Continuity and continuous multi-utility representations of nontotal preorders: some considerations concerning restrictiveness

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Abstract A continuous multi-utility fully represents a not necessarily total preorder on a topological space by means of a family of continuous increasing functions. While it is very attractive for obvious reasons, and therefore it has been applied in different contexts, such as expected utility for example, it is nevertheless very restrictive.

In this paper we first present some general characterizations of the existence of a continuous order-preserving function, and respectively a continuous multi-utility representation, for a preorder on a topological space. We then illustrate the restrictiveness associated to the existence of a continuous multi-utility representation, by referring both to appropriate continuity conditions which must be satisfied by a preorder admitting this kind of representation, and to the Hausdorff property of the quotient order topology corresponding to the equivalence relation induced by the preorder.

We prove a very restrictive result, which may concisely described as follows: the continuous multi-utility representability of all closed (or equivalently weakly continuous) preorders on a topological space is equivalent to the requirement according to which the quotient topology with respect to the equivalence corresponding to the coincidence of all continuous functions is discrete.

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1 Introduction

The necessity of considering *nontotal* (*incomplete*) *preferences* in order to deal with a more realistic framework dates back to the seminal paper of Aumann [3] published in 1962. Aumann pointed out that it is more appropriate not to assume that an individual may compare any two objects according with its own preferences, since "incomparability" may take place in some cases (see also Dubra et al. [26], Evren and Ok [29] and Ok [57]).

Clearly, when we deal with a nontotal binary relation, it is not possible to fully represent it by using only one function, as in the case of a total preorder. On the other hand, when the preference relation is defined on a topological space, continuity requirements of the representing functions naturally come into consideration. This is, needless to say, the spirit of the seminal famous papers by Debreu [24, 25] and Eilenberg [27], where general results about the existence of a *continuous utility function* for a *total preorder* on a topological space were presented.

Given a preorder $\precsim$ on a topological space (X,t), and the natural topology t_{nat} on the real line $\mathbb{R}$, we recall that a function $f : (X,\precsim,t) \to (\mathbb{R},\leq,t_{nat})$ is said to be a continuous order-preserving function for $\precsim$ if f is continuous on the topological space (X,t) and, for all points $x,y \in X$, $x \precsim y$ implies that $f(x) \leq f(y)$, and $x \prec y$ implies that $f(x) < f(y)$. Clearly, $\prec$ is the *strict par*t of the preorder $\precsim$.

Although an order-preserving function does not characterize a nontotal preorder, it is enough for many purposes, since it contains all the information concerning that preorder, which can be provided by a single real-valued function. So, for example, the maximization of an order-preserving function (when it is possible) for a preorder $\precsim$ on a set X leads to a *maximal element* x_0 of $(X,\precsim)$ (i.e., to a point $x_0 \in X$ such that $x_0 \prec z$ for no point $z \in X$).

It is worthwhile noticing that the mere existence of an order-preserving function for a binary relation $\precsim$ does not even imply transitivity of $\precsim$, but a weaker condition called *Suzumura consistency* (see Suzumura [61], Cato [22] and Bevilacqua et al. [8]).

Herden [33, 34, 35] introduced the concept of a *(decreasing) separable system* in order to derive very general conditions for the existence of a continuous order-preserving function for a nontotal preorder on a topological space. Actually, the notion of a decreasing separable system generalizes that of a *decreasing scale* in a *preordered topological space* (see Burgess and Fitzpatrick [20] and Johnson and Mandelker [39]).

Herden's efforts were addressed to the aim of unifying sparse but very significant results in the literature, concerning the existence of continuous order-preserving functions (*utility functions*) for total preorders. Classical examples of such results are the *Debreu's Theorem* (see Debreu [24, 25]), and respectively the *Eilenberg's Theorem* (see Eilenberg [27]), according to which every continuous total preorder on a second countable, and respectively on a connected and separable topological space admits a continuous utility representation.

An approach of this kind were actually initiated by Mehta [46, 47, 48, 49, 50, 51], who explored the possibility of recovering deep results by following the general framework of Nachbin [56]. Indeed, Nachbin first investigated in a systematic way the connection between Order and Topology. Mehta was able to establish very general conditions for the existence of a continuous *order-preserving function* for a preorder that may fail to be total on a topological space. The reader may also consult the book by Bridges and Mehta [19] for a miscellanea of theorems concerning the existence of continuous order isomorphisms. Other general results were presented, for example, by Beardon and Mehta [6, 7], and Herden and Mehta [36].

Incidentally, the concept of a *complete separable system on a topological space* has been recently used in order to present a characterization of *useful topologies* on a set X (i.e., topologies on X with respect to which all the continuous total preorders are representable by a continuous utility function). Indeed, Bosi and Herden [16, Theorem 3.1] showed that a topology t on a set X is useful if and only if the topology $t_{\mathscr{E}}$ induced (generated) by every *complete separable system* $\mathscr{E}$ on (X,t) is second countable. Other authors prefer the terminology *continuously representable* topology instead of useful topology (see e.g., Campión et al. [21]).

We recall that, from Herden [33, 34]], a family $\mathscr{E}$ of open subsets of a topological space (X,t) is said to be a *separable system on* (X,t) if there exist sets $E_1 \in \mathscr{E}$ and $E_2 \in \mathscr{E}$ such that $\overline{E_1} \subset E_2$, and for all sets $E_1 \in \mathscr{E}$ and $E_2 \in \mathscr{E}$ such that $\overline{E_1} \subset E_2$ there exists some set $E_3 \in \mathscr{E}$ such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$. If, for all sets $E \in \mathscr{E}$ and $E' \in \mathscr{E}$, at least one of the following conditions $E = E'$ or $\overline{E} \subset E'$ or $\overline{E'} \subset E$ holds, then $\mathscr{E}$ is said to be *complete* (see Bosi and Herden [16, Definition 2.2]). If X is endowed with a preorder $\precsim$, then we get the concept of a *complete decreasing separable system on a preordered topological space* $(X,\precsim,t)$ by simply requiring every set $E \in \mathscr{E}$ in the previous definition to be *decreasing*.

We recall that a preorder $\precsim$ on a topological space (X,t) is said to admit a *continuous multi-utility representation* if there exists a family $\mathscr{F}$ of (continuous) increasing real functions on the preordered topological space $(X,\precsim,t)$ such that, for all $x,y \in X$, $x \precsim y$ is equivalent to $f(x) \leq f(y)$ for all functions $f \in \mathscr{F}$. This kind of representation, whose main feature is to fully characterize the preorder, was first introduced by Levin [42], who called *functionally closed* a preorder admitting a continuous multi-utility representation on a topological space. Levin's fundamental theorem [42, 43], that using the notation of Evren and Ok [29, Theorem 1] states that every *closed preorder* (i.e., every preorder which is a closed subset of $X \times X$ with respect to the product topology $t \times t$ on $X \times X$ that is induced by t) on a locally and σ-compact Hausdorff space has a continuous multi-utility representation, still belongs to the most quoted theorems in Mathematical Utility Theory (cf. the literature that has been quoted in Bosi and Herden [14]).

In the framework of our approach, the particular relevance of closed preorders $\precsim$ on (X,t) is based, on the one hand, on the observation that a preorder $\precsim$ on (X,t) that has a continuous multi-utility representation must be closed (cf. Bosi and Herden [15, Proposition 2.1]). On the other hand, it is beyond any doubt that closed preorders are of particular interest in Mathematical Economics (cf., for instance, the literature that has been quoted by Evren and Ok [29], Bosi and Herden [14, 15], Minguzzi [53, 54] and many others). Indeed, in some standard textbooks on microeconomics (such as Mas-Colell, Whinston and Green [44, page 46]), the definitions of continuity of an (incomplete) preference relation and of a closed preference relation coincide. In combination with Proposition 2.1 in Bosi and Herden [15], these remarks on closed preorders suggest that the most fundamental problem in the theory of continuous multi-utility representations of preorders in some sense is the problem of precisely characterizing (determining) all topological spaces (Hausdorff spaces) (X,t) for which every closed preorder has a continuous multi-utility representation. A complete solution to this problem seems to be difficult. Indeed, since Levin's fundamental theorem [43], which we recalled above, no real progress towards a complete solution of the mentioned characterization problem has been made.

Levin's theorem only presents sufficient conditions for the existence of a continuous multi-utility representation. Therefore, in Bosi and Herden [15, Theorem 3.5] an effort has been made in order to also clarify up to which degree Levin's assumptions on the underlying topological space are really necessary.

