Positive solutions of indefinite logistic growth models with flux-saturated diffusion

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Dedicated, with esteem and friendship, to Professor Shair Ahmad for his 85th birthday

Abstract. This paper analyzes the quasilinear elliptic boundary value problem driven by the mean curvature operator

$$-\mathrm{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = \lambda a(x) f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

with the aim of understanding the effects of a flux-saturated diffusion in logistic growth models featuring spatial heterogeneities. Here, Ω is a bounded domain in $\mathbb{R}^N$ with a regular boundary $\partial\Omega$, $\lambda > 0$ represents a diffusivity parameter, a is a continuous weight which may change sign in Ω, and $f\colon [0,L] \to \mathbb{R}$, with $L > 0$ a given constant, is a continuous function satisfying $f(0) = f(L) = 0$ and $f(s) > 0$ for every $s \in \,]0, L[$. Depending on the behavior of f at zero, three qualitatively different bifurcation diagrams appear by varying λ. Typically, the solutions we find are regular as long as λ is small, while as a consequence of the saturation of the flux they may develop singularities when λ becomes larger. A rather unexpected multiplicity phenomenon is also detected, even for the simplest logistic model, $f(s) = s(L-s)$ and $a \equiv 1$, having no similarity with the case of linear diffusion based on the Fick-Fourier's law.

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1 Introduction

This paper analyzes the quasilinear elliptic problem

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \lambda a(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the diffusion is driven by the mean curvature operator. Here, $\lambda > 0$ is viewed as a parameter measuring diffusivity and

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(H_1^1) $\Omega \subset \mathbb{R}^N$ is a bounded domain, with a C^2 boundary $\partial\Omega$ in case $N \geq 2$;

(H_2^1) $a\colon \overline{\Omega} \to \mathbb{R}$ is a continuous function such that $\max_{\overline{\Omega}} a > 0$;

(H_3^1) $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying, for some constant $L > 0$, $f(0) = f(L) = 0$, and $f(s) > 0$ for every $s \in]0, L[$.

Assumption (H_2^1) on the weight a introduces spatial heterogeneities within the model and allows, but does not impose, that a changes sign in Ω. Assumption (H_3^1) basically requires that the reaction term af is of logistic-type. As well-known, logistic maps play a pivotal role in the modeling theory of various disciplines, with special prominence in biology, ecology, genetics; see, e.g., [7, 14, 15, 27, 28] and the extensive bibliographies therein. Unlike the classical theory based on the Fick-Fourier's law, where the flux depends linearly on ∇u, here the diffusion is governed by the bounded flux $\nabla u/\sqrt{1+|\nabla u|^2}$, which is approximately linear for small gradients but approaches saturation for large ones.

The aim of this work is, therefore, describing, understanding, and clarifying the effects of a flux-saturated diffusion in logistic growth models featuring spatial heterogeneities. This study is motivated by the investigations on reaction processes with saturating diffusion started in [33] and further carried out in [8, 20, 22, 34], in order to correct the non-physical gradient-flux relations at high gradients. This specific mechanism of diffusion, of which the mean curvature operator provides a paradigmatic example, may determine spatial patterns exhibiting abrupt transitions at the boundary or between adjacent profiles, up to the formation of discontinuities [4, 9, 10, 11, 12, 16, 18, 19, 23, 24, 25, 26, 35]. This makes the mathematical analysis of the problem (1.1) more delicate and sophisticated than the study of the corresponding semilinear model, the use of some tools of geometric measure theory being in particular required. It is an established fact indeed that the space of bounded variation functions is the natural setting for dealing with this problem. The precise notion of bounded variation solution of (1.1) used in this paper has been basically introduced in [3] and is recalled below for completeness.

Notation. Throughout this work, for every $v \in BV(\Omega)$, $Dv = D^a v\,\mathrm{d}x + D^s v$ is the Lebesgue-Nikodym decomposition of the Radon measure Dv in its absolutely continuous part $D^a v\,\mathrm{d}x$ and its singular part $D^s v$ with respect to the N-dimensional Lebesgue measure $\mathrm{d}x$ in $\mathbb{R}^N$, $|Dv|$ denotes the total variation of the measure Dv, and $\frac{Dv}{|Dv|}$ stands for the density of Dv with respect to its total variation. Further, $|\Omega|$ is the Lebesgue measure of Ω, while $\mathcal{H}_{N-1}$ represents the $(N-1)$-dimensional Hausdorff measure, and $|\partial\Omega|$ is the $\mathcal{H}_{N-1}$-measure of $\partial\Omega$. We refer to [2] for additional information. Moreover, for all functions $u, v\colon \Omega \to \mathbb{R}$, we write: $u \geq v$ if $\operatorname{ess\,inf}(u-v) \geq 0$; $u > v$ if $u \geq v$ and $\operatorname{ess\,sup}(u-v) > 0$; $u \gg v$ if, for a.e. $x \in \Omega$, $u(x) - v(x) \geq \operatorname{dist}(x, \partial\Omega)$. We also define $u \wedge v$ and $u \vee v$ by setting $(u \wedge v)(x) = \min\{u(x), v(x)\}$ and $(u \vee v)(x) = \max\{u(x), v(x)\}$ for a.e. $x \in \Omega$. Finally, we write u^+ for $u \vee 0$ and u^- for $-(u \wedge 0)$.

Definition 1.1. *By a* bounded variation solution *of* (1.1) *we mean a function* $u \in BV(\Omega)$*, with* $f(u) \in L^N(\Omega)$*, which satisfies*

$$\int_\Omega \frac{D^a u\, D^a \phi}{\sqrt{1+|D^a u|^2}}\,\mathrm{d}x + \int_\Omega \frac{Du}{|Du|}\frac{D\phi}{|D\phi|}|D^s\phi| + \int_{\partial\Omega} \operatorname{sgn}(u)\,\phi\,\mathrm{d}\mathcal{H}_{N-1} = \lambda \int_\Omega af(u)\phi\,\mathrm{d}x \tag{1.2}$$

for every $\phi \in BV(\Omega)$ *such that* $|D^s\phi|$ *is absolutely continuous with respect to* $|D^s u|$ *and* $\phi(x) = 0$ $\mathcal{H}_{N-1}$*-a.e. on the set* $\{x \in \partial\Omega\colon u(x) = 0\}$*. A bounded variation solution* u *is said* positive *if* $u > 0$.

Remark 1.1. It follows from [3, Section 3] that a function $u \in BV(\Omega)$, with $f(u) \in L^N(\Omega)$, is a bounded variation solution of (1.1) if and only if it satisfies the variatonal inequality

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_\Omega af(u)(v-u)\,\mathrm{d}x \quad \text{for all } v \in BV(\Omega), \tag{1.3}$$

where

$$\mathcal{J}(v) = \int_\Omega \big(\sqrt{1+|D^a v|^2} - 1\big)\mathrm{d}x + \int_\Omega |D^s v| + \int_{\partial\Omega} |v|\,\mathrm{d}\mathcal{H}_{N-1}.$$

Remark 1.2. If a bounded variation solution u of (1.1) belongs to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for some $p > N$, then it satisfies the differential equation in (1.1) for a.e. $x \in \Omega$ and the boundary condition for all $x \in \partial\Omega$. Therefore, u is a *strong solution* of (1.1). The L^p-regularity theory [17, Chapter 9] then entails that $u \in W^{2,q}(\Omega)$ for all $q > N$. Conversely, it is evident that any weak solution $u \in W_0^{1,1}(\Omega)$, and hence in particular any strong solution, is a bounded variation solution.

Remark 1.3. It is clear that, for any given $\lambda > 0$, $u = 0$ is a solution of (1.1), while $u = L$ is not. Indeed, if L were a solution, taking $\phi = 1$ as test function in (1.2) would yield $\int_{\partial\Omega} 1 \, d\mathcal{H}_{N-1} = |\partial\Omega| = 0$, which is a contradiction.

We are now going to present the main results obtained in this paper. Here, for the sake of clarity, our statements are set out in a simplified form, while referring to the subsequent sections for some variants or extensions thereof that rely on slightly more general but less neat conditions: for each result, the minimal needed assumptions will be specified in an appropriate remark placed just below the corresponding proof.

The first result only exploits the structural assumptions (H_1^1), (H_2^1), and (H_3^1). It provides us with the existence of a number $\lambda_* \geq 0$ such that, for all $\lambda > \lambda_*$, the problem (1.1) has a maximum solution u_λ, with $0 < u_\lambda < L$. The asymptotic behavior of u_λ, as $\lambda \to +\infty$, is described too, and the bifurcation of the solutions from the trivial line $\{(\lambda, 0) \colon \lambda \geq 0\}$ at the point $(0, 0)$ is ascertained in the case $\lambda_* = 0$. Figure 1 illustrates the admissible bifurcations diagrams.

Theorem 1.1. *Assume (H_1^1), (H_2^1), and (H_3^1). Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in \,]\lambda_*, +\infty[$ the problem* (1.1) *admits a maximum bounded variation solution u_λ, with $0 < u_\lambda < L$, which satisfies*

$$\lim_{\lambda \to +\infty} (\operatorname{ess\,sup} u_\lambda) = L. \tag{1.4}$$

Moreover, if $\lambda_* = 0$, then

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{BV} = 0. \tag{1.5}$$

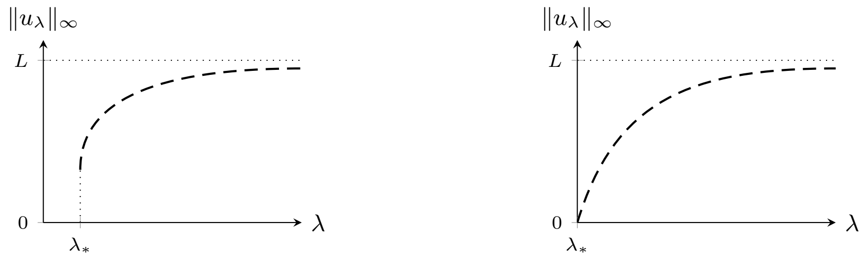

Figure 1: Admissible bifurcation diagrams for the problem (1.1) under the structural assumptions (H_1^1), (H_2^1), and (H_3^1), in case $\lambda_* > 0$ (left) or $\lambda_* = 0$ (right). Dashed curves indicate bounded variation solutions.

The specific features displayed by the bifurcation diagrams of the problem (1.1) are determined by the slope at 0 of the function f, as expressed by the following conditions:

(H_4^1) there exists $\lim_{s \to 0^+} \frac{f(s)}{s} = +\infty$ (*sublinear growth at* 0);

(H_5^1) there exists $\lim_{s \to 0^+} \frac{f(s)}{s} = \kappa \in \,]0, +\infty[$ (*linear growth at* 0);

(H_6^1) there exists $\lim_{s \to 0^+} \frac{f(s)}{s} = 0$ (*superlinear growth at* 0).

When f has a sublinear growth at zero, a bifurcation from the trivial line occurs at the point $(0, 0)$, and the existence of positive bounded variation solutions of the problem (1.1) is guaranteed for all $\lambda > 0$. In addition, positive strong solutions exist provided that λ is small enough.

Theorem 1.2. *Assume* (H_1^1), (H_2^1), (H_3^1), *and* (H_4^1). *Then for all* $\lambda > 0$ *the problem* (1.1) *admits at least one bounded variation solution* $u_\lambda \in BV(\Omega)$, *with* $0 < u_\lambda < L$, *which satisfies* (1.4) *and* (1.5). *Moreover, there exists* $\lambda^* > 0$ *such that, for all* $\lambda \in]0, \lambda^*[$, *solutions* u_λ *can be selected so that* $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *for any* $p > N$, *it is a strong solution and it satisfies*

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{W^{2,p}} = 0.$$

When f grows linearly at zero the bifurcation occurs from the trivial line at the point $(\lambda_1, 0)$, where λ_1 is the principal eigenvalue of the linear weighted problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\kappa\varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here, Ω satisfies (H_1^1), κ comes from (H_5^1), and a satisfies (H_2^1). It follows from [6] that λ_1 is positive and simple, with a positive eigenfunction φ_1. The L^p-regularity theory and a standard bootstrap argument entail that $\varphi_1 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $p > N$, while the strong maximum principle and the Hopf boundary point lemma yield $\varphi_1 \gg 0$. In this case the solvability of the problem (1.1) is guaranteed for all $\lambda > \lambda_1$. In addition, for λ close to λ_1 strong solutions do exist.

