CALENDAR YEAR EFFECT MODELING FOR CLAIMS RESERVING IN HGLM

Patrizia Gigante Liviana Picech Luciano Sigalotti

ABSTRACT

Claims reserving models are usually based on data recorded in run-off tables, according to the origin and the development years of the payments. The amounts on the same diagonal are paid in the same calendar year and are influenced by some common effects, e.g. claims inflation, that can induce dependence among payments. We introduce Hierarchical Generalized Linear Models (HGLM) with risk parameters related to the origin and the calendar years, in order to model the dependence both among payments of the same origin year and of the same calendar year. Besides the random effects, the linear predictor also includes fixed effects. All the parameters are estimated within the model by the h-likelihood approach. The prediction for the outstanding claims and an approximate formula to evaluate the mean square error of prediction are obtained. Moreover, a parametric bootstrap procedure is delineated to get an estimate of the predictive distribution of the outstanding claims. A Poisson-gamma HGLM with origin and calendar year effects is studied extensively and a numerical example is provided. We find that the estimates of the correlations can be significant for payments in the same calendar year and that the inclusion of calendar effects can determine a remarkable impact on the prediction uncertainty.

Keywords: Calendar year effects, Claims reserving, Dependence modeling, Hierarchical generalized linear models, Poisson-gamma model, Mean square error of prediction, Simulation.

1 Introduction

Claims reserving in non-life insurance is often based on run-off data where claim payments are recorded according to the origin year and the development year. The payments on the same diagonal of the run-off table share the same calendar/accounting year, hence they are subject to common effects connected with the year of payment, that can induce dependence among payments of the same calendar year. As a consequence, dependencies among payments of different origin years arise, whereas stochastic claims reserving models usually assume the independence. It emerges the need of adequately model such dependencies by taking account of the calendar year effects.

The calendar year effect is also generally referred to as claims inflation. Verbeek (1972)[21] and Taylor (1977)[19] introduce the separation method to separate claims inflation from the development pattern of the payments. Björkwall *et al.* (2010)[1] develop a bootstrap procedure for estimating the predictive distribution of the claims reserve and assessing the prediction error for the separation method. Jessen and Rietdorf (2011)[10] present two models with diagonal effects to account for claims inflation; the parameter estimation is based on the separation method and for the forecast of the diagonal effects an autoregressive process is used. Bohnert *et al.* (2016)[2] analyze the main driving factors for inflation in automobile insurance. They study the impact of claims inflation on claims reserving and extend the model in Björkwall *et al.* (2010)[1] by accounting for an extrapolation of future claims inflation using stochastic inflation models. Within the Chain-Ladder-type models the impact of claims inflation is studied e.g. in Brydon and Verrall (2009)[3] and in Kuang *et al.* (2011)[9].

Other contributions in literature mainly focus on modeling stochastic dependences among payments, caused by calendar year effects, in order to study the influence of such dependences on claims reserves and prediction errors. In this context, by assuming a Bayesian set-up, Wüthrich (2010)[23] studies a Bayes Chain Ladder model that allows for inference on calendar year random parameters. Within the same framework, Salzmann and Wüthrich (2012)[18] define a multivariate Bayes Chain Ladder model that enables modeling dependence along accounting years and study the sensitivities of claims reserves and prediction uncertainty as a function of a correlation parameter within accounting years. In the credibility framework, Bühlmann and Moriconi (2015)[5] develop a stochastic claims reserving model that extends the Bühlmann-Straub claims reserving model. Besides the risk parameters for the origin years, also risk parameters for the calendar years are considered, whereas the development pattern is assumed as given and equal to the Chain Ladder one.

We refer to the previous papers for further references on claims inflation modeling and calendar year effects.

The quoted studies show that the calendar year effects might be significant for the reserve estimate and can have a substantial impact on the evaluation of prediction uncertainty.

In this paper, we define a model that allows for the introduction of dependences along origin and payment years and, on the lines traced by the last-mentioned papers, we study the effects of such dependences on claims reserve evaluation, whereas we do not deal with the problem of claims inflation estimation and extrapolation.

In the Bayesian and credibility frameworks, the dependence induced by calendar year effects is modeled through diagonal risk parameters. An extension of Generalized Linear Models (GLM), the Hierarchical Generalized Linear Models (HGLMs) (Lee *et al.* (2006)[15]), allows including risk parameters in the model, by means of random effects in the linear predictor. Claims reserving in the HGLM framework is considered in Gigante *et al.* (2013a)[6], (2013b)[7] and (2016)[8]. The HGLM approach to claims reserving is also discussed in Verrall and Wüthrich (2015)[22].

We assume for the standardized payments a HGLM with risk parameters related to the origin and the calendar years. The two sets of risk parameters allow modeling the dependence among payments of the same origin year and of the same calendar year, respectively. Moreover, the parameters of the distributions of the random effects allow incorporating external information such as expert opinion. Besides the random effects, the linear predictor also includes fixed effects that can be used to model the claims development pattern. All the parameters are estimated within the model by maximum h-likelihood. We introduce a predictor for the outstanding claims and take advantage of an approximate formula, developed in Gigante *et al.* (2013a), to evaluate the mean square error of prediction (MSEP). The approximations are based on asymptotic results and the calculations are straightforward once the model estimates are available. The model is also appropriate for a simulation approach, so that, through a parametric bootstrap procedure, it is possible to get an estimate of the predictive distribution of the outstanding claims and to evaluate the MSEP. Moreover, simulation allows us to check the effect of the approximations in the formula of the MSEP.

In particular, we study a Poisson-gamma HGLM with random origin and calendar year effects, whereas the claims development pattern is modeled by fixed effect parameters. For this model we provide a numerical example. The parameters of the distributions of the origin year random effects are used to incorporate external information on the ultimate claims. The estimates of the covariances between payments show remarkable correlations for payments of the same calendar year. As in other studies, it results that the insertion of diagonal risk parameters has an effect on the claims reserve estimate and an even more remarkable effect on the MSEP.

The paper is organized as follows. In Section 2, we introduce the model assumptions. In Section 3 we recall the estimation procedure for the HGLMs based on the h-likelihood. Section 4 is devoted to the prediction problem and to the MSEP evaluation in claims reserving. In Section 5 a Poisson-gamma HGLM with calendar year effects is introduced. In Section 6, we develop a numerical example on the same data set used in Bühlmann and Moriconi (2015)[5] and Gigante *et al.* (2013b)[7], in order to make some comparisons of the results of the three models. Section 7 contains a simulation study. Finally, Section 8 concludes the paper.

2 Model assumptions

With respect to the claims of a portfolio, let $P_{i,j}$ be the incremental payments and $Y_{i,j} = P_{i,j}/\omega_{i,j}$ the payments standardized with respect to some known exposure measure $\omega_{i,j}$, where i denotes the origin year (e.g. accident year, underwriting year) and j the development year, $i, j = 0, \dots, t$. We assume that all claims are settled within $t + 1$ years and that a run-off triangle of data $y_{i,j} = p_{i,j}/\omega_{i,j}$, $i, j = 0, \dots, t$, $i + j \leq t$, is available.

For the random process $\{Y_{i,j}, i, j = 0, \dots, t\}$, we consider a mixture model depending on a vector of risk parameters related to the origin and the calendar (accounting, payment) years. The model belongs to the class of Hierarchical Generalized Linear Models. The assumptions, the estimation procedure and the prediction approach trace those in Gigante *et al.* (2013a)[6], (2013b)[7], (2016)[8], to which we refer for the technical details.

Let $(\boldsymbol{U},\boldsymbol{V}) = (U_0,\ldots,U_t,V_0,\ldots,V_{2t})$ the vector of the risk parameters, where U_i is related to the origin year i and V_{i+j} to the calendar year $i+j$.

Model Assumptions

(a1) Independence assumptions

The components of the risk parameter $(\boldsymbol{U},\boldsymbol{V}) = (U_0,\ldots,U_t,V_0,\ldots,V_{2t})$ are independent.

Conditionally on $(\boldsymbol{U},\boldsymbol{V})$, the response variables $Y_{i,j}$, $i,j = 0,\ldots,t$, are independent.

With respect to the risk parameters $(\boldsymbol{U},\boldsymbol{V})$, the conditional distribution of $Y_{i,j}$ only depends on U_i and V_{i+j}.

(a2) Distributional assumptions for the responses conditional on the risk parameters

The distribution of $Y_{i,j}|(U_i,V_{i+j}) = (u_i,v_{i+j})$ belongs to an Exponential Dispersion Family (EDF) with cumulant and variance functions b and V, respectively. So that

$$E[Y_{i,j}|(U_i,V_{i+j}) = (u_i,v_{i+j})] = \mu_{i,j},$$

$$var[Y_{i,j}|(U_i,V_{i+j}) = (u_i,v_{i+j})] = \frac{\phi_{i,j}}{\omega_{i,j}}V(\mu_{i,j}).$$

As it is quite natural, the weights are assumed equal to the exposure measures, however this is not necessary.