Continuous multi-utility representations were first deeply studied in the framework of *Expected Utility* with incomplete preferences (see the seminal paper by Dubra et al. [26], followed by other papers like Evren [28], Galaabaatar and Karni [30] and Gorno [32]), and later they also appear in other branches of Applied Mathematics like Game Theory and Welfare Economics (see e.g., Baucells and L. S. Shapley [5], and Banerjee and Dubey [4]).

However, the first systematic study of multi-utility representations in the general case is due to Evren and Ok [29], who presented different conditions for the existence of semicontinuous and continuous multi-utility representations.

A sufficient condition for the existence of a continuous multi-utility representation is presented in Bosi and Zuanon [18], based on the concept of an *extremely continuous* preorder introduced by Mashburn [45]. Typical topologies with respect to which an upper (lower) semicontinuous multi-utility representation exists have been recently presented in Bosi et al. [10].

Ok [57] studied finite (continuous) multi-utility representation, as well as Kaminski [41] and Yilmaz [63].

Minguzzi [53, 54] introduced the concept of a *continuous Richter-Peleg multi-utility representation* $\mathscr{F}$ of a preorder $\precsim$, which is a particular kind of continuous multi-utility representation where every function $f \in \mathscr{F}$ is a *Richter-Peleg utility function* for $\precsim$ (i.e., every function $f \in \mathscr{F}$ is order-preserving). Richter-Peleg multi-utilities have been recently studied by Alcantud et al. [2], who in particular were concerned with the case of a countable representation (see also Bevilacqua et al. [9]). Although very restrictive, the case of a countable (upper semi) continuous multi-utility is particularly favorable, since it automatically implies the existence of a countable continuous Richter-Peleg multi-utility. Indeed, from Alcantud et al. [2], if there exists a countable continuous multi-utility, then there also exists a countable continuous Richter-Peleg multi-utility.

Conditions for the existence of countable multi-utilities are also found in Kabanov and Lépinette [40], and in Bevilacqua et al. [9].

Nishimura and Ok [55] generalized multi-utility, which necessarily implies transitivity, to the nontransitive case. Indeed, for a not necessarily transitive binary relation $\lesssim$ on a set X, the following representation of $\lesssim$ can be considered to hold for all points $x,y \in X$: $[x \lesssim y \Leftrightarrow \sup_{\mathscr{F}\in\mathbb{F}} \inf_{f\in\mathscr{F}} (f(y)-f(x)) \geq 0,]$, where $\mathbb{F}$ is a set of sets $\mathscr{F}$ of real-valued functions f on X. While it is extremely nice, this kind of representation appears rather difficult to be modeled, at least according to our opinion.

Bosi et al. [11] generalized multi-utility by allowing *partial functions*, in order to also deal with nontransitive preferences. The idea of using partial functions is that of avoiding any unnecessary information, and to handle both incompleteness and intransitivity in a relatively easy way. Since transitivity is removed, typical nontransitive preference relations, like *interval orders* and *semiorders*, can be represented by using families of functions.

Let (X,t) be a topological space. In order for a preorder $\precsim$ on (X,t) to be representable by a continuous order-preserving real-valued function or else to admit a continuous multi-utility representation $\mathscr{F}$, it is necessary that for every pair $(x,y) \in \prec$ there exists a continuous increasing function $f_{xy} : (X,\precsim,t) \to (\mathbb{R},\leq,t_{nat})$ such that $f_{xy}(x) < f_{xy}(y)$. Therefore, in this paper $\precsim$ is considered to be *weakly continuous* if it satisfies the just defined monotony behavior, that is obviously equivalent to requiring for every pair $(x,y) \in \prec$ to exist a complete decreasing separable system $\mathscr{E}$ on X such that for every pair $(x,y) \in \prec$ there exist sets $E \subset \overline{E} \subset E'$ in $\mathscr{E}$ such that $x \in E$ and $y \notin E'$.

We recall that *weak continuity* of a preorder on a topological space was introduced by Bosi and Herden [12, 13] in order to discuss the *continuous analogue of the Szpilrain Theorem*, i.e., the identification of conditions under which a weakly continuous preorder $\precsim$ on a topological space (X,t) admits a *refinement* $\lesssim$ by a total and continuous preorder (in the sense that $\precsim \subset \lesssim$ and $\prec \subset <$).

In this paper, we focus our attention on the *Hausdorff property* of the quotient *order topology* $t^{\precsim}_{|\sim}$, which is implied by the existence of a continuous multi-utility representation for the preorder $\precsim$ on the topological space (X,t). The quotient is considered with respect to the *equivalence relation* $\sim$.

Here, the order topology $t^{\precsim}$ induced by a preorder $\precsim$ on X is the *coarsest* topology on X with respect to which the *strict lower section* and respectively *upper section* $l_{\precsim}(x) := \{z \in X \mid z \prec x\}$ and $r_{\precsim}(x) := \{z \in X \mid x \prec z\}$ are open subsets of X for every point $x \in X$. We prove (see Proposition 2.3 below) that, when the quotient set $X_{|\sim}$ consists of at least two points, in order for $t^{\precsim}_{|\sim}$ to be a Hausdorff topology on $X_{|\sim}$, it is necessary that the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$, where x runs through X, constitute a subbasis of $t^{\precsim}$. In practice, this means that there is no point $x \in X$ which is at the same time a *minimal* and a *maximal* element for $\precsim$ on X.

Theorem 2.23 presents the equivalence of different concepts of continuity concerning a total preorder $\precsim$ on a topological space (X,t), also by using the concept of a complete decreasing separable system. The important role of weak continuity of a preorder on a topological space is illustrated. In particular, Theorem 2.27 shows that the existence of a continuous order-preserving function $f : (X,\precsim,t) \to (\mathbb{R},\leq,t_{nat})$ is equivalent to the existence of a second countable topology t' on X, which is coarser than t and with respect to which $\precsim$ is weakly continuous.

Section 3 is devoted to the characterization of the existence of a continuous order-preserving function (see Theorem 3.1), and respectively the existence of a continuous multi-utility representation (see Theorem 3.2). The existence of a continuous Richter-Peleg multi-utility representation is considered in Theorem 3.3.

In Section 4 we present restrictive conditions, implied by the existence of a continuous multi-utility representation, which pose objective limitations to its applicability. Corollary 4.3 shows that if a preorder $\precsim$ on a topological space (X,t) admits a finite continuous Richter-Peleg multi-utility representation $\mathscr{F} = \{f_1, ..., f_n\}$, then the restriction of $\precsim$ to the components of (X,t) is total. This is a version of Proposition 5.2 in Alcantud et al. [2], who, based on a famous theorem by Schmeidler [60], showed that a nontrivial preorder on a connected topological space is total provided that it admits a finite continuous Richter-Peleg multi-utility representation.

Finally, Theorem 4.8 proves that the continuous multi-utility representability of all closed preorders on a topological space is equivalent to the continuous multi-utility representability of all weakly continuous preorders, and in turn to the requirement according to which the quotient topology with respect to the coincidence of all continuous functions is discrete. This very restrictive result is based on the fact that, given a topological space (X,t), the coarsest topology with respect to which every weakly continuous preorder $\precsim$ on (X,t) remains being continuous is actually the *weak topology* $\sigma(X, C(X,t,\mathbb{R}))$ of the real-valued continuous functions on (X,t), i.e. the coarsest topology with respect to which every real-valued continuous function $f \in C(X,t,\mathbb{R})$ remains being weakly continuous (see Lemma 4.7).

2 Notation and preliminary results

2.1 Basic definitions concerning preorders and their continuous representability

Definition 2.1 A *preorder* $\precsim$ on a nonempty set X is a *reflexive* and *transitive* binary relation on X. Denote by $\prec$ and $\sim$ the *strict part* and respectively the *symmetric part* of a preorder $\precsim$ on X (i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \precsim y)$ *and not*$(y \precsim x)$, and respectively $x \sim y$ if and only if $(x \precsim y)$ *and* $(y \precsim x)$).

From time to time, we shall write "$(x,y) \in \prec$" instead of "$x \prec y$". A preorder $\precsim$ on X is said to be *nontrivial* if there exist $x, y \in X$ such that $x \prec y$.

Denote by $\bowtie$ the *incomparability relation* associated with a preorder $\precsim$ on a set X (i.e., for all $x, y \in X$, $x \bowtie y$ if and only if *not*$(x \precsim y)$ *and not*$(y \precsim x)$).

