Polynomial goal programming and particle swarm optimization for enhanced indexation

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Abstract
Enhanced indexation is an investment strategy that aims to generate moderate and consistent excess returns with respect to a tracked benchmark index. In this work, we introduce an optimization approach where the risk of under-performing the benchmark is separated from the potential over-performance, and the Sharpe ratio measures the profitability of the active management. In addition, a cardinality constraint controls the number of active positions in the portfolio, while a turnover threshold limits the transaction costs. We adopt a polynomial goal programming approach to combine these objectives with the investor's preferences. An improved version of the particle swarm optimization algorithm with a novel constraint-handling mechanism is proposed to solve the optimization problem. A numerical example, where the Euro Stoxx 50 Index is used as the benchmark, shows that our method consistently produces larger returns, with reduced costs and risk exposition, than the standard indexing strategies over a 10-year backtesting period.

Keywords Enhanced indexation · Cardinality · Turnover constraint · Polynomial goal programming · Particle swarm optimization · Constraint handling

1 Introduction

During the last years, index-linked investing has become one of the most widely used vehicle for equity investors.

The standard index tracking (IT) strategy aims to replicate an index, assuming that the underlying market is efficient. Two alternatives are possible for this type of investment. In the so-called full tracking, an agent invests in each stock exactly the same proportion as it is in the benchmark index. Even if the exact replication is attained, this strategy is difficult to be realized in practice, because of its high transaction costs and possible liquidity crunches. To overcome these drawbacks, the so-called partial tracking strategy involves a limited number of stocks. Normally, it is executed by minimizing the tracking error between the return of the tracking portfolio and that of the index. For an extensive overview on the subject, one may refer to Affolter et al. (2016), Beasley et al. (2003), Benidis et al. (2018), Canakgoz and Beasley (2009), Guastaroba and Speranza (2012).

If the market is weakly efficient, enhanced indexation (EI) represents a promising portfolio design that combines the strengths of both passive and active management to generate moderate and consistent excess returns with respect to a mimicked benchmark index (DiBartolomeo 2000). Roughly speaking, the problem may be tackled by minimizing a function of the tracking error and, at the same time, maximizing the excess return of the portfolio. Several approaches and methods are described in the literature to construct an EI portfolio following this scheme. A detailed description of the early contributions to the problem is given in Canakgoz and Beasley (2009), while Bruni et al. 2015; Gnägi and Strub

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2018; Sharma et al. 2017; Xu et al. 2018; Zhao et al. 2019 concern the recent proposals.

Among the plethora of studies in this context, we now focus on those that are related to our work.

In Wu et al. (2007), the enhanced indexation strategy is formulated as a double-goal programming problem, in which one goal represents the desired return and the other is the desired tracking error. Li et al. (2011) propose a multi-objective optimization model for enhanced indexation that maximizes the excess return and, at the same time, minimizes the downside deviation between the portfolio and the benchmark. An immune-based multi-objective optimization algorithm is used to find the solutions. A heuristic approach that combines the kernel search with the ϵ-constraint method is proposed in Filippi et al. (2016), where the excess return is maximized and the absolute deviation is minimized. Guastaroba et al. (2016) propose two linear programming variants for the EI model using the Omega ratio, one with a fixed target point and another with a random target point. Thomaidis (2011) considers EI strategies, which focus on the probability of under-performing the benchmark. He handles the subjectivity of the investment targets by using the fuzzy set theory. The resulting cardinality-constrained portfolio selection problems are solved by three nature-inspired optimization techniques: simulated annealing, genetic algorithms and particle swarm optimization (PSO). An empirical analysis of the convergence properties of these algorithms shows the effectiveness of the evolutionary algorithms in reaching solutions in the vicinity of the optimum.

Other researchers propose to directly combine active and passive criteria in the EI model. Jorion (2003) jointly optimizes portfolio return, variance and tracking error. Vassiliadis et al. (2009) applies ant colony optimization to an active–passive approach under a downside risk framework. An active tracking formulation incorporating nonstandard criteria related to portfolio performance is proposed in Thomaidis (2010). Such targets/constraints are formulated within the framework of fuzzy mathematical multi-objective programming. Simulated annealing, genetic algorithms and PSO are used to solve the problem. The multi-objective active–passive approach in di Tollo et al. (2014) integrates the index tracking and the Markowitz model (Markowitz 1952). A hybrid metaheuristic that combines local search and quadratic programming is introduced to obtain an approximation of the Pareto set.

Recently, the problem of controlling the transaction costs has become of primary importance, due to its effect on the profitability of the index-linked strategies. Mezali and Beasley (2014) investigate two different index tracking models that account for fixed and variable transaction costs when constructing and/or rebalancing an index tracking portfolio. Additionally, they consider constraints limiting the number of stocks that can be bought/sold as well as limiting the total transaction cost that can be incurred. In Strub and Baumann (2018), the index tracking problem consists of rebalancing the composition of the index fund's tracking portfolio in response to new market information and cash deposits and withdrawals from investors. The index fund's tracking accuracy is maximized. Moreover, the strategy directly considers the trade-off between transaction costs and similarity in terms of normalized value development. Díaz et al. (2019) propose a hybrid model for solving the multi-period index tracking problem, which includes limits on the number of stocks, floor and ceiling constraints, diversification by sector, and transaction costs. Their model combines a genetic algorithm, used to select stocks, and a mixed-integer nonlinear programming, to estimate the weights.

We propose a novel enhanced index tracking strategy, based on the principles of the active–passive approach. On the one hand, our portfolio minimizes the number of periods in which it is below the benchmark and, at the same time, maximizes the number of periods in which it beats the market index. On the other hand, our model maximizes a risk-adjusted performance measure. We integrate the three criteria with the investor's preferences into a single function to minimize, using the polynomial goal programming approach (Deckro and Hebert 1988; Proelss and Schweizer 2014). A set of constraints from real-world practice is introduced. In particular, we consider cardinality, floor and ceiling constraints, and a turnover threshold to limit the costs in the rebalancing phase.

Since the PSO shows to be very promising in solving index-linked investment problems (Thomaidis 2011; Zhu et al. 2010), we deal with the inherent complexity of our optimization problem by using an ad hoc version of this algorithm, called IC-PSO. It uses a hybrid constraint-handling procedure, which combines a repair mechanism (Meghwani and Thakur 2017) with the domination principle (Deb 2000; Pulido and Coello 2004). Moreover, we introduce a multi-start perturbation procedure to prevent premature convergence and loss of diversity in the final stages of the algorithm.

In the literature, other techniques have already been proposed to overcome these drawbacks. For instance, Kaucic (2013) includes a re-initialization strategy based on two diversity measures in a variant of the PSO algorithm with an adaptive velocity based on the differential operator. In order to enhance the performance of PSO, Wang et al. (2013) propose a hybrid variant, which employs a diversity enhancing mechanism and neighborhood search strategies. A convergence speed controller is applied in Huang et al. (2019). Chowdhury et al. (2013) focus on mixed-discrete optimization problems. They integrate into the PSO algorithm an adaptive diversity-preservation technique, which provides a repulsion away from the best global solution in the case of continuous variables, and stochastically updates the discrete variables.

IC-PSO specifically addresses the stagnation issues of PSO when portfolio selection problems have to be solved, that is, the selection of stocks as well as the estimation of weights under real-world constraints. Two main steps are involved to this end. If the diversity level in the swarm falls below a given threshold, the candidate solutions are re-initialized. At the same time, if the convergence speed slows down, a swap operator (Krink et al. 2009) and an adaptive perturbation operator are applied to the best global solution to generate a new swarm. The hybrid constraint-handling technique guarantees a rapid convergence of the swarm toward the feasible region.

The rest of the paper is organized as follows. In Sect. 2, we introduce the proposed PGP-based multi-criteria index-linked investment problem. In Sect. 3, we design the IC-PSO algorithm. Section 4 first describes the computational experiments and introduces an investigation of the parameter setting for IC-PSO. Then, the performance of the PGP-based enhanced indexation is analyzed in comparison with other index-linked strategies. Finally, Sect. 5 concludes our work and provides some future research directions.

2 The optimization model

In Sect. 2.1, we introduce the structure and the constraints of the strategy. Then, the criteria considered for the portfolio design are presented in Sect. 2.2. Finally, Sect. 2.3 defines the procedure used to combine investment objectives and agent's preferences.

2.1 Investment framework

Let us consider a financial market with n stocks and represent a portfolio by a weight vector $\mathbf{x} = (x_1, \dots, x_n)^\top$, where $x_i \in \mathbb{R}$ denotes the proportion of capital invested in asset i, with $i = 1, \dots, n$, and the symbol "$^\top$" is the transpose operator. Let us assume to operate in a dynamic setting, where the portfolio positions are readjusted over time due to the market evolution, and fix an investment horizon of length h. If we observe the market over a time window $[0, T]$ and consider historical observations to be good predictors of the future, then the investment strategy we propose involves the following steps:

1. determine the optimal portfolio composition at time T on the basis of the market information over the period $[0, T]$;
2. maintain the portfolio composition unchanged for the next h trading periods, assuming the stocks selected at time T are still available at time $T + h$.

