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On large groups of symmetries of finite graphs embedded in spheres

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ABSTRACT

Let G be a finite group acting orthogonally on a pair (S^d, Γ) where Γ is a finite, connected graph of genus $g > 1$ embedded in the sphere S^d. The 3-dimensional case $d = 3$ has recently been considered in a paper by C. Wang, S. Wang, Y. Zhang and the present author where for each genus $g > 1$, the maximum order of an orientation-preserving G-action on a pair (S^3, Γ) is determined and the corresponding graphs Γ are classified (an upper bound for the order of G is $12(g-1)$). In the present paper, we consider arbitrary dimensions d and prove that the order of G is bounded above by a polynomial of degree $d/2$ in g if d is even and of degree $(d+1)/2$ if d is odd; moreover, the degree $d/2$ is best possible in even dimensions d. We discuss also the problem, given a finite graph Γ and its finite symmetry group, to find the minimal dimension of a sphere into which Γ embeds equivariantly as above.

Keywords: Groups acting on finite graphs; equivariant embeddings into spheres.

Mathematics Subject Classification 2010: 57S17, 57S25, 05C10

1. Introduction

We study large finite groups G of automorphisms of a finite, connected graph Γ which embeds smoothly into a sphere S^d of some dimension d such that the G-action on Γ extends to an orthogonal action of G on S^d. In other words, we study large finite groups G of orthogonal transformations of pairs (S^d, Γ) where Γ denotes a finite, connected graph smoothly embedded in a sphere S^d. All actions considered in the present paper will be faithful on both S^d and Γ, and all finite graphs Γ will be *hyperbolic*, i.e. connected, of genus $g > 1$ (the rank of its free fundamental group) and *without free edges* (edges with one vertex of valence 1; note that free edges can be deleted in a G-equivariant way without changing the genus of a graph); we allow closed and multiple edges.

The case of dimension $d = 3$ is considered in [7]. For a finite subgroup of the orthogonal group SO(4) acting on a pair (S^3, Γ), a regular neighborhood of Γ in

S^3 is a 3-dimensional handlebody V_g^3 of genus $g > 1$ on which G acts orientation-preservingly, and by [1, 8] there is the linear bound $|G| \leq 12(g-1)$ for orientation-preserving actions on 3-dimensional handlebodies. In [7], for each genus $g > 1$, the maximal possible order of a G-action on (S^3, Γ) is determined and the corresponding graphs Γ are classified (on the basis of analogous results in [6] for the case of closed surfaces embedded in S^3); the maximal possible order $12(g-1)$ is obtained only for finitely many values of g.

Concerning dimension $d = 4$, supppose that G acts orthogonally on a pair (S^4, Γ); now a regular neighborhood of Γ is a 4-dimensional handlebody V_g^4 whose boundary ∂V_g^3 is a connected sum $\sharp_g(S^2 \times S^1)$ of g copies of $S^2 \times S^1$. Finite group actions on such connected sums are considered in [12] whose results imply the quadratic upper bound $|G| \leq 24g(g-1)$, for $g \geq 15$; moreover, there does not exist a linear bound in g for the order of G.

In the following main result of the present paper, we consider the case of arbitrary dimensions $d \geq 4$.

Theorem 1. *Let G be a finite subgroup of the orthogonal group $\mathrm{O}(d+1)$ acting on a pair (S^d, Γ), for a finite hyperbolic graph of genus $g > 1$ embedded in S^d. Then, the order of G is bounded above by a polynomial of degree $d/2$ in g if d is even and of degree $(d+1)/2$ if d is odd. The degree $d/2$ is best possible in even dimensions, whereas in odd dimensions the optimal degree is either $(d-1)/2$ or $(d+1)/2$.*

So the optimal degree in odd dimensions remains open at present (except for $d = 3$, where it is $(d-1)/2 = 1$). The proof of Theorem 1 will be reduced to an analogous result in [2] about finite group actions on d-dimensional handlebodies (see Theorem 2 in Sec. 2).