(a3) Structural assumptions for the response variables

The expectations of the conditional standardized payments are given by

$$E[Y_{i,j}|(U_i,V_{i+j}) = (u_i,v_{i+j})] = \mu_{i,j} = g^{-1}(\boldsymbol{x}_{i,j}^T\boldsymbol{\beta} + w_{U,i} + w_{V,i+j}),$$

where $\boldsymbol{x}_{i,j}$ is a vector of covariates; $\boldsymbol{\beta}$ are the regression parameters, called *fixed effects*; $\boldsymbol{w} = (w_{U,0},\ldots,w_{U,t},w_{V,0},\ldots,w_{V,2t})$ are the *random effects*, with $w_{U,i} = g_U(u_i)$ and $w_{V,i+j} = g_V(v_{i+j})$. The functions g, g_U and g_V are strictly monotone with first and second order continuous derivatives.

(a4) Distributional assumptions for the risk parameters

Let $W_{U,i} = g_U(U_i)$ and $W_{V,i+j} = g_V(V_{i+j})$. We assume that the densities of $W_{U,i}$ and $W_{V,i+j}$ are

$$f_{W_{U,i}}(w) = \exp\left\{\frac{1}{\lambda_{U,i}}\left(\psi_{U,i}\theta_U - b_U(\theta_U)\right)\right\}c_U(\psi_{U,i},\lambda_{U,i}),$$
$$f_{W_{V,i+j}}(w) = \exp\left\{\frac{1}{\lambda_{V,i+j}}\left(\psi_{V,i+j}\theta_V - b_V(\theta_V)\right)\right\}c_V(\psi_{V,i+j},\lambda_{V,i+j}),$$

where: b_U, b_V are cumulant functions of EDFs; $\theta_U = b_U'^{-1}(g_U^{-1}(w))$, $\theta_V = b_V'^{-1}(g_V^{-1}(w))$; $\psi_{U,i}$, $\lambda_{U,i}$, $\psi_{V,i+j}$, $\lambda_{V,i+j}$ are parameters; $c_U(\psi_{U,i}, \lambda_{U,i})$, $c_V(\psi_{V,i+j}, \lambda_{V,i+j})$ are normalizing functions.

The above assumptions define a mixture model with mixing distribution the distribution of $\boldsymbol{W} = (\boldsymbol{W_U}, \boldsymbol{W_V}) = (W_{U,0}, \ldots, W_{U,t}, W_{V,0}, \ldots, W_{V,2t})$. Since the risk parameters are introduced in the regression structure through random effects, the model belongs to the class of mixed models.

The random parameters U_i and V_{i+j} take account of risk characteristics of the origin year i and the calendar year $i+j$, respectively. Such risk parameters allow for the modeling of dependence of the $Y_{i,j}$ related to origin year effects (e.g. correlation patterns among repeated payments of claims of the same origin year) and to accounting year effects (e.g. claims inflation). Note that, calendar year effects, such as claims inflation or, as pointed out in Taylor (1977), exogenous influences operating in experience years, influence all the payments in the same diagonal of the run-off table and introduce dependence also between different accident years.

In fact, the covariances of the response variables, for $(i,j) \neq (h,k)$, are

$$\begin{aligned} cov(Y_{i,j}, Y_{h,k}) &= E\left[cov(Y_{i,j}, Y_{h,k})|(\boldsymbol{U}, \boldsymbol{V})\right] + cov\left[E(Y_{i,j}|(\boldsymbol{U}, \boldsymbol{V}), E(Y_{h,k}|(\boldsymbol{U}, \boldsymbol{V})\right] \\ &= cov\left[g^{-1}(\boldsymbol{x}_{i,j}^T\boldsymbol{\beta} + W_{U,i} + W_{V,i+j}), g^{-1}(\boldsymbol{x}_{h,k}^T\boldsymbol{\beta} + W_{U,h} + W_{V,h+k})\right], \qquad (2.1) \end{aligned}$$

where the second equality follows from the conditional independence of the response variables, so that $cov(Y_{i,j}, Y_{h,k})|(\boldsymbol{U}, \boldsymbol{V}) = 0$. The last term is null if $i \neq h$ and $i+j \neq h+k$, due to the independence of the random effects, but it is not null otherwise, that is if the two responses refer to the same origin year, $i = h$, or to the same calendar year, $i+j = h+k$.

Coming back to the model specifications, if, in particular, g_U is the canonical link of b_U, that is $g_U = b_U'^{-1}$, then we have $\theta_U = w$ and the distribution of $W_{U,i} = b_U'^{-1}(U_i)$ belongs to the conjugate family of the EDF with cumulant b_U. In this case, under suitable hypotheses, the *hyperparameters* $\boldsymbol{\psi_U} = (\psi_{U,0}, \ldots, \psi_{U,t})$, $\boldsymbol{\lambda_U} = (\lambda_{U,0}, \ldots, \lambda_{U,t})$ are related to the moments of the risk parameter $\boldsymbol{U}$. Specifically, $\psi_{U,i} = E(U_i)$ (see e.g. Jewell (1974)[11]; Bühlmann, Gisler (2005)[4]) and, in Tweedie models with variance function $V(\mu) = \mu^p$, $\lambda_{U,i} = var(U_i)/E(U_i^p)$ (see Ohlsson, Johansson (2006)[17]). Similar considerations apply to $W_{V,i+j}$.

If $b = b_U = b_V$ and $g = g_U = g_V = b'^{-1}$, then the distributions of both $W_{U,i} = b'^{-1}(U_i)$ and $W_{V,i+j} = b'^{-1}(V_{i+j})$ are conjugate of the distribution of $Y_{i,j}|(U_i, V_{i+j}) = (u_i, v_{i+j})$. In this case, the HGLM is called *conjugate*.

In the following, we assume that the parameters $\boldsymbol{\psi_U} = (\psi_{U,0}, \ldots, \psi_{U,t})$, $\boldsymbol{\psi_V} = (\psi_{V,0}, \ldots, \psi_{V,2t})$ are given and that they are the expected values of the risk parameters. We remark that the values of the parameters $\boldsymbol{\psi_U}$, $\boldsymbol{\psi_V}$ can be used to incorporate external information into the model (see the example in Section 6).

We remark that the risk parameters are assumed to be independent. This assumption could be questionable particularly for the calendar year parameters, since there may be trends in the data due to calendar year effects. It can be accepted if we interpret the calendar year parameters as random variations around a trend and assume that any calendar year trend (e.g. economic inflation) was preliminarily removed from the data. Alternatively, we could incorporate in the model some trend estimates through the $\boldsymbol{\psi_V} = (\psi_{V,0}, \ldots, \psi_{V,2t})$. This would require side-estimates that would not be integral to the model, by using any available exogenous or collateral information, but possibly also informed by the model estimates of past calendar year effects. However, we note that in the case of superimposed inflation it could be difficult to detect a trend. Moreover, superimposed inflation may appear in a form that involves inherent serial correlation. Therefore, on the one hand, the independence assumption simplifies the model, but on the other, it determines some limitations.

3 Parameter estimation

In order to estimate the parameters in models with fixed and random effects, Lee, Nelder (1996) [13], Lee, Nelder (2001) [14], Lee *et al.* (2006) [15], introduced the *hierarchical log-likelihood.* In our problem the h-log-likelihood is the joint log-density evaluated at the data $\boldsymbol{y} = (y_{i,j}, i+j \leq t)$,

$$h = \log f_{(\boldsymbol{Y},\boldsymbol{W})} = l_{\boldsymbol{Y}|\boldsymbol{W}=\boldsymbol{w}} + l_{\boldsymbol{W_U}} + l_{\boldsymbol{W_V}}, \tag{3.1}$$

where $f_{(\boldsymbol{Y},\boldsymbol{W})}$ denotes the joint density of $(\boldsymbol{Y}, \boldsymbol{W_U}, \boldsymbol{W_V})$, $l_{\boldsymbol{Y}|\boldsymbol{W}=\boldsymbol{w}}$ the log-likelihood of $\boldsymbol{Y}|\boldsymbol{W} = \boldsymbol{w}$, which is equal to the log-likelihood of $\boldsymbol{Y}|(\boldsymbol{U}, \boldsymbol{V}) = (\boldsymbol{u}, \boldsymbol{v})$, $l_{\boldsymbol{W_U}}$ and $l_{\boldsymbol{W_V}}$ are the logarithms of the densites of $\boldsymbol{W_U}$ and $\boldsymbol{W_V}$.

