DOUBLE RESONANCE IN STURM-LIOUVILLE PLANAR BOUNDARY VALUE PROBLEMS

Andrea Sfecci

ABSTRACT. We provide some existence results for Sturm-Liouville boundary value problems associated with the planar differential system $Jz' = g(t,z) + r(t,z)$ where g is suitably controlled by the gradient of two positively homogeneous functions of degree 2 and r is sublinear with respect to the variable z at infinity. We study the existence of solutions when a double resonance phenomenon occurs by the introduction of Landesman-Lazer type conditions. Applications to scalar second order differential equations are given.

1. Introduction

For the scalar equation

$$(1.1) \qquad x'' + f(t,x) = 0$$

with periodic, Neumann or Dirichlet boundary conditions there have been several works concerning the existence of solutions under some nonresonance conditions.

The approach to resonance is a delicate problem and the most successful condition has been introduced by Landesman and Lazer, where the nonlinearity asymptotically lies between two eigenvalues of the linear differential equation,

see [9, 10]. In the case of asymmetric nonlinearities we mention [6, 8] for the periodic case and [15] for planar systems (see also [18, 19, 24] and the monograph [14] for further informations on this topic).

Even though double resonance phenomenon has been studied dealing with periodic boundary conditions, such a discussion has not been treated for Dirichlet and Neumann boundary conditions yet. In this paper we are going to present some existence results at double resonance for equation (1.1) when f satisfies

$$(1.2) \qquad 0 < \nu_1 \le \liminf_{x \to -\infty} \frac{f(t,x)}{x} \le \limsup_{x \to -\infty} \frac{f(t,x)}{x} \le \nu_2 \,,$$

$$(1.3) \qquad 0 < \mu_1 \le \liminf_{x \to +\infty} \frac{f(t,x)}{x} \le \limsup_{x \to +\infty} \frac{f(t,x)}{x} \le \mu_2$$

(see Theorem 4.3 below).

In such a situation the nonlinearity f "mimes" an asymmetric oscillator

$$x'' + \mu x^+ - \nu x^- = 0 \,,$$

where $x^+ = (|x| + x)/2$ and $x^- = (|x| - x)/2$. The previous scalar differential equation can be studied as a planar system of the type

$$(1.4) \qquad Jz' = \nabla V(z) \,, \quad z \in \mathbb{R}^2 \,,$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix and $V : \mathbb{R}^2 \to \mathbb{R}$ is a positively homogeneous C^1-function of degree 2, i.e. such that

$$0 < V(\lambda z) = \lambda^2 V(z) \,, \qquad \text{for every } \lambda > 0 \,, z \ne 0 \,.$$

For this reason, boundary value problems related to (1.4) present a particular interest in literature, see [2, 13, 16, 27] and the references therein.

In relation with the scalar second order differential equation (1.1), the Dirichlet boundary conditions $x(0) = x(T) = 0$ (DBC), the Neumann boundary conditions $x'(0) = x'(T) = 0$ (NBC) and the mixed boundary conditions $x'(0) = x(T) = 0$ (MBC) can be collected all together in a unique class of problems when we pass to consider planar systems as in (1.4). Indeed, we can ask a solution $z(t) = (x(t), y(t))$ to start and arrive at some points belonging to two lines in the plane:

$$(1.5) \qquad z(0) \in l_S \,, \qquad z(T) \in l_A \,,$$

where l_S is the *starting line* and l_A is the *arrival line*. In particular (DBC) is equivalent to the case $l_S = l_A = \{z = (x, y) \mid x = 0\}$, (NBC) is equivalent to the case $l_S = l_A = \{z = (x, y) \mid y = 0\}$ and (MBC) is equivalent to the case $l_S = \{z = (x, y) \mid y = 0\}$ and $l_A = \{z = (x, y) \mid x = 0\}$.

In [2, 16], the following class of problems, obtained as a perturbation of (1.4), is treated:

$$(1.6) \qquad \begin{cases} Jz' = \nabla V(z) + p(t, z) \,, \\ z(0) \in l_S \,, \qquad z(T) \in l_A \,, \end{cases}$$

where, for briefness, we say p is bounded and continuous.

Recalling that the unperturbed system (1.4) has an isochronous center of minimal period τ_V, and borrowing the definition from [2], we say that the unperturbed problem

$$(1.7) \qquad \begin{cases} Jz' = \nabla V(z) \,, \\ z(0) \in l_S \,, \qquad z(T) \in l_A \,. \end{cases}$$

is *resonant* if it has at least one nontrivial solution. As in the periodic case, if problem (1.7) is not resonant then a perturbed problem as in (1.6) admits a solution, cf. [16]. Conversely, if the unperturbed problem (1.7) is resonant, then the existence of a solution to problems as in (1.6) is ensured assuming an additional condition: in [2] the introduction of a Landesman-Lazer type assumptions provides an existence result. In these notes we continue the study performed in [2, 16] by Boscaggin, Fonda and Garrione. In particular, we are going to consider the wider class of problems

$$(1.8) \qquad \begin{cases} Jz' = g(t, z) + p(t, z) \,, \\ z(0) \in l_S \,, \qquad z(T) \in l_A \,, \end{cases}$$

where the function g is controlled by two positively homogeneous functions of degree 2, $V_1 \leq V_2$: More precisely, g satisfies

$$g(t, z) = \big(1 - \gamma(t, z)\big)\nabla V_1(z) + \gamma(t, z)\nabla V_2(z) \,,$$

where $\gamma : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$, with $0 \leq \gamma \leq 1$ and $p : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is sublinear with respect to the second variable. In Assumptions 2.1 and 3.1 below, concerning the functions g and p, we will require a Carathéodory type of regularity, so we will look for solutions of (1.8) belonging to the space $H^1([0, T], \mathbb{R}^2)$ satisfying the differential equation almost everywhere.

The study of existence of solutions to problem (1.8) is related to the study of perturbed asymmetric oscillators, e.g. differential equations as $x'' + \mu x^+ - \nu x^- + g(x) = e(t)$ where g and e are bounded continuous functions. In particular, (1.8) includes the scalar differential equation (1.1) where f satisfies (1.2) and (1.3), as a particular case. Such problems present a wide literature, see [3, 4, 5, 7, 8, 12, 13, 14, 20, 21, 22, 23] for a non exhaustive bibliography, apologizing for unavoidable missing references.

We recall that Drábek studied the possibility of introducing Landesman-Lazer type assumptions for a class of boundary value problems in [6, Section 4], focusing his attention on Dirichlet problems. In Section 4, we will present our main result applied to Dirichlet problems with *jumping* nonlinearities in the spirit of that paper.

Systems as in (1.8) have been investigated in [15] by Fonda and Garrione dealing with periodic boundary conditions (see also [11, 17]). In the periodic setting, the existence of solutions can be ensured if there exists a positive integer k such that

$$(1.9) \qquad \frac{T}{k+1} \le \tau_{V_2} \le \tau_{V_1} \le \frac{T}{k} \,,$$

where τ_{V_1} and τ_{V_2} are the periods of the solutions of system (1.4) choosing respectively $V = V_1$ and $V = V_2$.

In such a situation we can distinguish three situations: nonresonance, when we have the strict inequalities in (1.9); simple resonance, when we have a strict inequality and an equality in (1.9); double resonance, when two equalities hold in (1.9). In the presence of resonance we need to add additional assumptions. Landesman-Lazer type assumptions have been introduced in [15] (see also [8, 9, 10, 25, 26] for related results).

In this paper, dealing with system (1.8), we investigate all the three situations: nonresonance, simple resonance and double resonance. They will be treated in Section 3. The situations differ depending on the position of the value T in (1.8) with respect to a *resonance set* which will be introduced in (2.26).

The paper is organized as follows. In Section 2 we present some preliminary results: in Section 2.1 some properties of the autonomous system (1.4) are listed borrowing some notations from [2, 16], then in Section 2.2 we add a first perturbation presenting some properties of the solutions of system (1.8) in the *semi-autonomous* case $p \equiv 0$. We present the main Theorems 3.2 (nonresonance), 3.8 (simple resonance) and 3.9 (double resonance) in the successive Sections 3.1, 3.2 and 3.3, respectively. Finally, in Section 4, we present the applications of our theorems in the case of scalar equations (1.1) with an asymmetric nonlinearity, cf. Theorem 4.3.

In this paper, we will denote by $|\cdot|$ the Euclidean norm in $\mathbb{R}^2$ and we will use the complex notation for polar coordinates in the plane: $z = (x, y) = \rho e^{i\vartheta} = (\rho \cos\vartheta, \rho \sin\vartheta)$. Moreover, in order to well define the angle ϑ when we pass to polar coordinates, we will consider functions $z : I \to \mathbb{R}^2$ such that $z(t) \ne (0,0)$ for every $t \in I$. For briefness we call them *never-zero* functions.

2. Preliminaries

2.1. An autonomous isochronous planar system. In this section we recall some notations and contents from [2, 16]. Let us consider the planar system

$$(2.1) \qquad Jz' = \nabla V(z), \qquad z = (x,y) \in \mathbb{R}^2,$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is the standard symplectic matrix and $V : \mathbb{R}^2 \to \mathbb{R}$ is a C^1-function which is positively homogeneous of degree 2, i.e.

$$(2.2) \qquad 0 < V(\lambda z) = \lambda^2 V(z), \qquad \text{for every } \lambda > 0, z \neq 0.$$

Let us recall the validity of the Euler's formula: $\langle \nabla V(z) \mid z \rangle = 2V(z)$ for every $z \in \mathbb{R}^2$.

The origin is an isochronous center for system (2.1) of minimal period

$$(2.3) \qquad \tau_V = \int_0^{2\pi} \frac{d\theta}{2V(\cos\theta, \sin\theta)}$$

and all the solutions have the form $z(t) = C\varphi_V(t+\tau)$, with $C \geq 0$ and $\tau \in [0, \tau_V)$, where φ_V is a fixed nontrivial solution to (2.1). Without loss of generality we assume $V(\varphi_V(t)) \equiv \frac{1}{2}$ and $\varphi_V(0) = (0, y_0)$ with $y_0 > 0$.

Let us consider the following boundary condition

$$(2.4) \qquad z(0) \in l_S, \qquad z(T) \in l_A,$$

where l_S and l_A ("S" stands for *starting*, "A" for *arrival*) are lines through the origin of slope ζ_S and ζ_A, respectively. We mean that a line through the origin has slope $\zeta \in (-\pi/2, \pi/2]$ if it can be parametrized as $l : \mathbb{R} \to \mathbb{R}^2$, $l(s) = s(\cos\zeta, \sin\zeta)$.