The maximum order of a finite group of automorphisms of a finite hyperbolic graph of genus $g > 2$ is $2^g g!$ ([5]), obtained for a graph with one vertex and g closed edges (a "bouquet of g circles") whose automorphism group is isomorphic to the semidirect product $(\mathbb{Z}_2)^g \ltimes S_g$ (with normal subgroup $(\mathbb{Z}_2)^g$ on which the symmetric group S_g acts by permutation of coordinates). At present, we don't know the minimal dimension of a sphere which admits an equivariant embedding of the bouquet of g circles (i.e. invariant under an orthogonal action of $(\mathbb{Z}_2)^g \ltimes S_g$), see the question at the end of Sec. 2. For some other graphs with large symmetry groups instead, we determine this minimal dimension of an equivariant embedding in the examples of Sec. 2.

2. Proof of Theorem 1

See [2] for the following. A d-dimensional handlebody V_g^d of genus g can be defined as a regular neighborhood of a finite connected graph Γ of genus g embedded in S^d. Such a handlebody V_g^d can be uniformized by a Schottky group $\mathcal{S}_g$, a free group of rank g of Möbius transformations of S^{d-1} which extends naturally to the disk B^d ("Poincaré extension"); the interior of B^d is the Poincaré-model of hyperbolic space $\mathbb{H}^d$ on which Möbius transformations act as hyperbolic isometries. The handlebody V_g^d is obtained as the quotient $(B^d - \Lambda(\mathcal{S}_g))/\mathcal{S}_g$ where $\Lambda(\mathcal{S}_g) \subset S^{d-1}$ denotes the set of limit point of the action of $\mathcal{S}_g$ on B^d (a Cantor set), in particular $B^d - \Lambda(\mathcal{S}_g)$ is the universal covering of V_g^d. This gives the interior $\mathbb{H}^d/\mathcal{S}_g$ of V_g^d the structure of a complete hyperbolic manifold, and we say that the Schottky group $\mathcal{S}_g$ uniformizes the *hyperbolic handlebody* V_g^d. In particular, there is the notion of an isometry of such a hyperbolic handlebody meaning that it acts as an isometry on the interior of V_g^d; equivalently, each lift to the universal covering $B^d - \Lambda(\mathcal{S}_g)$ of V_g^d extends to a Möbius transformation of B^d.

The following is proved in [2].

Theorem 2. *Let G be a finite group of isometries of a hyperbolic handlebody V_g^d of dimension $d \geq 3$ and of genus $g > 1$ which acts faithfully on the fundamental group. Then, the order of G is bounded by a polynomial of degree $d/2$ in g if d is even and of degree $(d+1)/2$ if d is odd.*

Since a handlebody of dimension $d \geq 4$ admits S^1-actions, there is no upper bound for the order of finite group actions which are not faithful on the fundamental group (however, there is a Jordan-type bound for such actions, see [2, Corollary]). On the other hand, finite faithful actions on finite hyperbolic graphs are faithful on the fundamental group, that is injected into the outer automorphism group of the fundamental group ([11, Lemma 1]); conversely, it is observed in [9, p. 478] (as a version of the Nielsen realization problem for free groups) that every finite subgroup G of the outer automorphism group of a free group can be realized by an action of G on a finite graph.

Starting with the *Proof of Theorem* 1 now, let G be a finite group acting orthogonally on (S^d, Γ), for a finite hyperbolic graph Γ of genus $g > 1$ embedded in S^d. A G-invariant regular neighborhood of Γ in S^d is homeomorphic to a handlebody V_g^d of dimension d and genus g. Since we are assuming that the action of G on the graph Γ is faithful, by [11, Lemma 1] also the induced action of G on the fundamental group of Γ and, hence, of V_g^d is faithful and defines an injection of G into the outer automorphism groups of the fundamental groups of Γ and V_g^d. The first part of Theorem 1 is now a consequence of Theorem 2 and the following:

Proposition. *Let G be a finite group acting orthogonally on a pair (S^d, Γ), for a finite hyperbolic graph of genus $g > 1$ embedded in S^d, and hence, also on a handlebody V_g^d in S^d obtained as a G-invariant regular neighborhood of Γ. Then, V_g^d can be uniformized by a Schottky group such that V_g^d admits an isometric G-action (inducing the same injection into the outer automorphism group of the fundamental group as the original G-action).*

Very likely, the original G-action on V_g^d is in fact conjugate to an isometric G-action; however, in order to apply Theorem 2, we just need some isometric G-action on V_g^d which is faithful on the fundamental group, so we don't follow this here.

