An infinite-dimensional version of the Poincaré–Birkhoff theorem on the Hilbert cube

To the memory of Maria Gramegna (1887–1915)

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Abstract. We propose a version of the Poincaré–Birkhoff theorem for infinite-dimensional Hamiltonian systems, which extends a recent result by Fonda and Ureña [18]. The twist condition, adapted to a Hilbert cube, is spread on a sequence of approximating finite-dimensional systems. Some applications are proposed to pendulum-like systems of infinitely many ODEs. We also extend to the infinite-dimensional setting a celebrated theorem by Conley and Zehnder [8].

1 Introduction

The Poincaré–Birkhoff theorem was conjectured by Henri Poincaré in 1912, shortly before his death [31]. It was first stated for an area-preserving homeomorphism on an invariant planar annulus, assuming a *twist condition* at the boundary. A modern formulation of this original version, expressed in the covering space, reads as follows (see [6]).

Theorem 1. *Let $\varphi : \mathbb{R} \times [a, b] \to \mathbb{R} \times [a, b]$ be an area-preserving homeomorphism of the form*

$$\varphi(x, y) = (x + \vartheta(x, y), \rho(x, y)),$$

where the continuous functions $\vartheta(x, y)$ and $\rho(x, y)$ are 2π-periodic in their first variable x, with $\rho(x, a) = a$ and $\rho(x, b) = b$, for every $x \in \mathbb{R}$. Assume the boundary twist condition

$$\vartheta(x, a)\vartheta(x, b) < 0, \qquad \textit{for every } x \in \mathbb{R}.$$

Then φ has at least two fixed points in $[0, 2\pi[\times]a, b[$.

George Birkhoff in 1913 first gave a partial proof of the theorem [3], then started extending it to some mappings for which the invariance of the annulus is not required [4], and finally also proposed a version of the theorem in a higher dimensional setting [5].

For more than a century, a lot of effort has been devoted to generalize the theorem in both these directions. Indeed, on the one hand, the invariance of the domain turns out to be a serious obstacle in the applications to dynamical systems; along this line of research, several remarkable results have been obtained, making nowadays the theorem a very powerful tool when looking for periodic solutions of planar Hamiltonian systems. We refer to [10, 16, 23] for a review on the development of the planar theory, with special emphasis on the applications to ODEs; let us state here a version of the Poincaré–Birkhoff theorem, taken from [18], which can be employed in the planar Hamiltonian setting.

Theorem 2. *Consider the finite-dimensional Hamiltonian system*

$$u' = \frac{\partial H}{\partial v}(t,u,v)\,, \qquad v' = -\frac{\partial H}{\partial u}(t,u,v)\,, \tag{1}$$

where $H : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is T-periodic in t and 2π-periodic in u, continuous in (t,u,v) and continuously differentiable in (u,v). Let $\sigma \in \{-1,1\}$ and assume that every solution $w(t) = (u(t),v(t))$ of (1) with $v(0) \in [a,b]$ is defined for every $t \in [0,T]$ and satisfies

$$\begin{cases} v(0) = a & \Rightarrow \quad \sigma[u(T) - u(0)] < 0, \\ v(0) = b & \Rightarrow \quad \sigma[u(T) - u(0)] > 0. \end{cases}$$

Then, there exist at least two T-periodic solutions $w(t) = (u(t),v(t))$ of (1), such that

$$w(0) = (u(0),v(0)) \in [0,2\pi[\times\,]a,b[\,.$$

On the other hand, far fewer progresses have been made for the higher dimensional issue, which was considered by Birkhoff himself as an *outstanding question* [4, page 299]. Its study has led to some famous conjectures by Arnold [1] and eventually to the development of *symplectic geometry* [29]. By the use of monotonicity assumptions on the twist, some higher dimensional versions of the Poincaré–Birkhoff theorem have been given (see, e.g., [30]), but, so far, a genuine generalization has never been found.

Recently, however, Fonda and Ureña [18] provided an extension of Theorem 2 to Hamiltonian systems in any (even) finite dimension: considering a system like (1) with $(u,v) = (u_1,\dots,u_N,v_1,\dots,v_N) \in \mathbb{R}^{2N}$, a suitable twist condition on the boundary of the set $\prod_{k=1}^{N}[a_k, b_k]$, requiring a change of sign for $u_k(T) - u_k(0)$ when passing from $v_k(0) = a_k$ to $v_k(0) = b_k$, provides the existence of $N+1$ distinct T-periodic solutions (see [18, Theorem 6.2] for the precise statement). This twist condition clearly extends, in the higher dimensional case, the one proposed in Theorem 2.

Taking advantage of this result, in this paper we will provide for the first time an *infinite-dimensional version of the Poincaré–Birkhoff theorem*. This seems to be an ambitious task, since most of the compactness arguments used to prove the theorem in the finite-dimensional case could fail, of course. We will manage to overcome this difficulty by working on a *Hilbert cube* of the type $\prod_{k=1}^{\infty}[a_k, b_k]$ in the Hilbert space ℓ^2; the compactness of this set, together with a suitable formulation of the twist condition, will allow us to remedy the lack of compactness in the infinite-dimensional setting, finally getting the existence of *at least one* T-periodic solution of our (infinite-dimensional) Hamiltonian system, i.e., one fixed point of the associated Poincaré map.

The plan of the paper is as follows. In Section 2 we describe our infinite-dimensional setting and we prove our first main result (Theorem 3). The proof is based on a Galerkin-type approximation scheme: first, the main theorem in [18] is applied to a sequence of approximating finite-dimensional Hamiltonian systems, providing a corresponding sequence of periodic solutions; a compactness argument is then used to extract a subsequence converging to a periodic solution of the original infinite-dimensional system.

In Section 3 we provide some applications of our main result to systems of infinitely many second order ODEs, extending to the infinite-dimensional setting some well-known statements for pendulum-like scalar equations and systems.

In Section 4 we adopt a more abstract perspective, providing a further extension of Theorem 3; here, the twist condition takes a more general form, which can be seen as an infinite-dimensional generalization of the one introduced by Conley and Zehnder in [8, Theorem 3]. We also propose an infinite-dimensional extension of [8, Theorem 1], a result of the same authors providing an answer to a famous conjecture by Arnold. A final Appendix is devoted to the Hilbert cube and its main topological features.

This paper is dedicated to the memory of Maria Paola Gramegna who, at the beginning of the twentieth century, under the supervision of Giuseppe Peano, was one of the first pioneering mathematicians to prove the existence of solutions to infinite-dimensional differential systems [21]. She tragically died when she was 28 years old, victim of an earthquake.