Theorem 1.3. *Assume* (H_1^1), (H_2^1), (H_3^1), *and* (H_5^1). *Then for all* $\lambda > \lambda_1$ *the problem* (1.1) *admits at least one bounded variation solution* u_λ, *with* $0 < u_\lambda < L$, *which satisfies* (1.4). *Moreover, suppose that*

(H_7^1) *f is of class* C^2

and fix any $p > N$. *Then there exists a neighborhood* $\mathcal{U}$ *of* $(\lambda_1, 0)$ *in* $\mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *such that solutions* u_λ *can be selected so that* $(\lambda, u_\lambda) \in \mathcal{U}$, u_λ *is a strong solution and it satisfies*

$$\lim_{\lambda \to \lambda_1} \|u_\lambda\|_{W^{2,p}} = 0 \qquad \text{and} \qquad \lim_{\lambda \to \lambda_1} \frac{u_\lambda}{\|u_\lambda\|_{C^1}} = \varphi_1. \tag{1.6}$$

Finally, there exists $\eta > 0$ *such that the following assertions hold:*

(*i*) *if* $f''(0) < 0$, *then for all* $\lambda \in]\lambda_1, \lambda_1 + \eta[$ *there is at least one strong solution* $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *satisfying* (1.6)*;*

(*ii*) *if* $f''(0) > 0$, *then for all* $\lambda \in]\lambda_1 - \eta, \lambda_1[$ *there is at least one strong solution* $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *satisfying* (1.6).

Remark 1.4. For the standard logistic model $f(s) = s(L - s)$, the condition $f''(0) = -2 < 0$ holds and therefore the bifurcation is supercritical.

When f exhibits a superlinear growth at zero, the existence of multiple solutions can be detected if, for instance, conditions (H_2^1) and (H_6^1) are strengthened as follows. Let us set

$$\Omega^+ = \{x \in \Omega \colon a(x) > 0\}, \quad \Omega^- = \{x \in \Omega \colon a(x) < 0\}, \quad \Omega^0 = \{x \in \Omega \colon a(x) = 0\},$$

and replace (H_2^1) with

(H_8^1) $a \in C^2(\overline{\Omega})$, $\Omega^+ \neq \emptyset$, $\Omega^- \neq \emptyset$, $\Omega^0 = \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega$, and $\nabla a(x) \neq 0$ for all $x \in \Omega^0$,

as well as (H_6^1) with

(H_9^1) there exists $q > 1$, with $q < \frac{N+2}{N-2}$ if $N \geq 3$, such that

$$\lim_{s \to 0^+} \frac{f(s)}{s^q} = 1.$$

Then, for λ sufficiently large, the problem (1.1) has at least two positive bounded variation solutions, the smaller being strong.

Theorem 1.4. *Assume (H_1^1), (H_3^1), (H_8^1), and (H_9^1). Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in]\lambda_*, +\infty[$, the problem* (1.1) *admits at least one bounded variation solution u_λ and one strong solution $v_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for any $p > N$, such that $0 \ll v_\lambda < u_\lambda < L$. In addition, u_λ satisfies* (1.4)*, while v_λ satisfies*

$$\lim_{\lambda \to +\infty} \|v_\lambda\|_{W^{2,p}} = 0.$$

Figure 2 illustrates three qualitatively different bifurcation diagrams corresponding, respectively, to Theorems 1.2, 1.3, and 1.4.

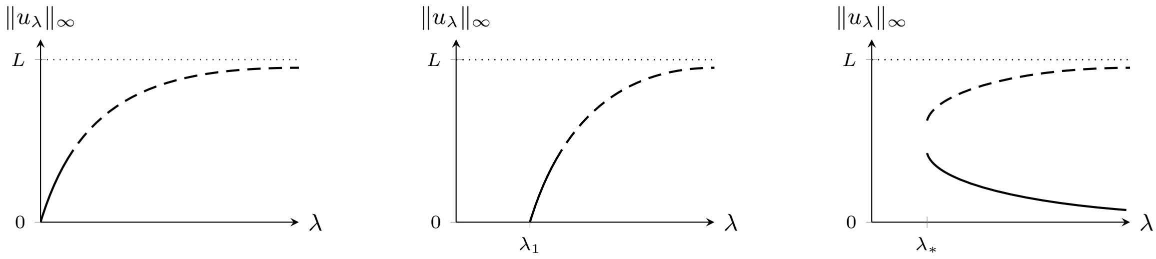

Figure 2: Admissible qualitative bifurcation diagrams for the problem (1.1), according to the growth of f at 0: either sublinear (left), or linear (center), or superlinear (right). Dashed curves indicate bounded variation solutions, solid curves represent strong solutions.

Unexpectedly enough, the existence of multiple solutions can always be detected in the standard logistic model, whenever the carrying capacity L is sufficiently large, even in the case where the weight function a is a positive constant (cf. Remark 1.5 below). We state such a multiplicity result for the simplest one-dimensional prototype of the problem (1.1), that is,

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a f(u) \quad \text{in }]0,1[, \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{1.7}$$

Theorem 1.5. *Assume (H_3^1),*

(H_{10}^1) *$a \in C^0([0,1])$ satisfies $a > 0$,*

and

(H_{11}^1) *there exist $r, R \in]0, L[$, with $r < R$, such that*

$$\frac{2F(r)}{r^2}(1 + \sqrt{1+r^2}) < \frac{F(R)}{R},$$

where $F(s) = \int_0^s f(t)\,\mathrm{d}t$ is the potential of f. Then there exist $\lambda_\sharp$ and $\lambda^\sharp$, with $0 \leq \lambda_\sharp < \lambda^\sharp$, such that for all $\lambda \in]\lambda_\sharp, \lambda^\sharp[$ the problem (1.7) *admits at least two bounded variation solutions u_λ, v_λ such that $0 < u_\lambda < v_\lambda < L$.*

It is worth stressing that the assumptions of Theorem 1.5 do not prevent f from being concave in $[0, L]$: this fact witnesses the peculiarity of this multiplicity result, which is specific of the quasilinear problem (1.1) and has no similarity at all with the semilinear case, where the concavity of f always guarantees the uniqueness of the positive solution, as proven in [5] even for sign-changing weights a.

Remark 1.5. For the standard logistic model, where $f(s) = s(L - s)$, condition (H_{11}^1) is satisfied if, for instance, $L > \frac{32}{3} \approx 10.67$.

Example 1.6. A numerical study of the problem (1.7), with $a \equiv 1$, $f(s) = s(L - s)$ and $L = 11 > \frac{32}{3}$, reveals the existence of three positive solutions in a (small) right neighborhood of the bifurcation point $\lambda_1 = \frac{\pi^2}{L} \approx 0.8972$, in particular at $\overline{\lambda} = 0.8975$, and of two positive solutions in a left neighborhood of λ_1. This is in complete agreement with (i) the bifurcation result stated in Theorem 1.3 and Remark 1.4, which predicts the bifurcation branch emanates from λ_1 pointing to the right; (ii) the multiplicity conclusions of Theorem 1.5, which guarantee the existence of two solutions in an interval of the λ-axis located on the left of λ_1. Hence a S-shaped bifurcation diagram is expected as shown by the picture on the left in Figure 3.

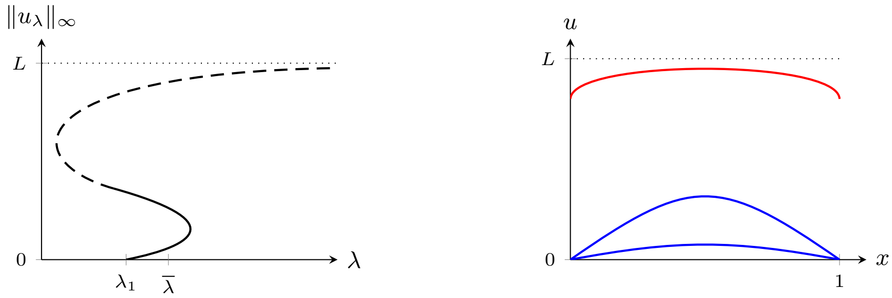

Figure 3: On the left, an admissible bifurcation diagram is depicted with reference to Example 1.6: the dashed curve indicates bounded variation solutions, the solid curve represents strong solutions. On the right, the profiles of the three detected solutions at $\lambda = \overline{\lambda}$ are shown: in blue the regular ones, in red the singular one.

The remainder of this paper is structured as follows. Section 2 is devoted to the proof of various statements concerning the existence and the asymptotic behavior of the positive bounded variation solutions of (1.1), under the sole structural conditions (H_1^1), (H_2^1), and (H_3^1); in particular, Theorem 1.1 is proven. Section 3 focuses on the discussion of the features displayed by the bifurcation diagrams of the problem (1.1) according to the slope at zero of the function f; here, some extensions, or variants, of Theorems 1.2, 1.3, and 1.4 are derived. Section 4 closes the paper by providing the proof of a more general version of the multiplicity result stated in Theorem 1.5.

2 Bounded variation solutions: existence and asymptotic behavior of the bifurcation branches

In this section we aim to prove Theorem 1.1, as well as some variants thereof, by using variational techniques in the space $BV(\Omega)$, in combination with the method of lower and upper solutions for mean curvature problems as first developed in [21] and independently in [29]. Henceforth, we endow the space $BV(\Omega)$ with the norm

$$\|v\|_{BV} = \int_\Omega |Dv| + \int_{\partial\Omega} |v| \,\mathrm{d}\mathcal{H}_{N-1},$$

which is equivalent to the usual one by [26, Proposition 2] and [2, Theorem 3.88]. Since we are looking for solutions u of (1.1) satisfying the condition $0 < u < L$, we can suppose, without loss of generality, that

$$f(s) = 0 \quad \text{for all } s \in \mathbb{R} \setminus [0, L].$$

We also set

$$F(s) = \int_0^s f(t)\,\mathrm{d}t \quad \text{for all } s \in \mathbb{R}.$$

Next, we introduce the action functional associated with the problem (1.1). Namely, for each $\lambda > 0$, we define $\mathcal{I}_\lambda \colon BV(\Omega) \to \mathbb{R}$ by

$$\mathcal{I}_\lambda(v) = \mathcal{J}(v) - \lambda \int_\Omega aF(v)\,\mathrm{d}x, \tag{2.1}$$

where $\mathcal{J} \colon BV(\Omega) \to \mathbb{R}$ is given by

$$\mathcal{J}(v) = \int_\Omega \sqrt{1 + |Dv|^2} - |\Omega| + \int_{\partial\Omega} |v|\,\mathrm{d}\mathcal{H}_{N-1},$$

having set

$$\int_\Omega \sqrt{1 + |Dv|^2} = \int_\Omega \sqrt{1 + |D^a v|^2}\,\mathrm{d}x + \int_\Omega |D^s v|.$$

We start by proving the existence of positive bounded variation solutions of (1.1) under the following conditions that weaken (H_1^1) and (H_2^1), respectively:

(H_1^2) $\Omega \subset \mathbb{R}^N$ is a bounded domain, with a boundary $\partial\Omega$ of class C^1 in case $N \geq 2$;

(H_2^2) $a \in L^\infty(\Omega)$ and there is a Caccioppoli set E of positive measure such that $\int_E a(x)\,\mathrm{d}x > 0$.

Proposition 2.1. *Assume (H_3^1), (H_1^2), and (H_2^2). Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in \,]\lambda_*, +\infty[$ the problem* (1.1) *admits a maximum bounded variation solution u_λ satisfying* $0 < u_\lambda < L$.

Proof. For later reference, the proof is split into three parts.

Step 1 : *For every $\lambda > 0$, there exists a global minimizer u_λ of $\mathcal{I}_\lambda$.* Fix $\lambda > 0$. From (H_2^2) and (H_3^1) we easily get, for all $v \in BV(\Omega)$,

$$\begin{aligned} \mathcal{I}_\lambda(v) &\geq \int_\Omega |Dv| - |\Omega| + \int_{\partial\Omega} |v|\,\mathrm{d}\mathcal{H}_{N-1} - \lambda \|a^+\|_{L^1} F(L) \\ &= \|v\|_{BV} - |\Omega| - \lambda \|a^+\|_{L^1} F(L). \end{aligned}$$

Therefore, $\mathcal{I}_\lambda$ is bounded from below and coercive. Let $(v_n)_n$ be a minimizing sequence. Since $(v_n)_n$ is bounded in $BV(\Omega)$, the compact embedding of $BV(\Omega)$ into $L^1(\Omega)$ implies that there exist a subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, and a function $u_\lambda \in BV(\Omega)$ such that $v_n \to u_\lambda$ in $L^1(\Omega)$ and a.e. in Ω. The lower semicontinuity of $\mathcal{J}_\lambda$ with respect to the L^1-convergence in $BV(\Omega)$ and the dominated convergence theorem easily yield

$$\mathcal{I}_\lambda(u_\lambda) \leq \liminf_{n \to +\infty} \mathcal{I}_\lambda(v_n) = \inf_{v \in BV(\Omega)} \mathcal{I}_\lambda(v),$$

that is, u_λ is a global minimizer of $\mathcal{I}_\lambda$.

