Positive solutions of superlinear indefinite prescribed mean curvature problems

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Dedicated, with friendship and esteem, to Julián López-Gómez on the occasion of his 60th birthday

Abstract. This paper analyzes the superlinear indefinite prescribed mean curvature problem

$$-\mathrm{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = \lambda a(x)h(u) \quad \text{in } \Omega, \qquad u=0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a regular boundary $\partial\Omega$, $h \in C^0(\mathbb{R})$ satisfies $h(s) \sim s^p$, as $s \to 0^+$, $p>1$ being an exponent with $p < \frac{N+2}{N-2}$ if $N \geq 3$, $\lambda > 0$ represents a parameter, and $a \in C^0(\overline{\Omega})$ is a sign-changing function. The main result establishes the existence of positive regular solutions when λ is sufficiently large, providing as well some information on the structure of the solution set. The existence of positive bounded variation solutions for λ small is further discussed assuming that h satisfies $h(s) \sim s^q$ as $s \to +\infty$, $q>0$ being such that $q < \frac{1}{N-1}$ if $N \geq 2$; thus, in dimension $N \geq 2$, the function h is not superlinear at $+\infty$, although its potential $H(s) = \int_0^s h(t)\,\mathrm{d}t$ is. Imposing such different degrees of homogeneity of h at 0 and at $+\infty$ is dictated by the specific features of the mean curvature operator.

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1 Introduction

Let us consider the following superlinear indefinite prescribed mean curvature problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda a(x)h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a C^2 boundary $\partial\Omega$, $h : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $\lim_{s\to 0^+} \frac{h(s)}{s^p} = 1$, the exponent $p > 1$ being such that $p < \frac{N+2}{N-2}$ if $N \geq 3$, $\lambda > 0$ represents a

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parameter, and $a\colon \overline{\Omega} \to \mathbb{R}$ is a continuous function which changes sign in Ω. Although in this work we are mainly concerned with the case $N \geq 2$, all our conclusions still hold if $N = 1$, even under weaker assumptions.

Elliptic boundary value problems involving the mean curvature operator play a pivotal role in the mathematical analysis of several physical or geometrical issues, such as capillarity phenomena for incompressible or compressible fluids, mathematical models in physiology or in electrostatics, flux-limited diffusion phenomena, prescribed mean curvature problems for cartesian surfaces in the Euclidean space: relevant references on these topics include [13, 14, 17, 18, 20, 21, 39].

In case $a \equiv 1$ a breakthrough result was obtained in [12], where the authors proved that problem (1.1) has at least one positive solution for all large $\lambda > 0$ by using a variational approach generalizing the Nehari method [32]. The discussion in [12] was motivated by some previous existence and non-existence results, obtained in [33, 40] and concerning radially symmetric solutions on balls. Conclusions similar to [12], or small extensions thereof, were later achieved in [29, 35, 36] without requiring that a is constant, so allowing spatial heterogeneities into the equation, but still assuming that it is positive and bounded away from zero. The existence of multiple nodal solutions was recently discussed in [10, 28]. In all these papers the approach was variational, in most cases still based on the Nehari method, and led to prove the existence of small regular, i.e., strong or classical, solutions of (1.1).

On the contrary, when a changes sign, basically no results were known in the literature about the existence of positive solutions for prescribed mean curvature problems, until the recent papers [23, 24, 25, 26, 27] which analyze the one-dimensional equation supplemented with Neumann, rather than Dirichlet, boundary conditions. Thus, this paper appears to be the first contribution where the existence of positive solutions for the superlinear indefinite Dirichlet problem (1.1) is faced in a genuine PDE setting. The presence in the existing literature of very few results about the quasilinear problem (1.1) is in sharp contrast with the wide number of works that are available in the semilinear setting, starting already in the early nineties with [1, 2, 4, 7, 8].

In order to give a flavor of our results, we produce here a special corollary of our main theorem which is stated as Theorem 2.2 in the next section. Let us set

$$\Omega^+ = \{x \in \Omega\colon a(x) > 0\}, \quad \Omega^0 = \{x \in \Omega\colon a(x) = 0\}, \quad \Omega^- = \{x \in \Omega\colon a(x) < 0\},$$

and

$$\mathscr{T} = \{(u, \lambda) \in W^{2,r}(\Omega) \times \,]0, +\infty[\colon u \text{ is a positive strong solution of (1.1) for some } \lambda > 0\},$$

where $r > N$ is fixed, and endow $\mathscr{T}$ with the topology of $C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in \left]0, 1 - \frac{N}{r}\right[$.

Proposition 1.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a C^2 boundary $\partial\Omega$ and let $h : \mathbb{R} \to \mathbb{R}$ be continuous and satisfy $\lim_{s\to 0^+} \frac{h(s)}{s^p} = 1$, the exponent $p > 1$ being such that $p < \frac{N+2}{N-2}$ if $N \geq 3$. Suppose that $a \in C^2(\overline{\Omega})$ satisfies*

$$\Omega^+ \neq \emptyset, \quad \Omega^- \neq \emptyset, \quad \Omega^0 = \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega, \quad \textit{and} \quad \nabla a(x) \neq 0 \textit{ for all } x \in \Omega^0. \tag{1.2}$$

Then, for any given $r > N$, there exists $\lambda^* \geq 0$ such that for all $\lambda > \lambda^*$ problem (1.1) has at least one positive solution $u \in W^{2,r}(\Omega)$. In addition, these solutions can be chosen so that the set $\mathscr{C} = \{(u, \lambda)\colon \lambda > \lambda^*\}$ is a connected component of $\mathscr{T}$ and

$$\lim_{\lambda \to +\infty} \max\{\|u\|_{W^{2,r}}\colon (u, \lambda) \in \mathscr{C}\} = 0.$$

In the sequel, we actually consider a more general problem than (1.1), that is, problem (2.1) below, where the right-hand side of the equation may further depend on the gradient of the solution, thus possibly causing the loss of the variational structure. We also replace the “thin” nodal set condition (1.2) on the weight a, first introduced in [7] in the semilinear case, with a condition allowing "thick" nodal sets, as discussed in [4, 16].

Our method of proof is based on interpreting the quasilinear problem (1.1), or respectively (2.1), when λ is large, as a small perturbation of a limiting semilinear problem for which the existence of a priori bounds for the possible positive solutions is known. Then, by relying on a fixed point index calculation as in [4] and by using a general Leray-Schauder continuation theorem on metric ANRs from [11], we can prove the existence of an unbounded connected component of the set of the positive solutions of problem (1.1), or respectively (2.1).

The final part of this paper, Section 4, is devoted to a brief discussion of the existence of positive solutions of problem (1.1) when λ is small: a situation that is not covered by Proposition 1.1, nor by Theorem 2.2. The main assumption here requires that $\lim_{s\to+\infty}\frac{h(s)}{s^q}=1$, the exponent $q>0$ being such that $q<\frac{1}{N-1}$ if $N\geq 2$; thus, except in dimension $N=1$, the function h is not superlinear at $+\infty$, although its potential $H(s)=\int_0^s h(t)\,\mathrm{d}t$ is. Since singular solution might now appear, as confirmed by simple explicit one-dimensional examples, we adapt to our context an approach which was developed in [22, 30, 35], in case a is positive, and which exploits non-smooth critical point theory in the space of bounded variation functions.

Applications of the results obtained in this work to the existence and the multiplicity of positive solutions of indefinite logistic growth models with flux-saturated diffusion are given in [38].

2 The main result

Let us consider the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=\lambda f(x,u,\nabla u)+g(x,u,\nabla u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\lambda>0$ is a parameter. We assume that

(H_1) $\Omega\subset\mathbb{R}^N$, with $N\geq 2$, is a bounded domain with boundary $\partial\Omega$ of class C^2;

(H_2) $f\colon\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a continuous function such that there exist $p>1$, with $p<\frac{N+2}{N-2}$ if $N\geq 3$, and $a\in C^0(\overline{\Omega})$ for which

$$\lim_{(s,\xi)\to(0,0)}\frac{f(x,s,\xi)}{|s|^{p-1}s}=a(x)\quad\text{uniformly in } \overline{\Omega};$$

(H_3) $g\colon\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ is a continuous function such that there exists $\mu\leq 0$ for which

$$\lim_{(s,\xi)\to(0,0)}\frac{g(x,s,\xi)}{s}=\mu\quad\text{uniformly in } \overline{\Omega}.$$

We suppose that the function a satisfies the following conditions, introduced in [4, 16]. Namely, we assume that

(H_4) $a\in C^2(\overline{\Omega})$;

(H_5) $\Omega^+=\{x\in\Omega\colon a(x)>0\}\neq\emptyset$, $\Omega^-=\{x\in\Omega\colon a(x)<0\}\neq\emptyset$, and $\Omega^0=\{x\in\Omega\colon a(x)=0\}$ is such that $\partial\Omega^0\subset\Omega$; the boundaries $\partial(\mathrm{int}\Omega^0)$, $\partial\Omega^+$, and $\partial\Omega^-$ are of class C^2; Ω^0 has a finite number of connected components, that we denote by D_i^+, D_j^-, and $D_k^\pm$.