2 The main result

In this section we state and prove our first main result, dealing with an infinite-dimensional Hamiltonian system on a separable real Hilbert space $\mathcal{H}$. Precisely, we consider the system

$$x' = \nabla_y H(t,x,y)\,, \qquad y' = -\nabla_x H(t,x,y)\,, \tag{2}$$

where $H : \mathbb{R} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is assumed to be T-periodic in the first variable, continuous in (t,x,y) and continuously differentiable in (x,y). More precisely, we assume that H is differentiable in every $z = (x,y) \in \mathcal{H} \times \mathcal{H}$, and $\nabla_z H = (\nabla_x H, \nabla_y H) : \mathbb{R} \times \mathcal{H} \times \mathcal{H} \to \mathcal{H} \times \mathcal{H}$ is continuous. Throughout the paper, solutions to (2) are meant in the classical sense, namely as continuously differentiable functions $z = (x,y) : I \to \mathcal{H} \times \mathcal{H}$, being $I \subset \mathbb{R}$ an interval, satisfying the differential equation pointwise; in particular, we will be interested in the existence of T-periodic solutions. Hamiltonian systems like (2) have been considered, e.g., in [2, 12]; we also mention the book [11] as a reference about the general theory of ODEs in infinite-dimensional spaces.

Let us introduce our structural framework. In the following, it will be convenient to identify the space $\mathcal{H}$ with ℓ^2, the space of real sequences $\xi = (\xi_k)_{k\geq 1}$ such that $\sum_{k=1}^{\infty} \xi_k^2 < \infty$, endowed with the usual scalar product

$$\langle \xi, \tilde{\xi} \rangle_{\ell^2} = \sum_{k=1}^{\infty} \xi_k \tilde{\xi}_k\,,$$

and the associated norm $\|\xi\|_{\ell^2} = \sqrt{\langle \xi, \xi \rangle_{\ell^2}}$. In this way, (2) can be thought as a system of infinitely many scalar ODEs,

$$\begin{cases} x_k' = \dfrac{\partial H}{\partial y_k}(t,(x_1,x_2,\dots),(y_1,y_2,\dots)), \\[2mm] y_k' = -\dfrac{\partial H}{\partial x_k}(t,(x_1,x_2,\dots),(y_1,y_2,\dots)), \end{cases} \qquad k = 1,2,\dots,$$

where $x = (x_1, x_2, \dots,)$ and $y = (y_1, y_2, \dots)$ belong to ℓ^2. We will also make use of the following standard notation: given $(\xi_1, \xi_2, \dots)$ and $(\tilde{\xi}_1, \tilde{\xi}_2, \dots)$ in ℓ^2,

$$\prod_{k=1}^{\infty} [\xi_k, \tilde{\xi}_k] := \{\chi = (\chi_1, \chi_2, \dots) \in \ell^2 \mid \xi_k \leq \chi_k \leq \tilde{\xi}_k\}.$$

First, we assume that $\nabla_z H(t,z)$ has at most linear growth in the variable z, namely:

($\mathcal{A}_1$) *there exists* $C > 0$ *such that*

$$\|\nabla_z H(t,z)\| \le C(1 + \|z\|), \quad \textit{for every } t \in [0,T], z \in \ell^2 \times \ell^2,$$

where the symbol $\|\cdot\|$ *denotes the usual norm in the product space.*

Second, we consider three sequences $(\tau_k)_k, (a_k)_k, (b_k)_k$ in ℓ^2, with

$$\tau_k > 0, \quad a_k \le 0 \le b_k \quad \text{and} \quad b_k - a_k > 0,$$

for every $k \ge 1$, and we define the two bounded subsets of ℓ^2

$$\mathbb{T}_\infty = \prod_{k=1}^{\infty} [0, \tau_k], \qquad \mathcal{D}_\infty = \prod_{k=1}^{\infty} [a_k, b_k].$$

With this notation, we assume the Lipschitz continuity condition

($\mathcal{A}_2$) *setting*

$$R = (\operatorname{diam}(\mathbb{T}_\infty \times \mathcal{D}_\infty) + 1)\, \mathrm{e}^{CT},$$

there exists a constant $L > 0$ *such that*

$$\|\nabla_z H(t, z_1) - \nabla_z H(t, z_2)\| \le L\|z_1 - z_2\|, \quad \textit{for every } t \in [0,T],\ z_1, z_2 \in \mathcal{B}_R,$$

where $\mathcal{B}_R \subset \ell^2 \times \ell^2$ *denotes the closed ball centered at* 0 *with radius* R *and* $C > 0$ *is the constant introduced in assumption* ($\mathcal{A}_1$).

Finally, to state the main result of this section we need to introduce the following Galerkin-type approximation scheme. Writing $\xi = (\xi_1, \xi_2, \dots) \in \ell^2$, for every integer $N \ge 1$ we define the projection $P_N : \ell^2 \to \mathbb{R}^N$ as

$$P_N(\xi_1, \xi_2, \dots) = (\xi_1, \xi_2, \dots, \xi_N),$$

and the immersion $I_N : \mathbb{R}^N \to \ell^2$ as

$$I_N(\eta_1, \eta_2, \dots, \eta_N) = (\eta_1, \eta_2, \dots, \eta_N, 0, 0, \dots).$$

Accordingly, we introduce the finite-dimensional approximating Hamiltonian function $H_N : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ by setting

$$H_N(t, u, v) = H(t, I_N u, I_N v)\,, \tag{3}$$

and we write the corresponding Hamiltonian system

$$u' = \nabla_v H_N(t, u, v)\,, \qquad v' = -\nabla_u H_N(t, u, v)\,, \tag{4}$$

where $u = (u_1, \dots, u_N)$, $v = (v_1, \dots, v_N) \in \mathbb{R}^N$. Notice that $\langle I_N u, I_N v\rangle_{\ell^2} = \sum_{i=1}^N u_i v_i$, so that the scalar product induced by ℓ^2 on $\mathbb{R}^N$ coincides with the usual Euclidean one, and the gradients in (4) are defined accordingly. In particular,

$$\begin{aligned} \nabla_u H_N(t,u,v) &= P_N \nabla_x H(t, I_N u, I_N v), \\ \nabla_v H_N(t,u,v) &= P_N \nabla_y H(t, I_N u, I_N v)\,. \end{aligned} \tag{5}$$

As a final notation, we set

$$\mathbb{T}_N = P_N \mathbb{T}_\infty = \prod_{k=1}^N [0, \tau_k], \qquad \mathcal{D}_N = P_N \mathcal{D}_\infty = \prod_{k=1}^N [a_k, b_k].$$

We are now in a position to state and prove our first main result, extending Theorem 2 to the infinite-dimensional setting.