The procedure can be repeated over time, updating the construction window of step 1 by eliminating the h oldest observations and including the h most recent ones. In this manner, the strategy schedules portfolio rebalancing at a regular calendar interval defined by the investment horizon h, starting from T Zhang and Maringer (2010).

In this study, we consider an investor who defines his optimal portfolios on the basis of the information conveyed by the $[0, T]$ construction window, his preferences and law guidelines.

For each time period $t \in [0, T]$, the investor has available the price of stock i, denoted by P_{it}, and the price of the benchmark, P_{bt}. Thus, the rate of return of stock i and of the benchmark at time t are computed as

$$R_{it} = \frac{P_{it} - P_{i,t-1}}{P_{i,t-1}}, \quad i = 1, \dots, n$$

and

$$R_{bt} = \frac{P_{bt} - P_{b,t-1}}{P_{b,t-1}},$$

respectively. The rate of return of a portfolio $\mathbf{x}$ at time t is the random variable

$$R_{pt}(\mathbf{x}) = \sum_{i=1}^{n} x_i R_{it} \tag{1}$$

having the probability distribution function induced by that of $(R_{1t}, \dots, R_{nt})$. Let us denote by $\mu = (\mu_1, \dots, \mu_n)^\top$ and C the vector of expected values and the covariance matrix of the rates of return of the risky assets. Assuming $R_{pt}(\mathbf{x})$ measurable in $(R_{1t}, \dots, R_{nt})$, its expected value is given by

$$\mu_p(\mathbf{x}) = \mathbf{x}^\top \mu \tag{2}$$

and its variance is

$$\sigma_p^2(\mathbf{x}) = \mathbf{x}^\top C \mathbf{x}. \tag{3}$$

In this study, we define the set of feasible, or admissible, portfolios through the following constraints.

1. The capital invested is equal to the capital available at time T. In terms of portfolio weights, this is equivalent to impose

$$\sum_{i=1}^{n} x_i = 1. \tag{4}$$

2. We require that an admissible portfolio exactly includes a predefined number K of stocks out of the n available

stocks. By introducing the binary variable δ_i to model the inclusion or exclusion of stock i in the portfolio as

$$\delta_i = \begin{cases} 0, & \text{if asset } i \text{ is not included in the portfolio} \\ 1, & \text{if asset } i \text{ is included in the portfolio} \end{cases} \quad (5)$$

the cardinality constraint can be written as

$$\sum_{i=1}^{n} \delta_i = K. \quad (6)$$

3. To avoid extreme positions and brokerage fees for very small orders, we impose minimum and maximum limits for the weights of the stocks in the portfolio. Let us denote by l_i and u_i the lower bound and the upper bound for the weight of stock i, with $0 < l_i < u_i \leq 1$, and then the buy-in thresholds are given by

$$\delta_i l_i \leq x_i \leq \delta_i u_i, \quad i = 1, \ldots, n. \quad (7)$$

Note that if a stock is not selected, no capital can be invested in it. Moreover, by assuming $l_i > 0$, short selling is not allowed.

4. Because of the transaction costs when we rebalance a portfolio, we include additional constraints involving the asset allocation changes compared to the current asset allocation. Let x_i be the portfolio weight of stock i at time $T + h$ and x_i^+ be the corresponding weight after rebalancing, and the turnover constraint can be expressed as

$$\sum_{i=1}^{n} \left| x_i^+ - x_i \right| \leq TR \quad (8)$$

where $TR \in [0, 1]$ is the maximum turnover rate. If TR is set to 0, then the rebalanced portfolio is equal to the initial one. As TR increases, the allowed turnover increases and, as a consequence, also the allowed costs for the new portfolio increase.

Remark 1 Equation (7) implicitly introduces cardinality constraints (Maringer and Oyewumi 2007). For instance, if we consider a common upper bound equal to u, then at least $K_{\min} = \left\lfloor \frac{1}{u} \right\rfloor$ stocks must be bought. At the same time, a common lower bound of l allows for at most $K_{\max} = \left\lceil \frac{1}{l} \right\rceil$ positive weights.

With an abuse of notation, let us denote by $\mathcal{X}$ the set of feasible portfolios. In the construction phase, $\mathcal{X}$ represents the solutions satisfying constraints (4), (6) and (7). If the optimization problem concerns the rebalancing phase, the feasible set also involves the constraint (8). The meaning of $\mathcal{X}$ will be clear from the context.

2.2 Passive and active investment criteria

In the literature, the tracking accuracy of a portfolio $\mathbf{x}$ with respect to a benchmark has been calculated as some function of the difference between the rate of return of the tracking portfolio, $R_{pt}(\mathbf{x})$, and that of the benchmark, R_{bt}. However, measures based only on the variance of this deviation, like the tracking error variance (Franks 1992), become zero even for tracking portfolios that do not satisfactorily replicate the movements of the benchmark, as is the case when the difference in returns from the benchmark is constant. A solution to this disadvantage is provided by the following family of measures (Beasley et al. 2003):

$$\mathrm{TE}(\mathbf{x}) = \frac{1}{T} \left(\sum_{t \in \mathcal{S}} \left| R_{pt}(\mathbf{x}) - R_{bt} \right|^{\alpha} \right)^{1/\alpha} \quad (9)$$

where $\alpha > 0$ is the power by which portfolio's rate of return deviations are penalized and $\mathcal{S}$ is a subset of the time window over which we compare $R_{pt}(\mathbf{x})$ and R_{bt}. In general, the choice $\alpha = 2$ guarantees good in-sample as well as out-of-sample performances (Takeda et al. 2013). The definition of $\mathcal{S}$ is related to the type of risk the investor is interested in. A significant number of studies tackle the index tracking problem by setting $\alpha = 2$ and $\mathcal{S} = [1, T]$; that is, $\mathcal{S}$ comprises all the time periods. Considering the previously defined set of feasible portfolios, $\mathcal{X}$, the passive strategy associated with these choices of α and $\mathcal{S}$ can be formulated as

$$\min_{\mathbf{x} \in \mathcal{X}} \frac{1}{T} \sqrt{\sum_{t=1}^{T} \left(R_{pt}(\mathbf{x}) - R_{bt} \right)^2}. \quad (10)$$

We can note that since the risk of the tracking portfolio is constrained to be as close as possible to the risk of the replicated index, Problem (10) maximizes the systematic risk, regardless of the total risk of the portfolio (Roll 1992). Moreover, investment opportunities, such as cross-sectional risk-return inversion, are not exploited in the optimization (Wurgler 2010).

To overcome these drawbacks, we introduce in the investment decision process other tracking measures, by modifying the time subset $\mathcal{S}$ in Eq. (9), as well as a performance measure from active management.

By defining $\mathcal{S}$ as the subset of periods in which the tracking portfolio under-performs the index, we can measure the downside risk of the tracking strategy by

$$\text{TE}^-(\mathbf{x}) = \frac{1}{T}\sqrt{\sum_{t=1}^{T}\left(\left(R_{bt} - R_{pt}(\mathbf{x})\right)^+\right)^2} \tag{11}$$

where $(a)^+ = \max(a, 0)$, for all $a \in \mathbb{R}$. In a similar manner, by taking into account only the periods in which the optimal portfolio outperforms the benchmark, we can calculate the upside potential profit of the strategy by

$$\text{TE}^+(\mathbf{x}) = \frac{1}{T}\sqrt{\sum_{t=1}^{T}\left(\left(R_{pt}(\mathbf{x}) - R_{bt}\right)^+\right)^2}. \tag{12}$$

In an attempt to further increase profits and control the portfolio total risk, we can consider the so-called Sharpe ratio, a performance measure defined as the expected rate of return of the portfolio in excess of the risk-free rate over the standard deviation of the rates of return of the portfolio itself (Sharpe 1966). According to Eqs. (2) and (3), the Sharpe ratio of portfolio $\mathbf{x}$ is calculated as

$$\text{SR}(\mathbf{x}) = \frac{\mu_p(\mathbf{x}) - R_f}{\sigma_p(\mathbf{x})} \tag{13}$$

where R_f denotes the risk-free rate. This measure evaluates the compensation earned by the investor per unit of both systematic and idiosyncratic risks (Caporin et al. 2014). Higher values of SR indicate more promising performance. However, when the expected excess rate of return is negative, the portfolio with higher standard deviation leads to a better Sharpe ratio, which is counter-intuitive. In this case, Israelsen et al. (2005) suggests to multiply, instead of dividing, the excess rate of return of the managed portfolio by its total risk. The resulting modified Sharpe ratio is

$$\text{MSR}(\mathbf{x}) = \frac{\mu_p(\mathbf{x}) - R_f}{\sigma_p(\mathbf{x})^{sign(\mu_p(\mathbf{x}))}} \tag{14}$$

where $sign(\cdot)$ represents the sign function. In line with the prescription of the Sharpe ratio, a higher value of MSR is preferred.

