

# FINITE GROUPS ACTING ON HYPERELLIPTIC 3-MANIFOLDS

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ABSTRACT. We consider 3-manifolds admitting the action of an involution such that its space of orbits is homeomorphic to  $S^3$ . Such involutions are called *hyperelliptic* as the manifolds admitting such an action. We consider finite groups acting on 3-manifolds and containing hyperelliptic involutions whose fixed-point set has  $r > 2$  components. In particular we prove that a simple group containing such an involution is isomorphic to  $PSL(2, q)$  for some odd prime power  $q$ , or to one of four other small simple groups.

## 1. INTRODUCTION

In the present paper, we consider *hyperelliptic 3-manifolds*; these are 3-manifolds which admit the action of a *hyperelliptic involution*, i.e. an involution with quotient space homeomorphic to  $S^3$ . In this paper 3-manifolds are all smooth, closed and orientable, and the actions of groups on 3-manifolds are smooth and orientation-preserving.

The analogous concept in dimension 2 is a classical research theme and many papers about hyperelliptic Riemann surfaces can be found in the literature. In particular the hyperelliptic involution of a Riemann surface of genus at least two is unique and central in the automorphism group of the hyperelliptic Riemann surface; as a consequence, the class of automorphism groups of hyperelliptic Riemann surfaces is very restricted (liftings of finite groups acting on  $S^2$ ).

In dimension 3 the presence of a hyperelliptic involution is equivalent to the fact that the 3-manifold is the 2-fold cover of  $S^3$  branched along a link (in brief *the 2-fold branched cover of a link*). In fact the fixed-point set of a hyperelliptic involution is a non-empty union of disjoint, simple, closed curves. If  $h$  is a hyperelliptic involution acting on  $M$ , the projection of the fixed-point set of  $h$  to  $M/\langle h \rangle \cong S^3$  is a link and  $M$  is the 2-fold branched cover of this link. On the other hand if  $M$  is a 2-fold branched cover of a link, the involution generating the deck transformation group is hyperelliptic. The class of 3-manifolds consisting of 2-fold branched covers of links has been extensively studied in the literature.

The aim in this paper is to understand which finite groups can act on a 3-manifold and contain a hyperelliptic involution; in particular we consider the case of finite simple groups. In contrast with dimension two, hyperelliptic involutions are not unique.

We remark that if  $M$  admits the action of a hyperelliptic involution  $h$  whose fixed-point set has  $r$  components, then the first  $\mathbb{Z}_2$ -homology group of  $M$  has rank  $r - 1$  (see, for example, [19, Sublemma 15.4]). Hence, a hyperelliptic 3-manifold determines the number of components of the fixed-point set of its hyperelliptic involutions and the situation of different numbers of components can be separately analyzed.

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*Key words and phrases.* 3-manifolds, finite groups of diffeomorphisms, hyperelliptic involutions, 2-fold branched coverings of links.

The case of hyperelliptic involutions with connected fixed-point set is particular, since a manifold admitting such an action is a  $\mathbb{Z}_2$ -homology 3-sphere. If  $G$  is a finite simple group containing a hyperelliptic involution with connected fixed-point set, then  $G$  is isomorphic to the alternating group  $\mathbb{A}_5$  [22]. This result is a consequence of a more general one that states that a simple group acting on a  $\mathbb{Z}_2$ -homology 3-sphere is isomorphic to a linear fractional group  $PSL(2, q)$  for some odd prime power  $q$  or to the alternating group  $\mathbb{A}_7$  [16]. The proof of this result is highly non-trivial since it is based on the Gorenstein-Walter classification of simple groups with dihedral Sylow 2-subgroups. We remark that each  $\mathbb{Z}_2$ -homology sphere is a rational homology 3-sphere, but for this broader class of 3-manifolds the situation changes completely: each finite group acts on a rational homology 3-sphere (see [7] for free actions and [2] for non-free actions). A complete classification of the finite groups acting on integer or  $\mathbb{Z}_2$ -homology 3-sphere is not known and appears to be difficult.

In the present paper we consider the case of  $r > 2$  components and prove the following theorem for simple groups.

**Theorem 1.** *If  $G$  is a finite simple group acting on a 3-manifold  $M$  and containing a hyperelliptic involution  $h$  such that its fixed-point set has  $r > 2$  components then  $G$  is isomorphic to a linear fractional group  $PSL(2, q)$ , for some odd prime power  $q$ , or to one of the following four groups: the projective special linear group  $PSL(3, 5)$ , the projective general unitary group  $PSU(3, 3)$ , the Mathieu group  $M_{11}$  and the alternating group  $\mathbb{A}_7$ .*

We remark that the list of possible groups provided by our theorem is very close to that obtained in the case of  $\mathbb{Z}_2$ -homology 3-spheres. In the proof of the theorem we use the Gorenstein-Walter classification of simple group with dihedral Sylow 2-subgroups and the Alperin-Brauer-Gorenstein classification of simple group with quasidihedral Sylow 2-subgroups.

It is worth mentioning that examples of groups containing a hyperelliptic involution are difficult to construct. The straightforward method consists of taking a finite group of symmetries of a link and lifting it to the 2-fold branched cover of the link. In this way we obtain groups with a central hyperelliptic involution. Moreover by the geometrization of 3-manifolds a finite group acting on  $S^3$  is conjugate to a finite subgroup of  $SO(4)$ , so we have a very restricted list of possibilities (see [6, 14]).

Other examples can be obtained by using Seifert spherical 3-manifolds. In particular the homology Poincaré sphere  $S^3/I_{120}^*$ , the octahedral manifold  $S^3/I_{48}^*$  and the quaternionic manifold  $S^3/Q_8$  admit an action of  $\mathbb{A}_5$  that includes a hyperelliptic involution (see [13] for the notations and the computation of the isometry groups and [14] for a method to compute the space of orbits of the involutions). The numbers of components of the fixed-point set of the hyperelliptic involutions in the examples are 1, 2 and 3, respectively.

At the moment  $\mathbb{A}_5$  is the only known simple group admitting an action on a 3-manifold and containing a hyperelliptic involution.