Step 2 : *For every $\lambda > 0$, u_λ is a bounded variation solution of* (1.1) *satisfying* $0 \leq u_\lambda < L$. From [30, Remark 2.2.] we know that any local minimizer u_λ of $\mathcal{I}_\lambda$ satisfies the variational inequality (1.3) and therefore by Remark 1.1 it is a bounded variation solution of (1.1).

Next, we show that $u_\lambda < L$. Taking $v = u_\lambda \wedge L$ as test function in (1.3) and observing that $v - u_\lambda = -(u_\lambda - L)^+$, we get

$$0 = \lambda \int_\Omega af(u_\lambda)\left(-(u_\lambda - L)^+\right)\,\mathrm{d}x \leq \mathcal{J}(u_\lambda \wedge L) - \mathcal{J}(u_\lambda).$$

Then, recalling that $u_\lambda \vee L = L + (u_\lambda - L)^+$ and using the lattice property proven in [21, Theorem 3.2] or [29, Proposition 2.2], we infer

$$\begin{aligned} 0 \le \mathcal{J}(u_\lambda \wedge L) - \mathcal{J}(u_\lambda) &\le \mathcal{J}(L) - \mathcal{J}(u_\lambda \vee L) \\ &= -\left(\int_\Omega \sqrt{1 + |D(u_\lambda - L)^+|^2} - |\Omega|\right) - \int_{\partial\Omega} (u_\lambda - L)^+ \, \mathrm{d}\mathcal{H}_{N-1} \le 0. \end{aligned}$$

This yields $(u_\lambda - L)^+ = 0$ and thus $u_\lambda \le L$. Moreover, we have that $u_\lambda < L$, because, by Remark 1.3, L is not a solution of (1.1).

At last, we prove that $u_\lambda \ge 0$. Taking $v = u_\lambda \vee 0$ as test function in (1.3), observing that $v - u_\lambda = u_\lambda^-$ and using again the lattice property, we obtain

$$\begin{aligned} 0 \le \mathcal{J}(u_\lambda \vee 0) - \mathcal{J}(u_\lambda) &\le \mathcal{J}(0) - \mathcal{J}(u_\lambda \wedge 0) \\ &= -\left(\int_\Omega \sqrt{1 + |Du_\lambda^-|^2} - |\Omega|\right) - \int_{\partial\Omega} u_\lambda^- \, \mathrm{d}\mathcal{H}_{N-1} \le 0. \end{aligned}$$

Hence we conclude that $u_\lambda^- = 0$ and thus $u_\lambda \ge 0$.

Step 3 : *For every $\lambda > \lambda_*$, there exists a maximum bounded variation solution w_λ of* (1.1) *such that* $0 < w_\lambda < L$. We first prove that the global minimizer u_λ of (1.1) satisfies $u_\lambda > 0$. To this end it is sufficient to show that, when λ is large enough, 0 is not a global minimizer of $\mathcal{I}_\lambda$. Thanks to (H_2^2), we can take $v = \chi_E \in BV(\Omega)$ in (2.1), where χ_E is the characteristic function of E. Denoting by $\mathrm{Per}(E, \Omega)$ the perimeter of E in Ω, we obtain

$$\mathcal{I}_\lambda(\chi_E) = \mathrm{Per}(E, \Omega) - \lambda F(L) \int_E a \, \mathrm{d}x.$$

Therefore, by setting

$$\lambda_* = \frac{\mathrm{Per}(E, \Omega)}{F(L) \int_E a \, \mathrm{d}x},$$

we infer that $\mathcal{I}_\lambda(0) = 0 > \mathcal{I}_\lambda(\chi_E)$, for every $\lambda > \lambda_*$.

Fix now $\lambda > \lambda_*$. Thanks to [21, Proposition 3.6] or to [29, Lemma 3.7] it is immediately checked that L is an upper bounded variation solution of (1.1), according to the definition given in [21, Section 3] or in [29, Section 2]. Hence, by [21, Theorem 3.4] or by [29, Theorem 2.4] there exists a maximum solution $w_\lambda \in BV(\Omega)$ of (1.1) with $u_\lambda \le w_\lambda < L$. Suppose by contradiction that there exists a solution $v_\lambda \in BV(\Omega)$ satisfying $0 < v_\lambda < L$ and $v_\lambda \not\le w_\lambda$. Since $\alpha = v_\lambda \vee w_\lambda$ is a lower bounded variation solution with $0 < \alpha < L$, by [21, Theorem 3.4] or by [29, Theorem 2.4] again, there should exist a solution $z_\lambda \in BV(\Omega)$ with $w_\lambda < \alpha \le z_\lambda < L$, which is a contradiction. Therefore w_λ is the maximum solution of (1.1) with $0 < w_\lambda < L$. This concludes the proof. □

Remark 2.1. From the proof of Proposition 2.1 it follows that the problem (1.1) has, for a given $\lambda > 0$, a solution $u_\lambda \in BV(\Omega)$, with $0 < u_\lambda < L$, if and only if there exists a function $\psi \in BV(\Omega)$ such that $0 < \psi < L$ and $\mathcal{I}_\lambda(\psi) < 0$. This in turn holds, for all large $\lambda > 0$, if and only if $\int_\Omega aF(\psi) \, \mathrm{d}x > 0$.

To complement the previous result, we investigate the asymptotic behavior of the maximum solutions $u_\lambda \in BV(\Omega)$ of (1.1) as $\lambda \to +\infty$. To this purpose, we assume the following condition

(H_3^2) $a \in L^\infty(\Omega)$ and there is an open set $\omega \subset \Omega$ such that $\operatorname{ess\,inf}_\omega a > 0$.

Assumption (H_3^2) obviously implies (H_2^2).

Proposition 2.2. *Assume (H_3^1), (H_1^2), and (H_3^2). Then the maximum bounded variation solution u_λ of* (1.1) *with $0 < u_\lambda < L$, which exists for all $\lambda \in \,]\lambda_*, +\infty[$ according to Proposition 2.1, further satisfies* (1.4).

Proof. We begin with the following simple consequence of assumption (H_3^1).

Claim 1. *There exist a global maximizer* $\sigma_M \in]0, L[$ *of* f *in* $[0, L]$ *and a sequence* $(\sigma_n)_n$ *in* $]\sigma_M, L[$ *such that*

$$\lim_{n\to+\infty} \sigma_n = L$$

and

$$f(s) \geq f(\sigma_n) \quad \text{for all } s \in]\sigma_M, \sigma_n[. \tag{2.2}$$

Indeed, the largest global maximizer $\sigma_M \in]0, L[$ of f in $[0, L]$ exists by (H_3^1). For each $n \geq 1$, let $\sigma_n \in \big]\sigma_M, L - \frac{L-\sigma_M}{n+1}\big]$ be the largest global minimizer of f in $\big]\sigma_M, L - \frac{L-\sigma_M}{n+1}\big]$. Assumption (H_3^1) implies that $f\big(L - \frac{L-\sigma_M}{n+1}\big) \to 0$ and hence that $\sigma_n \to L$, as $n \to +\infty$. Accordingly, $(\sigma_n)_n$ is the desired sequence.

Thanks to assumption (H_3^2) we can find constants $\varepsilon > 0$ and $\rho > 0$ such that $a(x) \geq \varepsilon$ for a.e. $x \in \omega_\rho$, where ω_ρ is the open ball of center x_0 and radius ρ. Without restriction we can suppose that $\overline{\omega_\rho} \subset \Omega$. Let (σ_M) be the largest global maximizer of f in $[0, L]$ and $(\sigma_n)_n$ be the sequence given by Claim 1. Fix n and, for simplifying notation, set $\sigma = \sigma_n$. Define also

$$\lambda_\star = \frac{N}{\varepsilon f(\sigma) \min\{\rho, \sigma - \sigma_M\}}.$$

Fix $\lambda > \lambda_\star$ and set $\tau = \frac{N}{\lambda \varepsilon f(\sigma)}$. Denote by ω_τ the open ball of center x_0 and radius τ. As $\tau \in]0, \rho[$, we have that $\overline{\omega_\tau} \subset \omega_\rho$. First, we define a function $v_1 \in W^{1,1}(\omega_\tau) \cap C^0(\overline{\omega_\tau}) \cap C^2(\omega_\tau)$ by

$$v_1(x) = \sigma - \tau + \sqrt{\tau^2 - |x - x_0|^2}.$$

Clearly, v_1 is a classical solution of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = \lambda \varepsilon f(\sigma) & \text{in } \omega_\tau, \\ u = v_1 & \text{on } \partial\omega_\tau. \end{cases}$$

It is immediately checked that v_1 also satisfies

$$\sigma_M < \sigma - \tau \leq \min v_1 < \max v_1 = \sigma. \tag{2.3}$$

Second, we define $v_2 \in W^{1,1}(\Omega \setminus \overline{\omega_\tau}) \cap C^0(\overline{\Omega} \setminus \omega_\tau) \cap C^2(\Omega \setminus \overline{\omega_\tau})$ by

$$v_2(x) = -1 - \sqrt{|x - x_0|^2 - \tau^2}.$$

The function v_2 is a classical solution of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = g(x) & \text{in } \Omega \setminus \overline{\omega_\tau}, \\ u = v_2 & \text{on } \partial(\Omega \setminus \overline{\omega_\tau}), \end{cases}$$

where $g \in L^\infty(\Omega)$ is given by

$$g(x) = \begin{cases} \dfrac{2(N-1)|x - x_0|^2 - N\tau^2}{(2|x - x_0|^2 - \tau^2)^{3/2}} & \text{if } x \in \Omega \setminus \overline{\omega_\tau}, \\ 0 & \text{if } x \in \overline{\omega_\tau}. \end{cases}$$

Clearly, we have that

$$\max v_2 = -1. \tag{2.4}$$

Third, we define a function v by

$$v(x) = \begin{cases} v_1(x) & \text{if } x \in \omega_\tau, \\ v_2(x) & \text{if } x \in \Omega \setminus \overline{\omega_\tau}. \end{cases}$$

It follows from [2, Theorem 3.84] that $v \in BV(\Omega)$. Let also $h \in L^\infty(\Omega)$ be defined by

$$h(x) = \begin{cases} \lambda\varepsilon f(\sigma) & \text{if } x \in \omega_\tau, \\ g(x) & \text{if } x \in \Omega \setminus \overline{\omega_\tau}. \end{cases}$$

Claim 2. *The function v is a bounded variation solution of*

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = h(x) & \text{in } \Omega, \\ u = v_2 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

We begin observing that, for all $x \in \partial\omega_\tau$, $v_1(x) > v_2(x)$ and

$$\frac{\nabla v_1(x) \cdot \nu_{\omega_\tau}(x)}{\sqrt{1+|\nabla v_1(x)|^2}} = -1, \qquad \frac{\nabla v_2(x) \cdot \nu_{\Omega\setminus\overline{\omega_\tau}}(x)}{\sqrt{1+|\nabla v_2(x)|^2}} = 1, \tag{2.6}$$

where $\nu_{\omega_\tau}(x)$ and $\nu_{\Omega\setminus\overline{\omega_\tau}}(x)$ are, respectively, the unit outer normals to ω_τ and to $\Omega \setminus \overline{\omega_\tau}$ at $x \in \partial\omega_\tau$. By [2, Theorem 3.84] we can write

$$Dv = D^a v \,\mathrm{d}x + D^s v = \nabla v \,\mathrm{d}x + (v_2 - v_1)\nu_{\omega_\tau} \,\mathrm{d}\mathcal{H}_{N-1}. \tag{2.7}$$

Take now a test function $\phi \in BV(\Omega)$ such that $|D^s\phi|$ is absolutely continuous with respect to $|D^s v|$ and $\phi(x) = 0$ $\mathcal{H}_{N-1}$-a.e. on the set $\{x \in \partial\Omega \colon v(x) = v_2(x)\}$. If we set $\phi_1 = \phi_{|\omega_\tau} \in W^{1,1}(\omega_\tau)$ and $\phi_2 = \phi_{|\Omega\setminus\overline{\omega_\tau}} \in W^{1,1}(\Omega \setminus \overline{\omega_\tau})$, then, by [2, Theorem 3.84] again, we have that

$$D\phi = D^a\phi \,\mathrm{d}x + D^s\phi = \nabla\phi \,\mathrm{d}x + (\phi_2 - \phi_1)\nu_{\omega_\tau} \,\mathrm{d}\mathcal{H}_{N-1}. \tag{2.8}$$