For later purpose, let us set

$$(2.5) \qquad \Delta\zeta = \begin{cases} \zeta_S - \zeta_A & \text{if } \zeta_S > \zeta_A \\ \zeta_S - \zeta_A + \pi & \text{if } \zeta_S \leq \zeta_A, \end{cases}$$

which is the smallest positive angle a solution covers moving from l_S to l_A, cf. Figure 1 (remember that solutions rotate clockwise).

Denote by

- $\tau_{0,V}$ the least nonnegative time such that $\varphi_V(\tau_{0,V}) \in l_S$,
- $\tau_{1,V}$ the least positive time such that $\varphi_V(\tau_{0,V} + \tau_{1,V}) \in l_A$,
- $\sigma_{1,V}$ the least nonnegative time such that $\varphi_V(\tau_{0,V} + \tau_{1,V} + \sigma_{1,V}) \in l_S$,
- $\tau_{2,V}$ the least positive time such that $\varphi_V(\tau_{0,V} + \tau_{1,V} + \sigma_{1,V} + \tau_{2,V}) \in l_A$,
- $\sigma_{2,V}$ the least nonnegative time such that $\varphi_V(\tau_{0,V} + \tau_{1,V} + \sigma_{1,V} + \tau_{2,V} + \sigma_{2,V}) \in l_S$.
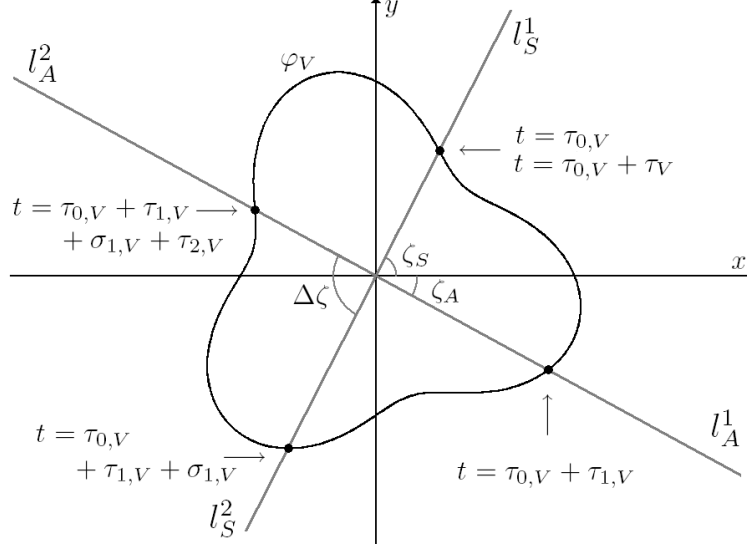
FIGURE 1. The notation of Section 2.1, a sketch in the case $\zeta_A < \zeta_S$.

Notice that, by definition,

$$(2.6) \qquad\qquad \tau_V = \tau_{1,V} + \sigma_{1,V} + \tau_{2,V} + \sigma_{2,V}$$

and in particular, as in (2.3), we have

$$(2.7)$$
$$\tau_{1,V} = \int_{\zeta_S - \Delta\zeta}^{\zeta_S} \frac{d\theta}{2V(\cos\theta, \sin\theta)}\,, \qquad \tau_{2,V} = \int_{\zeta_S + \pi - \Delta\zeta}^{\zeta_S + \pi} \frac{d\theta}{2V(\cos\theta, \sin\theta)}\,,$$
$$\sigma_{1,V} = \int_{\zeta_S - \pi}^{\zeta_S - \Delta\zeta} \frac{d\theta}{2V(\cos\theta, \sin\theta)}\,, \quad \sigma_{2,V} = \int_{\zeta_S}^{\zeta_S + \pi - \Delta\zeta} \frac{d\theta}{2V(\cos\theta, \sin\theta)}\,,$$

As a consequence, if l_S and l_A coincide, then $\sigma_{1,V} = \sigma_{2,V} = 0$.

In order to distinguish the two rays which l_A and l_S consist of respectively, we set (cf. Figure 1)

$$(2.8)$$
$$l_S^1 \ni \varphi_V(\tau_{0,V})\,, \qquad\qquad l_A^1 \ni \varphi_V(\tau_{0,V} + \tau_{1,V})\,,$$
$$l_S^2 \ni \varphi_V(\tau_{0,V} + \tau_{1,V} + \sigma_{1,V})\,, \quad l_A^2 \ni \varphi_V(\tau_{0,V} + \tau_{1,V} + \sigma_{1,V} + \tau_{2,V})\,.$$

The problem

$$(2.9) \qquad\qquad \begin{cases} Jz' = \nabla V(z)\,, \\ z(0) \in l_S\,, \qquad z(T) \in l_A\,, \end{cases}$$

is said to be *resonant* if it admits nontrivial solutions. Such a situation occurs if and only if T satisfies any one of the following identities, for a certain $k \in \mathbb{N}$,

$$(2.10) \qquad T \;=\; k\tau_V + \tau_{1,V}\,,$$

$$(2.11) \qquad T \;=\; k\tau_V + \tau_{1,V} + \sigma_{1,V} + \tau_{2,V}\,,$$

$$(2.12) \qquad T \;=\; k\tau_V + \tau_{2,V}\,,$$

$$(2.13) \qquad T \;=\; k\tau_V + \tau_{2,V} + \sigma_{2,V} + \tau_{1,V}\,,$$

and a nontrivial solution is given by

$$(2.14) \qquad\qquad \varphi_V(t + \tau_{0,V}) \qquad \text{if (2.10) or (2.11) holds,}$$

$$(2.15) \qquad \varphi_V(t + \tau_{0,V} + \tau_{1,V} + \sigma_{1,V}) \qquad \text{if (2.12) or (2.13) holds.}$$

Indeed, concerning the solutions of a resonant problem (2.9), we can distinguish different starting rays and arrival rays:

$$
\begin{aligned}
&\text{if (2.10) holds} &\Rightarrow\quad &\text{the solution starts from } l_S^1 \text{ and arrives on } l_A^1\,, \\
&\text{if (2.11) holds} &\Rightarrow\quad &\text{the solution starts from } l_S^1 \text{ and arrives on } l_A^2\,, \\
&\text{if (2.12) holds} &\Rightarrow\quad &\text{the solution starts from } l_S^2 \text{ and arrives on } l_A^2\,, \\
&\text{if (2.13) holds} &\Rightarrow\quad &\text{the solution starts from } l_S^2 \text{ and arrives on } l_A^1\,.
\end{aligned}
$$

2.2. Introducing a perturbation in the energy. We now focus our attention on the qualitative properties of solutions to the boundary value problem

$$(2.16) \qquad \begin{cases} Jz' = g(t,z)\,, \\ z(0) \in l_S\,, \qquad z(T) \in l_A\,, \end{cases}$$

where the function $g : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is suitably controlled by two Hamiltonians V_1 and V_2 as in the previous section. More precisely, we introduce the following.

ASSUMPTION 2.1. There exists a L^2-Carathéodory function $\gamma : [0,T] \times \mathbb{R}^2 \to [0,1]$ such that

$$g(t,z) = (1 - \gamma(t,z))\nabla V_1(z) + \gamma(t,z)\nabla V_2(z)\,,$$

where $V_1 \leq V_2$ are two positively homogeneous C^1-functions as in (2.2).

As in (2.6), we define the values

$$
\begin{array}{ccccccccc}
\tau_{V_2} &=& \tau_{1,V_2} &+& \sigma_{1,V_2} &+& \tau_{2,V_2} &+& \sigma_{2,V_2}\,, \\[2pt]
(2.17) \quad &&\text{\tiny I}\wedge && \text{\tiny I}\wedge && \text{\tiny I}\wedge && \text{\tiny I}\wedge \\[2pt]
\tau_{V_1} &=& \tau_{1,V_1} &+& \sigma_{1,V_1} &+& \tau_{2,V_1} &+& \sigma_{2,V_1}\,.
\end{array}
$$

Introducing polar coordinates $z = (x, y) = \rho e^{i\vartheta}$, the angular velocity of a *never-zero* solution of $Jz' = g(t, z)$ is given by

$$
\begin{aligned}
-\vartheta'(t) &= \frac{\langle Jz'(t) \mid z(t) \rangle}{|z(t)|^2} \\
&= (1 - \gamma(t, z(t)) \frac{\langle \nabla V_1(z(t)) \mid z(t) \rangle}{|z(t)|^2} + \gamma(t, z(t)) \frac{\langle \nabla V_2(z(t)) \mid z(t) \rangle}{|z(t)|^2} \\
&= 2(1 - \gamma(t, z(t))) V_1(\cos \vartheta(t), \sin \vartheta(t)) + 2\gamma(t, z(t)) V_2(\cos \vartheta(t), \sin \vartheta(t)),
\end{aligned}
$$

so that we obtain

$$(2.18) \qquad 0 < 2V_1(\cos \vartheta(t), \sin \vartheta(t)) \le -\vartheta'(t) \le 2V_2(\cos \vartheta(t), \sin \vartheta(t)).$$

By the previous computation, recalling (2.3), (2.7) and (2.8), a never-zero solution of $Jz' = g(t, z)$

$$(2.19) \qquad \text{moving from } l_S^1 \text{ to } l_A^1 \text{ spends a time } \Delta t_1 \in [\tau_{1, V_2}, \tau_{1, V_1}],$$

$$(2.20) \qquad \text{moving from } l_A^1 \text{ to } l_S^2 \text{ spends a time } \Delta t_2 \in [\sigma_{1, V_2}, \sigma_{1, V_1}],$$

$$(2.21) \qquad \text{moving from } l_S^2 \text{ to } l_A^2 \text{ spends a time } \Delta t_3 \in [\tau_{2, V_2}, \tau_{2, V_1}],$$

$$(2.22) \qquad \text{moving from } l_A^2 \text{ to } l_S^1 \text{ spends a time } \Delta t_4 \in [\sigma_{2, V_2}, \sigma_{2, V_1}],$$

$$(2.23) \qquad \text{completes a rotation around the origin in a time } \Delta t \in [\tau_{V_2}, \tau_{V_1}].$$

We now estimate the time spent by a solution of $Jz' = g(t, z)$ in order to move from a point of l_S^i ($i = 1, 2$) to a point of l_A, once it has covered an angle of at least $j\pi$ ($j \in \mathbb{N}$). Such a time belongs to a certain interval $[a_j^i, b_j^i]$, whose values can be computed using (2.19)-(2.23). When we do not distinguish between solutions departing from l_S^1 and l_S^2 the time belongs to a larger interval denoted by $I_j = [\alpha_j, \beta_j]$.