The proposed index-linked strategy aims to

1. reduce the number of periods in which the tracking portfolio is below the benchmark as well as the magnitude of these losses;
2. increase the number of periods in which the tracking portfolio beats the benchmark and the magnitude of these extra-rewards;
3. enhance the risk-adjusted performance of the portfolio.

Thus, investor's decisions are based on the following preference relation.

Definition 1 Let $\mathcal{X} \subset \mathbb{R}^n$ be the set of feasible portfolios and $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Then $\mathbf{x}$ (Pareto) dominates $\mathbf{y}$, in symbols $\mathbf{x} \prec \mathbf{y}$, if and only if $\text{TE}^-(\mathbf{x}) \leq \text{TE}^-(\mathbf{y})$, $\text{TE}^+(\mathbf{x}) \geq \text{TE}^+(\mathbf{y})$ and $\text{MSR}(\mathbf{x}) \geq \text{MSR}(\mathbf{y})$, with at least one strict inequality.

According to this definition, we say that a portfolio $\mathbf{x}^* \in \mathcal{X}$ is (Pareto) optimal if and only if it is nondominated with respect to $\mathcal{X}$, i.e., there does not exist another $\mathbf{x} \in \mathcal{X}$ that dominates $\mathbf{x}^*$. Thus, an efficient portfolio for this investment strategy is a Pareto optimal solution of the following multi-objective problem

$$\begin{aligned} &\text{minimize} \quad \left(\text{TE}^-(\mathbf{x}),\ -\text{TE}^+(\mathbf{x}),\ -\text{MSR}(\mathbf{x})\right) \\ &\text{subject to} \quad \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{15}$$

We consider the same set of criteria to guide the investment choices in both the construction and the rebalancing stages.

2.3 The polynomial goal programming procedure

A general way to solve problem (15) is by transforming it into a single-objective nonlinear programming problem. To this end, we adopt the polynomial goal programming (PGP) approach that aims at minimizing a polynomial expression of the deviations between the achievement of goals and their aspiration levels by accommodating investor's preferences (Deckro and Hebert 1988; Proelss and Schweizer 2014).

The procedure involves two steps. First, the optimal values of each objective are determined by separately solving the following optimization problems

$$\text{TE}_*^- = \min_{\mathbf{x}\in\mathcal{X}} \text{TE}^-(\mathbf{x}) \tag{16}$$

$$\text{TE}_*^+ = \max_{\mathbf{x}\in\mathcal{X}} \text{TE}^+(\mathbf{x}) \tag{17}$$

$$\text{MSR}_* = \max_{\mathbf{x}\in\mathcal{X}} \text{MSR}(\mathbf{x}). \tag{18}$$

Then, given the investor's subjective preferences toward the downside tracking error risk, denoted by λ_1, the upside potential tracking profit, λ_2, and the modified Sharpe ratio, λ_3, the portfolio optimization problem can be reformulated as

$$\begin{aligned} \text{minimize} \quad & Z = \left(1 + \frac{d_1}{\text{TE}_*^-}\right)^{\lambda_1} + \left(1 + \frac{d_2}{\text{TE}_*^+}\right)^{\lambda_2} \\ & \qquad + \left(1 + \left|\frac{d_3}{\text{MSR}_*}\right|\right)^{\lambda_3} \\ \text{subject to} \quad & \text{TE}^-(\mathbf{x}) - d_1 = \text{TE}_*^- \\ & \text{TE}^+(\mathbf{x}) + d_2 = \text{TE}_*^+ \\ & \text{MSR}(\mathbf{x}) + d_3 = \text{MSR}_* \\ & d_i \geq 0 \quad i = 1, 2, 3 \\ & \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{19}$$

This specification of the objective function ensures that it is monotonically increasing in the deviations from the theoretically optimal values of TE^-, TE^+ and MSR.

The higher the preference parameter λ_i, $i = 1, 2, 3$, the more relevant the corresponding objective is deemed by the investor. Thus, a preference parameter of zero would mean that the investor has no interest in optimizing the corresponding objective, while larger values would imply stronger interest in the objective.

In the experimental section, we will consider several sets of preferences to point out the features of this portfolio allocation framework for different investors attitudes.

3 Constrained particle swarm optimization

Section 3.1 introduces the fundamentals of swarm optimization and the constraint-handling mechanism we have developed. Section 3.2 presents some procedures designed to improve the constrained PSO algorithm.

3.1 Particle swarm optimization

Particle swarm optimization (PSO) is a population-based algorithm that performs multidimensional search by mimicking the movements of a bird flock, or a fish schooling, that searches for food (Eberhart and Kennedy 1995). The PSO mechanism is mainly based on the communication of information about good solutions through the swarm. In this manner, the particles will tend to move toward good areas in the search space (Wang et al. 2018).

For solving Problem (19), these ideas translates into the following adaptations.

At each iteration s, we consider a set of P candidate solutions, denoted by $\mathcal{P}$. The p-th element of $\mathcal{P}$ is defined by a pair of n-dimensional real vectors $\left(\mathbf{x}_p(s), \mathbf{v}_p(s)\right)$ such that:

- $\mathbf{x}_p(s) = \left(x_{p1}(s), \ldots, x_{pn}(s)\right)^\top$ represents the weights of the p-th portfolio to be optimized;
- $\mathbf{v}_p(s) = \left(v_{p1}(s), \ldots, v_{pn}(s)\right)^\top$ is called velocity, and conveys the information about possible changes in the composition of $\mathbf{x}_p$ for the next iteration.

For each weight vector $\mathbf{x}_p(s)$, the K assets with the highest weights enter the corresponding portfolio, while zero weight is assigned to the remaining $n-K$ assets. The binary variables δ_i, $i = 1, \ldots, n$, in (5) are thus implicitly handled, and the cardinality constraint is satisfied. Further, assuming $l_i > 0$ in Eq. (7) avoids the indecisiveness case represented by $\delta_i = 1$ and $x_i = 0$.

For all $p = 1, \ldots, P$, let us denote by $\mathbf{x}_p^{best}$ and $\mathbf{g}^{best}$ the most recent best portfolio related to candidate solution p and the overall best portfolio, respectively. Then, the update rule for the velocity vector $\mathbf{v}_p$ is given by

$$\mathbf{v}_p(s+1) = w\mathbf{v}_p(s) + c_1 r_1(\mathbf{x}_p^{best} - \mathbf{x}_p(s)) + c_2 r_2(\mathbf{g}^{best} - \mathbf{x}_p(s)) \tag{20}$$

where

- $w \in \mathbb{R}$, called inertia weight, controls the exploration and the exploitation steps of the algorithm by scaling the contribution of the current vector of weight changes;
- $\mathbf{x}_p^{best} - \mathbf{x}_p(s)$ represents the so-called cognitive component and quantifies how much displacement, starting at its current composition $\mathbf{x}_p(s)$, the candidate solution will need to reach its own best composition $\mathbf{x}_p^{best}$;
- $\mathbf{g}^{best} - \mathbf{x}_p(s)$, called social component, measures the distance of portfolio weights $\mathbf{x}_p(s)$ to the best composition $\mathbf{g}^{best}$ of the entire group of portfolios;
- c_1 and c_2 are positive acceleration coefficients used to weight the contribution of the cognitive and social components, respectively;
- r_1 and r_2 represent two random numbers generated by a uniform distribution on the interval [0, 1].

It can be noticed that if $w < 1$, $\mathbf{v}_p$ decreases over time until it is zero and $\mathcal{P}$ converges to an optimal portfolio, emphasizing the local search around the current search area. On the contrary, if $w > 1$, $\mathbf{v}_p$ increases and $\mathcal{P}$ diverges, emphasizing the global search. Moreover, if $c_1 > c_2$, then each portfolio presents a stronger attraction to its own best composition, resulting in a slow convergence, while if $c_1 < c_2$, it is most attracted to the global best portfolio composition, causing premature convergence.

The composition $\mathbf{x}_p(s)$ of the p-th portfolio in $\mathcal{P}$ is modified at iteration $s+1$ by the vector of weight changes $\mathbf{v}_p(s+1)$ according to the following updating rule

$$\mathbf{x}_p(s+1) = \mathbf{x}_p(s) + \mathbf{v}_p(s+1) \qquad p = 1, \ldots, P. \tag{21}$$

3.1.1 Constraint-handling procedure for the portfolio construction phase

In the construction phase, admissible portfolios have to satisfy cardinality, budget constraint and buy-in thresholds. In this case, we propose a repair operator that projects candidate solutions that do not meet constraints, onto the feasible region.