Thanks to (2.6), (2.7), and (2.8), we get

$$\begin{aligned} \int_{\omega_\tau} h\phi_1 \,\mathrm{d}x &= -\int_{\omega_\tau} \operatorname{div}\left(\frac{\nabla v_1}{\sqrt{1+|\nabla v_1|^2}}\right)\phi_1 \,\mathrm{d}x \\ &= -\int_{\partial\omega_\tau} \frac{\nabla v_1 \cdot \nu_{\omega_\tau}}{\sqrt{1+|\nabla v_1|^2}}\phi_1 \,\mathrm{d}\mathcal{H}_{N-1} + \int_{\omega_\tau} \frac{\nabla v_1 \cdot \nabla\phi_1}{\sqrt{1+|\nabla v_1|^2}} \,\mathrm{d}x \\ &= \int_{\partial\omega_\tau} \phi_1 \,\mathrm{d}\mathcal{H}_{N-1} + \int_{\omega_\tau} \frac{D^a v_1 \, D^a\phi_1}{\sqrt{1+|D^a v_1|^2}} \,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega\setminus\overline{\omega_\tau}} h\phi_2 \,\mathrm{d}x &= -\int_{\Omega\setminus\overline{\omega_\tau}} \operatorname{div}\left(\frac{\nabla v_2}{\sqrt{1+|\nabla v_2|^2}}\right)\phi_2 \,\mathrm{d}x \\ &= -\int_{\partial(\Omega\setminus\overline{\omega_\tau})} \frac{\nabla v_2 \cdot \nu_{\Omega\setminus\overline{\omega_\tau}}}{\sqrt{1+|\nabla v_2|^2}}\phi_2 \,\mathrm{d}\mathcal{H}_{N-1} + \int_{\Omega\setminus\overline{\omega_\tau}} \frac{\nabla v_2 \cdot \nabla\phi_2}{\sqrt{1+|\nabla v_2|^2}} \,\mathrm{d}x \\ &= -\int_{\partial\omega_\tau} \frac{\nabla v_2 \cdot \nu_{\Omega\setminus\overline{\omega_\tau}}}{\sqrt{1+|\nabla v_2|^2}}\phi_2 \,\mathrm{d}\mathcal{H}_{N-1} - \int_{\partial\Omega} \frac{\nabla v_2 \cdot \nu_\Omega}{\sqrt{1+|\nabla v_2|^2}}\phi_2 \,\mathrm{d}\mathcal{H}_{N-1} + \int_{\Omega\setminus\overline{\omega_\tau}} \frac{\nabla v_2 \cdot \nabla\phi_2}{\sqrt{1+|\nabla v_2|^2}} \,\mathrm{d}x \\ &= -\int_{\partial\omega_\tau} \phi_2 \,\mathrm{d}\mathcal{H}_{N-1} + \int_{\Omega\setminus\overline{\omega_\tau}} \frac{D^a v_2 \, D^a\phi_2}{\sqrt{1+|D^a v_2|^2}} \,\mathrm{d}x. \end{aligned}$$

Since

$$|D^s v| = |v_2 - v_1| \mathrm{d}\mathcal{H}^{N-1}, \qquad \frac{D^s v}{|D^s v|} = \frac{v_2 - v_1}{|v_2 - v_1|}\nu_{\omega_\tau},$$
$$|D^s \phi| = |\phi_2 - \phi_1| \mathrm{d}\mathcal{H}^{N-1}, \qquad \frac{D^s \phi}{|D^s \phi|} = \frac{\phi_2 - \phi_1}{|\phi_2 - \phi_1|}\nu_{\omega_\tau},$$

and $v = v_2$ on $\partial\Omega$, we can conclude that

$$\begin{aligned}\int_\Omega h\phi \,\mathrm{d}x &= \int_{\omega_\tau} h\phi_1 \,\mathrm{d}x + \int_{\Omega\setminus\overline{\omega_\tau}} h\phi_2 \,\mathrm{d}x = \int_{\partial\omega_\tau} (\phi_1 - \phi_2)\,\mathrm{d}\mathcal{H}_{N-1} + \int_\Omega \frac{D^a v \cdot D^a\phi}{\sqrt{1+|D^a v|^2}}\,\mathrm{d}x \\ &= \int_{\partial\omega_\tau} \frac{v_2 - v_1}{|v_2 - v_1|}\nu_{\omega_\tau} \cdot \frac{\phi_2 - \phi_1}{|\phi_2 - \phi_1|}\nu_{\omega_\tau} |\phi_2 - \phi_1|\,\mathrm{d}\mathcal{H}_{N-1} + \int_\Omega \frac{D^a v \cdot D^a\phi}{\sqrt{1+|D^a v|^2}}\,\mathrm{d}x \\ &= \int_\Omega \frac{D^s v}{|D^s v|} \cdot \frac{D^s\phi}{|D^s\phi|}|D^s\phi| + \int_{\partial\Omega} \mathrm{sgn}(v - v_2)\,\phi\,\mathrm{d}\mathcal{H}_{N-1} + \int_\Omega \frac{D^a v \cdot D^a\phi}{\sqrt{1+|D^a v|^2}}\,\mathrm{d}x.\end{aligned}$$

Therefore v is a bounded variation solution of (2.5) according to [3, Section 3]. This concludes the proof of Claim 2.

Let us now define a function $\ell\colon \Omega\times\mathbb{R}\to\mathbb{R}$ by

$$\ell(x,s) = \begin{cases} \lambda\varepsilon \min\{f(s), f(\sigma)\}\chi_{\omega_\tau}(x) + \lambda a(x) f(s)\chi_{\Omega\setminus\overline{\omega_\tau}}(x) & \text{if } s\geq 0, \\ -sg(x) & \text{if } -1 < s < 0, \\ g(x) & \text{if } s \leq -1, \end{cases} \tag{2.9}$$

where χ_{ω_τ} and $\chi_{\Omega\setminus\overline{\omega_\tau}}$ are the characteristic functions of ω_τ and of $\Omega\setminus\overline{\omega_\tau}$, respectively. The function ℓ satisfies the L^∞-Carathéodory conditions and, due to (2.2), (2.3), and (2.4),

$$\ell(x, v(x)) = h(x) \quad \text{for a.e. } x\in\Omega.$$

Consequently, v is a bounded variation solution of

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \ell(x,v) & \text{in } \Omega, \\ u = v_2 & \text{on } \partial\Omega. \end{cases}$$

Hence, by [3, Section 3] the function v also satisfies the variational inequality

$$\begin{aligned}\int_\Omega \sqrt{1+|Dw|^2} + \int_{\partial\Omega} |w - v_2|\,\mathrm{d}\mathcal{H}_{N-1} - \int_\Omega \sqrt{1+|Dv|^2} - \int_{\partial\Omega} |v - v_2|\,\mathrm{d}\mathcal{H}_{N-1} \\ \geq \int_\Omega \ell(x,v)(w - v)\,\mathrm{d}x\end{aligned} \tag{2.10}$$

for all $w\in BV(\Omega)$.

Claim 3. *The function $v\vee 0$ is a lower bounded variation solution of*

$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \ell(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Fix any $z\in BV(\Omega)$ such that $z\leq 0$. As $v(x) = v_2(x) < 0$ for all $x\in\partial\Omega$, we have that $|z(x)| = |v(x) + z(x)| - |v(x)|$ for $\mathcal{H}^{N-1}$-a.e. $x\in\partial\Omega$. Since v is a solution of (2.11), taking $v + z$ as test function in (2.10) yields

$$\begin{aligned}\int_\Omega \ell(x,v)z\,\mathrm{d}x &\le \int_\Omega \sqrt{1+|D(v+z)|^2} + \int_{\partial\Omega} |v+z-v_2|\,\mathrm{d}\mathcal{H}_{N-1} \\ &\quad - \int_\Omega \sqrt{1+|Dv|^2} - \int_{\partial\Omega} |v-v_2|\,\mathrm{d}\mathcal{H}_{N-1} \\ &= \int_\Omega \sqrt{1+|D(v+z)|^2} + \int_{\partial\Omega} |z|\,\mathrm{d}\mathcal{H}_{N-1} - \int_\Omega \sqrt{1+|Dv|^2} \\ &= \int_\Omega \sqrt{1+|D(v+z)|^2} + \int_{\partial\Omega} |v+z|\,\mathrm{d}\mathcal{H}_{N-1} - \int_\Omega \sqrt{1+|Dv|^2} - \int_{\partial\Omega} |v|\,\mathrm{d}\mathcal{H}_{N-1}.\end{aligned}$$

Further, as $\ell(x,0)=0$ for all $x\in\Omega$, we have that 0 is a solution of (2.11). Hence, according to [21] or [29], $v\vee 0$ is a lower bounded variation solution of (2.11). This concludes the proof of Claim 3.

Claim 4. *For any $\lambda>\lambda_\star$ there exists a lower bounded variation solution α of* (1.1) *such that* $0<\alpha<L$ *and* ess sup $\alpha\ge\sigma$*.* We know that $v\vee 0$ is a lower bounded variation solution of (2.11). Moreover, as $\ell(x,L)=0$ for all $x\in\Omega$, L is an upper bounded variation solution, but not a solution, of (2.11). Then, by [21] or by [29] there exists a solution α of (2.11) with $0<v\le\alpha<L$. From (2.9), we see that, for all $s\ge 0$ and a.e. $x\in\Omega$,

$$a(x)f(s)\ge\varepsilon\min\{f(s),f(\sigma)\}\chi_{\omega_\tau}(x)+a(x)f(s)\chi_{\Omega\setminus\overline{\omega_\tau}}(x)=\ell(x,s).$$

Therefore, we immediately infer from [21] or [29] that α is a lower bounded variation solution of (1.1) as well, thus concluding the proof of Claim 4.

We are now in position of concluding the proof. Indeed, for each $\eta\in\,]0,L[$ we can find $\sigma\in[L-\eta,L[$ and $\lambda_\star=\lambda_\star(\eta)>0$ such that, for all $\lambda>\lambda_\star$, there is a lower bounded variation solution α_λ of (1.1) with $\operatorname{ess\,sup}\alpha_\lambda\ge\sigma$. Hence, the maximum bounded variation solution u_λ of (1.1), which exists according to Proposition 2.1 for all $\lambda>\lambda_*$, must satisfy $u_\lambda\ge\alpha_\lambda$ and thus $\operatorname{ess\,sup}u_\lambda\ge\sigma\ge L-\eta$ for all $\lambda>\max\{\lambda_\star,\lambda_*\}$. Consequently, condition (1.4) is proven. ☐

We finally describe the behavior of the solutions $u_\lambda\in BV(\Omega)$ of (1.1), if any, as $\lambda\to 0^+$.

Proposition 2.3. *Assume* (H_3^1)*,* (H_1^2)*, and*

(H_4^2) *$a\in L^N(\Omega)$.*

Then any sequence $((\lambda_n,u_n))_n$ *of solutions of the problem* (1.1)*, with* $\lambda_n>0$*,* $0<u_n<L$*, for all* n*, and* $\lim\limits_{n\to+\infty}\lambda_n=0$*, satisfies*

$$\lim_{n\to+\infty}\|u_n\|_{BV}=0. \tag{2.12}$$

Proof. For any given n, taking $\phi=u_n$ as test function in (1.2), we get

$$\int_\Omega \frac{|D^a u_n|^2}{\sqrt{1+|D^a u_n|^2}}\,\mathrm{d}x+\int_\Omega |D^s u_n|+\int_{\partial\Omega}|u_n|\,\mathrm{d}\mathcal{H}_{N-1}=\lambda_n\int_\Omega af(u_n)u_n\,\mathrm{d}x.$$

Letting $n\to+\infty$, we find, as $\lambda_n\to 0$,

$$\lambda_n\int_\Omega af(u_n)\,u_n\,\mathrm{d}x\le\lambda_n\,\|a^+\|_{L^1}\|f(u_n)u_n\|_\infty\le\lambda_n\,\|a^+\|_{L^1}\,L\max_{s\in[0,L]}f(s)\to 0$$

and hence

$$\int_\Omega \frac{|D^a u_n|^2}{\sqrt{1+|D^a u_n|^2}}\,\mathrm{d}x+\int_\Omega |D^s u_n|+\int_{\partial\Omega}|u_n|\,\mathrm{d}\mathcal{H}_{N-1}\to 0. \tag{2.13}$$

Since each of the three terms of this sum tends to 0 as $n \to +\infty$, in particular, we have that

$$\int_\Omega \frac{|D^a u_n|^2}{\sqrt{1+|D^a u_n|^2}}\,\mathrm{d}x \to 0. \tag{2.14}$$

