Positive solutions of indefinite logistic growth models with flux-saturated diffusion

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Dedicated, with esteem and friendship, to Professor Shair Ahmad for his 85th birthday

Abstract. This paper analyzes the quasilinear elliptic boundary value problem driven by the mean curvature operator

$$-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \lambda a(x) f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

with the aim of understanding the effects of a flux-saturated diffusion in logistic growth models featuring spatial heterogeneities. Here, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a regular boundary $\partial\Omega$, $\lambda > 0$ represents a diffusivity parameter, $a$ is a continuous weight which may change sign in $\Omega$, and $f: [0, L] \to \mathbb{R}$, with $L > 0$ a given constant, is a continuous function satisfying $f(0) = f(L) = 0$ and $f(s) > 0$ for every $s \in [0, L]$. Depending on the behavior of $f$ at zero, three qualitatively different bifurcation diagrams appear by varying $\lambda$. Typically, the solutions we find are regular as long as $\lambda$ is small, while as a consequence of the saturation of the flux they may develop singularities when $\lambda$ becomes larger. A rather unexpected multiplicity phenomenon is also detected, even for the simplest logistic model, $f(s) = s(L - s)$ and $a \equiv 1$, having no similarity with the case of linear diffusion based on the Fick-Fourier’s law.

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1 Introduction

This paper analyzes the quasilinear elliptic problem

$$\begin{cases}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \lambda a(x) f(u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

(1.1)

where the diffusion is driven by the mean curvature operator. Here, $\lambda > 0$ is viewed as a parameter measuring diffusivity and

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(H\textsubscript{1} \textsuperscript{1}) \ \Omega \subset \mathbb{R}^N \text{ is a bounded domain, with a } C^2 \text{ boundary } \partial \Omega \text{ in case } N \geq 2; \\
(H\textsubscript{1} \textsuperscript{2}) \ a: \Omega \to \mathbb{R} \text{ is a continuous function such that } \max_{\Omega} a > 0; \\
(H\textsubscript{1} \textsuperscript{3}) \ f: \mathbb{R} \to \mathbb{R} \text{ is a continuous function satisfying, for some constant } L > 0, \ f(0) = f(L) = 0, \text{ and } f(s) > 0 \text{ for every } s \in [0, L].

Assumption (H\textsubscript{1} \textsuperscript{2}) on the weight \(a\) introduces spatial heterogeneities within the model and allows, but does not impose, that \(a\) changes sign in \(\Omega\). Assumption (H\textsubscript{1} \textsuperscript{3}) basically requires that the reaction term \(af\) is of logistic-type. As well-known, logistic maps play a pivotal role in the modeling theory of various disciplines, with special prominence in biology, ecology, genetics; see, e.g., [7, 14, 15, 27, 28] and the extensive bibliographies therein. Unlike the classical theory based on the Fick-Fourier's law, where the flux depends linearly on \(\nabla u\), here the diffusion is governed by the bounded flux \(\nabla u/\sqrt{1 + |\nabla u|^2}\), which is approximately linear for small gradients but approaches saturation for large ones.

The aim of this work is, therefore, describing, understanding, and clarifying the effects of a flux-saturated diffusion in logistic growth models featuring spatial heterogeneities. This study is motivated by the investigations on reaction processes with saturating diffusion started in [33] and further carried out in [8, 20, 22, 34], in order to correct the non-physical gradient-flux relations at high gradients. This specific mechanism of diffusion, of which the mean curvature operator provides a paradigmatic example, may determine spatial patterns exhibiting abrupt transitions at the boundary or between adjacent profiles, up to the formation of discontinuities [4, 9, 10, 11, 12, 16, 18, 19, 23, 24, 25, 26, 35]. This makes the mathematical analysis of the problem (1.1) more delicate and sophisticated than the study of the corresponding semilinear model, the use of some tools of geometric measure theory being in particular required. It is an established fact indeed that the space of bounded variation functions is the natural setting for dealing with this problem. The precise notion of bounded variation solution of (1.1) used in this paper has been basically introduced in [3] and is recalled below for completeness.

**Notation.** Throughout this work, for every \(v \in BV(\Omega)\), \(Dv = D^u v + D^s v\) is the Lebesgue-Nikodym decomposition of the Radon measure \(Dv\) in its absolutely continuous part \(D^a v\) and its singular part \(D^s v\) with respect to the \(N\)-dimensional Lebesgue measure \(dx\) in \(\mathbb{R}^N\), \(|Dv|\) denotes the total variation of the measure \(Dv\), and \(|\partial \Omega|\) stands for the density of \(Dv\) with respect to its total variation. Further, \([\Omega]\) is the Lebesgue measure of \(\Omega\), while \(H_{N-1}\) represents the \((N - 1)\)-dimensional Hausdorff measure, and \(|\partial \Omega|\) is the \(H_{N-1}\)-measure of \(\partial \Omega\). We refer to [2] for additional information. Moreover, for all functions \(u, v: \Omega \to \mathbb{R}\), we write: \(u \geq v\) if \(\text{ess inf}(u - v) \geq 0\); \(u > v\) if \(u \geq v\) and \(\text{ess sup}(u - v) > 0\); \(u \gg v\) if, for a.e. \(x \in \Omega\), \(u(x) - v(x) \geq \text{dist}(x, \partial \Omega)\). We also define \(u \wedge v\) and \(u \vee v\) by setting \((u \wedge v)(x) = \min\{u(x), v(x)\}\) and \((u \vee v)(x) = \max\{u(x), v(x)\}\) for a.e. \(x \in \Omega\). Finally, we write \(u^+\) for \(u \vee 0\) and \(u^-\) for \(-(u \wedge 0)\).

**Definition 1.1.** By a bounded variation solution of (1.1) we mean a function \(u \in BV(\Omega)\), with \(f(u) \in L^N(\Omega)\), which satisfies

\[
\int_{\Omega} \frac{D^a u D^a \phi}{\sqrt{1 + |D^a u|^2}} \, dx + \int_{\Omega} \frac{D u}{|D u|} \frac{D \phi}{|D \phi|} |D^s \phi| + \int_{\partial \Omega} \text{sgn}(u) \phi \, dH_{N-1} = \lambda \int_{\Omega} af(u) \phi \, dx \tag{1.2}
\]

for every \(\phi \in BV(\Omega)\) such that \(|D^s \phi|\) is absolutely continuous with respect to \(|D^a u|\) and \(\phi(x) = 0\) \(H_{N-1}\)-a.e. on the set \(\{x \in \partial \Omega: u(x) = 0\}\). A bounded variation solution \(u\) is said positive if \(u > 0\).

**Remark 1.1.** It follows from [3, Section 3] that a function \(u \in BV(\Omega)\), with \(f(u) \in L^N(\Omega)\), is a bounded variation solution of (1.1) if and only if it satisfies the variational inequality

\[
J(v) - J(u) \geq \lambda \int_{\Omega} af(u)(v - u) \, dx \quad \text{for all } v \in BV(\Omega), \tag{1.3}
\]

where

\[
J(v) = \int_{\Omega} (\sqrt{1 + |D^a v|^2} - 1) \, dx + \int_{\Omega} |D^s v| + \int_{\partial \Omega} |v| \, dH_{N-1}.
\]
Remark 1.2. If a bounded variation solution $u$ of (1.1) belongs to $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ for some $p > N$, then it satisfies the differential equation in (1.1) for a.e. $x \in \Omega$ and the boundary condition for all $x \in \partial \Omega$. Therefore, $u$ is a strong solution of (1.1). The $L^p$-regularity theory [17, Chapter 9] then entails that $u \in W^{2,q}(\Omega)$ for all $q > N$. Conversely, it is evident that any weak solution $u \in W^{1,1}_0(\Omega)$, and hence in particular any strong solution, is a bounded variation solution.

Remark 1.3. It is clear that, for any given $\lambda > 0$, $u = 0$ is a solution of (1.1), while $u = L$ is not. Indeed, if $L$ were a solution, taking $\phi = 1$ as test function in (1.2) would yield $\int_{\partial \Omega} 1 \, dH_{N-1} = |\partial \Omega| = 0$, which is a contradiction.

We are now going to present the main results obtained in this paper. Here, for the sake of clarity, our statements are set out in a simplified form, while referring to the subsequent sections for some variants or extensions thereof that rely on slightly more general but less neat conditions: for each result, the minimal needed assumptions will be specified in an appropriate remark placed just below the corresponding proof.

The first result only exploits the structural assumptions $(H^1_1)$, $(H^1_2)$, and $(H^1_3)$. It provides us with the existence of a number $\lambda_* \geq 0$ such that, for all $\lambda > \lambda_*$, the problem (1.1) has a maximum solution $u_\lambda$, with $0 < u_\lambda < L$. The asymptotic behavior of $u_\lambda$, as $\lambda \to +\infty$, is described too, and the bifurcation of the solutions from the trivial line $\{(\lambda, 0) : \lambda \geq 0\}$ at the point $(0, 0)$ is ascertained in the case $\lambda_* = 0$. Figure 1 illustrates the admissible bifurcations diagrams.

Theorem 1.1. Assume $(H^1_1)$, $(H^1_2)$, and $(H^1_3)$. Then there exists $\lambda_* \geq 0$ such that for all $\lambda \in [\lambda_*, +\infty[\) the problem (1.1) admits a maximum bounded variation solution $u_\lambda$, with $0 < u_\lambda < L$, which satisfies

$$\lim_{\lambda \to +\infty} (\operatorname{ess sup} u_\lambda) = L.$$ (1.4)

Moreover, if $\lambda_* = 0$, then

$$\lim_{\lambda \to 0^+} \|u_\lambda\|_{BV} = 0.$$ (1.5)

Figure 1: Admissible bifurcation diagrams for the problem (1.1) under the structural assumptions $(H^1_1)$, $(H^1_2)$, and $(H^1_3)$, in case $\lambda_* > 0$ (left) or $\lambda_* = 0$ (right). Dashed curves indicate bounded variation solutions.

The specific features displayed by the bifurcation diagrams of the problem (1.1) are determined by the slope at 0 of the function $f$, as expressed by the following conditions:

$(H^1_1)$ there exists $\lim_{s \to 0^+} \frac{f(s)}{s} = +\infty$ \hspace{1cm} (sublinear growth at 0);

$(H^1_2)$ there exists $\lim_{s \to 0^+} \frac{f(s)}{s} = \kappa \in ]0, +\infty[\) \hspace{1cm} (linear growth at 0);

$(H^1_3)$ there exists $\lim_{s \to 0^+} \frac{f(s)}{s} = 0$ \hspace{1cm} (superlinear growth at 0).

When $f$ has a sublinear growth at zero, a bifurcation from the trivial line occurs at the point $(0, 0)$,
and the existence of positive bounded variation solutions of the problem (1.1) is guaranteed for all \( \lambda > 0 \). In addition, positive strong solutions exist provided that \( \lambda \) is small enough.