We list the constants a_j^i, b_j^i, α_j, β_j ($i = 1, 2\,; j \in \mathbb{N}$) here below:

$$(2.24)$$

$$
\begin{aligned}
\alpha_{2k} &= \min\{a_{2k}^1, a_{2k}^2\}, & a_{2k}^1 &= k\tau_{V_2} + \tau_{1, V_2}, \\
& & a_{2k}^2 &= k\tau_{V_2} + \tau_{2, V_2}\,; \\
\beta_{2k} &= \max\{b_{2k}^1, b_{2k}^2\}, & b_{2k}^1 &= k\tau_{V_1} + \tau_{1, V_1}, \\
& & b_{2k}^2 &= k\tau_{V_1} + \tau_{2, V_1}\,; \\
\alpha_{2k+1} &= \min\{a_{2k+1}^1, a_{2k+1}^2\}, & a_{2k+1}^1 &= k\tau_{V_2} + \tau_{1, V_2} + \tau_{2, V_2} + \sigma_{1, V_2}, \\
& & a_{2k+1}^2 &= k\tau_{V_2} + \tau_{1, V_2} + \tau_{2, V_2} + \sigma_{2, V_2}\,; \\
\beta_{2k+1} &= \max\{b_{2k+1}^1, b_{2k+1}^2\}, & b_{2k+1}^1 &= k\tau_{V_1} + \tau_{1, V_1} + \tau_{2, V_1} + \sigma_{1, V_1}, \\
& & b_{2k+1}^2 &= k\tau_{V_1} + \tau_{1, V_1} + \tau_{2, V_1} + \sigma_{2, V_1}.
\end{aligned}
$$

Clearly, all the intervals $I_j = [\alpha_j, \beta_j]$ are well ordered in the following sense: $\alpha_j < \alpha_{j+1}$ and $\beta_j < \beta_{j+1}$ for every $j \in \mathbb{N}$. However, we can have $\alpha_{j+1} < \beta_j$:

more precisely we have

$$\beta_{2k} - \alpha_{2k+1} \geq k(\tau_{V_1} - \tau_{V_2}) - \tau_{V_2}\,, \qquad \beta_{2k+1} - \alpha_{2k+2} \geq k(\tau_{V_1} - \tau_{V_2}) - 2\tau_{V_2}\,,$$

for every index k. So, if $V_1 \neq V_2$, and in particular $\tau_{V_1} > \tau_{V_2}$, then there exists an index j_0 such that

$$(2.25) \qquad\qquad \alpha_{j+1} < \beta_j\,, \qquad \text{for every } j \geq j_0\,.$$

We introduce the *resonance set*

$$(2.26) \qquad\qquad \mathcal{I} = \bigcup_{j \in \mathbb{N}} I_j = \bigcup_{j \in \mathbb{N}} [\alpha_j\,,\, \beta_j]\,,$$

the interior of $\mathcal{I}$, denoted by $\overset{\circ}{\mathcal{I}}$, and

$$(2.27) \qquad\qquad \widetilde{\mathcal{I}} = \bigcup_{j \in \mathbb{N}} (\alpha_j\,,\, \beta_j)\,.$$

Notice that $\widetilde{\mathcal{I}} \subseteq \overset{\circ}{\mathcal{I}} \subseteq \mathcal{I}$ and so $\partial\mathcal{I} \subseteq \partial\widetilde{\mathcal{I}}$. Moreover, if $V_1 \neq V_2$, by (2.25), the resonance set $\mathcal{I}$ contains the unbounded interval $[\alpha_{j_0}, \infty)$.

In this paper we are going to treat the following situations:

- *Nonresonance*: $T \notin \mathcal{I}$, that is $\exists \kappa \in \mathbb{N}$ such that $\beta_\kappa < T < \alpha_{\kappa+1}$, or $T < \alpha_0$.
- *Simple resonance*: $T \in \partial\mathcal{I}$, that is $\exists \kappa \in \mathbb{N}$ such that $T = \alpha_\kappa$ or $T = \beta_\kappa$.
- *Double resonance*: $T \in \partial\widetilde{\mathcal{I}} \setminus \partial\mathcal{I}$, that is $\exists \kappa \in \mathbb{N}$ such that $T = \beta_\kappa = \alpha_{\kappa+1}$.

Let us now spend few words in order to explain how resonance occurs. For example, let us assume $g \equiv \nabla V_2$ and $T = \alpha_\kappa$, for a certain integer κ. If $T = \alpha_\kappa = a_\kappa^1$ then there exists an infinite number of solutions of (2.16) departing from ℓ_S^1. Conversely, if $T = \alpha_\kappa = a_\kappa^2$, then there exists an infinite number of solutions of (2.16) departing from ℓ_S^2. In general one or both the situations can arise. Hence, dealing with the resonance situations we will need to focus our attention on the indices $i = 1, 2$ satisfying the identities $\alpha_\kappa = a_\kappa^i$ and $\beta_\kappa = b_\kappa^i$, and, correspondingly, to solutions departing form the ray ℓ_S^i.

3. Main results

3.1. Nonresonance. In this section we consider the boundary value problem

$$(3.1) \qquad\qquad \begin{cases} Jz' = g(t, z) + p(t, z)\,, \\ z(0) \in l_S\,, \qquad z(T) \in l_A\,, \end{cases}$$

where $g : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ satisfies Assumption 2.1 and p has the following property.

Assumption 3.1. *The function* $p : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ *is a Carathéodory function such that, for every compact set* $K \subset \mathbb{R}^2$, $|p(t, z)| \leq \ell_K(t)$, *for a.e.* $t \in [0, T]$ *and* $z \in K$, *for a suitable* $\ell_K \in L^2([0, T], \mathbb{R})$. *Moreover,*

$$(3.2) \qquad \lim_{|z| \to \infty} \frac{p(t, z)}{|z|} = 0 \,,$$

uniformly for almost every $t \in [0, T]$.

Denote by $\Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ the flux of system

$$(3.3) \qquad J z' = g(t, z) + p(t, z) \,,$$

in particular $\Phi(\cdot, z_0)$ denotes the solution z of (3.3) such that $z(0) = \Phi(0, z_0) = z_0$. ([1]) We consider polar coordinates associated to never-zero solutions as follows. Let $z = \Phi(\cdot, z_0) : \mathcal{U} \to \mathbb{R}^2$ be the solution of (3.3) such that $z(0) = \Phi(0, z_0) = z_0$ and assume that $\Phi(t, z_0) \neq (0, 0)$ for every $t \in \mathcal{U}$, then we can find two continuous function $\mathscr{R}(\cdot, z_0) : \mathcal{U} \to \mathbb{R}^+$ and $\Theta(\cdot, z_0) : \mathcal{U} \to \mathbb{R}$ such that

$$(3.4) \qquad \Phi(t, z_0) = \mathscr{R}(t, z_0)\big(\cos \Theta(t, z_0) \,, \, \sin \Theta(t, z_0) \big) \,.$$

For definiteness, we choose the value $\Theta(0, z_0) \in [0, 2\pi)$. In what follows, we shall often use the *covered-angle* function

$$(3.5) \qquad \Delta\Theta(t, z_0) = \Theta(0, z_0) - \Theta(t, z_0) \geq 0 \,.$$

Our first result generalizes [16, Theorem 3.1]:

Theorem 3.2 Nonresonance. *Consider problem* (3.1), *where* g *satisfies Assumption 2.1 and* p *satisfies Assumption 3.1. Assume* $T \notin \mathcal{I}$, *where* $\mathcal{I}$ *is the resonance set introduced in* (2.26).

Then there exists at least one solution of (3.1).

Proof. If $T \notin \mathcal{I}$, then we have $T \in (\beta_{j-1}, \alpha_j)$ for a certain $j \in \mathbb{N}$ (set $\beta_{-1} = 0$). In particular we can find a small $\varepsilon_0 > 0$ such that

$$(3.6) \qquad T \in \big(\beta_{j-1} + \varepsilon_0, \alpha_j - \varepsilon_0 \big) \,.$$

Introducing polar coordinates $z = \rho e^{i\vartheta}$, the angular velocity of solutions of (3.3) is given by

$$-\vartheta'(t) = \frac{\langle\, g(t, z(t)) + p(t, z(t)) \mid z(t) \,\rangle}{|z(t)|^2}$$

and we can compute

$$(3.7) \qquad 2V_1(\cos\vartheta(t), \sin\vartheta(t)) + e(t) \leq -\vartheta'(t) \leq 2V_2(\cos\vartheta(t), \sin\vartheta(t)) + e(t) \,,$$

([1]) We can assume, without loss of generality, the uniqueness of the solutions to the Cauchy problems. Indeed, by standard arguments, all the results in this paper can be obtained with a limit procedure introducing a sequence of approximating nonlinearities having such a uniqueness property.

where
$$e(t) = \frac{\langle\, p(t, z(t)) \mid z(t) \,\rangle}{|z(t)|^2} \,.$$

Then, from (2.19)–(2.23), choosing $\varepsilon > 0$ sufficiently small to guarantee

(3.8) $$\varepsilon(2j+1) < \varepsilon_0 \,,$$

we can find $R_1 > 0$ such that a solution of (3.3), satisfying $|z(t)| \geq R_1$ for every $t \in [0, T]$, rotates clockwise and

(3.9) moving from l_S^1 to l_A^1 spends a time $\Delta t_1 \in (\tau_{1,V_2} - \varepsilon \,,\, \tau_{1,V_1} + \varepsilon)\,,$

(3.10) moving from l_A^1 to l_S^2 spends a time $\Delta t_2 \in (\sigma_{1,V_2} - \varepsilon \,,\, \sigma_{1,V_1} + \varepsilon)\,,$

(3.11) moving from l_S^2 to l_A^2 spends a time $\Delta t_3 \in (\tau_{2,V_2} - \varepsilon \,,\, \tau_{2,V_1} + \varepsilon)\,,$

(3.12) moving from l_A^2 to l_S^1 spends a time $\Delta t_4 \in (\sigma_{2,V_2} - \varepsilon \,,\, \sigma_{2,V_1} + \varepsilon)\,,$

(3.13) completes a rotation around the origin in
$$\text{a time } \Delta t \in (\tau_{V_2} - 4\varepsilon \,,\, \tau_{V_1} + 4\varepsilon)\,.$$

We now focus our attention on the radial velocity of never-zero solutions. The explicit formula is given by
$$\rho'(t) = \frac{\langle\, z'(t) \mid z(t) \,\rangle}{|z(t)|}$$

and, from Assumptions 2.1 and 3.1, we can find a function $M \in L^2([0, T], \mathbb{R})$ such that

(3.14) $$\begin{aligned} |\rho'(t)| &\leq &|g(t, z(t)) + p(t, z(t))| \\ &\leq &M(t)(1 + |z(t)|) = M(t)(1 + \rho(t))\,. \end{aligned}$$

We claim that, for every $r_1 > 0$ there exists $r_2 > r_1$ such that, if a solution z of (3.3) satisfies $|z(t_1)| = r_1$ for a certain $t_1 \in [0, T]$, then $|z(t)| < r_2$ for every $t \in [0, T]$ (cf. [15, Lemma 2.2]).