Proof of the Proposition. The group G acts as a group of automorphisms of the finite graph Γ. Let $\tilde{\Gamma}$ be the universal covering tree of Γ; the group of all lifts of all elements of G to $\tilde{\Gamma}$ defines a group E of automorphisms of the tree $\tilde{\Gamma}$ and a group extension $1 \to F_g \hookrightarrow E \to G \to 1$, where F_g denotes the universal covering group, a free group of rank g isomorphic to the fundamental group of Γ.

By possibly subdividing edges by a new vertex, we can assume that G acts on Γ without inversions of edges; then the quotient $\bar{\Gamma} = \Gamma/G$ is again a finite graph. Choose a maximal tree in $\bar{\Gamma}$ and lift it isomorphically first to Γ and then to $\tilde{\Gamma}$, then lift also the remaining edges of $\bar{\Gamma}$ to Γ and $\tilde{\Gamma}$. Associating to the vertices and edges of $\bar{\Gamma}$ the stabilizers in G or E of the lifted vertices and edges in Γ and $\tilde{\Gamma}$, this defines a finite graph of finite groups $(\bar{\Gamma}, \mathcal{G})$, with inclusions of the edge groups into the adjacent vertex groups. The fundamental group $\pi_1(\bar{\Gamma}, \mathcal{G})$ of the finite graph of finite groups $(\bar{\Gamma}, \mathcal{G})$ is the iterated free product with amalgamation and HNN-extension of the vertex groups amalgamated over the edge groups, first taking the iterated free product with amalgamation over the chosen maximal tree of Γ and then associating an HNN-generator to each of the remaining edges. By the standard theory of groups acting on trees, graphs of groups and their fundamental groups (see [3, 4] or [10]), the extension E is isomorphic to the fundamental group $\pi_1(\bar{\Gamma}, \mathcal{G})$ of $(\bar{\Gamma}, \mathcal{G})$ and we have a group extension

$$1 \to F_g \hookrightarrow E = \pi_1(\bar{\Gamma}, \mathcal{G}) \to G \to 1.$$

We will assume in the following that the graph of groups $(\bar{\Gamma}, \mathcal{G})$ has no *trivial edges*, i.e. no edges with two different vertices such that the edge group coincides with one of the two vertex groups (by collapsing trivial edges, i.e. by amalgamating the two vertex groups into a single vertex group). We will realize the extension $E = \pi_1(\bar{\Gamma}, \mathcal{G})$ as a group of isometries of hyperbolic space $\mathbb{H}^d$, the interior of the d-ball B^d, or equivalently as a group of Möbius transformations of B^d or $S^{d-1} = \partial B^d$, such that the subgroup F_g is realized by a Schottky group $\mathcal{S}_g$. Then, G acts as a group of isometries of the hyperbolic handlebody $V_g^d = (B^d - \Lambda(\mathcal{S}_g))/\mathcal{S}_g$ proving the Proposition. The realization of $E = \pi_1(\bar{\Gamma}, \mathcal{G})$ is by standard combination methods as described in [2, p. 247] and [9, p. 479–482]. As an illustration, we discuss the case of a graph of groups $(\bar{\Gamma}, \mathcal{G})$ with a single, nonclosed edge. We lift the edge to an edge B of Γ, with vertices A_1 and A_2. The corresponding edge and vertex groups of $(\bar{\Gamma}, \mathcal{G})$ (which we denote by the same letters) are defined as the stabilizers of the edge B and its vertices in the group G acting orthogonally on $\Gamma \subset S^d$, and $E = \pi_1(\bar{\Gamma}, \mathcal{G})$ is the free product with amalgamation $A_1 *_B A_2$.

For $i = 1$ and 2, the vertex group $A_i \subset G$ acts orthogonally on S^d and fixes the corresponding vertex A_i of the edge B of Γ. A regular invariant neighborhood of the fixed point A_i is a ball B_i^d with an action of A_i. The edge group B is a subgroup of both A_1 and A_2 which fixes both vertices A_1 and A_2 and pointwise the edge B. We can assume that the intersection $\partial B_1^d \cap \partial B_2^d$ of the two boundary spheres S^{d-1} is nonempty and hence a sphere S^{d-2}. We conjugate A_2 by the reflection in this sphere S^{d-2} and obtain a group A_2' fixing the vertex A_1; since the reflection commutes with each element of B, both A_1 and A_2' act on B_1^d now with a common subgroup B.