A key ingredient in the proof of Theorem 1 is the analysis of the Sylow 2-subgroups; in particular we prove the following proposition which is interesting in its own right.

**Proposition 1.** *Let  $G$  be a finite 2-group acting on a 3-manifold and containing a hyperelliptic involution whose fixed-point set has  $r > 2$  components. The subgroup generated by all hyperelliptic involutions in  $G$  has order two or is dihedral.*

Since by Thurston's Orbifold Geometrization theorem each hyperelliptic involution of a hyperbolic 3-manifold is conjugate to an isometry and the isometry groups of hyperbolic 3-manifolds are finite, we can easily derive from Proposition 1 the following corollary.

**Corollary 1.** *There are at most three inequivalent links with at least 3 components which have the same hyperbolic 2-fold branched cover.*

This upper bound was already claimed in [15] where a proof based on a result contained in [18] was given; however, in the proof in [18] we found a gap that seems not easy to fill. In the present paper the proof of Corollary 1 is independent of the result in [18]. We remark that in [15] three is proved to be the optimal bound.

In this paper also the case of arbitrary groups is considered. If the hyperelliptic involution has a fixed-point set with odd number of components our methods can be extended: in Section 5 we prove a theorem which describes the structure of an arbitrary group acting on a 3-manifold and containing a hyperelliptic involution whose fixed-point set has  $r$  components with  $r > 2$  and odd.

We remark that the case  $r = 2$  remains completely open. Considering our approach, one of the main ingredients is the analysis of the centralizers of the hyperelliptic involutions but, if the components of the fixed-point set are two, this analysis is much more difficult.

From now on  $G$  is a finite group acting smoothly and orientation preservingly on a closed, orientable, smooth 3-manifold  $M$ .

**Organization of the paper.** In Section 2, we consider centralizers of hyperelliptic involutions. In Section 3, we prove several properties of finite 2-groups containing a hyperelliptic involution whose fixed-point set has  $r > 2$  components. In Section 4 we prove Theorem 1. Finally, in Section 5 we consider arbitrary groups containing a hyperelliptic involution whose fixed-point set has  $r$  components with  $r > 2$  and  $r$  odd.

## 2. CENTRALIZERS OF HYPERELLIPTIC INVOLUTIONS

In this section we collect some properties about the elements of  $G$  commuting with a hyperelliptic involution, in particular the centralizer of a hyperelliptic involution is of very special type.

**Remark 1.** Let  $K$  be a simple closed curve in  $M$  and let  $I$  be the subgroup of  $G$  fixing  $K$  setwise. Since  $G$  is finite, we can choose a  $G$ -invariant Riemannian metric on  $M$  with respect to which  $G$  acts by isometries. An element  $f$  of  $I$  can act on  $K$  either as rotation or as a reflection. In the first case we call  $f$  a  $K$ -rotation, while in the second case a  $K$ -reflection. The elements acting trivially on  $K$  are considered  $K$ -rotations. The subgroup of  $K$ -rotations is abelian of rank at most two and its index in  $I$  is at most two. The conjugate of a  $K$ -rotation by a  $K$ -reflection is the inverse of the  $K$ -rotation. We can conclude that  $I$  is isomorphic to a subgroup of a semidirect product  $\mathbb{Z}_2 \rtimes (\mathbb{Z}_n \times \mathbb{Z}_m)$ , for some non-negative integers  $n$  and  $m$ , where  $\mathbb{Z}_2$  operates on the normal subgroup  $\mathbb{Z}_n \times \mathbb{Z}_m$  by sending each element to its inverse.

**Proposition 2.** *If  $h \in G$  is a hyperelliptic involution then the quotient  $C_G(h)/\langle h \rangle$  is isomorphic to a finite subgroup of  $\mathrm{SO}(4)$ .*

*Proof.* The group  $C_G(h)/\langle h \rangle$  acts on  $M/\langle h \rangle$  that is homeomorphic to  $S^3$ . The Thurston orbifold geometrization theorem (see [3]) and the spherical space form conjecture for free actions on  $S^3$  proved by Perelman (see [17]) imply that every finite group of diffeomorphisms of the 3-sphere is conjugate to a finite subgroup of  $SO(4)$ .  $\square$

The two following technical lemmas concerning elements commuting with hyperelliptic involutions are used several times in the paper.

**Lemma 1.** *Let  $h_1, h_2$  be two commuting hyperelliptic involutions of  $G$  and  $E = \langle h_1, h_2 \rangle$ . Then  $C_G(E)/\langle h_1 \rangle$  is isomorphic to a subgroup of  $\mathbb{Z}_2 \times (\mathbb{Z}_n \times \mathbb{Z}_m)$ , for some non-negative integers  $n$  and  $m$ , where  $\mathbb{Z}_2$  operates on the normal subgroup  $\mathbb{Z}_n \times \mathbb{Z}_m$  by sending each element to its inverse.*

*Proof.* The group  $C_G(E)/\langle h_1 \rangle$  acts on the 3-sphere  $M/\langle h_1 \rangle$  and contains in its center  $h_2\langle h_1 \rangle$ . Since  $h_2$  has non-empty fixed-point set on  $M$ , the fixed-point set of  $h_2\langle h_1 \rangle$  on  $M/\langle h_1 \rangle$  is a simple closed curve fixed by  $C_G(E)/\langle h_1 \rangle$  setwise. The claim follows from Remark 1.  $\square$

**Lemma 2.** [18, Lemma 1] *Let  $h$  be a hyperelliptic involution in  $G$ , and suppose that the fixed-point set of  $h$  has  $r > 2$  components. Let  $x$  be an element of  $G$  different from  $h$  that commutes with  $h$ . Then the fixed-point set of  $x$  has also  $r$  components if and only if  $x$  is a strong inversion of the fixed-point set of  $h$  (that is  $x$  acts as a reflection on each component of the fixed-point set of  $h$ ).*

### 3. FINITE 2-GROUPS CONTAINING HYPERELLIPTIC INVOLUTIONS

In this section we analyse geometric and algebraic properties of 2-groups acting on a 3-manifold and containing a hyperelliptic involution whose fixed point set has  $r > 2$  components. If  $g$  is an element of  $G$ , we denote by  $Fix(g)$  the fixed-point set of  $g$ .