Indeed, we can set

(3.15) $$r_2 := 1 + \big(r_1 + \sqrt{T}\|M\|_2\big)\, e^{\sqrt{T}\|M\|_2}\,,$$

where M was introduced in (3.14). Let us consider a solution z of (3.3) as in the claim and assume by contradiction that there exists $t_2 \in [0, T]$ such that $|z(t_2)| = r_2$ and $r_1 < |z(t)| < r_2$ for every $t \in (t_1, t_2)$ (possibly with $t_2 < t_1$). From (3.14), applying Gronwall's Lemma, we get

$$|z(t)| \leq \big(r_1 + \sqrt{T}\|M\|_2\big)\, e^{\left|\int_{t_0}^t M(s)\,ds\right|}\,,$$

for every $t \in (t_1, t_2)$. So $|z(t_2)| < r_2$, giving a contradiction. The claim is thus proved.

The previous claim provides the validity of the following proposition as a counterpart.

PROPOSITION 3.3. *For every $r_1 > 0$ there exists $r_2 > r_1$, defined as in (3.15), such that if a solution of (3.3) satisfies $|z(\bar{t})| \geq r_2$ for a certain $\bar{t} \in [0, T]$, then $|z(t)| \geq r_1$ for every $t \in [0, T]$.*

When r_1 is equal to R_1 as defined after Equation (3.8), let R_2 be the corresponding r_2 value. In this way, any solution z of (3.3), with $|z(\bar{t})| \geq R_2$ for a certain $\bar{t} \in [0, T]$, satisfies $|z(t)| \geq R_1$ for every $t \in [0, T]$, so that (3.9)-(3.13) hold. In particular, if we want to estimate the time needed by such a solution in order to cover a certain angle, we simply have to sum the values in (3.9)-(3.13). So, recalling the definition of a_j^i and b_j^i given in (2.24), the previous computation gives the following proposition.

PROPOSITION 3.4. *If $\bar{z} \in l_S^i$ ($i = 1, 2$), with $|\bar{z}| \geq R_2$, then the solution $\Phi(\cdot, \bar{z})$, in order to cover the angle $j_0 \pi + \Delta\zeta$, needs a time*

$$\tau \in \left(a_{j_0}^i - (2j_0 + 1)\varepsilon, b_{j_0}^i + (2j_0 + 1)\varepsilon \right).$$

Hence, if $(a_{j_0}^i - (2j_0+1)\varepsilon, b_{j_0}^i + (2j_0+1)\varepsilon) \cap [0, T] = \varnothing$, then the angle $j_0\pi + \Delta\zeta$ cannot be covered by solutions departing from a point $\bar{z} \in l_S^i$, with $|\bar{z}| \geq R_2$, in the time interval $[0, T]$. Conversely, if $(a_{j_0}^i - (2j_0+1)\varepsilon, b_{j_0}^i + (2j_0+1)\varepsilon) \subset [0, T]$, then a solution departing from a point $\bar{z} \in l_S^i$, with $|\bar{z}| \geq R_2$, must cover such an angle, in the time interval $[0, T]$.

From the validity of (3.6), the choice of ε as in (3.8), and the definition of β_{j-1} and α_j given in (2.24), we have

$$b_{j-1}^i + (2j - 1)\varepsilon < T < a_j^i - (2j + 1)\varepsilon, \qquad (i = 1, 2).$$

Hence, we have the following proposition.

PROPOSITION 3.5. *For every $R > R_2$, setting $z^1 = R(\cos\zeta_S, \sin\zeta_S) \in l_S^1$ and $z^2 = -R(\cos\zeta_S, \sin\zeta_S) \in l_S^2$, then the covered-angle function in (3.5) satisfies*

$$\Delta\Theta(T, z^1) \in \left((j-1)\pi + \Delta\zeta, \, j\pi + \Delta\zeta \right).$$

$$\Delta\Theta(T, z^2) \in \left((j-1)\pi + \Delta\zeta, \, j\pi + \Delta\zeta \right).$$

In particular, $\Phi(T, z^1)$ and $\Phi(T, z^2)$ belongs to different connected components of $\mathbb{R}^2 \setminus l_A$.

By standard argument (cf. Figure 2) the curve $\eta : [-R, R] \to \mathbb{R}^2$, with $R > R_2$, defined as $\eta(\sigma) = \Phi\big(T, \sigma(\cos\zeta_S, \sin\zeta_S)\big)$ intersects the line l_A for a certain $\bar{\sigma}$ and this concludes the proof of Theorem 3.2.

3.2. Simple resonance. In this section we consider the case $T \in \partial\mathcal{I}$, where $\mathcal{I}$ was introduced in (2.26). In such a situation $\beta_{j-1} < T = \alpha_j$ or $\beta_{j-1} = T < \alpha_j$ for a certain $j \in \mathbb{N}$.

We can distinguish the following cases:

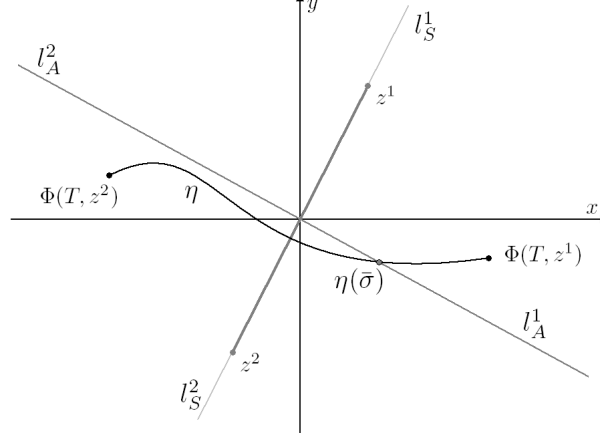(R1) $T = \alpha_j = a_j^1 < a_j^2,$

FIGURE 2. A sketch on Proposition 3.5 and an idea of the proof of Theorem 3.2.

(R2)
$$T = \alpha_j = a_j^2 < a_j^1 \,,$$

(R3)
$$T = \alpha_j = a_j^1 = a_j^2 \,,$$

(R4)
$$T = \beta_{j-1} = b_{j-1}^1 > b_{j-1}^2 \,,$$

(R5)
$$T = \beta_{j-1} = b_{j-1}^2 > b_{j-1}^1 \,,$$

(R6)
$$T = \beta_{j-1} = b_{j-1}^1 = b_{j-1}^2 \,.$$

In order to obtain the existence of a solution to (3.1) we need to introduce a Landesman-Lazer type condition. To this aim, let us define the functions

(3.16)
$$\mathcal{G}_i(t,z) = g(t,z) + p(t,z) - \nabla V_i(z) \,, \qquad (i = 1,2)$$

and introduce the following hypotheses:

(F−): There exists $\eta \in L^1([0,T], \mathbb{R})$ such that

$$\langle\, \mathcal{G}_1(t, \lambda z) \mid z \,\rangle \geq \eta(t) \,,$$

for almost every $t \in [0,T]$, for every $z \in \mathbb{R}^2$ satisfying $|z| \leq 1$ and for every $\lambda > 1$.

(F+): There exists $\eta \in L^1([0,T], \mathbb{R})$ such that

$$\langle\, \mathcal{G}_2(t, \lambda z) \mid z \,\rangle \leq \eta(t) \,,$$

for almost every $t \in [0,T]$, for every $z \in \mathbb{R}^2$ satisfying $|z| \leq 1$ and for every $\lambda > 1$.

The previous hypotheses are plainly satisfied if the function p is bounded. Indeed, we can compute

$$\langle \, \mathcal{G}_1(t, \lambda z) \mid z \, \rangle = \langle \, \gamma(t, \lambda z)\big(\nabla V_2(\lambda z) - \nabla V_1(\lambda z)\big) + p(t, \lambda z) \mid z \, \rangle$$
$$= 2\lambda \gamma(t, \lambda z)\big(V_2(z) - V_1(z)\big) + \langle \, p(t, \lambda z) \mid z \, \rangle \geq -|p(t, \lambda z)|$$

and

$$\langle \, \mathcal{G}_2(t, \lambda z) \mid z \, \rangle = \langle \, \big(1 - \gamma(t, \lambda z)\big)\big(\nabla V_1(\lambda z) - \nabla V_2(\lambda z)\big) + p(t, \lambda z) \mid z \, \rangle$$
$$= 2\lambda\big(1 - \gamma(t, \lambda z)\big)\big(V_1(z) - V_2(z)\big) + \langle \, p(t, \lambda z) \mid z \, \rangle \leq |p(t, \lambda z)| \,,$$

for a.e. $t \in [0, T]$, every $z \in \mathbb{R}^2$ satisfying $|z| \leq 1$ and every $\lambda > 1$.

Finally, we introduce the functions

$$(3.17) \qquad \mathcal{J}^-(\theta) = \int_0^T \liminf_{(\lambda, \omega) \to (+\infty, \theta)} \langle \, \mathcal{G}_1(t, \lambda\varphi_{V_1}(t + \omega)) \mid \varphi_{V_1}(t + \omega) \, \rangle \, dt \,,$$

$$(3.18) \qquad \mathcal{J}^+(\theta) = \int_0^T \limsup_{(\lambda, \omega) \to (+\infty, \theta)} \langle \, \mathcal{G}_2(t, \lambda\varphi_{V_2}(t + \omega)) \mid \varphi_{V_2}(t + \omega) \, \rangle \, dt \,.$$

The Landesman-Lazer type assumptions we need to require can be summarized as follows.