Hence, we can represent Ω^0 in the form

$$\Omega^0 = \bigcup_i D_i^+ \ \cup \ \bigcup_j D_j^- \ \cup \ \bigcup_k D_k^\pm,$$

where the components D_i^+, D_j^-, and $D_k^\pm$ are supposed to satisfy:

(H_6) for each i, $\partial D_i^+ \subset \overline{\Omega^+}$ and there exist $\gamma_{1,i} > 0$, a neighborhood U_i^+ of ∂D_i^+, and $\alpha_i^+ : \overline{U_i^+} \to]0, +\infty[$ such that

$$a(x) = \alpha_i^+(x) \operatorname{dist}(x, \partial D_i^+)^{\gamma_{1,i}} \quad \text{for all } x \in \Omega^+ \cap U_i^+;$$

(H_7) for each j, $\partial D_j^- \subset \overline{\Omega^-}$ and there exist $\gamma_{2,j} > 0$, a neighborhood U_j^- of ∂D_j^-, and $\alpha_j^- : \overline{U_j^-} \to]-\infty, 0[$ such that

$$a(x) = \alpha_j^-(x) \operatorname{dist}(x, \partial D_j^-)^{\gamma_{2,j}} \quad \text{for all } x \in \Omega^- \cap U_j^-;$$

(H_8) for each k, the following alternative holds

$(H_{8.1})$ if $\operatorname{int}(D_k^\pm) = \emptyset$, then
- $\partial D_k^\pm = \Gamma_k$ are of class C^2;
- there exist $\gamma_{3,k} > 0$, a neighborhood U_k^+ of Γ_k, and $\alpha_k^+ : \overline{U_k^+} \to]0, +\infty[$ such that
$$a(x) = \alpha_k^+(x) \operatorname{dist}(x, \Gamma_k)^{\gamma_{3,k}} \quad \text{for all } x \in \Omega^+ \cap U_k^+; \tag{2.2}$$
- there exist $\gamma_{4,k} > 0$, a neighborhood U_k^- of Γ_k, and $\alpha_k^- : \overline{U_k^-} \to]-\infty, 0[$ such that
$$a(x) = \alpha_k^-(x) \operatorname{dist}(x, \Gamma_k)^{\gamma_{4,k}} \quad \text{for all } x \in \Omega^- \cap U_k^-; \tag{2.3}$$

$(H_{8.2})$ if $\operatorname{int}(D_k^\pm) \neq \emptyset$, then
- $\partial D_k^\pm = \Gamma_k^+ \cup \Gamma_k^-$, with $\Gamma_k^+ \cap \Gamma_k^- = \emptyset$, $\Gamma_k^+ \subset \overline{\Omega^+}$, $\Gamma_k^- \subset \overline{\Omega^-}$ of class C^2;
- there exist $\gamma_{3,k} > 0$, a neighborhood U_k^+ of Γ_k^+, and $\alpha_k^+ : \overline{U_k^+} \to]0, +\infty[$ satisfying condition (2.2);
- there exist $\gamma_{4,k} > 0$, a neighborhood U_k^- of Γ_k^-, and $\alpha_k^- : \overline{U_k^-} \to]-\infty, 0[$ satisfying condition (2.3).

Let us define

$$D^+ = \bigcup_i D_i^+, \quad D^- = \bigcup_j D_j^-, \quad D^\pm = \bigcup_k D_k^\pm. \tag{2.4}$$

The set D^+ (respectively, D^-) is constituted by the connected components D_i^+ (respectively, D_j^-) of Ω^0, that are surrounded by regions of positivity (respectively, negativity) of a. Instead, $D^\pm$ is constituted by the connected components D_j^- of Ω^0, that are in between a region of positivity and one of negativity of a. $D^\pm$ can be either a "thin" nodal set, like when assuming condition (1.2), or a "thick" nodal set, that is, of positive measure. An example of an admissible nodal configuration for the function a is provided in Figure 1.

Remark 2.1. Let $a \in C^2(\overline{\Omega})$ be a sign-changing function satisfying condition (1.2). In this case, D^+, D^-, and $\operatorname{int}(D^\pm)$ are all empty sets, and assumption $(H_{8.1})$ holds. Indeed, let Γ_k be a connected component of Ω^0. Then, condition (2.2) is satisfied taking $\gamma_{1,k} = 1$ and $\alpha_k^+ : U_k^+ \to]0, +\infty[$ defined by

$$\alpha_k^+(x) = \begin{cases} -|\nabla a(x)| & \text{if } x \in U_k^+ \setminus \Gamma_k, \\ \dfrac{a(x)}{\operatorname{dist}(x, \Gamma_k)} & \text{if } x \notin U_k^+ \setminus \Gamma_k, \end{cases}$$

where U_k^+ is a suitable tubular neighborhood of Γ_k. Condition (2.3) can be verified similarly.

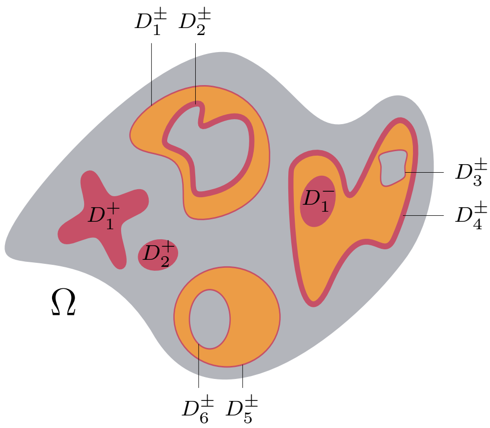


Figure 1: Example of an admissible nodal configuration for a weight a. The sets Ω^+, Ω^0, and Ω^- are respectively the union of the gray, the red, and the yellow regions. The connected components of Ω^0 are $D^+ = \bigcup_{i=1}^2 D_i^+$, $D^\pm = \bigcup_{k=1}^6 D_k^\pm$, and $D^- = D_1^-$.

Notation. For any function $u\colon \overline{\Omega} \to \mathbb{R}$, we write

- $u \geq 0$ if $\operatorname{ess\,inf} u \geq 0$
- $u > 0$ if $u \geq 0$ and $\operatorname{ess\,sup} u > 0$,
- $u \gg 0$ if, for all $x \in \Omega$, $u(x) > 0$ and, for all $x \in \partial\Omega$, either $u(x) > 0$, or both $u(x) = 0$ and $\limsup_{t\to 0^-} \frac{u(x+t\nu)}{t} < 0$, where $\nu = \nu(x)$ is the unit outer normal to Ω at $x \in \partial\Omega$.

Definition 2.1. *By a solution of* (2.1) *we mean a function $u \in W^{2,q}(\Omega)$ for some $q > N$, which satisfies the equation a.e. in Ω and the boundary condition everywhere on $\partial\Omega$. If $u > 0$, we say that u is positive and, if $u \gg 0$, we say that u is strictly positive.*

We also define the solution set of problem (2.1) by

$$\mathscr{S} = \{(u, \lambda) \in W^{2,q}(\Omega) \times]0, +\infty[\colon u \text{ is a strictly positive solution of (2.1) for some } \lambda > 0\},$$

where $q > N$ is given; $\mathscr{S}$ is endowed with the topology $C^{1,\gamma}(\overline{\Omega})$ for some fixed $\gamma \in \left]0, 1 - \frac{N}{q}\right[$.

Theorem 2.2. *Assume (H_1)–(H_8) and fix $q > N$ and $\gamma \in \left]0, 1 - \frac{N}{q}\right[$. Then, there exist a constant $\lambda^* \geq 0$ and a connected component $\mathscr{C}$ of $\mathscr{S}$ such that $\operatorname{proj}_{\mathbb{R}} \mathscr{C} =]\lambda^*, +\infty[$ and*

$$\lim_{\lambda\to+\infty} \max\{\|u\|_{W^{2,q}} \colon (u, \lambda) \in \mathscr{C}\} = 0.$$

Remark 2.3. Choosing $f(x, s) = a(x)h(s)$ and $g(s) = 0$ problem (1.1) becomes a special case of (2.1). Hence, also thanks to Remark 2.1, Proposition 1.1 follows.

Remark 2.4. Conditions (H_2) and (H_3) are rather general and allow to include in (2.1) some classes of anisotropic mean curvature equations considered in [9]. It is worthy to point out that problem (2.1) does not have in general a variational structure, so that the use of critical point theory is ruled out.

3 Proof of Theorem 2.2

Throughout this section we always suppose that conditions (H_1), (H_2), and (H_3) hold. We also henceforth fix exponents $q > N$ and $\gamma \in \left]0, 1 - \frac{N}{q}\right[$. It is clear that, with this choice, the Sobolev space $W^{2,q}(\Omega)$ is compactly embedded into the Hölder space $C^{1,\gamma}(\overline{\Omega})$.