Theorem 3. *Let $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ hold and assume further that:*

- *the Hamiltonian function H is τ_k-periodic in the variable x_k, for every $k \geq 1$;*
- *there exists a sequence $(\sigma_k)_k$ in $\{-1, 1\}$ such that, for every sufficiently large integer N, if $w(t) = (u(t), v(t))$ is a solution of (4) with $v(0) \in \partial\mathcal{D}_N$, then, for every $k = 1, \dots, N$,*

$$\begin{cases} v_k(0) = a_k \quad \Rightarrow \quad \sigma_k[u_k(T) - u_k(0)] < 0, \\ v_k(0) = b_k \quad \Rightarrow \quad \sigma_k[u_k(T) - u_k(0)] > 0. \end{cases} \tag{6}$$

Then, there exists a T-periodic solution $z(t) = (x(t), y(t))$ of (2), such that

$$z(0) = (x(0), y(0)) \in \mathbb{T}_\infty \times \mathcal{D}_\infty.$$

Remark 4. Some comments about Theorem 3 are in order. As in the classical version of the Poincaré–Birkhoff Theorem, the assumption of periodicity in the x_k-variables for the Hamiltonian H implies that the natural phase space for system (2) looks like the product of the infinite-dimensional "torus" $\mathbb{T}_\infty$ with the infinite-dimensional "cube" $\mathcal{D}_\infty$. The key point in our infinite-dimensional setting is that both these sets are compact. Indeed, since $(\tau_k)_k, (a_k)_k, (b_k)_k$ belong to ℓ^2, both $\mathbb{T}_\infty$ and $\mathcal{D}_\infty$ are homeomorphic to the *Hilbert cube* $[0,1]^{\mathbb{N}}$, whose compactness follows from Tychonoff's Theorem (see the final Appendix for further details).
Referring to the twist condition, it is worth noticing that the set $\mathcal{D}_\infty$ has empty interior, since it is a compact subset of an infinite-dimensional space.

Hence, each of its points is a boundary point and thus a twist-type assumption on $\partial\mathcal{D}_\infty$ would hardly be satisfied. In our statement, the twist condition (6) is indeed required on a sequence of finite-dimensional approximating systems, and this seems to be a convenient choice also for the applications.

Proof. Throughout the proof, it will be convenient to make use of the projection operator on the product spaces, namely $\mathcal{P}_N : \ell^2 \times \ell^2 \to \mathbb{R}^N \times \mathbb{R}^N$, defined as

$$\mathcal{P}_N(x, y) = (P_N x, P_N y) = (x_1, \ldots, x_N, y_1, \ldots, y_N).$$

We also define the operator $\mathcal{I}_N : \mathbb{R}^N \times \mathbb{R}^N \to \ell^2 \times \ell^2$ as

$$\mathcal{I}_N(u, v) = (I_N u, I_N v) = ((u_1, \ldots, u_N, 0, \ldots), (v_1, \ldots, v_N, 0, \ldots)),$$

and we set

$$\mathfrak{P}_N = \mathcal{I}_N \circ \mathcal{P}_N : \ell^2 \times \ell^2 \to \ell^2 \times \ell^2,$$

in such a way that

$$\mathfrak{P}_N(x, y) = ((x_1, \ldots, x_N, 0, \ldots), (y_1, \ldots, y_N, 0, \ldots)).$$

We first prove some preliminary estimates on the solutions of the Cauchy problems associated with (2), whose integral formulation reads as

$$z(t) = z(0) + \int_0^t \begin{pmatrix} \nabla_y H(s, z(s)) \\ -\nabla_x H(s, z(s)) \end{pmatrix} ds. \tag{7}$$

Using the linear growth assumption $(\mathcal{A}_1)$ and Gronwall's lemma, it is easily checked that, if a solution $z(t)$ is defined on $[0, T_0]$ for some $T_0 \in]0, T]$, then it satisfies the estimate

$$\|z(t)\| \le (1 + \|z(0)\|)\, \mathrm{e}^{CT_0}, \qquad \text{for every } t \in [0, T_0].$$

In particular, if $z(0) \in \mathbb{T}_\infty \times \mathcal{D}_\infty$, it follows that $z(t) \in \mathcal{B}_R$ for every $t \in [0, T_0]$, with R as in $(\mathcal{A}_2)$. By the Lipschitz continuity on $\mathcal{B}_R$, we thus have that $z(t)$ is actually defined on the whole interval $[0, T]$, is therein unique and belongs to $\mathcal{B}_R$ for every $t \in [0, T]$. The same argument shows that, for any solution $w(t) = (u(t), v(t))$ of (4) satisfying $w(0) \in \mathbb{T}_N \times \mathcal{D}_N$, it holds that $\mathcal{I}_N w(t) \in \mathcal{B}_R$ for every $t \in [0, T]$ (notice that $\nabla_w H_N$ straightly satisfies the finite-dimensional counterparts of assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, with the same constants).

As a consequence of the above proved global existence, together with the twist condition (6), we can apply [18, Theorem 6.2] to obtain, for every large enough integer N, a T-periodic solution $w^N(t) = (u^N(t), v^N(t))$ of (4) with $w^N(0) \in \mathbb{T}_N \times \mathcal{D}_N$. Moreover, in view of the above estimates, $\mathcal{I}_N w^N(t) \in \mathcal{B}_R$ for every $t \in [0,T]$.

Let now $z_0^N = \mathcal{I}_N w^N(0)$; we thus have a sequence $(z_0^N)_N$ in $\mathbb{T}_\infty \times \mathcal{D}_\infty$. By the discussion in Remark 4, the set $\mathbb{T}_\infty \times \mathcal{D}_\infty$ is compact in $\ell^2 \times \ell^2$, so that there exists a subsequence, still denoted by $(z_0^N)_N$, which converges to some $z_0 \in \mathbb{T}_\infty \times \mathcal{D}_\infty$. In view of the arguments at the beginning of the proof, the solution $z(t)$ of (2) starting from $z(0) = z_0$ is uniquely defined on $[0,T]$; we are going to show that $z(t)$ is T-periodic, thus completing the proof of the theorem.

Indeed, we will prove that

$$\mathcal{I}_N w^N(t) \to z(t)\,, \quad \text{uniformly for every } t \in [0,T]\,,$$

this being enough since the uniform limit of T-periodic functions is a T-periodic function. To this end, we fix $\varepsilon > 0$, and we define $\varepsilon' = \varepsilon/2\mathrm{e}^{LT}$, being $L > 0$ as in assumption $(\mathcal{A}_2)$. Writing

$$\|z(t) - \mathcal{I}_N w^N(t)\| \leq \|z(t) - \mathfrak{P}_N z(t)\| + \|\mathfrak{P}_N z(t) - \mathcal{I}_N w^N(t)\|,$$

we are led to estimate each summand separately. As for the first one, since $\mathfrak{P}_N \to \mathrm{Id}$ in the space $\mathcal{L}(\ell^2)$ of bounded linear operators on ℓ^2 and $z(t) \in \mathcal{B}_R$ for every $t \in [0,T]$, for N large enough it holds that

$$\|z(t) - \mathfrak{P}_N z(t)\| \leq \varepsilon', \quad \text{for any } t \in [0,T].$$