We do not explain here in detail the estimation approach, since it can be obtained by simple adaption of the procedures described in Gigante *et al.* (2013a)[6], (2016)[8]. We just outline it and remark some specific aspects of the current model.

If in addition to $\boldsymbol{\psi_U}$, $\boldsymbol{\psi}_V$ and $\boldsymbol{\omega} = (\omega_{i,j}, i, j = 0, \ldots, t)$, also the dispersion parameters $\boldsymbol{\phi} = (\phi_{i,j}, i, j = 0, \ldots, t)$, related to the standardized payments, and $\boldsymbol{\lambda}_U$, $\boldsymbol{\lambda}_V$, related to the risk parameters, are known, ignoring irrelevant constant terms, we get

$$\begin{aligned} h(\boldsymbol{\beta}, \boldsymbol{w}; \boldsymbol{\phi}, \boldsymbol{\lambda_U}, \boldsymbol{\lambda}_V; \boldsymbol{y}, \boldsymbol{\psi_U}, \boldsymbol{\psi}_V, \boldsymbol{\omega}) &= \sum_{i,j:i+j\leq t} \frac{\omega_{i,j}}{\phi_{i,j}} \left[y_{i,j}\theta_{i,j} - b(\theta_{i,j}) \right] \\ + \sum_{i=0}^{t} \frac{1}{\lambda_{U,i}} \left[\psi_{U,i}\theta_{U,i} - b_U(\theta_{U,i}) \right] &+ \sum_{i,j:i+j\leq 2t} \frac{1}{\lambda_{V,i+j}} \left[\psi_{V,i+j}\theta_{V,i+j} - b_V(\theta_{V,i+j}) \right], \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} &\theta_{i,j} = b'^{-1}(g^{-1}(\boldsymbol{x}_{i,j}^T\boldsymbol{\beta} + w_{U,i} + w_{V,i+j}), \\ &\theta_{U,i} = b_U'^{-1}(g_U^{-1}(w_{U,i})), \\ &\theta_{V,i+j} = b_V'^{-1}(g_V^{-1}(w_{V,i+j})). \end{aligned}$$

The h-log-likelihood (3.2) can be viewed as the log-likelihood of an *augmented* GLM for the data $\boldsymbol{y}$ and *pseudo-data* $\boldsymbol{\psi_U}$, $\boldsymbol{\psi_V}$, with weights $\omega_{i,j}/\phi_{i,j}$, $i + j \leq t$, $1/\lambda_{U,i}$, $i = 0, \ldots, t$, $1/\lambda_{V,i+j}$, $i + j \leq 2t$, respectively, and dispersion parameter 1.

Notice that in order to interpret (3.2) as the log-likelihood of a genuine GLM, we should have $b = b_U = b_V$ and $g = g_U = g_V$.

The augmented GLM has the following regression structure

$$\begin{array}{ll} \eta_{i,j} = g(\mu_{i,j}) = \boldsymbol{x}_{i,j}^T\boldsymbol{\beta} + w_{U,i} + w_{V,i+j}, & i+j \leq t, \\ \eta_{U,i} = g_U(u_i) = w_{U,i}, & i = 0, \dots, t, \\ \eta_{V,i+j} = g_V(v_{i+j}) = w_{V,i+j}, & i+j \leq 2t. \end{array}$$

The maximum h-log-likelihood estimates of the fixed and random effects $\boldsymbol{\delta} = (\boldsymbol{\beta}^T, \boldsymbol{w}^T)^T$ are the solutions of the system

$$\left\{ \begin{array}{rcl} \partial h / \partial \boldsymbol{\beta} & = & \mathbf{0} \\ \partial h / \partial \boldsymbol{w} & = & \mathbf{0}. \end{array} \right.$$

It is easy to verify that the above conditions imply that the estimate of $w_{V,i+j}$, with $i + j > t$, coincides with $g_V(\psi_{V,i+j})$:

$$\hat{w}_{V,i+j} = g_V(\psi_{V,i+j}), \qquad i+j > t. \tag{3.3}$$

The system can be solved by the Iterative Weighted Least Squares algorithm. The inverse $\boldsymbol{\mathcal{I}}(\hat{\boldsymbol{\delta}})^{-1}$ of the Fisher information matrix of the augmented GLM, evaluated at the estimate $\hat{\boldsymbol{\delta}}$, is an estimate of the variance-covariance matrix

$$var \left[\begin{array}{c} \tilde{\boldsymbol{\beta}} \\ \tilde{\boldsymbol{w}} - \boldsymbol{W} \end{array} \right],$$

where $\tilde{\boldsymbol{\beta}}$, $\tilde{\boldsymbol{w}}$ are the estimators of the fixed and random effects. The estimator $\tilde{\boldsymbol{w}}$ of the parameter $\boldsymbol{w}$ is a predictor of the random vector $\boldsymbol{W}$. In particular, by (3.3), the maximum h-log-likelihood estimator of $w_{V,i+j}$, with $i+j > t$, is $\tilde{w}_{V,i+j} = g_V(\psi_{V,i+j})$.

In this way, by estimating the augmented GLM, we get estimates of the model parameters and of the standard errors of their estimators.

The HGLMs have been extended to quasi-HGLMs, allowing for the possibility of specifying only the first two moments of the distributions of the conditional responses and/or the risk parameters. Moreover, by following the Extended Quasi-Likelihood approach proposed by Nelder and Pregibon (1987) [16], also the dispersion parameters can be estimated and they can have their own regression structures. The parameters of such models can be estimated through an algorithm in which four interconnected GLMs are fitted iteratively: the above augmented GLM to obtain the fixed and random effects for given dispersion parameters, and three suitable GLMs with gamma distributed responses to obtain the regression parameters of the dispersion components, given the fixed and random effects.

The delineated estimation process allows obtaining the estimates of the fixed effects $\hat{\boldsymbol{\beta}}$, of the origin year effects $\hat{u}_i$, $i = 0, \dots, t$, of the calendar year effects $\hat{v}_{i+j}$, $i+j = 0, \dots, 2t$, and of the variance-covariance matrix of the parameter estimators.

4 Reserve prediction and prediction error

In order to predict the outstanding claims and evaluate the quality of the prediction, as usually done, we restrict ourselves to considering the exposures related to the origin years only, ω_i. Moreover, we assume that the dispersion parameters are constant, denoted by ϕ, λ_U, λ_V.

Let

$$R_i = \sum_{i=t-i+1}^{t} P_{i,j} = \sum_{i=t-i+1}^{t} \omega_i Y_{i,j},$$

denote the outstanding claims of the origin year i, $i = 1, \ldots, t$, and

$$R = \sum_{i=1}^{t} R_i = \sum_{i,j:i+j>t} P_{i,j} = \sum_{i,j:i+j>t} \omega_i Y_{i,j}.$$

the total outstanding claims.

The conditional expectation of R, at time t, is

$$E(R|\mathcal{D}_t) = \sum_{i,j:i+j>t} \omega_i E(Y_{i,j}|\mathcal{D}_t), \tag{4.1}$$

where $\mathcal{D}_t = \{Y_{i,j}, i + j \leq t\}$.

By the tower property of the conditional expectation and the conditional independence of the $Y_{i,j}$, given $(\boldsymbol{U}, \boldsymbol{V})$, we get

$$E(Y_{i,j}|\mathcal{D}_t) = E\left[E(Y_{i,j}|\mathcal{D}_t, \boldsymbol{U}, \boldsymbol{V})|\mathcal{D}_t\right] = E\left[g^{-1}(\boldsymbol{x}_{i,j}^T\boldsymbol{\beta} + W_{U,i} + W_{V,i+j})|\mathcal{D}_t\right]. \tag{4.2}$$

Now, we assume that the parameter estimates $\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{w}}$ and the corresponding estimators $\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{w}}$ provide estimates and estimators of the linear predictors $\boldsymbol{x}_{ij}^T\boldsymbol{\beta} + w_{U,i} + w_{V,i+j}$, also for $i + j > t$. Note that this does not allow considering the payment year as a categorical covariate in $\boldsymbol{x}_{i,j}$.

As a predictor for $Y_{i,j}$, $i+j > t$, we consider the following estimator of $E(Y_{i,j}|\mathcal{D}_t)$

$$\tilde{Y}_{i,j} = g^{-1}(\boldsymbol{x}_{i,j}^T\tilde{\boldsymbol{\beta}} + \tilde{w}_{U,i} + \tilde{w}_{V,i+j}),$$

where $\tilde{\boldsymbol{\beta}}, \tilde{\boldsymbol{w}}_U, \tilde{\boldsymbol{w}}_V$ are the maximum h-log-likelihood estimators. As noted in Section 3, the predictors of the random effects related to future calendar years are

$$\tilde{w}_{V,i+j} = g_V(\psi_{V,i+j}), \qquad i + j = t + 1, \ldots, 2t.$$

Hence, future diagonal effects are forecast as functions of hyperparameters associated with the distributions of diagonal risk parameters; these forecasts are certain because the hyperparameters are fixed. We note that also in Bühlmann, Moriconi (2015) [5] the estimators of the random parameters related to future calendar years are assumed to be certain, given by the a priori expected values of such parameters. Here, this is implied by the HGLM estimation approach, since they are the maximum h-log-likelihood estimators.