ASSUMPTION 3.6 LANDESMAN-LAZER TYPE ASSUMPTIONS.

If (R1) holds, assume (F+) and $\quad \mathcal{J}^+(\tau_{0,V_2}) < 0 \,,$

if (R2) holds, assume (F+) and $\quad \mathcal{J}^+(\tau_{0,V_2} + \tau_{1,V_2} + \sigma_{1,V_2}) < 0 \,,$

if (R3) holds, assume (F+) and $\quad$ both $\mathcal{J}^+(\tau_{0,V_2}) < 0$ and

$$\mathcal{J}^+(\tau_{0,V_2} + \tau_{1,V_2} + \sigma_{1,V_2}) < 0 \,;$$

if (R4) holds, assume (F−) and $\quad \mathcal{J}^-(\tau_{0,V_1}) > 0 \,,$

if (R5) holds, assume (F−) and $\quad \mathcal{J}^-(\tau_{0,V_1} + \tau_{1,V_1} + \sigma_{1,V_1}) > 0 \,,$

if (R6) holds, assume (F−) and $\quad$ both $\mathcal{J}^-(\tau_{0,V_1}) > 0$ and

$$\mathcal{J}^-(\tau_{0,V_1} + \tau_{1,V_1} + \sigma_{1,V_1}) > 0 \,.$$

In the case of *simple resonance* only one of the alternatives (R1)-(R6) is satisfied, and so we need to check only one of the alternatives in Assumption 3.6.

REMARK 3.7. In [2] the case $V_1 = V_2$ has been studied. In such a situation $a_j^i = b_j^i$ for every $j \in \mathbb{N}$ and $i \in \{1, 2\}$. In particular Theorem 2.1 and Theorem 2.3-2 in [2] treat the case "j is even", while Theorem 2.3-1 and Theorem 2.3-3 the case "j is odd".

For example, in [2, Theorem 2.1] the situations (17),(18),(19) or (20) correspond respectively to (R1), (R4), (R3) and (R6).

Moreover, in [2] the function p is asked to be bounded. The weaker hypothesis on p introduced in Assumption 3.1, forces us to introduce the additional hypotheses (F±). This possibility was already suggested in [2], referring to [1]. Conditions (F±) are the natural extension to the case $V_1 \neq V_2$ of the ones suggested in [1, Proposition 4.1].

The existence result in the case of simple resonance is the following.

THEOREM 3.8 SIMPLE RESONANCE. *Consider problem* (3.1), *where p satisfies Assumption 3.1 and g satisfies Assumption 2.1. Assume $T \in \partial \mathcal{I}$, where $\mathcal{I}$ was introduced in* (2.26) *and that Landesman-Lazer Assumption 3.6 is fulfilled.*
Then there exists at least one solution of (3.1).

PROOF. Let us start assuming that (R1) holds, in particular $\beta_{j-1} < T = \alpha_j$.
Let us consider $z^2 \in l_S^2$ and the solution $\Phi(\cdot, z^2)$ of (3.3). By Proposition 3.4, if $|z^2|$ is sufficiently large then

$$(3.19) \qquad \Delta\Theta(T, z^2) \in \left((j-1)\pi + \Delta\zeta \,,\, j\pi + \Delta\zeta\right).$$

In fact, $b_{j-1}^2 < T < a_j^2$ holds.
Let us now consider $z^1 \in l_S^1$ and the solution $\Phi(\cdot, z^1)$ of (3.3). Arguing similarly, if $|z^1|$ is sufficiently large then

$$(3.20) \qquad \Delta\Theta(T, z^1) \in \left((j-1)\pi + \Delta\zeta \,,\, (j+1)\pi + \Delta\zeta\right).$$

Notice that the interval is larger since $b_{j-1}^1 < T = a_j^1$. We need to prove that the situation $\Delta\Theta(T, z^1) \in \left[j\pi + \Delta\zeta \,,\, (j+1)\pi + \Delta\zeta\right)$ is forbidden. Once this claim is proved, we will obtain the existence of $R > 0$ with the property explained in Proposition 3.5, thus permitting us to conclude the proof of the theorem as in the previous section.

We argue by contradiction and suppose the existence of a sequence $(z_n^0)_n \subset l_S^1$, with $|z_n^0| \to \infty$, such that

$$(3.21) \qquad \Delta\Theta(T, z_n^0) \in \left[j\pi + \Delta\zeta \,,\, (j+1)\pi + \Delta\zeta\right).$$

Set $z_n(t) = \Phi(t, z_n^0)$ (in particular $\|z_n\|_\infty \to \infty$ as $n \to \infty$ by construction), and introduce the sequence

$$(3.22) \qquad w_n = \frac{z_n}{\|z_n\|_\infty} = \frac{\Phi(\cdot, z_n^0)}{\|\Phi(\cdot, z_n^0)\|_\infty}$$

consisting of solutions to

$$(3.23) \qquad \begin{cases} Jw_n' = (1 - \Gamma_n(t))\nabla V_1(w_n) + \Gamma_n(t)\nabla V_2(w_n) + p_n(t) \,, \\ w_n(0) \in l_S^1 \,, \end{cases}$$

where $\Gamma_n(t) := \gamma(t, w_n(t)\|z_n\|_\infty)$ and $p_n(t) := \dfrac{p(t, w_n(t)\|z_n\|_\infty)}{\|z_n\|_\infty}$. Notice that $\|w_n\|_\infty = 1$ and so $\|w_n\|_2 \le \sqrt{T}$. From Assumption 3.1, for every $\delta > 0$, we can find a function $\ell_\delta \in L^2([0,T], \mathbb{R})$ such that

$$|p(t,z)| \le \delta|z| + \ell_\delta(t) .$$

In particular we have

$$(3.24) \qquad |p_n(t)| \le \delta|w_n(t)| + \frac{\ell_\delta(t)}{\|z_n\|_\infty} .$$

Since w_n solves the differential equation in (3.23), then $(w_n')_n$ is bounded in $L^2([0,T], \mathbb{R}^2)$, thus giving us that $(w_n)_n$ is bounded in $H^1([0,T], \mathbb{R}^2)$.

Then, there exists a function $w \in H^1([0,T], \mathbb{R}^2)$, such that (up to a subsequence) $w_n \rightharpoonup w$ weakly in $H^1([0,T], \mathbb{R}^2)$ and $w_n \to w$ uniformly. In particular $w \ne 0$ and $w(0) \in l_S^1$.

Moreover, the sequence $(\Gamma_n)_n$ is bounded in $L^2([0,T], \mathbb{R})$ and converges weakly to a certain $\Gamma \in L^2([0,T], \mathbb{R})$, up to a subsequence. The sequence $(\Gamma_n)_n$ is contained in the closed convex subset $\{q \in L^2([0,T], \mathbb{R}) \mid 0 \le q(t) \le 1 \text{ a.e. in } [0,T]\}$, so that we have also $0 \le \Gamma \le 1$ a.e. in $[0,T]$.

Concerning the sequence $(p_n)_n$, from the validity of (3.24), we can prove that $\limsup_n \|p_n\|_2 \le 2\delta\sqrt{T}$, for every $\delta > 0$. So, we have $p_n \to 0$ in $L^2([0,T], \mathbb{R}^2)$.

Then, passing to the weak limit in (3.23) we get

$$(3.25) \qquad \begin{cases} Jw' = (1 - \Gamma(t))\nabla V_1(w) + \Gamma(t)\nabla V_2(w) , \\ w(0) \in l_S^1 . \end{cases}$$

We claim that $\Gamma = 1$ a.e. in $[0,T]$.

From (3.21), setting $w_n^0 = w_n(0) = \dfrac{z_n^0}{\|z_n\|_\infty}$ and using polar coordinates $w_n = \varrho_n e^{i\theta_n}$, we get $\theta_n(0) - \theta_n(T) \in \big[j\pi + \Delta\zeta , (j+1)\pi + \Delta\zeta \big)$. Hence, setting $w = \varrho e^{i\theta}$,

$$(3.26) \qquad \theta(0) - \theta(T) \in \big[j\pi + \Delta\zeta , (j+1)\pi + \Delta\zeta \big] ,$$

so that there exists $\bar{t} \in [0,T]$ such that $\theta(0) - \theta(\bar{t}) = j\pi + \Delta\zeta$. Hence, integrating

$$-\frac{\theta'(t)}{2V_2(\cos\theta(t), \sin\theta(t))} \le 1$$

in the interval $[0, \bar{t}]$ we get

$$(3.27) \qquad \int_{\zeta_S - (j\pi + \Delta\zeta)}^{\zeta_S} \frac{d\theta}{2V_2(\cos\theta, \sin\theta)} = a_j^1 \le \bar{t} .$$

By hypothesis $a_j^1 = T$, hence $\bar{t} = T$ holds.

We have proved that $w(T) \in l_A$ and w covers the angle $j\pi + \Delta\zeta$ in the interval $[0,T]$.

Let us now parametrize w using the polar coordinates induced by φ_{V_2}:

$$(3.28) \qquad w(t) = r(t)\varphi_{V_2}(t + \omega(t)).$$

A standard computation provides

$$(3.29) \qquad r'(t) = -r(t)(1 - \Gamma(t))\langle \nabla V_1(\varphi_{V_2}(t + \omega(t))) \mid \varphi'_{V_2}(t + \omega(t))\rangle,$$

$$(3.30) \qquad \omega'(t) = (1 - \Gamma(t))(2V_1(\varphi_{V_2}(t + \omega(t))) - 1).$$

Notice that $\omega(0) = \tau_{0,V_2}$ and $\omega(T) = \tau_{0,V_2} + a_j^1 - T = \tau_{0,V_2}$, so

$$(3.31) \qquad 0 = \int_0^T \omega'(t)\,dt = \int_0^T (1 - \Gamma(t))(2V_1(\varphi_{V_2}(t + \omega(t))) - 1)\,dt.$$

Recalling that $V_1 \leq V_2$, we get $2V_1(\varphi_{V_2}) - 1 \leq 2V_2(\varphi_{V_2}) - 1 = 0$ so

$$(1 - \Gamma(t))(2V_1(\varphi_{V_2}(t + \omega(t))) - 1) \leq 0 \qquad \text{a.e. in } [0, T].$$

Hence, from (3.31), we necessarily have

$$(3.32) \qquad (1 - \Gamma(t))(2V_1(\varphi_{V_2}(t + \omega(t))) - 1) = 0 \qquad \text{a.e. in } [0, T],$$

in particular, using (3.30), $\omega = \tau_{0,V_2}$ a.e. in $[0, T]$.