First of all, since large values of weight changes may cause candidate solutions to leave the domain boundaries (see Eberhart and Shi 2001), we clamp vectors $\mathbf{v}_p$ to lie in a predefined interval. The vector of weight changes for the candidate solution p on dimension i at iteration s becomes:

$$v_{pi}(t) = \begin{cases} v_i^{\min}, & \text{if } v_{pi}(s) < v_i^{\min} \\ v_i^{\max}, & \text{if } v_{pi}(s) > v_i^{\max} \\ v_{pi}(s), & \text{otherwise} \end{cases} \tag{22}$$

for $p = 1, \dots, P$ and $i = 1, \dots, n$, where $v_i^{\min}$ and $v_i^{\max}$ are minimum and maximum percentages of capital that are allowed to be moved for asset i, respectively. In this paper, we follow Shi and Eberhart (1998) and set the maximum weight change equal to a fraction δ of the distance between the bounds of the search space, that is, $v_i^{\max} = \delta(u_i - l_i)$, with $0 < \delta < 1$. The minimum weight change is then defined as $v_i^{\min} = -v_i^{\max}$.

When a portfolio weight goes beyond one of its boundaries due to (21), we modify the candidate solution as follows:

1. the portfolio weight takes the value of that boundary (either the lower or the upper boundary);
2. the corresponding perturbation component in $\mathbf{v}_p$ is scaled by a random number generated by a uniform distribution on the interval [0, 1], and multiplied by -1 so that it searches in the opposite direction.

In this manner, we obtain a set of candidate portfolios that satisfy floor and ceiling constraints. Finally, the full investment condition (4) is dealt with the repair transformations recommended in Meghwani and Thakur (2017) as follows. Let us consider a portfolio $\mathbf{x} = (x_1, \dots, x_n)^\top$ such that

1. $l_i \le x_i \le u_i$ for all $i \in I_+$,
2. $\sum_{i \in I_+} l_i < 1$,
3. $\sum_{i \in I_+} u_i > 1$,

with $I_+ = \{i = 1, \dots, n \mid x_i > 0\}$, $|I_+| = K$. Then, we define the portfolio $\tilde{\mathbf{x}} = (\tilde{x}_1, \dots, \tilde{x}_n)^\top \in \mathcal{X}$ as $\tilde{x}_i = 0$ for $i \notin I_+$, and, for all $i \in I_+$,

$$\tilde{x}_i = \begin{cases} l_i + \dfrac{(x_i - l_i)}{\sum_{j \in I_+}(x_j - l_j)}\left(1 - \displaystyle\sum_{j \in I_+} l_j\right), \\ \qquad \text{if } \displaystyle\sum_{j \in I_+} x_j > 1 \text{ and } i \in I_+ \\ x_i, \quad \text{if } \displaystyle\sum_{j \in I_+} x_j = 1 \\ u_i - \dfrac{(u_i - x_i)}{\sum_{j \in I_+}(u_j - x_j)}\left(\displaystyle\sum_{j \in I_+} u_j - 1\right), \\ \qquad \text{if } \displaystyle\sum_{j \in I_+} x_j < 1 \text{ and } i \in I_+. \end{cases} \tag{23}$$

It can be noticed that $\tilde{\mathbf{x}}$ preserves the invested positions of $\mathbf{x}$, satisfies the constraints $l_i \le \tilde{x}_i \le u_i$, for all $i \in I_+$, and $\sum_{j=1}^n \tilde{x}_j = 1$.

To select the vectors $\mathbf{x}_p^{best}$, $p = 1, \dots, P$, and $\mathbf{g}^{best}$ in each iteration, we use the following comparison rules. If the current portfolio composition $\mathbf{x}_p(s)$ attains a worst objective function value than the best portfolio composition vector in memory, $\mathbf{x}_p^{best}$, the latter is kept; otherwise, the current vector of weights replaces the one in memory; if they assume the same objective function value, then we select one of them randomly. A similar idea is also used to update $\mathbf{g}^{best}$, but now, the previous $\mathbf{g}^{best}$ is compared to each $\mathbf{x}_p^{best}$. For minimization problems, these rules can be expressed as

$$\mathbf{x}_p^{best} = \begin{cases} \mathbf{x}_p^{best}, & \text{if } F\left(\mathbf{x}_p(s)\right) > F\left(\mathbf{x}_p^{best}\right) \\ \mathbf{x}_p^{best}, & \text{if } F\left(\mathbf{x}_p(s)\right) = F\left(\mathbf{x}_p^{best}\right) \text{ and } r > 0.5 \\ \mathbf{x}_p(s), & \text{otherwise} \end{cases} \tag{24}$$

and

$$\mathbf{g}^{best} = \begin{cases} \mathbf{g}^{best}, & \text{if } F\left(\mathbf{x}_p^{best}\right) > F\left(\mathbf{g}^{best}\right) \\ \mathbf{g}^{best}, & \text{if } F\left(\mathbf{x}_p^{best}\right) = F\left(\mathbf{g}^{best}\right) \text{ and } \tilde{r} > 0.5 \\ \mathbf{x}_p^{best}, & \text{otherwise} \end{cases} \tag{25}$$

where F is the objective function at hand, r and $\tilde{r}$ are two random numbers generated by a uniform distribution on the interval [0, 1].[1]

3.1.2 Constraint-handling procedure for the portfolio rebalancing phase

Since the rebalancing phase takes into account the turnover constraint as well as the constraints involved in the construction phase, we propose to update the weight vectors $\mathbf{x}_p$, $p = 1, \dots, P$, by integrating into the previous repair mechanism the following set of feasibility rules based on the domination principle (Deb 2000):

1. when two feasible portfolios are compared, that is, both the portfolios meet (8), the one with the best objective function value is chosen;
2. when a feasible portfolio is compared with an infeasible portfolio, that is, a portfolio with a turnover higher than the maximum turnover rate, the first one is chosen;

[1] For maximization problems, it suffices to replace $>$ with $<$ in (24) and (25).

3. when two infeasible portfolios are compared, the one with the lowest constraint violation is chosen.

According to these rules, portfolios that are feasible in terms of the turnover are always considered better than the infeasible ones. When two feasible portfolios are considered, they are only compared on the basis of their objective function values by means of (24) to identify $\mathbf{x}_p^{best}$ and through (25) to select $\mathbf{g}^{best}$, respectively. In the case of two infeasible portfolios, the comparison is based on the turnover violations, irrespective of the objective function values. This rule aims at pushing infeasible portfolios to the feasible region.

The resulting variant of the PSO algorithm, involving these constraint-handling techniques, will be denoted by C-PSO.

3.2 Improvements to the C-PSO algorithm

In C-PSO, the decision variables $\mathbf{x}_{pi}$, with $i = 1, \ldots, n$, indicate if a security has been selected and, at the same time, the amount invested in it. Hence, stagnation and premature convergence can occur in two forms. It may happen that the update rule (20) pushes each portfolio weight vector $\mathbf{x}_p$ toward both its best composition $\mathbf{x}_p^{best}$ and the overall best portfolio $\mathbf{g}^{best}$. It means that the values of the cognitive and social components become small. Once these best candidate solutions get stuck in local optima, all portfolios in $\mathcal{P}$ will quickly converge to those points. Alternatively, it may also happen that the same set of assets is selected for all the elements of $\mathcal{P}$. In this case, PSO moves over only a specific subspace of the search region and is unable to include other assets in the composition of the candidate solutions.

To prevent these issues, we consider the following diversity measure based on the relative distance of the portfolios $\mathbf{x}_p$, $p = 1, \ldots, P$, from the overall best portfolio $\mathbf{g}^{best}$ (Kaucic 2013):

$$\delta_1(s) = \max_{p \in \mathcal{P}} \frac{\left\| \mathbf{x}_p(s) - \mathbf{g}^{best} \right\|}{\operatorname{diam}(\mathcal{X})} \tag{26}$$

where $\|\cdot\|$ represents the Euclidean distance and $\operatorname{diam}(\mathcal{X}) = \sqrt{\sum_{i=1}^{n} (u_i - l_i)^2}$ is the diameter of the smallest hypercube that contains the feasible set $\mathcal{X}$. If $\delta_1(s)$ is smaller than a predefined threshold ϵ_{δ_1}, with $0 < \epsilon_{\delta_1} < 1$, stagnation occurs. The candidate solutions $\left(\mathbf{x}_p(s), \mathbf{v}_p(s)\right)$ are then re-initialized, but the current $\mathbf{x}_p^{best}$ and $\mathbf{g}^{best}$ individuals are unchanged. In this manner, the generated candidate solutions will move toward the potential neighborhood of the optimal solution found so far, increasing the global search capabilities of the algorithm.

This procedure, however, may slow down the convergence speed of the PSO. According to Huang et al. (2019), we can measure the magnitude of this effect by

$$\delta_2(s) = \left| \frac{F\left(\mathbf{g}^{best}(s-\tau)\right) - F\left(\mathbf{g}^{best}(s)\right)}{F\left(\mathbf{g}^{best}(s-\tau)\right)} \right| \tag{27}$$

where $\mathbf{g}^{best}(s-\tau)$ and $\mathbf{g}^{best}(s)$ represent the overall best solutions found at iteration $s-\tau$ and s, respectively. If the improvement in the objective function value $\delta_2(s)$ is less than a given threshold ϵ_{δ_2}, with $\epsilon_{\delta_2} > 0$, it means that the PSO algorithm cannot make any relevant progress in τ iterations. To improve the quality of the solutions in this case, we have designed the perturbation procedure reported in Algorithm 1.