3.1 A perturbation scheme

We start observing that, by setting $\rho = \lambda^{1/1-p}$ and $u = \rho v$, problem (2.1) can be written in the equivalent form

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \big(\delta_{ij} - \theta_{ij}(\nabla v, \rho)\big)\partial_{x_i x_j} v + \omega v = \big(\omega + a(x)|v|^{p-1} + \mu + \mathfrak{h}(x, v, \nabla v, \rho)\big) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where, for $i, j \in \{1, \dots, N\}$, δ_{ij} is the Kronecker's delta, $\theta_{ij} \colon \mathbb{R}^N \times \,]0, +\infty[\, \to \mathbb{R}$ and $\mathfrak{h} \colon \Omega \times \mathbb{R} \times \mathbb{R}^N \times \,]0, +\infty[\, \to \mathbb{R}$ are continuous functions defined by

$$\theta_{ij}(\xi, \rho) = \frac{\rho^2 \xi_i \xi_j}{1 + \rho^2 |\xi|^2}$$

and

$$\begin{aligned} \mathfrak{h}(x, s, \xi, \rho) = &\frac{\rho^2|\xi|^2}{1 + \sqrt{1 + \rho^2|\xi|^2}} \Big(\rho^{1-p} f(x, \rho s, \rho\xi) + g(x, \rho s, \rho\xi) \\ &+ \rho^{-p}\big(f(x, \rho s, \rho\xi) - a(x)\rho^p |s|^{p-1} s\big) + \rho^{-1}\big(g(x, \rho s, \rho\xi) - \mu\rho s\big)\Big). \end{aligned}$$

Note that, for all $i, j \in \{1, \dots, N\}$,

$$\lim_{\rho \to 0} \theta_{ij}(\xi, \rho) = 0$$

uniformly on any compact subset of $\mathbb{R}^N$ and, thanks to conditions (H_2) and (H_3),

$$\lim_{\rho \to 0} \mathfrak{h}(x, s, \xi, \rho) = 0$$

uniformly on $\overline{\Omega} \times K$, where K is any compact subset of $\mathbb{R} \times \mathbb{R}^N$. Hence, we can extend $\mathfrak{h}$ by continuity to $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \{0\}$, by setting

$$\mathfrak{h}(x, s, \xi, 0) = 0 \quad \text{in } \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

Thus, if $\rho = 0$, problem (3.1) reads

$$\begin{cases} -\Delta v = a(x)|v|^{p-1} v + \mu v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Therefore, problem (3.1) can be viewed as a perturbation of problem (3.2).

Remark 3.1. We point out that any differential operator that admits a decomposition such as in (3.1) could be considered in place of the mean curvature operator.

Next, we introduce the cone

$$\mathscr{P} = \{u \in C^{1,\gamma}(\overline{\Omega}) \colon u \geq 0\}, \tag{3.3}$$

that we endow with the topology of $C^{1,\gamma}(\overline{\Omega})$.

We also fix an arbitrary open bounded subset $\mathscr{O}$ of $\mathscr{P}$; the precise choice of the set $\mathscr{O}$ that we will need to accomplish the proof of Theorem 2.2 is specified in the statement of Lemma 3.8 below.

Pick $\rho^* > 0$ such that

$$\sup\left\{ \sum_{i,j=1}^{N} \|\theta_{ij}(\nabla v,\rho)\|_\infty \colon v \in \overline{\mathscr{O}},\ \rho \in [0,\rho^*] \right\} \le \frac{1}{2} \tag{3.4}$$

and set

$$\omega^* = \sup\left\{ \|av^{p-1} + \mu + \mathfrak{h}(\cdot, v, \nabla v, \rho)\|_\infty \colon v \in \overline{\mathscr{O}},\ \rho \in [0,\rho^*] \right\}. \tag{3.5}$$

Then, for each $\rho \in [0,\rho^*]$ and $\omega \in \,]\omega^*, +\infty[$, we denote by $\mathcal{T}_\omega \colon \overline{\mathscr{O}} \times [0,\rho^*] \to C^{1,\gamma}(\overline{\Omega})$ the operator that sends each $v \in \overline{\mathscr{O}}$ and $\rho \in [0,\rho^*]$ onto the unique solution $w \in W^{2,q}(\Omega)$ of the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \big(\delta_{ij} - \theta_{ij}(\nabla v,\rho)\big)\partial_{x_i x_j} w + \omega w = \big(\omega + a(x)|v|^{p-1} + \mu + \mathfrak{h}(x, v, \nabla v, \rho)\big)v & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Remark 3.2. For each $v \in \overline{\mathscr{O}}$ and $\rho \in [0,\rho^*]$, the linear differential operator

$$-\sum_{i,j=1}^{N} \big(\delta_{ij} - \theta_{ij}(\nabla v,\rho)\big)\partial_{x_i x_j} w$$

is uniformly elliptic, because condition (3.4) implies that

$$\frac{1}{2}|\xi|^2 \le |\xi|^2 - \sum_{i,j=1}^{N} \theta_{ij}(\nabla v,\rho)\xi_i\xi_j \le \frac{3}{2}|\xi|^2$$

for all $\xi \in \mathbb{R}^N$ and $\rho \in [0,\rho^*]$. Therefore, the operator $\mathcal{T}_\omega$ is well-defined for every $\omega \ge 0$.

Remark 3.3. $\mathcal{T}_\omega(\cdot,0)$ is the operator that sends each $v \in \overline{\mathscr{O}}$ onto the unique solution $w \in W^{2,q}(\Omega)$ of the problem

$$\begin{cases} -\Delta w + \omega w = (\omega + \mu)v + a(x)|v|^{p-1}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

The following result establishes the most relevant properties of the operator $\mathcal{T}_\omega$.

Lemma 3.4. *Assume (H_1)–(H_3). Let $\mathscr{O}$ be an open bounded subset of the cone $\mathscr{P}$ defined by (3.3). Then, for each $\omega \in \,]\omega^*, +\infty[$, the operator $\mathcal{T}_\omega \colon \overline{\mathscr{O}} \times [0,\rho^*] \to C^{1,\gamma}(\overline{\Omega})$ is continuous, compact, and maps $\overline{\mathscr{O}} \times [0,\rho^*]$ into $\mathscr{P}$. Moreover, $\mathcal{T}_\omega$ satisfies*

$$\lim_{\rho\to 0}\Big(\max_{v\in\overline{O}} \|\mathcal{T}_\omega(v,\rho) - \mathcal{T}_\omega(v,0)\|_{C^{1,\gamma}}\Big) = 0. \tag{3.8}$$

Proof. Let us fix $\omega \in \,]\omega^*, +\infty[$. We first prove that there exists a constant $M > 0$ such that

$$\|\mathcal{T}_\omega(v,\rho)\|_{W^{2,q}} < M \tag{3.9}$$

for all $v \in \overline{\mathscr{O}}$ and $\rho \in [0,\rho^*]$. Indeed, by the boundedness of $\overline{\mathscr{O}}$ in $C^1(\overline{\Omega})$ there exists a constant $c_1 > 0$ such that

$$\|\big(\omega + av^{p-1} + \mu + \mathfrak{h}(\cdot, v, \nabla v, \rho)\big)\|_\infty < c_1$$

for all $v \in \overline{\mathscr{O}}$ and $\rho \in [0,\rho^*]$. Thus, by [41, Theorem 3.28] there exists a constant $c_2 > 0$ such that

$$\|w\|_{W^{2,q}} = \|\mathcal{T}_\omega(v,\rho)\|_{W^{2,q}} < c_2.$$

We notice that c_1 depend only upon N, q, Ω, and the moduli of continuity of the coefficients θ_{ij}, which are controlled by the bound in $C^{1,\gamma}(\overline{\Omega})$ of the functions $v \in \overline{\mathscr{O}}$. Thus, estimate (3.9) is proven.

The compactness of the operator $\mathcal{T}_\omega$ follows from (3.9) and the compact embedding of $W^{2,q}(\Omega)$ into $C^{1,\gamma}(\overline{\Omega})$.

In order to prove the continuity of the operator $\mathcal{T}_\omega$, let us consider two sequences $(v_n)_n$ in $\overline{\mathscr{O}}$ and $(\rho_n)_n$ in $[0,\rho^*]$ such that $v_n \to v \in \overline{\mathscr{O}}$ in $C^{1,\gamma}(\overline{\Omega})$ and $\rho_n \to \rho$. Then, we take any subsequence $\big(v_{n_k}\big)_k$ of $(v_n)_n$ and $\big(\rho_{n_k}\big)_k$ of $(\rho_n)_n$, and set $w_{n_k} = \mathcal{T}_\omega(v_{n_k},\rho_{n_k})$. From estimate (3.9), we infer the existence of a further subsequence $\big(w_{n_{k_\ell}}\big)_\ell$, denoted by $(w_\ell)_\ell$ for convenience, and of a function $w \in W^{2,q}(\Omega)$ such that $w_\ell \to w$ weakly in $W^{2,q}(\Omega)$ and strongly in $C^{1,\gamma}(\overline{\Omega})$. Denoting $\big(v_{n_{k_\ell}}\big)_\ell$ by $(v_\ell)_\ell$, we have that $\theta_{ij}(\nabla v_\ell,\rho_\ell) \to \theta_{ij}(\nabla v,\rho)$ in $L^\infty(\Omega)$, for all $i,j \in \{1,\dots,N\}$, and hence

$$-\sum_{i,j=1}^{N} \left(\delta_{ij} - \theta_{ij}(\nabla v_\ell,\rho)\right)\partial_{x_i x_j} w_\ell \to -\sum_{i,j=1}^{N} \left(\delta_{ij} - \theta_{ij}(\nabla v,\rho)\right)\partial_{x_i x_j} w$$

weakly in $L^q(\Omega)$ and

$$\big(\omega + a v_\ell^{p-1} + \mu + \mathfrak{h}(\cdot, v_\ell, \nabla v_\ell, \rho_\ell)\big)v_\ell \to \big(\omega + a v^{p-1} + \mu + \mathfrak{h}(\cdot, v, \nabla v, \rho)\big)v$$

in $L^\infty(\Omega)$. Therefore, w is the solution of (3.6), that is, $w = \mathcal{T}_\omega(v,\rho)$. This concludes the proof of the continuity of the operator $\mathcal{T}_\omega$.