**Theorem 1.2.** Assume \((H_1^1), (H_2^1), (H_3^1),\) and \((H_4^1)\). Then for all \( \lambda > 0 \) the problem (1.1) admits at least one bounded variation solution \( u_\lambda \in BV(\Omega) \), with \( 0 < u_\lambda < L \), which satisfies (1.4) and (1.5). Moreover, there exists \( \lambda^* > 0 \) such that, for all \( \lambda \in [0, \lambda^*] \), solutions \( u_\lambda \) can be selected so that \( u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) for any \( p > N \), it is a strong solution and it satisfies

\[
\lim_{\lambda \to 0^+} \|u_\lambda\|_{W^{2,p}} = 0.
\]

When \( f \) grows linearly at zero the bifurcation occurs from the trivial line at the point \((\lambda_1, 0)\), where \( \lambda_1 \) is the principal eigenvalue of the linear weighted problem

\[
\begin{aligned}
-\Delta \varphi &= \lambda a(x) \kappa \varphi \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Here, \( \Omega \) satisfies \((H_1^1)\), \( \kappa \) comes from \((H_2^1)\), and \( a \) satisfies \((H_3^1)\). It follows from [6] that \( \lambda_1 \) is positive and simple, with a positive eigenfunction \( \varphi_1 \). The \( L^p \)-regularity theory and a standard bootstrap argument entail that \( \varphi_1 \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) for all \( p > N \), while the strong maximum principle and the Hopf boundary point lemma yield \( \varphi_1 \gg 0 \). In this case the solvability of the problem (1.1) is guaranteed for all \( \lambda > \lambda_1 \). In addition, for \( \lambda \) close to \( \lambda_1 \) strong solutions do exist.

**Theorem 1.3.** Assume \((H_1^1), (H_2^1), (H_3^1),\) and \((H_4^1)\). Then for all \( \lambda > \lambda_1 \) the problem (1.1) admits at least one bounded variation solution \( u_\lambda \), with \( 0 < u_\lambda < L \), which satisfies (1.4). Moreover, suppose that

\((H_7^1)\) \( f \) is of class \( C^2 \)

and fix any \( p > N \). Then there exists a neighborhood \( U \) of \((\lambda_1, 0)\) in \( \mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) such that solutions \( u_\lambda \) can be selected so that \( (\lambda, u_\lambda) \in U \), \( u_\lambda \) is a strong solution and it satisfies

\[
\lim_{\lambda \to \lambda_1} \|u_\lambda\|_{W^{2,p}} = 0 \quad \text{and} \quad \lim_{\lambda \to \lambda_1} \frac{u_\lambda}{\|u_\lambda\|_{C^1}} = \varphi_1. \tag{1.6}
\]

Finally, there exists \( \eta > 0 \) such that the following assertions hold:

(i) if \( f''(0) < 0 \), then for all \( \lambda \in (\lambda_1, \lambda_1 + \eta) \) there is at least one strong solution \( u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) satisfying (1.6);

(ii) if \( f''(0) > 0 \), then for all \( \lambda \in (\lambda_1 - \eta, \lambda_1) \) there is at least one strong solution \( u_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) satisfying (1.6).

**Remark 1.4.** For the standard logistic model \( f(s) = s(L - s) \), the condition \( f''(0) = -2 < 0 \) holds and therefore the bifurcation is supercritical.

When \( f \) exhibits a superlinear growth at zero, the existence of multiple solutions can be detected if, for instance, conditions \((H_5^1)\) and \((H_6^1)\) are strengthened as follows. Let us set

\[
\Omega^+ = \{x \in \Omega: a(x) > 0\}, \quad \Omega^- = \{x \in \Omega: a(x) < 0\}, \quad \Omega^0 = \{x \in \Omega: a(x) = 0\},
\]

and replace \((H_5^1)\) with

\((H_7^1)\) \( a \in C^2(\overline{\Omega}) \), \( \Omega^+ \neq 0 \), \( \Omega^- \neq 0 \), \( \Omega^0 = \overline{\Omega^+} \cap \overline{\Omega^-} \subset \Omega \), and \( \nabla a(x) \neq 0 \) for all \( x \in \Omega^0 \),

as well as \((H_8^1)\) with
there exists \( q > 1 \), with \( q < \frac{N+2}{N-2} \) if \( N \geq 3 \), such that
\[
\lim_{s \to 0^+} \frac{f(s)}{s^q} = 1.
\]

Then, for \( \lambda \) sufficiently large, the problem (1.1) has at least two positive bounded variation solutions, the smaller being strong.

**Theorem 1.4.** Assume \((H_1^1),\ (H_1^3),\ (H_3^1)\), and \((H_3^2)\). Then there exists \( \lambda_* \geq 0 \) such that for all \( \lambda \in [\lambda_*, +\infty) \), the problem (1.1) admits at least one bounded variation solution \( u_\lambda \) and one strong solution \( v_\lambda \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \), for any \( p > N \), such that \( 0 < v_\lambda < u_\lambda < L \). In addition, \( u_\lambda \) satisfies (1.4), while \( v_\lambda \) satisfies
\[
\lim_{\lambda \to +\infty} \|v_\lambda\|_{W^{2,p}} = 0.
\]

Figure 2 illustrates three qualitatively different bifurcation diagrams corresponding, respectively, to Theorems 1.2, 1.3, and 1.4.

Figure 2: Admissible qualitative bifurcation diagrams for the problem (1.1), according to the growth of \( f \) at 0: either sublinear (left), or linear (center), or superlinear (right). Dashed curves indicate bounded variation solutions, solid curves represent strong solutions.

Unexpectedly enough, the existence of multiple solutions can always be detected in the standard logistic model, whenever the carrying capacity \( L \) is sufficiently large, even in the case where the weight function \( a \) is a positive constant (cf. Remark 1.5 below). We state such a multiplicity result for the simplest one-dimensional prototype of the problem (1.1), that is,
\[
\begin{aligned}
&-\left( \frac{u'}{\sqrt{1 + (u')^2}} \right)' = \lambda af(u) & \text{in } [0,1],
\end{aligned}
\]

\[ u(0) = 0, \quad u(1) = 0. \quad (1.7) \]

**Theorem 1.5.** Assume \((H_1^1)\),
\( (H_1^0) \ a \in C^0([0,1]) \) satisfies \( a > 0 \),

and
\( (H_1^1) \) there exist \( r, R \in [0,L] \), with \( r < R \), such that
\[
\frac{2F(r)}{r^2} (1 + \sqrt{1 + r^2}) < \frac{F(R)}{R},
\]
where \( F(s) = \int_0^s f(t) \, dt \) is the potential of \( f \). Then there exist \( \lambda_2, \lambda_3 \), with \( 0 < \lambda_2 < \lambda_3 \), such that for all \( \lambda \in [\lambda_2, \lambda_3] \) the problem (1.7) admits at least two bounded variation solutions \( u_\lambda, v_\lambda \) such that \( 0 < u_\lambda < v_\lambda < L \).
It is worth stressing that the assumptions of Theorem 1.5 do not prevent \( f \) from being concave in \([0, L]\); this fact witnesses the peculiarity of this multiplicity result, which is specific of the quasilinear problem (1.1) and has no similarity at all with the semilinear case, where the concavity of \( f \) always guarantees the uniqueness of the positive solution, as proven in [5] even for sign-changing weights \( a \).

**Remark 1.5.** For the standard logistic model, where \( f(s) = s(L - s) \), condition \((H_{11})\) is satisfied if, for instance, \( L > \frac{\pi^2}{4} \approx 10.67 \).

**Example 1.6.** A numerical study of the problem (1.7), with \( a \equiv 1 \), \( f(s) = s(L - s) \) and \( L = 11 > \frac{\pi^2}{4} \), reveals the existence of three positive solutions in a (small) right neighborhood of the bifurcation point \( \lambda_1 = \frac{\pi^2}{4} \approx 0.8972 \), in particular at \( \lambda = 0.8975 \), and of two positive solutions in a left neighborhood of \( \lambda_1 \). This is in complete agreement with (i) the bifurcation result stated in Theorem 1.3 and Remark 1.4, which predicts the bifurcation branch emanates from \( \lambda_1 \) pointing to the right; (ii) the multiplicity conclusions of Theorem 1.5, which guarantee the existence of two solutions in an interval of the \( \lambda \)-axis located on the left of \( \lambda_1 \). Hence a S-shaped bifurcation diagram is expected as shown by the picture on the left in Figure 3.

![Figure 3: On the left, an admissible bifurcation diagram is depicted with reference to Example 1.6: the dashed curve indicates bounded variation solutions, the solid curve represents strong solutions. On the right, the profiles of the three detected solutions at \( \lambda = \lambda_1 \) are shown: in blue the regular ones, in red the singular one.](image)

The remainder of this paper is structured as follows. Section 2 is devoted to the proof of various statements concerning the existence and the asymptotic behavior of the positive bounded variation solutions of (1.1), under the sole structural conditions \((H_{11})\), \((H_{12})\), and \((H_{13})\); in particular, Theorem 1.1 is proven. Section 3 focuses on the discussion of the features displayed by the bifurcation diagrams of the problem (1.1) according to the slope at zero of the function \( f \); here, some extensions, or variants, of Theorems 1.2, 1.3, and 1.4 are derived. Section 4 closes the paper by providing the proof of a more general version of the multiplicity result stated in Theorem 1.5.

## 2 Bounded variation solutions: existence and asymptotic behavior of the bifurcation branches

In this section we aim to prove Theorem 1.1, as well as some variants thereof, by using variational techniques in the space \( BV(\Omega) \), in combination with the method of lower and upper solutions for mean curvature problems as first developed in [21] and independently in [29]. Henceforth, we endow the space \( BV(\Omega) \) with the norm

\[
\|v\|_{BV} = \int_{\Omega} |Dv| + \int_{\partial \Omega} |v| d\mathcal{H}^{N-1},
\]
which is equivalent to the usual one by [26, Proposition 2] and [2, Theorem 3.88]. Since we are looking for solutions \( u \) of (1.1) satisfying the condition \( 0 < u < L \), we can suppose, without loss of generality, that

\[
f(s) = 0 \quad \text{for all } s \in \mathbb{R} \setminus [0, L].
\]

We also set

\[
F(s) = \int_0^s f(t) \, dt \quad \text{for all } s \in \mathbb{R}.
\]

Next, we introduce the action functional associated with the problem (1.1). Namely, for each \( \lambda > 0 \), we define \( \mathcal{I}_\lambda : BV(\Omega) \to \mathbb{R} \) by

\[
\mathcal{I}_\lambda(v) = \mathcal{J}(v) - \lambda \int_{\Omega} a F(v) \, dx,
\]

where \( \mathcal{J} : BV(\Omega) \to \mathbb{R} \) is given by

\[
\mathcal{J}(v) = \int_{\Omega} \sqrt{1 + |Dv|^2} - |\Omega| + \int_{\partial \Omega} |v| \, d\mathcal{H}_{N-1},
\]

having set

\[
\int_{\Omega} \sqrt{1 + |Dv|^2} = \int_{\Omega} \sqrt{1 + |D^a v|^2} \, dx + \int_{\Omega} |D^s v|.
\]

We start by proving the existence of positive bounded variation solutions of (1.1) under the following conditions that weaken \((H^1_1)\) and \((H^2_1)\), respectively:

\((H^2_1)\) \( \Omega \subset \mathbb{R}^N \) is a bounded domain, with a boundary \( \partial \Omega \) of class \( C^1 \) in case \( N \geq 2 \);

\((H^2_2)\) \( a \in L^\infty(\Omega) \) and there is a Caccioppoli set \( E \) of positive measure such that \( \int_E a(x) \, dx > 0 \).