A preorder $\precsim$ is said to be *total* if, for all $x, y \in X$, either $x \precsim y$ or $y \precsim x$ (i.e., $\bowtie = \emptyset$).

Clearly, the symmetric part $\sim$ associated to any preorder $\precsim$ on X is an *equivalence relation* on X (i.e,, $\sim$ is *reflexive*, *symmetric* and *transitive*).

We denote by $\precsim_{|\sim}$ the *quotient order* on the *quotient set* $X_{|\sim}$ (i.e., for all $x,y \in X$, $[x] \precsim_{|\sim} [y]$ if and only if $x \precsim y$, where $[x] = \{z \in X : z \sim x\}$ is the *indifference class* associated to $x \in X$).

Let $(X, \precsim)$ be an arbitrarily chosen *preordered set*. We define, for every point $x \in X$, the following subsets of X:

$$d_{\precsim}(x) := \{z \in X \mid z \precsim x\},\ i_{\precsim}(x) := \{z \in X \mid x \precsim z\},$$

$$l_{\precsim}(x) := \{z \in X \mid z \prec x\},\ r_{\precsim}(x) := \{z \in X \mid x \prec z\}.$$

For any pair $(x,y) \in X \times X$ such that $(x,y) \in \prec$, we shall denote by $]x,y[$ the (maybe empty) *open interval* defined as $]x,y[:= r_{\precsim}(x) \cap l_{\precsim}(y)$.

A pair $(x,y) \in \prec$ is said to be a *jump* in $(X, \precsim)$ if $]x,y[= \emptyset$.

Let us now present the basic definition of the *order topology* corresponding to a preorder $\precsim$ on a set X.

Definition 2.2 The *order topology* $t^{\precsim}$ on X associated with a preorder $\precsim$ on X is defined to be the coarsest topology on X for which the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$ are open.

In order to avoid artificial and superfluous considerations, we can assume for the moment that the *quotient order topology* $t_{|\sim}^{\precsim_{|\sim}}$ (which in the sequel will be denoted by $t_{|\sim}^{\precsim}$ for the sake of convenience) to be a *Hausdorff topology* on $X_{|\sim}$. For underlining the importance of this assumption and for later use we still notice that in case that $X_{|\sim}$ contains at least two elements the following necessary condition for $t_{|\sim}^{\precsim}$ to be Hausdorff holds.

Indeed, the following proposition holds (see also Ward [62, Lemma 2]).

Proposition 2.3 *Let $\precsim$ be a preorder on X. Then the following assertion holds.*

SB: *In order for $t_{|\sim}^{\precsim}$ to be a Hausdorff topology on $X_{|\sim}$, it is necessary that the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$, where x runs through X, constitute a subbasis of $t^{\precsim}$.*

Proof. In order to prove the above condition **SB**, it suffices to show that

$$\bigcup_{z \in X} (l_{\precsim}(z) \cup r_{\precsim}(z)) = X.$$

Consider any point $x \in X$. Since we assume that $X_{|\sim}$ contains at least two elements, there exists some point $y \in X$ such that *not*$(y \sim x)$. The fact that $t_{|\sim}^{\precsim}$ is a Hausdorff topology on $X_{|\sim}$ implies the existence of two points $y_1, x_1 \in X$ such that either $[y] \in l_{\precsim_{|\sim}}([y_1])$, $[x] \in r_{\precsim_{|\sim}}([x_1])$, $l_{\precsim_{|\sim}}([y_1]) \cap r_{\precsim_{|\sim}}([x_1]) = \emptyset$, or else $[x] \in l_{\precsim_{|\sim}}([x_1])$, $[y] \in r_{\precsim_{|\sim}}([y_1])$, $l_{\precsim_{|\sim}}([x_1]) \cap r_{\precsim_{|\sim}}([y_1]) = \emptyset$. Hence, either $x \in r_{\precsim}(x_1)$ or $x \in l_{\precsim}(x_1)$, and the thesis follows. □

Definition 2.4 A point $x_0 \in X$ is said to be a *maximal* (*minimal*) element for a preorder $\precsim$ on a set X if $r_{\precsim}(x_0) = \emptyset$ ($l_{\precsim}(x_0) = \emptyset$).

Remark 2.5 It is immediate to check that the above condition **SB** is equivalent to the condition requiring that, for every point $x \in X$, either $l_{\precsim}(x)$ or $r_{\precsim}(x)$ is nonempty (i.e., no point $x \in X$ is at the same time a minimal and a maximal element for $\precsim$ on X).

Definition 2.6 A preorder $\precsim$ on a topological space (X,t) is said to be *continuous* if $l_{\precsim}(x) = \{z \in X \mid z \prec x\}$ and $r_{\precsim}(x) = \{z \in X \mid x \prec z\}$ are both open subsets of X for every $x \in X$.

Definition 2.7 A real-valued function u on a preordered set $(X,\precsim)$ is said to be

(i) *isotonic* or *increasing* if, for all $x,y \in X$,

$$x \precsim y \Rightarrow u(x) \leq u(y);$$

(ii) a *weak utility for* $\prec$ if, for all $x,y \in X$,

$$x \prec y \Rightarrow u(x) < u(y);$$

(iii) *strictly isotonic* or *order-preserving* if it is both increasing and a weak utility for $\prec$.

Strictly isotonic functions on $(X,\precsim)$ are also called *Richter-Peleg representations* of $\precsim$ in the economic literature (see e.g. Peleg [58] and Richter [59]).

Definition 2.8 A preorder $\precsim$ on a topological space (X,t) is said to be

(i) *closed* if it is a closed subset of $X \times X$ with respect to the product topology $t \times t$ on $X \times X$ that is induced by t;

(ii) *semi-closed* if $d_{\precsim}(x) = \{z \in X \mid z \precsim x\}$ and $i_{\precsim}(x) = \{z \in X \mid x \precsim z\}$ are both closed subsets of X for every $x \in X$;

(iii) *weakly continuous* if for every pair $(x,y) \in \prec$ there exists a continuous and increasing real-valued function f_{xy} on X such that $f_{xy}(x) < f_{xy}(y)$.

It is immediate to check that every closed preorder is also semi-closed, while the converse is not true.

Remark 2.9 We notice that for other authors, actually, a preorder is continuous if it is semi-closed (see e.g. Bridges and Mehta [19, Definition 1.6.1]). Our choice is presently suggested by our definition of the order topology (see Definition 2.2 above).

Definition 2.10 A real-valued function u on a totally preordered set $(X,\precsim)$ is said to be a *utility function* for $\precsim$ if, for all $x,y \in X$,

$$x \precsim y \Leftrightarrow u(x) \leq u(y).$$

An order-preserving function for a total preorder is necessarily a utility function. Clearly, a utility function characterizes a total preorder, while this is not the case of an order-preserving function for a nontotal preorder. On the other hand, in the general case of a nontotal preorder, an order-preserving function provides the greatest amount of information concerning the preorder which can be furnished by a real-valued function.

Definition 2.11 A preorder $\precsim$ on a topological space (X,t) is said to have a *continuous multi-utility representation* if there exists a family $\mathscr{F}$ of increasing and continuous functions $f : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that

$$\precsim = \{(x,y) \in X \times X \mid \forall f \in \mathscr{F}\,(f(x) \leq f(y))\}$$

or, equivalently, if there exists for every pair $(x,y) \in X \times X$ such that $not(y \precsim x)$ some continuous and increasing function $f_{xy} : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{xy}(x) < f_{xy}(y)$.

The above fundamental Definition 2.8, (iii), of a weakly continuous preorder on a topological space is justified by the following proposition, whose immediate proof is left to the reader.

Proposition 2.12 *Let $\precsim$ be a preorder on a topological space (X,t). If either there exists a continuous order-preserving function $f : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ or $\precsim$ admits a continuous multi-utility representation $\mathscr{F}$, then $\precsim$ is weakly continuous.*

The following proposition appears as Proposition 2.1 in Bosi and Herden [15].

Proposition 2.13 *Let $\precsim$ be a preorder on (X,t) that admits a continuous multi-utility representation. Then $\precsim$ is a closed preorder on (X,t).*

The consideration of the Hausdorff property referred to the quotient order topology is justified by the following simple proposition. Needless to say, we assume that the quotient space $X_{|\sim}$ has at least two elements.