As for the second summand, we first pass to the integral formulations of (2) and (4), namely (7) and

$$w^N(t) = w^N(0) + \int_0^t \begin{pmatrix} \nabla_v H_N(s, w^N(s)) \\ -\nabla_u H_N(s, w^N(s)) \end{pmatrix} ds,$$

and we use standard properties of the Riemann integral so as to obtain

$$\|\mathfrak{P}_N z(t) - \mathcal{I}_N w^N(t)\| \leq \|\mathfrak{P}_N z(0) - \mathcal{I}_N w^N(0)\| +$$

$$+ \int_0^t \left\| \mathfrak{P}_N \begin{pmatrix} \nabla_y H(s, z(s)) \\ -\nabla_x H(s, z(s)) \end{pmatrix} - \mathcal{I}_N \begin{pmatrix} \nabla_v H_N(s, w^N(s)) \\ -\nabla_u H_N(s, w^N(s)) \end{pmatrix} \right\| ds.$$

Now, since by definition $\mathcal{I}_N w^N(0) = z_0^N = \mathfrak{P}_N z_0^N$ and $\|\mathfrak{P}_N\|_{\mathcal{L}(\ell^2)} \leq 1$, for N sufficiently large it holds that

$$\|\mathfrak{P}_N z(0) - \mathcal{I}_N w^N(0)\| \leq \|z(0) - z_0^N\| \leq \varepsilon'.$$

On the other hand, using (5) we rewrite the integral term as

$$\int_0^t \left\| \mathfrak{P}_N \begin{pmatrix} \nabla_y H(s, z(s)) \\ -\nabla_x H(s, z(s)) \end{pmatrix} - \mathfrak{P}_N \begin{pmatrix} \nabla_y H(s, \mathcal{I}_N w^N(s)) \\ -\nabla_x H(s, \mathcal{I}_N w^N(s)) \end{pmatrix} \right\| ds$$

which in turn can be estimated by

$$L \int_0^t \|z(s) - \mathcal{I}_N w^N(s)\| \, ds,$$

using again the fact that $\|\mathfrak{P}_N\|_{\mathcal{L}(\ell^2)} \leq 1$, together with the Lipschitz condition $(\mathcal{A}_2)$, and recalling that $z(t)$ and $\mathcal{I}_N w^N(t)$ belong to $\mathcal{B}_R$, for every $t \in [0, T]$. Summing up, for every $t \in [0, T]$ and every large enough N, it holds that

$$\|z(t) - \mathcal{I}_N w^N(t)\| \leq \varepsilon \mathrm{e}^{-LT} + L \int_0^t \|z(s) - \mathcal{I}_N w^N(s)\| \, ds.$$

By Gronwall's Lemma we get

$$\|z(t) - \mathcal{I}_N w^N(t)\| \leq \varepsilon, \qquad \text{for every } t \in [0, T],$$

whence the conclusion. □

Remark 5. Let us notice that [18, Theorem 6.2], used in the proof of our main result, actually gives the existence of $N+1$ distinct T-periodic solutions to (4). Therefore, under the assumptions of Theorem 3, it would be natural to conjecture the existence of *infinitely many* T-periodic solutions of (2). This however seems to be out of reach within our Galerkin-type approximation argument, since multiplicity may be lost when passing to the limit.

3 Some examples of applications

In this section, we give a possible application of Theorem 3 to an infinite-dimensional second order system of ODEs. Precisely, we consider a system of the type

$$x_k'' + \frac{\partial \mathcal{V}}{\partial x_k}(t, x_1, \dots, x_k, \dots) = e_k(t), \qquad k = 1, 2, \dots, \tag{8}$$

where $\mathcal{V}(t, x_1, \dots, x_k, \dots)$ is T-periodic in the variable t and τ_k-periodic in each variable x_k, while $e_k(t)$ is a T-periodic forcing term with zero mean, i.e.,

$$\int_0^T e_k(t) \, dt = 0, \qquad k = 1, 2, \dots \tag{9}$$

Such a setting is motivated by the classical result for pendulum-like scalar equations [17, 20, 27], together with its several generalizations to finite-dimensional systems [7, 13, 15, 18, 19, 22, 24, 25, 28, 32, 34]. Our next result will then represent a possible infinite-dimensional extension.

To enter the functional setting of Section 2, some care is required. Precisely, we suppose that $\mathcal{V} : \mathbb{R} \times \ell^2 \to \mathbb{R}$ is continuous in all its variables and continuously differentiable with respect to $x = (x_1, x_2, \dots) \in \ell^2$; moreover, we require that the map

$$e : \mathbb{R} \to \ell^2, \qquad t \mapsto e(t) = (e_1(t), e_2(t), \dots),$$

is well-defined and continuous. Due to these assumptions, (8) can be rewritten in a compact way as

$$x'' + \nabla_x \mathcal{V}(t, x) = e(t). \tag{10}$$

Solutions to (10) will then be meant as C^2-functions $x : \mathbb{R} \to \ell^2$ satisfying the equation pointwise.

We are now ready to state the main result of this section.

Theorem 6. *In the above setting, suppose further that* $(\tau_k)_k$ *belongs to* ℓ^2. *Moreover, assume that:*

$(\mathcal{V}_1)$ *there exists* $(M_k)_k$ *in* ℓ^2 *such that, for every* $k \geq 1$,

$$\left| \frac{\partial \mathcal{V}}{\partial x_k}(t, x) \right| \leq M_k\,, \qquad \text{for every } (t, x) \in [0, T] \times \ell^2;$$

$(\mathcal{V}_2)$ *for every* $\rho > 0$, *there exists* $L_\rho > 0$ *such that*

$$\|\nabla_x \mathcal{V}(t, x) - \nabla_x \mathcal{V}(t, \tilde{x})\|_{\ell^2} \leq L_\rho \|x - \tilde{x}\|_{\ell^2}\,, \text{ for every } t \in [0, T],\ x, \tilde{x} \in B_\rho,$$

where B_ρ *denotes the closed ball in* ℓ^2, *centered at* 0 *with radius* ρ.

Then, system (8) *has a* T*-periodic solution.*

Proof. As a first step, we rewrite system (10) as

$$x_k' = y_k + \int_0^t e_k(s)\, ds\,, \quad y_k' = -\frac{\partial \mathcal{V}}{\partial x_k}(t, x_1, \dots, x_k, \dots)\,, \quad k = 1, 2, \dots;$$

it is easily checked that such a system possesses a Hamiltonian structure, with Hamiltonian function $H : \mathbb{R} \times \ell^2 \times \ell^2 \to \mathbb{R}$ given by

$$H(t, x, y) = \sum_{k=1}^{\infty} \left(\frac{y_k^2}{2} + y_k \int_0^t e_k(s)\, ds \right) + \mathcal{V}(t, x_1, \dots, x_k, \dots).$$

Notice that H is well-defined, is τ_k-periodic in each variable x_k and, thanks to the zero mean value condition (9), is T-periodic in the variable t. Moreover, both the assumptions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$ of the previous section are satisfied. Indeed, since

$$\nabla_z H(t,z) = \left(\nabla_x \mathcal{V}(t,x), y + \int_0^t e(s)\, ds \right),$$

assumption $(\mathcal{A}_2)$ follows plainly from $(\mathcal{V}_2)$. On the other hand, assumption $(\mathcal{V}_1)$ yields

$$\|\nabla_z H(t,z)\|^2 \leq \sum_{k=1}^{\infty} M_k^2 + 2 \left(\|y\|_{\ell^2}^2 + \int_0^T \|e(s)\|_{\ell^2}^2\, ds \right),$$

for every $t \in [0,T]$ and $z = (x,y) \in \ell^2 \times \ell^2$, implying that $(\mathcal{A}_1)$ holds true.

