Prospect Theory Based Portfolio Optimization Problem with Imprecise Forecasts

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In this paper we propose a novel interval optimization approach for portfolio selection when imprecise forecasts are available. We consider investors acting their choices according to the prospect theory, where scenarios are provided in the form of approximate numbers. The resulting constrained nonlinear interval optimization problem is converted into two nonlinear programming problems using a total order relation between intervals. Static and dynamic analysis of portfolios involving assets from the Croatian market illustrate the potential of the method with respect to the standard procedure.

Key Words: prospect theory, random sets, interval orders, interval optimization, Croatian stock market
JEL *Classification:* C61, C63, G11, G15, G17

Introduction

The relationship between the theory of financial markets and the rational behavior of an individual has been discussed for many years. In particular, many mathematical models have been developed which take into account both the uncertainty deriving from the investments in a certain number of assets (portfolio theory) and the subjective risk aversion of a single investor (utility theory).

The modern utility theory began with Von Neumann and Morgenstern (1947). Their representation theorem asserts that any individual whose preferences satisfy some given axioms has a utility (or value) function, which is concave in case of risk aversion (i.e., refusal of a fair gamble with zero expected value takes place).

Modern portfolio theory (MPT) was introduced by Markowitz (1952; 1959). He adopted a quadratic utility function as a reasonable approximation of a rational investor's behavior with risk-return tradeoffs.

Bawa (1975) and Fishburn (1977) gave proofs that mean-lower partial moment models can implement Von Neumann and Morgenstern utility functions and, at the same time, can be easily related to first, second and third stochastic dominance. In these models the wealth of the investor is replaced by a return rate below a desirable target. A few years later Fishburn and Kochenberger (1979) and Holthausen (1981) introduced their upper and lower partial moment models (UPM-LPM), namely an extension of the previous models.

The prospect theory (PT) was developed in order to find an answer to the concerns arising from the paradoxes of Allais (1953) and Ellsberg (1961). Starting from UPM-LPM models, Kahneman and Tversky (1979) added the notion of a distorted probability, i.e. a nonlinear transformation of the probability scale which possibly overweights small probabilities and underweights moderate and high probabilities.

Since PT does not always satisfy stochastic dominance and it is not easily extended to prospects with a large number of outcomes, both problems can be solved by cumulative prospect theory (CPT), in which the rank-dependent (or cumulative) functional is considered (Tversky and Kahneman 1992). In recent years several nonlinear programming problems have been developed in order to define optimal portfolios achieving the maximum of the utility functions, both in PT and in CPT. De Giorgi, Hens, and Meyer (2007) developed an algorithm that finds the best asset allocation in PT, overcoming the difficulties caused by non-differentiability and non-concavity of the value function. On the other hand, Hens and Mayer (2014) maximized CPT along the Markowitz's mean-variance efficient frontier. They found that the difference between the two methods is negligible in case of normally distributed returns but becomes considerable if asset allocation data for pension funds are considered.

Differently from the aforementioned works on PT and CPT, we consider an extended model for the financial market in which securities have random interval payoffs, but with a fixed underlying probability space as explained in You (2013). The uncertainties are not from probabilities measures but from realizations of random variables. Thus, a random interval is here interpreted as an imprecise perception of a random variable (Miranda, Couso, and Gil 2005) and can be roughly defined as an interval whose two endpoints are random variables.

To the best of our knowledge, this is the first attempt to integrate imprecise forecasts as random intervals into the PT-based portfolio selection model. Moreover, from a computational point of view, we provide a tractable formulation of the resulting interval optimization problem by means of two nonlinear programming problems. Solutions are detected on the basis of a genetic algorithm, whose efficiency in terms of both robustness and computational costs is also investigated. Static and dynamic analysis are conducted for both standard and interval-based PT portfolios on eight assets from the Croatian stock index. Results illustrate the potential of the proposed model.

The paper is organized as follows. The next section introduces some basic concepts which will represent the building-blocks of the developed model. The standard PT portfolio selection problem is described in the third section with asset returns modeled by means of random variables. The fourth section introduces the model with imprecise forecasts given as interval numbers. A numerical analysis is provided in the fifth section to verify the performance of the suggested optimal portfolios. The sixth section 6 concludes the paper with a summary and some remarks.

Background on Interval Analysis

INTERVAL NUMBERS

Following You (2013), an extended model for the financial market is considered in which securities have random interval payoffs with a fixed underlying probability space. In this environment, the uncertainties are not from probabilities measures but from realizations of random variables. This uncertain, imprecise and incomplete information can thus be incorporated into the portfolio optimization process by expressing data and/or parameters as intervals instead of single values. An adequate algebraic and probabilistic setting has to be defined in order to properly introduce the decision maker actions.

Definition 1. *An interval number, denoted as $\widetilde{a}$, is a bounded and closed subset of $\mathbb{R}$ given by*

$$\widetilde{a} = [a^l, a^u] \overset{def}{=} \left\{x \in \mathbb{R} \mid a^l \leqslant x \leqslant a^u\right\} \tag{1}$$

where $a^l, a^u \in \mathbb{R}$, with $a^l \leqslant a^u$, are the lower and the upper bounds of $\widetilde{a}$, respectively.

This representation of an interval number $\widetilde{a}$ is called endpoints (shortly EP) form. The set of all interval numbers on $\mathbb{R}$ is denoted as $\mathcal{K}_c(\mathbb{R})$.

Remark 1. *Note that if $a^l = a^u$ then $\widetilde{a}$ reduces to a real number.*

The sum of two interval numbers and the product of an interval number by a scalar are defined in terms of the corresponding Minkowski set-theoretic operations.

Definition 2. *For every $\widetilde{a} = [a^l, a^u]$, $\widetilde{b} = [b^l, b^u]$ in $\mathcal{K}_c(\mathbb{R})$ and $\gamma \in \mathbb{R}$, we have*

$$\widetilde{a} + \widetilde{b} \overset{def}{=} \left\{a + b \mid a \in \widetilde{a},\ b \in \widetilde{b}\right\} = [a^l + b^l, a^u + b^u]. \tag{2}$$

and

$$\gamma * \widetilde{a} \overset{def}{=} \{\gamma a \mid a \in \widetilde{a}\} = \begin{cases} [\gamma a^l, \gamma a^u] & \text{if } \gamma \geqslant 0 \\ [\gamma a^u, \gamma a^l] & \text{if } \gamma < 0. \end{cases} \tag{3}$$

The space $\mathcal{K}_c(\mathbb{R})$ with its arithmetic can be embedded onto the closed convex cone $\mathbb{R} \times [0, +\infty)$ of $\mathbb{R}^2$ by means of the so called t-vector function (Corral, Gil, and Gil 2011):

$$\begin{aligned} t \colon \mathcal{K}_c(\mathbb{R}) &\to \mathbb{R}^2 \\ \widetilde{a} &\mapsto (a^c, a^w) \end{aligned} \tag{4}$$

which maps an interval number $\widetilde{a}$ to an ordered pair of real numbers representing its center a^c and its radius a^w. With an abuse of notation, we will follow the customary of identify $\mathcal{K}_c(\mathbb{R})$ with its copy in $\mathbb{R}^2$. In this manner, a second characterization of an interval number is possible.

Definition 3. *An interval number $\widetilde{a} \in \mathcal{K}_c(\mathbb{R})$ is said to be in midpoint-radius (MR) form if it is encoded as the following vector of $\mathbb{R}^2$*

$$\widetilde{a} = (a^c, a^w) \overset{def}{=} \left(\frac{a^u + a^l}{2}, \frac{a^u - a^l}{2}\right) \tag{5}$$

where a^c denotes the center of the interval and a^w is its half-width.