Proposition 2.14 *Let $\precsim$ be a preorder on a topological space (X,t) and assume that $\precsim$ admits a continuous multi-utility representation. Then $(X_{|\sim}, t^{\precsim}_{|\sim})$ is a Hausdorff space.*

Proof. Let $\precsim$ have a continuous multi-utility representation and let $x \in X$ and $y \in X$ be arbitrarily chosen points such that $not(y \precsim x)$. Then there exists a continuous and increasing function $f_{xy} : (X, \precsim, t^{\precsim}) \longrightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{xy}(x) < f_{xy}(y)$. Therefore, for every real number $\alpha \in]f_{xy}(x), f_{xy}(y)[$, $f_{xy}^{-1}]-\infty, \alpha[$ and $f_{xy}^{-1}]\alpha, +\infty[$ are disjoint $t^{\precsim}_{|\sim}$-open sets containing $[x]$ and $[y]$, respectively. Hence, the desired conclusion follows. □

By putting together Proposition 2.3 and Proposition 2.14, we obviously get the following result.

Proposition 2.15 *If a preorder $\precsim$ on a topological space (X,t) admits a continuous multi-utility representation $\mathscr{F}$, then the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$, where x runs through X, constitute a subbasis of $t^{\precsim}$.*

Definition 2.16 A preorder $\precsim$ on a topological space (X,t) is said to have a *continuous Richter-Peleg multi-utility representation* if there exists a continuous multi-utility representation $\mathscr{F}$ for $\precsim$ such that every function $f \in \mathscr{F}$ is order-preserving for $\precsim$.

Remark 2.17 It has been already observed (see Alcantud et al. [2, Remark 2.3]) that a (continuous) Richter-Peleg multi-utility representation for a preorder $\precsim$ characterizes the strict part $\prec$ of $\precsim$, i.e., for all $x, y \in X$,

$$x \prec y \Leftrightarrow \forall f \in \mathscr{F}\,(f(x) < f(y)).$$

The proof of the following proposition is contained in the proof of Proposition 5.2 in Alcantud et al. [2]. We include it separately for reader's convenience and for further use.

Proposition 2.18 *If a preorder $\precsim$ on a topological space (X,t) admits a finite continuous Richter-Peleg multi-utility representation $\mathscr{F} = \{f_1, ..., f_n\}$, then $\precsim$ is continuous.*

Proof. Just consider that, due to Remark 2.17, the sets

$$\begin{aligned}
l_{\precsim}(x) &= \{z \in X \mid z \prec x\} = \{z \in X \mid f_i(z) < f_i(x), \text{ for all } i \in \{1, ..., n\}\} = \\
&= \bigcap_{i=1}^{n} f_i^{-1}(]-\infty, f_i(x)[), \\
r_{\precsim}(x) &= \{z \in X \mid x \prec z\} = \{z \in X \mid f_i(x) < f_i(z), \text{ for all } i \in \{1, ..., n\}\} = \\
&= \bigcap_{i=1}^{n} f_i^{-1}(]f_i(x), +\infty[),
\end{aligned}$$

are open for every $x \in X$ as a consequence of the continuity of every function f_i $(i = 1, ..., n)$. □

2.2 Decreasing separable systems and continuity of preorders

Definition 2.19 If $\precsim$ is a preorder on X, then a subset D of X is said to be *decreasing*, resp. *increasing*, if $d_{\precsim}(x) \subset D$, resp. $i_{\precsim}(x) \subset D$, for all $x \in D$.

Definition 2.20 (Herden [33, 34]) A family $\mathscr{E}$ of open decreasing subsets of X is said to be a *decreasing separable system on* $(X, \precsim, t)$ if it satisfies the following conditions:

DS1: There exist sets $E_1 \in \mathscr{E}$ and $E_2 \in \mathscr{E}$ such that $\overline{E_1} \subset E_2$.

DS2: For all sets $E_1 \in \mathscr{E}$ and $E_2 \in \mathscr{E}$ such that $\overline{E_1} \subset E_2$ there exists some set $E_3 \in \mathscr{E}$ such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$.

If, for all sets $E \in \mathscr{E}$ and $E' \in \mathscr{E}$, at least one of the following conditions $E = E'$ or $\overline{E} \subset E'$ or $\overline{E'} \subset E$ holds, then $\mathscr{E}$ is said to be *complete*.

The concept of a complete separable system generalizes that of a *decreasing scale* in a preordered topological space (see Burgess and Fitzpatrick [20] and Johnson and Mandelker [39]). Decreasing scales, which have been widely used for providing characterizations of the existence of continuous real-valued order-preserving functions (see , e.g., Alcantud et al. [1] and Bosi and Mehta [17]) have the disadvantage to be countable.

The reader may notice that, given a complete decreasing separable system $\mathscr{E}$ on X, the inclusion $E \subsetneqq E'$ for any two sets $E \in \mathscr{E}$ and $E' \in \mathscr{E}$ implies that $\overline{E} \subset E'$. In the remainder of this paper we shall always use this observation without extra hints.

Nevertheless, we must mention that in the arbitrary case, however, it cannot be concluded that the inclusion $E \subsetneqq E'$ implies that $\overline{E} \subset E'$. In the arbitrary case (that will not be considered here) we, therefore, replace $\mathscr{E}$ by

$$\mathscr{E}^- := \mathscr{E} \setminus \{E' \in \mathscr{E} | \exists E \in \mathscr{E} \, (E \subsetneqq E' \wedge \overline{E} \cap X \setminus E' \neq \emptyset)\}.$$

Because of the above considerations, the authors would like to take this opportunity of pointing out that it does not mean any loss of generality just to consider in the remainder of this paper complete decreasing separable systems.

Remark 2.21 The particular relevance of (complete) decreasing separable systems on $(X, \precsim, t)$ is given by the following two fundamental relations between continuous increasing real-valued functions on $(X, \precsim, t)$ and decreasing separable systems on $(X, \precsim, t)$ (cf. Herden [33] and Herden and Pallack [37, Lemma 3.6]).

1. Let f be a continuous increasing real-valued function on $(X, \precsim, t)$. Then

$$\mathscr{E} := \{f^{-1}(]-\infty, q[) \mid q \in \mathbb{Q}\}$$

is a complete decreasing separable system on $(X, \precsim, t)$.

2. Let $\mathscr{E}$ be a (complete) decreasing separable system on $(X, \precsim, t)$. Then $\mathscr{E}$ induces a continuous increasing real-valued function on $(X, \precsim, t)$ by at first defining inductively a function $q \longrightarrow E_q$ from $[0,1] \cap \mathbb{Q} \longrightarrow \mathscr{E}$ in such a way that $\overline{E_p} \subset E_q$ whenever $p < q$, in order to then define a continuous increasing real-valued function f on $(X, \precsim, t)$ by setting

$$f(x) := \begin{cases} \inf\{q \in [0,1] \cap \mathbb{Q} \mid x \in E_q\}, \text{ if } x \in \bigcup_{q \in [0,1] \cap \mathbb{Q}} E_q, \\ 1, \qquad \text{otherwise,} \end{cases}$$

for all $x \in X$.

Remark 2.22 In case that $\precsim$ is the identity relation on X we merely speak of a separable system on X. Then the conditions **DS1** and **DS2** are abbreviated by **S1** and **S2**.

The following theorem holds, presenting different conditions all equivalent to the continuity of a total preorder on a topological space.