Algorithm 1: Perturbation procedure

Input : $\mathbf{g}^{best}, l, u, s, s_{max}$, operator
Output: $\hat{\mathbf{x}}_p(s)$

Set $\hat{\mathbf{x}}_p(s) = \mathbf{g}^{best}$
if operator *is swap* **then**
  $a \to \left\{i : x_{pi}(s) = 0\right\}$
  $b \to \left\{i : l_i \le x_{pi}(s) \le u_i\right\}$
  $\hat{x}_{pa}(s) = l_a + \frac{x_{pb}(s) - l_b}{u_b - l_b}(u_a - l_a)$
  $\hat{x}_{pb}(s) = 0$
else
  for $i = 1$ *to* n **do**
    if $g_i^{best} > 0$ **then**
      Set $d = \left(1 - \frac{s}{s_{max}}\right)(u_i - l_i)$
      $lb = \max\left(g_i^{best} - d, l_i\right)$
      $ub = \min\left(g_i^{best} + d, u_i\right)$
      $\hat{x}_{pi}(s) \to [lb, ub]$
    end
  end
end

Assuming that the higher-quality solutions are near the global optimal solutions with higher probability than the lower-quality ones (Huang et al. 2019), we generate a novel population of P portfolios in the neighborhood of $\mathbf{g}^{best}$. Half of the candidate solutions are generated by applying the swap operator proposed in Krink et al. (2009) to $\mathbf{g}^{best}$. Let a and b be two randomly chosen positions in $\mathbf{g}^{best}$, such that $g_a^{best} = 0$ and $l_b \le g_b^{best} \le u_b$, then the new p-th weight vector, $\hat{\mathbf{x}}_p(s)$, is defined componentwise as

$$\hat{x}_{pi}(s) = \begin{cases} g_i^{best}, & \text{if } i \ne a \text{ and } i \ne b \\ l_a + \dfrac{g_b^{best} - l_b}{u_b - l_b}(u_a - l_a), & \text{if } i = a \\ 0, & \text{if } i = b. \end{cases} \tag{28}$$

The remaining portfolios are the result of the following procedure, similar to that proposed in Coello et al. (2004). Given $I_+^g = \{i = 1, \ldots, n \mid g_i^{best} > 0\}$, for all $i \in I_+^g$, we define

$$W_i^s = \left[g_i^{best} - \left(1 - \frac{s}{s_{\max}}\right)(u_i - l_i),\ g_i^{best} + \left(1 - \frac{s}{s_{\max}}\right)(u_i - l_i) \right]$$

where s is the current iteration number and $s_{\max}$ represents the maximum allowed number of iterations. Then, $\hat{x}_{pi}(s)$ is randomly generated from the interval $W_i^s \cap [l_i, u_i]$. For $i \notin I_+^g$, we set $\hat{x}_{pi}(s) = 0$. By narrowing the range over time, this scheme improves the exploratory behavior around $\mathbf{g}^{best}$.

Remark 2 The swap operator provides an outward drift to explore un-reached regions. The new candidate solution maintains K active positions, satisfies quantity constraints for a and removes asset b from the portfolio. In general, the second operator can produce infeasible portfolios. The hybrid constraint-handling procedure we have previously described is, thus, fundamental to move these portfolios toward the feasible region.

Finally, we update the population of candidate solutions $\mathbf{x}_p$ with the corresponding $\hat{\mathbf{x}}_p$, maintaining unchanged $\mathbf{v}_p$, $\mathbf{x}_p^{best}$ and $\mathbf{g}^{best}$. The conditions related to (26) and (27) are applied periodically, every τ iterations, to avoid unnecessary calculations and, at the same time, to exploit the convergence speed capabilities of the standard PSO algorithm.

The pseudocode of the improved C-PSO algorithm that incorporates the proposed multi-start perturbation procedure, henceforth IC-PSO, is given in Algorithm 2. It can be implemented to solve Problem (19) when rebalancing is requested. With suitable changes, IC-PSO can also be used to identify the aspired levels in Problems (16)–(18) as well as to find a solution for the construction phase of Problem (19).

4 Experimental analysis and discussion

Section 4.1 presents the data set and describes the structure of the experiments. In Sect. 4.2, we investigate the capabilities of the developed algorithm. First, we identify a set of suitable configurations for the parameters. Successively, we show the benefits of IC-PSO with respect to C-PSO. Section 4.3 analyzes the performance of the PGP-based investment approach by varying the vector of preferences.

Algorithm 2: IC-PSO

Input : investment objective $F, K, l, u, TR, \mathbf{x}^+, \underline{S}, \underline{P}, P, \delta, \epsilon_{\delta_1}, \epsilon_{\delta_2}, s_{max}$
Output: $\mathbf{g}^{best}$

Set $s = 0$
Initialize $\mathbf{x}_p(s)$ and $\mathbf{v}_p(s)$, with $p = 1, \ldots, P$
Set $\mathbf{x}_1(s) = \mathbf{x}^+$
Clamp $\mathbf{v}_p(s)$ using Eq. (22)
Repair $\mathbf{x}_p(s)$ using Eqs. (23)
Calculate turnover violation using Eq. (8)
Calculate $F(\mathbf{x}_p(s))$
Set $\mathbf{x}_p^{best} = \mathbf{x}_p(0)$
Find $\mathbf{g}^{best}$ using Eq. (25)
while $s < s_{max}$ **do**
    s = s + 1
    for $p = 1$ *to* P **do**
        Update $\mathbf{v}_p(s)$ using Eqs. (20) and (22)
        Update $\mathbf{x}_p(s)$ using Eqs. (21)
        for $i = 1$ *to* n **do**
            if $\mathbf{x}_{pi}(s) > 0 \wedge \mathbf{x}_{pi}(s) < l_i$ **then**
                Set $\mathbf{x}_{pi}(s) = l_i$
                $r \rightarrow [0, 1]$
                Set $\mathbf{v}_{pi}(s) = -r\mathbf{v}_{pi}(s)$
            else if $\mathbf{x}_{pi}(s) > u_i$ **then**
                Set $\mathbf{x}_{pi}(s) = u_i$
                $r \rightarrow [0, 1]$
                Set $\mathbf{v}_{pi}(s) = -r\mathbf{v}_{pi}(s)$
        end
        Clamp $\mathbf{v}_p(s)$ using Eq. (22)
        Repair $\mathbf{x}_p(s)$ using Eqs. (23)
        Calculate turnover violation using Eq. (8)
        Calculate $F(\mathbf{x}_p(s))$
        Update $\mathbf{x}_p^{best}$ using Eq. (24)
        Update $\mathbf{g}^{best}$ using Eq. (25)
    end
    if $s \bmod \tau = 0$ **then**
        if $\delta_1(s) < \epsilon_{\delta_1}$ **then**
            Re-initialize $\mathbf{x}_p(s)$
            Update $\mathbf{x}_p(s)$ using Algorithm 1
            Repair $\mathbf{x}_p(s)$ using Eqs. (23)
            Calculate turnover violation using Eq. (8)
            Calculate $F(\mathbf{x}_p(s))$
            Update $\mathbf{x}_p^{best}$ using Eq. (24)
            Update $\mathbf{g}^{best}$ using Eq. (25)
        end
        if $\delta_2(s) < \epsilon_{\delta_2}$ **then**
            Update $\mathbf{x}_p(s)$ using Algorithm 1
            Repair $\mathbf{x}_p(s)$ using Eqs. (23)
            Calculate turnover violation using Eq. (8)
            Calculate $F(\mathbf{x}_p(s))$
            Update $\mathbf{x}_p^{best}$ using Eq. (24)
            Update $\mathbf{g}^{best}$ using Eq. (25)
        end
    end
end

Table 1 List of stocks forming the investible universe

No.	Company	No.	Company
1	ADIDAS	25	SOCIETE GENERALE
2	KONINKLIJKE AHOLD DELHAIZE	26	IBERDROLA
3	AIR LIQUIDE	27	ING GROEP
4	AIRBUS	28	INTESA SANPAOLO
5	ALLIANZ	29	INDITEX
6	ANHEUSER-BUSCH INBEV	30	KERING
7	ASML HOLDING	31	L'OREAL
8	AXA	32	LVMH
9	BANCO SANTANDER	33	MUENCHENER RUCK.
10	BASF	34	NOKIA
11	BAYER	35	ORANGE
12	BBV.ARGENTARIA	36	PHILIPS ELTN. KONINKLIJKE
13	BMW	37	SAFRAN
14	BNP PARIBAS	38	SANOFI
15	CRH	39	SAP
16	DAIMLER	40	SCHNEIDER ELECTRIC
17	DANONE	41	SIEMENS
18	DEUTSCHE POST	42	TELEFONICA
19	DEUTSCHE TELEKOM	43	TOTAL
20	ENEL	44	WFD UNIBAIL RODAMCO STAPLED UNITS
21	ENGIE	45	UNILEVER DUTCH CERT.
22	ENI	46	VINCI
23	ESSILORLUXOTTICA	47	VIVENDI
24	FRESENIUS	48	VOLKSWAGEN PREF.