By means of Eqn. (5), we can easily move from EP to MR encoding and vice versa. In particular, for every $\widetilde{a} \in \mathcal{K}_c(\mathbb{R})$, we have that

$$(a^c, a^w) = \{x \in \mathbb{R} \mid a^c - a^w \leq x \leq a^c + a^w\} = [a^l, a^u].$$

From these observations it emerges that the former encoding is suitable to introduce algebraic properties of intervals while the latter can be used to exhibit and explicitly manipulate the uncertainty in interval numbers.

RANDOM INTERVALS

We can introduce random intervals by exploiting the EP encoding of interval numbers as follows.

Definition 4. *Let* $(\Omega, \mathfrak{F}, P)$ *be a probability space. A multi-valued mapping* $\Gamma\colon \Omega \to \mathcal{K}_c(\mathbb{R})$, *given by* $\Gamma(\omega) = [\inf \Gamma(\omega), \sup \Gamma(\omega)]$, *where* $\inf \Gamma,\ \sup \Gamma\colon \Omega \to \mathbb{R}$ *are two real-valued functions such that* $\inf \Gamma \leq \sup \Gamma$ *almost surely, is said a random interval if* $\inf \Gamma$ *and* $\sup \Gamma$ *are real-valued random variables.*

A notion associated to the concept of random interval is the following.

Definition 5. *Let* $\Gamma\colon \Omega \to \mathcal{K}_c(\mathbb{R})$ *be a random interval. A random variable* $X\colon \Omega \to \mathbb{R}$ *is said a (measurable) selection of* Γ *if* X *is measurable and* $X(\omega) \in \Gamma(\omega)$ *for all* $\omega \in \Omega$.

The set of all measurable selections of Γ is denoted by $\mathcal{S}(\Gamma)$.

We assume that a random interval Γ represents an incomplete knowledge about the outcomes of a given random variable X. Thus, all the information we have available is that X is a measurable selection of Γ. Accordingly, let $\mathbb{E}(X)$ represents the Lebesgue expectation of a random variable X, the random interval corresponding to an imprecise/incomplete perception of $X(\omega)$ for all $\omega \in \Omega$ is the so-called Aumann expectation (Aumann 1965) and is defined as follows.

Definition 6. *Let* $(\Omega, \mathfrak{F}, P)$ *be a probability space and* $\Gamma\colon \Omega \to \mathcal{K}_c(\mathbb{R})$ *be a random interval such that all its selections are integrable, i.e.* $X \in L^1(\Omega, \mathfrak{F}, P)$ *for all* $X \in \mathcal{S}(\Gamma)$. *The interval number* $\mathbb{E}(\Gamma)$ *defined as*

$$\mathbb{E}(\Gamma) = [\mathbb{E}(\inf \Gamma), \mathbb{E}(\sup \Gamma)]$$

where $\inf \Gamma$ *and* $\sup \Gamma$ *are the two random variables specified in Definition 4, is called the expected (or mean) value of* Γ *in Aumann's sense.*

Remark 2. *In the definition of* $\mathbb{E}(\Gamma)$, *the set of all measurable selections* $\mathcal{S}(\Gamma)$ *is replaced by the subset of all integrable selections.*

The Aumann expectation is coherent with interval arithmetic (Molchanov 2005) and inherits many valuable probabilistic and statistical properties from expectation of a real-valued random variable, such as the satisfaction of the strong law of large numbers (Artstein and Vitale 1975). The next proposition, in particular, summarizes some results that will be used in the next sections to formalize the notion of expected interval (rate of) return and other related notions.

Proposition 1. *Let* $(\Omega, \mathfrak{F}, P)$ *be a probability space. The Aumann mean of a random interval satisfy the following properties:*

i) if Γ *is a random interval such that* $\Gamma(\Omega) = \{\widetilde{a}_1, \ldots, \widetilde{a}_n\}$ *and* $\{\Omega_i\}_{i=1}^n$ *is a partition of* Ω, *with* $\Omega_i = \Gamma^{-1}(\widetilde{a}_i)$, $i = 1, \ldots, n$, *then*

$$\mathbb{E}(\Gamma) = \sum_{i=1}^{n} P(\Omega_i) * \widetilde{a}_i;$$

ii) for every $\alpha, \beta \in \mathbb{R}$, $\widetilde{a} \in \mathcal{K}_c(\mathbb{R})$ *and* Γ, Υ *random intervals, then*

$$\mathbb{E}(\alpha * \Gamma + \beta * \Upsilon + \widetilde{a}) = \alpha * \mathbb{E}(\Gamma) + \beta * \mathbb{E}(\Upsilon) + \widetilde{a}.$$

Remark 3. *We have limited the presentation to the* $\mathcal{K}_c(\mathbb{R})$ *space, omitting the exposition for the general n-dimensional case, in order to avoid useless cumbersome notations since the results are almost the same.*

INTERVAL EXTENSION OF A POINT-VALUED FUNCTION

Now we explain how an interval extension of a point-valued function to an interval-valued mapping can be constructed in order to develop an interval model. The exposition specializes the arguments in Hickey (2001) to the continuous case, since only this type of functions will be handled in the next sections. Noting that a continuous point-valued function maps compact sets into compact sets, we can state the following definition for a multi-valued mapping extending a function.

Definition 7. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be a continuous point-valued function. The natural interval extension of* f *is the multi-valued mapping* $\widehat{f}: \mathcal{K}_c(\mathbb{R}) \to \mathcal{K}_c(\mathbb{R})$ *given by*

$$\widehat{f}(\widetilde{\mathbf{x}}) \overset{def}{=} \begin{cases} \{f(\mathbf{x}) \mid \mathbf{x} \in \widetilde{\mathbf{x}} \cap \mathrm{dom}(f)\}, & \text{if } \widetilde{\mathbf{x}} \cap \mathrm{dom}(f) \neq \emptyset \\ \emptyset, & \text{otherwise} \end{cases} \tag{6}$$

where dom(f) *is the domain of* f.

The natural interval extension can be straightforwardly computed in the following case.

Lemma 2. *Let* $f: \mathbb{R} \to \mathbb{R}$ *be any monotone continuous point-valued function and assume* $\widetilde{x} \cap dom(f) = [x^l, x^u]$ *is non-empty, then it holds*

$$\widehat{f}(\widetilde{x}) = \begin{cases} [f(x^l), f(x^u)] & \text{if } f \text{ is increasing} \\ [f(x^u), f(x^l)] & \text{if } f \text{ is decreasing.} \end{cases} \tag{7}$$

The proof of this result is an immediate consequence of Eqn. 6 and of the Weierstrass' theorem.