We obtain the following predictor for the total outstanding claims

$$\tilde{R} = \sum_{i,j:i+j>t} \omega_i g^{-1}(\boldsymbol{x}_{i,j}^T\tilde{\boldsymbol{\beta}} + \tilde{w}_{U,i} + \tilde{w}_{V,i+j}), \tag{4.3}$$

and the reserve estimate

$$\hat{R} = \sum_{i,j:i+j>t} \omega_i g^{-1}(\boldsymbol{x}_{i,j}^T \hat{\boldsymbol{\beta}} + \hat{w}_{U,i} + \hat{w}_{V,i+j}). \tag{4.4}$$

As a measure of prediction uncertainty, we use the conditional mean square error of prediction which takes account of the fluctuations of the outstanding claims around the predictor $\tilde{R}$. As in previous papers on claims reserve evaluation in the HGLM approach (see Gigante et al. (2013a)[6], (2013b)[7], (2016)[8]), we use an approximate formula for the MSEP based on the following decomposition

$$\begin{aligned} MSEP_{R|\mathcal{D}_t}(\tilde{R}) &= E\left[(R-\tilde{R})^2|\mathcal{D}_t\right] = var(R|\mathcal{D}_t) + \Big(E(R|\mathcal{D}_t) - \tilde{R}\Big)^2 \\ &= E[var(R|\boldsymbol{U},\boldsymbol{V})|\mathcal{D}_t] + var[E(R|\boldsymbol{U},\boldsymbol{V})|\mathcal{D}_t] + \Big(E(R|\mathcal{D}_t) - \tilde{R}\Big)^2. \end{aligned} \tag{4.5}$$

In the current model, the estimate becomes

$$\begin{aligned} \widehat{MSEP}_{R|\mathcal{D}_t}(\tilde{R}) = &\sum_{i,j:i+j>t} \frac{\hat{\phi}}{\omega_i} V\left(g^{-1}(\boldsymbol{x}_{i,j}^T \hat{\boldsymbol{\beta}} + \hat{w}_{U,i} + \hat{w}_{V,i+j})\right) \\ &+ \left\{\boldsymbol{J}_r(\boldsymbol{w})\boldsymbol{H}_{22}^{-1}\boldsymbol{J}_r(\boldsymbol{w})^T\right\}|_{\hat{\boldsymbol{\delta}}} + \left\{\boldsymbol{J}_f(\boldsymbol{\beta})\boldsymbol{G}^{-1}\boldsymbol{J}_f(\boldsymbol{\beta})^T\right\}|_{\hat{\boldsymbol{\delta}}}\,, \end{aligned} \tag{4.6}$$

where $\boldsymbol{J}_r$ and $\boldsymbol{J}_f$ denote the Jacobian matrices of the functions

$$\begin{aligned} r(\boldsymbol{w}) &= \sum_{i,j:i+j>t} \omega_i g^{-1}(\boldsymbol{x}_{i,j}^T \boldsymbol{\beta} + w_{U,i} + w_{V,i+j}), \\ f(\boldsymbol{\beta}) &= \sum_{i,j:i+j>t} \omega_i g^{-1}(\boldsymbol{x}_{i,j}^T \boldsymbol{\beta} + \tilde{w}_{U,i}(\boldsymbol{\beta}) + \tilde{w}_{V,i+j}(\boldsymbol{\beta})), \end{aligned}$$

with $\tilde{\boldsymbol{w}}(\boldsymbol{\beta})$ denoting the maximum h-log-likelihood estimator of $\boldsymbol{w}$ obtained for given $\boldsymbol{\beta}$. The Jacobian matrix of $\tilde{\boldsymbol{w}}(\boldsymbol{\beta})$ is given by $-\boldsymbol{H}_{22}^{-1}\boldsymbol{H}_{12}^T$ (Lee, Nelder (1996, Appendix C)[13], Lee, Ha (2010)[12]). The matrices $\boldsymbol{H}_{22}^{-1}$, $\boldsymbol{G}^{-1}$ and $\boldsymbol{H}_{12}^T$ are obtained from the Fisher information matrix of the augmented GLM and its inverse

$$\boldsymbol{\mathcal{I}}(\boldsymbol{\delta}) = \begin{bmatrix} \boldsymbol{H}_{11} & \boldsymbol{H}_{12} \\ \boldsymbol{H}_{12}^T & \boldsymbol{H}_{22} \end{bmatrix}, \qquad \boldsymbol{\mathcal{I}}(\boldsymbol{\delta})^{-1} = \begin{bmatrix} \boldsymbol{G}^{-1} & \boldsymbol{F} \\ \boldsymbol{F}^T & \boldsymbol{C} \end{bmatrix}, \tag{4.7}$$

where $\boldsymbol{H}_{11}$ denotes the block of the second derivatives of the h-log-likelihood with respect to $\boldsymbol{\beta}$, $\boldsymbol{H}_{22}$ denotes the block of the second derivatives with respect to $\boldsymbol{w}$ and $\boldsymbol{H}_{12}$ the block of the mixed derivatives.

The estimate of the conditional MSEP (4.5) takes account of the variability in the estimates of both the regression parameters $\boldsymbol{\beta}$ and the random effects $\boldsymbol{w}$. However, it does not allow for the variability in the dispersion parameter estimates, $\hat{\boldsymbol{\gamma}}_\phi$, $\hat{\boldsymbol{\gamma}}_{\lambda_U}$ and $\hat{\boldsymbol{\gamma}}_{\lambda_V}$. An insight into this aspect can be obtained from the standard errors estimated through the Fisher information matrices of the GLMs used to estimate such parameters.

In conclusion of this section, we would like to stress the usefulness of a closed formula for the MSEP, easy to calculate, although approximate, to get an evaluation of the quality of the reserve prediction. Alternatively, a simulation approach can be used, which allows getting much more information on the predictive distribution. However, we remark that in mixture models simulation techniques are often computationally demanding, because they require repeated estimation of the model parameters on the basis of re-sampled data. In Section 7 we illustrate a simulation approach to get estimates of the MSEP and of the predictive distribution of the outstanding loss liabilities.

5 Poisson-gamma HGLM with calendar year effects

As an example of a model that falls within the scope of the paper, we consider an extension of the Poisson-gamma HGLM in Gigante *et al.* (2013b) [7], obtained by adding random calendar year effects.

More precisely, in the model assumptions of Section 2, we consider the following specifications.

The response variables $Y_{i,j}$ are the incremental payments $P_{i,j}$ (unstandardized).

In $(a2)$, the conditional distribution of $Y_{i,j}|(U_i, V_{i+j}) = (u_i, v_{i+j})$ is overdispersed Poisson, with constant dispersion parameter ϕ and weight 1. Hence we have a quasi-HGLM.

In $(a3)$, the link function g and the functions g_U, g_V, that transform the risk parameters, are the logarithm. It follows that we obtain a multiplicative model for the conditional expected value $\mu_{i,j}$ of $Y_{i,j}$,

$$E[Y_{i,j}|(U_i, V_{i+j})] = \exp(\beta_j + \log(U_i) + \log(V_{i+j})) = e^{\beta_j} U_i V_{i+j},$$

where the fixed effects are related to the development years only.

In $(a4)$, the distributions of $W_{U,i} = \log(U_i)$ and $W_{V,i+j} = \log(V_{i+j})$ are conjugate of the Poisson EDF, with constant dispersion parameters, that is

$$f_{W_{U,i}}(w) = \exp\left\{\frac{1}{\lambda_U}\left(\psi_{U,i} w - \exp(w)\right)\right\} c_U(\psi_{U,i}, \lambda_U),$$
$$f_{W_{V,i+j}}(w) = \exp\left\{\frac{1}{\lambda_V}\left(\psi_{V,i+j} w - \exp(w)\right)\right\} c_V(\psi_{V,i+j}, \lambda_V).$$

Hence, U_i and V_{i+j} are gamma distributed with $E(U_i) = \psi_{U,i}$, $E(V_{i+j}) = \psi_{V,i+j}$, assumed to be given, and $var(U_i) = \psi_{U,i}\lambda_U$, $var(V_{i+j}) = \psi_{V,i+j}\lambda_V$.

We discuss some aspects of the model.