Possibly passing to a subsequence, still labeled by n, it follows from (2.14) that $(D^a u_n)_n$ converges to 0 a.e. in Ω. Thus, the Severini-Egorov's theorem implies that, for every $\varepsilon > 0$, there exists a measurable subset $S_\varepsilon \subset \Omega$, with $|S_\varepsilon| < \varepsilon$, such that $D^a u_n \to 0$ uniformly on $\Omega \setminus S_\varepsilon$. Fix $\varepsilon > 0$. Then, there exists $\bar{n}$ such that, for each $n > \bar{n}$, $|D^a u_n| < \varepsilon$ a.e. in $\Omega \setminus S_\varepsilon$ and, possibly reducing the size of the set S_ε, $|D^a u_n| \geq \varepsilon$ a.e. in S_ε. Hence, we obtain

$$\begin{aligned}
\int_\Omega \frac{|D^a u_n|^2}{\sqrt{1+|D^a u_n|^2}}\,\mathrm{d}x &= \int_{\Omega\setminus S_\varepsilon} \frac{|D^a u_n|^2}{\sqrt{1+|D^a u_n|^2}}\,\mathrm{d}x + \int_{S_\varepsilon} \frac{|D^a u_n|}{\sqrt{1+|D^a u_n|^2}}|D^a u_n|\,\mathrm{d}x \\
&\geq \frac{1}{\sqrt{1+\varepsilon^2}} \int_{\Omega\setminus S_\varepsilon} |D^a u_n|^2\,\mathrm{d}x + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \int_{S_\varepsilon} |D^a u_n|\,\mathrm{d}x \\
&= \frac{1}{\sqrt{1+\varepsilon^2}} \|D^a u_n\|^2_{L^2(\Omega\setminus S_\varepsilon)} + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \|D^a u_n\|_{L^1(S_\varepsilon)} \\
&\geq \frac{c_\varepsilon}{\sqrt{1+\varepsilon^2}} \|D^a u_n\|^2_{L^1(\Omega\setminus S_\varepsilon)} + \frac{\varepsilon}{\sqrt{1+\varepsilon^2}} \|D^a u_n\|_{L^1(S_\varepsilon)}
\end{aligned}$$

where $c_\varepsilon > 0$ is the embedding constant of $L^2(\Omega \setminus S_\varepsilon)$ into $L^1(\Omega \setminus S_\varepsilon)$. This estimate allows to conclude, thanks to (2.14), that $(D^a u_n)_n$ converges to 0 in both $L^1(\Omega \setminus S_\varepsilon)$ and $L^1(S_\varepsilon)$, and thus in $L^1(\Omega)$. Since by (2.13)

$$\int_\Omega |D^s u_n| + \int_{\partial\Omega} |u_n|\,\mathrm{d}\mathcal{H}_{N-1} \to 0,$$

as $n \to +\infty$, we can therefore conclude that (2.12) holds. □

Proof of Theorem 1.1. As (H_1^1) implies (H_1^2) and (H_2^1) implies (H_3^2), combining Propositions 2.1, 2.2, and 2.3 yields Theorem 1.1. □

Remark 2.2. From the above proof it follows that Theorem 1.1 still holds replacing (H_1^1) with (H_1^2) and (H_2^1) with (H_3^2).

3 Prescribing different growth conditions at zero

In this section we discuss the existence and the multiplicity of solutions of the problem (1.1) by imposing one of the growth conditions on f at zero expressed by (H_4^1), or (H_5^1), or (H_6^1).

3.1 Sublinear growth

In this subsection we establish two results from which Theorem 1.2 will eventually be inferred. The first statement guarantees the existence of a solution for any $\lambda > 0$ under a generalized form of condition (H_4^1).

Proposition 3.1. *Assume* (H_3^1), (H_1^2), (H_3^2), *and*

(H_1^3) $\displaystyle\limsup_{s\to 0^+} \frac{F(s)}{s^2} = +\infty.$

Then for all $\lambda > 0$ *the problem* (1.1) *admits at least one bounded variation solution* $u_\lambda \in BV(\Omega)$, *which can be selected so as to satisfy* $0 < u_\lambda < L$, (1.4)*, and* (1.5).

Proof. It is convenient here to suppose that $f(s) = 0$ for all $s \in \mathbb{R} \setminus [0, L]$. For any given $\lambda > 0$, the existence of a global minimizer u_λ of $\mathcal{I}_\lambda$, satisfying $0 \le u_\lambda < L$, is guaranteed by the first two steps of the proof of Theorem 2.1. Hence, according to Remark 2.1, in order to establish that $u_\lambda > 0$, it is sufficient to find a function $\psi \in BV(\Omega)$ such that $\mathcal{I}_\lambda(\psi) < 0$. We first notice that, by assumption (H_1^3), there exists a sequence $(s_n)_n$ in $]0, L[$ such that

$$\lim_{n\to+\infty} s_n = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{F(s_n)}{s_n^2} = +\infty.$$

Next, we pick an open set ω_1 such that $\overline{\omega_1} \subset \omega$, with ω defined in (H_3^2), and a function $w \in H_0^1(\Omega)$ such that $w(x) \ge 0$ in Ω, $w(x) = 0$ in $\Omega \setminus \omega$ and $w(x) = 1$ in ω_1. Hence, we have that

$$\begin{aligned} \mathcal{I}_\lambda(s_n w) &= \frac{1}{2}\int_\omega |s_n \nabla w|^2 \,\mathrm{d}x - \lambda \int_{\omega_1} aF(s_n w)\,\mathrm{d}x - \lambda \int_{\omega\setminus\omega_1} aF(s_n w)\,\mathrm{d}x \\ &\le s_n^2 \left(\frac{1}{2}\int_\omega |\nabla w|^2 \,\mathrm{d}x - \lambda \frac{F(s_n)}{s_n^2} \int_{\omega_1} a \,\mathrm{d}x \right) < 0, \end{aligned}$$

for all large n. This implies that $\mathcal{I}_\lambda(u_\lambda) < 0$ and thus $u_\lambda > 0$. Finally, we observe that $\mathcal{I}_\lambda(u_\lambda) < 0$ yields

$$\mathcal{J}(u_\lambda) < \lambda \int_\Omega aF(u_\lambda)\,\mathrm{d}x \le \lambda \|a\|_{L^1} F(L)$$

and then

$$\begin{aligned} &\frac{1}{2}\int_\Omega \frac{|D^a u_\lambda|^2}{\sqrt{1+|D^a u_\lambda|^2}}\,\mathrm{d}x + \int_\Omega |D^s u_\lambda| + \int_{\partial\Omega} |u_\lambda| \,\mathrm{d}\mathcal{H}_{N-1} \\ &\quad \le \int_\Omega \frac{|D^a u_\lambda|^2}{1+\sqrt{1+|D^a u_\lambda|^2}}\,\mathrm{d}x + \int_\Omega |D^s u_\lambda| + \int_{\partial\Omega} |u_\lambda| \,\mathrm{d}\mathcal{H}_{N-1} = \mathcal{J}(u_\lambda) \to 0, \quad \text{as } \lambda \to 0^+. \end{aligned}$$

Hence, arguing as in the proof of Proposition 2.3, we see that

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{BV} = 0.$$

The last conclusion,

$$\lim_{\lambda\to+\infty} (\operatorname{ess\,sup} u_\lambda) = L,$$

follows from Proposition 2.2. □

The next result yields the existence of strong solutions for λ sufficiently small.

Proposition 3.2. *Assume (H_1^1), (H_3^1), (H_3^2), and (H_1^3). Then there exists $\lambda^* \in]0, +\infty]$ such that for all $\lambda \in]0, \lambda^*[$ the problem (1.1) admits at least one strong solution $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for all $p > N$, satisfying $0 < u_\lambda < L$ and*

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{W^{2,p}} = 0.$$

Proof. From [30, Theorem 3.1] we infer the existence of $\lambda^* \in]0, +\infty]$ such that for every $\lambda \in]0, \lambda^*[$ there is $u_\lambda \in C^{1,\gamma}(\overline{\Omega}) \cap W_0^{1,1}(\Omega)$, for some $\gamma \in]0, 1[$, such that $u_\lambda > 0$,

$$\int_\Omega \frac{\nabla u \cdot \nabla \phi}{\sqrt{1+|\nabla u|^2}}\,\mathrm{d}x = \lambda \int_\Omega af(u)\phi\,\mathrm{d}x, \qquad \text{for all } \phi \in C_0^\infty(\Omega),$$

and

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{C^1(\overline{\Omega})} = 0.$$

The L^p-regularity theory implies that $u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for all $p > N$, and

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{W^{2,p}} = 0.$$

This ends the prof. □

Proof of Theorem 1.2. As (H_2^1) implies (H_3^2) and (H_4^1) implies (H_1^3), Propositions 3.1 and 3.2 yield Theorem 1.2. □

Remark 3.1. From the above proof it follows that Theorem 1.2 still holds replacing (H_2^1) with (H_3^2) and (H_4^1) with (H_1^3).

3.2 Linear growth

In this subsection we provide a proof of Theorem 1.3 as a consequence of two slightly more general results stated below as Propositions 3.3 and 3.4. The basic assumption of Proposition 3.3 is

(H_2^3) there exists $\displaystyle\lim_{s \to 0^+} \frac{2F(s)}{s^2} = \kappa \in]0, +\infty[$,

generalizing condition (H_5^1). Once the constant κ is assigned by (H_2^3), we assume (H_1^1) and

(H_3^3) $a \in L^\infty(\Omega)$ is such that $\operatorname{ess\,sup}_\Omega a > 0$.

Then, we respectively denote by λ_1 and φ_1 the principal eigenvalue and the principal eigenfunction of the linear weighted problem

$$\begin{cases} -\Delta\varphi = \lambda a(x)\kappa\varphi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It follows from [6] that $\lambda_1 > 0$, λ_1 is simple and $\varphi_1 > 0$. As already observed, the L^p-regularity theory and a standard bootstrap argument then entail that $\varphi_1 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ for all $p > N$, while the strong maximum principle and the Hopf boundary point lemma yield $\varphi_1 \gg 0$.

Proposition 3.3. *Assume (H_1^1), (H_3^1), (H_2^3), and (H_3^3). Then for all $\lambda > \lambda_1$ the problem* (1.1) *admits at least one bounded variation solution u_λ, satisfying $0 < u_\lambda < L$.*

Proof. It is convenient here to suppose that $f(s) = 0$ for all $s \in \mathbb{R} \setminus [0, L]$. Fix any $\lambda > \lambda_1$. By Remark 2.1, it is enough to find a function $\psi \in BV(\Omega)$ such that $\mathcal{I}_\lambda(\psi) < 0$. Assumption (H_2^3) implies that there is a sequence $(s_n)_n$ in $]0, L[$ such that

$$\lim_{n \to +\infty} s_n = 0 \qquad \text{and} \qquad \lim_{n \to +\infty} \frac{F(s_n)}{s_n^2} = \kappa$$

and hence

$$\lim_{k \to 0^+} \frac{2F(s_n\varphi_1(x))}{s_n^2\varphi_1(x)^2} = \kappa \quad \text{uniformly in } x \in \Omega.$$

This yields

$$\begin{aligned} \lim_{n \to +\infty} \int_\Omega \left(\frac{|\nabla\varphi_1|^2}{1 + \sqrt{1 + s_n^2|\nabla\varphi_1|^2}} - \lambda a \frac{F(s_n\varphi_1)}{s_n^2\varphi_1^2}\varphi_1^2 \right) \mathrm{d}x &= \frac{1}{2}\int_\Omega \left(|\nabla\varphi_1|^2 - \lambda\kappa a\varphi_1^2 \right) \mathrm{d}x \\ &= \frac{1}{2}\int_\Omega \left(1 - \frac{\lambda}{\lambda_1} \right) |\nabla\varphi_1|^2 \,\mathrm{d}x < 0. \end{aligned}$$

We therefore conclude that

$$\mathcal{I}(s_n\varphi_1) = s_n^2 \int_\Omega \left(\frac{|\nabla\varphi_1|^2}{1 + \sqrt{1 + s_n^2|\nabla\varphi_1|^2}} - \lambda a \frac{F(s_n\varphi_1)}{s_n^2\varphi_1^2}\varphi_1^2 \right) \mathrm{d}x < 0,$$

for all large n. □

Remark 3.2. It is evident from the above proof that in place of (H_2^3) one can assume the existence of a constant $\kappa \in]0,+\infty[$ and of a sequence $(s_n)_n$ in $]0,L[$ such that

$$\lim_{n\to+\infty} s_n = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \frac{2F(s_n)}{s_n^2} = \kappa \in]0,+\infty[\,.$$

The next result guarantees the existence of small positive strong solutions u of (1.1). Fix $p > N$ and introduce the set

$$\mathcal{S} = \{(\lambda, u) \in \mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \colon \lambda > 0 \text{ and } u > 0 \text{ is a strong solution of (1.1)}\} \cup \{(\lambda_1, 0)\}.$$

Since $(\lambda, 0)$ solves (1.1) for all $\lambda \in \mathbb{R}$, we look for positive solutions bifurcating from the line of the trivial solutions by the Crandall-Rabinowitz theorem [13]. Namely, the following local bifurcation result holds.