Identifying B_1^d with the standard ball B^d, with the center A_1 corresponding to the origin 0, we have orthogonal actions of A_1 and A_2' on B^d with a common subgroup B. The orthogonal action of B on B^d fixes two diametral points in ∂B^d which are not fixed by any other element of A_1 and A_2' (corresponding to the intersection of the edge B with ∂B_1^d), and hence, B fixes pointwise the hyperbolic line L in hyperbolic space $\mathbb{H}^d$ (the interior of B^d) connecting these two diametral points. We conjugate A_2' by a reflection in a hyperbolic hyperplane orthogonal of the line L far from the origin $0 \in B^d$. The group generated by A_1 and the reflected group A_2'' is then isomorphic to the free product with amalgamation $A_1 *_B A_2''$ and realizes $E = A_1 *_B A_2$ as a group of hyperbolic isometries of $\mathbb{H}^d$, or equivalently of Möbius transformations of $S^{d-1} = \partial B^d$ (by standard combination methods, see [2, p. 247] and [9, p. 480] for more details and some figures).

In a similar way, inductively edge by edge in finitely many steps, one realizes the group $E = \pi_1(\bar{\Gamma}, \mathcal{G})$ also in the general case (see again [9, p. 479–482]).

This concludes the proof of the Proposition and, by Theorem 2, also of the first statement of Theorem 1. □

For the second statement of Theorem 1, we construct an infinite series of orthogonal actions of finite groups G on finite graphs Γ embedded in S^d which realize the lower bounds for the polynomial degrees in Theorem 1.

Example 1. For $k > 1$, let $G = C_1 \times \ldots \times C_k \cong (\mathbb{Z}_m)^k$, of order $n = m^k$, be the product of k cyclic groups $C_i \cong \mathbb{Z}_m$ of order m. Choose an orthogonal action of G on $\mathbb{R}^{2k}$ as follows. Decomposing $\mathbb{R}^{2k} = P_1 \times \ldots \times P_k$ as the product of k orthogonal planes P_i, each C_i acts on P_i faithfully by rotations and trivially on the $k-1$ orthogonal planes. This G-action on $\mathbb{R}^{2k}$ extends to an orthogonal G-action on the one-point compactification S^{2k} of $\mathbb{R}^{2k}$, with two global fixed points 0 and ∞.

We consider a graph Γ in S^{2k} with two vertices 0 and ∞ and km connecting edges divided into k groups of m edges. We embed the first group of m edges into P_1 such that C_1 permutes these edges cyclically, then the next m edges into P_2, etc., defining an orthogonal action of G on the graph Γ embedded in S^{2k}. The graph Γ has genus $g = km - 1$, hence

$$|G| = m^k = (g+1)^k / k^k,$$

which is a polynomial of degree $k = d/2$ in g.

Suppose that $m > 2$; then, $d = 2k$ is the minimal dimension of a sphere which admits a G-equivariant embedding of the graph Γ. In fact, in such an embedding into a sphere S^d, the group $G \cong (\mathbb{Z}_m)^k$ has a global fixed point and, hence, acts orthogonally on the boundary S^{d-1} of an invariant regular neighborhood of a global fixed point, and $d-1=2k-1$ is the minimal dimension of a sphere with a faithful orthogonal action of $(\mathbb{Z}_m)^k$; equivalently, the minimal dimension of a faithful, real, linear representation of $(\mathbb{Z}_m)^k$ is $2k$.

In odd dimensions $d=2k+1$, we extend the orthogonal action of G on $\mathbb{R}^{2k}$ to an orthogonal action on $\mathbb{R}^{2k+1}$, trivial on the last coordinate, and then proceed as before; we get a polynomial of degree $k=(d-1)/2$ in g for the order of G. As noted in the Introduction, the optimal degree in dimension $d=3$ is $(d-1)/2=1$; for odd dimensions $d>3$, the optimal degree is either $(d-1)/2$ or $(d+1)/2$ but at present we do not know which of these two values occurs.

This completes the proof of Theorem 1.

We present some other infinite series of finite orthogonal group actions on finite graphs embedded in spheres which realize the minimal dimension of such an embedding.