**Proposition 2.1.** Assume \((H^1_1)\), \((H^2_1)\), and \((H^2_2)\). Then there exists \( \lambda_* > 0 \) such that for all \( \lambda \in [\lambda_*, +\infty[ \) the problem (1.1) admits a maximum bounded variation solution \( u_\lambda \) satisfying \( 0 < u_\lambda < L \).

**Proof.** For later reference, the proof is split into three parts.

**Step 1:** For every \( \lambda > 0 \), there exists a global minimizer \( u_\lambda \) of \( \mathcal{I}_\lambda \). Fix \( \lambda > 0 \). From \((H^2_2)\) and \((H^2_1)\) we easily get, for all \( v \in BV(\Omega) \),

\[
\mathcal{I}_\lambda(v) \geq \int_{\Omega} |Dv| - |\Omega| + \int_{\partial \Omega} |v| \, d\mathcal{H}_{N-1} - \lambda \|a^+\|_{L^1} F(L)
= \|v\|_{BV} - |\Omega| - \lambda \|a^+\|_{L^1} F(L).
\]

Therefore, \( \mathcal{I}_\lambda \) is bounded from below and coercive. Let \( (v_n)_n \) be a minimizing sequence. Since \( (v_n)_n \) is bounded in \( BV(\Omega) \), the compact embedding of \( BV(\Omega) \) into \( L^1(\Omega) \) implies that there exist a subsequence of \( (v_n)_n \), still denoted by \( (v_n)_n \), and a function \( u_\lambda \in BV(\Omega) \) such that \( v_n \to u_\lambda \) in \( L^1(\Omega) \) and a.e. in \( \Omega \). The lower semicontinuity of \( \mathcal{J}_\lambda \) with respect to the \( L^1 \)-convergence in \( BV(\Omega) \) and the dominated convergence theorem easily yield

\[
\mathcal{I}_\lambda(u_\lambda) \leq \liminf_{n \to +\infty} \mathcal{I}_\lambda(v_n) = \inf_{v \in BV(\Omega)} \mathcal{I}_\lambda(v),
\]

that is, \( u_\lambda \) is a global minimizer of \( \mathcal{I}_\lambda \).

**Step 2:** For every \( \lambda > 0 \), \( u_\lambda \) is a bounded variation solution of (1.1) satisfying \( 0 \leq u_\lambda < L \). From [30, Remark 2.2.] we know that any local minimizer \( u_\lambda \) of \( \mathcal{I}_\lambda \) satisfies the variational inequality (1.3) and therefore by Remark 1.1 it is a bounded variation solution of (1.1).

Next, we show that \( u_\lambda < L \). Taking \( v = u_\lambda \wedge L \) as test function in (1.3) and observing that \( v - u_\lambda = -(u_\lambda - L)^+ \), we get

\[
0 = \lambda \int_{\Omega} a f(u_\lambda) \left(-(u_\lambda - L)^+\right) \, dx \leq \mathcal{J}(u_\lambda \wedge L) - \mathcal{J}(u_\lambda).
\]
Then, recalling that $u_\lambda \vee L = L + (u_\lambda - L)^+$ and using the lattice property proven in [21, Theorem 3.2] or [29, Proposition 2.2], we infer

$$0 \leq J(u_\lambda \wedge L) - J(u_\lambda) \leq J(L) - J(u_\lambda \vee L)$$

$$= - \left( \int_{\Omega} \sqrt{1 + |D(u_\lambda - L)^+|^2} - |\Omega| \right) - \int_{\partial\Omega} (u_\lambda - L)^+ \, dH_{N-1} \leq 0.$$

This yields $(u_\lambda - L)^+ = 0$ and thus $u_\lambda \leq L$. Moreover, we have that $u_\lambda < L$, because, by Remark 1.3, $L$ is not a solution of (1.1).

At last, we prove that $u_\lambda \geq 0$. Taking $v = u_\lambda \vee 0$ as test function in (1.3), observing that $v - u_\lambda = u_\lambda^-$ and using again the lattice property, we obtain

$$0 \leq J(u_\lambda \vee 0) - J(u_\lambda^-) \leq J(0) - J(u_\lambda \wedge 0)$$

$$= - \left( \int_{\Omega} \sqrt{1 + |D(u_\lambda^-)|^2} - |\Omega| \right) - \int_{\partial\Omega} u_\lambda^- \, dH_{N-1} \leq 0.$$

Hence we conclude that $u_\lambda^- = 0$ and thus $u_\lambda \geq 0$.

**Step 3:** For every $\lambda > \lambda_*$, there exists a maximum bounded variation solution $w_\lambda$ of (1.1) such that $0 < w_\lambda < L$. We first prove that the global minimizer $u_\lambda$ of (1.1) satisfies $u_\lambda > 0$. To this end it is sufficient to show that, when $\lambda$ is large enough, 0 is not a global minimizer of $\mathcal{I}_\lambda$. Thanks to $(H_2^2)$, we can take $v = \chi_E \in BV(\Omega)$ in (2.1), where $\chi_E$ is the characteristic function of $E$. Denoting by $\text{Per}(E, \Omega)$ the perimeter of $E$ in $\Omega$, we obtain

$$\mathcal{I}_\lambda(\chi_E) = \text{Per}(E, \Omega) - \lambda F(L) \int_E a \, dx.$$

Therefore, by setting

$$\lambda_* = \frac{\text{Per}(E, \Omega)}{F(L) \int_E a \, dx},$$

we infer that $\mathcal{I}_\lambda(0) < \mathcal{I}_\lambda(\chi_E)$, for every $\lambda > \lambda_*$.\[\square\]

**Remark 2.1.** From the proof of Proposition 2.1 it follows that the problem (1.1) has, for a given $\lambda > 0$, a solution $u_\lambda \in BV(\Omega)$, with $0 < u_\lambda < L$, if and only if there exists a function $\psi \in BV(\Omega)$ such that $0 < \psi < L$ and $\mathcal{I}_\lambda(\psi) < 0$. This in turn holds, for all large $\lambda > 0$, if and only if $\int_{\Omega} aF(\psi) \, dx > 0$.

To complement the previous result, we investigate the asymptotic behavior of the maximum solutions $u_\lambda \in BV(\Omega)$ of (1.1) as $\lambda \to +\infty$. To this purpose, we assume the following condition

$$(H_2^2) \quad a \in L^\infty(\Omega) \text{ and there is an open set } \omega \subset \Omega \text{ such that } \text{ess inf}_\omega a > 0.$$ 

Assumption $(H_2^2)$ obviously implies $(H_2^2)$.

**Proposition 2.2.** Assume $(H_1^2)$, $(H_2^2)$, and $(H_3^2)$. Then the maximum bounded variation solution $u_\lambda$ of (1.1) with $0 < u_\lambda < L$, which exists for all $\lambda \in [\lambda_*, +\infty]$ according to Proposition 2.1, further satisfies (1.4).
Proof. We begin with the following simple consequence of assumption \((H^3_3)\).

**Claim 1.** There exist a global maximizer \(\sigma_M \in [0, L]\) of \(f\) in \([0, L]\) and a sequence \((\sigma_n)\) in \([\sigma_M, L]\) such that

\[
\lim_{n \to +\infty} \sigma_n = L
\]

and

\[
f(s) \geq f(\sigma_n) \quad \text{for all } s \in [\sigma_M, \sigma_n].
\] (2.2)

Indeed, the largest global maximizer \(\sigma_M \in [0, L]\) of \(f\) in \([0, L]\) exists by \((H^3_1)\). For each \(n \geq 1\), let \(\sigma_n \in [\sigma_M, L - \frac{L - \sigma_M}{n+1}]\) be the largest global minimizer of \(f\) in \([\sigma_M, L - \frac{L - \sigma_M}{n+1}]\). Assumption \((H^3_3)\) implies that \(f(L - \frac{L - \sigma_M}{n+1}) \to 0\) and hence that \(\sigma_n \to L\), as \(n \to +\infty\). Accordingly, \((\sigma_n)\) is the desired sequence.

Thanks to assumption \((H^3_2)\) we can find constants \(\varepsilon > 0\) and \(\rho > 0\) such that \(a(x) \geq \varepsilon\) for a.e. \(x \in \omega_\rho\), where \(\omega_\rho\) is the open ball of center \(x_0\) and radius \(\rho\). Without restriction we can suppose that \(\overline{\omega_\rho} \subset \Omega\).

Let \((\sigma_M)\) be the largest global maximizer of \(f\) in \([0, L]\) and \((\sigma_n)\) be the sequence given by Claim 1. Fix \(n\) and, for simplifying notation, set \(\sigma = \sigma_n\). Define also

\[
\lambda_\varepsilon = \frac{N}{\varepsilon f(\sigma) \min \{\rho, \sigma - \sigma_M\}}.
\]

Fix \(\lambda > \lambda_\varepsilon\) and set \(\tau = \frac{N}{\varepsilon f(\sigma)}\). Denote by \(\omega_\tau\) the open ball of center \(x_0\) and radius \(\tau\). As \(\tau \in [0, \rho]\), we have that \(\overline{\omega_\tau} \subset \omega_\rho\). First, we define a function \(v_1 \in W^{1,1}(\omega_\tau) \cap \mathcal{C}^0(\overline{\omega_\tau}) \cap \mathcal{C}^2(\omega_\tau)\) by

\[
v_1(x) = \sigma - \tau + \sqrt{\tau^2 - |x - x_0|^2}.
\]

Clearly, \(v_1\) is a classical solution of

\[
\begin{aligned}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= \lambda \varepsilon f(\sigma) \quad \text{in } \omega_\tau, \\
u &= v_1 \quad \text{on } \partial \omega_\tau.
\end{aligned}
\]

It is immediately checked that \(v_1\) also satisfies

\[
\sigma_M < \sigma - \tau \leq \min v_1 < \max v_1 = \sigma.
\] (2.3)

Second, we define \(v_2 \in W^{1,1}(\Omega \setminus \overline{\omega_\tau}) \cap \mathcal{C}^0(\overline{\Omega \setminus \omega_\tau}) \cap \mathcal{C}^2(\Omega \setminus \overline{\omega_\tau})\) by

\[
v_2(x) = -1 - \sqrt{|x - x_0|^2 - \tau^2}.
\]