The covariances of the response variables are (see (2.1))

$$
cov(Y_{i,j}, Y_{h,k}) = \begin{cases} var(Y_{i,j}) & i = h, i+j = h+k \\ \\ e^{\beta_j} e^{\beta_k} var(U_i) E(V_{i+j}) E(V_{h+k}) & i = h, i+j \neq h+k \\ \\ e^{\beta_j} e^{\beta_k} E(U_i) E(U_h) var(V_{i+j}) & i \neq h, i+j = h+k \\ \\ 0 & i \neq h, i+j \neq h+k \end{cases} \tag{5.1}
$$

where

$$
\begin{aligned}
var(Y_{i,j}) &= E\left[var(Y_{i,j}|U_i, V_{i+j})\right] + var\left[E(Y_{i,j}|U_i, V_{i+j})\right] \\
&= E\left[\phi e^{\beta_j} U_i V_{i+j}\right] + var\left[e^{\beta_j} U_i V_{i+j}\right] \\
&= \phi e^{\beta_j} E(U_i) E(V_{i+j}) + (e^{\beta_j})^2 \left[E(U_i^2) E(V_{i+j}^2) - E(U_i)^2 E(V_{i+j})^2\right].
\end{aligned} \tag{5.2}
$$

The correlations of payments of the same origin year or of the same calendar year are all positive. Given $\boldsymbol{\beta}$ and $\boldsymbol{\psi}_U$, the greater the variance of U_i, the greater the covariances of payments of origin year i. Analogously, given $\boldsymbol{\beta}$ and $\boldsymbol{\psi}_V$, the greater the variance of V_{i+j}, the greater the covariances of payments of calendar year $i+j$.

As for the parameter estimates, similarly as in Gigante et al. (2013b) [7], it can be proved that they satisfy the following conditions

$$
\begin{cases} exp(\hat{\beta}_j) = \frac{\sum_{i=0}^{t-j} y_{i,j}}{\sum_{i=0}^{t-j} \hat{u}_i \hat{v}_{i+j}} & j = 0, \ldots, t \\ \\ \hat{u}_i = \hat{z}_{U,i} \frac{\sum_{j=0}^{t-i} y_{i,j}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j) \hat{v}_{i+j}} + (1 - \hat{z}_{U,i}) \psi_{U,i} & i = 0, \ldots, t \\ \\ \hat{v}_k = \hat{z}_{V,k} \frac{\sum_{i+j=k} y_{i,j}}{\sum_{i+j=k} \exp(\hat{\beta}_j) \hat{u}_i} + (1 - \hat{z}_{V,k}) \psi_{V,k} & k = 0, \ldots, t \\ \\ \hat{v}_k = \psi_{V,k} & k = t+1, \ldots, 2t \end{cases} \tag{5.3}
$$

with

$$
\hat{z}_{U,i} = \frac{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j) \hat{v}_{i+j}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j) \hat{v}_{i+j} + \hat{\phi}/\hat{\lambda}_U} \tag{5.4}
$$

and

$$
\hat{z}_{V,k} = \frac{\sum_{i+j=k} \exp(\hat{\beta}_j) \hat{u}_i}{\sum_{i+j=k} \exp(\hat{\beta}_j) \hat{u}_i + \hat{\phi}/\hat{\lambda}_V}. \tag{5.5}
$$

Therefore, the risk parameter estimates follow a sort of credibility formula, in that they are mixtures of the a priori expected values and of estimates based on the available data. The weights depend on the ratios $\hat{\phi}/\hat{\lambda}_U$ and $\hat{\phi}/\hat{\lambda}_V$ of the estimates of the dispersion parameters.

By exploiting the expression of the origin year effect estimates, $\hat{u}_i$, the reserve estimate $\hat{R}_i$ is given by a mixture of two components: a Chain Ladder-type reserve, $\hat{R}_i^{CL-type}$, that is based on the last cumulative payments, and a Bornhuetter-Ferguson-type reserve, $\hat{R}_i^{BF-type}$, that takes account of the external estimate $\psi_{U,i}$. In fact, it is easy to verify that

$$\begin{aligned}\hat{R}_i &= \hat{u}_i \sum_{j=t-i+1}^{t} \exp(\hat{\beta}_j)\hat{v}_{i+j} \\ &= \left(\hat{z}_{U,i} \frac{\sum_{j=0}^{t-i} y_{i,j}}{\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)\hat{v}_{i+j}} + (1-\hat{z}_{U,i})\psi_{U,i} \right) \sum_{j=t-i+1}^{t} \exp(\hat{\beta}_j)\hat{v}_{i+j} \\ &= \hat{z}_{U,i}\hat{R}_i^{CL-type} + (1-\hat{z}_{U,i})\hat{R}_i^{BF-type}, \end{aligned} \tag{5.6}$$

where

$$\hat{R}_i^{CL-type} = \frac{(1-\hat{b}_{i,t-i})}{\hat{b}_{i,t-i}} \sum_{j=0}^{t-i} y_{i,j}, \qquad \hat{R}_i^{BF-type} = \psi_{U,i} \left(\sum_{j=0}^{t} \exp(\hat{\beta}_j)\hat{v}_{i+j} \right) (1-\hat{b}_{i,t-i}),$$

with

$$\hat{b}_{i,h} = \frac{\sum_{j=0}^{h} \exp(\hat{\beta}_j)\hat{v}_{i+j}}{\sum_{j=0}^{t} \exp(\hat{\beta}_j)\hat{v}_{i+j}}, \qquad h = 0, \dots, t. \tag{5.7}$$

Note that $\hat{b}_{i,h}$, $h = 0, \dots, t$, can be interpreted as the prediction of the claims development pattern of origin year i,

$$b_{i,h} = \frac{E(C_{i,h})}{E(C_{i,t})} = \frac{\sum_{j=0}^{h} \exp(\beta_j)\psi_{V,i+j}}{\sum_{j=0}^{t} \exp(\beta_j)\psi_{V,i+j}}, \qquad h = 0, \dots, t,$$

where $C_{i,j}$ denotes the cumulative payments in cell (i, j).

As it is well known, in the Chain Ladder method the outstanding loss liabilities of origin year i are given by

$$R_i^{CL} = C_{i,t-i} \frac{(1-b_{t-i})}{b_{t-i}},$$

where $b_h = 1/(f_h f_{h+1} \dots f_{t-1})$ denotes the claims development pattern calculated from the Chain Ladder development factors f_j, $j = 0, \dots, t-1$ (see Wüthrich and Merz (2008)[24], (2.1)). So that the term $\hat{R}_i^{CL-type}$ in (5.6) can be related to this reserve, but note that in our model the claims development pattern depends on the origin year and its estimate is also affected by the external estimates $\psi_{U,i}$, $\psi_{V,i+j}$. On the other hand, in the Bornhuetter-Ferguson method the outstanding loss liabilities of origin year i are given by

$$R_i^{BF} = \mu_i(1-b_{t-i}),$$

where μ_i represents an external estimate of the ultimate claims of origin year i and again b_h denotes the claims development pattern, usually calculated from the Chain Ladder factors. Hence the term $\hat{R}_i^{BF-type}$ can be related to a Bornhuetter-Ferguson reserve, if $\psi_{U,i}$ is seen as an external estimate of the ultimate claims of origin year i, that is corrected by taking account of the run-off data through the multiplicative term $\sum_{j=0}^{t} \exp(\hat{\beta}_j)\hat{v}_{i+j}$.

6 Numerical results

In this section we describe the results of a numerical example for the Poisson-gamma HGLM with random origin and calendar year effects described in Section 5.

The data are the incremental payments and the prior ultimate claims provided in Table 2.5 and in Table 2.4 in Wüthrich and Merz (2008)[24]. The same data have been used in several papers so that some comparisons can be made. In particular, they have been used in Gigante *et al.* (2013b) [7] to illustrate a Poisson-gamma HGLM with random origin year effects and in Bühlmann and Moriconi (2015) [5] to illustrate a credibility model with random origin year and diagonal effects.

As for the expected values of the risk parameters, we assume $\psi_{U,i} = E(U_i)$ equal to the prior ultimate claims reported in Table 1, so that the external information on the ultimate claims is incorporated into the model, and $\psi_{V,i+j} = E(V_{i+j}) = 1$. Therefore, we assume that there is no trend in the expected calendar year effects. As noted in Bühlmann and Moriconi (2015) [5] and remarked in Section 2, this assumption would request to remove preliminarily any calendar year trend (e.g. economic inflation) from the data. It follows that the expected values of the unconditional payments are $E(Y_{i,j}) = \exp(\beta_j)\psi_{U,i}$. Hence, the expected ultimate claim amount of origin year i, $E(C_{i,t}) = \psi_{U,i} \sum_{j=0}^{t} \exp(\beta_j)$, is assumed to be proportional to the external information on the ultimate claims $\psi_{U,i}$ and, according to the usual parameter interpretation, $E(Y_{i,j})$ is the product of the expected ultimate claim amount of origin year i and the proportion of such amount paid in development year j.