Theorem 2.23 *Let $(X, \precsim, t)$ be a totally preordered topological space. Then the following conditions are equivalent:*

1. *$\precsim$ is continuous;*
2. *The order topology $t^{\precsim}$ is coarser than t;*
3. *$\precsim$ is semi-closed;*
4. *$\precsim$ is closed;*
5. *For all $x, y \in X$ such that not$(y \precsim x)$ there exist an open decreasing subset U_x of X containing x and an open increasing subset U_y of X containing y such that $U_x \cap U_y = \emptyset$;*
6. *$\prec$ is an open subset of $X \times X$ considered with the product topology $t \times t$;*
7. *$d_{\precsim}(x) = \{y \in X | y \precsim x\}$ is a closed subset of X and $l_{\precsim}(x) = \{y \in X | y \prec x\}$ is an open subset of X for every point $x \in X$;*
8. *$i_{\precsim}(x) = \{z \in X | x \precsim z\}$ is a closed subset of X and $r_{\precsim}(x) = \{z \in X | x \prec z\}$ is an open subset of X for every point $x \in X$;*
9. *$\precsim$ is weakly continuous;*
10. *For every pair $(x, y) \in \prec$ a decreasing separable system $\mathscr{E}_{xy}$ on X can be chosen in such a way that there exist sets $E \subset \overline{E} \subset E'$ in $\mathscr{E}_{xy}$ such that $x \in E$ and $y \notin E'$;*
11. *For every pair$(x, y) \in \prec$ a complete decreasing separable system $\mathscr{E}_{xy}$ on X can be chosen in such a way that there exist sets $E \subset \overline{E} \subset E'$ in $\mathscr{E}_{xy}$ such that $x \in E$ and $y \notin E'$.*

Proof. The equivalence of conditions 1, 2 ,3, 4, 6 was proved by Bridges and Mehta [19, Proposition 1.6.2]. The equivalence of conditions 4 and 5 appears in Ward [62, Lemma 1]. The equivalence of conditions 3, 7 and 8 is obvious. The equivalence of conditions 3 and 9 was proved by Herden and Pallack [38, Lemma 2.2]. Finally, the equivalence of conditions 9, 10 and 11 comes from Remark 2.21. □

Let $\mathbb{S}_C(X)$ be the set of all complete separable systems $\mathscr{E}$ on X that contain X (the reader may verify that the assumption X to be contained in $\mathscr{E}$ does not mean any loss of generality). In Bosi and Herden [16], $\mathbb{S}_C(X)$ has been endowed with the preorder $\precsim_{\mathbb{S}}$ that is defined by considering for every complete separable system $\mathscr{E} \in \mathbb{S}_C(X)$ the topology $t_{\mathscr{E}}$ on X that is generated by $\mathscr{E}$ (i.e., $\mathscr{E}$ is a subbasis of $t_{\mathscr{E}}$), in order to then set

$$\mathscr{E} \precsim_{\mathbb{S}} \mathscr{L} \Leftrightarrow t_{\mathscr{E}} \subset t_{\mathscr{L}}$$

for all complete separable systems $\mathscr{E} \in \mathbb{S}_C(X)$ and $\mathscr{L} \in \mathbb{S}_C(X)$.

Let $\mathbb{S}_C(X)_{\sim_{\mathbb{S}}}$ be the set of indifference (equivalence) classes of $\precsim_{\mathbb{S}}$ and let, in addition, $\mathbb{P}_{\trianglelefteq}(X)$ be the set of all total continuous preorders on (X,t).

Then the following fundamental proposition holds (cf. Bosi and Herden [16, Proposition 3.2]).

Proposition 2.24 *There exists a one-to-one correspondence between* $\mathbb{P}_{\trianglelefteq}(X)$ *and* $\mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}$.

Proposition 2.24 leads us immediately to a first solution of the problem of characterizing in a simple way all *useful* (*continuously representable*) topologies on X, i. e. all topologies on X having the property according to which all their total continuous preorders are continuously representable by a utility function (cf. Bosi and Herden [16, Theorem 3.1] and Herden [35, Corollary 2.1]).

Proposition 2.25 *The following assertions are equivalent:*

(i) *t is useful.*

(ii) *For every complete separable system $\mathscr{E} \in \mathbb{S}_C(X)$, the topology $t_{\mathscr{E}}$ generated by $\mathscr{E}$ is second countable.*

One immediately verifies that Proposition 2.25 is a common generalization of **ET** (Eilenberg utility representation Theorem (Eilenberg [27])) and **DT** (Debreu utility representation Theorem (Debreu [24, 25])), which read as follows.

ET: *Every connected and separable topology t on X is useful.*

DT: *Every second countable topology t on X is useful.*

Herden and Pallack [38, Theorem 2.15] proved the following generalization of the Debreu utility representation Theorem.

Theorem 2.26 *Let $\precsim$ be a weakly continuous preorder on a second countable topological space (X,t). Then there exists a continuous order-preserving function* $f : (X,\precsim,t) \to (\mathbb{R},\leq,t_{nat})$.

Based on Theorem 2.26, we can present a characterization of the existence of a continuous order-preserving function, which utilizes weak continuity.

Theorem 2.27 *Let $\precsim$ be a preorder on a topological space (X,t). Then the following conditions are equivalent:*

(i) *There exists a continuous order-preserving function* $f : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$.

(ii) *There exists a second countable topology* t' *on* X*, which is coarser than* t *and with respect to which* $\precsim$ *is weakly continuous.*

Proof. (i) $\Rightarrow$ (ii). Let $f : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$ be a continuous order-preserving function, and consider the topology t' on X which is generated by the families $\{f^{-1}(]-\infty, q[)\}_{q \in \mathbb{Q}}$ and $\{f^{-1}(]q, +\infty[)\}_{q \in \mathbb{Q}}$, in order to immediately verify that t' is second countable, coarser than t and such that $\precsim$ is weakly continuous with respect to t'.

(ii) $\Rightarrow$ (i). Apply Theorem 2.26 to $\precsim$ on (X, t'), and just observe that a continuous order-preserving function $f : (X, \precsim, t') \to (\mathbb{R}, \leq, t_{nat})$ is continuous on (X, t). □

Definition 2.28 A preorder $\precsim$ on a set X is said to be *Cantor-separable* if there exists a countable subset Z of X such that for all $(x, y) \in \prec$ there exists $z \in Z$ such that $x \prec z \prec y$.

The following theorem, which is closely related to the utility representation theorem of Peleg [58], illustrates the adequateness of the definition of weak continuity of a preorder (see the above Definition 2.8, (iii)), which was introduced by Bosi and Herden [12, 13]).

By the way, the reader may recall that Peleg [58], who was one of the first concerned with continuous representability of arbitrary preorders (actually, irreflexive and transitive binary relations) on (X, t) instead of only total preorders, when proving his continuous utility theorem, has taken advantage of the fact that a Cantor-separable preorder $\precsim$ on (X, t) is weakly continuous, provided that it satisfies the following properties:

P1: $l_{\precsim}(x)$ is open for every $x \in X$;

P2: $\overline{l_{\precsim}(x)} \subset l_{\precsim}(y)$ for every pair $(x, y) \in \prec$.

The reader may also consult Herden [33, Remark 4.1].

Proposition 2.29 *The following assertions hold:*

(i) *A Cantor-separable preorder* $\precsim$ *on* (X, t) *or, equivalently, a preorder* $\precsim$ *on* (X, t) *that has no jumps and which satisfies one of the conditions 7 or 8 of Theorem 2.23 is weakly continuous.*

(ii) *A preorder* $\precsim$ *on* (X, t) *is weakly continuous provided that it is both continuous and semi-closed.*

Proof. A proof of assertion (i) is implicit in our proof of assertion (ii). In addition, the original proof of Peleg also can be applied in order to prove assertion (i). Hence, it suffices to concentrate on the proof of assertion (ii). Let, therefore, some pair $(x, y) \in \prec$ be arbitrarily chosen. Then the continuity of $\precsim$ will follow if we are able to prove the existence of a decreasing separable system $\mathscr{E}_{xy}$ on X such that $x \in E$ and $y \notin E$ for every set $E \in \mathscr{E}_{xy}$. We, thus, set $\mathscr{E}_{xy} := \{l_{\precsim}(z) | x \prec z \precsim y\}$ in order to then show that $\mathscr{E}_{xy}$ is the desired decreasing separable system on X satisfying the property that $x \in E$ and $y \notin E$ for every set $E \in \mathscr{E}_{xy}$. The definition of $\mathscr{E}_{xy}$ implies that we only must show $\mathscr{E}_{xy}$ to be a decreasing separable system on X. Hence, we distinguish between the following two cases.

Case 1: There exists some $z \in X$ such that $x \prec z \prec y$. Then we may conclude that $l_{\precsim}(z) \subset \overline{l_{\precsim}(z)} \subset d_{\precsim}(z) \subset l_{\precsim}(y)$ and the validity of condition **DS1** is guaranteed.

Case 2: There exists no $z \in X$ such that $x \prec z \prec y$. In this case $r_{\precsim}(x) \cup l_{\precsim}(y) = i_{\precsim}(x) \cup d_{\precsim}(y)$ is an open and closed subset of X (of course, the equation $r_{\precsim}(x) \cup l_{\precsim}(y) = i_{\precsim}(x) \cup d_{\precsim}(y)$ holds whenever $x \prec y$; the emptiness of the interval $]x,y[$ is not needed). But since $]x,y[= \emptyset$ we may conclude that $l_{\precsim}(y) = (i_{\precsim}(x) \cup d_{\precsim}(y)) \setminus r_{\precsim}(x)$ is open and closed, which also implies the validity of condition **DS1**.