The function \(v_2\) is a classical solution of

\[
\begin{aligned}
-\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) &= g(x) \quad \text{in } \Omega \setminus \overline{\omega_\tau}, \\
u &= v_2 \quad \text{on } \partial(\Omega \setminus \overline{\omega_\tau}),
\end{aligned}
\]

where \(g \in L^\infty(\Omega)\) is given by

\[
g(x) = \begin{cases} 2(N - 1)|x - x_0|^2 - N \tau^2 & \text{if } x \in \Omega \setminus \overline{\omega_\tau}, \\ \frac{2(N - 1)|x - x_0|^2 - N \tau^2}{(2|x - x_0|^2 - \tau^2)^{3/2}} & \text{if } x \in \overline{\omega_\tau}. \end{cases}
\]

Clearly, we have that \(\max v_2 = -1\). (2.4)
Third, we define a function \( v \) by
\[
v(x) = \begin{cases} 
  v_1(x) & \text{if } x \in \omega_r, \\
  v_2(x) & \text{if } x \in \Omega \setminus \overline{\omega_r}.
\end{cases}
\]

It follows from [2, Theorem 3.84] that \( v \in BV(\Omega) \). Let also \( h \in L^\infty(\Omega) \) be defined by
\[
h(x) = \begin{cases} 
  \lambda x f(\sigma) & \text{if } x \in \omega_r, \\
  g(x) & \text{if } x \in \Omega \setminus \overline{\omega_r}.
\end{cases}
\]

**Claim 2.** The function \( v \) is a bounded variation solution of
\[
\begin{align*}
& -\text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = h(x) \quad \text{in } \Omega, \\
& u = v_2 \quad \text{on } \partial \Omega.
\end{align*}
\]  
(2.5)

We begin observing that, for all \( x \in \partial \omega_r \), \( v_1(x) > v_2(x) \) and
\[
\frac{\nabla v_1(x) \cdot \nu_{\omega_r}(x)}{\sqrt{1 + |\nabla v_1(x)|^2}} = -1, \quad \frac{\nabla v_2(x) \cdot \nu_{\Omega \setminus \overline{\omega_r}}(x)}{\sqrt{1 + |\nabla v_2(x)|^2}} = 1,
\]
(2.6)

where \( \nu_{\omega_r}(x) \) and \( \nu_{\Omega \setminus \overline{\omega_r}}(x) \) are, respectively, the unit outer normals to \( \omega_r \) and to \( \Omega \setminus \overline{\omega_r} \) at \( x \in \partial \omega_r \). By [2, Theorem 3.84] we can write
\[
Dv = D^a v \, dx + D^s v = \nabla v \, dx + (v_2 - v_1) \nu_{\omega_r} \, dH_{\mathbb{N} - 1}.
\]
(2.7)

Take now a test function \( \phi \in BV(\Omega) \) such that \( |D^s \phi| \) is absolutely continuous with respect to \( |D^a v| \) and \( \phi(x) = 0 \mid_{H_{\mathbb{N} - 1} \text{-a.e.}} \) on the set \( \{ x \in \partial \Omega: v(x) = v_2(x) \} \). If we set \( \phi_1 = \phi_{\omega_r} \in W^{1,1}(\omega_r) \) and \( \phi_2 = \phi_{\Omega \setminus \overline{\omega_r}} \in W^{1,1}(\Omega \setminus \overline{\omega_r}) \), then, by [2, Theorem 3.84] again, we have that
\[
D\phi = D^a \phi \, dx + D^s \phi = \nabla \phi \, dx + (\phi_2 - \phi_1) \nu_{\omega_r} \, dH_{\mathbb{N} - 1}.
\]
(2.8)

Thanks to (2.6), (2.7), and (2.8), we get
\[
\int_{\omega_r} h \phi_1 \, dx = -\int_{\omega_r} \text{div} \left( \frac{\nabla v_1}{\sqrt{1 + |\nabla v_1|^2}} \right) \phi_1 \, dx \\
= -\int_{\partial \omega_r} \frac{\nabla v_1 \cdot \nu_{\omega_r}}{\sqrt{1 + |\nabla v_1|^2}} \phi_1 \, dH_{\mathbb{N} - 1} + \int_{\omega_r} \frac{\nabla v_1 \cdot \nabla \phi_1}{\sqrt{1 + |\nabla v_1|^2}} \, dx
\]
and
\[
\int_{\Omega \setminus \overline{\omega_r}} h \phi_2 \, dx = -\int_{\Omega \setminus \overline{\omega_r}} \text{div} \left( \frac{\nabla v_2}{\sqrt{1 + |\nabla v_2|^2}} \right) \phi_2 \, dx \\
= -\int_{\partial (\Omega \setminus \overline{\omega_r})} \frac{\nabla v_2 \cdot \nu_{\Omega \setminus \overline{\omega_r}}}{\sqrt{1 + |\nabla v_2|^2}} \phi_2 \, dH_{\mathbb{N} - 1} + \int_{\Omega \setminus \overline{\omega_r}} \frac{\nabla v_2 \cdot \nabla \phi_2}{\sqrt{1 + |\nabla v_2|^2}} \, dx
\]
\[
= -\int_{\partial \omega_r} \frac{\nabla \phi_2}{\sqrt{1 + |\nabla \phi_2|^2}} \, dH_{\mathbb{N} - 1} + \int_{\Omega \setminus \overline{\omega_r}} \frac{D^a v_2 D^a \phi_2}{\sqrt{1 + |D^a v_2|^2}} \, dx.
\]
Since
\[ |D^s v| = |v_2 - v_1|d\mathcal{H}^{N-1}, \quad \frac{D^s v}{|D^s v|} = \frac{v_2 - v_1}{|v_2 - v_1|}\nu_\omega, \]
and \( v = v_2 \) on \( \partial\Omega \), we can conclude that
\[
\int_\Omega h\phi \, dx = \int_\omega h\phi_1 \, dx + \int_{\partial\omega_\tau} h\phi_2 \, dx = \int_{\partial\omega_\tau} (\phi_1 - \phi_2) \, d\mathcal{H}^{N-1} + \int_\Omega \frac{D^s v \cdot D^s \phi}{\sqrt{1 + |D^s v|^2}} \, dx
\]
\[
= \int_{\partial\omega_\tau} \frac{v_2 - v_1}{|v_2 - v_1|}\nu_\omega \cdot \frac{\phi_2 - \phi_1}{|\phi_2 - \phi_1|} \nu_\omega |\phi_2 - \phi_1| \, d\mathcal{H}^{N-1} + \int_\Omega \frac{D^s v \cdot D^s \phi}{\sqrt{1 + |D^s v|^2}} \, dx
\]
\[
= \int_{\Omega} \frac{D^s v}{|D^s v|} \frac{D^s \phi}{|D^s \phi|} + \int_{\partial\Omega} \text{sgn}(v - v_2) \phi \, d\mathcal{H}^{N-1} + \int_\Omega \frac{D^s v \cdot D^s \phi}{\sqrt{1 + |D^s v|^2}} \, dx.
\]
Therefore \( v \) is a bounded variation solution of (2.5) according to [3, Section 3]. This concludes the proof of Claim 2.

Let us now define a function \( \ell: \Omega \times \mathbb{R} \to \mathbb{R} \) by
\[
\ell(x, s) = \begin{cases} 
\lambda \min \{f(s), f(\sigma)\} \chi_{\omega_\tau}(x) + \lambda a(x)f(\sigma) \chi_{\Omega \setminus \overline{\omega_\tau}}(x) & \text{if } s \geq 0, \\
sgn(s) & \text{if } -1 < s < 0, \\
g(x) & \text{if } s \leq -1,
\end{cases}
\] (2.9)
where \( \chi_{\omega_\tau} \) and \( \chi_{\Omega \setminus \overline{\omega_\tau}} \) are the characteristic functions of \( \omega_\tau \) and of \( \Omega \setminus \overline{\omega_\tau} \), respectively. The function \( \ell \) satisfies the \( L^\infty \)-Carathéodory conditions and, due to (2.2), (2.3), and (2.4),
\[
\ell(x, v(x)) = h(x) \quad \text{for a.e. } x \in \Omega.
\]
Consequently, \( v \) is a bounded variation solution of
\[
\begin{cases} 
- \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \ell(x, v) & \text{in } \Omega, \\
u = v_2 & \text{on } \partial\Omega.
\end{cases}
\]
Hence, by [3, Section 3] the function \( v \) also satisfies the variational inequality
\[
\int_\Omega \sqrt{1 + |Dw|^2} + \int_{\partial\Omega} |w - v_2| \, d\mathcal{H}^{N-1} - \int_\Omega \sqrt{1 + |Dv|^2} - \int_{\partial\Omega} |v - v_2| \, d\mathcal{H}^{N-1}
\geq \int_\Omega \ell(x, v)(w - v) \, dx
\] (2.10)
for all \( w \in BV(\Omega) \).

Claim 3. The function \( v \lor 0 \) is a lower bounded variation solution of
\[
\begin{cases} 
- \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \ell(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega.
\end{cases}
\] (2.11)
Fix any \( z \in BV(\Omega) \) such that \( z \leq 0 \). As \( v(x) = v_2(x) < 0 \) for all \( x \in \partial\Omega \), we have that \( |z(x)| = |v(x) + z(x)| - |v(x)| \) for \( \mathcal{H}^{N-1} \)-a.e. \( x \in \partial\Omega \). Since \( v \) is a solution of (2.11), taking \( v + z \) as test function
in (2.10) yields
\[
\int_\Omega \ell(x,v)z \, dx \leq \int_\Omega \sqrt{1 + |D(v + z)|^2} + \int_{\partial \Omega} |v + z - v_2| \, d\mathcal{H}^{N-1} \\
- \int_\Omega \sqrt{1 + |Dv|^2} - \int_{\partial \Omega} |v - v_2| \, d\mathcal{H}^{N-1} \\
= \int_\Omega \sqrt{1 + |D(v + z)|^2} + \int_{\partial \Omega} |z| \, d\mathcal{H}^{N-1} - \int_\Omega \sqrt{1 + |Dv|^2} \\
= \int_\Omega \sqrt{1 + |D(v + z)|^2} + \int_{\partial \Omega} |v + z| \, d\mathcal{H}^{N-1} - \int_\Omega \sqrt{1 + |Dv|^2} - \int_{\partial \Omega} |v| \, d\mathcal{H}^{N-1}.
\]

Further, as \( \ell(x,0) = 0 \) for all \( x \in \Omega \), we have that 0 is a solution of (2.11). Hence, according to [21] or [29], \( v \vee 0 \) is a lower bounded variation solution of (2.11). This concludes the proof of Claim 3.

Claim 4. For any \( \lambda > \lambda_* \) there exists a lower bounded variation solution \( \alpha \) of (1.1) such that \( 0 < \alpha < L \) and \( \text{ess sup} |\alpha| \geq \lambda \). We know that \( v \vee 0 \) is a lower bounded variation solution of (2.11). Moreover, as \( \ell(x,L) = 0 \) for all \( x \in \Omega \), \( L \) is an upper bounded variation solution, but not a solution, of (2.11). Then, by [21] or by [29] there exists a solution \( \alpha \) of (2.11) with \( 0 < v \leq \alpha < L \). From (2.9), we see that, for all \( s \geq 0 \) and a.e. \( x \in \Omega \),
\[
a(x)f(s) \geq \varepsilon \min\{f(s),f(\sigma)\} \chi_{\omega_{s}}(x) + a(x)f(s)\chi_{\Omega\setminus\omega}(x) = \ell(x,s).
\]
Therefore, we immediately infer from [21] or [29] that \( \alpha \) is a lower bounded variation solution of (1.1) as well, thus concluding the proof of Claim 4.