Proposition 3.4. *Assume (H_1^1), (H_3^1), (H_5^1), (H_7^1), (H_3^3), and fix $p > N$. Then there exists a neighborhood $\mathcal{U}$ of $(\lambda_1, 0)$ in $\mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and functions*

$$\chi \colon]-1,1[\to \mathbb{R}, \qquad \psi \colon]-1,1[\to \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \colon \int_\Omega u\varphi_1 \,\mathrm{d}x = 0 \right\}$$

of class C^1 such that

$$\chi(0) = \lambda_1, \qquad \psi(0) = 0,$$

and

$$\mathcal{S} \cap \mathcal{U} = \big\{(\lambda, u) \colon \lambda = \chi(t),\ u = t(\varphi_1 + \psi(t)),\ t \in [0,1[\,\big\}.$$

Supposing, in addition, that

(H_4^3) *f is of class C^3,*

the following assertions hold:

- (i) *if either $f''(0) > 0$ or, otherwise, $f''(0) = 0$ and $\frac{f'''(0)}{f'(0)} > -\frac{\int_\Omega |\nabla\varphi_1|^4 \,\mathrm{d}x}{\int_\Omega |\nabla\varphi_1|^2\varphi_1^2 \,\mathrm{d}x}$, then the bifurcation of positive solutions is subcritical,*
- (ii) *if either $f''(0) < 0$ or, otherwise, $f''(0) = 0$ and $\frac{f'''(0)}{f'(0)} < -\frac{\int_\Omega |\nabla\varphi_1|^4 \,\mathrm{d}x}{\int_\Omega |\nabla\varphi_1|^2\varphi_1^2 \,\mathrm{d}x}$, then the bifurcation of positive solutions is supercritical.*

Proof. Fix $p > N$ and define the operator $\mathcal{F} \colon \mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ by setting

$$\mathcal{F}(\lambda, u) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) + \lambda a f(u).$$

It is clear that $(\lambda, u) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfies $\mathcal{F}(\lambda, u) = 0$ if and only if u is a strong solution of (1.1) for some $\lambda > 0$. By combining the results in [36, Chapter II, Section 4] with the continuity, from $W^{1,p}(\Omega)$ to $L^p(\Omega)$, of the linear operators which map any function u onto its weak partial derivative $\partial_i u$, with $i = 1, \ldots, N$, we infer that $\mathcal{F}$ is of class C^2 under (H_7^1) and, respectively, of class C^3 under (H_4^3).

The partial derivatives of $\mathcal{F}$ relevant to the present proof are produced below. For all $(\lambda, u) \in \mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $v, w, z \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, there hold:

$$\partial_\lambda \mathcal{F}(\lambda, u) = a f(u),$$

$$\partial_u \mathcal{F}(\lambda, u)[v] = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla u|^2}} - \frac{\nabla u \cdot \nabla v}{\left(\sqrt{1 + |\nabla u|^2}\right)^3} \nabla u\right) + \lambda a f'(u) v,$$

$$\partial_{u\lambda}\mathcal{F}(\lambda,u)[v] = af'(u)v,$$

$$\begin{aligned}\partial_{uu}\mathcal{F}(\lambda,u)[v][w] = \operatorname{div}\Big(&-\frac{\nabla u\cdot\nabla w}{(\sqrt{1+|\nabla u|^2})^3}\nabla v - \frac{\nabla w\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla u - \frac{\nabla u\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla w \\ &+ 3\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u\Big) + \lambda a f''(u)vw,\end{aligned}$$

$$\begin{aligned}\partial_{uuu}\mathcal{F}(\lambda,u)[v][w][z] = \operatorname{div}\Big(&-\frac{\nabla z\cdot\nabla w}{(\sqrt{1+|\nabla u|^2})^3}\nabla v - \frac{\nabla w\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla z - \frac{\nabla z\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla w \\ &+ 3\frac{(\nabla u\cdot\nabla z)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla v + 3\frac{(\nabla w\cdot\nabla v)\,(\nabla z\cdot\nabla u)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u \\ &+ 3\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^5}\nabla w + 3\frac{(\nabla z\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u \\ &+ 3\frac{(\nabla u\cdot\nabla v)\,(\nabla z\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u + 3\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla z \\ &- 15\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^7}\nabla u\Big) + \lambda a f'''(u)vwz.\end{aligned}$$

Let us set

$$\mathcal{L} = \partial_u\mathcal{F}(\lambda_1,0) = \Delta + \lambda_1 a\kappa\mathcal{I} \qquad \text{and} \qquad \mathcal{M} = \partial_{u\lambda}\mathcal{F}(\lambda_1,0) = a\kappa\mathcal{I},$$

where $\kappa = f'(0)$ and $\mathcal{I}$ is the identity operator. It is clear that $\mathcal{L}$ is a Fredholm operator with index 0, having kernel

$$N(\mathcal{L}) = \operatorname{span}\{\varphi_1\},$$

and range

$$R(\mathcal{L}) = \left\{u \in L^p(\Omega)\colon \int_\Omega u\varphi_1\,\mathrm{d}x = 0\right\}.$$

Further, the transversality condition

$$\mathcal{M}[\varphi_1] = a\kappa\varphi_1 \notin R(\mathcal{L})$$

is satisfied, because

$$\begin{aligned}A &= \int_\Omega \mathcal{M}[\varphi_1]\varphi_1\,\mathrm{d}x = \int_\Omega a\kappa\varphi_1^2\,\mathrm{d}x \\ &= \lambda_1^{-1}\int_\Omega -\Delta\varphi_1\varphi_1\,\mathrm{d}x = \lambda_1^{-1}\int_\Omega |\nabla\varphi_1|^2\,\mathrm{d}x > 0.\end{aligned}$$

Hence, the Crandall-Rabinowitz theorem [13, Theorem 1.7] yields the existence of a neighborhood $\mathcal{U}$ of $(\lambda_1,0)$ in $\mathbb{R}\times W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ and of functions

$$\chi\colon\,]-1,1[\, \to \mathbb{R}, \qquad \psi\colon\,]-1,1[\, \to \left\{u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)\colon \int_\Omega u\varphi_1\,\mathrm{d}x = 0\right\}$$

of class C^1 such that

$$\chi(0) = \lambda_1, \qquad \psi(0) = 0,$$

and

$$\mathcal{S}\cap\mathcal{U} = \big\{(\lambda,u)\colon \lambda = \chi(t),\ u = t(\varphi_1+\psi(t)),\ t \in\,]-1,1[\,\big\}.$$

We further infer from [13, Theorems 1.7 and 1.18] (see also [1, Chapter 5.4]) that

$$\lambda = \chi(t) = \lambda_1 - \frac{B}{A}\, t + o(t),$$

where

$$\begin{aligned} B =& \frac{1}{2}\int_\Omega \mathcal{F}_{uu}(\lambda_1, 0)[\varphi_1][\varphi_1]\varphi_1 \,\mathrm{d}x = \frac{1}{2}\int_\Omega \lambda_1 a f''(0)\varphi_1^3 \,\mathrm{d}x \\ =& \frac{1}{2}\frac{f''(0)}{\kappa}\int_\Omega -\Delta\varphi_1 \varphi_1^2 \,\mathrm{d}x = \frac{f''(0)}{\kappa}\int_\Omega |\nabla\varphi_1|^2 \varphi_1 \,\mathrm{d}x. \end{aligned}$$

Thus, the bifurcation of positive solutions is subcritical if $f''(0) > 0$, while it is supercritical if $f''(0) < 0$. In case $f''(0) = 0$ and f satisfying (H_4^3), we define

$$\begin{aligned} C =& \frac{1}{3}\int_\Omega \mathcal{F}_{uuu}(\lambda_1, 0)[\varphi_1][\varphi_1][\varphi_1]\varphi_1 \,\mathrm{d}x \\ =& \frac{1}{3}\int_\Omega \left(-3\,\mathrm{div}(|\nabla\varphi_1|^2\nabla\varphi_1) + \lambda_1 a f'''(0)\varphi_1^3\right)\,\varphi_1 \,\mathrm{d}x \\ =& \frac{1}{3}\int_\Omega \left(-3\,\mathrm{div}(|\nabla\varphi_1|^2\nabla\varphi_1)\,\varphi_1 - \frac{f'''(0)}{\kappa}\Delta\varphi_1\varphi_1^3\right)\,\mathrm{d}x \\ =& \int_\Omega \left(|\nabla\varphi_1|^4 + \frac{f'''(0)}{\kappa}|\nabla\varphi_1|^2\varphi_1^2\right)\,\mathrm{d}x\,, \end{aligned}$$

and we get, as $B = 0$,

$$\lambda = \chi(t) = \lambda_1 - \frac{1}{2}\frac{C}{A}\, t^2 + o(t^2).$$

Thus, the bifurcation is subcritical if

$$\int_\Omega \left(|\nabla\varphi_1|^4 + \frac{f'''(0)}{\kappa}|\nabla\varphi_1|^2\varphi_1^2\right)\,\mathrm{d}x > 0,$$

while it is supercritical if

$$\int_\Omega \left(|\nabla\varphi_1|^4 + \frac{f'''(0)}{\kappa}|\nabla\varphi_1|^2\varphi_1^2\right)\,\mathrm{d}x < 0.$$

This ends the proof. □

Proof of Theorem 1.3. Since (H_2^1) implies (H_3^3), combining Propositions 2.2, 3.3, and 3.4 yields Theorem 1.3. □

Remark 3.3. From the above proof it follows that Theorem 1.3 still holds replacing (H_2^1) with (H_3^3).

3.3 Superlinear growth

The aim of this subsection is providing a proof of Theorem 1.4, by combining Proposition 2.2 with Proposition 3.5 below, that has been recently proven in [32]. The following assumptions are here considered:

(H_5^3) $a \in C^2(\overline{\Omega})$;

(H_6^3) $\Omega^+ = \{x \in \Omega\colon a(x) > 0\} \neq \emptyset$, $\Omega^- = \{x \in \Omega\colon a(x) < 0\} \neq \emptyset$, and $\Omega^0 = \{x \in \Omega\colon a(x) = 0\}$ is such that $\partial\Omega^0 \subset \Omega$; the boundaries $\partial(\mathrm{int}\Omega^0)$, $\partial\Omega^+$ and $\partial\Omega^-$ are of class C^2; Ω^0 has a finite number of connected components, that we denote by D_i^+, D_j^- and $D_k^\pm$.

Hence, we can represent Ω^0 in the form

$$\Omega^0 = \bigcup_i D_i^+ \;\cup\; \bigcup_j D_j^- \;\cup\; \bigcup_k D_k^\pm,$$

where the components D_i^+, D_j^- and $D_k^\pm$ are supposed to satisfy:

(H_7^3) for each i, $\partial D_i^+ \subset \overline{\Omega^+}$ and there exist $\gamma_{1,i} > 0$, a neighborhood U_i^+ of ∂D_i^+ and $\alpha_i^+ : \overline{U_i^+} \to]0, +\infty[$ such that

$$a(x) = \alpha_i^+(x)\,\mathrm{dist}(x, \partial D_i^+)^{\gamma_{1,i}} \quad \text{for all } x \in \Omega^+ \cap U_i^+;$$

(H_8^3) for each j, $\partial D_j^- \subset \overline{\Omega^-}$ and there exist $\gamma_{2,j} > 0$, a neighborhood U_j^- of ∂D_j^- and $\alpha_j^- : \overline{U_j^-} \to]-\infty, 0[$ such that

$$a(x) = \alpha_j^-(x)\,\mathrm{dist}(x, \partial D_j^-)^{\gamma_{2,j}} \quad \text{for all } x \in \Omega^- \cap U_j^-;$$

(H_9^3) for each k, the following alternative holds

($H_{9.1}^3$) if $\mathrm{int}(D_k^\pm) = \emptyset$, then

- $\partial D_k^\pm = \Gamma_k$ are of class C^2;
- there exist $\gamma_{3,k} > 0$, a neighborhood U_k^+ of Γ_k and $\alpha_k^+ : \overline{U_k^+} \to]0, +\infty[$ such that

$$a(x) = \alpha_k^+(x)\,\mathrm{dist}(x, \Gamma_k)^{\gamma_{3,k}} \quad \text{for all } x \in \Omega^+ \cap U_k^+; \tag{3.1}$$

- there exist $\gamma_{4,k} > 0$, a neighborhood U_k^- of Γ_k and $\alpha_k^- : \overline{U_k^-} \to]-\infty, 0[$ such that

$$a(x) = \alpha_k^-(x)\,\mathrm{dist}(x, \Gamma_k)^{\gamma_{4,k}} \quad \text{for all } x \in \Omega^- \cap U_k^-; \tag{3.2}$$

($H_{9.2}^3$) if $\mathrm{int}(D_k^\pm) \neq \emptyset$, then

- $\partial D_k^\pm = \Gamma_k^+ \cup \Gamma_k^-$, with $\Gamma_k^+ \cap \Gamma_k^- = \emptyset$, $\Gamma_k^+ \subset \overline{\Omega^+}$, $\Gamma_k^- \subset \overline{\Omega^-}$ of class C^2;
- there exist $\gamma_{3,k} > 0$, a neighborhood U_k^+ of Γ_k^+ and $\alpha_k^+ : \overline{U_k^+} \to]0, +\infty[$ satisfying condition (3.1);
- there exist $\gamma_{4,k} > 0$, a neighborhood U_k^- of Γ_k^- and $\alpha_k^- : \overline{U_k^-} \to]-\infty, 0[$ satisfying condition (3.2).