Example 2. (i) Let Γ be the complete graph with $d+2$ vertices, or the 1-skeleton of a $(d+1)$-simplex. We embed the regular $(d+1)$-simplex into B^{d+1}, with vertices in $S^d=\partial B^{d+1}$, and project its edges radially to S^d; this defines an embedding of Γ into S^d. The automorphism group of the graph Γ is the symmetric group S_{d+2} which extends to an orthogonal action on the sphere S^d. Again d is the minimal dimension of such an equivariant embedding since it is the minimal dimension of a sphere with a faithful, orthogonal action of S_{d+2} (equivalently, the minimal dimension of a faithful, real, linear representation of the symmetric group S_{d+2} is $d+1$).

(ii) Let Γ' be a graph with two vertices and $d+2$ edges. Then, Γ' has an embedding into S^{d+1} which is "dual" to the embedding of Γ into $S^d\subset S^{d+1}$ in part (i) in the following sense: the sphere S^d separates S^{d+1} into two balls B^{d+1}; the two vertices of Γ' are the centers of these two balls, and each of the $d+2$ connecting edges intersects exactly one of the $d+2$ faces of the projected $(d+1)$-simplex in S^d in its center.

The symmetric group S_{d+2} acts orthogonally on (S^{d+1},Γ') with two global fixed points (the double suspension of the action of S_{d+2} on S^d), in particular it acts on S^d and it follows as in (i) that $d+1$ is the minimal dimension of a sphere with an equivariant embedding of Γ'.

(iii) Let Γ be the 1-skeleton of the $(d+1)$-dimensional hypercube now, with a projection to $S^d=\partial B^{d+1}$ as in (i). The graph Γ has an automorphism group $(\mathbb{Z}_2)^{d+1}$ which extends to an orthogonal action on S^d (inversion of coordinates), and d is the minimal dimension of a sphere with an orthogonal action of $(\mathbb{Z}_2)^{d+1}$ (the minimal dimension of a faithful, real, linear representation of $(\mathbb{Z}_2)^{d+1}$ is $d+1$).

Dualizing as in (ii) we get a graph Γ' in S^{d+1} with two vertices connected by $2(d+1)$ edges (each intersecting exactly one of the faces of the hypercube), with an action of $(\mathbb{Z}_2)^{d+1}$ with two global fixed points, and $d+1$ is the minimal dimension of such an embedding (since d is the minimal dimension of an orthgonal action of $(\mathbb{Z}_2)^{d+1}$ on a sphere). Note that this realizes the minimal dimension in the case

$m = 2$ of the group $(\mathbb{Z}_2)^k$ in Example 1, replacing each plane P_i with a rotation by a line with a reflection.

Question. As noted in the introduction, the maximum order of a finite group of automorphisms of a finite hyperbolic graph of genus $g > 2$ is $2^g g!$ ([5]), obtained for a graph Γ with a single vertex and g closed edges whose automorphism group is the semidirect product $(\mathbb{Z}_2)^g \ltimes S_g$ (the symmetric group S_g acts by permutation of the edges, the normal subgroup $(\mathbb{Z}_2)^g$ by inversion of the edges). What is the minimal dimension of a sphere with an equivariant embedding of this graph?

Note that the graph Γ embeds equivariantly into $\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 = \mathbb{R}^{2g}$, by embedding each of the g closed edges into a different plane $\mathbb{R}^2$ (the unique vertex corresponds to the origin of $\mathbb{R}^{2g}$ and also of each plane $\mathbb{R}^2$); the group $(\mathbb{Z}_2)^g$ acts by reflections in lines through the origin on the g planes, inverting the g embedded closed edges, the symmetric group S_g acts by permutation of the planes. Then, Γ admits an equivariant embedding also into the one-point compactification S^{2g} of $\mathbb{R}^{2g}$. On the other hand, the minimal dimension of a sphere with an orthogonal action of $(\mathbb{Z}_2)^g \ltimes S_g$ is $g-1$ (since g is the minimal dimension of a faithful, real, linear representation of $(\mathbb{Z}_2)^g$). It seems reasonable, however, that the minimal dimension of an equivariant embedding of Γ into a sphere is $2g$ (as described above), but this minimal dimension, between $g - 1$ and $2g$, remains open at present.

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