As we will see more precisely in the following sections, the value function suggested by Tversky and Kahneman (1992) has a power function form. Moreover, the same value function with the same parameterization has been adopted by De Giorgi, Hens, and Meyer (2007) to evaluate financial portfolios. Following these two studies, we tackle the portfolio selection problem when agents exhibit preferences in line with prospect theory and have imprecise information about the behavior of the market. Hence, we are interested to evaluate the natural extension of the power function. In particular, since the parameters of the value function in Tversky and Kahneman (1992) are rational, we analyze the case with rational exponents, i.e. $v(x) = x^{\frac{r}{s}}$, where r, s are coprime positive integers. Without loss of generality, we assume $\widetilde{x} \cap \mathrm{dom}(f) = \widetilde{x}$ and define the r-th power of an interval $\widetilde{x} = [x^l, x^u]$ as

$$\widetilde{x}^r \overset{def}{=} \begin{cases} \left[(x^l)^r, (x^u)^r\right] & \text{if } r \text{ is odd or } x^l \geqslant 0 \\ \left[(x^u)^r, (x^l)^r\right] & \text{if } r \text{ is even and } x^u \leqslant 0 \\ \left[0, \max\left\{(x^l)^r, (x^u)^r\right\}\right] & \text{if } r \text{ is even and } x^l \leqslant 0 \leqslant x^u \end{cases} \tag{8}$$

and its s-th root as

$$\widetilde{x}^{\frac{1}{s}} \overset{def}{=} \begin{cases} \left[(x^l)^{\frac{1}{s}}, (x^u)^{\frac{1}{s}}\right] & \text{if } s \text{ is odd or } x^l \geqslant 0 \\ \left[0, (x^u)^{\frac{1}{s}}\right] & \text{if } s \text{ is even and } x^l \leqslant 0 \leqslant x^u \\ \emptyset & \text{if } s \text{ is even and } x^u < 0. \end{cases} \tag{9}$$

Accordingly, the natural extension of the power function with rational exponents may be obtained by combining Eqns. (8) and (9) as follows

$$\widetilde{v}(\widetilde{x}) = \widetilde{x}^{\frac{r}{s}} \overset{def}{=} (\widetilde{x}^r)^{\frac{1}{s}}. \tag{10}$$

ORDER RELATIONS FOR INTERVAL NUMBERS

Mathematical programming involving interval numbers can be considered as optimization problems with uncertain or imprecise information in the objective function coefficients and/or constraints. Thereby, the preference relations for interval numbers play an important role to select the best alternative.

From a set-theoretic point of view, the set inclusion '$\subseteq$' represents a first example of partial order that can be used in decision-making problems involving interval numbers. However, since it fails to order pairs of intervals that are disjoint or overlapping, its use is limited. Several alternative approaches have been proposed in literature to fulfill these shortcomings. Depending on the methods used to define them, these order definitions can be divided in the following four groups (Karmakar and Bhunia 2012): general definitions of interval ranking that exploit the EP and MR characterizations, orderings that depend on some particular indices or specified functions, interval rankings depending on probabilistic or fuzzy concept and diagrammatic representations.

In this paper, the preference relation due to Hu and Wang (2006) is considered since, from one hand, it exploits the metric structure of the $\mathcal{K}_c(\mathbb{R})$ embedding onto $\mathbb{R}^2$, and, from the other hand, it is one of the most suitable for ranking interval numbers according to the findings in Karmakar and Bhunia (2012).

Definition 8. *Let* $\widetilde{a} = [a^l, a^u] = (a^c, a^w)$ *and* $\widetilde{b} = [b^l, b^u] = (b^c, b^w)$ *be two interval numbers in* $\mathcal{K}_c(\mathbb{R})$*, then the Hu and Wang's (shortly, HW) relation is given by*

$$\widetilde{a} \preccurlyeq \widetilde{b} \Longleftrightarrow (a^c < b^c) \ \vee \ (a^c = b^c \ \wedge \ a^w \geqslant b^w)\,. \tag{11}$$

Furthermore,

$$\widetilde{a} \prec \widetilde{b} \Longleftrightarrow \widetilde{a} \preccurlyeq \widetilde{b} \ \wedge \ \widetilde{a} \neq \widetilde{b} \tag{12}$$

defines the HW strict order relation on $\mathcal{K}_c(\mathbb{R})$.

Remark 4. *The HW ordering for* $\mathcal{K}_c(\mathbb{R})$ *is defined in terms of the corresponding order relation on the MR-coordinate space in such a way that between two intervals, the one with higher center or, if they present the same midpoint, with smaller width, is preferred.*

The Portfolio Selection Model under Prospect Theory

The financial market is modelled by a probability space $(\Omega, \mathfrak{F}, P)$ and consists of n risky assets, indexed from 1 to n. Agents allocate their wealth over a one-period investment horizon according to the following table of scenarios

$$\begin{pmatrix} \mathbf{r}_1 & \dots & \mathbf{r}_S \\ p_1 & \dots & p_S \end{pmatrix} \quad \text{with } \sum_{s=1}^{S} p_s = 1 \text{ and } p_s \geqslant 0 \ \forall s \tag{13}$$

where S represents the number of involved scenarios, $\mathbf{r}_s = (r_{1s}, \dots, r_{ns})^T$ is the n-vector of rates of return for the s-th scenario and p_s is the associated probability of occurrence, $s = 1, \dots, S$. In this manner, the expected rate of return for the i-th security can be computed as the mean rate of return over the S scenarios, i.e.

$$\mathbb{E}(r_i) = \sum_{s=1}^{S} p_s r_{is}.$$

Now, in order to formulate the asset allocation problem, let x_i be the weight of the i-th asset in the portfolio and impose on weights the constraints

$$\sum_{i=1}^{n} x_i = 1$$

and $x_i \geqslant 0$ for all i, that means all budget is invested and short-sales are not allowed. The set of all feasible portfolios satisfying these conditions is denoted by $\mathcal{X}$. Each $\mathbf{x} \in \mathcal{X}$ defines a random variable that represents the portfolio rate of return with an expected value expressed in terms of the scenario realizations. More specifically, denoting the portfolio rate of return for a fixed $\mathbf{x} \in \mathcal{X}$ under the s-th scenario by

$$r_s^p \stackrel{def}{=} \mathbf{x}^T \mathbf{r}_s = \sum_{i=1}^{n} x_i r_{is} \quad \text{for } i = 1, \ldots, S, \tag{14}$$

it holds that the expected rate of return of the portfolio is

$$\mathbb{E}(r^p) = \sum_{i=1}^{n} x_i \mathbb{E}(r_i) = \sum_{i=1}^{n} x_i \sum_{s=1}^{S} p_s r_{is} = \sum_{s=1}^{S} p_s \sum_{i=1}^{n} x_i r_{is} = \sum_{s=1}^{S} p_s r_s^p. \tag{15}$$

A market participant is said to be a PT-investor, if she/he operates consistently with prospect theory. This means that the decisions related to investments are articulated on the basis of the following three assumptions. First, outcomes are evaluated in comparison to a certain benchmark rather then an absolute final wealth. This behavior is modeled by a reference point, which divides outcomes into gains and losses. The reference level of wealth, r^{ref}, may be represented by a target wealth fixed at the beginning of the investment period, or by an expected wealth, or by the value of a given weighted index of the random assets (Pirvu and Schulze 2012). Second, reactions toward probable gains and losses are different and the corresponding prospect value is decided by means of an S-shaped function that is concave for gains and convex and steeper for losses. This value function models the loss aversion and reference dependence besides the risk aversion for gains. Third, the PT-investor does not use physical outcome probabilities for the investment decisions, instead he considers probabilities distorted by a weighting function. Typically, this distortion overestimates low probabilities.

We now formalize the PT-portfolio selection problem by explicitly introducing the features just described. The value function and the weighting function proposed in Kahneman and Tversky (1979) and Tversky and Kahneman (1992) are implemented. More specifically, a piecewise power value function is employed, which can be formulated in our context as

$$v(\mathbf{x}|\mathbf{r}_s; r^{ref}) = \begin{cases} \left(\mathbf{x}^T\mathbf{r}_s - r^{ref}\right)^\alpha & \text{if } \mathbf{x}^T\mathbf{r}_s \geqslant r^{ref} \\ -\beta(r^{ref} - \mathbf{x}^T\mathbf{r}_s)^\alpha & \text{if } \mathbf{x}^T\mathbf{r}_s < r^{ref} \end{cases} \tag{16}$$

where r^{ref} represents a reference rate of return for the investment, α denotes the risk aversion parameter and β is the loss aversion parameter. As previously mentioned, we follow Tversky and Kahneman (1992) and De Giorgi, Hens, and Meyer (2007) and set $\alpha = 0.88$ and $\beta = 2.25$. Figure 1, in the plot on the left, displays $v(x)$ with this parameter selection in the case $r^{ref} = 0$.