We point out also that, when $\rho = 0$ a stronger conclusion can be achieved. Indeed, the L^p-regularity theory [41, Section 3.7.3] implies that, if $v_n \to v$ in L^∞, then $\mathcal{T}_\omega(v_n,0) \to \mathcal{T}_\omega(v,0)$ strongly in $W^{2,q}(\Omega)$.

To show that the operator $\mathcal{T}_\omega$ maps $\overline{\mathscr{O}} \times [0,\rho^*]$ into $\mathscr{P}$, we actually prove that $\mathcal{T}_\omega(\cdot,\rho)$ is strongly positive, for each $\rho \in [0,\rho^*]$. Indeed, if we take $v \in \overline{\mathscr{O}}$ such that $v > 0$, the choice $\omega > \omega^*$ and condition (3.5) imply that

$$\big(\omega + a v^{p-1} + \mu + \mathfrak{h}(\cdot, v, \nabla v, \rho)\big)v > 0.$$

Hence, by the strong maximum principle [41, Theorem 3.27], the solution w of (3.6) satisfies $w(x) > 0$ for all $x \in \Omega$ and, by the Hopf boundary point lemma [41, Theorem 3.26], $\frac{\partial w}{\partial \nu}(x) < 0$ for all $x \in \partial\Omega$. Thus, we conclude that $\mathcal{T}_\omega(v,\rho) = w \gg 0$.

At last, we prove that (3.8) holds. Assume, by contradiction, that there exist sequences $(v_n)_n$ in $\overline{\mathscr{O}}$ and $(\rho_n)_n$ in $[0,\rho^*]$, with $\rho_n \to 0$, such that

$$\inf_n \ \|\mathcal{T}_\omega(v_n,\rho_n) - \mathcal{T}_\omega(v_n,0)\|_{C^{1,\gamma}} > 0. \tag{3.10}$$

Since $(v_n)_n$ is bounded in $C^{1,\gamma}(\overline{\Omega})$, we can suppose, possibly passing to a subsequence still denoted by $(v_n)_n$, that $(v_n)_n$ converges in $C^1(\overline{\Omega})$ to some $v \in C^1(\overline{\Omega})$. On the other hand, there is a constant $c_3 > 0$ such that, for all n,

$$|\nabla v_n(x) - \nabla v_n(y)| \le \|v_n\|_{C^{1,\gamma}}|x-y|^\gamma \le c_3|x-y|^\gamma$$

for every $x,y \in \overline{\Omega}$. Thus, passing to the limit, we find

$$|\nabla v(x) - \nabla v(y)| \le c_3|x-y|^\gamma$$

for every $x,y \in \overline{\Omega}$, that is, $v \in C^{1,\gamma}(\overline{\Omega})$. Set, for convenience, $w_n = \mathcal{T}_\omega(v_n,\rho_n)$. From (3.9), it follows the existence of a subsequence $\big(w_{n_k}\big)_k$ of $(w_n)_n$ and of a function $w \in W^{2,q}(\Omega)$ such that $w_{n_k} \to w$ weakly in $W^{2,q}(\Omega)$ and strongly in $C^{1,\gamma}(\overline{\Omega})$. Since $\theta_{ij}(\nabla v_{n_k},\rho_{n_k}) \to 0$ in $L^\infty(\Omega)$ for all $i,j \in \{1,\dots,N\}$, we get

$$-\sum_{i,j=1}^{N} \left(\delta_{ij} - \theta_{ij}(\nabla v_{n_k},\rho_{n_k})\right)\partial_{x_i x_j} w_{n_k} \to -\Delta w$$

weakly in $L^q(\Omega)$ and

$$\big(\omega + av_{n_k}^{p-1} + \mu + \mathfrak{h}(\cdot, v_{n_k}, \nabla v_{n_k}, \rho_{n_k})\big)v_{n_k} \to \big(\omega + \mu)v + av^p$$

in $L^\infty(\Omega)$. Therefore, w is the solution of (3.7), that is, $w = \mathcal{T}_\omega(v, 0)$, and

$$w_{n_k} = \mathcal{T}_\omega(v_{n_k}, \rho_{n_k}) \to w = \mathcal{T}_\omega(v, 0)$$

in $C^{1,\gamma}(\overline{\Omega})$. Moreover, as $v_{n_k} \to v$ in $L^\infty(\Omega)$, we have

$$\mathcal{T}_\omega(v_{n_k}, 0) \to \mathcal{T}_\omega(v, 0)$$

strongly in $W^{2,q}(\Omega)$ and hence in $C^{1,\gamma}(\overline{\Omega})$. Then, we conclude that

$$\lim_{k\to+\infty} \|\mathcal{T}_\omega(v_{n_k}, \rho_{n_k}) - \mathcal{T}_\omega(v_{n_k}, 0)\|_{C^{1,\gamma}} = 0,$$

contradicting (3.10). □

3.2 A priori estimates

In this section, we collect some results obtained in [4, 16] concerning the semilinear problem

$$\begin{cases} -\Delta v = \sigma v + a(x)|v|^{p-1}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.11}$$

where $\sigma \in \mathbb{R}$ is a given constant, the weight function a satisfies (H_4) and the exponent p, coming from condition (H_2), is such that

(H_8) $p > 1$, with $p < \frac{N+2}{N-2}$ if $N \geq 3$.

We start by recalling a non-existence result.

Lemma 3.5 ([4, Theorem 3.3])**.** *Assume (H_1), (H_4), (H_5), and (H_9). Then, there exists $\sigma^* > 0$ such that, for all $\sigma \geq \sigma^*$, problem (3.11) has no solution $v > 0$.*

Remark 3.6. Without loss of generality, we will always suppose that $\sigma^* > \lambda_1$, where λ_1 is the principal eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Combining the blow-up argument in [19] with some Liouville-type theorems in [4, 16] yields the following result.

Lemma 3.7 ([4, 16, 19])**.** *Assume (H_1) and (H_4)–(H_9). Let $\Sigma \subset \mathbb{R}$ be any bounded interval. Then, there exists a constant $c > 0$ such that every solution $v > 0$ of (3.11), for any $\sigma \in \Sigma$, satisfies*

$$\|v\|_{C^{1,\gamma}} < c. \tag{3.12}$$

Proof. We provide a sketch of the proof to show how the results in [4, 16, 19] have to be exploited to prove that, given a bounded interval $\Sigma \subset \mathbb{R}$, there exists a constant $c > 0$ such that every solution $v > 0$ of (3.11), for any $\sigma \in \Sigma$, satisfies

$$\|v\|_\infty < c. \tag{3.13}$$

Estimate (3.12) then follows by the L^p-regularity theory, possibly taking a larger constant.

In order to get (3.13), we suppose by contradiction that there exists a sequence of solutions $(v_n)_n$ of

$$\begin{cases} -\Delta v_n = \sigma_n v_n + a(x)|v_n|^{p-1}v_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

with $v_n > 0$ and $\sigma_n \in \Sigma$ for all n, such that

$$\lim_{n\to+\infty} \max_{\overline{\Omega}} v_n = +\infty.$$

We can assume, by [16, Lemma 3.2], that $\max_{\overline{\Omega}} v_n = v_n(x_n)$ for some $x_n \in \overline{\Omega^-} \cup \overline{\Omega^+}$ and, by compactness, that

$$\lim_{n\to+\infty} x_n = x_0 \in \overline{\Omega^-} \cup \overline{\Omega^+}.$$

A contradiction is then reached by showing that $x_0 \notin \overline{\Omega^-} \cup \overline{\Omega^+}$, for a suitable partition of $\overline{\Omega^-} \cup \overline{\Omega^+}$ obtained from (2.4), namely, by proving that none of the following cases may occur: (i) $x_0 \in \Omega^- \cup \Omega^+$, (ii) $x_0 \in \partial\Omega$, (iii) $x_0 \in \partial D^+$, (iv) $x_0 \in \partial D^-$, (v) $x_0 \in \partial D^\pm$.

To rule out case (i), we first apply [16, Lemma 3.8] on each compact subset of Ω^- and get $x_0 \notin \Omega^-$. On the other hand, the classical blow-up arguments given in [19] imply that $x_0 \notin \Omega^+$. This way, we conclude that $x_0 \notin \Omega^- \cup \Omega^+$.

To exclude case (ii), we notice that, by conditions (H_1) and (H_5), the weight a satisfies either $a(x) < 0$ for all $x \in \partial\Omega$, or $a(x) > 0$ for all $x \in \partial\Omega$. If we suppose that the first alternative holds, then $x_0 \notin \partial\Omega$ follows by applying [16, Lemma 3.9]. The argument in [16] involves some comparison results stated in [15] which are still valid in our framework. Namely, we split the set $\{x \in \Omega\colon \min\{a(x), 0\} = 0\}$ into two components $\bigcup_j D_j^-$ and $\Omega^+ \cup \bigcup_i D_i^+ \cup \bigcup_k D_k^\pm$ and proceed as in the proof of [16, Lemma 3.9] by using [15, Lemma 2.3 and Lemma 2.6]. Also the second alternative leads to the same conclusion exploiting the results of [19]. Thus, we find that anyhow $x_0 \notin \partial\Omega$.

If we suppose that (iii) holds, then $x_0 \in \partial D_i^+$ for some i, as the sets D_i^+ are disjoint by (H_5). This is in contradiction with [16, Lemma 3.5], provided that (H_6) holds. Hence, we infer that $x_0 \notin \partial D_i^+$, for each i, and so $x_0 \notin \partial D^+$.