To conclude the proof, we thus need to find two sequences $(a_k)_k, (b_k)_k$ in ℓ^2 such that the twist condition (6) holds true. To this end, we set

$$a_k = -2M_k T, \qquad b_k = 2M_k T,$$

and, for N sufficiently large, we consider the finite-dimensional system

$$u_k' = v_k + \int_0^t e_k(s)\, ds\,, \quad v_k' = -\frac{\partial \mathcal{V}}{\partial u_k}(t, u_1, \ldots, u_N, 0, \ldots), \quad k = 1, \ldots, N,$$

which is readily verified to be the finite-dimensional approximation (4). Integrating the equations, we immediately see that, if $v_k(0) = a_k$, then $v_k(t) < 0$ for every $t \in [0,T]$, whence $u_k(T) - u_k(0) < 0$, using (9) once more. Symmetrically, if $v_k(0) = b_k$, then $v_k(t) > 0$ for every $t \in [0,T]$, so that $u_k(T) - u_k(0) > 0$. Theorem 3 thus applies, giving the conclusion. □

Example 1. As a first example of application of Theorem 6, we consider a system like (10), where

$$\mathcal{V}(t,x) = -\sum_{k=1}^{+\infty} \frac{c_k}{\omega_k} \cos(\omega_k x_k) \cos(\omega_{k+1} x_{k+1}),$$

with $c_k > 0$ and $\omega_k > 0$, for every $k \geq 1$. We have the cyclically coupled system

$$x_k'' + \left[\frac{c_{k-1}\omega_k}{\omega_{k-1}} \cos(\omega_{k-1} x_{k-1}) + c_k \cos(\omega_{k+1} x_{k+1}) \right] \sin(\omega_k x_k) = e_k(t),\ k = 1, 2, \ldots$$

where we have formally set $c_0 = 0$ and $\omega_0 = 1$. Assuming that the sequences

$$(c_k)_k, \qquad \left(\frac{1}{\omega_k}\right)_k, \qquad \left(\frac{c_{k-1}\omega_k}{\omega_{k-1}}\right)_k$$

all belong to ℓ^2 (e.g., we could take $c_k = 1/k$ and $\omega_k = k$), we can apply Theorem 6, so that a T-periodic solution exists.

Example 2. Another example can be obtained if we now identify ℓ^2 with the space of sequences $(\xi_k)_k$ where k ranges from $-\infty$ to $+\infty$, i.e., with $\ell^2(\mathbb{Z})$. Defining

$$\mathcal{V}(t,x) = -\sum_{k=-\infty}^{+\infty} \frac{1}{\omega_k}\cos(\omega_k x_k)\Big(c_k' \cos(\omega_{k-1}x_{k-1}) + c_k'' \cos(\omega_{k+1}x_{k+1})\Big),$$

with $c_k', c_k'' > 0$ and $\omega_k > 0$ for every integer k, we have the system

$$x_k'' + [\alpha_k \cos(\omega_{k-1}x_{k-1}) + \beta_k \cos(\omega_{k+1}x_{k+1})]\sin(\omega_k x_k) = e_k(t), \quad k \in \mathbb{Z},$$

where

$$\alpha_k = \frac{c_k'\omega_{k-1} + c_{k-1}''\omega_k}{\omega_{k-1}}, \qquad \beta_k = \frac{c_k''\omega_{k+1} + c_{k+1}'\omega_k}{\omega_{k+1}}.$$

If we assume that all the sequences $(c_k)_k$, $(\omega_k^{-1})_k$, $(\alpha_k)_k$, $(\beta_k)_k$ belong to $\ell^2(\mathbb{Z})$ (e.g., taking $c_k' = c_k'' = (|k|+1)^{-1}$ and $\omega_k = |k|+1$), we can suitably adapt Theorem 6 to this new setting, and conclude that a T-periodic solution exists.

We would like to consider now a system like

$$\vartheta_k'' + \gamma_k \frac{\partial \mathcal{W}}{\partial \vartheta_k}(t, \vartheta_1, \dots, \vartheta_k, \dots) = f_k(t), \qquad k = 1, 2, \dots, \tag{11}$$

where $\gamma_k > 0$ for every $k \geq 1$, assuming that $\mathcal{W}$ is T-periodic in the variable t and 2π-*periodic in each variable* ϑ_k. In this case, if

$$\sum_{k=1}^{\infty} \frac{1}{\gamma_k} < +\infty,$$

then it is easy to see that the change of variables $x_k = \vartheta_k/\sqrt{\gamma_k}$ leads back to the setting of system (8), with $\mathcal{V}(t, x_1, \dots, x_k, \dots) = \mathcal{W}(t, \vartheta_1, \dots, \vartheta_k, \dots)$ and $e_k(t) = f_k(t)/\gamma_k$ (the k-th period will now be $\tau_k = 2\pi/\sqrt{\gamma_k}$). To make such a procedure rigorous, we need to settle equation (11) in the Hilbert space of weighted ℓ^2-summable sequences

$$\ell_w^2 = \left\{ (\xi_k)_k \;\middle|\; \sum_{k=1}^{\infty} \frac{\xi_k^2}{\gamma_k} < \infty \right\},$$

endowed with the scalar product

$$\langle \xi, \tilde{\xi} \rangle_{\ell^2_w} = \sum_{k=1}^{\infty} \frac{\xi_k \tilde{\xi}_k}{\gamma_k},$$

and the corresponding norm $\|\xi\|_{\ell^2_w} = \sqrt{\langle \xi, \xi \rangle_{\ell^2_w}}$. Indeed, assuming $\mathcal{W} : \mathbb{R} \times \ell^2_w \to \mathbb{R}$ to be continuously differentiable in ϑ, by the definition of the inner product in ℓ^2_w one has

$$\nabla_\vartheta \mathcal{W}(t, \vartheta) = \left(\gamma_1 \frac{\partial \mathcal{W}}{\partial \vartheta_1}(t, \vartheta), \gamma_2 \frac{\partial \mathcal{W}}{\partial \vartheta_2}(t, \vartheta), \dots \right).$$

Hence, system (11) can be briefly written as

$$\vartheta'' + \nabla_\vartheta \mathcal{W}(t, \vartheta) = f(t),$$

where of course the map $t \mapsto f(t) = (f_1(t), f_2(t), \dots)$ is supposed to be well-defined and continuous with values in ℓ^2_w, and a solution is meant to be a C^2-function $\vartheta : I \to \ell^2_w$, where $I \subset \mathbb{R}$ is an interval, which satisfies the equation pointwise. We then have the following.