In order to now finish the proof of the proposition take sets $l_{\precsim}(u) \in \mathscr{E}_{xy}$ and $l_{\precsim}(v) \in \mathscr{E}_{xy}$ such that $\overline{l_{\precsim}(u)} \subset l_{\precsim}(v)$ be arbitrarily chosen. Our arguments that have been used in order to do the second case allow us to assume without loss of generality that there exists some $w \in X$ such that $u \prec w \prec v$. Hence, our arguments that just have been used above apply. This statement already completes the proof of the proposition. $\square$

We must still mention that the concept of a continuous preorder on (X,t) up to now, however, has not completely been clarified in the literature. Indeed, some authors, in particular, Mas-Colell et al. [44] or Gerasimou [31] identify continuity of a preorder $\precsim$ on (X,t) with its closedness.

One immediately verifies (see the equivalence "4 $\Leftrightarrow$ 9" of Theorem 2.23), that in case that $\precsim$ is total, $\precsim$ is closed if and only if $\precsim$ is weakly continuous. But weak continuity of $\precsim$ and the property $\precsim$ to be a closed subset of the product space $(X \times X, t \times t)$ are not equivalent, in general, as the following example shows.

The following example of a closed and not weakly continuous preorder was presented by Herden and Pallack [38].

Example 2.30 Let $X := \mathbb{R}$. Then we consider the topology t on X that contains the empty set and the sets $X \setminus F$ where F runs through the empty set and all finite subsets of X. In addition, we choose the preorder

$$\precsim := \{(x,x) | x \in \mathbb{R}\} \cup \{1,2\}.$$

Then $\precsim$ is obviously a closed subset of the product space $(X \times X, t \times t)$. Since the intersection of any two open set in nonempty, we have that every continuous real-valued function f on (X,t) is constant. Therefore, there cannot exist any continuous function $f_{12} : (X, \precsim, t)) \rightarrow (\mathbb{R}, \leq, t_{nat})$ such that $f_{12}(1) < f_{12}(2)$, and this implies that $\precsim$ is not weakly continuous.

Remark 2.31 A slight extension of Schmeidler's proof in his well-known paper published in 1971 (see Schmeidler [60]) implies a variant of his theorem, which guarantees that a preorder $\precsim$ which is both continuous and semi-closed on a topological space (X,t) and which satisfies the additional property of the quotient topology $t_{|\sim}^{\precsim}$ being Hausdorf is such that the restriction of $\precsim$ to the components of (X,t) is total. The proof of this version of Schmeidler theorem will be completely presented in the last section of this paper, In opinion of the authors, this theorem underlines the relative strength of requiring both continuity and semi-closedness.

Despite Remark 2.31, we have that assertion (ii) of Proposition 2.29 cannot be improved. Indeed, we now present an example of a preorder $\precsim$ on (X,t) which satisfies, except for the assumption $r_{\precsim}(x)$ to be open for every point $x \in X$, any of the assumptions the validity of which is postulated by continuity and semi-closedness, but, nevertheless, fails to be continuous.

Example 2.32 Let $(X,t) := (\mathbb{R}, t_{nat})$ as underlying topological space. The preorder $\precsim$ to be considered on (X,t) is defined by setting

$$\begin{aligned}\precsim &:= \{(x,x) | x \in X\} \cup \{(x,y) \in X \times X | x \leq y \leq 0 \vee 1 \leq x \leq y \vee x \leq 0 \wedge 1 \leq y\} \\ &\cup \ \{(z,y) \in X \times X | 0 < z < y \wedge 1 \leq y\}.\end{aligned}$$

A direct verification implies that for every $x \in X$ both sets $d_{\precsim}(x)$ and $i_{\precsim}(x)$ are closed. In addition, it follows that $l_{\precsim}(x)$ is open for every $x \in X$. Let us now assume that there exists some complete decreasing separable system $\mathscr{E}_{01}$ on X that could be chosen in such a way that $0 \in E$ and $1 \notin E$ for every set $E \in \mathscr{E}_{01}$. Then the assumption every set $E \in \mathscr{E}_{01}$ to be the union of sets $l_{\precsim}(z)$ $(0 < z \leq 1)$ does not mean any loss of generality. Since the interval $]0,1[$ is empty our arguments that have been applied in the second case of the proof of Proposition 2.29 imply that in order for $\mathscr{E}_{01}$ to satisfy condition **DS1** it is necessary $l_{\precsim}(1)$ to be closed. But $l_{\precsim}(1) \subsetneqq \overline{l_{\precsim}(1)} = d_{\precsim}(1)$ and we are done.

3 Characterization of continuous representations

Based on the concept of a complete decreasing separable system, the observations following Definition 2.20, and the above Remark 2.21, we can easily deduce the following variant of Theorem 3.2 in Herden and Pallack [38], providing a characterization of the existence of a continuous order-preserving function for a not necessarily total preorder on a topological space.

Theorem 3.1 *let $\precsim$ be a preorder on a topological space (X,t). Then the following conditions are equivalent:*

(i) *There exists a continuous order-preserving function $f : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$.*

(ii) *There exists a countable complete decreasing separable system $\mathscr{E}$ on $(X, \precsim, t)$ such that for every pair $(x,y) \in \prec$ there exist sets $E, E' \in \mathscr{E}$ such that $\overline{E} \subset E'$, $x \in E$ and $y \in X \setminus E'$.*

(iii) *There exists a countable family* $\{\mathscr{E}_n\}_{n\in\mathbb{N}}$ *of complete decreasing separable systems on* $(X,\precsim,t)$ *such that for every pair* $(x,y)\in\prec$ *there exists some* $n\in\mathbb{N}$ *such that* $x\in E$ *and* $y\in X\setminus E$ *for every* $E\in\mathscr{E}_n$.

We are now interested in continuous multi-utility representation. For every $x\in X$ we denote by $\mathscr{E}(x)$ the family of all (complete) decreasing separable systems $\mathscr{E}$ on $(X,\precsim,t)$ such that $x\in\bigcap_{E\in\mathscr{E}}E$ for all $\mathscr{E}\in\mathscr{E}(x)$.

We are fully prepared for solving the problem of characterizing all preorders $\precsim$ on (X,t) that have a continuous multi-utility representation (cf. also Evren and Ok [29, Theorem 0 and Theorem 4], where the equivalence "(i) $\Leftrightarrow$ (ii)" has been proved).

Theorem 3.2 *Let* $\precsim$ *be a preorder on a topological space* (X,t)*. Then the following assertions are equivalent:*

(i) $\precsim$ *has a continuous multi-utility representation.*

(ii) $x\precsim y$ *whenever for all* $\mathscr{E}\in\mathscr{E}(y)$ *and all* $E\in\mathscr{E}$ *the inclusion* $x\in E$ *holds.*

(iii) *For every pair* $(x,y)\in X\times X$ *such that not*$(y\precsim x)$ *there exists some* $\mathscr{E}\in\mathscr{E}(x)$ *such that* $x\in E$ *and* $y\in X\setminus\overline{E}$ *for all* $E\in\mathscr{E}$.

(iv) $d_{\precsim}(x)=\bigcap_{\mathscr{E}\in\mathscr{E}(x)}\bigcap_{E\in\mathscr{E}}E$ *for all* $x\in X$.

Proof. (i) $\Rightarrow$ (ii): Assertion (ii) means that $x\precsim y$, whenever $f(x)\leq f(y)$ for all continuous increasing real-valued functions f on $(X,\precsim,t)$. This observation already guarantees the validity of the implication "(i) $\Rightarrow$ (ii)".

(ii) $\Rightarrow$ (iii): Since *not*$(y\precsim x)$, assertion (ii) implies the existence of some decreasing separable system $\mathscr{E}\in\mathscr{E}(x)$ for which there exists some $E\in\mathscr{E}$ such that $x\in E$ and $y\in X\setminus E$. Therefore, we distinguish between the following two cases.

Case 1: $E=\bigcap_{E'\in\mathscr{E}}E'$. In this case the equation $\bigcap_{E'\in\mathscr{E}}E'=\bigcap_{E'\in\mathscr{E}}\overline{E'}$ implies that $E=\overline{E}$. Then $\{E\}$ is a separable system on $(X,\precsim,t)$ that belongs to $\mathscr{E}(x)$ and has the desired property that $x\in E$ and $y\in X\setminus\overline{E}$ for all $E\in\{E\}$.