Let us now focus our attention on the radial velocity formula (3.29). We are going to prove that $r' = 0$ almost everywhere in $[0, T]$.

Let us consider $t_0 \in [0, T]$ such that $\Gamma(t_0) < 1$ (the situation is trivial if $\Gamma(t_0) = 1$). By (3.32) we necessarily have $2V_1(\varphi_{V_2}(t_0 + \omega(t_0))) = 1$. Recalling that $V_2 \geq V_1$ and $2V_2(\varphi_{V_2}) \equiv 1$, we find that t_0 is a minimum of the function $\mathcal{V}(t) = V_2(\varphi_{V_2}(t + \omega(t))) - V_1(\varphi_{V_2}(t + \omega(t)))$, precisely $\mathcal{V}(t_0) = 0$, so that $\mathcal{V}'(t_0) = 0$. Since V_2 is constant along φ_{V_2} we get $\frac{d}{dt}V_1(\varphi_{V_2}(\cdot + \omega))\big|_{t=t_0} = 0$ and consequently $\langle \nabla V_1(\varphi_{V_2}(t_0 + \omega(t_0))) \mid \varphi'_{V_2}(t_0 + \omega(t_0))\rangle = 0$, giving $r'(t_0) = 0$.

We have proved that $w = C\varphi_{V_2}(t + \tau_{0,V_2})$ for a certain constant $C > 0$, so that $\Gamma = 1$ a.e. in $[0, T]$.

Let us consider again the sequence $(w_n)_n$ introduced in (3.22) and the polar coordinates $w_n = \varrho_n e^{i\theta_n}$. We have

$$(3.33) \qquad \int_{\theta_n(T)}^{\theta_n(0)} \frac{d\theta}{2V_2(\cos\theta, \sin\theta)} \geq T,$$

for large indices n by the validity of (3.21): indeed, the integral provides the time spent by a solution of the system $Jz' = \nabla V_2(z)$ to cover the angular sector between $\theta_n(0) = \zeta_S$ and $\theta_n(T) \leq \zeta_S - (j\pi + \Delta\zeta)$, while T is the time spent to cover the (not larger) angular sector between ζ_S and $\zeta_S - (j\pi + \Delta\zeta)$.

The angular speed of w_n is given by

$$
\begin{aligned}
-\theta'_n(t) &= \frac{\langle\, Jw'_n(t) \mid w_n(t)\,\rangle}{|w_n(t)|^2} = \frac{\langle\, Jz'_n(t) \mid z_n(t)\,\rangle}{|z_n(t)|^2} \\
&= 2V_2(\cos\theta_n(t), \sin\theta_n(t)) \\
&\qquad + \frac{\langle\, g(t, z_n(t)) + p(t, z_n(t)) - \nabla V_2(z_n(t)) \mid z_n(t)\,\rangle}{|z_n(t)|^2}\,,
\end{aligned}
$$

thus giving

$$
\int_{\theta_n(T)}^{\theta_n(0)} \frac{d\theta}{2V_2(\cos\theta, \sin\theta)} = T + \int_0^T \frac{\langle\, \mathcal{G}_2(t, z_n(t)) \mid z_n(t)\,\rangle}{2V_2(z_n(t))}\, dt\,,
$$

where $\mathcal{G}_2(t, z) = g(t, z) + p(t, z) - \nabla V_2(z)$. By (3.33), we obtain

$$
\mathcal{X}_n := \int_0^T \frac{\langle\, \mathcal{G}_2(t, z_n(t)) \mid z_n(t)\,\rangle}{2V_2(z_n(t))} \geq 0
$$

for sufficiently large indices n. We parametrize the solutions z_n in the polar coordinates induced by φ_{V_2}:

$$
z_n(t) = r_n(t)\varphi_{V_2}(t + \omega_n(t))\,.
$$

So, we obtain

$$
\mathcal{X}_n = \int_0^T \frac{\langle\, \mathcal{G}_2\big(t, r_n(t)\varphi_{V_2}(t + \omega_n(t))\big) \mid \varphi_{V_2}(t + \omega_n(t))\,\rangle}{r_n(t)}\,.
$$

From (3.22) and recalling that $w_n \to w = C\varphi_{V_2}(t + \tau_{0,V_2})$ uniformly, we have $\frac{r_n(t)}{\|z_n\|_\infty} \to C$ and $\omega_n \to \tau_{0,V_2}$. Then

$$
\begin{aligned}
0 &\leq \limsup_{n \to +\infty} \|z_n\|_\infty \mathcal{X}_n \\
&\leq \int_0^T \limsup_{n \to +\infty} \frac{\langle\, \mathcal{G}_2(t, r_n(t)\varphi_{V_2}(t + \omega_n(t))) \mid \varphi_{V_2}(t + \omega_n(t))\,\rangle}{\frac{r_n(t)}{\|z_n\|_\infty}}\, dt \\
&\leq \frac{1}{C}\int_0^T \limsup_{(\lambda,\omega)\to(+\infty,\tau_{0,V_2})} \langle\, \mathcal{G}_2(t, \lambda\varphi_{V_2}(t + \omega)) \mid \varphi_{V_2}(t + \omega)\,\rangle\, dt
\end{aligned}
$$

(we can apply Fatou's lemma since (F+) holds).

The last inequality, using (3.18), can be rewritten as $\mathcal{J}^+(\tau_{0,V_2}) \geq 0$ which contradicts Assumption 3.6.

We have proved the theorem if (R1) holds.

Let us now spend few words in order to explain how to adapt the proof in the other situations.

Assume (R2). In such a situation the role of l_S^1 and l_S^2 is switched. So, the "good estimate" in (3.19) is easily obtained for $z^1 \in l_S^1$, with $|z^1|$ sufficiently large, while the "bad estimate" (3.20) occurs treating $z^2 \in l_S^2$. In this case, we assume by contradiction the existence of a sequence $(z_n^0)_n \subset l_S^2$, with $|z_n^0| \to \infty$,

satisfying (3.21). The proof can be plainly adapted, but we underline the main differences: the starting angle $\Theta(0, z_n^0) = \zeta_S$ is replaced by $\Theta(0, z_n^0) = \zeta_S + \pi$ and the constant a_j^1 is replaced by a_j^2. In particular, (3.27) becomes

$$\int_{\zeta_S + \pi - (j\pi + \Delta\zeta)}^{\zeta_S + \pi} \frac{d\theta}{2V_2(\cos\theta, \sin\theta)} = a_j^2 \leq \bar{t}.$$

Finally, the limit function is now $w(t) = C\varphi_{V_2}(t + \tau_{0,V_2} + \tau_{1,V_2} + \sigma_{1,V_2})$.

Assume now (R3). Under this hypothesis, we get the "bad estimate" (3.20) both for $z_0 \in l_S^1$ and $z_0 \in l_S^2$ with $|z_0|$ large. So, the proof is a *gluing* of cases (R1) and (R2). The same reasoning holds for (R6): the proof will follow by the ones of cases (R4) and (R5) we are going to provide.

Let us consider (R4). The validity of (3.19) is given when we treat solutions $\Phi(\cdot, z_0)$ of (3.3) with $z_0 \in l_S^2$ with $|z_0|$ sufficiently large. However, solutions $\Phi(\cdot, z_0)$ of (3.3) with $z_0 \in l_S^1$ satisfies (no more (3.20), but)

$$(3.34) \qquad \Delta\Theta(T, z_0) \in \big((j-2)\pi + \Delta\zeta, \, j\pi + \Delta\zeta\big).$$

and we need to forbid the situation $\Delta\Theta(T, z_0) \in \big((j-2)\pi + \Delta\zeta, \, (j-1)\pi + \Delta\zeta\big]$. Arguing as above, we can consider a diverging sequence $(z_n^0)_n \subset \ell_S^1$ such that

$$\Delta\Theta(T, z_n^0) \in \big((j-2)\pi + \Delta\zeta, \, (j-1)\pi + \Delta\zeta\big].$$

We introduce similarly the sequence $(w_n)_n$, and prove that it converges to a solution w of (3.25). Now, we claim that $\Gamma = 0$, a.e. in $[0, T]$. In this case (3.26) is replaced by

$$(3.35) \qquad \theta(0) - \theta(T) \in \big[(j-2)\pi + \Delta\zeta, \, (j-1)\pi + \Delta\zeta\big].$$

Then, we introduce $\bar{t} > T$ such that $\theta(0) - \theta(\bar{t}) = (j-1)\pi + \Delta\zeta$ and compute, since $-\frac{\theta'(t)}{2V_1(\cos\theta(t), \sin\theta(t))} \geq 1$, the validity of

$$\int_{\zeta_S - ((j-1)\pi + \Delta\zeta)}^{\zeta_S} \frac{d\theta}{2V_1(\cos\theta, \sin\theta)} = b_j^1 \geq \bar{t}$$

bringing us to the conclusion $T = b_j^1 \geq \bar{t} \geq T$, hence $T = \bar{t}$.