**Lemma 3.** *Suppose that  $G$  is an elementary abelian 2-group. If  $G$  contains a hyperelliptic involution  $h$  such that  $Fix(h)$  has  $r > 2$  components, then  $G$  contains at most three hyperelliptic involutions which generate a subgroup of rank at most two. If the group generated by the hyperelliptic involutions has rank two, then it contains all the elements of  $G$  whose fixed-point set has  $r$ -components.*

*Proof.* We denote by  $K$  one of the components of  $Fix(h)$ . We can suppose that  $G$  is generated by involutions whose fixed-point set has  $r$ -components. By Lemma 2 each element in  $G$  leaves  $K$  invariant. Remark 1 implies that  $G$  has rank at most three. If  $G$  has rank one or two the thesis trivially holds, thus we can suppose that  $G$  has rank three.

Since  $G$  has rank three,  $G$  contains also two involutions different from  $h$  acting on  $K$  as rotations; if we denote one of these rotations by  $y$ , the other one is  $hy$ . By Lemma 2, the fixed-point set of  $y$  and  $hy$  can not have  $r$  components. If  $h$  is the only hyperelliptic involution in  $G$  the thesis trivially holds, thus we can assume the existence of  $t$ , a hyperelliptic involution different from  $h$ . By Lemma 2 hyperelliptic involutions in  $G$  distinct from  $h$  are  $K$ -reflections. The  $K$ -reflections in  $G$  are  $t, th, ty$  and  $thy$ . If the fixed-point set of  $ty$  had  $r$ -components (in particular if it was hyperelliptic), then  $ty$  and  $y$  should be strong inversions of  $Fix(t)$  but this is impossible since  $Fix(y)$  does not have  $r$  components. The same holds for  $thy$ . This implies that all the involutions whose fixed-point set has  $r$  components are contained in the group generated by  $t$  and  $h$ ; the hyperelliptic involutions are at most three.  $\square$

**Lemma 4.** *Suppose that  $G$  is a 2-group containing a hyperelliptic involution  $h$  such that  $Fix(h)$  has  $r > 2$  components, then there exists  $x$  an involution (not necessarily hyperelliptic) such that  $G = C_G(x)$  and  $Fix(x)$  has  $r$  components.*

*Proof.* This Lemma is implied by the proof of Theorem 1 in [18]. We give here the proof for the sake of completeness.

Suppose that  $h$  is not central in  $G$  and define  $N$  the subgroup of  $C_G(h)$  generated by all involutions whose fixed-point set has exactly  $r$  components. By Lemma 2, the non-trivial elements of  $N$  act as reflections on  $Fix(h)$ , thus  $N$  fixes setwise all the components of  $Fix(h)$  and Remark 1 applies to  $N$ . Since  $C_G(h)$  is a proper subgroup of  $G$ , by [20, Theorem 1.6.] there exists an element  $f$  of  $G \setminus C_G(h)$  normalizing  $C_G(h)$ . The element  $t := fhf^{-1}$  is a hyperelliptic involution commuting with  $h$ , thus  $t$  is in  $N$ . Moreover  $t$  is central in  $C_G(h)$  and by Remark 1, the subgroup  $N$  is elementary abelian of rank two or three.

By Lemma 3 we obtain that  $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  and we will show that  $G = C_G(th)$ . We remark that  $f$  normalizes  $N$  and, since its order is a power of two, it centralizes  $th$ . We obtain that  $C_G(th)$  contains properly  $C_G(t) = C_G(h)$ .

Denote by  $g$  an element normalizing  $C_G(th)$ ; the element  $gthg^{-1}$  is in the center of  $C_G(th)$ , so it commutes with  $t$  and  $h$ . The fixed point set of  $gthg^{-1}$  has  $r$  components, thus  $gthg^{-1}$  is contained in  $N$ . If  $gthg^{-1} = h$  or  $gthg^{-1} = t$ , we obtain  $C_G(th) = C_G(t) = C_G(h)$  getting a contradiction. The only possibility is  $gthg^{-1} = th$ . By [20, Theorem 1.6.] we obtain  $G = C_G(th)$ . The fixed-point set of  $th$  has  $r$  components, for  $th$  is a strong inversion of  $Fix(h)$ ; this concludes the proof of the Lemma. □

We remark that in this paper we include between the dihedral groups the elementary abelian group of rank 2.

**Proposition 3.** *Suppose that  $G$  is a 2-group containing a hyperelliptic involution  $h$  such that  $Fix(h)$  has  $r > 2$  components. If  $H$  is the subgroup of  $G$  generated by the hyperelliptic involutions in  $G$ , then  $G$  either has order two or is dihedral.*

*Proof.* By Lemma 4, the group  $G$  contains a central involution  $x$  such that  $Fix(x)$  has  $r$ -components. By Lemma 2 a hyperelliptic involution in  $G$  either coincides with  $x$  or is a strong inversion of  $Fix(x)$ , so any hyperelliptic involution leaves invariant each component of  $Fix(x)$ . Let  $K$  be one of the components of  $Fix(x)$ . A hyperelliptic involution that is different from  $x$  is a  $K$ -reflection and each element in  $H$  leaves  $K$  setwise invariant. By Remark 1 we obtain that  $H$  is a subgroup of a semidirect product  $\mathbb{Z}_2 \ltimes (\mathbb{Z}_{2^a} \times \mathbb{Z}_{2^b})$ , where  $a$  and  $b$  are positive integers.