Similarly, if (iv) holds, then $x_0 \in \partial D_j^-$ for some j. Now, assuming (H_7), a contradiction is reached by applying [16, Lemma 3.6]. This yields $x_0 \notin \partial D_j^-$, for each j, and so $x_0 \notin \partial D^-$.

At last, if (v) is assumed true, then $x_0 \in \partial D_k^\pm$ for some k. Now, if $\mathrm{int} D_k^\pm = \emptyset$, then condition $(H_{8.1})$ implies that $x_0 \in \partial D_k^\pm = \Gamma_k$, contradicting the result in [16, Lemma 3.4]. On the other hand, if $\mathrm{int} D_k^\pm \neq \emptyset$, then condition $(H_{8.2})$ implies that $x_0 \in \partial D_k^\pm = \Gamma_k^+ \cup \Gamma_k^-$. Applying [16, Lemma 3.3 and Lemma 3.6] leads to a contradiction. Thus, we see that $x_0 \notin \Gamma_k^+ \cup \Gamma_k^-$ for each k and, in turn, $x_0 \notin \partial D^\pm$. □

3.3 Fixed point index calculation

Let us fix $\omega \in \,]\omega^{**} - \mu, +\infty[$, where

$$\omega^{**} = \max\{\omega^*, p\,\|a^-\|_\infty\, c^{p-1}\} > 0, \tag{3.14}$$

the number ω^* being defined by (3.5), the constant c coming from Lemma 3.7, for the choice $\Sigma = \{\mu\}$, and a^- denoting the negative part of the function a. Let us rewrite problem (3.2) in the form

$$\begin{cases} -\Delta v + \omega v = (\mu + \omega)v + a(x)|v|^{p-1}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.15}$$

Next, let us embed (3.15) into the problem

$$\begin{cases} -\Delta v + \omega v = (\sigma+\omega)v + a(x)|v|^{p-1}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

depending on the parameter $\sigma \in \Sigma = [\mu, \sigma^*]$, with $\sigma^* > \lambda_1$ being defined in Lemma 3.5 and Remark 3.6.

The previous discussion implies that, for $\sigma = \mu$, problem (3.16) is precisely (3.11) and, for $\sigma = \sigma^*$, problem (3.16) has no solution $v > 0$. Notice also that, as $\omega^{**} - \mu \geq 0$, we have that $\sigma + \omega \geq 0$, for all $\sigma \in \Sigma$.

Let $\mathcal{K}_\omega \colon L^\infty(\Omega) \to W^{2,q}(\Omega)$ be the operator which sends any function $z \in L^\infty(\Omega)$ onto the unique solution $w \in W^{2,q}(\Omega)$ of the problem

$$\begin{cases} -\Delta w + \omega w = z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases}$$

The strong maximum principle and the Hopf boundary lemma imply that $\mathcal{K}_\omega$ is strongly positive, that is, if $z > 0$, then $\mathcal{K}_\omega(z) \gg 0$.

From now on, we denote by $\mathscr{B}_r$ the open ball in $\mathscr{P}$ of center 0 and radius $r > 0$, i.e.,

$$\mathscr{B}_r = \{v \in \mathscr{P} \colon \|v\|_{C^{1,\gamma}} < r\}.$$

Let us take $r = c$, with c coming from Lemma 3.7, and define the operator $\mathcal{S}_\omega \colon \overline{\mathscr{B}_c} \times \Sigma \to C^{1,\gamma}(\overline{\Omega})$ by

$$\mathcal{S}_\omega(v,\sigma) = \mathcal{K}_\omega((\sigma+\omega)v + av^p).$$

The L^p-regularity theory, the boundedness of $\mathscr{B}_c$, and the compact embedding of $W^{2,q}(\Omega)$ into $C^{1,\gamma}(\overline{\Omega})$ imply that the operator $\mathcal{S}_\omega$ is continuous and compact. Moreover, for each $\sigma \in \Sigma$, $\mathcal{S}_\omega(\cdot,\sigma)$ is strongly increasing. Indeed, if $v_1, v_2 \in \overline{\mathscr{B}_c}$ satisfy $v_1 < v_2$, then, thanks to $\omega > \omega^{**} - \mu$, we easily get

$$(\sigma+\omega)(v_2 - v_1) + a(v_2^p - v_1^p) > \big(\omega^{**} - p\|a^-\|_\infty c^{p-1}\big)(v_2 - v_1) \geq 0.$$

Hence, by the strong positivity of $\mathcal{K}_\omega$, it follows that $\mathcal{S}_\omega(v_1,\sigma) \gg \mathcal{S}_\omega(v_2,\sigma)$. In particular, since $\mathcal{S}_\omega(0,\sigma) = 0$, we have that $\mathcal{S}_\omega(\cdot,\sigma)$ is strongly positive for each $\sigma \in \Sigma$, that is, $\mathcal{S}_\omega(v,\sigma) \gg 0$, for every $v \in \overline{\mathscr{B}_c}$ with $v > 0$.

Since $\mathscr{P}$ is a non-empty closed convex subset of $C^{1,\gamma}(\overline{\Omega})$, $\mathscr{B}_c$ is open in $\mathscr{P}$, and by Lemma 3.7, $\mathcal{S}_\omega(\cdot,\sigma)$ has no fixed point on $\partial\mathscr{B}_c$ for any $\sigma \in \Sigma$, then the fixed point index

$$\operatorname{ind}\big(\mathcal{S}_\omega(\cdot,\sigma), \mathscr{B}_c, \mathscr{P}\big)$$

is well-defined and independent of $\sigma \in \Sigma$. We refer to [3, Section 11] for the properties of the fixed point index we are going to use in the sequel.

Lemma 3.8. *Assume* (H_1)*–*(H_8) *and fix* $\omega > \omega^{**} - \mu$, *with* ω^{**} *defined by* (3.14). *Then, there exists* $R > 0$ *such that, setting* $\mathscr{O} = \mathscr{B}_c \setminus \overline{\mathscr{B}_R}$,

$$\operatorname{ind}\big(\mathcal{S}_\omega(\cdot,\mu), \mathscr{O}, \mathscr{P}\big) = -1.$$

Proof. The argument follows similar patterns as [4, Theorem 7.4]. Let us observe that, for every $\sigma \in \Sigma$, there exists the right derivative

$$\partial^+\mathcal{S}_\omega(0,\sigma)[v] = \lim_{\tau\to 0^+} \frac{\mathcal{S}_\omega(\tau v,\sigma)}{\tau} = (\sigma+\omega)\mathcal{K}_\omega(v),$$

for all $v \in \mathscr{B}_c$. Moreover, $\partial^+ \mathcal{S}_\omega(0,\sigma) = (\sigma+\omega)\mathcal{K}_\omega$ has an eigenvalue κ, with an eigenfunction $\varphi > 0$, if and only if

$$(\sigma+\omega)\mathcal{K}_\omega(\varphi) = \kappa\varphi. \tag{3.17}$$

Note that $\kappa > 0$, because $\sigma > 0$, $\varphi > 0$, and $\mathcal{K}_\omega$ is strongly positive. Since (3.17) is equivalent to

$$\begin{cases} -\Delta\varphi = \left(\frac{\sigma+\omega}{\kappa} - \omega\right)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

we conclude that $\frac{\sigma+\omega}{\kappa} - \omega = \lambda_1$, that is,

$$\kappa = \frac{\sigma+\omega}{\lambda_1+\omega},$$

where λ_1 is the principal eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. This in turn implies that

$$\kappa > 1, \quad \text{if } \sigma = \sigma^*,$$

as, according to Remark 3.6, $\sigma^* > \lambda_1$, and

$$\kappa < 1, \quad \text{if } \sigma = \mu,$$

as $\mu \leq 0 < \lambda_1$. Therefore, from [3, Lemma 13.1], we infer that, for all small $r > 0$, both $\mathcal{S}_\omega(\cdot,\sigma^*)$ and $\mathcal{S}_\omega(\cdot,\mu)$ have no fixed points on $\partial\mathscr{B}_r$ and, moreover,

$$\mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\sigma^*), \mathscr{B}_r, \mathscr{P}\big) = 0$$

and

$$\mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\mu), \mathscr{B}_r, \mathscr{P}\big) = 1.$$

Hence, Lemma 3.5 and the excision property yield, for all small $r > 0$,

$$\mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\sigma^*), \mathscr{B}_c, \mathscr{P}\big) = \mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\sigma^*), \mathscr{B}_r, \mathscr{P}\big) = 0.$$

Lemma 3.7 and the homotopy invariance property imply

$$\mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\mu), \mathscr{B}_c, \mathscr{P}\big) = \mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\sigma^*), \mathscr{B}_c, \mathscr{P}\big).$$

Finally, the additivity property allows to conclude that there exists $R \in \,]0,c[$ such that, by setting $\mathscr{O} = \mathscr{B}_c \setminus \mathscr{B}_R$, we have

$$\mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\mu), \mathscr{O}, \mathscr{P}\big) = \mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\mu), \mathscr{B}_c, \mathscr{P}\big) - \mathrm{ind}\big(\mathcal{S}_\omega(\cdot,\mu), \mathscr{B}_R, \mathscr{P}\big) = -1. \qquad \square$$