Corollary 7. *In the above setting, suppose further that* $\int_0^T f_k(t)\, dt = 0$ *for every* $k \geq 1$ *and that*

$(\mathcal{W}_1)$ *there exists a constant* $M > 0$ *such that, for every* $k \geq 1$,

$$\left| \gamma_k \frac{\partial \mathcal{W}}{\partial \vartheta_k}(t, \vartheta) \right| \leq M, \qquad \text{for every } (t, \vartheta) \in [0, T] \times \ell^2_w;$$

$(\mathcal{W}_2)$ *for every* $\rho > 0$, *there exists* $L_\rho > 0$ *such that*

$$\|\nabla_\vartheta \mathcal{W}(t, \vartheta) - \nabla_\vartheta \mathcal{W}(t, \tilde{\vartheta})\|_{\ell^2_w} \leq L_\rho \|\vartheta - \tilde{\vartheta}\|_{\ell^2_w}, \text{ for every } t \in [0, T],\ \vartheta, \tilde{\vartheta} \in B_\rho,$$

where B_ρ *denotes the closed ball in* ℓ^2_w, *centered at* 0 *with radius* ρ.

Then, system (11) *has a* T*-periodic solution.*

As a possible application, referring to Example 1 above and taking $\gamma_k = k^2$, $c_k = 1/k$ and $\omega_k = k$, for every $k \geq 1$, we can deal with the system

$$\vartheta_k'' + [q_k \cos \vartheta_{k-1} + \cos \vartheta_{k+1}] \sin \vartheta_k = f_k(t), \quad k = 1, 2, \dots,$$

where $q_1 = 0$ and

$$q_k = \left(\frac{k}{k-1} \right)^2, \quad k = 2, 3, \dots.$$

4 A further generalization

In this section, we propose a further infinite-dimensional extension of the Poincaré–Birkhoff theorem, which will include Theorem 3 as a special case. Such a generalization will be given on the lines of the N-dimensional version proved by Fonda and Ureña in [18, Theorem 6.1], which we briefly recall below (in a slightly simplified version). In the following, by a *convex body* $D \subset \mathbb{R}^N$ we mean the closure of a non-empty, open, convex and bounded set; accordingly, we denote by $\mathcal{N}(v)$ the corresponding outer normal cone at the point $v \in \partial D$, namely, the set

$$\mathcal{N}(v) = \left\{ \zeta \in \mathbb{R}^N \mid \langle \zeta, v - v' \rangle_{\mathbb{R}^N} \geq 0, \ \text{ for every } v' \in D \right\}.$$

Theorem 8. *[18, Theorem 6.1]. Let $\{b_1, \dots, b_N\}$ be a basis of $\mathbb{R}^N$ and consider the finite-dimensional Hamiltonian system*

$$u' = \nabla_v H(t,u,v)\,, \qquad v' = -\nabla_u H(t,u,v)\,, \tag{12}$$

where $H : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t, continuous in (t,u,v), continuously differentiable in (u,v) and such that, for $k = 1, \dots, N$,

$$H(t, u + b_k, v) = H(t,u,v), \qquad \textit{for every } t \in [0,T], \ (u,v) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Let $\mathbb{A}$ be a regular and symmetric $N \times N$ matrix and let $D \subset \mathbb{R}^N$ be a convex body. Furthermore, assume that every solution $w(t) = (u(t), v(t))$ of (12) with $v(0) \in D$ is defined for every $t \in [0,T]$, and

$$v(0) \in \partial D \ \Rightarrow \ \langle u(T) - u(0), \mathbb{A}\zeta \rangle_{\mathbb{R}^N} > 0, \quad \textit{for every } \zeta \in \mathcal{N}(v(0)) \setminus \{0\}. \tag{13}$$

Then, there exist at least $N+1$ distinct T-periodic solutions $w(t) = (u(t), v(t))$ of (12) such that

$$w(0) = (u(0), v(0)) \in \mathcal{T}_N \times D,$$

where $\mathcal{T}_N = \left\{ \sum_{k=1}^N \alpha_k b_k \mid 0 \leq \alpha_k \leq 1 \right\}$.

Condition (13) was inspired by a similar one previously considered by Conley and Zehnder [8]; in the particular case when $D = \prod_{k=1}^N [a_k, b_k]$ and $\mathbb{A}$ is a diagonal matrix, it contains the twist condition appearing in the statement of [18, Theorem 6.2] (we recall that such a theorem was used in the proof of Theorem 3). For other types of twist conditions, we refer to [14, 18].

Let us now provide an infinite-dimensional version of Theorem 8. Given a separable real Hilbert space $\mathcal{H}$ with Hilbert basis $(e_k)_k$, we consider the Hamiltonian system

$$x' = \nabla_y H(t,x,y), \qquad y' = -\nabla_x H(t,x,y), \tag{14}$$

where $H : \mathbb{R} \times \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is T-periodic in the first variable, continuous in (t, x, y) and continuously differentiable in $z = (x, y)$. Similarly as in Section 2, we further assume that:

$(\mathcal{A}_1')$ *there exists* $C > 0$ *such that*

$$\|\nabla_z H(t, z)\| \leq C(1 + \|z\|), \quad \textit{for every } t \in [0, T],\ z \in \mathcal{H} \times \mathcal{H}.$$

Moreover, given a non-empty, convex and compact set $\mathcal{D} \subset \mathcal{H}$ and a sequence $(\tau_k)_k \in \ell^2$, with $\tau_k > 0$ for every $k \geq 1$, we define the bounded subset of ℓ^2

$$\mathcal{T}_\infty = \left\{ \sum_{k=1}^{\infty} \alpha_k e_k \mid 0 \leq \alpha_k \leq \tau_k \right\},$$

and we assume that:

$(\mathcal{A}_2')$ *setting*

$$R = (\mathrm{diam}(\mathcal{T}_\infty \times \mathcal{D}) + 1)\, \mathrm{e}^{CT},$$

there exists a constant $L > 0$ *such that*

$$\|\nabla_z H(t, z_1) - \nabla_z H(t, z_2)\| \leq L\|z_1 - z_2\|, \quad \textit{for every } t \in [0, T],\ z_1, z_2 \in \mathcal{B}_R,$$

where $\mathcal{B}_R \subset \mathcal{H} \times \mathcal{H}$ *denotes the closed ball centered at* 0 *with radius* R.

Finally, for a strictly increasing sequence of positive integers $(p_N)_N$, we set

$$X_N = \mathrm{span}\{e_1, \ldots, e_{p_N}\},$$

and denote by $\Pi_N : \mathcal{H} \to X_N$ the corresponding orthogonal projection. With these preliminaries, we have the following result.