By contradiction, we suppose that  $H$  is neither cyclic nor dihedral. Since  $H$  is not cyclic, it contains  $t$  a  $K$ -reflection that is hyperelliptic. We obtain that  $H \cong \mathbb{Z}_2 \ltimes (\mathbb{Z}_{2^a} \times \mathbb{Z}_{2^b})$ , with  $a \geq 1$  and  $b \geq 1$  and  $x$  is contained in  $H$ . We denote by  $g$  and  $f$  two generators of the subgroup of  $K$ -rotations, respectively of order  $2^a$  and  $2^b$ . We recall that in  $H$  we have four conjugacy classes of  $K$ -reflections:  $\{tf^k g^h \mid k \text{ odd and } h \text{ even}\}$ ,  $\{tf^k g^h \mid k \text{ even and } h \text{ odd}\}$ ,  $\{tf^k g^h \mid k \text{ even and } h \text{ even}\}$  and  $\{tf^k g^h \mid k \text{ odd and } h \text{ odd}\}$ .

If  $a = b = 1$  then  $H$  is an elementary abelian group of rank three and Lemma 3 gives a contradiction.

Then we suppose that  $a > 1$  and  $b > 1$ . The elements  $t$ ,  $tg^{2^{a-1}}$  and  $tf^{2^{b-1}}$  are all hyperelliptic, for they are conjugate in  $H$ , and generate an elementary abelian subgroup of rank three. This gives a contradiction by Lemma 3.

Finally we assume that  $a = 1$  and  $b > 1$ . Since the elements  $t$  and  $tf^{2^{b-1}}$  are conjugate, also  $tf^{2^{b-1}}$  is hyperelliptic. Consider the group generated by  $t$ ,  $f^{2^{b-1}}$  and  $g$ , which is an elementary abelian group of rank three. By Lemma 3 the only other involution in this group, that can be hyperelliptic, is  $f^{2^{b-1}}$ . Thus the elements of type  $tf^k g$  with  $k$  even can not be hyperelliptic. Consider now the group generated by  $tf$ ,  $f^{2^{b-1}}$  and  $g$ . If  $tf$  is hyperelliptic, then so is  $tf^{2^{b-1}+1}$ . By Lemma 3 the involutions in the conjugacy class of  $tf g$  can not be hyperelliptic. In this case the group generated by  $t$  and  $f$  is dihedral and contains all the hyperelliptic involutions. If  $tf g$  is hyperelliptic, we can reverse the roles and we obtain that all the hyperelliptic rotations are contained in the dihedral subgroup generated by  $t$  and  $f g$ .  $\square$

**Lemma 5.** *Suppose that  $N$  is a 2-subgroup of  $G$  generated by hyperelliptic involutions, then the quotient  $M/N$  is homeomorphic to  $S^3$ .*

*Proof.* By Proposition 3 the group  $N$  is either cyclic of order two or dihedral. In the first case the thesis trivially holds. In the second case  $N$  is generated by two hyperelliptic involutions that we denote by  $t$  and  $s$ . Let  $2^a$  be the order of  $ts$  (i.e.  $2^{a+1}$  is the order of  $N$ ). We consider the following subnormal series of subgroups of  $N$ :

$$N_1 = \langle t \rangle \subset N_2 = \langle N_1, t(ts)^{2^{a-1}} \rangle \subset \dots \subset N_a = \langle N_{a-1}, t(ts)^2 \rangle \subset N = \langle N_a, s \rangle$$

All the involutions in  $N$ , but one that is central in  $N$ , are conjugate either to  $t$  or to  $s$ , thus all the involutions, but possibly one, are hyperelliptic. Since  $t$  is hyperelliptic  $M/N_1$  is homeomorphic to  $S^3$ . We recall that, by the positive solution of the Smith Conjecture, if an involution acts with non-empty fixed-point set on  $S^3$ , then its action is linear and the underlying topological space of the quotient orbifold of  $S^3$  by this involution is again a 3-sphere. Since  $N_1$  is normal in  $N_2$ , the quotient group  $N_2/N_1$  induces an action on  $M/N_1$ . In particular  $t(ts)^{2^{a-1}}$  projects to an involution acting faithfully on  $M/N_1$  and generating  $N_2/N_1$ . As the fixed-point set of  $t(ts)^{2^{a-1}}$  is non-empty, so is the fixed-point set of the projected involution. We can obtain  $M/N_2$  as a double quotient  $(M/N_1)/(N_2/N_1)$  and, since  $N_2/N_1$  is generated by an involution not acting freely,  $M/N_2$  is again homeomorphic to  $S^3$ . We can iteratively apply this argument to each group  $N_i$  of the subnormal series, and finally we get the thesis.  $\square$

We recall that the 2-rank of a finite group is the maximum of the ranks of elementary abelian 2-subgroups. The following proposition assures us that the 2-rank of a finite group containing a hyperelliptic involution is small.

**Proposition 4.** *If  $G$  contains a hyperelliptic involution  $h$  such that  $\text{Fix}(h)$  has  $r > 2$  components, then the 2-rank of  $G$  is at most four.*

*Proof.* We denote by  $S$  a Sylow 2-subgroup of  $G$  and by  $E$  one elementary abelian subgroup of maximal order. Since the conjugate of a hyperelliptic involution is hyperelliptic, we can suppose that  $S$  contains  $h$  by the Sylow Theorems. Let  $H$  be the normal subgroup of  $S$

generated by all the hyperelliptic involutions in  $S$ . By Proposition 3 we obtain that  $H$  is cyclic of order two or dihedral. The subgroup  $E$  projects to an elementary abelian group acting on  $M/H$ , which is isomorphic to  $EH/H \cong E/E \cap H$ . Since by Lemma 5 the quotient  $M/H$  is homeomorphic to  $S^3$ , the group  $E/E \cap H$  admits a faithful action on  $S^3$  and hence has rank at most three (see [16, Propositions 2 and 3].) The group  $E \cap H$  has rank at most two. If  $E \cap H$  is cyclic, we are done. If  $E \cap H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  then  $E$  contains a hyperelliptic involution  $t$  (at most one involution in  $H$  can be non-hyperelliptic). By using the action induced by  $E$  to  $M/\langle t \rangle$ , which is homeomorphic to  $S^3$ , we can conclude also in this case that the rank of  $E$  is at most four. □

#### 4. FINITE SIMPLE GROUPS CONTAINING HYPERELLIPTIC INVOLUTIONS

In this section we prove Theorem 1. We consider first the following algebraic definitions:

**Definition 1.** Let  $F$  be a finite group with two subgroups  $T$  and  $C$  such that  $C \leq T \leq F$ .

- (1)  $C$  is said to be *weakly closed* in  $T$ , if  $C^f \leq T$  implies  $C^f = C$ , for each  $f \in F$ .
- (2)  $C$  is said to be *strongly closed* in  $T$ , if  $C^f \cap T \leq C$ , for each  $f \in F$ .
- (3) Let  $I$  be the set of the involutions of  $C$ ; the subgroup  $C$  is said to be *strongly involution closed* in  $T$ , if  $I^f \cap T \subset C$ , for each  $f \in F$ .