3.4 Branches of positive solutions

We are now in position of concluding the proof of Theorem 2.2. Since $\mathcal{S}_\omega(\cdot,\mu) = \mathcal{T}_\omega(\cdot,0)$ in $\overline{\mathscr{O}}$, we infer that

$$\mathrm{ind}\big(\mathcal{T}_\omega(\cdot,0), \mathscr{O}, \mathscr{P}\big) = -1.$$

Condition (3.8) of Lemma 3.4 guarantees that there exists $\rho^{**} \in \,]0,\rho^*]$ such that $\mathcal{T}_\omega(\cdot,\rho)$ has no fixed points on $\partial\mathscr{O}$, for all $\rho \in [0,\rho^{**}]$. Hence, the homotopy invariance property of the fixed point index implies that

$$\mathrm{ind}\big(\mathcal{T}_\omega(\cdot,\rho), \mathscr{O}, \mathscr{P}\big) = -1,$$

for all $\rho \in [0, \rho^{**}]$. Then, the Leray-Schauder continuation theorem, as formulated in [11, Theorem 1] in the frame of metric ANRs, yields the existence of a maximal closed connected set

$$\mathscr{E} \subseteq \{(v, \rho) \in \mathscr{P} \times]0, \rho^{**}] \colon v = \mathcal{T}_\omega(v, \rho)\}$$

such that $\mathrm{proj}_{\mathbb{R}} \mathscr{E} =]0, \rho^{**}]$. The change of variables

$$u = \rho v, \quad \lambda = \rho^{1-p}$$

defines a homeomorphism of $\mathscr{P} \times]0, \rho^{**}[$ onto $\mathscr{P} \times]\lambda^*, +\infty[$, where $\lambda^* = (\rho^{**})^{1-p}$. Hence, the set

$$\mathscr{C} = \{(u, \lambda) \in \mathscr{P} \times]\lambda^*, +\infty[\colon (\lambda^{\frac{1}{1-p}} u, \lambda^{\frac{1}{1-p}}) \in \mathscr{E}\}$$

is still closed and connected and satisfies the condition

$$\mathrm{proj}_{\mathbb{R}} \mathscr{C} =]\lambda^*, +\infty[.$$

It is clear that all pairs $(u, \lambda) \in \mathscr{C}$ are solutions of (2.1) with $u \gg 0$. In addition, from estimate (3.8), we have that

$$\lim_{\lambda \to +\infty} \max\{\|u\|_{W^{2,q}} \colon (u, \lambda) \in \mathscr{C}\} = 0.$$

This ends the proof of Theorem 2.2.

4 Additional results and questions

A natural question suggested by our main result is whether the existence of positive solutions can be guaranteed, under the conditions of Theorem 2.2, even when the parameter $\lambda > 0$ is small. In this section we give a short account of what may happen in this case. In particular, we can give a positive answer for the model problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \lambda a(x) h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

in the setting of bounded variation solutions, under the following assumptions:

(H_{10}) $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded domain with a Lipschitz boundary $\partial\Omega$;

(H_{11}) $h \colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $h(0) = 0$ and there exists a constant $q > 0$, with $q < \frac{1}{N-1}$, if $N \geq 2$,

$$\lim_{s \to +\infty} \frac{H(s)}{s^{q+1}} = 1$$

and

$$\lim_{s \to +\infty} \frac{(q+1)H(s) - sh(s)}{s} = 0 \tag{4.2}$$

where $H(s) = \int_0^s h(t)\,\mathrm{d}t$;

(H_{12}) $a \in L^\infty(\Omega)$ and there exists a Caccioppoli set E such that

$$\int_E a \,\mathrm{d}x > 0.$$

Remark 4.1. Condition (H_{11}) implies that there exists a constant $C > 0$ such that

$$|h(s)| \le C(1 + |s|^q) \qquad \text{for all } s \ge 0, \tag{4.3}$$

and

$$|H(s)| \le C(1 + |s|^{q+1}) \quad \text{for all } s \ge 0. \tag{4.4}$$

Hence, in dimension $N \ge 2$, the function h is sublinear at $+\infty$, although its potential H is superlinear at $+\infty$. Of course, condition (H_{11}) does not prevent h to be superlinear at 0 and thus to satisfies the assumptions of Theorem 2.2.

Notation. For any Radon measure μ, we denote by $\mu = \mu^a + \mu^s$ the Lebesgue decomposition in its absolutely continuous part and its singular part with respect to the N-dimensional Lebesgue measure in $\mathbb{R}^N$, $|\mu|$ stands for the total variation of μ, and $\frac{\mu}{|\mu|}$ indicates the density of μ with respect to its total variation. Further, $\mathcal{H}_{N-1}$ represents the $(N-1)$-dimensional Hausdorff measure. We refer to [5] for definitions and results concerning bounded variation functions.

Definition 4.1 ([6])**.** *By a bounded variation solution of* (4.1) *we mean a function $u \in BV(\Omega)$ which satisfies*

$$\int_\Omega \frac{D^a u\, D^a \phi}{\sqrt{1+|D^a u|^2}}\,\mathrm{d}x + \int_\Omega \frac{Du}{|Du|}\frac{D\phi}{|D\phi|}|D^s\phi| + \int_{\partial\Omega} \operatorname{sgn}(u)\,\phi\,\mathrm{d}\mathcal{H}_{N-1} = \lambda \int_\Omega a h(u)\phi\,\mathrm{d}x \tag{4.5}$$

for every $\phi \in BV(\Omega)$ such that $|D^s\phi|$ is absolutely continuous with respect to $|D^s u|$ and $v(x) = 0$ $\mathcal{H}_{N-1}$-a.e. on the set $\{x \in \partial\Omega \colon u(x) = 0\}$. If, in addition, $u > 0$, we say that u is a positive bounded variation solution of (4.1)*.*

Considering bounded variation solutions is motivated by the possible occurrence of singular solutions of problem (4.1) even in dimension $N = 1$, as witnessed by the following example, where for small $\lambda > 0$ the solutions of (4.1) may exhibit jump discontinuities at the nodal points of the weight a as well as at the boundary points.

Example 4.2. Let us consider the 1-dimensional problem

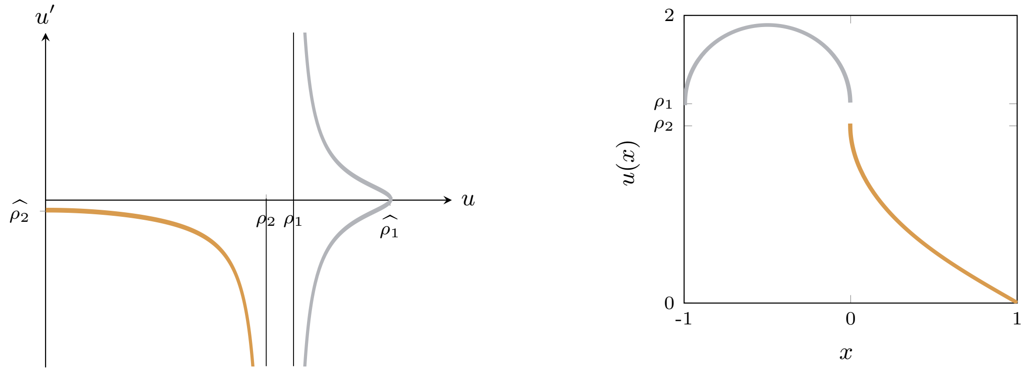
$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = \lambda a(x) u^q \quad \text{in} \quad]-1,1[, \\ u(-1) = 0, \quad u(1) = 0, \end{cases} \tag{4.6}$$

where $q > 0$ is fixed, the weight $a\colon [-1,1] \to \mathbb{R}$ is defined by $a(x) = -\operatorname{sgn}(x)$, and $\lambda > 0$ is a parameter. Then, there exists $\lambda_* > 0$ such that for each $\lambda \in]0,\lambda_*[$ problem (4.6) has at least one positive bounded variation solution $u \in BV(-1,1)$, which is not regular (see Figure 2). Namely, for all $\lambda \in]0,\lambda_*[$, there exist $\rho_1 = \rho_1(\lambda)$, $\rho_2 = \rho_2(\lambda)$, $\widehat{\rho_2} = \widehat{\rho_2}(\rho_2)$, with $\rho_1 > \rho_2 > 0 > \widehat{\rho_2}$, and a positive solution $u \in BV(-1,1)$ of (4.6) satisfying the conditions:

(i) $u \in C^2(]-1,0[)$ and $u \in C^2(]0,1])$;

(ii) $\lim_{x\to -1^+}(u(x),u'(x)) = (\rho_1,+\infty)$ and $\lim_{x\to 0^-}(u(x),u'(x)) = (\rho_1,-\infty)$;

(iii) $\lim_{x\to 0^+}(u(x),u'(x)) = (\rho_2,+\infty)$ and $\lim_{x\to 1}(u(x),u'(x)) = (0,\widehat{\rho_2})$.



(a) Trajectory of the solution of (4.7) through $(0, \widehat{\rho_1})$ (gray) and trajectory of the solution of (4.8) through $(\widehat{\rho_2}, 0)$ (yellow).

(b) Graph of a solution $u \in BV(-1, 1)$ of (4.6) with jump discontinuities at -1 and 0.

Figure 2: Analysis of (4.6) for $q = 1$ and $\lambda = 1.1$.