4.1 Data description and problem setting

Our investible universe is represented by the 48 stocks listed in Table 1, while the Euro Stoxx 50 Index plays the role of the benchmark. The corresponding daily closing prices have been downloaded from Datastream and cover the period from 07/07/2005 to 21/06/2019, for a total of 3642 observations. Figure 1 shows the evolution of the prices of the benchmark for this time frame. The investment strategies are based on the rolling window procedure described in Sect. 2.1. A training period of 250 days is used to determine the optimal portfolios, and 21 days represent the investment horizon. For each estimation window, the expected rates of return of the stocks are calculated as sample estimates. Because of its effectiveness in reducing the estimation error as well as in lowering the out-of-sample variance of the portfolios, we have adopted the Bayesian shrinkage estimator proposed in Ledoit and Wolf (2003) for the covariance matrix.

We consider portfolios with 20 stocks. The buy-in thresholds for the stocks weights are $l_i = 0.02$ and $u_i = 0.30$, for $i = 1, \dots, n$, respectively. When rebalancing is considered, the maximum turnover rate TR is set to 10%.

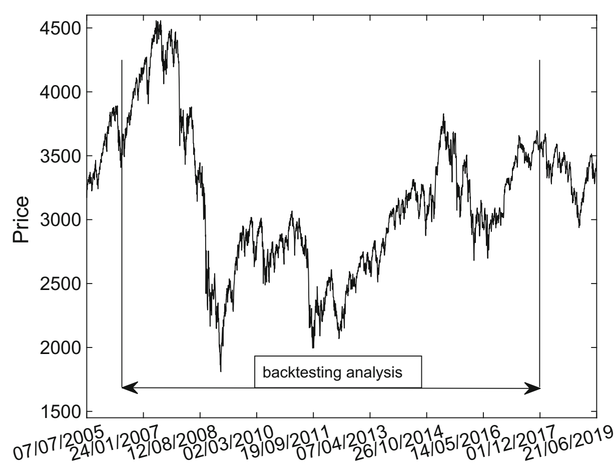

Fig. 1 Prices of the Euro Stoxx 50 Index from 07/07/2005 to 21/06/2019

The six vectors of preferences listed in Table 2 provide some representative investor attitudes toward the criteria considered in the PGP model. The first four types of investors exploit two out of three criteria. Agents A_1 and A_2 are more conservative, giving double importance to the downside tracking error risk, while agents A_3 and A_4 are more aggres-

Table 2 Investors considered in Problem (19) with the corresponding preferences toward the downside tracking error risk (λ_1), the upside potential tracking profit (λ_2) and the modified Sharpe ratio (λ_3)

Parameter	Value					
Agent	A_1	A_2	A_3	A_4	A_5	A_6
λ_1	2	2	1	1	2	3
λ_2	0	1	0	2	1	2
λ_3	1	0	2	0	1.5	1

Table 3 Values of the C-PSO parameters used in the experimental analysis

Parameter	Value	Parameter	Value
w	0.4	P	30
c_1	1.5	δ	0.07
c_2	2	$s_{\max}$	5000

Table 4 End date of the time frames used as estimation windows for the parameter tuning, divided by market phase and investment type

Market phase	Investment type	End date
Uptrend	Construction	06/07/2007
	Rebalancing	06/08/2007
Downtrend	Construction	03/11/2008
	Rebalancing	02/12/2008
Sideways/broadening trends	Construction	31/12/2014
	Rebalancing	29/01/2015

Table 5 Average rankings achieved by the Friedman test for the IC-PSO algorithm in the two investment phases

Investment phase	τ	ϵ_{δ_1}	ϵ_{δ_2}	Rankings
Construction	300	0.0005	0.0010	19.2778
Rebalancing	300	0.0010	0.0010	19.3889

sive, doubling the attention for the other criteria. Moreover, agents A_1 and A_3 manage the active risk-return performance in terms of the modified Sharpe ratio, while agents A_2 and A_4 make their decisions on the basis of the upside potential tracking information. The last two agents consider all the criteria in acting their decisions, with particular attention to control the downside risk.

4.2 Algorithmic analysis

The algorithms have been implemented in MATLAB R2019a and are run on a Workstation with Intel Core i9 with 3.3 GHz and 16 GB RAM.

4.2.1 Parameter settings

Since the hybrid constraint-handling procedure in C-PSO does not require any setup, we focus on only the parameters of the PSO, that is, the inertia weight w, the acceleration coefficients c_1 and c_2, the population size P, the fraction δ used to clamp the velocity update in Eq. (22), and the number of iterations $s_{\max}$. Based on the sensitivity analysis of Shi and Eberhart (1998), we consider the parameters values reported in Table 3. The procedure designed to improve C-PSO has three parameters to be tuned, namely the cycle parameter τ, the stagnation threshold ϵ_{δ_1} and the quality threshold ϵ_{δ_2}. To guarantee a suitable compromise between rapid convergence and re-initialization, we select τ in the set $\{100, 300, 500\}$. For the stagnation parameter, the analysis has focused on the values $\{0.0001, 0.0005, 0.001\}$, while the quality improvement parameter is checked in $\{0.0001, 0.001, 0.01\}$. With these parameter choices, we aim at promoting the local search nearby the optimal solutions. At the same time, implementing re-initialization from time to time guarantees adequate global search capabilities.

The effect of these parameters on IC-PSO is analyzed by testing the optimization problems related to the three aspired levels and the six instances of Problem (19). To take into account possible time dependence, three sets of data windows have been randomly selected. As given in Table 4, the first data set corresponds to an uptrend phase, the second refers to a downtrend phase, and the third involves sideways/broadening trends. Thus, the test set includes 27 optimization problems for the construction phase. The best solutions are then used as initial portfolios for the following 27 optimization problems regarding the rebalancing phase. To check the robustness of the results, 30 runs for each test problem are used.

Table 5 shows the best parameter configurations for the two optimization phases, based on the average ranking of the Friedman test (Derrac et al. 2011). The only difference between the two setups is for the stagnation threshold level. This is due to the turnover constraint, used in the rebalancing phase, that reduces the set of feasible portfolios with respect to the construction phase. As a consequence, IC-PSO aims to increase the number of re-initializations, in order to devote more time to the global search. This is attained by doubling ϵ_{δ_1}.

4.2.2 Performance evaluation of the algorithms

To show the merits of IC-PSO over C-PSO, the nine test functions related to Problem (19) are carried out for 10 dates randomly selected from 07/07/2005 to 21/06/2019. The configurations of the algorithms in the experimental study

Table 6 Wilcoxon signed-rank test results for the construction phase at different dates

Experiment	End date	R^+	R^-	p value
1	18/06/2009	43	2	0.0059
2	24/03/2010	44	1	0.0039
3	29/07/2011	45	0	0.0020
4	23/12/2011	45	0	0.0020
5	18/09/2012	45	0	0.0020
6	16/08/2013	43	2	0.0059
7	30/04/2015	45	0	0.0020
8	01/07/2015	44	1	0.0039
9	13/10/2015	44	1	0.0039
10	14/02/2019	44	1	0.0039

Table 7 Wilcoxon signed-rank test results for the rebalancing phase at different dates

Experiment	End date	R^+	R^-	p value
1	17/07/2009	45	0	0.0020
2	22/04/2010	45	0	0.0020
3	29/08/2011	45	0	0.0020
4	23/01/2012	45	0	0.0020
5	17/10/2012	45	0	0.0020
6	16/09/2013	45	0	0.0020
7	29/05/2015	45	0	0.0020
8	30/07/2015	45	0	0.0020
9	11/11/2015	45	0	0.0020
10	15/03/2019	42	3	0.0098

correspond to the best alternatives found in the previous section.

Their performance are compared in terms of the Wilcoxon signed-rank test (Derrac et al. 2011). Table 6 reports the results for the average values over 30 runs regarding the construction phase. Similarly, the results for the rebalancing phase over 30 runs are given in Table 7. In both cases, the second column provides the end date of the corresponding estimation window, R^+ is the sum of ranks for the problems in which IC-PSO outperformed C-PSO, R^- denotes the sum of the ranks for the opposite, and the last column shows the resultant p-values. It can be seen that IC-PSO is significantly better than C-PSO in all the experiments.

As an example to support these findings, we focus on the role that the improvement procedure has on the behavior of PSO for agent A_6 in Experiment 10. Figure 2 points out the rapid convergence of C-PSO with respect to IC-PSO in the construction phase. In particular, as shown in Fig. 3, it can be seen that the re-initialization step increases the diversity of solutions in IC-PSO and, at the same time, the local search technique improves the solution quality even in the last iterations. Figures 4 and 5 exhibit similar effects for the rebalancig phase. As shown in Fig. 6, the feasibility rules handle the turnover constraint efficiently.

Similar conclusions can be inferred for the other agents and test sets and, thus, are omitted.