Once proved that w cover the angle $(j-1)\pi + \Delta\zeta$ in the interval $[0, T]$, we can introduce the polar coordinates induced by φ_{V_1}:

$$(3.36) \qquad w(t) = r(t)\varphi_{V_1}(t + \omega(t)).$$

A standard computation provides

$$(3.37) \qquad r'(t) = -r(t)\Gamma(t)\big\langle \nabla V_2(\varphi_{V_1}(t + \omega(t))) \mid \varphi'_{V_1}(t + \omega(t)) \big\rangle,$$

$$(3.38) \qquad \omega'(t) = \Gamma(t)(2V_2(\varphi_{V_1}(t + \omega(t))) - 1).$$

Again $\omega(0) = \omega(T) = \tau_{0,V_1}$ and we can similarly conclude that $\Gamma = 0$ almost everywhere in $[0, T]$, thus obtaining $w(t) = C\varphi_{V_1}(t + \tau_{0,V_1})$ for a suitable positive constant C. Then, introducing standard polar coordinates for the sequence $(w_n)_n$, we can prove that

$$\int_{\theta_n(T)}^{\theta_n(0)} \frac{d\theta}{2V_1(\cos\theta, \sin\theta)} \leq T$$

for large indices n. Similarly as above we can compute

$$\int_{\theta_n(T)}^{\theta_n(0)} \frac{d\theta}{2V_1(\cos\theta, \sin\theta)} = T + \int_0^T \frac{\langle \mathcal{G}_1(t, z_n(t)) \mid z_n(t) \rangle}{2V_1(z_n(t))} =: T + \mathcal{X}_n\,,$$

where $\mathcal{G}_1(t, z) = g(t, z) + p(t, z) - \nabla V_1(z)$ and $\mathcal{X}_n \leq 0$ for sufficiently large indices n. Parameterizing the solutions z_n in the polar coordinates induced by φ_{V_1}, i.e. $z_n(t) = r_n(t)\varphi_{V_1}(t + \omega_n(t))$ we obtain

$$
\begin{aligned}
0 &\geq \liminf_{n\to+\infty} \|z_n\|_\infty \mathcal{X}_n \\
&\geq \int_0^T \liminf_{n\to+\infty} \frac{\langle \mathcal{G}_1(t, r_n(t)\varphi_{V_1}(t + \omega_n(t))) \mid \varphi_{V_1}(t + \omega_n(t)) \rangle}{\frac{r_n(t)}{\|z_n\|_\infty}} \, dt \\
&\geq \frac{1}{C} \int_0^T \liminf_{(\lambda,\omega)\to(+\infty,\tau_{0,V_1})} \langle \mathcal{G}_1(t, \lambda\varphi_{V_1}(t + \omega)) \mid \varphi_{V_1}(t + \omega) \rangle \, dt\,.
\end{aligned}
$$

(we can apply Fatou's lemma since (F–) holds).

Finally, the last inequality, using (3.17), can be rewritten as $\mathcal{J}^-(\tau_{0,V_2}) \leq 0$ which contradicts Assumption 3.6.

This prove the case (R4).

The modification needed to prove the case (R5) from the previous situation, are similar to the ones provided when (R2) holds.

3.3. Double resonance. In this section we consider the case $T \in \partial\widetilde{\mathcal{I}} \setminus \partial\mathcal{I}$, where $\mathcal{I}$ and $\widetilde{\mathcal{I}}$ were introduced in (2.26) and (2.27). In particular, $T = \beta_{j-1} = \alpha_j$ for a certain $j \in \mathbb{N}$.

In such a situation one among the alternatives (R1)–(R3) is fulfilled and one among (R4)–(R6). We thus need to introduce a double Landesman-Lazer type condition in order to find solutions to (3.1). Roughly speaking, the validity of two of the requirements in Assumption 3.6 is necessary.

Theorem 3.9 Double resonance. *Consider the problem* (3.3)*, where g satisfies Assumption 2.1 and p satisfies Assumption 3.1. Assume* $T \in \partial\widetilde{\mathcal{I}} \setminus \partial\mathcal{I}$ *and that Landesman-Lazer Assumptions 3.6 are fulfilled.*

Then there exists at least one solution of (3.1)*.*

PROOF. We focus our attention on the situation which presents all the difficulties: we thus assume that both (R3) and (R6) hold.

Let us consider $z_0 \in l_S = l_S^1 \cup l_S^2$ and the solution $\Phi(\cdot, z_0)$ of (3.3). If $|z_0|$ is sufficiently large, we have (no more (3.20) or (3.34), but)

$$(3.39) \qquad \Delta\Theta(T, z_0) \in \big((j-2)\pi + \Delta\zeta, (j+1)\pi + \Delta\zeta\big).$$

We need to avoid the situation

$$\Delta\Theta(T, z_0) \in \big((j-2)\pi + \Delta\zeta, (j-1)\pi + \Delta\zeta\big] \cup \big[j\pi + \Delta\zeta, (j+1)\pi + \Delta\zeta\big).$$

for every $z_0 \in l_S$ such that $|z_0|$ is sufficiently large.

We assume the existence of four diverging sequences: $(z_{n,1}^0)_n \subset \ell_S^1$, $(z_{n,2}^0)_n \subset \ell_S^1$, $(z_{n,3}^0)_n \subset \ell_S^2$, $(z_{n,4}^0)_n \subset \ell_S^2$, such that, for every $n \in \mathbb{N}$,

$$\begin{aligned}
\Delta\Theta(T, z_{n,1}^0) &\in \big((j-2)\pi + \Delta\zeta, (j-1)\pi + \Delta\zeta\big], \\
\Delta\Theta(T, z_{n,2}^0) &\in \big[j\pi + \Delta\zeta, (j+1)\pi + \Delta\zeta\big), \\
\Delta\Theta(T, z_{n,3}^0) &\in \big((j-2)\pi + \Delta\zeta, (j-1)\pi + \Delta\zeta\big], \\
\Delta\Theta(T, z_{n,4}^0) &\in \big[j\pi + \Delta\zeta, (j+1)\pi + \Delta\zeta\big).
\end{aligned}$$

For all the four sequences, thanks to the validity of the Landesman-Lazer Assumption 3.6, adapting the proof of Theorem 3.8 we will get a contradiction.

In this way, we obtain the existence of $R > 0$ such that, if $z_0 \in l_S^2 \cup l_S^1$ and $|z_0| > R$ then the solution $\Phi(\cdot, z_0)$ of (3.3) satisfies

$$\Delta\Theta(T, z_0) \in \big((j-1)\pi + \Delta\zeta, j\pi + \Delta\zeta\big).$$

Hence, we are in the situation of Proposition 3.5 and we can conclude as in the proof of Theorem 3.2.

4. Applications to asymmetric nonlinearities

In this section we focus our attention on scalar differential equations

$$(4.1) \qquad x'' + f(t, x) = 0,$$

where f is a continuous function satisfying

$$(4.2) \qquad 0 < \nu_1 \le \liminf_{x \to -\infty} \frac{f(t,x)}{x} \le \limsup_{x \to -\infty} \frac{f(t,x)}{x} \le \nu_2,$$

$$(4.3) \qquad 0 < \mu_1 \le \liminf_{x \to +\infty} \frac{f(t,x)}{x} \le \limsup_{x \to +\infty} \frac{f(t,x)}{x} \le \mu_2,$$

uniformly with respect to t (we assume f to be continuous just to simplify the argument).

We refer to [2, Section 3] for a comparison with the case $\nu = \nu_1 = \nu_2$, $\mu = \mu_1 = \mu_2$. In this section we extend the results presented there. In particular, we will focus our attention only on the Dirichlet problems. The case

of problems with Neumann boundary conditions or mixed boundary conditions $x'(0) = x(T) = 0$, which are treated in [2] too, is left to the reader as an exercise, for briefness.

Setting $z(t) = (x(t), x'(t))$, we can write equation (4.1) in the form of a planar system as in (1.8) where $2V_i = x'(t)^2 + \mu_i(x^+(t))^2 + \nu_i(x^-(t))^2$. The planar system $Jz' = \nabla V_i(z)$, is nothing else but the asymmetric oscillator

$$x'' + \mu_i x^+ - \nu_i x^- = 0,$$

and admits periodic solutions of period $\tau_{V_i} = \dfrac{\pi}{\sqrt{\mu_i}} + \dfrac{\pi}{\sqrt{\nu_i}}$ of the form $z(t) = (x(t), x'(t))$ with $x(t) = C\phi_{\mu_i,\nu_i}(t + t_0)$ where $C \in \mathbb{R}^+$, $t_0 \in \mathbb{R}$ and

$$\phi_{\mu_i,\nu_i}(t) := \begin{cases} \dfrac{1}{\sqrt{\mu_i}} \sin\left(\sqrt{\mu_i}t\right) & t \in \left[0, \dfrac{\pi}{\sqrt{\mu_i}}\right], \\[2mm] \dfrac{1}{\sqrt{\nu_i}} \sin\left(\sqrt{\nu_i}\left(\dfrac{\pi}{\sqrt{\mu_i}} - t\right)\right) & t \in \left[\dfrac{\pi}{\sqrt{\mu_i}}, \dfrac{\pi}{\sqrt{\mu_i}} + \dfrac{\pi}{\sqrt{\nu_i}}\right]. \end{cases}$$

Concerning the problem with Dirichlet boundary conditions, we can compute the constants introduced in Section 2.1:

$$\tau_{0,V_i} = \sigma_{1,V_i} = \sigma_{2,V_i} = 0, \quad \tau_{1,V_i} = \frac{\pi}{\sqrt{\mu_i}}, \quad \tau_{2,V_i} = \frac{\pi}{\sqrt{\nu_i}}.$$

In particular, the scalar problem

$$(4.4) \qquad \begin{cases} x'' + \mu_i x^+ - \nu_i x^- = 0, \\ x(0) = 0 = x(T), \end{cases}$$

has nontrivial solutions if one of the following identities holds for a certain $k \in \mathbb{N}$:

$$(4.5) \qquad T = \alpha_{i,k} := \pi \left[(k+1)\frac{1}{\sqrt{\mu_i}} + k\frac{1}{\sqrt{\nu_i}}\right],$$

$$(4.6) \qquad T = \beta_{i,k} := \pi \left[k\frac{1}{\sqrt{\mu_i}} + (k+1)\frac{1}{\sqrt{\nu_i}}\right],$$

$$(4.7) \qquad T = \gamma_{i,k} := \pi(k+1)\left[\frac{1}{\sqrt{\mu_i}} + \frac{1}{\sqrt{\nu_i}}\right].$$

They are indeed equivalent to (2.10)-(2.13). In particular, ϕ_{μ_i,ν_i} solves (4.4) when (4.5) or (4.7) holds, while

$$\psi_{\mu_i,\nu_i}(t) := \phi_{\mu_i,\nu_i}\left(t + \pi/\sqrt{\mu_i}\right)$$

solves it when (4.6) or (4.7) holds.