A scenario-based portfolio rate of return r_s^p is called a gain if it is greater than or equal to r^{ref}, otherwise is said a loss. The distortion of outcome probabilities for gains and losses is thus modelled by means of the following probability weighting function

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \tag{17}$$

with $0 < \gamma \leqslant 1$. In line with Kahneman and Tversky (1979) and Tversky and Kahneman (1992), we consider $\gamma = 0.65$. Figure 1, in the right chart, displays $w(p)$ with this parameter selection and the no-distorted probability function, represented by $w(p)$ with $\gamma = 1$. From this comparison, we can see how the nonlinear transformation of the probability scale (17) overweights small probabilities and underweights moderate and high probabilities. Finally, the PT-investor formulates her/his investment decisions according to the solution of the following nonlinear optimization problem

$$\begin{aligned} \max \quad & f(\mathbf{x}|\mathbf{r}_1, \ldots, \mathbf{r}_S; r^{ref}) = \sum_{s=1}^{S} \pi_s v\left(\mathbf{x}|\mathbf{r}_s; r^{ref}\right) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n. \end{aligned} \tag{18}$$

The Interval Based PT Portfolio Optimization Model

The market is the same as that described in the previous section, i.e. with the same risky assets, indexed from 1 to n, and the same probability space $(\Omega, \mathfrak{F}, P)$. The novelty now is that rates of return are modeled by random intervals instead of random variables in order to represent imprecise and incomplete knowledge about the future dynamics of the market.

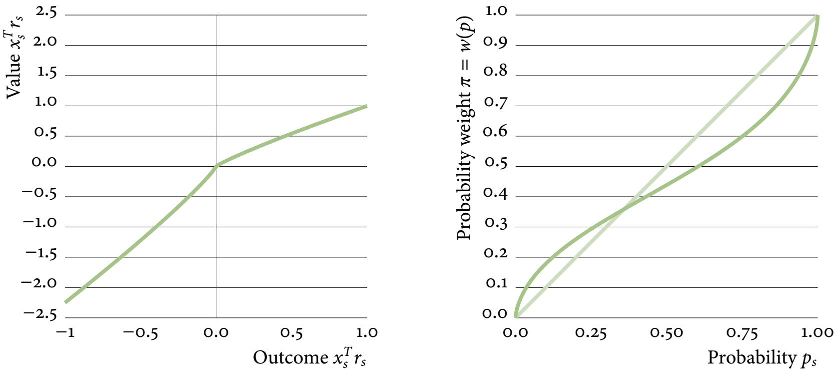

FIGURE 1 Value Function with $\alpha = 0.88, \beta = 2.25$ and $r^{ref} = 0$ (Left) and Weighting Function with $\gamma = 0.65$ (Right) as in Tversky and Kahneman (1992)

The interval counterpart of the financial quantities entering Problem (18) are to be established. More specifically, noting that the sum of random intervals is a random interval (Molchanov 2005), we can give the following definitions for the interval rate of return of a portfolio and for its expected value, respectively.

Definition 9. *Let $\widetilde{R}_i = \left[R_i^l, R_i^u\right]$ be the random interval rate of return of the i-th asset, $i = 1, \ldots, n$, with Aumann mean $\mathbb{E}(\widetilde{R}_i) = \left[\mathbb{E}(R_i^l), \mathbb{E}(R_i^u)\right]$. The interval rate of return of the portfolio with weights $(x_1, \ldots, x_n)^T$ in the n-dimensional simplex $\mathcal{X}$ is the random interval*

$$\widetilde{R}^p = \left[R^{p,l}, R^{p,u}\right] \stackrel{def}{=} \sum_{i=1}^{n} x_i \widetilde{R}_i = \left[\sum_{i=1}^{n} x_i R_i^l, \sum_{i=1}^{n} x_i R_i^u\right] \tag{19}$$

with Aumann mean given by

$$\mathbb{E}(\widetilde{R}^p) = \left[\mathbb{E}\left(R^{p,l}\right), \mathbb{E}\left(R^{p,u}\right)\right] = \left[\sum_{i=1}^{n} x_i \mathbb{E}(R_i^l), \sum_{i=1}^{n} x_i \mathbb{E}(R_i^u)\right]. \tag{20}$$

Remark 5. *If we remove the assumption of no-short selling,* (19) *and* (20) *are not true in general due to* (2).

We assume the investor operates her/his decisions on the basis of the following table of interval scenarios:

$$\begin{pmatrix} \widetilde{\mathbf{r}}_1 & \ldots & \widetilde{\mathbf{r}}_S \\ p_1 & \ldots & p_S \end{pmatrix} \quad \text{with} \sum_{s=1}^{S} p_s = 1 \text{ and } p_s \geqslant 0 \ \forall s \tag{21}$$

where $\widetilde{\mathbf{r}}_s = (\widetilde{r}_{1s}, \ldots, \widetilde{r}_{ns})^T$ is the n-vector of interval rates of return for the

s-th scenario, $s = 1, \ldots, S$, and p_s is the associated probability of occurrence for scenario s.

Remark 6. *The process of interval scenarios generation for the rate of return of asset i can be reduced to the construction of scenarios for the jointly distributed random variables $\left(R_i^l, R_i^u\right)$, representing the lower and the upper endpoints of the random interval $\widetilde{R}_i$, taking into account also the dependence on the other random interval asset rates of return. Moreover, we do not assume the outcomes of interval scenarios be disjoint, i.e. $\widetilde{r}_{is'} \cap \widetilde{r}_{is''} \neq \emptyset$, for $s', s'' \in \{1, \ldots, S\}$, in order to offer a larger degree of freedom in modeling uncertainty (Zhu, Ji, and Li 2015). In the empirical examples, for instance, a PCA-based method is used to generate point-valued scenarios and, successively, a perturbation technique is integrated in order to obtain interval scenarios. However other more complex techniques are also possible, like moment-matching and Monte Carlo methods, to adequately represent the distributions of asset rates of return (for a complete review of standard method for scenario generation, the interested reader may consult Mitra and Di Domenica 2010).*

The portfolio interval rate of return under the s-th scenario can thus be defined as

$$\widetilde{r}_s^p = \left[r_s^{p,l}, r_s^{p,u}\right] \stackrel{def}{=} \left[\sum_{i=1}^{n} x_i r_{is}^l, \sum_{i=1}^{n} x_i r_{is}^u\right]. \tag{22}$$

Similar to the case of random variables, it is easy to show that if

$$\mathbb{E}(R_i^l) = \sum_{s=1}^{S} p_s r_{is}^l \quad \text{and} \quad \mathbb{E}(R_i^u) = \sum_{s=1}^{S} p_s r_{is}^u,$$

for $i = 1, \ldots, n$, the expected (in the Aumann's sense) interval rate of return of the portfolio in Eqn. (20) can be directly evaluated in terms of the scenarios as

$$\mathbb{E}(\widetilde{R}^p) = \left[\sum_{s=1}^{S} p_s r_s^{p,l}, \sum_{s=1}^{S} p_s r_s^{p,u}\right]. \tag{23}$$

An analogous result can be obtained with the MR interval encoding.