If  $S$  is a Sylow 2-subgroup of  $G$ , the subgroup of  $S$  generated by hyperelliptic involution is weakly closed in  $S$  and is cyclic or dihedral by Proposition 3.

The groups containing a dihedral 2-subgroup weakly closed in the Sylow 2-subgroup were studied by Fukushima in [8]. We will use his result to strongly restrict the possible simple groups containing a hyperelliptic involution with fixed-point set with  $r > 2$  components.

The other two definitions (strongly closed and strongly involution closed subgroups) will be used in the next session where the case of arbitrary groups will be considered.

In the proof of Theorem 1 we will find simple groups with Sylow 2-subgroups that are dihedral, semidihedral or wreath products  $\mathbb{Z}_{2^m} \wr \mathbb{Z}_2$ .

We recall that a semidihedral group of order  $2^n$  with  $n \geq 4$  has the following presentation:

$$\langle s, f \mid s^2 = f^{2^{n-1}} = 1, sfs = f^{2^{n-2}-1} \rangle.$$

We remark that in literature these groups are also called quasidihedral. For the basic properties of semidihedral groups see [1, Lemma 1].

The wreath product  $\mathbb{Z}_{2^m} \wr \mathbb{Z}_2$  is a semidirect product  $(\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}) \rtimes \mathbb{Z}_2$  where  $\mathbb{Z}_2$  acts on a given couple of generators of  $\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$  exchanging them.

*Proof.* Let  $S$  be a Sylow 2-subgroup of  $G$  containing  $h$  and let  $D$  be the subgroup of  $S$  generated by all the hyperelliptic involutions of  $S$ . If  $h$  was the only hyperelliptic involution in  $S$ , then by Glauberman's  $Z^*$ -theorem  $h$  would be in the center of  $G$ , and this is impossible since  $G$  is simple. So we can suppose that  $G$  contains at least two hyperelliptic involutions. Then,  $D$  is weakly closed and dihedral by Proposition 3. By [8, Theorem 1], since  $G$  is simple, we get that either  $G$  is isomorphic to  $PSU(3, 4)$ , or  $S$  is dihedral, semidihedral or a wreath product  $\mathbb{Z}_{2^m} \wr \mathbb{Z}_2$ .

If  $G \cong PSU(3, 4)$  there is a unique class of involutions, that should be all hyperelliptic, and  $S$  should contain an elementary abelian subgroup of rank 4 (see [5]); this contradicts Proposition 3.

When  $S$  is dihedral we can conclude by using the Gorenstein-Walter classification of finite simple groups with dihedral Sylow 2-subgroups [11].

If  $S$  is quasidihedral we can use the result of Alperin, Brauer and Gorenstein [1] and we obtain that  $G$  is isomorphic to  $PSL(3, q)$  for some odd prime power  $q$  such that  $q \equiv -1 \pmod{4}$  or to  $PSU(3, q)$  for some odd prime power  $q$  such that  $q \equiv 1 \pmod{4}$  or to  $M_{11}$ . To exclude most part of these groups we can use the structure of the centralizer of the involutions which for groups of Lie type is described in [10, Table 4.5.2.]. We remark that in these groups there exists a unique conjugacy class of involutions and so each involution is hyperelliptic. By Proposition 2 and the classification of the finite subgroups of  $SO(4)$  (see [6] or [14]) we know that a non-abelian simple section of any centralizer of an involution is isomorphic to  $A_5$ . Analyzing [10, Table 4.5.2.] we can conclude that the only possible groups are  $PSL(3, 5)$ ,  $PSU(3, 3)$  and  $M_{11}$ .

Suppose finally that  $S$  is a wreath product  $\mathbb{Z}_{2^m} \wr \mathbb{Z}_2$ . Let  $s, t \in S$  be elements of order  $2^m$  and 2, respectively, such that  $S_0 := \langle s, s^t \rangle \cong \mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}$  and  $S = S_0 \langle t \rangle$ . Note that an element  $\mu = s_0 t \in S \setminus S_0$  has order 2 if and only if  $s_0^t = s_0^{-1}$ , that is  $s_0 \in \langle s^{-1} s^t \rangle$ . Thus, the involutions in  $S \setminus S_0$  are exactly  $2^m$  and  $S$  contains  $2^m + 3$  involutions. Since  $G$  is simple, by Corollary 2G, Proposition 2F<sub>a</sub> and Proposition 2B<sub>1</sub> in [4], all the involutions of  $S$  are conjugate in  $G$ . Hence they are all hyperelliptic, and they generate a dihedral subgroup by Proposition 3, of order at most  $2^{m+1}$ . It follows that the involutions in  $S$  are at most  $2^m + 1$ , a contradiction.  $\square$

## 5. THE CASE OF HYPERELLIPTIC INVOLUTIONS WITH FIXED-POINT SET WITH ODD NUMBER OF COMPONENTS

If we drop the hypothesis that  $G$  is simple, the conditions given by Fukushima's result on the structure of  $G$  are too weak to lead us to an effective restriction of the possible groups. On the other hand if we suppose that the number of components of the fixed-point set of hyperelliptic involutions is odd, we are able to prove the existence of a dihedral 2-subgroup strongly involution closed and we can use the stronger results in [9] and [12].