We are now in position of concluding the proof. Indeed, for each \( \eta \in [0, L] \) we can find \( \sigma \in [L - \eta, L] \) and \( \lambda_* = \lambda_*(\eta) > 0 \) such that, for all \( \lambda > \lambda_* \), there is a lower bounded variation solution \( \alpha_\lambda \) of (1.1) with \( \text{ess sup} \alpha_\lambda \geq \sigma \). Hence, the maximum bounded variation solution \( u_\lambda \) of (1.1), which exists according to Proposition 2.1 for all \( \lambda > \lambda_* \), must satisfy \( u_\lambda \geq \alpha_\lambda \) and thus \( \text{ess sup} u_\lambda \geq \sigma \geq L - \eta \) for all \( \lambda > \max\{\lambda_*, \lambda_*\} \). Consequently, condition (1.4) is proven.

We finally describe the behavior of the solutions \( u_\lambda \in BV(\Omega) \) of (1.1), if any, as \( \lambda \to 0^+ \).

**Proposition 2.3.** Assume \((H_1^2), (H_3^2)\), and
\((H_4^2)\) \( a \in L^N(\Omega) \).

Then any sequence \( (\lambda_n, u_n) \) of solutions of the problem (1.1), with \( \lambda_n > 0, \ 0 < u_n < L \), for all \( n \), and
\[
\lim_{n \to +\infty} \lambda_n = 0,
\]

satisfies
\[
\lim_{n \to +\infty} \|u_n\|_{BV} = 0.
\] (2.12)

**Proof.** For any given \( n \), taking \( \phi = u_n \) as test function in (1.2), we get
\[
\int_\Omega \frac{|D^s u_n|^2}{\sqrt{1 + |D^s u_n|^2}} \, dx + \int_\Omega |D^s u_n| \, dx + \int_{\partial \Omega} |u_n| \, d\mathcal{H}^{N-1} = \lambda_n \int_\Omega a f(u_n) u_n \, dx.
\]

Letting \( n \to +\infty \), we find, as \( \lambda_n \to 0 \),
\[
\lambda_n \int_\Omega a f(u_n) u_n \, dx \leq \lambda_n \|a^+\|_{L^1} \|f(u_n) u_n\|_{L^\infty} \leq \lambda_n \|a^+\|_{L^1} \max_{s \in [0,L]} f(s) \to 0
\]
and hence
\[
\int_\Omega \frac{|D^s u_n|^2}{\sqrt{1 + |D^s u_n|^2}} \, dx + \int_\Omega |D^s u_n| \, dx \to 0. \quad (2.13)
\]
Then for all $c$, Assumption 3.1 guarantees the existence of a solution for any growth conditions $H(x)$.

Proposition 3.1. Assume $H(x)$, $H_1$, $H_2$, and $H_3$. Then $f(s)$ tends to $0$ as $|s|\to 0^+$.

Remark 2.2. From the above proof it follows that Theorem 1.1 still holds replacing $H_1$ with $H_2$ and $H_1$ with $H_3$.

3 Prescribing different growth conditions at zero

In this section we discuss the existence and the multiplicity of solutions of the problem (1.1) by imposing one of the growth conditions on $f$ at zero expressed by $H_1$, or $H_2$, or $H_3$.

3.1 Sublinear growth

In this subsection we establish two results from which Theorem 1.2 will eventually be inferred. The first statement guarantees the existence of a solution for any $\lambda > 0$ under a generalized form of condition $H_1$.

Proposition 3.1. Assume $H_1$, $H_2$, and $H_3$. Then $f(s)$ tends to $0$ as $|s|\to 0^+$.

Then for all $\lambda > 0$ the problem (1.1) admits at least one bounded variation solution $u_\lambda \in BV(\Omega)$, which can be selected so as to satisfy $0 < u_\lambda < L$, (1.4), and (1.5).
Proof. It is convenient here to suppose that \( f(s) = 0 \) for all \( s \in \mathbb{R} \setminus [0, L] \). For any given \( \lambda > 0 \), the existence of a global minimizer \( u_\lambda \) of \( \mathcal{I}_\lambda \), satisfying \( 0 \leq u_\lambda < L \), is guaranteed by the first two steps of the proof of Theorem 2.1. Hence, according to Remark 2.1, in order to establish that \( u_\lambda > 0 \), it is sufficient to find a function \( \psi \in BV(\Omega) \) such that \( \mathcal{I}_\lambda(\psi) < 0 \). We first notice that, by assumption \((H_i^1)\), there exists a sequence \( (s_n)_n \) in \([0, L]\) such that

\[
\lim_{n \to +\infty} s_n = 0 \quad \text{and} \quad \lim_{n \to +\infty} \frac{F(s_n)}{s_n^2} = +\infty.
\]

Next, we pick an open set \( \omega_1 \) such that \( \partial \Omega \subset \omega \), with \( \omega \) defined in \((H_i^2)\), and a function \( w \in H^1(\Omega) \) such that \( w(x) \geq 0 \) in \( \Omega \), \( w(x) = 0 \) in \( \Omega \setminus \omega \) and \( w(x) = 1 \) in \( \omega_1 \). Hence, we have that

\[\mathcal{I}_\lambda(s_n w) = \frac{1}{2} \int_\omega |s_n \nabla w|^2 \, dx - \lambda \int_{\omega_1} a F(s_n w) \, dx - \lambda \int_{\omega \setminus \omega_1} a F(s_n w) \, dx \leq s_n^2 \left( \frac{1}{2} \int_\omega |\nabla w|^2 \, dx - \lambda \frac{F(s_n)}{s_n^2} \int_{\omega_1} a \, dx \right) < 0,\]

for all large \( n \). This implies that \( \mathcal{I}_\lambda(u_\lambda) < 0 \) and thus \( u_\lambda > 0 \). Finally, we observe that \( \mathcal{I}_\lambda(u_\lambda) < 0 \) yields

\[\mathcal{J}(u_\lambda) < \lambda \int_\Omega a F(u_\lambda) \, dx \leq \lambda \|a\|_{L^1} F(L)\]

and then

\[\frac{1}{2} \int_\Omega \frac{|D^3 u_\lambda|^2}{\sqrt{1 + |D^2 u_\lambda|^2}} \, dx + \int_\Omega |D^3 u_\lambda| + \int_{\partial \Omega} |u_\lambda| \, dH_{N-1} \leq \int_\Omega \frac{|D^3 u_\lambda|^2}{\sqrt{1 + |D^2 u_\lambda|^2}} \, dx + \int_\Omega |D^3 u_\lambda| + \int_{\partial \Omega} |u_\lambda| \, dH_{N-1} = \mathcal{J}(u_\lambda) \to 0, \quad \text{as } \lambda \to 0^+.\]

Hence, arguing as in the proof of Proposition 2.3, we see that

\[\lim_{\lambda \to 0^+} \|u_\lambda\|_{BV} = 0.\]

The last conclusion,

\[\lim_{\lambda \to +\infty} (\text{ess sup } u_\lambda) = L,\]

follows from Proposition 2.2.

The next result yields the existence of strong solutions for \( \lambda \) sufficiently small.

**Proposition 3.2.** Assume \((H_1^1), (H_i^1), (H_i^2), (H_i^3)\). Then there exists \( \lambda^* \in [0, +\infty) \) such that for all \( \lambda \in [0, \lambda^*] \) the problem (1.1) admits at least one strong solution \( u_\lambda \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \), for all \( p > N \), satisfying \( 0 < u_\lambda < L \) and

\[\lim_{\lambda \to 0^+} \|u_\lambda\|_{W^{2,p}} = 0.\]

**Proof.** From [30, Theorem 3.1] we infer the existence of \( \lambda^* \in [0, +\infty) \) such that for every \( \lambda \in [0, \lambda^*] \) there is \( u_\lambda \in C^{1,\gamma}(\overline{\Omega}) \cap W^{1,1}_0(\Omega) \), for some \( \gamma \in [0, 1] \), such that \( u_\lambda > 0 \),

\[\int_\Omega \frac{\nabla u \cdot \nabla \phi}{\sqrt{1 + |\nabla u|^2}} \, dx = \lambda \int_\Omega a f(u) \phi \, dx, \quad \text{for all } \phi \in C^\infty_0(\Omega),\]

and

\[\lim_{\lambda \to 0^+} \|u_\lambda\|_{C^1(\overline{\Omega})} = 0.\]
The $L^p$-regularity theory implies that $u_\lambda \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, for all $p > N$, and
\[
\lim_{\lambda \to 0^+} \|u_\lambda\|_{W^{2,p}} = 0.
\]
This ends the prof.\qed

Proof of Theorem 1.2. As $(H^2_1)$ implies $(H^3_2)$ and $(H^4_1)$ implies $(H^3_1)$, Propositions 3.1 and 3.2 yield Theorem 1.2.\qed

Remark 3.1. From the above proof it follows that Theorem 1.2 still holds replacing $(H^2_1)$ with $(H^2_2)$ and $(H^4_1)$ with $(H^3_1)$.

### 3.2 Linear growth

In this subsection we provide a proof of Theorem 1.3 as a consequence of two slightly more general results stated below as Propositions 3.3 and 3.4. The basic assumption of Proposition 3.3 is

$(H^3_2)$ there exists \( \lim_{s \to 0^+} \frac{2F(s)}{s^2} = \kappa \in [0, +\infty[ \),

generalizing condition $(H^3_1)$. Once the constant $\kappa$ is assigned by $(H^3_2)$, we assume $(H^3_1)$ and $(H^3_3)$ $a \in L^\infty(\Omega)$ is such that $\text{ess sup}_\Omega a > 0$.

Then, we respectively denote by $\lambda_1$ and $\varphi_1$ the principal eigenvalue and the principal eigenfunction of the linear weighted problem

\[
\begin{align*}
-\Delta \varphi &= \lambda a(x) \kappa \varphi \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

It follows from [6] that $\lambda_1 > 0$, $\lambda_1$ is simple and $\varphi_1 > 0$. As already observed, the $L^p$-regularity theory and a standard bootstrap argument then entail that $\varphi_1 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ for all $p > N$, while the strong maximum principle and the Hopf boundary point lemma yield $\varphi_1 > 0$.

**Proposition 3.3.** Assume $(H^3_1)$, $(H^3_3)$, $(H^3_2)$, and $(H^3_3)$. Then for all $\lambda > \lambda_1$ the problem (1.1) admits at least one bounded variation solution $u_\lambda$, satisfying $0 < u_\lambda < L$.