In this financial environment, the PT investor articulates her/his choices relative to an interval reference rate of return, $\widetilde{r}^{ref} = [r^{ref,l}, r^{ref,u}]$, on the basis of the natural extension of the piecewise power function (16). Gains and losses are now defined on the basis of the preference relations (11) and (12): an interval rate of return is called a gain if it is preferred to the interval reference point, conversely, it represents a loss. Maintaining the same parameter setting of the PT model, i.e. $\alpha = 0.88 = 22/25$ and $\beta = 2.25$, the interval extension of the value function (16) is defined as

$$\widetilde{v}(\mathbf{x}|\widetilde{\mathbf{r}}_s;\widetilde{r}^{ref}) = \begin{cases} \left(\widetilde{r}_s^p - \widetilde{r}^{ref}\right)^{\alpha} & \text{if } \widetilde{r}_s^p \succcurlyeq \widetilde{r}^{ref} \\ -\beta(\widetilde{r}^{ref} - \widetilde{r}_s^p)^{\alpha} & \text{if } \widetilde{r}_s^p \prec \widetilde{r}^{ref} \end{cases} \tag{24}$$

where $\widetilde{r}_s^p$ is defined in (22).

According to the interpretation of a random interval as an imprecise perception of a random variable, we assume that the weighting function, describing the process of distortion for the probabilities of occurrence of a given scenario, depends solely on the center of the interval rates of return, and not on their width. Thus, the decision weights for our interval extension can also be evaluated by means of the function (17) of the PT model, with the same parameterizations. Finally, the PT-investor that takes into account also imprecise forecasts for her/his investment decisions has to solve the following nonlinear interval-valued programming problem

$$\begin{aligned} \max \quad & V(\mathbf{x}|\widetilde{\mathbf{r}}_1,\ldots,\widetilde{\mathbf{r}}_S;\widetilde{r}^{ref}) = \sum_{s=1}^{S} \pi_s \widetilde{v}(\mathbf{x}|\widetilde{\mathbf{r}}_s;\widetilde{r}^{ref}) \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n \end{aligned} \tag{25}$$

where 'max' is interpreted as the most preferred interval value for $V(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots,\widetilde{\mathbf{r}}_S;\widetilde{r}^{ref})$ with respect to the order relation (11).

Remark 7. *If $\widetilde{r}^{ref}$ is a degenerate interval, i.e. has null radius, the HW ordering reduces to the $\leqslant$ order for the interval centers. Moreover, if no imprecision is assumed in forecasts, Problem* (25) *reduces to the standard PT portfolio selection Problem* (18).

Definition 10. *A point $\mathbf{x}^* \in \mathcal{X}$ is an optimal solution of Problem* (25) *if there does not exist another point $\mathbf{x} \in \mathcal{X}$ such that $V(\mathbf{x}|\widetilde{\mathbf{r}}_1,\ldots,\widetilde{\mathbf{r}}_S;\widetilde{r}^{ref}) \prec V(\mathbf{x}^*|\widetilde{\mathbf{r}}_1,\ldots,\widetilde{\mathbf{r}}_S;\widetilde{r}^{ref})$.*

The next result permits to convert the nonlinear constrained interval optimization problem (25) into two nonlinear programming problems. In this manner we are able to solve our interval portfolio selection problem with standard nonlinear optimization solvers.

Proposition 3. *$\mathbf{x}^* \in \mathbb{R}^n$ is an optimal solution of the constrained nonlinear interval optimization problem* (25) *with respect to the HW order relation if* *and only if* $\mathbf{x}^*$ *is a solution of the two scalar nonlinear optimization problems*

$$\begin{aligned} max \quad & V^c(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref}) \\ s.t. \quad & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geqslant 0, \quad j = 1, \ldots, n \end{aligned} \tag{26}$$

and

$$\begin{aligned} min \quad & V^w(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref}) \\ s.t. \quad & \mathbf{x} \in \{\mathbf{y} \mid \mathbf{y} \text{ is a solution of } (26)\} \end{aligned} \tag{27}$$

where $V^c(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref})$ *and* $V^w(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref})$ *are the center and the radius of the interval* $V(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref})$*, respectively.*

Proof. Let $\mathbf{x}^* \in \mathbb{R}^n$ be a solution of Problem (25) with respect to the HW order relation. This implies that for any $\mathbf{x} \in \mathcal{X}$ we have

$$V(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref}) = \left(V^c(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref}), V^w(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref})\right)$$

with $V^c(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref}) \leq V^c(\mathbf{x}^*|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref})$. Thus, $\mathbf{x}^*$ solves (26). If in particular $V^c(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref}) = V^c(\mathbf{x}^*|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref})$, by the HW order relation definition, we also have $V^w(\mathbf{x}|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref}) \geq V^w(\mathbf{x}^*|\widetilde{\mathbf{r}}_1, \ldots, \widetilde{\mathbf{r}}_s; \widetilde{r}^{ref})$ and we conclude that $\mathbf{x}^*$ satisfies (27).
The converse is verified by reverting this process. Thus, the theorem remains proved. □

Illustrative Examples

DATA DESCRIPTION

The experiments have been based on data relative to the Croatia Zagreb Stock Exchange index (CROBEX). The investment universe comprises the following 8 assets: Adris Grupa d.d. (ADRS), Atlantic Grupa d.d. (ATGR), Ericsson Nikola Tesla d.d. (ERNT), HT d.d. (HT), INA d.d. (INA), Konar Elektroindustrija d.d. (KOEI), Kras d.d. (KRAS), Ledo d.d. (LEDO), Podravka d.d. (PODR) and Valamar Riviera d.d. (RIVP). The time series include weekly closing prices covering the period from 20/04/2009 to 23/06/2016 for a total of 356 observations. The quotations are taken from `http://zse.hr`. In figure 2, the evolution of CROBEX prices over the investment period is displayed in the left plot and the corresponding weekly rates of return are represented in the plot on the right. We have selected this time frame according to the findings in Pesa and Brajkovic (2016), in order to avoid the structural break in the Croatian economy

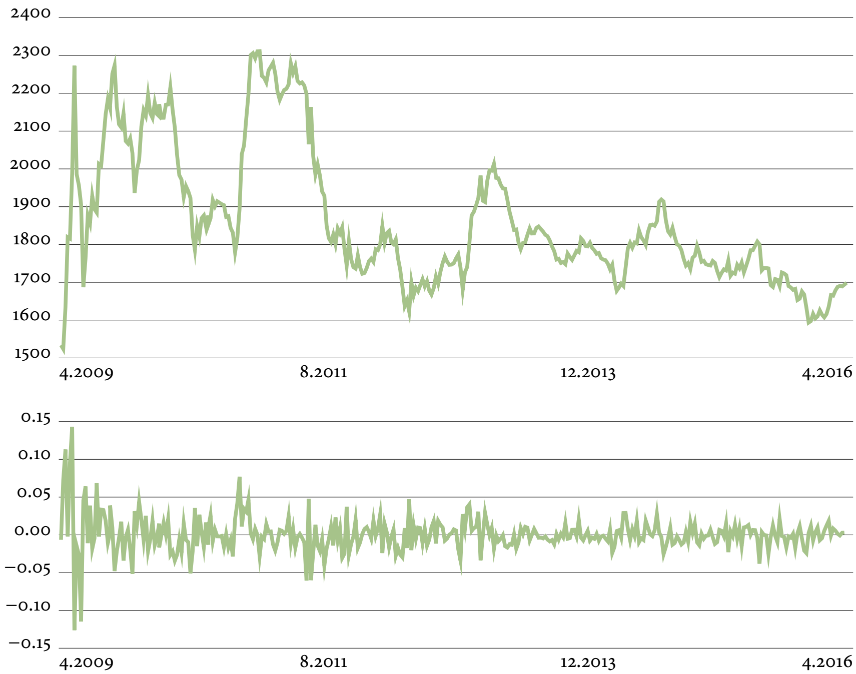

FIGURE 2 CROBEX Prices for the Period 20 April 2009 to 23 June 2016 (Top) and the Corresponding Rates of Return (Bottom)