The proof is based on the analysis of the geometry associated to the phase-portraits in the plane (u, u') of the equations

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \lambda u^q, \tag{4.7}$$

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = -\lambda u^q. \tag{4.8}$$

When $r > ((q+1)/\lambda)^{1/(q+1)}$, all the orbits of (4.7) through $(r, 0)$ are unbounded in u' and such that $u > \mathfrak{r}_\lambda^+(r)$ with $\mathfrak{r}_\lambda^+(r) = \left(r^{q+1} - \frac{(q+1)}{\lambda}\right)^{1/(q+1)}$. Then, we look for a solution of (4.7) that starting at $(r, 0)$ takes the time $T_\lambda^+(r) = 1/2$ to reach $(\mathfrak{r}_\lambda^+(r), +\infty)$. The time-map associated with (4.7) is given by

$$T_\lambda^+(r) = \int_{\mathfrak{r}_\lambda^+(r)}^{r} \frac{1+\frac{\lambda}{q+1}(s^{q+1}-r^{q+1})}{\sqrt{1-\left(1+\frac{\lambda}{q+1}(s^{q+1}-r^{q+1})\right)^2}}\, \mathrm{d}s.$$

We notice that for $r_1 = ((q+1)/\lambda)^{1/(q+1)}$ there exists $\lambda_* > 0$ such that $T_{\lambda_*}^+(r_1) = 1/2$. Moreover, for every $\lambda \in \,]0, \lambda_*[$, we have that $T_\lambda^+(r_1) > T_{\lambda^*}^+(r_1)$. Let us fix $\lambda \in \,]0, \lambda_*[$. Since $T_\lambda^+(r_1) > 1/2$ and $\lim_{r\to+\infty} T_\lambda^+(r) = 0$, there exists $\widehat{\rho_1} > r_1$ such that $T_\lambda^+(\widehat{\rho_1}) = 1/2$. Thus, by taking $\rho_1 = \mathfrak{r}_\lambda^+(\widehat{\rho_1})$, it follows that the solution u_1 of problem (4.7) with initial values $u(-1/2) = \widehat{\rho_1}$, $u(-1/2) = 0$, satisfies condition (ii).

For every $r > 0$, all the orbits of (4.8) through $(0, -r)$ are unbounded in u' and such that $|u| < \mathfrak{r}_\lambda^-(r)$, where $\mathfrak{r}_\lambda^-(r) = ((q+1)/\lambda\sqrt{1+r^2})^{1/(q+1)}$ Now, we look for a solution of (4.8) that starting at $(0, -r)$ takes the time $T_\lambda^-(r) = 1$ to reach $(\mathfrak{r}_\lambda^-(r), -\infty)$. The time-map associated with (4.8) is given by

$$T_\lambda^-(r) = \int_0^{\mathfrak{r}_\lambda^-(r)} \frac{\frac{1}{\sqrt{1+r^2}} - \frac{\lambda}{(q+1)}s^{q+1}}{\sqrt{1-\left(\frac{1}{\sqrt{1+r^2}} - \frac{\lambda}{(q+1)}s^{q+1}\right)^2}}\, \mathrm{d}s.$$

We see that, for any given $\lambda > 0$, $\lim_{r\to0^+} T_\lambda^-(r) = +\infty$ and $\lim_{r\to-\infty} T_\lambda^-(r) = 0$. Hence there exists $\widehat{\rho_2} = \widehat{\rho_2}(\lambda) < 0$ such that $T_\lambda^-(\widehat{\rho_2}) = 1$. By taking $\rho_2 = \mathfrak{r}_\lambda^-(\widehat{\rho_2})$ it follows that the solution u_2 of problem (4.8), with terminal values $u(1) = 0$, $u(1) = \widehat{\rho_2}$, satisfies condition (iii).

Finally, we can prove that the function $u \in BV(-1,1)$ defined by $u(x) = u_1(x)$ in $]-1,0[$ and $u(x) = u_2(x)$ in $]0,1[$ is a positive bounded variation solution of (4.6) according to Definition 4.5.

The next result provides the existence, for all small $\lambda > 0$, of bounded variation solutions of problem (4.1).

Theorem 4.3. *Assume* (H_{10})–(H_{12}). *Then, there exists* $\lambda_* > 0$ *such that, for all* $\lambda \in]0, \lambda_*[$, *problem* (4.1) *has at least one positive solution* $u \in BV(\Omega)$. *In addition, these solutions* $u = u_\lambda$ *can be chosen so that*

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{L^{q+1}} = +\infty. \tag{4.9}$$

Proof. We provide a quick proof by skipping some details because it follows relatively standard patterns. The approach is variational and is based on the application of a version of a mountain pass lemma for non-smooth functionals similar to [37, Lemma 3.7]. Let us first introduce the functional $\mathcal{J} \colon BV(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{J}(v) = \int_\Omega \sqrt{1 + |Dv|^2} + \int_{\partial\Omega} |v| \, \mathrm{d}\mathcal{H}_{N-1},$$

where

$$\int_\Omega \sqrt{1 + |Dv|^2} = \int_\Omega \sqrt{1 + |D^a v|^2} \, \mathrm{d}x + \int_\Omega |D^s v|.$$

Since we are looking for positive solutions, we can replace the function h at the right hand side of the equation in (4.1) with $\tilde{h}$, defined by $\tilde{h}(s) = h(s^+)$ for all $s \in \mathbb{R}$. For simplicity, we will still denote $\tilde{h}$ by h. Let us define the functional $\mathcal{I} \colon BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}(v) = \mathcal{J}(v) - \lambda \int_\Omega aH(v) \, \mathrm{d}x.$$

According to [6], a function $u \in BV(\Omega)$ satisfies (4.5) if and only if

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_\Omega ah(u)(v - u) \, \mathrm{d}x \quad \text{for all } v \in BV(\Omega). \tag{4.10}$$

It is convenient to endow the space $BV(\Omega)$ with the norm

$$\|v\|_{BV} = \int_\Omega |Dv| + \int_{\partial\Omega} |v| \, \mathrm{d}\mathcal{H}_{N-1},$$

which is equivalent to the usual one by [31, Proposition 2] and [5, Theorem 3.88].

The proof consists of four steps.

Step 1 : *Mountain pass geometry.* Let us first show that there exist constants $R > 0$, $\lambda_* > 0$, and $\eta > 0$ such that, for all $\lambda \in]0, \lambda_*[$,

$$\inf_{v \in S_R} \mathcal{I}(v) \geq \mathcal{I}(0) + \eta, \tag{4.11}$$

where $S_R = \{v \in BV(\Omega) \colon \|v\|_{BV} = R\}$. Indeed, from Jensen's inequality we infer that, for all $v \in BV(\Omega)$,

$$\begin{aligned}
\mathcal{J}(v) &\geq \operatorname{meas}(\Omega) \sqrt{1 + \left(\frac{\|D^a v\|_{L^1}}{\operatorname{meas}(\Omega)} \right)^2} + \int_\Omega |D^s v| + \int_{\partial\Omega} |v| \, \mathrm{d}\mathcal{H}_{N-1} \\
&\geq \operatorname{meas}(\Omega) + \frac{1}{2} \left(\|D^a v\|_{L^1} + \int_\Omega |D^s v| + \int_{\partial\Omega} |v| \, \mathrm{d}\mathcal{H}_{N-1} - \operatorname{meas}(\Omega) \right) \\
&= \mathcal{J}(0) + \frac{1}{2} \left(\|v\|_{BV} - \operatorname{meas}(\Omega) \right).
\end{aligned}$$

Hence, condition (4.4) and the continuous embedding of $BV(\Omega)$ into $L^{q+1}(\Omega)$ yield the existence of a constant $C > 0$ such that, for all $v \in S_R$,

$$\begin{aligned}\mathcal{I}(v) &\geq \mathcal{J}(0) + \frac{1}{2}\left(R - \mathrm{meas}(\Omega)\right) - \lambda \|a\|_\infty \int_\Omega |H(v)| \, \mathrm{d}x \\ &\geq \mathcal{J}(0) + \frac{1}{2}\left(R - \mathrm{meas}(\Omega)\right) - \lambda \|a\|_\infty \, C(1 + R^{q+1}).\end{aligned}$$

Then, by taking $R > \mathrm{meas}(\Omega)$, condition (4.11) clearly follows for a suitable choice of λ_*.

Next, we see that, for any $\lambda > 0$, there exists $u_1 \in BV(\Omega)$, with $\|u_1\|_{BV} > R$, such that

$$\mathcal{I}(u_1) < \mathcal{I}(0).$$

Indeed, denoting by χ_E the characteristic function of the Caccioppoli set E considered in (H_{12}), one simply takes $u_1 = k\chi_E$ with $k > 0$ sufficiently large.