Case 2: There exists some $E'\in\mathscr{E}$ such that $\overline{E'}\subsetneqq E$. Now $\mathscr{E}'=\{E''\in\mathscr{E}\mid\overline{E''}\subset E'\}$ is a separable system on $(X,\precsim,t)$ that belongs to $\mathscr{E}(x)$ and has the desired property that $x\in E''$ and $y\in X\setminus\overline{E''}$ for all $E''\in\mathscr{E}'$.

(iii) $\Rightarrow$ (iv): It is clear that, regardless the validity of condition (iii), we have that $d_{\precsim}(x)\subset\bigcap_{\mathscr{E}\in\mathscr{E}(x)}\bigcap_{E\in\mathscr{E}}E$ for all $x\in X$. Conversely, if for some $x\in X$ and $z\in X$ it happens that $z\notin d_{\precsim}(x)\Leftrightarrow$ *not*$(z\precsim x)$, then from condition (iii) there exists some $\mathscr{E}\in\mathscr{E}(x)$ such that $x\in E$ and $z\in X\setminus\overline{E}$ for all $E\in\mathscr{E}$, so that $z\notin\bigcap_{\mathscr{E}\in\mathscr{E}(x)}\bigcap_{E\in\mathscr{E}}E$.

(iv) $\Rightarrow$ (i): In order to prove that $\precsim$ admits a continuous multi-utility representation, we may concentrate on the situation that points $x\in X$ and $y\in X$ such that *not*$(y \precsim x)$ or, equivalently, $y \notin d_{\precsim}(x)$ have been chosen. Then similar arguments as the ones that have been applied in the proof of the implication "(ii) $\Rightarrow$ (iii)" guarantee the existence of some separable system $\mathscr{E} = \{E_q\}_{q \in [0,1] \cap \mathbb{Q}}$ on $(X, \precsim, t)$ such that $x \in \bigcap_{q \in [0,1] \cap \mathbb{Q}} E_q$ and $y \in X \setminus \bigcup_{q \in [0,1] \cap \mathbb{Q}} E_q = X \setminus \bigcup_{q \in [0,1] \cap \mathbb{Q}} \overline{E_q}$. With the help of these equations we may define a continuous decreasing function $f : (X, \precsim, t) \longrightarrow ([0,1], \leq, t_{nat})$ by setting

$$f(z) := \begin{cases} \inf\{q \in [0,1] \cap \mathbb{Q} \mid z \in E_q\}, \text{ if } z \in \bigcup_{q \in [0,1] \cap \mathbb{Q}} E_q, \\ 1, \qquad \text{otherwise,} \end{cases}$$

for all $z \in X$. The definition of f implies that $f(x) = 0 < 1 = f(y)$. This consideration finishes the proof of the theorem. □

Alcantud et al. [2, Proposition 3.2] proved that there exists a continuous Richter-Peleg multi-utility representation for a preorder on a topological space if and only if there exist both a continuous multi-utility representation and a continuous order-preserving function for the preorder. Therefore, from Theorem 3.1 and Theorem 3.2, we immediately get the following characterization of the existence of a continuous Richter-Peleg multi-utility representation.

Theorem 3.3 *Let $\precsim$ be a preorder on a topological space (X,t). Then the following assertions are equivalent:*

(i) *$\precsim$ has a continuous Richter-Peleg multi-utility representation.*

(ii) *The following conditions hold:*

a. *For every pair $(x,y) \in X \times X$ such that not$(y \precsim x)$ there exists some $\mathscr{E} \in \mathscr{E}(x)$ such that $x \in E$ and $y \in X \setminus \overline{E}$ for all $E \in \mathscr{E}$;*

b. *There exists a countable complete decreasing separable system $\mathscr{E}$ on $(X, \precsim, t)$ such that for every pair $(x,y) \in \prec$ there exists sets $E, E' \in \mathscr{E}$ such that $\overline{E} \subset E'$, $x \in E$ and $y \in X \setminus E'$.*

4 Restrictive results concerning continuous multi-utility representations

Schmeidler [60] proved the following famous theorem.

Theorem 4.1 (Schmeidler [60]) *Let $\precsim$ be a nontrivial preorder on a connected topological space (X,t). If, for every $x \in X$, the sets $d_{\precsim}(x)$ and $i_{\precsim}(x)$ are closed and the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$ are open, then the preorder $\precsim$ is total.*

In combination with the results of the preceding section, in this section we first want to present the following more general version of Schmeidler's Theorem.

Theorem 4.2 *Let $(X,\precsim,t)$ be a preordered topological space that satisfies the properties $t^{\precsim}_{|\sim}$ to be Hausdorff, and $\precsim$ to be both continuous and semi-closed. Then the restriction of $\precsim$ to the components of (X,t) is total.*

Proof. Let $C \subset X$ be a component of (X,t) and let the point $x \in C$ be arbitrarily chosen. Since $t^{\precsim}_{|\sim}$ is Hausdorff it follows, with help of condition **SB** of Proposition 2.3, that at least one of the sets $l_{\precsim}(x)$ or $r_{\precsim}(x)$ is not empty. We, thus, must distinguish between the cases $l_{\precsim}(x) \neq \emptyset$ and $r_{\precsim}(x) = \emptyset$, $l_{\precsim}(x) = \emptyset$ and $r_{\precsim}(x) \neq \emptyset$ and $l_{\precsim}(x) \neq \emptyset$ and $r_{\precsim}(x) \neq \emptyset$. Since all these cases can be handled by analogous arguments it suffices to discuss the case that $l_{\precsim}(x)$ as well as $r_{\precsim}(x)$ is not empty. Let, therefore, $\mathscr{O}(x)$ be the collection of all open intervals $]y,z[$ that contain x. Then we set $O_x := \bigcup_{]y,z[\in \mathscr{O}(x)}]y,z[$ in order to distinguish between the cases $C \cap O_x = \emptyset$ and $C \cap O_x \neq \emptyset$. In the first case we may conclude that $C = [x]$ and we are done. In the second case it follows that there exists some point $y \in C$ or some point $z \in C$ such that $y \prec x$ or $x \prec z$. Since $\precsim$ is assumed to be semi-closed these inequalities guarantee, however, that the sub-space $(C, \precsim_{|C}, t_{|C})$ of $(X,\precsim,t)$ satisfies the assumptions of Schmeidler's Theorem. Hence, the restriction $\precsim_{|C}$ of $\precsim$ to C is total. □

From Proposition 2.18 and Theorem 4.2, we immediately obtain the following restrictive result concerning the existence of finite continuous Richter-Peleg multi-utility representations.

Corollary 4.3 *Let $\precsim$ be a preorder on a topological space (X,t), which admits a finite continuous Richter-Peleg multi-utility representation $\mathscr{F} = \{f_1,...,f_n\}$. Then the restriction of $\precsim$ to the components of (X,t) is total.*

A preorder $\precsim$ on (X,t) which has a continuous multi-utility representation must be both closed and continuous. The following three problems are therefore particularly important.

Problem 1: Determine all topological spaces (X,t) having the property that all their closed preorders are weakly continuous.

Problem 2: Determine all topological spaces (X,t) having the property that all their weakly continuous preorders are closed.

Problem 3: Determine all topological spaces (X,t) having the property that all their weakly continuous preorders admit a continuous multi-utility representation.

The first problem has been analyzed, at least partially, in Bosi and Herden [15]. The following definition is found in Nachbin [56].

Definition 4.4 A preorder $\precsim$ on a topological space (X,t) is said to be *normal* if for any two disjoint closed decreasing, respectively increasing subsets A and B of X there exist disjoint open decreasing, respectively increasing subsets U and V of X such that $A \subset U$ and $B \subset V$.

For example, the following results hold (see Bosi and Herden [15, Corollary 3.5 and Theorem 3.4]).

Theorem 4.5 *Let (X,t) be a connected metrizable space. Then the following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on (X,t) admits a continuous multi-utility representation.*

(ii) *(X,t) is locally compact and second countable.*

Theorem 4.6 *Let (X,t) be a Hausdorff space. Then the following assertions are equivalent:*

(i) *Every closed preorder $\precsim$ on (X,t) admits a continuous multi-utility representation.*

(ii) *Every closed preorder $\precsim$ on (X,t) is normal.*

As regards problems 2 and 3 above, surprisingly, the common solution of both problems is possible in a very satisfactory and restrictive way.