during 2009 due to both the political crisis after Prime Minister's resignation for corruption cases and the subprime mortgages crisis in USA. The descriptive statistics for assets and index rate of return series are given in table 1. They reveal that, except for ERNT and HT, all the other assets have comparable values in terms of the first four moments, with a mean rate of return of about 0.26% (considerably greater than the 0.06% of the market), a standard deviation of about 3.35% (greater than the 2.33% of the market), positive skewness and kurtosis above 6. Instead, ERNT and HT present mean rates of return below the market (0.03% and −0.08%, respectively) and negative skewness.

The distributional characteristics of the rates of return are moreover analysed by means of the following set of tests: the Jarque-Bera, the Lilliefors and the Shapiro-Wilk tests are used to infer the assumption of normality; the Ljung-Box test has been adopted to verify the presence of autocorrelation; while the Engle's LM test checks the conditional heteroskedasticity effect in residuals up to the second order. Table 2 reports the corresponding statistics and p-values. The results confirm the non-

TABLE 1 Descriptive Statistics

Assets	Min.	Max.	Mean	Std. dev.	Skewness	Kurtosis
ADRS	−0.1459	0.1692	0.0030	0.0345	0.6215	7.9480
ATGR	−0.0743	0.1230	0.0020	0.0265	1.1175	6.2788
ERNT	−0.1811	0.1681	0.0003	0.0363	−0.3381	8.9328
HT	−0.1208	0.0900	−0.0008	0.0232	−0.5558	7.7699
KOEI	−0.1846	0.1538	0.0027	0.0294	0.2155	10.6474
KRAS	−0.1042	0.2326	0.0023	0.0340	1.5576	11.9749
LEDO	−0.2205	0.1833	0.0033	0.0384	0.4427	8.8284
PODR	−0.1213	0.2979	0.0024	0.0382	1.7176	13.8118
CROBEX	−0.1262	0.1429	0.0006	0.0233	0.5878	12.4975

NOTES Relative to the weekly rates of return from 27 April 2009 to 23 June 2016 for a total of 355 observations: Adris Grupa d.d. (ADRS), Atlantic Grupa d.d. (ATGR), Ericsson Nikola Tesla d.d. (ERNT), HT d.d. (HT), INA d.d. (INA), Konar Elektroindustrija d.d. (KOEI), Kras d.d. (KRAS), Ledo d.d. (LEDO), Podravka d.d. (PODR), Valamar Riviera d.d. (RIVP) and CROBEX index.

normality of assets rates of return and indicate serial autocorrelation in all time series, except for KOEI. The ARCH effect is present in 5 of the 8 time series (ADRS, ATGR, KOEI, KRAS, LEDO and PODR), thus the current rates of return of these assets are affected by spillover effects due to the rates of return of previous periods.

GENETIC ALGORITHM BASED OPTIMIZATION

In this subsection we describe the numerical methods implemented for both the generation of (interval) scenarios and for solving Problems (18) and (25).

As the distribution of asset rates of return is unknown, we make no assumption regarding either their joint and their marginal distributions instead we adopt the sampling procedure based on principal component analysis (PCA) developed by Topaloglou, Vladimirou, and Zenios (2002) to generate standard scenarios. It works as follows. First, a sufficient number of principal components (PC) for each asset is retained in order to capture most of the variability of historical samples. In the experiments we fixed a lower threshold for the represented historical variability equals to 80%. Second, the range of each PC is partitioned into several subintervals and the ratio of the number of samples within each subinterval to the total number of samples is used to represent the probability associated to that subinterval. Scenarios for each PC are then constructed with the

TABLE 2 Statistical Tests for Normality, Autocorrelation and Conditional Heteroskedasticity

Assets	Jarque-Bera	Lilliefors	Shapiro-Wilk	Ljung-Box	ARCH(2)
ADRS	384.9961**	0.0961**	0.9103**	23.0440	50.8836**
	(0.0010)	(0.0010)	(0.0000)	(0.2866)	(0.0000)
ATGR	232.9138**	0.1094**	0.9277**	30.8236	15.7314**
	(0.0010)	(0.0010)	(0.0000)	(0.0576)	(0.0004)
ERNT	527.3964**	0.1245**	0.8874**	17.2229	1.2951
	(0.0010)	(0.0010)	(0.0000)	(0.6385)	(0.5233)
HT	354.8119**	0.0771**	0.9245**	16.8107	1.3620
	(0.0010)	(0.0010)	(0.0000)	(0.6652)	(0.5061)
KOEI	867.8049**	0.1040**	0.8943**	51.8879**	45.2857**
	(0.0010)	(0.0010)	(0.0000)	(0.0001)	(0.0000)
KRAS	1334.9987**	0.1307**	0.8647**	23.4527	8.0778*
	(0.0010)	(0.0010)	(0.0000)	(0.2671)	(0.0176)
LEDO	514.0789**	0.1307**	0.8888**	18.5401	13.1084
	(0.0010)	(0.0010)	(0.0000)	(0.5519)	(0.0014)**
PODR	1903.6417**	0.0994**	0.8901**	31.1953	8.4204
	(0.0010)	(0.0010)	(0.0000)	(0.0527)	(0.0148)*

NOTES Relative to the weekly rates of return from 27 April 2009 to 23 June 2016 for a total of 355 observations. The p-values corresponding to the test statistics are reported in parentheses, ** and * denote rejection of the null hypothesis at 1% and 5% significance levels, respectively. ARCH(2) is the Engle's LM test for the ARCH effect in the residuals up to the second order.

midpoints of subintervals and the associated probabilities. Since the PCs are independent, the joint scenarios are given as the Cartesian product of scenarios of individual PCs. Finally, an inverse linear transformation derived by PCA provides the scenarios of asset rates of return. As indicated by the authors, in order to mitigate estimation risks, we revise the asset rates of return under each scenario by adding a Bayes-Stein correction term.

Interval-valued scenarios are obtained in this paper by applying to the point-valued scenarios the perturbation method proposed in Zhu, Ji, and Li (2015). More specifically, let $\mathbf{r}_s = (r_{1s}, \ldots, r_{ns})^t$ denote the n-vector of asset rates of return under the s-th scenario, $s = 1, \ldots, S$, then the corresponding perturbed interval-valued scenario is defined as

$$\widetilde{\mathbf{r}}_s = \left[r_{1s} - 1.96\frac{\widehat{\sigma}_1}{\sqrt{T}}, r_{1s} + 1.96\frac{\widehat{\sigma}_1}{\sqrt{T}} \right] \times \dots$$
$$\times \left[r_{ns} - 1.96\frac{\widehat{\sigma}_n}{\sqrt{T}}, r_{ns} + 1.96\frac{\widehat{\sigma}_n}{\sqrt{T}} \right],$$

where $\widehat{\sigma}_i$ is the standard deviation of rates of return for the i-th asset estimated by the T historical samples used to generate the traditional scenarios. In the MR form it can be compactly rewritten as

$$\widetilde{\mathbf{r}}_s = \left(\mathbf{r}_s, 1.96\frac{\widehat{\sigma}}{\sqrt{T}} \right)$$

with $\widehat{\sigma} = (\widehat{\sigma}_1, \dots, \widehat{\sigma}_n)^t$. In this case, the interval portfolio rate of return under the s-th scenario (22) becomes

$$\widetilde{\mathbf{r}}_s^p = \left(\sum_{i=1}^{n} x_i r_{is}, \frac{1.96}{\sqrt{T}} \sum_{i=1}^{n} x_i \widehat{\sigma}_i \right) \tag{28}$$

for all $\mathbf{x} \in \mathcal{X}$ and $s = 1, \dots, S$.