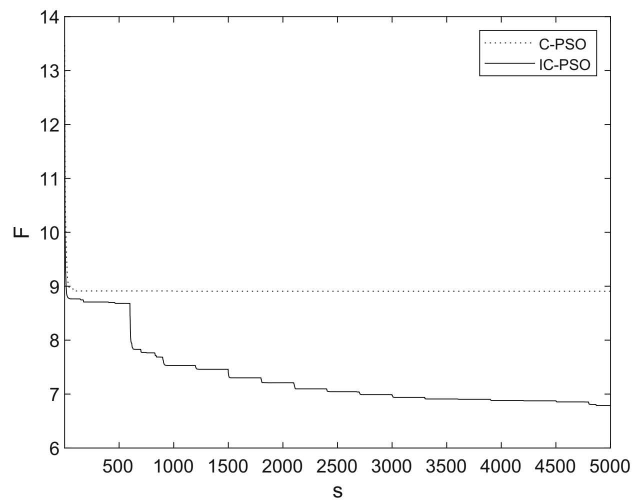

Fig. 2 Comparison of C-PSO and IC-PSO in solving Problem (19) for agent A_6 at 14/02/2019 (construction phase), in terms of the best objective function values. The results are averaged over 30 runs

4.3 Performance evaluation of the strategies

As in the previous sections, the out-of-sample analysis uses a rolling window strategy with 250 days for the estimation window and 21 days for the investment horizon. The backtesting period runs from 23/06/2006 to 27/12/2017 and covers 142 consecutive portfolio rebalancing phases (see Fig. 1). The performance of the PGP-based models is tested with respect to two passive strategies, namely the index tracking model, given by Problem (10), and the portfolio that minimizes the downside tracking error (11).

The following measures are considered in the comparisons. We compute the wealth of a portfolio at the t-th day as in (Benidis et al. 2018)

$$W_t = W_{t-1}(1 + R_{pt}) - c_t(x_t, x_{t-1}) \tag{29}$$

where c_t represents the transaction cost function that depends on the current and previous portfolios, denoted by x_t and x_{t-1}, respectively. In particular, we define c_t as in Beraldi et al. (2019). Its characteristics are reported in Table 8.

We propose to measure the effect of the costs on the available capital over the out-of-sample period by

$$C = \frac{1}{N_{reb}} \sum_{t=1}^{N_{reb}} \frac{c_t(x_t, x_{t-1})}{W_{t-1}} \tag{30}$$

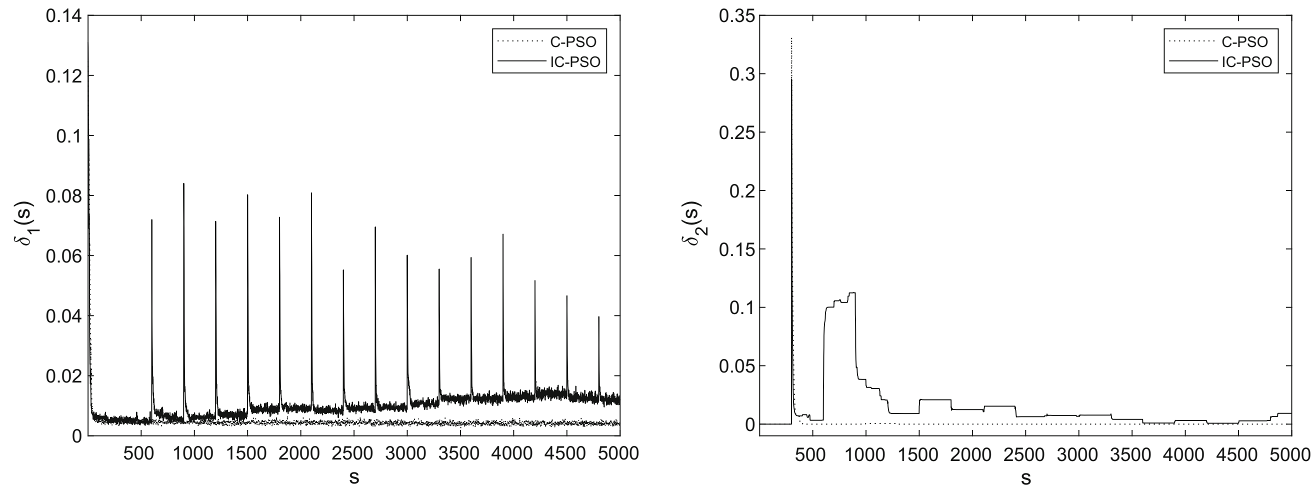

Fig. 3 Comparison of C-PSO and IC-PSO in solving Problem (19) for agent A_6 at 14/02/2019 (construction phase), in terms of the population diversity δ_1 (on the left) and objective function improvement δ_2 (on the right). The results are averaged over 30 runs

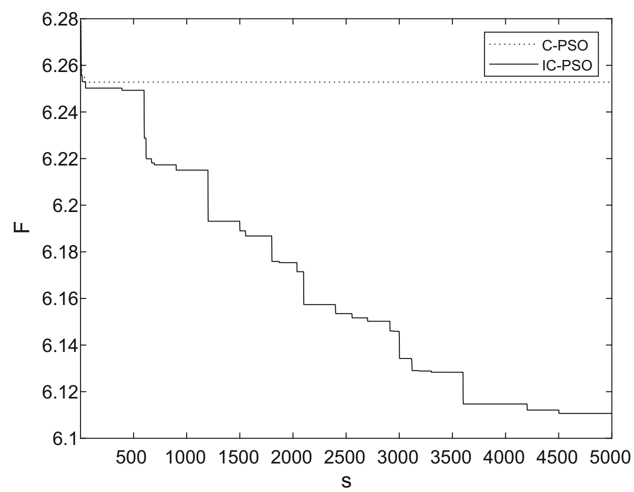

Fig. 4 Comparison of C-PSO and IC-PSO in solving Problem (19) for agent A_6 at 15/03/2019 (rebalancing phase), in terms of the best objective function values. The results are averaged over 30 runs

where W_0 represents the initial wealth, x_0 is the initial portfolio, and N_{reb} is the number of times the portfolio is rebalanced. In the experiments, W_0 is set equal to 1,000,000€ for all the investment strategies.

As suggested in Sharpe et al. (1995), to compare the profitability of the investments, we use the so-called compound annual growth rate (CAGR), which is calculated as

$$CAGR = \left(\frac{W_T}{W_0}\right)^{250/T} - 1 \quad (31)$$

where T is the number of days in the investment period and W_T is the final wealth.

For a rigorous assessment of the risk-adjusted performance, we consider two standard measures: the out-of-sample Sharpe ratio (Sharpe 1966) and the Rachev ratio (Biglova et al. 2004). The first is defined as the ratio between the mean of the out-of-sample portfolio returns and their standard deviation. The second is specified as the ratio between the average of the best $\beta\%$ returns of a portfolio and that of the worst $\alpha\%$ returns. In this study, we have set both the parameters α and β equal to 10, as in Bruni et al. (2015). In general, the out-of-sample Sharpe ratio is more focused to describe the central part of the portfolio return distribution, while the Rachev ratio stresses the behavior of the distribution on the tails. Moreover, we analyze the capability of the strategies to track the benchmark by evaluating the correlation and the average difference between the annualized out-of-sample returns of the portfolios and of the market index.

Tables 9, 10 and 11 list the results of the comparisons for different values of the portfolio size. Over the same time period, the CAGR of the Euro Stoxx 50 is 0.04%, its Sharpe ratio is 11.56%, and the Rachev ratio is 97.26%. Overall, the strategies present very low costs, with C varying between 0.08% and 0.15%. Moreover, there is no clear relationship between performance and portfolio cardinality. Notwithstanding, all the PGP-based portfolios beat the buy & hold strategy as well as the corresponding passive portfolios in terms of CAGR and risk-adjusted performance measures.

The high values of the correlations show that the PGP-based portfolios are able to replicate the benchmark trend. At the same time, the values of the average differences, which are higher than those of the passive strategies, show that the PGP-based models are able to generate moderate and consistent excess returns with respect to the benchmark.