**Proof.** It is convenient here to suppose that $f(s) = 0$ for all $s \in \mathbb{R} \setminus [0, L]$. Fix any $\lambda > \lambda_1$. By Remark 2.1, it is enough to find a function $\psi \in BV(\Omega)$ such that $I_\lambda(\psi) < 0$. Assumption $(H^3_2)$ implies that there is a sequence $(s_n)_n$ in $[0, L[$ such that

\[
\lim_{n \to +\infty} s_n = 0 \quad \text{and} \quad \lim_{n \to +\infty} \frac{F(s_n)}{s_n^2} = \kappa
\]

and hence

\[
\lim_{k \to 0^+} \frac{2F(s_n \varphi_1(x))}{s_n^2 \varphi_1(x)^2} = \kappa \quad \text{uniformly in } x \in \Omega.
\]

This yields

\[
\lim_{n \to +\infty} \int_{\Omega} \left( \frac{\|\nabla \varphi_1^2}{1 + \sqrt{1 + s_n^2 |\nabla \varphi_1|^2}} - \lambda a \frac{F(s_n \varphi_1)}{s_n^2 \varphi_1^2} \varphi_1^2 \right) \, dx = \frac{1}{2} \int_{\Omega} (|\nabla \varphi_1|^2 - \lambda \kappa a \varphi_1^2) \, dx
\]

\[
= \frac{1}{2} \int_{\Omega} \left( 1 - \frac{\lambda}{\lambda_1} \right) |\nabla \varphi_1|^2 \, dx < 0.
\]

We therefore conclude that

\[
I(s_n \varphi_1) = s_n^2 \int_{\Omega} \left( \frac{\|\nabla \varphi_1^2}{1 + \sqrt{1 + s_n^2 |\nabla \varphi_1|^2}} - \lambda a \frac{F(s_n \varphi_1)}{s_n^2 \varphi_1^2} \varphi_1^2 \right) \, dx < 0,
\]

for all large $n$.\qed
Remark 3.2. It is evident from the above proof that in place of \((H^3_2)\) one can assume the existence of a constant \(\kappa \in [0, +\infty]\) and of a sequence \((s_n)_{n}\) in \([0, L]\) such that

\[
\lim_{n \to +\infty} s_n = 0 \quad \text{and} \quad \frac{2F(s_n)}{s_n^2} = \kappa \in [0, +\infty].
\]

The next result guarantees the existence of small positive strong solutions \(u\) of (1.1). Fix \(p > N\) and introduce the set

\[
S = \{(\lambda, u) \in \mathbb{R} \times W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) : \lambda > 0 \text{ and } u > 0 \text{ is a strong solution of (1.1)}\} \cup \{(\lambda_1, 0)\}.
\]

Since \((\lambda, 0)\) solves (1.1) for all \(\lambda \in \mathbb{R}\), we look for positive solutions bifurcating from the line of the trivial solutions by the Crandall-Rabinowitz theorem [13]. Namely, the following local bifurcation result holds.

**Proposition 3.4.** Assume \((H^1_1), (H^1_2), (H^2_1), (H^2_2)\), and fix \(p > N\). Then there exists a neighborhood \(\mathcal{U}\) of \((\lambda_1, 0)\) in \(\mathbb{R} \times W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\) and functions

\[
\chi : [-1, 1] \to \mathbb{R}, \quad \psi : [-1, 1] \to \left\{ u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) : \int_{\Omega} u \varphi_1 \, dx = 0 \right\}
\]

of class \(C^1\) such that

\[
\chi(0) = \lambda_1, \quad \psi(0) = 0,
\]

and

\[
S \cap \mathcal{U} = \{(\lambda, u) : \lambda = \chi(t), \ u = t(\varphi_1 + \psi(t)), \ t \in [0, 1]\}.
\]

Supposing, in addition, that \((H^3_3)\) is of class \(C^3\),

the following assertions hold:

(i) if either \(f''(0) > 0\) or, otherwise, \(f''(0) = 0\) and \(\frac{f'''(0)}{f''(0)} > -\frac{\int_{\Omega} |\nabla \varphi_1|^4 \, dx}{\int_{\Omega} |\nabla \varphi_1|^2 \varphi_1^2 \, dx}\), then the bifurcation of positive solutions is subcritical,

(ii) if either \(f''(0) < 0\) or, otherwise, \(f''(0) = 0\) and \(\frac{f'''(0)}{f''(0)} < -\frac{\int_{\Omega} |\nabla \varphi_1|^4 \, dx}{\int_{\Omega} |\nabla \varphi_1|^2 \varphi_1^2 \, dx}\), then the bifurcation of positive solutions is supercritical.

**Proof.** Fix \(p > N\) and define the operator \(\mathcal{F} : \mathbb{R} \times W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega)\) by setting

\[
\mathcal{F}(\lambda, u) = \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) + \lambda uf(u).
\]

It is clear that \((\lambda, u) \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\) satisfies \(\mathcal{F}(\lambda, u) = 0\) if and only if \(u\) is a strong solution of (1.1) for some \(\lambda > 0\). By combining the results in [36, Chapter II, Section 4] with the continuity, from \(W^{1,p}(\Omega)\) to \(L^p(\Omega)\), of the linear operators which map any function \(u\) onto its weak partial derivative \(\partial_i u\), with \(i = 1, \ldots, N\), we infer that \(\mathcal{F}\) is of class \(C^2\) under \((H^1_1)\) and, respectively, of class \(C^3\) under \((H^3_3)\).

The partial derivatives of \(\mathcal{F}\) relevant to the present proof are produced below. For all \((\lambda, u) \in \mathbb{R} \times W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\) and \(v, w, z \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)\), there hold:

\[
\partial_{\lambda} \mathcal{F}(\lambda, u) = af(u),
\]

\[
\partial_u \mathcal{F}(\lambda, u)[v] = \text{div} \left( \frac{\nabla v}{\sqrt{1 + |\nabla u|^2}} \right) - \frac{\nabla u \cdot \nabla v}{(\sqrt{1 + |\nabla u|^2})^3} \nabla u + \lambda af'(u) v,
\]
\[ \partial_{u\lambda} F(\lambda, u)[v] = a f'(u)v, \]

\[ \partial_{u\lambda} F(\lambda, u)[v][w] = \text{div} \left( -\frac{\nabla u \cdot \nabla w}{(\sqrt{1 + |\nabla u|^2})^3} \nabla v - \frac{\nabla w \cdot \nabla v}{(\sqrt{1 + |\nabla u|^2})^3} \nabla u - \frac{\nabla u \cdot \nabla v}{(\sqrt{1 + |\nabla u|^2})^3} \nabla w \right. \]

\[ \left. + \frac{3}{(\sqrt{1 + |\nabla u|^2})^5} \left( \nabla u \cdot \nabla v \right) \left( \nabla u \cdot \nabla w \right) \nabla \lambda + \frac{3}{(\sqrt{1 + |\nabla u|^2})^5} \left( \nabla u \cdot \nabla w \right) \nabla \lambda \right) + \lambda a f''(u)v w, \]

\[ \partial_{uu\lambda} F(\lambda, u)[v][w][z] = \text{div} \left( -\frac{\nabla z \cdot \nabla w}{(\sqrt{1 + |\nabla u|^2})^3} \nabla v - \frac{\nabla w \cdot \nabla v}{(\sqrt{1 + |\nabla u|^2})^3} \nabla z - \frac{\nabla z \cdot \nabla v}{(\sqrt{1 + |\nabla u|^2})^3} \nabla w \right. \]

\[ \left. + \frac{3}{(\sqrt{1 + |\nabla u|^2})^5} \left( \nabla u \cdot \nabla z \right) \left( \nabla u \cdot \nabla w \right) \nabla \lambda + \frac{3}{(\sqrt{1 + |\nabla u|^2})^5} \left( \nabla z \cdot \nabla v \right) \nabla \lambda \right) + \lambda a f''(u)v wz. \]

Let us set

\[ \mathcal{L} = \partial_{u\lambda} F(\lambda_1, 0) = \Delta + \lambda_1 \alpha \kappa \mathcal{I} \quad \text{and} \quad \mathcal{M} = \partial_{u\lambda} F(\lambda_1, 0) = a \kappa \mathcal{I}, \]

where \( \kappa = f'(0) \) and \( \mathcal{I} \) is the identity operator. It is clear that \( \mathcal{L} \) is a Fredholm operator with index 0, having kernel

\[ N(\mathcal{L}) = \text{span}\{\varphi_1\}, \]

and range

\[ R(\mathcal{L}) = \left\{ u \in L^p(\Omega) : \int_{\Omega} u \varphi_1 \, dx = 0 \right\}. \]

Further, the transversality condition

\[ \mathcal{M}[\varphi_1] = a \kappa \varphi_1 \not\in R(\mathcal{L}) \]

is satisfied, because

\[ A = \int_{\Omega} \mathcal{M}[\varphi_1] \varphi_1 \, dx = \int_{\Omega} a \kappa \varphi_1^2 \, dx \]

\[ = \lambda_1^{-1} \int_{\Omega} -\Delta \varphi_1 \varphi_1 \, dx = \lambda_1^{-1} \int_{\Omega} |\nabla \varphi_1|^2 \, dx > 0. \]

Hence, the Crandall-Rabinowitz theorem [13, Theorem 1.7] yields the existence of a neighborhood \( \mathcal{U} \) of \( (\lambda_1, 0) \) in \( \mathbb{R} \times W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) and of functions

\[ \chi : ]-1, 1[ \rightarrow \mathbb{R}, \quad \psi : ]-1, 1[ \rightarrow \left\{ u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) : \int_{\Omega} u \varphi_1 \, dx = 0 \right\} \]

of class \( C^1 \) such that

\[ \chi(0) = \lambda_1, \quad \psi(0) = 0, \]

and

\[ \mathcal{S} \cap \mathcal{U} = \{ (\lambda, u) : \lambda = \chi(t), \ u = t(\varphi_1 + \psi(t)), \ t \in ]-1, 1[ \}. \]
We further infer from [13, Theorems 1.7 and 1.18] (see also [1, Chapter 5.4]) that
\[ \lambda = \chi(t) = \lambda_1 - \frac{B}{A} t + o(t), \]
where
\[ B = \frac{1}{2} \int_\Omega F_{uuu}(\lambda_1, 0) [\varphi_1] [\varphi_1] [\varphi_1] \varphi_1 \, dx = \frac{1}{2} \int_\Omega \lambda_1 a f'''(0) \varphi_1^3 \, dx \]
\[ = \frac{1}{2} \frac{f''(0)}{\kappa} \int_\Omega -\Delta \varphi_1 \varphi_1^2 \, dx = \frac{f''(0)}{\kappa} \int_\Omega |\nabla \varphi_1|^2 \varphi_1 \, dx. \]
Thus, the bifurcation of positive solutions is subcritical if \( f''(0) > 0 \), while it is supercritical if \( f''(0) < 0 \).
In case \( f''(0) = 0 \) and \( f \) satisfying \((H_3^4)\), we define
\[ C = \frac{1}{3} \int_\Omega F_{uuu}(\lambda_1, 0) [\varphi_1] [\varphi_1] [\varphi_1] \varphi_1 \, dx \]
\[ = \frac{1}{3} \int_\Omega (-3 \text{div}(|\nabla \varphi_1|^2 \nabla \varphi_1) + \lambda_1 a f'''(0) \varphi_1^3) \varphi_1 \, dx \]
\[ = \frac{1}{3} \int_\Omega (-3 \text{div}(|\nabla \varphi_1|^2 \nabla \varphi_1) \varphi_1 - \frac{f''(0)}{\kappa} \Delta \varphi_1 \varphi_1^3) \, dx \]
\[ = \int_\Omega \left( |\nabla \varphi_1|^4 + \frac{f''(0)}{\kappa} |\nabla \varphi_1|^2 \varphi_1^2 \right) \, dx, \]
and we get, as \( B = 0 \),
\[ \lambda = \chi(t) = \lambda_1 - \frac{1}{2} \frac{C}{A} t^2 + o(t^2). \]
Thus, the bifurcation is subcritical if
\[ \int_\Omega \left( |\nabla \varphi_1|^4 + \frac{f''(0)}{\kappa} |\nabla \varphi_1|^2 \varphi_1^2 \right) \, dx > 0, \]
while it is supercritical if
\[ \int_\Omega \left( |\nabla \varphi_1|^4 + \frac{f''(0)}{\kappa} |\nabla \varphi_1|^2 \varphi_1^2 \right) \, dx < 0. \]
This ends the proof.