For detecting optimal solutions to Problems (18) and (26)–(27), due to non-differentiability and non-concavity of the objectives, we propose a procedure involving an evolutionary optimization technique, the so-called genetic algorithm (GA). This is a population based stochastic search method implementing the Darwinian principle of 'the survival of the fittest' and natural genetics (Goldberg 1989). The algorithm starts with an initial population of candidate solutions (called individuals) where each individual is represented using some form of encoding as a chromosome. These chromosomes are evaluated for their fitness and those with the highest value are selected in the population for reproduction. The selected individuals are then manipulated by two genetic operators, called crossover and mutation. The crossover is applied to create offspring from a pair of selected chromosomes while mutation is used to promote little modification/change in the offspring. The repeated applications of genetic operators to the relatively fit chromosomes result in an increase in the average fitness of the population over generation and identification of improved solutions to the problem under investigation. This process is applied iteratively until the termination criterion is satisfied. To implement the GA the following basic components are then to be considered: algorithm parameters (population size, probability of crossover and probability of mutation), chromosome representation, initialization of population, evaluation of fitness function, candidate selection process and genetic operators (crossover, mutation and elitism).

TABLE 3 Parameter Setting for the Considered GA

Parameter name	Value	Parameter name	Value
Generations	300	Crossover probability	0.50
Population size	100	Mutation scale	0.50
Seeding size	1	Mutation shrink	0.75
Elite size	1		

In this paper a variant of the algorithm by Kaucic (2012) is implemented. A real coding representation is adopted, i.e. a dimensional vector is used as a chromosome to represent a candidate optimal portfolio and an elite strategy is considered to clone the best individual from one generation to the next. The initial population is generated according to the following procedure for uniform vector generation over a simplex (Rubinstein 1982):

Step 1. Generate n random numbers from exponential distribution with parameter $\theta = 1$, i.e. $\lambda_i \sim \exp(1)$, $i = 1, \dots, n$.

Step 2. Apply the following formula and deliver $\mathbf{x} = (x_1, \dots, x_n)^t$ as a vector distributed uniformly on $\mathcal{X}$:

$$\mathbf{x} = (x_1, \dots, x_n)^t = \left(\frac{\lambda_1}{\sum_{i=1}^{n} \lambda_i}, \dots, \frac{\lambda_n}{\sum_{i=1}^{n} \lambda_i} \right)^t.$$

Selection is made by the stochastic universal sampling. Uniform crossover is used to avoid the positional and distributional bias that may prevent the production of good solutions. The Gaussian mutation operator modifies offsprings using the Gaussian distribution. A control on the composition of each offspring is then included to guarantee its feasibility. The parameter setting is listed in table 3.

Moreover, for the dynamic portfolio analysis, in order to generate solutions consistent over time, we implement a population seeding such that the best individual from the previous optimized population is copied in the initial population of the current optimization period.

STATIC AND DYNAMIC PORTFOLIO ANALYSIS

We test the flexibility and the efficiency of the proposed model with respect to the standard PT portfolio optimization procedure from both a static and a dynamic point of view.

The static portfolio analysis, in particular, focuses on the reaction of the PT and IPT optimal portfolios to changes in the reference point and

TABLE 4 Rates of Return and Corresponding Weights for the Optimal Portfolios to the PT-Based Problem (18) for Different Levels of the Reference Point r^{ref}

r^{ref}	0	0.0005	0.0010	0.0015
obj value	0.0013	0.0014	0.0014	0.0014
ADRS	0.2968	0.3044	0.3051	0.3051
ATGR	0.2154	0.2377	0.2982	0.2982
ERNT	0.0749	0.0378	0.0165	0.0165
HT	0.1440	0.0872	0.1122	0.1122
KOEI	0.0831	0.1408	0.1038	0.1038
KRAS	0.0725	0.1044	0.0820	0.0820
LEDO	0.0583	0.0579	0.0335	0.0335
PODR	0.0540	0.0298	0.0492	0.0492

reference interval, respectively. To this end, for the standard PT investor, the reference level r^{ref} is assumed to vary in the set $\{0, 0.0005, 0.0010, 0.0015\}$, while, for the PT investor with imprecise information, we assume that both center and radius of $\widetilde{r}^{ref}$ vary, with $r^{ref,c}$ coming from $\{0, 0.0005, 0.0010, 0.0015\}$ and $r^{ref,w}$ from $\{0, 0.0010, 0.0030, 0.0050\}$. For these experiments all data covering the period from 27/04/2009 to 23/06/2016 are used. 4 PCS are able to capture 81% of the variability and 3465 point-valued scenarios have been generated. The associated interval-valued scenarios are then constructed on the basis of (28). Due to the complexity of the objective landscape in (18) and in (25), we implement 50 simulations for each experiment and the solution with the maximum objective value has been identified as the optimal portfolio.

Table 4 reports the results for the experiments associated to the PT model. The 'obj value' row lists the rates of return reached by the optimal solutions and the remaining rows show the corresponding portfolio weights. It can be observed that the optimal rate of return is equal to 13 basis points when the reference point, r^{ref}, is set to 0, while it is equal to 14 basis points in the remaining cases. Moreover, for the last two experiments, GA detects the same optimal portfolio. These findings suggest that investment decisions are affected only marginally by the reference point when its magnitude is too large. In terms of prospect theory, it means that when agents present too optimistic expectations, portfolio constraints and market conditions play a crucial role in determining optimal portfolios.

The analysis for the interval-based PT model is more complex, involv-

TABLE 5 Interval Rates of Return for the Optimal Portfolios to the Interval PT-Based Problem (25) for Different Levels of the Reference Interval $\widetilde{r}^{ref}$

IPT		$r^{ref,c}$			
		0	5	10	15
$r^{ref,w}$	0	(15, 34)	(14, 33)	(15, 33)	(14, 33)
	10	(15, 43)	(14, 44)	(14, 43)	(14, 43)
	30	(14, 64)	(15, 64)	(14, 64)	(15, 64)
	50	(14, 81)	(15, 85)	(15, 85)	(14, 85)

NOTES Intervals are expressed in MR form.

ing 16 experiments. In the paper we limit to report results for the interval rates of return, omitting the portfolio weights, however these data are available upon request. Table 4 shows the interval rates of return for the identified optimal portfolios to Problems (26)–(27) in MR form to facilitate comparisons with the standard PT case. It emerges that the midpoints do not depend neither from the center of the reference interval nor its radius. We can not reach the same conclusion for the radii of the interval rates of return, since they increase proportionally to the width of the reference interval. Moreover, similar to the PT model, the composition of the optimal portfolios remains quite stable in all experiments.

The dynamic portfolio analysis is based on a sliding window procedure which focuses on the last six months of observations, from 16/11/2015 to 16/05/2016 (26 weeks). Here we test the following three aspects of the investment process for the proposed interval-valued PT-based portfolio optimization model:

i) the ability to identify profitable solutions;
ii) the level of diversification of these solutions;
iii) how the portfolio strategy can be expensive over time.