Let us define

$$D^+ = \bigcup_i D_i^+, \quad D^- = \bigcup_j D_j^-, \quad D^\pm = \bigcup_k D_k^\pm.$$

The set D^+ (respectively, D^-) is constituted by the connected components D_i^+ (respectively, D_j^-) of Ω^0, that are surrounded by regions of positivity (respectively, negativity) of a. Instead, $D^\pm$ is constituted by the connected components D_j^- of Ω^0, that are in between a region of positivity and one of negativity of a. $D^\pm$ can be either a "thin" nodal set, like when assuming condition (H_8^1), or a "thick" nodal set, that is, of positive measure. An example of an admissible nodal configuration for the function a is provided by Figure 4.

Remark 3.4. Let $a \in C^2(\overline{\Omega})$ be a sign-changing function satisfying condition (H_8^1). Then, as already observed in [32], D^+, D^-, and $\mathrm{int}(D^\pm)$ are all empty sets, and assumption ($H_{9.1}^3$) holds.

Proposition 3.5. *[32, Theorem 2.2] Assume (H_1^1), (H_9^1), (H_5^3), (H_6^3), (H_7^3), (H_8^3), and (H_9^3). Then there exists $\lambda_* > 0$ such that for all $\lambda \in]0, \lambda_*[$ the problem* (1.1) *admits at least one strong solution $v_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, for any $p > N$, satisfying $v_\lambda \gg 0$ and*

$$\lim_{\lambda \to +\infty} \|v_\lambda\|_{W^{2,p}} = 0.$$

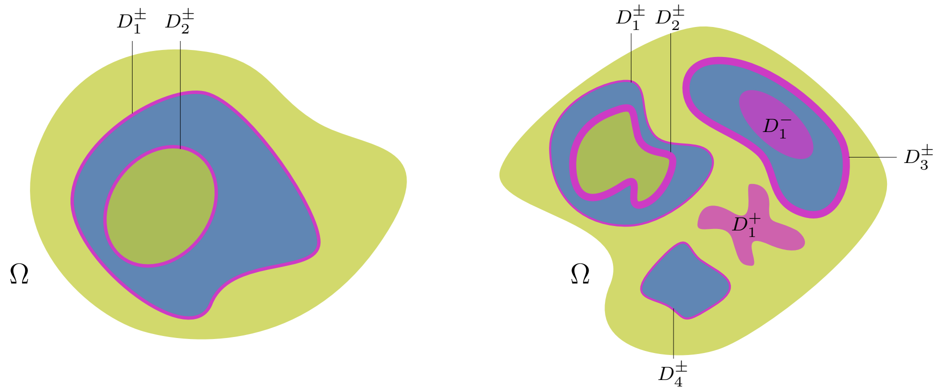

Figure 4: Examples of admissible nodal configurations for the weight a: the union of the green, the purple and the blue regions are respectively the sets Ω^+, Ω^0 and Ω^-. On the left, $\Omega^0 = \bigcup_{k=1}^2 D_k^\pm$ satisfies the assumptions of both Theorem 1.4 and Proposition 3.5. On the right, $\Omega^0 = D_1^+ \cup D_1^- \cup \bigcup_{k=1}^4 D_k^\pm$ satisfies the assumptions of Proposition 3.5.

Proof of Theorem 1.4. From Propositions 2.2 and 3.5, as well as Remark 3.4, we know that, for any given $p > N$, there exists $\lambda_* > 0$ such that, for all $\lambda > \lambda_*$, the problem (1.1) admits a maximum bounded variation solution u_λ and one strong solution $v_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying $0 < u_\lambda, v_\lambda < L$,

$$\lim_{\lambda\to+\infty} (\operatorname{ess\,sup} u_\lambda) = L \qquad \text{and} \qquad \lim_{\lambda\to+\infty} (\operatorname{ess\,sup} v_\lambda) = 0.$$

Hence we infer that $v_\lambda < u_\lambda$, provided λ is large enough. Thus Theorem 1.4 is proven. □

Remark 3.5. From the above proof it follows that Theorem 1.4 still holds replacing (H_8^1) with (H_5^3), (H_6^3), (H_7^3), (H_8^3), and (H_9^3).

4 A peculiar multiplicity result

We prove here a more general version of Theorem 1.5 where the positivity and the continuity assumptions on the weight a are dropped.

Proposition 4.1. *Assume*

(H_1^4) $f\colon [0, L] \to \mathbb{R}$, *with* $L > 0$ *a given constant, is a continuous function satisfying* $f(0) = f(L) = 0$ *and* $f(s) > 0$ *for every* $s \in\,]0, L[$,

(H_2^4) $a \in L^1(0, 1)$ *and satisfies* $\int_0^1 a\,\mathrm{d}x > 0$,

and

(H_3^4) *there exist* $r, R \in\,]0, L[$, *with* $r < R$, *such that*

$$\|a^+\|_{L^1} \frac{2F(r)}{r^2}\bigl(1 + \sqrt{1 + r^2}\bigr) < \left(\int_0^1 a\,\mathrm{d}x\right) \frac{F(R)}{R}.$$

Then there exist $\lambda_\sharp, \lambda^\sharp \in\,]0, +\infty[$, *with* $\lambda_\sharp < \lambda^\sharp$, *such that for all* $\lambda \in\,]\lambda_\sharp, \lambda^\sharp[$ *the problem* (1.7) *admits at least two bounded variation solutions* u_λ, v_λ *such that* $0 < u_\lambda < v_\lambda < L$.

Proof. The proof relies on a counterpart for the problem (1.1) of a mountain pass lemma for non-smooth functionals stated in [31, Lemma 3.7]. It is convenient here too to suppose that $f(s) = 0$ for all $s \in \mathbb{R}\setminus[0, L]$. For any given $\lambda > 0$ we introduce the functionals $\mathcal{J}$, j, $\mathcal{I}\colon BV(0,1) \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{J}(v) &= \int_0^1 \left(\sqrt{1+|D^a v|^2} - 1\right) \mathrm{d}x + \int_0^1 |D^s v| + |v(0)| + |v(1)|,\\
j(v) &= \frac{\|D^a v\|_{L^1}^2}{1+\sqrt{1+\|D^a v\|_{L^1}^2}} + \int_0^1 |D^s v| + |v(0)| + |v(1)|,\\
\mathcal{I}_\lambda(v) &= \mathcal{J}(v) - \lambda \int_0^1 aF(v)\,\mathrm{d}x.
\end{aligned}$$

By the Jensen's inequality we see that

$$\mathcal{J}(v) \geq j(v) \quad \text{for all } v \in BV(0,1).$$

According to Remark 1.1, a function $u \in BV(0,1)$ is a bounded variation solution of (1.7) if and only if

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_0^1 af(u)(v-u)\,\mathrm{d}x \quad \text{for all } v \in BV(0,1). \tag{4.1}$$

We endow the space $BV(0,1)$ with the norm

$$\|v\|_{BV} = |D^a v\|_{L^1} + \int_0^1 |D^s v| + |v(0)| + |v(1)|,$$

which, as already observed in Section 2, is equivalent to the standard one.

Step 1 : *Mountain pass geometry.* Let $r, R > 0$ be the constants introduced in assumption (H_3^4). Define

$$\mathcal{B}_r = \{v \in BV(0,1)\colon \|v\|_{BV} = r\}.$$

We first show that there exist constants $\lambda_\sharp, \lambda^\sharp > 0$, with $\lambda_\sharp < \lambda^\sharp$ such that, for each $\lambda \in]\lambda_\sharp, \lambda^\sharp[$,

$$\inf_{v \in \mathcal{B}_r} \mathcal{I}_\lambda(v) > 0 = \mathcal{I}_\lambda(0). \tag{4.2}$$

Elementary calculations show that the function $\zeta\colon [0, +\infty[\to \mathbb{R}$

$$\zeta(\xi) = \frac{\xi^2}{1+\sqrt{1+\xi^2}} - \xi$$

is decreasing and hence, for all $\xi \in [0, r]$,

$$\frac{\xi^2}{1+\sqrt{1+\xi^2}} + r - \xi \geq \frac{r^2}{1+\sqrt{1+r^2}}.$$

Hence, we infer that, for all $v \in \mathcal{B}_r$

$$j(v) = \frac{\|D^a v\|_{L^1}^2}{1+\sqrt{1+\|D^a v\|_{L^1}^2}} + r - \|D^a v\|_{L^1} \geq \frac{r^2}{1+\sqrt{1+r^2}}.$$

On the other hand, we have that, for all $v \in \mathcal{B}_r$,

$$\|v\|_\infty \leq \|v\|_{BV} = r$$

and hence, as F is increasing,

$$\|F(v)\|_\infty \le F(\|v\|_\infty) \le F(r).$$

Thus, we can conclude that, for all $v \in \mathcal{B}_r$,

$$\begin{aligned}\mathcal{I}_\lambda(v) = \mathcal{J}(v) - \lambda \int_0^1 aF(v)\,\mathrm{d}x &\ge j(v) - \lambda\|a^+\|_{L^1}\|F(v)\|_\infty \\ &\ge \frac{r^2}{1+\sqrt{1+r^2}} - \lambda\|a^+\|_{L^1}F(r).\end{aligned}$$

By using (H_2^4), we can take

$$\lambda^\sharp < \Big(\|a^+\|_{L^1}\frac{F(r)}{r^2}\big(1+\sqrt{1+r^2}\big)\Big)^{-1}.$$

Hence, condition (4.2) holds for each $\lambda \in \,]0, \lambda^\sharp[$.

Next, using (H_2^4) again, we can take $\lambda_\sharp > 0$ such that

$$\lambda_\sharp > \bigg(\Big(\int_0^1 a\,\mathrm{d}x\Big)\frac{F(R)}{2R}\bigg)^{-1},$$

and so we obtain, for each $\lambda > \,]\lambda_\sharp, +\infty[$,

$$\mathcal{I}_\lambda(R) = 2R - \lambda\int_0^1 aF(R)\,\mathrm{d}x = 2R - \lambda\Big(\int_0^1 a\,\mathrm{d}x\Big)F(R) < 0 = \mathcal{I}_\lambda(0). \tag{4.3}$$

Note that assumption (H_3^4) implies that

$$\bigg(\Big(\int_0^1 a\,\mathrm{d}x\Big)\frac{F(R)}{2R}\bigg)^{-1} < \Big(\|a^+\|_{L^1}\frac{F(r)}{r^2}\big(1+\sqrt{1+r^2}\big)\Big)^{-1}.$$

In particular, $\lambda_\sharp, \lambda^\sharp$ can be chosen so as to satisfy

$$\bigg(\Big(\int_0^1 a\,\mathrm{d}x\Big)\frac{F(R)}{2R}\bigg)^{-1} < \lambda_\sharp < \lambda^\sharp < \Big(\|a^+\|_{L^1}\frac{F(r)}{r^2}\big(1+\sqrt{1+r^2}\big)\Big)^{-1}.$$

Therefore, for each $\lambda \in \,]\lambda_\sharp, \lambda^\sharp[$, conditions (4.2) and (4.3) hold, with $\|R\|_{BV} = 2R > r$, thus displaying the desired mountain pass geometry of the functional $\mathcal{I}_\lambda$.