**Proof of Theorem 1.3.** Since \((H_2^1)\) implies \((H_3^4)\), combining Propositions 2.2, 3.3, and 3.4 yields Theorem 1.3.

**Remark 3.3.** From the above proof it follows that Theorem 1.3 still holds replacing \((H_2^1)\) with \((H_3^4)\).

### 3.3 Superlinear growth

The aim of this subsection is providing a proof of Theorem 1.4, by combining Proposition 2.2 with Proposition 3.5 below, that has been recently proven in [32]. The following assumptions are here considered:

\( (H_2^3) \) \( a \in C^2(\overline{\Omega}); \)

\( (H_3^k) \) \( \Omega^+ = \{ x \in \Omega: a(x) > 0 \} \neq \emptyset, \Omega^- = \{ x \in \Omega: a(x) < 0 \} \neq \emptyset, \) and \( \Omega^0 = \{ x \in \Omega: a(x) = 0 \} \) is such that \( \partial \Omega^0 \subset \Omega; \) the boundaries \( \partial(\text{int}\Omega^0), \partial\Omega^+ \) and \( \partial\Omega^- \) are of class \( C^2; \) \( \Omega^0 \) has a finite number of connected components, that we denote by \( D_i^+, D_j^- \) and \( D_k^\pm \).
Hence, we can represent $\Omega^0$ in the form

$$\Omega^0 = \bigcup_i D^+_i \cup \bigcup_j D^-_j \cup \bigcup_k D^\pm_k,$$

where the components $D^+_i$, $D^-_j$ and $D^\pm_k$ are supposed to satisfy:

(H$^1_2$) for each $i$, $\partial D^+_i \subset \overline{\Omega^+}$ and there exist $\gamma_{1,i} > 0$, a neighborhood $U^+_i$ of $\partial D^+_i$ and $\alpha^+_i : \overline{U^+_i} \to ]0, +\infty[$ such that

$$a(x) = \alpha^+_i(x) \text{dist}(x, \partial D^+_i)^{\gamma_{1,i}} \quad \text{for all } x \in \Omega^+ \cap U^+_i;$$

(H$^2_2$) for each $j$, $\partial D^-_j \subset \overline{\Omega^-}$ and there exist $\gamma_{2,j} > 0$, a neighborhood $U^-_j$ of $\partial D^-_j$ and $\alpha^-_j : \overline{U^-_j} \to ]-\infty, 0[$ such that

$$a(x) = \alpha^-_j(x) \text{dist}(x, \partial D^-_j)^{\gamma_{2,j}} \quad \text{for all } x \in \Omega^- \cap U^-_j;$$

(H$^3_2$) for each $k$, the following alternative holds

(H$^3_{2,1}$) if $\text{int}(D^\pm_k) = \emptyset$, then
- $\partial D^\pm_k = \Gamma_k$ are of class $C^2$;
- there exist $\gamma_{3,k} > 0$, a neighborhood $U^+_k$ of $\Gamma_k$ and $\alpha^+_k : \overline{U^+_k} \to ]0, +\infty[$ such that

$$a(x) = \alpha^+_k(x) \text{dist}(x, \Gamma_k)^{\gamma_{3,k}} \quad \text{for all } x \in \Omega^+ \cap U^+_k;$$

(H$^3_{2,2}$) if $\text{int}(D^\pm_k) \neq \emptyset$, then
- $\partial D^\pm_k = \Gamma^+_k \cup \Gamma^-_k$, with $\Gamma^+_k \cap \Gamma^-_k = \emptyset$, $\Gamma^+_k \subset \overline{\Omega^+}$, $\Gamma^-_k \subset \overline{\Omega^-}$ of class $C^2$;
- there exist $\gamma_{3,k} > 0$, a neighborhood $U^+_k$ of $\Gamma^+_k$ and $\alpha^+_k : \overline{U^+_k} \to ]0, +\infty[$ satisfying condition (3.1);
- there exist $\gamma_{4,k} > 0$, a neighborhood $U^-_k$ of $\Gamma^-_k$ and $\alpha^-_k : \overline{U^-_k} \to ]-\infty, 0[$ satisfying condition (3.2).

Let us define

$$D^+ = \bigcup_i D^+_i, \quad D^- = \bigcup_j D^-_j, \quad D^\pm = \bigcup_k D^\pm_k.$$ 

The set $D^+$ (respectively, $D^-$) is constituted by the connected components $D^+_i$ (respectively, $D^-_j$) of $\Omega^0$, that are surrounded by regions of positivity (respectively, negativity) of $a$. Instead, $D^\pm$ is constituted by the connected components $D^\pm_k$ of $\Omega^0$, that are in between a region of positivity and one of negativity of $a$. $D^\pm$ can be either a “thin” nodal set, like when assuming condition (H$^3_k$), or a “thick” nodal set, that is, of positive measure. An example of an admissible nodal configuration for the function $a$ is provided by Figure 4.

**Remark 3.4.** Let $a \in C^2(\overline{\Omega})$ be a sign-changing function satisfying condition (H$^1_k$). Then, as already observed in [32], $D^+$, $D^-$, and $\text{int}(D^\pm)$ are all empty sets, and assumption (H$^3_{0,1}$) holds.

**Proposition 3.5.** [32, Theorem 2.2] Assume (H$^1_1$), (H$^2_2$), (H$^3_1$), (H$^3_2$), (H$^3_3$), (H$^3_4$), (H$^3_5$), and (H$^3_6$). Then there exists $\lambda_*>0$ such that for all $\lambda \in ]0, \lambda_*[$ the problem (1.1) admits at least one strong solution $v_\lambda \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, for any $p > N$, satisfying $v_\lambda \geq 0$ and

$$\lim_{\lambda \to +\infty} \|v_\lambda\|_{W^{2,p}} = 0.$$
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Figure 4: Examples of admissible nodal configurations for the weight \( a \): the union of the green, the purple and the blue regions are respectively the sets \( \Omega^+ \), \( \Omega^0 \) and \( \Omega^- \). On the left, \( \Omega^0 = \bigcup_{k=1}^2 D_k^+ \) satisfies the assumptions of both Theorem 1.4 and Proposition 3.5. On the right, \( \Omega^0 = D_1^+ \cup D_2^- \cup \bigcup_{k=1}^4 D_k^\pm \) satisfies the assumptions of Proposition 3.5.

Proof of Theorem 1.4. From Propositions 2.2 and 3.5, as well as Remark 3.4, we know that, for any given \( p > N \), there exists \( \lambda^* > 0 \) such that, for all \( \lambda > \lambda^* \), the problem (1.1) admits a maximum bounded variation solution \( u_\lambda \) and one strong solution \( v_\lambda \in W^{2,p}(\Omega) \setminus W^{1,p}_0(\Omega) \) satisfying

\[
0 < u_\lambda < v_\lambda < L
\]

\[
\lim_{\lambda \to +\infty} \left( \text{ess sup } u_\lambda \right) = L \quad \text{and} \quad \lim_{\lambda \to +\infty} \left( \text{ess sup } v_\lambda \right) = 0.
\]

Hence we infer that \( v_\lambda < u_\lambda \), provided \( \lambda \) is large enough. Thus Theorem 1.4 is proven.

Remark 3.5. From the above proof it follows that Theorem 1.4 still holds replacing \((H_0^1), (H_0^2), (H_0^3), (H_0^4), \) and \((H_0^5)\).

4 A peculiar multiplicity result

We prove here a more general version of Theorem 1.5 where the positivity and the continuity assumptions on the weight \( a \) are dropped.

Proposition 4.1. Assume

\((H_1^1)\) \( f : [0,L] \to \mathbb{R} \), with \( L > 0 \) a given constant, is a continuous function satisfying \( f(0) = f(L) = 0 \) and \( f(s) > 0 \) for every \( s \in ]0,L[\),

\((H_2^2)\) \( a \in L^1(0,1) \) and satisfies \( \int_0^1 a \, dx > 0 \),

and

\((H_3^3)\) there exist \( r, R \in ]0,L[ \), with \( r < R \), such that

\[
\|a^+\|_{L^1} \frac{2F(r)}{r^2} \left( 1 + \sqrt{1 + r^2} \right) < \left( \int_0^1 a \, dx \right) \frac{F(R)}{R}.
\]

Then there exist \( \lambda_2, \lambda^2 \in ]0, +\infty[ \), with \( \lambda_2 < \lambda^2 \), such that for all \( \lambda \in ]\lambda_2, \lambda^2[ \) the problem (1.7) admits at least two bounded variation solutions \( u_\lambda, v_\lambda \) such that \( 0 < u_\lambda < v_\lambda < L \).
Proof. The proof relies on a counterpart for the problem (1.1) of a mountain pass lemma for non-smooth functionals stated in [31, Lemma 3.7]. It is convenient here too to suppose that $f(s) = 0$ for all $s \in \mathbb{R}\setminus[0,L].$

For any given $\lambda > 0$ we introduce the functionals $\mathcal{J}, j, \mathcal{I}: BV(0,1) \to \mathbb{R}$ defined by

$$\mathcal{J}(v) = \int_0^1 (\sqrt{1 + |D^n v|^2} - 1) \, dx + \int_0^1 |D^s v| + |v(0)| + |v(1)|,$$

$$j(v) = \frac{\|D^n v\|_{L^1}^2}{1 + \sqrt{1 + \|D^n v\|_{L^1}^2}} + \int_0^1 |D^s v| + |v(0)| + |v(1)|,$$

$$\mathcal{I}_\lambda(v) = \mathcal{J}(v) - \lambda \int_0^1 a F(v) \, dx.$$

By the Jensen’s inequality we see that

$$\mathcal{J}(v) \geq j(v) \quad \text{for all} \quad v \in BV(0,1).$$

According to Remark 1.1, a function $u \in BV(0,1)$ is a bounded variation solution of (1.7) if and only if

$$\mathcal{J}(v) - \mathcal{J}(u) \geq \lambda \int_0^1 a f(u)(v - u) \, dx \quad \text{for all} \quad v \in BV(0,1). \quad (4.1)$$

We endow the space $BV(0,1)$ with the norm

$$\|v\|_{BV} = \|D^n v\|_{L^1} + \int_0^1 |D^s v| + |v(0)| + |v(1)|,$$

which, as already observed in Section 2, is equivalent to the standard one.