Theorem 9. *Let* $(\mathcal{A}_1')$ *and* $(\mathcal{A}_2')$ *hold and assume further that:*

- *for every* $k \geq 1$*, the Hamiltonian* H *satisfies the periodicity assumption*

$$H(t, x + \tau_k e_k, y) = H(t, x, y), \quad \textit{for every } t \in [0, T], (x, y) \in \mathcal{H} \times \mathcal{H};$$

- *there exists an invertible self-adjoint operator* $A \in \mathcal{L}(\mathcal{H})$*, satisfying* $A(X_N) \subset X_N$ *for every* N*, such that the following condition holds true: for every sufficiently large integer* $N \geq 1$*, if* $w(t) = (u(t), v(t)) \in X_N \times X_N$ *is a solution of*

$$u' = \Pi_N \nabla_y H(t, u, v), \qquad v' = -\Pi_N \nabla_x H(t, u, v), \tag{15}$$

with $v(0) \in \partial_{X_N}(\mathcal{D} \cap X_N)$*, then*

$$\langle u(T) - u(0), A\zeta \rangle > 0, \quad \textit{for every } \zeta \in \mathcal{N}_{\mathcal{D} \cap X_N}(v(0)) \setminus \{0\}. \tag{16}$$

Then, there exists a T-periodic solution $z(t) = (x(t), y(t))$ of (14) *such that*

$$z(0) = (x(0), y(0)) \in \mathcal{T}_\infty \times \mathcal{D}.$$

Remark 10. A bit of caution in considering condition (16) is needed. Indeed, it is implicitly assumed that, for every N sufficiently large, the set $\mathcal{D} \cap X_N$ is a convex body with respect to the relative topology of the finite-dimensional subspace X_N (for example, the theorem will not be applicable if $\mathcal{H} = \ell^2$ with the usual Hilbert basis and $\mathcal{D} = \prod_{k=1}^{M}[0, 1/k] \times \{0\} \times \{0\} \times \ldots$, since $\mathcal{D} \cap X_N$ has empty interior in X_N when $N > M$). Having this in mind, if $\mathcal{D} \cap X_N$ is a convex body, $\partial_{X_N}(\mathcal{D} \cap X_N)$ and $\mathcal{N}_{\mathcal{D}\cap X_N}(v)$ denote the boundary and the normal cone in X_N at v, respectively.

Proof. We just give a sketch of the proof, since it is similar to the one of Theorem 3. Defining $H_N : \mathbb{R} \times X_N \times X_N \to \mathbb{R}$ as the restriction of H to $\mathbb{R} \times X_N \times X_N$, it can be seen that

$$\nabla_u H_N(t, u, v) = \Pi_N \nabla_x H(t, u, v), \qquad \nabla_v H_N(t, u, v) = \Pi_N \nabla_y H(t, u, v),$$

and all the assumptions of Theorem 8 are satisfied. Hence, there is a T-periodic solution $w^N(t)$ of (15) satisfying $w^N(0) \in (\mathcal{T}_\infty \cap X_N) \times (\mathcal{D} \cap X_N)$. By compactness, there is a subsequence, still denoted by $(w^N(0))_N$, which converges to some $z_0 \in \mathcal{T}_\infty \times \mathcal{D}$. The solution $z(t)$ of (14) starting from $z(0) = z_0$ is uniquely defined on $[0, T]$ by $(\mathcal{A}_1')$ and $(\mathcal{A}_2')$, and the same argument used in the proof of Theorem 3 can be applied, showing that $z(t)$ is T-periodic. □

Let us show how Theorem 3 follows from Theorem 9. Let $\mathcal{H} = \ell^2$, with its usual Hilbert basis $(e_k)_k$, and set $p_N = N$ and

$$\mathcal{D} = \prod_{k=1}^{\infty}[a_k, b_k] = \mathcal{D}_\infty.$$

In this case,

$$\mathcal{D} \cap X_N = \prod_{k=1}^{N}[a_k, b_k] \times \{0\} \times \{0\} \times \cdots$$

is a convex body in X_N for every N and its normal cone at

$$v = (v_1, \ldots, v_N, 0, 0, \ldots) \in \partial_{X_N}(\mathcal{D} \cap X_N)$$

is given by

$$\mathcal{N}_{\mathcal{D}\cap X_N}(v) = \mathcal{I}_1(v_1) \times \mathcal{I}_2(v_2) \times \cdots \times \mathcal{I}_N(v_N) \times \{0\} \times \{0\} \times \cdots,$$

where, for $k \geq 1$,

$$\mathcal{I}_k(v_k) = \begin{cases} (-\infty, 0) & \text{if } v_k = a_k \\ (0, +\infty) & \text{if } v_k = b_k \\ \{0\} & \text{if } v_k \in (a_k, b_k). \end{cases}$$

Then, defining the bounded self-adjoint operator $A : \ell^2 \to \ell^2$ by

$$Ae_k = \sigma_k e_k, \qquad \text{for every } k \geq 1,$$

it is immediately checked that the twist condition (16) holds true. Since $\mathcal{D}$ is convex and compact, as already remarked, the conclusion follows.

As a final remark, we notice that, using Theorem 9, we can also extend Theorem 3 to a "vector Hilbert cube" framework. Precisely, we can replace the intervals $[a_k, b_k]$ by convex bodies $D_k \subset \mathbb{R}^{d_k}$ having arbitrary finite dimension $d_k \geq 1$; the assumption that $(a_k)_k, (b_k)_k$ belong to ℓ^2 with $a_k \leq 0 \leq b_k$ is accordingly replaced by

$$(\operatorname{diam} D_k)_k \text{ belongs to } \ell^2, \text{ and } 0 \in D_k.$$

By minor modifications of the arguments in the Appendix, we see that the set

$$\mathcal{D} = \prod_{k=1}^{\infty} D_k$$

is a convex compact subset of the space ℓ^2. On this set, a natural twist condition, generalizing (6), can be stated on the lines of the one in Theorem 8. More precisely, setting $\mathcal{D}_N = \prod_{k=1}^{N} D_k$ and writing any vector $\eta \in \mathbb{R}^{p_N}$, with $p_N = d_1 + \ldots + d_N$, as $\eta = (\vec{\eta}_1, \ldots, \vec{\eta}_N)$, with $\vec{\eta}_k \in \mathbb{R}^{d_k}$, we require the following:

- *for every $k \geq 1$, there exists a symmetric and regular $d_k \times d_k$ matrix $\mathbb{A}_k$ such that, for every sufficiently large integer N, if $w(t) = (u(t), v(t)) \in \mathbb{R}^{p_N} \times \mathbb{R}^{p_N}$ is a solution of*

$$u' = \nabla_v H_N(t, u, v), \qquad v' = -\nabla_u H_N(t, u, v), \tag{17}$$

with $v(0) \in \partial \mathcal{D}_N$, then

$$\sum_{k=1}^{N} \langle \vec{u}_k(T) - \vec{u}_k(0), \mathbb{A}_k \vec{\zeta}_k \rangle > 0, \qquad \textit{for every } \zeta \in \mathcal{N}_{\mathcal{D}_N}(v(0)) \setminus \{0\}.$$