Scenarios are developed recursively by using a sliding window of 104 weeks of rates of return, which are updated as the process move on, by removing the first data and by adding the most recent information.

Figure 3 compares the evolution of the values of portfolios obtained by the interval-based PT strategy with those produced by the buy & hold strategy and by the standard PT portfolio selection model, respectively. It is assumed that the initial value is 10,000 kn at the starting date for all the investment strategies. We see that apart the last week of 2015 and the first week of 2016, when the buy & hold strategy is the most valuable, in the remaining weeks the PT models prevail. In particular, during Febru-

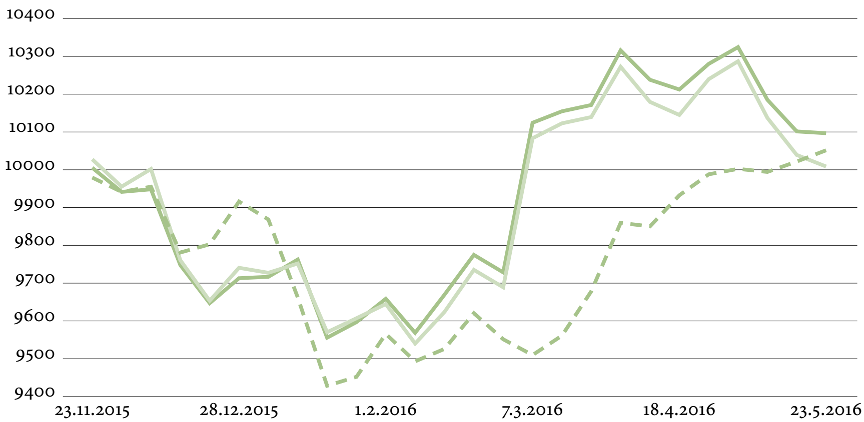

FIGURE 3 Evolution of the Optimal Portfolio Values for the Buy & Hold Strategy (Dashed), PT (Light) and Interval-Based PT (Dark) Models over the Period from 23 November 2015 to 23 May 2016; Initial Portfolio Value Fixed to 10,000 kn.

ary and March 2016 the PT models rise from the lowest to the highest levels (from around 9,500 kn to around 10,350 kn), while the buy & hold strategy remains below 9,800 kn. Qualitatively, the proposed model with interval data overperforms the standard model.

Diversification is useful to reduce risks and provide protection against extreme events by ensuring that one is not overly exposed to individual occurrences. In this paper we measure the degree of diversification of a given portfolio by means of the so-called diversification index (DI) defined by Woerheide and Persson (1993) as

$$\mathrm{DI}(t) = 1 - \sum_{i=1}^{n} x_{i,t}^2$$

where $x_{i,t}$ is the weight of the i-th asset in the portfolio at time t. A greater value of DI implies a greater diversification.

Trading cost is another crucial aspect to be taken into account when an investment strategy has to be evaluated. We consider the turnover, a measure of the changes in the portfolio composition from one period to the next, as an indirect estimate of trading costs due to the difficulty in directly modelling them. It is given by

$$\mathrm{turnover}(t) = \sum_{i=1}^{n} \left| x_{i,t} - x_{i,t-1} \right| .$$

The grater is the turnover and more expensive is the investment strategy.

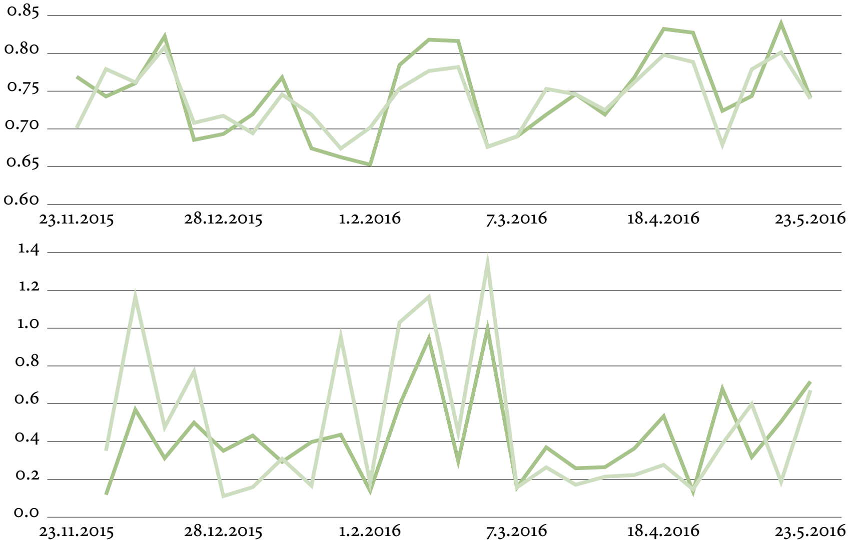

FIGURE 4 Diversification Index (top) and Turnover (bottom) of the Optimal Portfolio Values for PT (Light) and Interval-Based PT (Dark) Models over the Period from 23 November 2015 to 23 May 2016.

TABLE 6 Summary Statistics for the Buy & Hold Strategy, PT and Interval-Based PT Models over the Period from 23 November 2015 to 23 maXy 2016

Item	Buy & Hold	PT	IPT
Initial wealth (kn)	10000	10000	10000
Final wealth (kn)	10072	10009	10097
Rate of return (annualized %)	1.44	1.18	1.94
Diversification index (mean %)	—	74.08	74.60
Turnover (mean %)	—	47.70	42.72

In figure 4, we qualitatively compare, on the left chart, the diversification effect and, on the right, the turnover for the two PT investment strategies over time. Table 6 provides the mean of DI and turnover for the six months analyzed. Summing up, we conclude that the interval-based strategy present greater diversification and lower turnover with respect to the standard PT-based strategy.

Conclusions

In this paper we propose a novel interval optimization approach for the PT-based portfolio selection problem. The principal idea is to represent imprecise/incomplete information as random sets. An extension of the standard PT model is proposed that exploits interval analysis to define the interval counterpart of the value function. A model to generate interval-valued scenarios from point-valued ones is analyzed. The resulting constrained nonlinear interval optimization problem is converted into two nonlinear programming problems using a total order relation between intervals.

The flexibility and the efficiency of the proposed model are then compared with those of the standard PT portfolio selection procedure from both a static and a dynamic point of view in a set of experiments involving 8 assets from the Croatian market. A real-coded GA is developed to generate the solutions. Results indicate that the proposed model, incorporating imprecise knowledge, is robust with respect to the changes in the reference levels and is able to produce investment strategies that provide at the same time higher diversification and lower turnover than the standard PT model, which uses point-valued data.

These findings are very promising although there are several directions of improvement for the proposed model to manage imprecise and incomplete information in behavioural portfolios. First, our analysis covered a limited number of assets during a particular economic phase. Thus, we plan to involve more assets from different markets, covering a larger time window, in order to provide more conclusive results about the properties and capabilities of the model. A comparison with the classical mean-variance approach can be done to highlight the differences with the standard portfolio selection procedure. Second, we focus on an extension of portfolio optimization in a prospect theory environment, however, it is also possible to define a suitable interval counterpart of the model taking into account preferences based on cumulative prospect theory. This is of interest because of its consistency with first order stochastic dominance. Some details will indeed be given in a forthcoming paper. Third, the addition of uncertainty about forecasts has been modelled through random intervals, an alternative is represented by modelling returns as random variables with specific distributional assumptions and adding a degree of uncertainty relative to the parameters of the distributions.

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