Now we come to the model estimate (for implementation, we have developed our own code in SAS).

In Table 1 are reported the estimates $\exp(\hat{\beta}_j)$, $j = 0, \dots, t$, $\hat{u}_i = \exp(\hat{w}_{U,i})$, $i = 0, \dots, t$, and $\hat{v}_k = \exp(\hat{w}_{V,k})$, $k = i + j \leq t$, whereas, for $k = i + j > t$, $\hat{v}_k = 1$. It is easy to check that the development year factors are rather similar to the Chain Ladder ones, in particular for low development years. The parameters related to the origin year effect are quite close to their expected values. As for the parameters related to the calendar year effect, several of them are sensibly different from 1 (the expected value), in particular for the calendar years 1, 2, 8, 9. This indicates that such effect is appreciable in the data. As noted above, for the sake of comparison with Bühlmann and Moriconi (2015) [5], we have set the parameters $\psi_{V,i+j}$ equal to 1, hence we have assumed that any trend was already removed from the data. On the contrary, the estimates suggest the presence of a trend in the data, that should be accounted for through the $\psi_{V,i+j}$ or by inserting in the fixed part of the model a regression component related to the payment year.

The estimates of the dispersion parameters are $\hat{\phi} = 12,281$, $\hat{\lambda}_U = 5,269$ and $\hat{\lambda}_V = 0.00503$. It follows that, the estimates of the coefficients of variation of the risk parameters U_i, $(\lambda_U / E(U_i))^{1/2}$, are about 2% and the estimates of the coefficients of variation of the risk parameters V_{i+j} are about 7%. The rather high coefficients of variation for the calendar year effects suggest that a model with random effects for such component is suitable for this data.

We point out that the model has not been selected through a validation procedure, but it has been chosen to derive some comparisons with the example developed on the same dataset in Gigante *et al.* (2013b), in order to appreciate the effect of

i,j,k	$\exp(\hat{\beta}_j)$	$\psi_{U,i}$	$\hat{u}_i$	$\hat{v}_k$	$\hat{z}_{U,i}$	$\hat{z}_{V,k}$
0	0.5190	11,653,101	11,827,546	0.9776	0.2804	0.7155
1	0.2565	11,367,306	11,271,388	1.1045	0.2927	0.7844
2	0.0620	10,962,965	11,064,095	1.0884	0.2874	0.7933
3	0.0203	10,616,762	10,653,721	1.0395	0.2796	0.7922
4	0.0138	11,044,881	11,062,856	1.0097	0.2743	0.7964
5	0.0067	11,480,700	11,497,398	1.0098	0.2697	0.8026
6	0.0051	11,413,572	11,391,764	0.9581	0.2601	0.8046
7	0.0011	11,126,527	10,943,022	0.9563	0.2540	0.8011
8	0.0011	10,986,548	10,893,966	0.9365	0.2365	0.7990
9	0.0015	11,618,437	11,665,042	0.9195	0.1699	0.8051

Table 1. *Parameter estimates*

the calendar year components. However, the graphs of the studentized deviance residuals in Figure 1 do not show remarkable model failures.

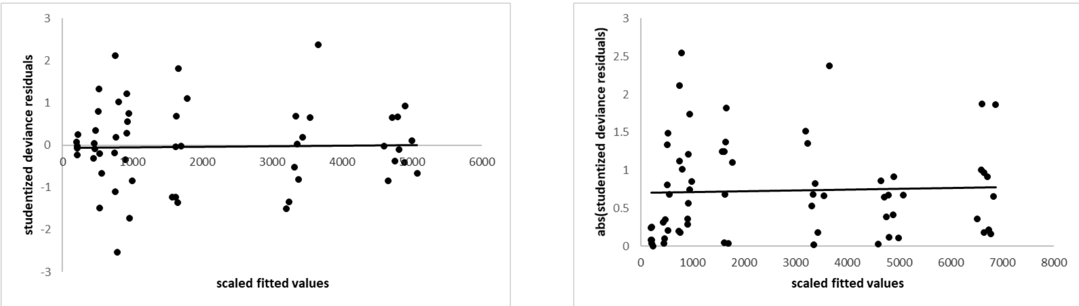

Figure 1. *Residuals versus fitted*

The weights (5.4), (5.5) are reported in the last two columns of Table 1; $\hat{z}_{U,i}$, $i = 0, \ldots, t$, show a decreasing trend with increasing i, whereas $\hat{z}_{V,k}$, $k = 0, \ldots, t$, show an increasing trend with increasing k. Therefore, in general, the more the available data in the run-off triangle, the higher the weights. Note the quite high values of the weights for the calendar year effects, for $k = 0, \ldots, t$: this entails that the parameter estimates $\hat{v}_k$ are nearer to the observed component, $\frac{\sum_{i+j=k} y_{i,j}}{\sum_{i+j=k} \exp(\hat{\beta}_j)\hat{u}_i}$, than to the expected value, $\psi_{V,k}$, in line with the above remark on the presence of a calendar year effect in the data. This happens because, in the weights $\hat{z}_{V,k}$, $\hat{\phi}/\hat{\lambda}_V = 2,441,202$ is low with respect to $\sum_{i+j=k} \exp(\hat{\beta}_j)\hat{u}_i$, $k = 0, \ldots, t$. On the contrary, the high value of $\hat{\phi}/\hat{\lambda}_U = 2.33$ with respect to $\sum_{j=0}^{t-i} \exp(\hat{\beta}_j)\hat{v}_{i+j}$, $i = 0, \ldots, t$, implies that the weights $\hat{z}_{U,i}$ are not so high.

We observe that in this example the two random components act so that, if the total observed payment of one origin year, $\sum_{j=0}^{t-i} y_{i,j}$, is greater than the estimate of its expected value, $\hat{E}(C_{i,t-i}) = \psi_{U,i} \sum_{j=0}^{t-i} \exp(\hat{\beta}_j)\psi_{V,i+j}$, then also the prediction of the same payment, $\hat{C}_{i,t-i} = \hat{u}_i \sum_{j=0}^{t-i} \exp(\hat{\beta}_j)\hat{v}_{i+j}$, is greater than the expected value. Moreover, the predictions entail a smoothing effect on the latest observed diagonal values, as can be seen in Figure 2.

The HGLM reserve predictions, $\hat{R}_i = \hat{u}_i \sum_{j=t-i+1}^{t} \exp(\hat{\beta}_j)$, and prediction errors,

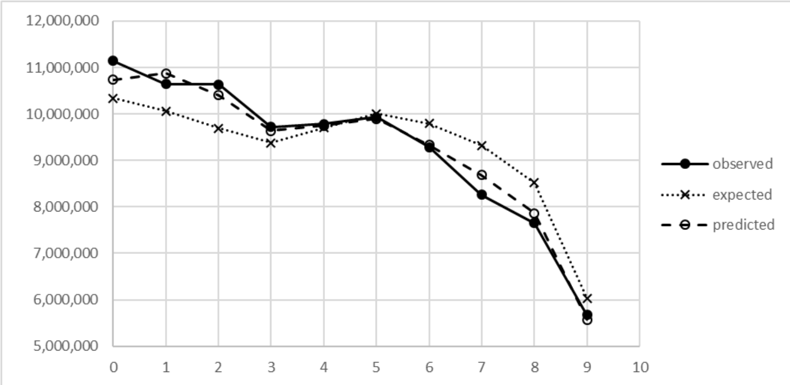

Figure 2. *Latest observed diagonal*

given by the square roots of the MSEPs, are reported in Table 2. We note that, as usual, there is considerable uncertainty in the reserve estimates in the earlier origin years and then the relative prediction errors decrease. The prediction error for the whole reserve as a percentage of the claims reserve is about 7.7%. As already remarked, the conditional MSEP estimate allows for the variability in the estimates of the fixed effects $\boldsymbol{\beta}$ and also in the random effects $\boldsymbol{w} = (w_{U,0}, \ldots, w_{U,t}, w_{V,0}, \ldots, w_{V,2t})$, not in the dispersion parameter estimates. The process error is sensibly higher than the estimation error in all origin years, except in the first one.