Before stating the corresponding theorem, the reader may recall that, when we consider the space $C(X,t,\mathbb{R})$ of all continuous real-valued function on the topological space (X,t), the *weak topology* on X, $\sigma(X,C(X,t,\mathbb{R}))$, is the coarsest topology on X satisfying the property that every continuous real-valued function on (X,t) remains being continuous. Two points $x,y \in X$ are considered as being *equivalent* if $f(x) = f(y)$ for all functions $f \in C(X,t,\mathbb{R})$. For two equivalent points $x,y \in X$, we write $x \sim_{C(X,t,\mathbb{R})} y$ ($x \sim_C y$ for the sake of brevity).

It is well known that $(X_{|\sim_C}, \sigma(X,C(X,t,\mathbb{R}))_{|\sim_C})$ is a completely regular Hausdorff-space (cf., for instance, Cigler and Reichel [23, Satz 10, page 101]). It is clear that $(X_{|\sim_C}, \sigma(X,C(X,t,\mathbb{R}))_{|\sim_C})$ is the quotient space of $\sigma(X,C(X,t,\mathbb{R}))$ that is induced by the equivalence relation $\sim_C$.

The following lemma holds true.

Lemma 4.7 *The coarsest topology on X satisfying the property that all weakly continuous preorders on (X,t) remain being continuous is $\sigma(X,C(X,t,\mathbb{R}))$. (Of course, this assertion is equivalent to the statement that a preorder $\precsim$ on (X,t) is weakly continuous if and only if it is weakly continuous with respect to $\sigma(X,C(X,t,\mathbb{R}))$).*

Proof. Although the validity of this assertion appears somewhat surprisingly, its trueness is trivial. Indeed, since weak continuity of a preorder $\precsim$ on (X,t) is described by continuous (increasing) real-valued functions, its validity is immediate (the reader may notice that this phenomenon underlines once more the appropriateness of the concept of a weakly continuous preorder on (X,t)). □

We are now ready to prove the very restrictive result according to which the continuous multi-utility representability of all closed preorders on a topological space is equivalent to the continuous multi-utility representability of all weakly continuous preorders, and in turn to the requirement according to which the quotient topology with respect to the coincidence of all continuous functions is discrete

Theorem 4.8 *Let (X,t) be a topological space. The following assertions are equivalent:*

(i) *Every weakly continuous preorder $\precsim$ on (X,t) has a continuous multi-utility representation.*

(ii) *Every weakly continuous preorder $\precsim$ on (X,t) is closed.*

(iii) *$t_{|\sim_C}$ is the discrete topology on $X_{|\sim_C}$.*

Proof. (i) $\Rightarrow$ (ii): We already know that a preorder $\precsim$ on (X,t) that has a continuous multi-utility representation must be closed. Hence, nothing remains to be shown.

(ii) $\Rightarrow$ (iii): Let (X,t) be a topological space for which every weakly continuous preorder $\precsim$ is closed. Then the properties of the defined equivalence relation "$\sim_C$" on X imply that every weakly continuous preorder $\precsim$ on $X_{|\sim_C}$ is closed. Therefore, we may identify the topological spaces (X,t) and $(X_{|\sim_C}, t_{|\sim_C})$. This means that we may assume, in the remainder of the proof of the implication "(ii) $\Rightarrow$ (iii)", (X,t) to be a Hausdorff-space. In order to now prove the validity of the implication it, thus, suffices to show that there exists no point $y \in X$ such that the singleton $\{y\}$ is not an open subset of X. This will be done by contraposition. We, therefore, assume, in contrast, that there exists at least one point $y \in X$ such that $\{y\}$ is not an open subset of X and proceed by arbitrarily choosing some point $z \in X \setminus \{y\}$. Since (X,t) is a Hausdorff-space, it follows that $\{y\}$ as well as $\{z\}$ are closed subsets of X, which implies that $D := \{y,z\}$ is a closed subset of X. The inclusion $(X, \sigma(X, C(X,t,\mathbb{R})) \subset (X,t)$ allows to conclude, in addition, that every continuous function $f : (X, \sigma(X, C(X,t,\mathbb{R})) \rightarrow ([0,1], t_{nat})$ is a continuous function $f : (X,t) \rightarrow ([0,1], t_{nat})$. Since $(X, \sigma(X, C(X,t,\mathbb{R}))$ is completely regular, there exists for every point $x \in X \setminus D$ a continuous function $f_x : (X,t) \rightarrow ([0,1], t_{nat})$ such that $f_x(x) = 0$ and $f_x(D) = \{1\}$. Hence, the preorder $\precsim$ on (X,t) that is defined by setting

$$\precsim := \{(v,v) | v \in X\} \cup \{(x,y) | x \in X \setminus D\} \cup \{(x,z) | x \in X \setminus D\}$$

is continuous. Assertion (ii), thus, implies that $\precsim$ is a closed subset of $X \times X$. Hence, it follows that there exist open subsets U and V of X such that $y \in U$ and $z \in V$ and, moreover, $U \times V \cap \precsim = \emptyset$. Since y is assumed to be not an open subset of X, we may conclude that U contains at least one point u that is different from y. Because of the definition of $\precsim$, this means, however, that $(u,z) \in U \times V \cap \precsim$. This contradiction proves assertion (iii).

(iii) $\Rightarrow$ (i): Assertion (iii) implies that $\sigma(X,C(X,t,\mathbb{R}))_{|\sim_C} = t_{|\sim_C}$. Hence, the space $(X_{\sim_C}, \sigma(X,C(X,t,\mathbb{R}))_{|\sim_C})$ is discrete, which allows us to conclude that every weakly continuous preorder $\precsim$ on $(X_{\sim_C}, \sigma(X,C(X,t,\mathbb{R}))_{|\sim_C})$ admits a continuous multi-utility representation. It, thus, follows that every weakly continuous preorder $\precsim$ on $(X,\sigma(X,C(X,t,\mathbb{R}))$ has a continuous multi-utility representation. This means that we may apply Lemma 4.7 in order to conclude that also every weakly continuous preorder $\precsim$ on (X,t) admits a continuous multi-utility representation, which finishes the proof of the implication. □

Remark 4.9 Clearly, the equivalent assertions (i), (ii) and (iii) of Theorem 4.8 are also equivalent to any of the following (equivalent) assertions:

(iv) Every weakly continuous preorder $\precsim$ on $(X,\sigma(X,C(X,t,\mathbb{R}))$ has a continuous multi-utility representation;

(v) Every weakly continuous preorder $\precsim$ on $(X,\sigma(X,C(X,t,\mathbb{R}))$ is closed;

(vi) $\sigma(X,C(X,t,\mathbb{R}))_{\sim_C}$ is the discrete topology on $X_{\sim_C}$.

5 Conclusions

In this paper we have presented some general results concerning the existence of continuous representations of nontotal preorders on a topological space. The corresponding characterizations are mainly based on the concept of a *complete decreasing separable system* in a preordered topological space, which was introduced and widely studied by Herden [33, 34, 35].

We have focused our attention on continuity-like conditions which are necessary for the existence of a continuous order-preserving function and respectively a continuous multi-utility representation.

In particular, we have taken into consideration the property of *weak continuity*. Following the terminology introduced by Bosi and Herden [12, 13], a preorder on a topological space (X,t) is weakly continuous if for every pair $(x,y) \in \prec$ there exists a continuous and increasing real-valued function f_{xy} on X such that $f_{xy}(x) < f_{xy}(y)$.

We have presented some results which illustrate the restrictiveness of the continuous multi-utility representation, which nevertheless has been presented in the past as the best kind of continuous representation under incompleteness of the preference relation. To be precise, by using considerations according to which the quotient order topology is a Hausdorff topology, we have proven a variant of a famous theorem by Schmeidler [60]. Indeed, we have shown that if a continuous multi-utility representation exists for a preorder whose strict lower and upper sections are all open, then the preorder is total on each component. Further, we have proven that if a finite continuous Richter-Peleg multi-utility representation exists, then the preorder is total on every component.

Finally, using classical considerations concerning the *weak topology*, we have shown that the continuous multi-utility representability of all closed preorders (or equivalently weakly continuous preorders) on a topological space is equivalent to the requirement according to which the quotient topology with respect to the equivalence corresponding to the coincidence of all continuous functions is discrete.

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