Step 2 : Existence of almost critical points. Henceforth, we fix $\lambda \in \,]\lambda_\sharp, \lambda^\sharp[$. Then, we set

$$\Gamma = \{\gamma \in C^0([0,1], BV(0,1))\colon \gamma(0) = 0,\ \gamma(1) = R\}.$$

From Step 1, we infer that

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{I}_\lambda(\gamma(t)) > \max\{\mathcal{I}_\lambda(0),\ \mathcal{I}_\lambda(R)\} > 0.$$

Then, from a variant of [31, Lemma 3.7] valid for the functional $\mathcal{I}_\lambda$, there exist sequences $(v_n)_n$ in $BV(0,1)$ and $(\varepsilon_n)_n$ in $\mathbb{R}$ with

$$\lim_{n\to+\infty} \varepsilon_n = 0 \tag{4.4}$$

such that, for every n,

$$c_\lambda - \frac{1}{n} \le \mathcal{I}_\lambda(v_n) \le c_\lambda + \frac{1}{n} \tag{4.5}$$

and

$$\mathcal{J}(v) - \mathcal{J}(v_n) \geq \lambda \int_0^1 af(v_n)(v - v_n)\,\mathrm{d}x + \varepsilon_n \|v - v_n\|_{BV} \quad \text{for all } v \in BV(0,1). \tag{4.6}$$

Step 3 : *Estimates on the almost critical points.* By (4.5) the sequence $(v_n)_n$ satisfies, for every n,

$$\|v_n\|_{BV} - 1 \leq \mathcal{J}(v_n) \leq \lambda \int_0^1 aF(v_n)\,\mathrm{d}x + c_\lambda + 1 \leq \lambda^\sharp \|a^+\|_{L^1} F(L) + c_\lambda + 1$$

and thus

$$\sup_n \|v_n\|_{BV} < +\infty. \tag{4.7}$$

Step 4 : *Existence of a positive bounded variation solution* u_λ *with* $\mathcal{I}_\lambda(u_\lambda) = c_\lambda > 0$*.* By the compact embedding of $BV(0,1)$ into $L^1(0,1)$, there exist a subsequence of $(v_n)_n$, still denoted by $(v_n)_n$, and $u_\lambda \in BV(0,1)$ such that $\lim_{n\to+\infty} v_n = u_\lambda$ in $L^1(0,1)$ and a.e. in $[0,1]$. By passing to the inferior limit in (4.6) and using the lower semicontinuity of $\mathcal{J}$ with respect to the L^1-convergence in $BV(0,1)$, as well as the dominated convergence theorem, we obtain

$$\begin{aligned}\mathcal{J}(v) - \lambda \int_0^1 af(u_\lambda)v\,\mathrm{d}x &= \mathcal{J}(v) - \lambda \lim_{n\to+\infty} \int_0^1 af(v_n)v\,\mathrm{d}x \\ &\geq \liminf_{n\to+\infty} \mathcal{J}(v_n) - \lambda \lim_{n\to+\infty} \int_0^1 af(v_n)v_n\,\mathrm{d}x \geq \mathcal{J}(u_\lambda) - \lambda \int_0^1 af(u_\lambda)u_\lambda\,\mathrm{d}x,\end{aligned}$$

for all $v \in BV(0,1)$. Hence, condition (4.1) holds and thus u_λ is a solution of (1.7).

Next we prove that $\mathcal{I}_\lambda(u_\lambda) = c_\lambda$ by showing that

$$\lim_{n\to+\infty} \mathcal{I}_\lambda(v_n) = \mathcal{I}_\lambda(u_\lambda) \tag{4.8}$$

and using (4.5). The dominated convergence theorem implies that

$$\lim_{n\to+\infty} \int_0^1 aF(v_n)\,\mathrm{d}x = \int_0^1 aF(u_\lambda)\,\mathrm{d}x.$$

Hence, to prove (4.8) it is enough to verify that

$$\lim_{n\to+\infty} \mathcal{J}(v_n) = \mathcal{J}(u_\lambda).$$

The lower semicontinuity of $\mathcal{J}$ with respect to the L^1-convergence yields

$$\liminf_{n\to+\infty} \mathcal{J}(v_n) \geq \mathcal{J}(u_\lambda).$$

On the other hand, taking the solution u_λ as test function in (4.6), we get, for all n,

$$\mathcal{J}(v_n) \leq \mathcal{J}(u_\lambda) - \lambda \int_0^1 af(v_n)(u_\lambda - v_n)\,\mathrm{d}x - \varepsilon_n \|u_\lambda - v_n\|_{BV}.$$

Passing to the superior limit and using the dominated convergence theorem again, together with (4.4) and (4.7), we infer that

$$\limsup_{n\to+\infty} \mathcal{J}(v_n) \leq \mathcal{J}(u_\lambda).$$

Step 5 : *Existence of a positive bounded variation solution* w_λ *with* $\mathcal{I}_\lambda(w_\lambda) < 0$*.* From the proof of Proposition 2.1 it is apparent that we only need showing that the functional $\mathcal{I}_\lambda$ attains negative values if $\lambda \in]\lambda_\sharp, \lambda^\sharp[$. This is indeed guaranteed by (4.3). Then the global minimizer of $\mathcal{I}_\lambda$ provides us with a solution $w_\lambda \neq u_\lambda$.

The same argument developed in the proof of Proposition 2.1 shows that $0 < u_\lambda, w_\lambda < L$. Further, since $u_\lambda \neq w_\lambda$ and L is an upper bounded variation solution, but not a solution, we have that $0 < u_\lambda, w_\lambda < u_\lambda \vee w_\lambda < L$ and $u_\lambda \vee w_\lambda$ is a lower bounded variation solution. Hence we infer from [21] or [29] the existence of a solution $v_\lambda > u_\lambda$. This concludes the proof of Proposition 4.1. □

Proof of Theorem 1.5. Notice that (H_3^1), (H_{10}^1), and (H_{11}^1) imply (H_1^4), (H_2^4), and (H_3^4), respectively. Thus, Theorem 1.5 is directly inferred from Proposition 4.1. □

Remark 4.1. The case where Ω is an arbitrary bounded interval $]c, d[$ can be easily handled via the change of variables

$$\xi = \frac{x-c}{d-c}, \quad v(\xi) = \frac{1}{d-c} u\big(c + (d-c)\xi\big),$$

which transforms the problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a f(u) & \text{in }]c, d[, \\ u(c) = 0, \quad u(d) = 0, \end{cases}$$

into

$$\begin{cases} -\left(\dfrac{v'}{\sqrt{1+(v')^2}}\right)' = \lambda \tilde{a} \tilde{f}(v) & \text{in }]0, 1[, \\ v(0) = 0, \quad v(1) = 0, \end{cases}$$

where

$$\tilde{a}(\xi) = a\big(c + (d-c)\xi\big), \quad \tilde{f}(s) = (d-c) f\big((d-c)s\big).$$

Remark 4.2. It is worth observing that if a vanishes on the boundary of its domain and it satisfies the regularity condition specified in [23, 24] at its nodal points, then all solutions of (1.7) are strong solutions. This topic will be discussed in detail elsewhere.

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References

[1] A. Ambrosetti, G. Prodi, A primer of nonlinear analysis, vol. 34 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1993.

[2] L. Ambrosio, N. Fusco, D. Pallara, Functions of bounded variation and free discontinuity problems, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[3] G. Anzellotti, The Euler equation for functionals with linear growth, Trans. Amer. Math. Soc. 290 (2) (1985) 483–501.

[4] D. Bonheure, P. Habets, F. Obersnel, P. Omari, Classical and non-classical solutions of a prescribed curvature equation, J. Differential Equations 243 (2) (2007) 208–237.

[5] K. J. Brown, P. Hess, Stability and uniqueness of positive solutions for a semi-linear elliptic boundary value problem, Differential Integral Equations 3 (2) (1990) 201–207.

[6] K. J. Brown, S. S. Lin, On the existence of positive eigenfunctions for an eigenvalue problem with indefinite weight function, J. Math. Anal. Appl. 75 (1980) 112–120.

[7] R. S. Cantrell, C. Cosner, Spatial ecology via reaction-diffusion equations, Wiley Series in Mathematical and Computational Biology, John Wiley & Sons, Ltd., Chichester, 2003.

[8] A. Chertock, A. Kurganov, P. Rosenau, On degenerate saturated-diffusion equations with convection, Nonlinearity 18 (2) (2005) 609–630.

[9] C. Corsato, C. De Coster, N. Flora, P. Omari, Radial solutions of the Dirichlet problem for a class of quasilinear elliptic equations arising in optometry, Nonlinear Anal. 181 (2019) 9–23.

[10] C. Corsato, C. De Coster, F. Obersnel, P. Omari, A. Soranzo, A prescribed anisotropic mean curvature equation modeling the corneal shape: a paradigm of nonlinear analysis, Discrete Contin. Dyn. Syst. Ser. S 11 (2) (2018) 213–256.

[11] C. Corsato, C. De Coster, P. Omari, The Dirichlet problem for a prescribed anisotropic mean curvature equation: existence, uniqueness and regularity of solutions, J. Differential Equations 260 (5) (2016) 4572–4618.

[12] C. Corsato, P. Omari, F. Zanolin, Subharmonic solutions of the prescribed curvature equation, Commun. Contemp. Math. 18 (3) (2016) 1550042, 33 pp.

[13] M. G. Crandall, P. H. Rabinowitz, Bifurcation from simple eigenvalues, J. Functional Analysis 8 (1971) 321–340.

[14] P. C. Fife, Mathematical aspects of reacting and diffusing systems, vol. 28 of Lecture Notes in Biomathematics, Springer-Verlag, Berlin-New York, 1979.

[15] W. H. Fleming, A selection-migration model in population genetics, J. Math. Biol. 2 (3) (1975) 219–233.

[16] C. Gerhardt, Boundary value problems for surfaces of prescribed mean curvature, J. Math. Pures Appl. (9) 58 (1) (1979) 75–109.

[17] D. Gilbarg, N. S. Trudinger, Elliptic partial differential equations of second order, Classics in Mathematics, Springer-Verlag, Berlin, 2001, reprint of the 1998 edition.

[18] E. Giusti, Boundary value problems for non-parametric surfaces of prescribed mean curvature, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 3 (3) (1976) 501–548.

[19] E. Giusti, Minimal surfaces and functions of bounded variation, vol. 80 of Monographs in Mathematics, Birkhäuser Verlag, Basel, 1984.

[20] A. Kurganov, P. Rosenau, On reaction processes with saturating diffusion, Nonlinearity 19 (1) (2006) 171–193.

[21] V. K. Le, On a sub-supersolution method for the prescribed mean curvature problem, Czechoslovak Math. J. 58 (133) (2) (2008) 541–560.

[22] J. D. Logan, Biological invasions with flux-limited dispersal, Math. Sci. Res. J. 7 (2) (2003) 47–62.

[23] J. López-Gómez, P. Omari, Characterizing the formation of singularities in a superlinear indefinite problem related to the mean curvature operator, J. Differential Equations (2020) https://doi.org/10.1016/j.jde.2020.01.015.

[24] J. López-Gómez, P. Omari, Regular versus singular solutions in a quasilinear indefinite problem with an asymptotically linear potential, Adv. Nonlinear Stud. (2020) https://doi.org/10.1515/ans-2020-2083.

[25] U. Massari, M. Miranda, Minimal surfaces of codimension one, vol. 91 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 1984, Notas de Matemática [Mathematical Notes], 95.

[26] M. Miranda, Dirichlet problem with L^1 data for the non-homogeneous minimal surface equation, Indiana Univ. Math. J. 24 (1974/75) 227–241.

[27] J. D. Murray, Mathematical biology. I, vol. 17 of Interdisciplinary Applied Mathematics, 3rd ed., Springer-Verlag, New York, 2002, An introduction.

[28] J. D. Murray, Mathematical biology. II, vol. 18 of Interdisciplinary Applied Mathematics, 3rd ed., Springer-Verlag, New York, 2003, Spatial models and biomedical applications.

[29] F. Obersnel, P. Omari, Existence and multiplicity results for the prescribed mean curvature equation via lower and upper solutions, Differential Integral Equations 22 (9-10) (2009) 853–880.

[30] F. Obersnel, P. Omari, Positive solutions of the Dirichlet problem for the prescribed mean curvature equation, J. Differential Equations 249 (7) (2010) 1674–1725.

[31] F. Obersnel, P. Omari, S. Rivetti, Asymmetric Poincaré inequalities and solvability of capillarity problems, J. Funct. Anal. 267 (3) (2014) 842–900.

[32] P. Omari, E. Sovrano, Positive solutions of superlinear indefinite prescribed mean curvature problems, Commun. Contemp. Math. (2020) in press.

[33] P. Rosenau, Free-energy functionals at the high-gradient limit, Physical Review A 41 (4) (1990) 2227–2230.

[34] P. Rosenau, Tempered diffusion: a transport process with propagating fronts and inertial delay, Phys. Rev. A 46 (1992) 7371–4.

[35] R. Temam, Solutions généralisées de certaines équations du type hypersurfaces minima, Arch. Rational Mech. Anal. 44 (1971/72) 121–156.

[36] T. Valent, Boundary value problems of finite elasticity, vol. 31 of Springer Tracts in Natural Philosophy, Springer-Verlag, New York, 1988.

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