**5.1. Existence of a strongly involution closed dihedral subgroup.** We begin with an algebraic remark.

**Remark 2.** Let  $F$  be a finite group. Suppose that  $N$  is a normal subgroup of  $F$  and  $P$  is a  $p$ -subgroup of  $F$ . If the order of  $N$  is coprime to  $p$ , then the normalizer of the projection of  $P$  to  $F/N$  is the projection of the normalizer of  $P$  in  $F$ , that is

$$\mathcal{N}_{F/N}(PN/N) = (\mathcal{N}_F(P)N)/N.$$

The inclusion  $\supseteq$  holds trivially. We prove briefly the other inclusion. Let  $fN$  be an element of  $\mathcal{N}_{F/N}(PN/N)$ , then  $P^f \subseteq PN$ . Both  $P^f$  and  $P$  are Sylow  $p$ -subgroups of  $PN$  and by the second Sylow Theorem they are conjugate by an element  $tn \in PN$ . We obtain that  $P^{f^{tn}} = P$ , and hence  $f \in \mathcal{N}_F(P)N$ . By the same argument we can prove that two  $p$ -groups

$P$  and  $U$  are conjugate in  $F$  if and only if  $PN/N$  and  $UN/N$  are conjugate in  $F/N$ . Moreover the analogous following formula holds for centralizers:

$$\mathcal{C}_{F/N}(PN/N) = \mathcal{C}_F(P)N/N.$$

The inclusion  $\supseteq$  holds trivially again. If  $fN \in \mathcal{C}_{F/N}(PN/N)$  then  $fN$  normalizes  $(PN)/N$  and we can suppose by the first part of the remark that  $f \in \mathcal{N}_F(P)$ . Moreover for each  $t \in P$  there exists  $n \in N$  such that  $t^f = tn$ . Since  $t^f \in P$  and  $P \cap N$  is trivial,  $n = 1$  for each  $n$  and  $f \in \mathcal{C}_F(P)$ .

**Proposition 5.** *Let  $S$  be a Sylow 2-subgroup of  $G$  containing a hyperelliptic involution  $h$  such that  $\text{Fix}(h)$  has  $r > 2$  components. If  $r$  is an odd number, then either  $h$  is the unique hyperelliptic involution contained in  $S$  or the subgroup  $B$  generated by all the involutions of  $S$  whose fixed-point sets have  $r$  components is dihedral and strongly involution closed in  $S$ .*

*Proof.* We may assume that there are at least two hyperelliptic involutions in  $G$ ; we denote by  $D$  the subgroup of  $S$  generated by the hyperelliptic involutions. The group  $D$  is dihedral by Proposition 3 and  $D \subseteq B$ . We denote by  $2^{n+1}$  the order of  $D$ . If  $B = D$  then the fixed-point set of each involution in  $B$  has  $r$ -components and we are done. So assume that  $B \neq D$ . Write  $D = \langle f, h \rangle$ , where  $f$  generates a cyclic subgroup of order  $2^n$  and  $h$  is a hyperelliptic involution inverting  $f$ . We denote by  $L$  the fixed-point set of  $f^{2^{n-1}}$ . We remark that  $f^{2^{n-1}}$  can be hyperelliptic or not, but since  $f^{2^{n-1}}$  can be seen as a product of two commuting hyperelliptic involutions, in any case the fixed-point set  $L$  has  $r$  components by Lemma 2.

**Claim:** *For every involution  $t \in B \setminus D$  whose fixed-point set has  $r$  components,  $t$  acts as a strong inversion on each component of  $L$  and  $\langle t, D \rangle$  is dihedral.*

By Lemma 2 all the elements of  $D$  fix setwise each component of  $L$ . If  $n > 1$  the element  $t$  centralizes  $f^{2^{n-1}}$ ; if  $n = 1$  the element  $t$  centralizes at least one involution in  $D$  and we can suppose that  $f^{2^{n-1}}$  commutes with  $t$ . Since  $t$  has order two and  $r$  is odd,  $t$  fixes setwise one of the components of  $L$ , which we denote by  $K$ . If  $t$  acted as a rotation on  $K$  it would commute with the hyperelliptic involution; in particular  $f^{2^{n-1}}$ ,  $t$  and  $h$  would generate an elementary group of rank three contradicting Lemma 3, so  $t$  acts on  $K$  as a reflection and each element of  $\langle t, D \rangle$  fixes setwise the simple closed curve  $K$ . By Remark 1 the group  $\langle t, D \rangle$  is either dihedral or isomorphic to  $\mathbb{Z}_2 \times \mathbb{D}_{2^n}$ . If  $\langle t, D \rangle$  was not dihedral,  $t$  would commute with a hyperelliptic involution different from  $f^{2^{n-1}}$  and  $\langle t, D \rangle$  contains an elementary subgroup of rank three contradicting Lemma 3.

It remains to prove that  $t$  acts as a strong inversion on each component of  $L$ . The involution  $t$  projects to  $\bar{t}$ , an involution acting on the quotient  $M/D$ . Since by Lemma 5  $M/D$  is homeomorphic to  $S^3$ , the involution  $\bar{t}$  has non-empty and connected fixed-point set. Since  $\langle t, D \rangle$  is dihedral, the only non trivial element in  $D$  leaving invariant  $\text{Fix}(t)$  is  $f^{2^{n-1}}$ . Since  $r$  is odd  $f^{2^{n-1}}$  leaves invariant at least a component of  $\text{Fix}(t)$ , then the only possibility to get a connected fixed-point set for  $\bar{t}$  is that the  $f^{2^{n-1}}$  acts as a strong inversion on each component of  $\text{Fix}(t)$ . Since each component of  $L$  can intersect  $\text{Fix}(\bar{t})$  in at most two points, we obtain that  $t$  acts as a strong inversion on each component of  $L$ . This concludes the proof of the Claim.