Origin year	Reserve	Prediction error	%	Process error	%	Estimation error	%
1	16,389	20,295	123.8	14,238	86.9	14,462	88.2
2	27,841	24,917	89.5	18,553	66.6	16,633	59.7
3	38,434	27,926	72.7	21,797	56.7	17,456	45.4
4	96,297	41,488	43.1	34,712	36.0	22,722	23.6
5	176,998	54,905	31.0	47,280	26.7	27,912	15.8
6	332,200	73,887	22.2	65,533	19.7	34,128	10.3
7	540,715	93,593	17.3	84,637	15.7	39,953	7.4
8	1,213,470	146,811	12.1	134,907	11.1	57,911	4.8
9	4,291,646	355,320	8.3	329,211	7.7	133,687	3.1
Total	6,733,989	521,451	7.7	437,300	6.5	284,042	4.2

Table 2. *Reserve and prediction error estimates*

It is interesting to compare these results with the reserve and prediction error estimates obtained in Gigante *et al.* (2013b)[7] in the Poisson-gamma HGLM without calendar year effects, reported in Table 3. We note that, with respect to the current model, the reserves and the prediction errors are underestimated. In fact, the reserve in Table 2 is about 8% higher than that in Table 3 and even more relevant is the difference between the prediction errors, about 24% higher in Table 2. Actually, as noted in several papers (e.g. Wüthrich (2010)[23], Salzmann and Wüthrich (2012)[18], Bühlmann and Moriconi (2015)[5]), the inclusion of random diagonal effects can be significant especially for the evaluation of the prediction uncertainty. The higher prediction errors are implied by a more appropriate dependence modeling of the incremental payments. In this regard, we have estimated the coefficients of correlation of the couples of payments from (5.1) and (5.2), by plugging-in the parameter estimates. The correlations for payments related to the same origin year are rather low, they vary from 0.005 to 0.05. In Figure 3*(a)*, we have plotted the correlation coefficients of $(Y_{0,0}, Y_{0,j})$, $j = 1, \ldots, t$. The correlations of $(Y_{0,h}, Y_{0,j})$, $j = h+1, \ldots, t$, are lower than those of $(Y_{0,0}, Y_{0,j})$, but they show a pattern that is similar to the one in the figure, from h on. Similar patterns are found for the other origin years. Differently, the correlations for payments of the same calendar year are quite high in the first development years: the correlation coefficients of $(Y_{i,0}, Y_{i-1,1})$ are about 0.58, the correlation coefficients of $(Y_{i,0}, Y_{i-2,2})$ and $(Y_{i-1,1}, Y_{i-2,2})$ vary from 0.33 to 0.38. Then the correlation coefficients decrease. For example, in Figure 3*(b)* we have plotted the following correlation coefficients of the last observed calendar year $corr(Y_{9,0}, Y_{9-j,j})$, $j = 1, \ldots, 9$. The pattern could be explained by the type of data that are from a motor insurance portfolio where most of the claims are paid within the second development year, as can also be seen from the development factors reported in Table 1. The model takes account of such correlations in the reserve evaluation.

Origin year	Reserve	Prediction error	%
1	15,199	21,082	138.7
2	26,125	26,155	100.1
3	34,857	28,674	82.3
4	86,623	42,357	48.9
5	159,377	55,987	35.1
6	294,565	74,221	25.2
7	470,703	92,566	19.7
8	1,086,682	142,204	13.1
9	4,061,356	312,042	7.7
Total	6,235,487	419,505	6.7

Table 3. *Reserve and prediction error estimates without calendar year effects*

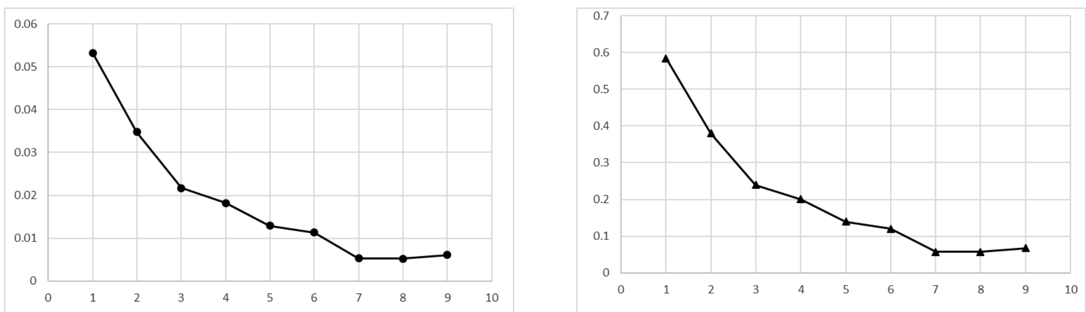

Figure 3. *(a)* $corr(Y_{0,0}, Y_{0,j})$, $j = 1, \ldots, 9$ *(b)* $corr(Y_{9,0}, Y_{9-j,j})$, $j = 1, \ldots, 9$

The two components, CL-type and BF-type in (5.6), of the reserve predictions and the still-to-come factors $1 - \hat{b}_{i,t-i}$, with $\hat{b}_{i,t-i}$ given in (5.7), are reported in

Table 4. Note that the CL-type and BF-type reserves are rather close. This happens because, differently than the genuine Chain Ladder and Bornhuetter-Ferguson reserves, both of them combine the external estimates and the run-off data. As a difference to Gigante *et al.* (2013b)[7], the current HGLM reserves are closer to the BF-type reserves, than to the CL-type. This is explained by the weights $\hat{z}_{U,i}$ that are all lower than 30%. However, the other comments in the quoted paper still apply here. In fact, as already noted, the still-to-come factors show that, in this portfolio, most of the claim amount for a given origin year is paid within the first two development years; the estimate of the factor $\sum_{j=0}^{t} \exp(\beta_j)$ is 0.8869, lower than 1, hence the estimate of the expected ultimate claims $\hat{E}(C_{i,t}) = \psi_{U,i} \sum_{j=0}^{t} \exp(\hat{\beta}_j)$ is lower than the external estimate $\psi_{U,i}$; since, as remarked in other papers, such external estimates are quite conservative for the portfolio under consideration, the HGLM estimates update the external data according to the run-off data.

Origin year	$\hat{R}_i^{CL-type}$	$\hat{R}_i^{BF-type}$	$1 - \hat{b}_{i,t-i}$
1	16,052	16,529	0.0015
2	28,472	27,587	0.0027
3	38,777	38,300	0.0040
4	96,711	96,140	0.0098
5	177,694	176,741	0.0176
6	330,390	332,836	0.0344
7	514,082	549,782	0.0586
8	1,180,174	1,223,783	0.1337
9	4,375,391	4,274,499	0.4353
Total	6,757,743	6,736,197	

Table 4. *CL-type and BF-type components of the HGLM reserves*

Finally, we report in Table 5 the reserve predictions and the prediction errors of the total portfolio obtained by the HGLM model and by the homogeneous and inhomogeneous credibility models in Bühlmann and Moriconi (2015)[5]. Although the models are different, even in the structures of the expected conditional payments, some comparison can be made. The HGLM reserve is intermediate between the two credibility reserves. Note that, similarly to the inhomogeneous credibility model, we assume that the expected values of the origin year effects, $\psi_{U,i}$, are not estimated from the data and are set equal to the external estimates of the ultimate claims. But, on the other hand, as noted above, the estimates of the development year parameters have the effect to correct such external estimates. The correction term is 0.8869 that can be compared with the "collective correction factor", $\hat{\mu}_0 = 0.8820$, that adjusts the a priori estimates of the ultimate claims in the homogeneous credibility model. The prediction error in the HGLM is higher than in both of the credibility models. The difference can be explained by the fact that in these models the development pattern is not estimated within the model and the Chain Ladder one is used. Hence the estimation error connected with this component is not accounted for in the prediction error. In fact, the prediction errors in the credibility models are close to the process error component in the HGLM (see Table 2).

Model	Reserve	Prediction error	%
HGLM	6,733,989	521,451	7.7
Homogeneous credibility model	6,416,109	426,609	6.6
Inhomogeneous credibility model	7,002,087	407,426	5.8

Table 5. *Model comparisons*

7 Simulation study

In this section, we illustrate the evaluation of the MSEP and of the predictive distribution of the outstanding loss liabilities by means of a simulation approach.

If the distributions of U_i, V_{i+j} and $Y_{i,j}|(U_i, V_{i+j}) = (u_i, v_{i+j})$ in Section 3 are suitable for simulation, the MSEP can be estimated by the usual parametric bootstrap procedure defined by the following algorithm. We note that the estimation of the MSEP obtained via simulation can also be used to empirically test the effect of the approximations considered in Section 3.

Step 1 *Parameter estimation*

From the original data, given $\psi_{U,i} = E(U_i)$, $i = 0, \ldots, t$, and $\psi_{V,i+j} = E(V_{i+j})$, $i+j = 0, \ldots, 2t$, calculate the HGLM estimates of the parameters $(\hat{\boldsymbol{\beta}}, \hat{\phi}, \hat{\lambda}_U, \hat{\lambda}_V)$.

