-decreasing subset $O_{x,y}$ of X such that $x \in O_{x,y}$ and $y \in X \setminus O_{x,y}$. Observe that, since (X, τ) is T_1, we have that $I(x)$ is the union of finitely many closed sets, and therefore, it is itself closed. Define $V(x) = O_{x,y} \setminus I(x)$ to immediately notice that $V(x)$ is open. Finally, it must be $V(x) \precsim y$ since $y \prec z \in V(x) \subset O_{x,y}$ implies that $y \in O_{x,y}$ since $O_{x,y}$ is $\prec$-decreasing, and this is contradictory. This consideration completes the proof. □

We finish this section by presenting a representation result concerning quasi upper semicontinuous preorders.

Theorem 2.11 *Let $\precsim$ be preorder on a second countable topological space (X, τ). Then the following conditions are equivalent:*

(i) *There exists an upper semicontinuous weak utility u for the strict part $\prec$ of $\precsim$;*
(ii) *$\precsim$ is quasi upper semicontinuous.*

Proof The implication (i) $\Rightarrow$ (ii) is clear (see Remark 2.5). To prove the implication (ii) $\Rightarrow$ (i), denote by $\mathcal{B} = \{B_n : n \in \mathbb{N}^+\}$ a countable base of (X, τ), and define, for all $n \in \mathbb{N}^+$,

$$B_n^0 = B_n \cup \bigcup_{z \in B_n} l^0(z),$$

where the sets $l^0(z)$ are defined in condition (iii) of Proposition 2.4. It is immediate to check that B_n^0 is a $\prec$-decreasing subset of X for every $n \in \mathbb{N}^+$. Consider, for every $n \in \mathbb{N}^+$, the upper semicontinuous real-valued function on (X, τ) defined as follows:

$$u_n(z) := \begin{cases} 0 & \text{if} \quad z \in B_n^0 \\ 1 & \text{if} \quad z \in X \setminus B_n^0 \end{cases},$$

to easily observe that $u := \sum_{n \in \mathbb{N}^+} 2^{-n} u_n$ is an upper semicontinuous weak utility for $\prec$. So the proof is complete. □

Since a compact metric space (X, d) is separable, and therefore, second countable, we arrive at the following corollary, whose easy proof is left to the reader.

Corollary 2.12 *Let $\precsim$ be a quasi upper semicontinuous preorder on a compact metric space (X, d). Then there exists a maximal element x_0 which is obtained by maximizing an upper semicontinuous weak utility u for the strict part $\prec$ of $\precsim$.*

3 Existence of maximal elements

We present the main result of this paper, which guarantees the existence of a maximal element for a quasi upper semicontinuous preorder on a compact topological space. We recall just before that a *multi-utility representation* of a preorder $\precsim$ on a set X (see e.g., the seminal paper of Evren and Ok 2011) is a family U of functions $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$, with the property that for each $x, y \in X$,

$$x \precsim y \Leftrightarrow [u(x) \leq u(y), \text{ for all } u \in U]. \quad (1)$$

A preorder $\precsim$ on a set X is said to be *nontrivial* if there are two elements $x, y \in X$ such that $x \prec y$.

A classical and widely used result, whose attribution is nevertheless unknown, states that there exists a maximal element x_0 for every upper semicontinuous of type 2 preorder $\precsim$ on a compact topological space (X, τ) (see e.g., the application of paragraph 2 in Evren and Ok (2011), who used the concept of an upper semicontinuous multi-utility representation to prove this theorem). Now we have that the following more general theorem holds.

Theorem 3.1 *Let $\precsim$ be a nontrivial quasi upper semicontinuous preorder on a compact topological space (X, τ). Then there is a maximal element x_0 for $\precsim$.*

Proof Since there exists an upper semicontinuous of type 2 preorder $\lesssim$ on (X, τ) such that $\prec \subset <$, and $\lesssim$ has a maximal element x_0 from the popular theorem we mentioned above, we have that x_0 is also a maximal element for $\precsim$. □

Corollary 3.2 *Let $\precsim$ be a preorder on a set X. Then the following conditions are equivalent:*

- (i) *There exists a maximal element x_0 for $\precsim$.*
- (ii) *There exists a compact topology τ on X such that $\precsim$ is quasi upper semicontinuous with respect to τ.*

Proof (ii) $\Rightarrow$ (i). Immediate by Theorem 3.1. (i) $\Rightarrow$ (ii). If there exists a maximal element x_0 for $\precsim$, then from Alcantud (2002, Theorem 4), there exists a compact topology τ on X such that $\precsim$ is upper semicontinuous of type 1 with respect to τ. From Remark 2.5, we have that $\precsim$ is in particular quasi upper semicontinuous on (X, τ). This consideration finishes the proof. □

4 Conclusions

In this paper, we have presented a new concept of upper semicontinuity of a preorder on a topological space, namely quasi upper semicontinuity. Such a concept is inherited from utility theory and appears more suitable than the other notions of semicontinuty to deal with maximal elements of preorders on compact spaces. Quasi upper semicontinuity of a preorder is naturally associated to the existence of an upper semicontinuous weak utility for its strict part. We have presented a condition under which quasi upper semicontinuous preorders are also transfer weakly upper continuous.

Acknowledgements Financial support by Istituto Nazionale di Alta Matematica "F. Severi" (Italy) is gratefully acknowledged.

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