Of course, in (17) we mean the truncated Hamiltonian H_N as the vectorial analogue of the one in (3), namely,

$$H_N(t, u, v) = H(t, (\vec{u}_1, \dots, \vec{u}_N, 0, \dots), (\vec{v}_1, \dots, \vec{v}_N, 0, \dots)).$$

To see that the above framework enters the statement of Theorem 9, it is enough to choose the usual Hilbert basis in the space $\mathcal{H} = \ell^2$, and to define $A \in \mathcal{L}(\ell^2)$ as the diagonal operator

$$A = \begin{pmatrix} \mathbb{A}_1 & 0 & 0 & \cdots \\ 0 & \mathbb{A}_2 & 0 & \cdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \end{pmatrix}.$$

Let us now investigate the case when the Hamiltonian function $H(t, x, y)$, besides being τ_k-periodic in each variable x_k, is also periodic in some of the variables y_k. This situation has been considered in the finite-dimensional case in [14, Theorem 12], where it was shown that, if the Hamiltonian is periodic in $x_1, \dots, x_N$ and in $y_1, \dots, y_M$, adding a twist condition on the complementary $(N-M)$-dimensional space one obtains $N + M + 1$ distinct T-periodic solutions. In the case when $M = N$, i.e., when the Hamiltonian is periodic in all variables, the twist condition is not necessary any more, and one gets $2N + 1$ periodic solutions: this is a famous theorem by Conley and Zehnder [8, Theorem 1] partially solving a conjecture by Arnold.

By the techniques introduced in this paper, it is possible to deal with various situations where the Hamiltonian function, defined on an infinite-dimensional separable Hilbert space, is also periodic in all variables x_k and in some of the variables y_k, maybe also an infinite number of them. To be brief, we will only consider here the case when the Hamiltonian is periodic in all variables, similarly as in [8, Theorem 1].

Theorem 11. *Let the Hamiltonian function $H(t, x, y)$ be τ_k-periodic in each variable x_k, and $\hat{\tau}_k$-periodic in each variable y_k, where $(\tau_k)_k$ and $(\hat{\tau}_k)_k$ are two positive sequences in ℓ^2. Accordingly, define*

$$\mathcal{T}_\infty = \left\{ \sum_{k=1}^{\infty} \alpha_k e_k \mid 0 \le \alpha_k \le \tau_k \right\}, \quad \widehat{\mathcal{T}}_\infty = \left\{ \sum_{k=1}^{\infty} \alpha_k e_k \mid 0 \le \alpha_k \le \hat{\tau}_k \right\},$$

and assume conditions $(\mathcal{A}_1')$ and $(\mathcal{A}_2')$, with $\mathcal{D}$ replaced by $\widehat{\mathcal{T}}_\infty$. Then, there exists a T-periodic solution of (14).

Proof. All the finite-dimensional reductions (15) of our Hamiltonian system have a T-periodic solution $w^N(t) = (u^N(t), v^N(t))$: this can be deduced from [8, 24, 34]. By the periodicity of the Hamiltonian function, we can assume that $w^N(0) \in \mathcal{T}_\infty \times \widehat{\mathcal{T}}_\infty$, and the compactness of this set allows us to conclude along the lines of the proof of Theorem 3. □

Appendix: the Hilbert cube

The Hilbert cube is defined as the set

$$\mathfrak{C} = [0,1] \times [0,1] \times \ldots = [0,1]^{\mathbb{N}},$$

with the usual product topology (that is, the topology generated by all the Cartesian products of open sets in every component space, only finitely many of which can be proper subsets). In Functional Analysis, however, the name Hilbert cube is usually attributed to the closed convex subset of ℓ^2 defined by

$$\mathcal{C} = \prod_{k=1}^{\infty} \left[0, \frac{1}{k}\right].$$

Here, however, the topology is the one inherited by the metric topology on ℓ^2; with this choice, it can be seen (see [9, pp. 164-165]) that the map

$$\mathcal{C} \to \mathfrak{C}, \qquad (\xi_k)_k \mapsto (k\xi_k)_k$$

is a homeomorphism. As a consequence, $\mathcal{C}$ is compact, since the compactness of $\mathfrak{C}$ just follows from Tychonoff's Theorem. Even more, it can be seen (see [33, Theorem 2.3.3]) that every compact convex subset of a Banach space is linearly homeomorphic to a closed convex subset of $\mathcal{C}$. Hence, the Hilbert cube $\mathcal{C}$ turns out to be a natural choice when trying to prove fixed point theorems in an infinite-dimensional setting (cf. [26]).

In this paper, we made use of sets of the type

$$\mathcal{D}_\infty = \prod_{k=1}^{\infty} [a_k, b_k] \subset \ell^2,$$

where $(a_k)_k, (b_k)_k$ belong to ℓ^2, and $a_k \leq 0 \leq b_k$ for any $k \geq 1$. It is easily verified that, whenever $b_k - a_k > 0$ for every $k \geq 1$, the set $\mathcal{D}_\infty$ is homeomorphic to $\mathcal{C}$ via the affine map

$$\mathcal{C} \to \mathcal{D}_\infty, \qquad (\xi_k)_k \mapsto (a_k + k(b_k - a_k)\xi_k)_k,$$

and hence is compact. However, for the reader's convenience, we prove here below its compactness in a self-contained way (relying only on well-known properties of the metric topology of ℓ^2).

Proof. Being ℓ^2 a metric space and $\mathcal{D}_\infty$ a closed set, it is enough to prove that $\mathcal{D}_\infty$ is totally bounded, namely that for every $\epsilon > 0$ there exist $\xi^1, \ldots, \xi^n \in \ell^2$ such that

$$\mathcal{D}_\infty \subset \bigcup_{i=1}^{n} \mathcal{B}_\epsilon(\xi^i),$$

where $\mathcal{B}_\epsilon(\xi^i)$ is the open ball centered at ξ^i having radius equal to ϵ. Thus, let us fix $\epsilon > 0$. Correspondingly, there exists $N \geq 1$ such that

$$\|P_N\xi - \xi\|_{\ell^2}^2 = \sum_{k=N+1}^{\infty} \xi_k^2 \leq \sum_{k=N+1}^{\infty} (a_k^2 + b_k^2) < \frac{\epsilon^2}{4}\,, \tag{18}$$

for every $\xi = (\xi_1, \xi_2, \ldots) \in \mathcal{D}_\infty$. On the other hand, since $I_N P_N \mathcal{D}_\infty$ is (finite-dimensional and hence) compact, it is totally bounded, so that there exist $\xi^1, \ldots, \xi^n \in \ell^2$ with

$$I_N P_N \mathcal{D}_\infty \subset \bigcup_{i=1}^{n} \mathcal{B}_{\epsilon/2}(\xi^i). \tag{19}$$

Combining (18) and (19), the conclusion straightly follows. □

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