Step 2 *Simulation*

Assume that the stochastic process

$$\{U_0, \ldots, U_t, V_0, \ldots, V_{2t}, Y_{0,0}, \ldots, Y_{t,t})\},$$

follows the estimated distribution, hence the model parameters are $(\boldsymbol{\psi}_U, \boldsymbol{\psi}_V, \hat{\boldsymbol{\beta}}, \hat{\phi}, \hat{\lambda}_U, \hat{\lambda}_V)$.

For $b = 1, \ldots, B$,

- simulate, independently, the random effects $u_0^{(b)}, \ldots, u_t^{(b)}$, $v_0^{(b)}, \ldots, v_{2t}^{(b)}$ from the respective distributions with parameters $(\psi_{U,i}, \hat{\lambda}_U)$, $i = 0, \ldots, t$, $(\psi_{V,i+j}, \hat{\lambda}_V)$, $0 \leq i + j \leq 2t$;
- simulate $y_{i,j}^{(b)}$ for the upper triangle, $i + j \leq t$, and $y_{i,j}^{*(b)}$ for the lower triangle, $i + j > t$, from the respective overdispersed Poisson conditional distributions with parameters $(\hat{\beta}_j, u_i^{(b)}, v_{i+j}^{(b)}, \hat{\phi})$.

Step 3 *Parameter estimation from simulated triangles*

For $b = 1, \ldots, B$,

- from the simulated upper triangle $y_{i,j}^{(b)}$, $i+j \leq t$, given $(\boldsymbol{\psi}_U, \boldsymbol{\psi}_V)$, estimate the parameters $(\hat{\boldsymbol{\beta}}^{(b)}, \hat{\phi}^{(b)}, \hat{\lambda}_U^{(b)}, \hat{\lambda}_V^{(b)})$ and the predictions of the random effects $\hat{\boldsymbol{w}}_U^{(b)}, \hat{\boldsymbol{w}}_V^{(b)}$;
- evaluate the predicted outstanding claims $\hat{R}^{(b)}$, by (4.4), and the estimated mean square error of prediction, $\widehat{MSEP}^{(b)}$, by (4.5) and (4.6);

– calculate the simulated outstanding claims i.e. the sum of the simulated payments in the lower triangle $R^{*(b)} = \sum_{i,j:i+j>t} \omega_i y^{*(b)}_{i,j}$.

Step 4 *MSEP estimation*

Let $MSEP_{est} = \frac{1}{B}\sum_{b=1}^{B} \widehat{MSEP}^{(b)}$ the average of the estimated MSEPs and $MSEP_{sim} = \frac{1}{B}\sum_{b=1}^{B}(R^{*(b)} - \hat{R}^{(b)})^2$ the MSEP estimated via simulation.

Simulation also allows obtaining an estimate of the predictive distribution of the outstanding claims, from which we can get estimates of the expected loss liabilities, of the coefficient of variation and of the quantiles. For this purpose, in Step 3 of the above algorithm, for any $b = 1, \ldots, B$, simulate M lower triangles $y^{**(b,m)}_{i,j}$, $i + j > t$, $m = 1, \ldots, M$, from the process with distribution given by the estimated parameters $(\boldsymbol{\psi}_U, \boldsymbol{\psi}_{V,i+j}, \hat{\boldsymbol{\beta}}^{(b)}, \hat{\phi}^{(b)}, \hat{\lambda}^{(b)}_U, \hat{\lambda}^{(b)}_V)$. The estimate of the outstanding claims distribution is then given by the empirical distribution function of the simulated values $R^{**(b,m)} = \sum_{i,j:i+j>t} \omega_i y^{**(b,m)}_{i,j}$, $b = 1, \ldots, B, m = 1, \ldots, M$.

A simulation study has been conducted for the numerical example in Section 6. In Table 6 we report the values of the square root of the average of the estimated MSEPs, $MSEP_{est}$, and of the $MSEP_{sim}$ estimated via simulation, in 20,000 simulations. As noted above, $MSEP_{sim}$ can be compared with $MSEP_{est}$ to appreciate the effect of the approximations in (4.6). The relative differences are very low, for the total reserve lower than 1%, the higher difference is 1.6% for the reserve of origin year 4. The two estimated root MSEP are also very close to the prediction errors of the reserve estimators obtained from the original data in Table 2. We can say that, for this model and these data, the approximate formula for the MSEP derived in Section 4 performs quite well.

Origin year	$MSEP^{1/2}_{est}$	$MSEP^{1/2}_{sim}$
1	20,109	20,283
2	24,550	24,351
3	27,838	27,538
4	41,483	40,824
5	54,937	54,791
6	74,131	74,267
7	94,789	95,027
8	147,944	148,781
9	355,084	359,460
Total	520,535	525,669

Table 6. *Prediction errors*

To get estimates of the predictive distributions of the outstanding loss liabilities, for each of the $B = 20,000$ simulated run-off triangles, we have simulated $M = 10$ lower triangles. Some characteristic values of the distributions are reported in Table 7. The means are very close to the HGLM reserve predictions in Table 2. The standard errors can be compared with the prediction errors of the reserves. The distribution of the total liability is slightly skewed to the right, the skewness coefficient is about 0.16. As expected, and in line with the results in Table 2, the coefficients of variation are decreasing with increasing origin years. The percentiles or Value-at-Risk at high confidence levels allow obtaining additional information on the distributions. They can also be used to assess a risk adjustment component to be added to the estimate of the outstanding loss liabilities to evaluate the liability for incurred claims as requested by the IFRS 17 accounting standard. We note that the percentiles of the distribution of the total liability are near to those of the normal with same mean and variance. Therefore, as highlighted in Taylor (2000)[20], for practical purposes the normal approximation could be acceptable for the whole reserve. However, as expected, it could be critical for the single accident years, in particular for the less recent ones.

Origin year	Mean	Std	$VaR_{75\%}$	$VaR_{90\%}$	$VaR_{95\%}$	$VaR_{99\%}$	$CV\%$
1	16,435	20,098	26,586	44,067	56,556	83,371	122.3
2	27,654	24,493	41,118	60,833	74,295	103,844	88.6
3	38,780	27,684	54,692	76,143	90,339	120,775	71.4
4	97,270	41,407	122,705	151,997	170,743	210,091	42.6
5	178,220	54,934	212,688	250,186	274,024	322,863	30.8
6	334,633	74,245	382,333	431,396	461,743	523,393	22.2
7	551,977	95,397	613,718	675,752	714,074	790,420	17.3
8	1,226,380	149,904	1,323,935	1,420,112	1,478,786	1,597,240	12.2
9	4,277,567	360,496	4,510,545	4,738,896	4,882,555	5,169,551	8.4
Total	6,748,915	524,390	7,090,591	7,421,755	7,632,194	8,046,769	7.8

Table 7. *Characteristic values of the predictive distributions*

8 Conclusions

We have introduced HGLMs that allow for the modeling of calendar year effects in claims reserving in order to take account of the dependences among payments, due to such effects.

We obtain the prediction of the outstanding claims and an approximate analytical formula for the MSEP, easy to compute once the model estimates are available. The MSEP takes account of the process risk and, for the estimation risk, of variability in the regression parameters and random effects. The model provides estimates of the correlations between payments.

We have studied in detail an overdispersed Poisson-gamma HGLM with random effects related to the origin and the calendar years. It has been applied to a motor insurance liability data set of the actuarial literature. The results have confirmed the relevance of calendar year effects. In fact, the estimates of covariances show remarkable correlations between payments of the same calendar year, made in the first development years. The inclusion of calendar year effects determines a remarkable increment of the MSEP, with respect to other models in which the dependence among payments in the same calendar year is ignored.

Moreover, a simulation approach has been considered to estimate the predictive distribution and to check the impact of the approximations in the MSEP formula. We have found, in our example, that the impact of the approximations is moderate.

So, the analytical formula for the MSEP appears particularly useful to get insights on the quality of the reserve prediction.

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PATRIZIA GIGANTE
Dipartimento di Scienze Economiche, Aziendali, Matematiche e Statistiche 'B. de Finetti'
University of Trieste
Piazzale Europa 1, 34127 Trieste, Italy
E-Mail: patrizia.gigante@deams.units.it

LIVIANA PICECH (Corresponding author)
Dipartimento di Scienze Economiche, Aziendali, Matematiche e Statistiche 'B. de Finetti'
University of Trieste
Piazzale Europa 1, 34127 Trieste, Italy
E-Mail: liviana.picech@deams.units.it

LUCIANO SIGALOTTI
Dipartimento di Scienze Economiche e Statistiche
University of Udine
Via Tomadini 30, 33100 Udine, Italy
E-Mail: luciano.sigalotti@uniud.it