By the previous Claim and Lemma 2, all the generating involutions of  $B$  leave invariant all the components of  $L$ , in particular  $K$ ; therefore  $B$  is isomorphic to a subgroup of a semidirect product  $\mathbb{Z}_2 \times (\mathbb{Z}_{2^a} \times \mathbb{Z}_{2^b})$  where  $\mathbb{Z}_2$  operates on the normal subgroup  $\mathbb{Z}_{2^a} \times \mathbb{Z}_{2^b}$  by sending each element to its inverse (see Remark 1). Assume by contradiction that  $B$  is not dihedral. By Lemma 2, the subgroup  $B$  is not elementary abelian. Then the center of  $B$  contains an involution  $c$  not contained in  $D$  and  $c$  centralizes an element of order at least 4. It follows that  $c$  acts as a non-trivial rotation on every component of  $K$  and hence  $ch$  acts as a strong inversion on  $K$ . By Lemma 3 we get that  $ch$  has  $r$  components and the Claim yields that  $\langle D, ch \rangle = D \times \langle c \rangle$  is dihedral: a contradiction. Since  $B$  is dihedral, the fixed-point set of each involution in  $B$  has  $r$  components and  $B$  is strongly involution closed.  $\square$

**5.2. Structure theorem for arbitrary groups.** We recall some algebraic definitions we use in this subsection. A finite group  $Q$  is *quasisimple* if it is perfect (the abelianised group is trivial) and the factor group  $Q/\mathcal{Z}(Q)$  of  $Q$  by its centre  $\mathcal{Z}(Q)$  is a non-abelian simple group (see [21, chapter 6.6]). A group  $E$  is *semisimple* if it is perfect and the factor group  $E/\mathcal{Z}(E)$  is a direct product of non-abelian simple groups. A semisimple group  $E$  is a central product of quasisimple groups which are uniquely determined. Any finite group  $F$  has a unique maximal semisimple normal subgroup  $\mathcal{E}(F)$  (maybe trivial), which is characteristic in  $F$ . The subgroup  $\mathcal{E}(F)$  is called the *layer* of  $F$  and the quasisimple factors of  $\mathcal{E}(F)$  are called the *components* of  $F$ .

The maximal normal nilpotent subgroup of a finite group  $F$  is called the *Fitting subgroup* and we denote it by  $\mathcal{F}(F)$ . The Fitting subgroup commutes elementwise with the layer of  $F$ . The normal subgroup generated by  $\mathcal{E}(F)$  and by  $\mathcal{F}(F)$  is called the *generalised Fitting subgroup* and we denote it by  $\mathcal{F}^*(F)$ . The generalised Fitting subgroup has the important property to contain its centraliser in  $F$ , which thus coincides with the centre of  $\mathcal{F}^*(F)$ . For further properties of the generalised Fitting subgroup see [21, Section 6.6].

If  $F$  is a finite group we denote by  $\mathcal{O}(F)$  the maximal normal subgroup of odd order of  $F$ .

**Proposition 6.** *Let  $S$  be a Sylow 2-subgroup of  $G$  containing a hyperelliptic involution  $h$  such that  $\text{Fix}(h)$  has  $r > 2$  components and let  $B$  be the subgroup of  $S$  generated by the involutions whose fixed-point sets have  $r$  components and let  $N$  be the subgroup normally generated by  $B$ . If  $r$  is an odd number, then one of the following conditions holds:*

- (1)  *$S$  contains a unique hyperelliptic involution, the subgroup  $\langle h, \mathcal{O}(G) \rangle / \mathcal{O}(G)$  lies in the centre of  $G/\mathcal{O}(G)$  and  $G/\langle h, \mathcal{O}(G) \rangle$  is isomorphic to a quotient of a finite subgroup of  $\text{SO}(4)$ ;*
- (2)  *$S$  contains at least two hyperelliptic involutions, the Sylow 2-subgroup of  $N$  is dihedral and one of these two situations occurs:*
  - (a)  $N = B\mathcal{O}(N)$
  - (b)  $N/\mathcal{O}(N)$  is isomorphic to  $\mathbb{A}_7$ ,  $\text{PGL}(2, q)$  for an odd prime power  $q$ , or  $\text{PSL}(2, q)$  for an odd prime power  $q > 5$ .

*Proof.* If  $h$  is the only hyperelliptic involution in  $S$ , then we can use the Glauberman's  $Z^*$ -theorem and obtain that  $h\mathcal{O}$  is contained in the center of  $G/\mathcal{O}$ . By Remark 2 and Proposition 2, we get that (1) holds. We assume that  $S$  contains at least two hyperelliptic involutions. By Proposition 5, the subgroup  $B$  is a strongly involution closed dihedral

subgroup of  $G$ . By [12, Theorem 3.1] we obtain that either  $B$  is strongly closed in  $S$  or  $B$  is contained with index 2 in a strongly closed semidihedral subgroup  $Q$ .

We show that the latter case does not occur. Suppose by contradiction that  $B$  is a maximal subgroup in a strongly closed semidihedral subgroup  $Q$  and denote by  $c$  the unique central involution of  $Q$ . Then  $Fix(c)$  has  $r$  components by Lemma 4 and there exists at least another hyperelliptic involution that is not central in  $Q$ . All the non central involutions of  $Q$  are conjugate (see [1, Lemma 1]); so every non central involution of  $Q$  is hyperelliptic and, by Lemma 2, acts as a reflection on each component of  $Fix(c)$ . Let  $f$  be an element of order 4 in  $Q$  not generating a normal subgroup of  $Q$ . We note that  $f$  leaves invariant  $Fix(c)$ . Since  $Fix(c)$  has an odd number of components,  $f$  fixes setwise one component of  $Fix(c)$ . By Remark 1 the element  $f$  acts as a rotation on the fixed component; this implies that  $f$  and any non central involution in  $Q$  generate an abelian or dihedral group and this is impossible.