Step 2 : *Existence of almost critical points.* Henceforth, we fix $\lambda \in \,]0, \lambda_*[$. Let us set $\Gamma = \{\gamma \in C^0([0,1], BV(\Omega))\colon \gamma(0) = 0,\ \gamma(1) = u_1\}$. From *Step* 1, we infer that

$$c_I = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \mathcal{I}(\gamma(t)) > \max\{\mathcal{I}(0),\ \mathcal{I}(u_1)\}.$$

Note that $c_I \geq \mathcal{I}(0) + \eta$, where $\eta > 0$ is independent of $\lambda \in \,]0, \lambda_*[$. Then, from the counterpart of [37, Lemma 3.7] for $\mathcal{I}$, there exist $(v_n)_n$ in $BV(\Omega)$ and $(\varepsilon_n)_n$ in $\mathbb{R}$ with

$$\lim_{n \to +\infty} \varepsilon_n = 0$$

such that, for every n,

$$c_I - \frac{1}{n} \leq \mathcal{I}(v_n) \leq c_I + \frac{1}{n} \tag{4.12}$$

and

$$\mathcal{J}(v) - \mathcal{J}(v_n) \geq \lambda \int_\Omega a h(v_n)(v - v_n) \, \mathrm{d}x + \varepsilon_n \|v - v_n\|_{BV} \quad \text{for all } v \in BV(\Omega). \tag{4.13}$$

Step 3 : *Estimates on the almost critical points.* The sequence $(v_n)_n$ satisfies

$$\sup_n \|v_n\|_{BV} < +\infty. \tag{4.14}$$

Indeed, from *Step* 2, taking $v = 2v_n$ as test function in (4.13) and observing that $\mathcal{J}(2v_n) \leq 2\mathcal{J}(v_n)$, we have

$$-\mathcal{J}(v_n) \leq -\lambda \int_\Omega a h(v_n)(v_n) \, \mathrm{d}x - \varepsilon_n \|v_n\|_{BV}. \tag{4.15}$$

Moreover, from (4.12), we get

$$(q+1)\mathcal{J}(v_n) \leq \lambda \int_\Omega a(q+1) H(v_n) \, \mathrm{d}x + (q+1)(c_I + \tfrac{1}{n}). \tag{4.16}$$

Summing up (4.15) and (4.16), we obtain

$$q\mathcal{J}(v_n) \leq \lambda \int_\Omega a\Big((q+1)H(v_n) - h(v_n)v_n\Big) \, \mathrm{d}x + (q+1)(c_I + \tfrac{1}{n}) - \varepsilon_n \|v_n\|_{BV}.$$

Thanks to condition (4.2) in (H_{11}) and the embedding of $BV(\Omega)$ into $L^1(\Omega)$, for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that, for all n,

$$q\|v_n\|_{BV} \leq q\mathcal{J}(v_n) \leq (\lambda\|a\|_\infty\varepsilon - \varepsilon_n)\,\|v_n\|_{BV} + \lambda\|a\|_\infty c_\varepsilon + (q+1)(c_I + \tfrac{1}{n}).$$

By taking $\varepsilon > 0$ sufficiently small, we conclude that (4.14) holds.

Step 4 : *Existence of a positive bounded variation solution.* By the compact embedding of $BV(\Omega)$ into $L^{q+1}(\Omega)$, there exist a subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, and $u \in BV(\Omega)$ such that $\lim_{n\to+\infty} v_n = u$ in $L^{q+1}(\Omega)$ and a.e. in Ω. By passing to the inferior limit in (4.13) and using the lower semicontinuity of $\mathcal{J}$ with respect to the L^1-convergence in $BV(\Omega)$, we obtain

$$\begin{aligned}\mathcal{J}(v) - \lambda\int_\Omega ah(u)v\,\mathrm{d}x &= \mathcal{J}(v) - \lambda\lim_{n\to+\infty}\int_\Omega ah(v_n)v\,\mathrm{d}x\\ &\geq \liminf_{n\to+\infty}\mathcal{J}(v_n) - \lambda\lim_{n\to+\infty}\int_\Omega ah(v_n)v_n\,\mathrm{d}x \geq \mathcal{J}(u) - \lambda\int_\Omega ah(u)u\,\mathrm{d}x,\end{aligned}$$

for all $v \in BV(\Omega)$. Hence, condition (4.10) holds and thus u is a solution of (4.1).

Next, we show that $u \geq 0$. By using the lattice property stated in [34, Proposition 2.1], we easily get from (4.10)

$$0 = \lambda\int_\Omega ah(-u^-)u^-\,\mathrm{d}x = \lambda\int_\Omega ah(u)(u^+ - u)\,\mathrm{d}x \leq \mathcal{J}(u^+) - \mathcal{J}(u) \leq \mathcal{J}(0) - \mathcal{J}(-u^-).$$

Since 0 is a solution of (4.1), we also have $\mathcal{J}(0) - \mathcal{J}(-u^-) \leq 0$. Hence, we infer

$$\mathcal{J}(0) = \mathcal{J}(-u^-) = \int_\Omega \sqrt{1+|Du^-|^2} + \int_{\partial\Omega}|u^-|\,\mathrm{d}\mathcal{H}_{N-1}.$$

This yields $u^- = 0$. Taking u as test function in (4.13), we notice that

$$\limsup_{n\to\infty}\mathcal{J}(v_n) \leq \mathcal{J}(u).$$

Thus, by the lower semicontinuity of $\mathcal{J}$ with respect to the L^1-convergence, we obtain

$$\lim_{n\to\infty}\mathcal{J}(v_n) = \mathcal{J}(u),$$

and so

$$\lim_{n\to\infty}\mathcal{I}(v_n) = \mathcal{I}(u).$$

Hence, we derive from (4.12) that $\mathcal{I}(u) = c_I$. As $c_I > \mathcal{I}(0)$, we conclude that $u > 0$.

Step 5 : *Behavior of solutions as* $\lambda \to 0^+$. In order to prove that the solutions $u = u_\lambda$ can be chosen so that (4.9) holds, we suppose by contradiction that, for every n, there exist $\lambda_n \in\,]0, \lambda_*[$ and a solution u_n of (4.1) such that

$$\lim_{n\to+\infty}\lambda_n = 0, \quad \sup_n\|u_n\|_{L^{q+1}} < +\infty, \quad \text{and} \quad \inf_n\mathcal{I}(u_n) \geq \mathcal{I}(0) + \eta. \tag{4.17}$$

From (4.5) and (4.3), we can find a possibly larger constant $C > 0$ such that, for all n,

$$\mathcal{J}(0) \leq \mathcal{J}(u_n) \leq \lambda_n\int_\Omega ah(u_n)u_n\,\mathrm{d}x + \mathcal{I}(0) \leq \lambda_n\|a\|_\infty C(\|u_n\|^{q+1}_{L^{q+1}} + 1) + \mathcal{J}(0).$$

Hence, letting $n \to +\infty$ and using the first two relations in (4.17), we obtain

$$\lim_{n\to+\infty} \mathcal{J}(u_n) = \mathcal{J}(0).$$

Similarly, from (4.4) we derive

$$\mathcal{I}(u_n) = \mathcal{J}(u_n) - \lambda_n \int_\Omega aH(u_n)\,\mathrm{d}x \le \mathcal{J}(u_n) + \lambda_n \|a\|_\infty C(\|u_n\|_{L^{q+1}}^{q+1} + 1).$$

Letting $n \to +\infty$ and using again the first two relations in (4.17), we infer

$$\lim_{n\to+\infty} \mathcal{I}(u_n) = \mathcal{J}(0) = \mathcal{I}(0),$$

which is in contradiction with the third relation in (4.17). This concludes the proof. □

Combining Theorem 2.2 and Theorem 4.3 yields the following result.

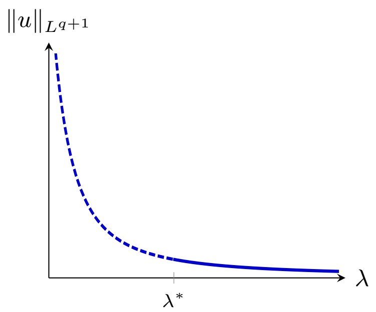
Corollary 4.4. *Assume* (H_1), (H_4)–(H_8), (H_{11}), *and*

(H_{13}) *there exist* $p > 1$, *with* $p < \frac{N+2}{N-2}$, *if* $N \ge 3$, *and* $k > 0$ *such that*

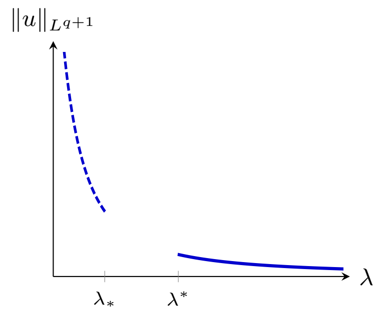
$$\lim_{s\to 0} \frac{h(s)}{|s|^{p-1}s} = k.$$

Then, there exist $\lambda_* > 0$ *and* $\lambda^* > 0$ *such that, for all* $\lambda \in \,]0, \lambda_*[\, \cup \,]\lambda^*, +\infty[$, *problem* (4.1) *has at least one positive solution* $u \in BV(\Omega)$.

Remark 4.5. It is worth noticing that in dimension $N = 1$ we can prove that problem (4.1) has at least one positive bounded variation solution for any given $\lambda > 0$. Indeed, due to the continuous embedding of $BV(\Omega)$ into $L^\infty(\Omega)$, one sees that inequality (4.11) holds without any restriction on the parameter $\lambda > 0$; in particular, one can take $\lambda_* > \lambda^*$. Hence, it follows that, for all $\lambda \in \,]0, +\infty[$, there exists a positive bounded variation solution of problem (4.1), which is regular for $\lambda > \lambda^*$. It remains an open question to ascertain whether the same conclusion holds true in higher dimension as well (see Figure 3).



(a) Case $N = 1$.



(b) Case $N \ge 2$.

Figure 3: Bifurcation diagrams for problem (4.1). Dotted lines indicate singular solutions and solid lines stand for regular solutions.

Remark 4.6. In dimension $N = 1$ the regularity of all solutions of problem (4.1) may be proven along the lines of [25], placing suitable condition on the behavior of the weight a near its nodal points. This matter will be discussed in detail elsewhere.

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