Figure 7 reports a graphical comparison of the values of the wealth for the best PGP-based model, namely the one using the preference vector A_6 with 15 stocks, with the best passive strategy, given by the index tracking model with 15

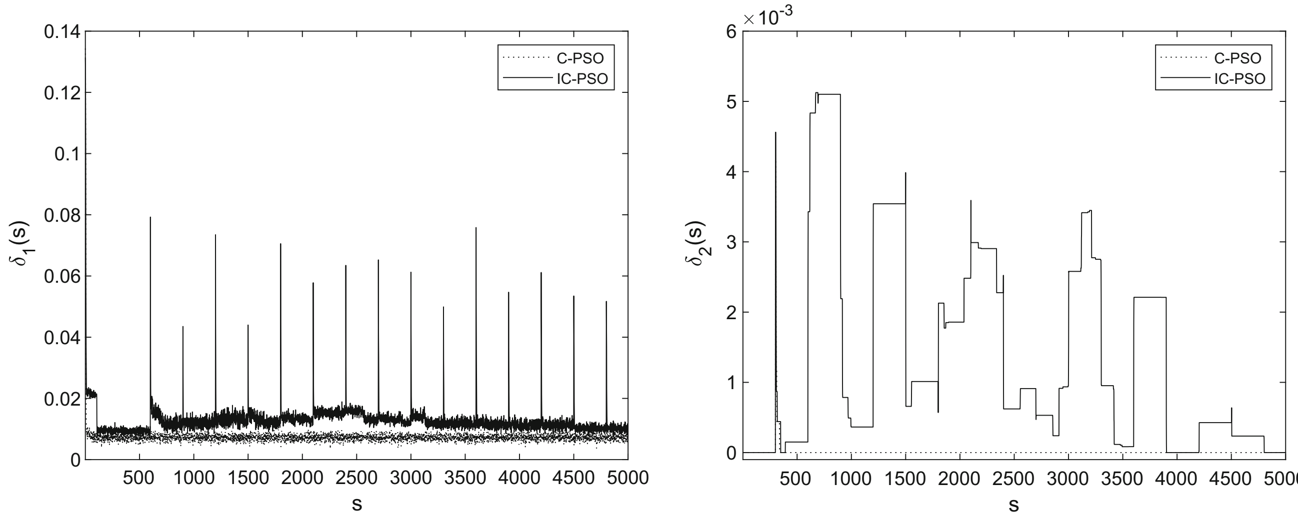

Fig. 5 Comparison of C-PSO and IC-PSO in solving Problem (19) for agent A_6 at 15/03/2019 (rebalancing phase), in terms of the population diversity δ_1 (on the left) and objective function improvement δ_2 (on the right). The results are averaged over 30 runs

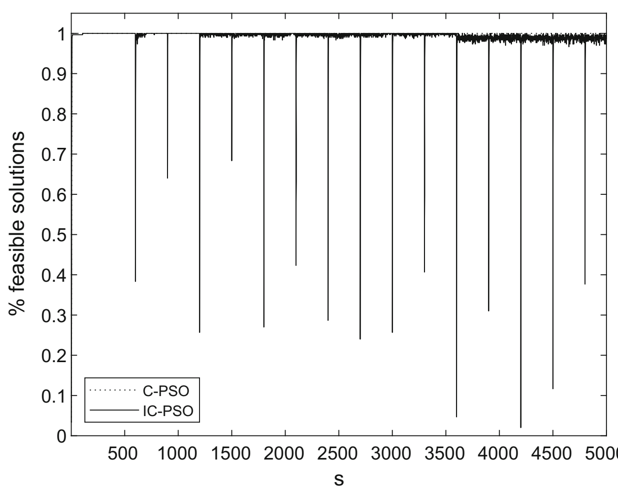

Fig. 6 Comparison of C-PSO and IC-PSO in solving Problem (19) for agent A_6 at 15/03/2019 (rebalancing phase) in terms of the percentage of feasible solutions in the set of candidate solutions $\mathcal{P}$. The results are averaged over 30 runs

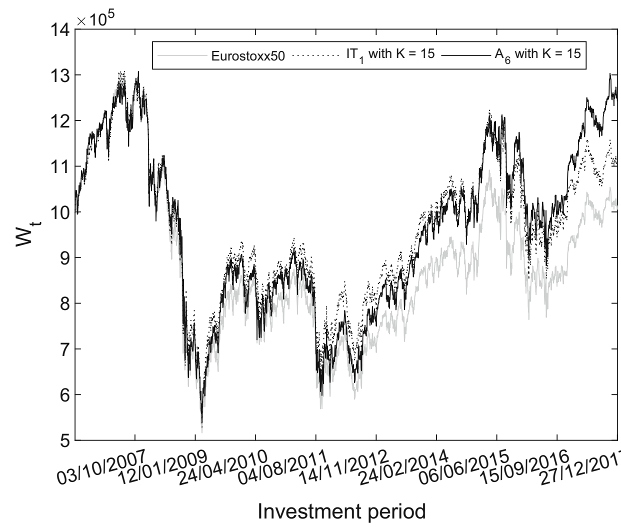

Fig. 7 Wealth evolution of different index-linked investment strategies from 23/06/2006 to 27/12/2017

Table 8 Structure of the transaction costs

Trading segment (€)	Fixed fee (€)	Proportional cost (%)
0–7999	40	0
8000–49,999	0	0.5
50,000–99,999	0	0.4
100,000–199,999	0	0.25
$\geq$ 200,000	400	0

constituents. Finally, to test whether the corresponding out-of-sample mean returns of the two strategies over the different market phases are statistically different, we apply a one-sided t test. The null hypothesis is that the two strategies have the same mean return, the alternative is that the PGP-based model has a greater value. The results listed in Table 12 show that the PGP-based portfolio behaves as the passive strategy in downtrend and sideways phases. However, when the market grows up, it is able to consistently outperform the tracking portfolio.

5 Concluding remarks

We have introduced an EI strategy involving the downside risk and the upside potential profit of the tracking portfolio, and its Sharpe ratio as criteria to optimize. The investor's

Table 9 Out-of-sample results for the index tracking strategy (10), denoted by IT_1, the strategy that minimizes the down side risk (11), denoted by IT_2, and the six instances of Problem (19). All the portfolios have cardinality $K = 15$ and refer to the investment period from 03/10/2007 to 27/12/2017

	IT_1	IT_2	A_1	A_2	A_3	A_4	A_5	A_6
C (%)	0.08	0.10	0.09	0.10	0.10	0.12	0.10	0.10
CAGR (%)	0.79	0.33	1.56	1.37	1.72	0.84	1.48	1.82
Sharpe ratio (%)	14.96	13.02	18.12	17.41	18.75	15.25	17.77	19.29
Rachev ratio (%)	98.90	98.66	97.79	98.03	97.65	97.33	97.63	97.88
Correlation (%)	98.28	98.30	98.23	98.30	98.02	97.99	98.05	98.13
Average diff. (%)	0.85	0.40	1.46	1.38	1.49	0.94	1.36	1.75

Table 10 Out-of-sample results for the index tracking strategy (10), denoted by IT_1, the strategy that minimizes the down side risk (11), denoted by IT_2, and the six instances of Problem (19). All the portfolios have cardinality $K = 20$ and refer to the investment period from 03/10/2007 to 27/12/2017

	IT_1	IT_2	A_1	A_2	A_3	A_4	A_5	A_6
C (%)	0.10	0.11	0.10	0.11	0.11	0.12	0.11	0.11
CAGR (%)	0.03	– 0.01	1.07	0.66	1.25	0.52	1.09	0.89
Sharpe (%)	11.74	11.58	16.04	14.40	16.70	13.87	16.09	15.28
Rachev (%)	98.65	98.63	98.63	98.69	98.51	98.44	98.51	98.93
Correlation (%)	98.42	98.51	98.67	98.66	98.52	98.53	98.60	98.53
Average diff. (%)	0.10	0.07	1.02	0.70	1.10	0.61	1.01	0.86

Table 11 Out-of-sample results for the index tracking strategy (10), denoted by IT_1, the strategy that minimizes the down side risk (11), denoted by IT_2, and the six instances of Problem (19). All the portfolios have cardinality $K = 30$ and refer to the investment period from 03/10/2007 to 27/12/2017

	IT_1	IT_2	A_1	A_2	A_3	A_4	A_5	A_6
C (%)	0.14	0.15	0.14	0.15	0.15	0.15	0.15	0.15
CAGR (%)	0.71	0.74	1.08	0.74	1.35	1.06	1.6	1.16
Sharpe (%)	14.58	14.76	16.10	14.71	17.21	16.10	16.89	16.48
Rachev (%)	98.04	98.13	98.56	98.35	98.42	99.04	98.64	98.64
Correlation (%)	98.83	98.86	98.85	98.91	98.79	98.82	98.84	98.85
Average diff. (%)	0.75	0.79	1.06	0.77	1.26	1.10	1.22	1.15

Table 12 Results of the one-sided t test for the out-of-sample mean returns of the PGP-based model with the preference vector A_6 and $K = 15$ and those of the index tracking model with $K = 15$ in different market phases

Time period	Market phase	p value
23/06/2006–21/07/2011	Downtrend and sideways	0.6654
22/07/2011–27/12/2017	Uptrend	0.0481

preferences have been used to combine these objectives by polynomial goal programming. The resulting nonlinear problem has been solved by a novel version of the particle swarm optimization algorithm, called IC-PSO. This procedure uses a multi-start perturbation procedure to improve the search capabilities of the original algorithm. In addition, IC-PSO handles the portfolio constraints by a hybrid technique that combines a repair mechanism with the domination principle.

The computational analysis on a real-world case study showed the effectiveness of IC-PSO and, at the same time, the backtesting pointed out the benefits of the proposed EI portfolio with respect to the passive strategies.

Future research work on the topic includes the study of more types of preferences, eventually by defining a dynamic setting, dependent on the economic cycle. We also plan further investigations of the profitability of our PGP-based strategy for other international indexes of stocks and fixed income securities.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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