Therefore we have that  $B$  is strongly closed in  $S$  and by the main theorem of [12] cases (2a) or (2b) hold, or  $N/\mathcal{O}(N) \cong PSU(3, 4)$ . In this case, it is straightforward to see that  $D$  is an elementary abelian group of order 4, the subgroup  $D$  coincides with the center of a Sylow 2-subgroup  $T$  of  $N$  and  $T/Z(T)$  is elementary abelian of rank 4, contradicting Lemma 1.  $\square$

**Proposition 7.** *Let  $S$  be a Sylow 2-subgroup of  $G$  containing a hyperelliptic involution  $h$  such that  $Fix(h)$  has  $r > 2$  components and  $B$  be the subgroup of  $S$  generated by the involutions with fixed-point set with  $r$  components. If  $r$  is odd and  $S$  contains at least two hyperelliptic involutions, then one of the two following conditions holds:*

- (1)  $G$  is solvable,  $B\mathcal{O}(G)$  is normal in  $G$  and  $G/B\mathcal{O}(G)$  is isomorphic to a quotient of a finite subgroup of  $SO(4)$ ;
- (2)  $G/\mathcal{O}(G)$  contains a unique component  $Q/\mathcal{O}(G)$  isomorphic either to  $\mathbb{A}_7$  or to  $PSL(2, q)$  with  $q$  an odd prime power, and the Fitting subgroup of  $G/\mathcal{O}(G)$  is cyclic or dihedral. The quotient  $G/Q$  is a solvable group with cyclic Sylow  $p$ -subgroups for each  $p$  odd.

*Proof.* Let  $N$  be the subgroup normally generated by  $B$  in  $G$ . We remark that  $\mathcal{O}(N) = \mathcal{O}(G) \cap N$ . Cases (2a) or (2b) in Proposition 6 hold.

First we suppose that case (2a) holds, that is  $N = B\mathcal{O}(N)$ . In particular  $B\mathcal{O}(N)$  and  $B\mathcal{O}(G)$  are normal in  $G$ . By Remark 2 and isomorphisms Theorems, the group  $G/B\mathcal{O}(G)$  is isomorphic to the quotient of  $\mathcal{N}_G(B)$  by  $B$ ; by Lemma 5 the group  $\mathcal{N}_G(B)/B$  is isomorphic to a finite subgroup of  $SO(4)$ . Since the automorphism group of a dihedral groups is solvable, the quotient  $\mathcal{N}_G(B)/\mathcal{C}_G(B)$  is solvable too. Moreover, the group  $\mathcal{C}_G(B)$  commutes elementwise with the elementary abelian 2-subgroup of rank two of  $B$  generated by two hyperelliptic involutions. By Lemma 1, the subgroup  $\mathcal{C}_G(B)$  is solvable. Hence  $\mathcal{N}_G(B)$  is solvable and in this case we are done.

Suppose now to be in the case (2b) of Proposition 6. If  $H$  is a subgroup of  $G$ , we denote by  $\overline{H}$  the projection  $H\mathcal{O}(G)/\mathcal{O}(G)$  of  $H$  to  $G/\mathcal{O}(G)$ ; in particular  $\overline{G}$  is  $G/\mathcal{O}(G)$ . By the structure of  $\overline{N}$  it follows that  $\overline{G}/\mathcal{C}_{\overline{G}}(\overline{N})$  has a unique simple section isomorphic either to  $\mathbb{A}_7$  or to  $PSL(2, q)$ . On the other hand, by Remark 2,  $\mathcal{C}_{\overline{G}}(\overline{N}) \leq \mathcal{C}_{\overline{G}}(\overline{B}) \cong \mathcal{C}_G(B)/\mathcal{C}_G(B) \cap \mathcal{O}(G)$  which is solvable by Lemma 1. This implies that the layer of  $\overline{G}$  is isomorphic either to  $\mathbb{A}_7$  or to  $PSL(2, p^n)$ .

Consider now the Fitting subgroup  $\mathcal{F}(\overline{G})$  of  $\overline{G}$  which is a 2-group by definition of  $\mathcal{O}(G)$ . We remark that  $\mathcal{F}(\overline{G}) \cap \mathcal{E}(\overline{G}) = \{1\}$ . The subgroup  $\mathcal{F}(\overline{G})$  commutes with an elementary

2-subgroup of rank 2 generated by two projections of hyperelliptic involutions and contained in  $\mathcal{E}(\overline{G})$ . By Remark 2 the subgroup  $\mathcal{F}(\overline{G})$  is the projection of a 2-subgroup of  $G$  commuting with an elementary 2-subgroup of rank 2 generated by two hyperelliptic involutions. By Lemma 1 the subgroup  $\mathcal{F}(\overline{G})$  is cyclic or dihedral. The product  $\mathcal{F}(\overline{G}) \times \mathcal{E}(\overline{G})$  is the generalized Fitting subgroup of  $\overline{G}$ , and hence the quotient of  $\overline{G}$  by the center of generalized Fitting subgroup is isomorphic to a subgroup of  $\text{Aut}(\mathcal{F}(\overline{G})) \times \text{Aut}(\mathcal{E}(\overline{G}))$ . This implies that (2) holds. □

We summarize in the following theorem the results contained in Propositions 6 and 7.

**Theorem 2.** *Let  $S$  be a Sylow 2-subgroup of  $G$  containing a hyperelliptic involution  $h$  such that  $\text{Fix}(h)$  has  $r > 2$  components. If  $r$  is odd one of the following cases occurs:*

- (1) *the subgroup  $\langle h, \mathcal{O}(G) \rangle / \mathcal{O}(G)$  lies in the centre of  $G / \mathcal{O}(G)$  and  $G / \langle h, \mathcal{O}(G) \rangle$  is isomorphic to a quotient of a finite subgroup of  $\text{SO}(4)$ ;*
- (2)  *$G$  is solvable and there exists a dihedral 2-subgroup  $B$  such that  $B\mathcal{O}(G)$  is normal in  $G$  and  $G / B\mathcal{O}(G)$  is isomorphic to a quotient of a finite subgroup of  $\text{SO}(4)$ ;*
- (3)  *$G / \mathcal{O}(G)$  contains a normal subgroup  $E \times F$  containing its centralizer; the subgroup  $E$  is isomorphic either to  $\mathbb{A}_7$  or to  $\text{PSL}(2, q)$  with  $q$  an odd prime number, and  $F$  is cyclic or dihedral. The quotient of  $(G / \mathcal{O}(G)) / E \times F$  is a solvable group with cyclic Sylow  $p$ -subgroups for each  $p$  odd.*

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