**Step 1: Mountain pass geometry.** Let $r, R > 0$ be the constants introduced in assumption $(H^4_4).$ Define

$$\mathcal{B}_r = \{v \in BV(0,1) : \|v\|_{BV} = r\}.$$

We first show that there exist constants $\lambda_2, \lambda_1 > 0,$ with $\lambda_2 < \lambda_1$ such that, for each $\lambda \in [\lambda_2, \lambda_1],$

$$\inf_{v \in \mathcal{B}_r} \mathcal{I}_\lambda(v) > 0 = \mathcal{I}_\lambda(0). \quad (4.2)$$

Elementary calculations show that the function $\zeta: [0, +\infty] \to \mathbb{R}$

$$\zeta(\xi) = \frac{\xi^2}{1 + \sqrt{1 + \xi^2}} - \xi$$

is decreasing and hence, for all $\xi \in [0, r],$

$$\frac{\xi^2}{1 + \sqrt{1 + \xi^2}} + r - \xi \geq \frac{r^2}{1 + \sqrt{1 + r^2}}.$$

Hence, we infer that, for all $v \in \mathcal{B}_r$

$$j(v) = \frac{\|D^n v\|_{L^1}^2}{1 + \sqrt{1 + \|D^n v\|_{L^1}^2}} + r - \|D^n v\|_{L^1} \geq \frac{r^2}{1 + \sqrt{1 + r^2}}.$$

On the other hand, we have that, for all $v \in \mathcal{B}_r,$

$$\|v\|_{\infty} \leq \|v\|_{BV} = r$$
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and hence, as $F$ is increasing,
\[ \|F(v)\|_\infty \leq F(\|v\|_\infty) \leq F(r). \]

Thus, we can conclude that, for all $v \in B_r$,
\[
I_\lambda(v) = J(v) - \lambda \int_0^1 aF(v) \, dx \geq j(v) - \lambda\|a^+\|_{L^1} \|F(v)\|_\infty \geq \frac{r^2}{1 + \sqrt{1 + r^2}} - \lambda\|a^+\|_{L^1} F(r).
\]

By using $(H_4')$, we can take \( \lambda^\sharp < \left(\|a^+\|_{L^1} \frac{F(r)}{r^2} (1 + \sqrt{1 + r^2})\right)^{-1}. \)

Hence, condition (4.2) holds for each \( \lambda \in [0, \lambda^\sharp]. \)

Next, using $(H_2')$ again, we can take \( \lambda^\sharp > 0 \) such that
\[
\lambda^\sharp > \left(\left(\int_0^1 a \, dx\right) \frac{F(R)}{2R}\right)^{-1},
\]
and so we obtain, for each \( \lambda > \lambda^\sharp, +\infty[, \)
\[
I_\lambda(R) = 2R - \lambda \int_0^1 aF(R) \, dx = 2R - \lambda \left(\int_0^1 a \, dx\right) F(R) < 0 = I_\lambda(0).
\] (4.3)

Note that assumption $(H_4')$ implies that
\[
\left(\left(\int_0^1 a \, dx\right) \frac{F(R)}{2R}\right)^{-1} < \left(\|a^+\|_{L^1} \frac{F(r)}{r^2} (1 + \sqrt{1 + r^2})\right)^{-1}.
\]

In particular, \( \lambda^\sharp, \lambda^\sharp \) can be chosen so as to satisfy
\[
\left(\left(\int_0^1 a \, dx\right) \frac{F(R)}{2R}\right)^{-1} < \lambda^\sharp < \lambda^\sharp < \left(\|a^+\|_{L^1} \frac{F(r)}{r^2} (1 + \sqrt{1 + r^2})\right)^{-1}.
\]

Therefore, for each \( \lambda \in [\lambda^\sharp, \lambda^\sharp], \) conditions (4.2) and (4.3) hold, with \( \|R\|_{BV} = 2R > r, \) thus displaying the desired mountain pass geometry of the functional \( I_\lambda. \)

**Step 2: Existence of almost critical points.** Henceforth, we fix \( \lambda \in (\lambda^\sharp, +\infty[). \) Then, we set
\[
\Gamma = \{\gamma \in C^0([0, 1], BV(0, 1)) : \gamma(0) = 0, \gamma(1) = R\}.
\]

From Step 1, we infer that
\[
c_\lambda = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(0), I_\lambda(R)\} > 0.
\]

Then, from a variant of [31, Lemma 3.7] valid for the functional \( I_\lambda, \) there exist sequences \( (v_n)_n \) in \( BV(0, 1) \) and \( (\varepsilon_n)_n \) in \( \mathbb{R} \) with
\[
\lim_{n \to +\infty} \varepsilon_n = 0 \quad (4.4)
\]
such that, for every \( n, \)
\[
c_\lambda - \frac{1}{n} \leq I_\lambda(v_n) \leq c_\lambda + \frac{1}{n} \quad (4.5)
\]
and
\[ J(v) - J(v_n) \geq \lambda \int_0^1 af(v_n)(v - v_n) \, dx + \varepsilon_n \|v - v_n\|_{BV} \quad \text{for all } v \in BV(0, 1). \tag{4.6} \]

**Step 3:** Estimates on the almost critical points. By (4.5) the sequence \((v_n)\) satisfies, for every \(n\),
\[ \|v_n\|_{BV} - 1 \leq J(v_n) \leq \lambda \int_0^1 aF(v_n) \, dx + c_\lambda + 1 \leq \lambda^2 \|a^+\|_{L^1} F(L) + c_\lambda + 1 \]
and thus
\[ \sup_n \|v_n\|_{BV} < +\infty. \tag{4.7} \]

**Step 4:** Existence of a positive bounded variation solution \(u_\lambda\) with \(I_\lambda(u_\lambda) = c_\lambda > 0\). By the compact embedding of \(BV(0, 1)\) into \(L^1(0, 1)\), there exist a subsequence of \((v_n)\), still denoted by \((v_n)\), and \(u_\lambda \in BV(0, 1)\) such that \(\lim_{n \to +\infty} v_n = u_\lambda\) in \(L^1(0, 1)\) and a.e. in \([0, 1]\). By passing to the inferior limit in (4.6) and using the lower semicontinuity of \(J\) with respect to the \(L^1\)-convergence in \(BV(0, 1)\), as well as the dominated convergence theorem, we obtain
\[ J(v) - \lambda \int_0^1 af(u_\lambda)v \, dx = J(v) - \lambda \lim_{n \to +\infty} \int_0^1 af(v_n)v \, dx \geq \liminf_{n \to +\infty} J(v_n) - \lambda \lim_{n \to +\infty} \int_0^1 af(v_n)v_n \, dx \geq J(u_\lambda) - \lambda \int_0^1 af(u_\lambda)u_\lambda \, dx, \]
for all \(v \in BV(0, 1)\). Hence, condition (4.1) holds and thus \(u_\lambda\) is a solution of (1.7).

Next we prove that \(I_\lambda(u_\lambda) = c_\lambda\) by showing that
\[ \lim_{n \to +\infty} I_\lambda(v_n) = I_\lambda(u_\lambda) \tag{4.8} \]
and using (4.5). The dominated convergence theorem implies that
\[ \lim_{n \to +\infty} \int_0^1 aF(v_n) \, dx = \int_0^1 aF(u_\lambda) \, dx. \]
Hence, to prove (4.8) it is enough to verify that
\[ \lim_{n \to +\infty} J(v_n) = J(u_\lambda). \]
The lower semicontinuity of \(J\) with respect to the \(L^1\)-convergence yields
\[ \liminf_{n \to +\infty} J(v_n) \geq J(u_\lambda). \]
On the other hand, taking the solution \(u_\lambda\) as test function in (4.6), we get, for all \(n\),
\[ J(v_n) \leq J(u_\lambda) - \lambda \int_0^1 af(v_n)(u_\lambda - v_n) \, dx - \varepsilon_n \|u_\lambda - v_n\|_{BV}. \]
Passing to the superior limit and using the dominated convergence theorem again, together with (4.4) and (4.7), we infer that
\[ \limsup_{n \to +\infty} J(v_n) \leq J(u_\lambda). \]

**Step 5:** Existence of a positive bounded variation solution \(w_\lambda\) with \(I_\lambda(w_\lambda) < 0\). From the proof of Proposition 2.1 it is apparent that we only need showing that the functional \(I_\lambda\) attains negative values
if $\lambda \in ]\lambda_1, \lambda_1^\sharp[$. This is indeed guaranteed by (4.3). Then the global minimizer of $I_\lambda$ provides us with a solution $w_\lambda \neq u_\lambda$.

The same argument developed in the proof of Proposition 2.1 shows that $0 < u_\lambda, w_\lambda < L$. Further, since $u_\lambda \neq w_\lambda$ and $L$ is an upper bounded variation solution, but not a solution, we have that $0 < u_\lambda, w_\lambda < u_\lambda \vee w_\lambda < L$ and $u_\lambda \vee w_\lambda$ is a lower bounded variation solution. Hence we infer from [21] or [29] the existence of a solution $v_\lambda > u_\lambda$. This concludes the proof of Proposition 4.1.

Proof of Theorem 1.5. Notice that $(H^3_1), (H^3_{10})$, and $(H^3_{11})$ imply $(H^3_4), (H^3_2)$, and $(H^3_3)$, respectively. Thus, Theorem 1.5 is directly inferred from Proposition 4.1.

Remark 4.1. The case where $\Omega$ is an arbitrary bounded interval $]c, d[$ can be easily handled via the change of variables

$$
\xi = \frac{x - c}{d - c}, \quad v(\xi) = \frac{1}{d - c}u(c + (d - c)\xi),
$$

which transforms the problem

$$
\begin{cases}
-\left(\frac{u'}{\sqrt{1 + (u')^2}}\right)' = \lambda af(u) \quad \text{in } ]c, d[, \\
u(c) = 0, \quad u(d) = 0,
\end{cases}
$$

into

$$
\begin{cases}
-\left(\frac{v'}{\sqrt{1 + (v')^2}}\right)' = \lambda \tilde{a}\tilde{f}(v) \quad \text{in } ]0, 1[, \\
v(0) = 0, \quad v(1) = 0,
\end{cases}
$$

where

$$
\tilde{a}(\xi) = a(c + (d - c)\xi), \quad \tilde{f}(s) = (d - c)f((d - c)s).
$$

Remark 4.2. It is worth observing that if $a$ vanishes on the boundary of its domain and it satisfies the regularity condition specified in [23, 24] at its nodal points, then all solutions of (1.7) are strong solutions. This topic will be discussed in detail elsewhere.

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References


Logistic growth models with flux-saturated diffusion