Let us now focus our attention on the Dirichlet problem

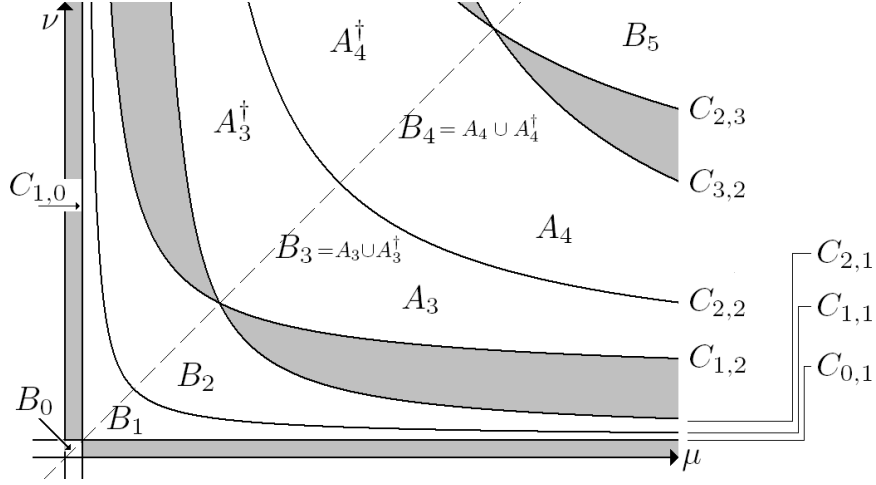$$(4.8) \qquad \begin{cases} x'' + f(t,x) = 0, \\ x(0) = 0 = x(T), \end{cases}$$

FIGURE 3. The Dancer-Fučík spectrum for the Dirichlet problem. The open rectangle $\mathcal{R} = (\mu_1, \mu_2) \times (\nu_1, \nu_2)$ cannot intersect the forbidden regions in gray and the spectrum lines $C_{a,b}$. However, the vertices of $\mathcal{R}$ can intersect the spectrum (resonance situation).

where f satisfies (4.2) and (4.3). The resonance set $\mathcal{I}$ in (2.26) is now

$$\mathcal{I} = \bigcup_{k \geq 0} [\min\{\alpha_{2,k}, \beta_{2,k}\}, \max\{\alpha_{1,k}, \beta_{1,k}\}] \cup [\gamma_{2,k}, \gamma_{1,k}] \,.$$

The resonance set $\mathcal{I}$ is useful when we want to investigate resonance phenomena when T varies and the constants $\mu_1, \mu_2, \nu_1, \nu_2$ are fixed a priori. On the contrary, fixing T, we can study resonance when the other constants change using the set Σ_D known as the Dancer-Fučík spectrum associated to (4.4). We recall that, for a fixed T, the set Σ_D collects all the couples (μ, ν) in the first quadrant $Q = (\mathbb{R}^+)^2$ satisfying one among (4.5)-(4.7), and it consists of an infinite number of curves, see Figure 3,

$$C_{a,b} = \left\{ (\mu, \nu) \in Q \mid \frac{a}{\sqrt{\mu}} + \frac{b}{\sqrt{\nu}} = \frac{T}{\pi} \right\} \,,$$

where

$$(a, b) \in \Gamma := \{(a, b) \in \mathbb{N}^2 \mid a + b > 0 \,, |a - b| \leq 1\}$$
$$= \{(0, 1), (1, 0), (1, 1), (1, 2), (2, 1), (2, 2), \ldots\} \,.$$

Notice that the curves $C_{k,k+1}$ and $C_{k+1,k}$ intersect in the point (λ_k, λ_k), with $\lambda_k = (\pi(2k + 1)/T)^2$. We can write

$$\Sigma_D = \bigcup_{(a,b) \in \Gamma} C_{a,b} \,.$$

Let us introduce the sets

$$A_0 = \{(\mu,\nu) \in Q \mid \mu \geq \nu \,, \pi > T\sqrt{\mu}\,\}$$

and, for every integer $j \geq 1$,

$$A_{2j-1} = \left\{ (\mu,\nu) \in Q \,\Big|\, \mu \geq \nu \,;\, j\left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right) - \frac{\pi}{\sqrt{\mu}} < T < j\left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right) \right\},$$

$$A_{2j} = \left\{ (\mu,\nu) \in Q \,\Big|\, \mu \geq \nu \,;\, j\left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right) < T < j\left(\frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\nu}}\right) + \frac{\pi}{\sqrt{\mu}} \right\}.$$

Then we define $A_k^\dagger = \{(\mu,\nu) \in Q \mid (\nu,\mu) \in A_k\}$, which is the reflection of A_k with respect to the line $\mu = \nu$. Finally, define $B_k = A_k \cup A_k^\dagger$ and $B = \bigcup_{k \geq 0} B_k$, see Figure 3.

Concerning the resonance phenomenon, we can summarize the possible situations by checking the position of the open rectangle $\mathcal{R} = (\mu_1,\mu_2) \times (\nu_1,\nu_2)$ with respect to the Dancer-Fučík spectrum as follows:

- *Nonresonance:* $\overline{\mathcal{R}} \subset B$,
- *Resonance:* $\mathcal{R} \subset B$ and $\overline{\mathcal{R}} \cap \Sigma_D \neq \varnothing$. In particular, in such a situation, one (*simple resonance*) or both (*double resonance*) the points (μ_1,ν_1) and (μ_2,ν_2) belong to Σ_D.

In particular, we need to avoid the situation "$\mathcal{R}$ *intersects* $Q \setminus (B \cup \Sigma_D)$" (roughly speaking, $\mathcal{R}$ cannot intersect the gray-colored region in Figure 3).

Let us here recall the results by Drábek in [6] for a comparison. There, equation (4.1) is treated assuming (4.2) and (4.3); a double resonance situation is considered but introducing a single Landesman-Lazer condition: indeed the author asked the validity of both (4.2) and (4.3) with *strict inequality on one side in a set of positive measure*, thus "escaping from resonance" from that side.

We now focus our attention on the Landesman-Lazer conditions we need to introduce. Hypotheses (F+) and (F−) becomes (cf. [1, Corollary 4.1]):

(f−): There exists $\eta \in L^1([0,T],\mathbb{R})$ such that

$$\mathrm{sgn}(x)\big(f(t,x) - \mu_1 x^+ + \nu_1 x^-\big) \geq \eta(t)\,,$$

for every $t \in [0,T]$, and every $x \in \mathbb{R}$.

(f+): There exists $\eta \in L^1([0,T],\mathbb{R})$ such that

$$\mathrm{sgn}(x)\big(f(t,x) - \mu_2 x^+ + \nu_2 x^-\big) \leq \eta(t)\,,$$

for every $t \in [0,T]$, and every $x \in \mathbb{R}$.

Then, we define the values

$$\mathcal{A}^-(\zeta) := \int_{\{\zeta>0\}} \left(\liminf_{x\to+\infty} f(t,x) - \mu_1 x \right) \zeta(t)\,dt +$$
$$\int_{\{\zeta<0\}} \left(\limsup_{x\to-\infty} f(t,x) - \nu_1 x \right) \zeta(t)\,dt\,,$$

$$\mathcal{A}^+(\zeta) := \int_{\{\zeta>0\}} \left(\limsup_{x\to+\infty} f(t,x) - \mu_2 x \right) \zeta(t)\,dt +$$
$$\int_{\{\zeta<0\}} \left(\liminf_{x\to-\infty} f(t,x) - \nu_2 x \right) \zeta(t)\,dt\,.$$

Collecting all the possible situations in a single statement we can summarize the Landesman-Lazer Assumption 3.6 for the Dirichlet problem (4.8) as follows.

ASSUMPTION 4.1. Assume $\mathcal{R} \subset B$ and

if $(\mu_1, \nu_1) \in C_{h,h+1}$ let   (f–) holds and $\mathcal{A}^-(\psi_{\mu_1,\nu_1}) > 0$,

if $(\mu_1, \nu_1) \in C_{h+1,h}$ let   (f–) holds and $\mathcal{A}^-(\phi_{\mu_1,\nu_1}) > 0$,

if $(\mu_1, \nu_1) \in C_{h,h}$ let   (f–) holds and both

               $\mathcal{A}^-(\phi_{\mu_1,\nu_1}) > 0$ and $\mathcal{A}^-(\psi_{\mu_1,\nu_1}) > 0$,

if $(\mu_2, \nu_2) \in C_{k,k+1}$ let   (f+) holds and $\mathcal{A}^+(\psi_{\mu_2,\nu_2}) < 0$,

if $(\mu_2, \nu_2) \in C_{k+1,k}$ let   (f+) holds and $\mathcal{A}^+(\phi_{\mu_2,\nu_2}) < 0$,

if $(\mu_2, \nu_2) \in C_{k,k}$ let   (f+) holds and both

               $\mathcal{A}^+(\phi_{\mu_2,\nu_2}) < 0$ and $\mathcal{A}^+(\psi_{\mu_2,\nu_2}) < 0$.

REMARK 4.2. Concerning the previous assumption, if $\mathcal{R} \subset B_{2j}$ for a positive integer j then only the case $(\mu_1, \nu_1) \in C_{j,j}$ can hold, and similarly we can have $(\mu_2, \nu_2) \in C_{j,j+1}$ if $\mu \leq \nu$ or $(\mu_2, \nu_2) \in C_{j+1,j}$ if $\mu \geq \nu$. On the other hand, if $\mathcal{R} \subset B_{2j-1}$ for a positive integer j then only the case $(\mu_2, \nu_2) \in C_{j,j}$ can hold, and similarly we can have $(\mu_1, \nu_1) \in C_{j,j-1}$ if $\mu \leq \nu$ or $(\mu_1, \nu_1) \in C_{j-1,j}$ if $\mu \geq \nu$.

Let us conclude this paper with the existence theorem for the Dirichlet problem.

THEOREM 4.3. *Consider the problem* (4.8) *where the continuous function f satisfies* (4.2) *and* (4.3).
If Assumption 4.1 holds, then there exists at least one solution of (4.8).

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References

[1] A. Boscaggin and M. Garrione, Resonance and rotation numbers for planar Hamiltonian systems: Multiplicity results via the Poincaré-Birkhoff theorem, Nonlinear Anal. 74 (2011), 4166–4185.

[2] A. Boscaggin and M. Garrione, Resonant Sturm-Liouville boundary value problems in differential systems in the plane, Z. Anal. Anwend. 35 (2016), 41–59.

[3] E.N. Dancer, Boundary value problems for weakly nonlinear ordinary differential equations, Bull. Austral. Math. Soc. 15 (1976), 321–328.

[4] E.N. Dancer, Proof of the results in "Boundary value problems for weakly nonlinear ordinary differential equations", Rend. Istit. Mat. Univ. Trieste 42 (2010), 31–57.

[5] Y. Dong, On the solvability of asymptotically positively homogeneous equations with Sturm-Liouville boundary value conditions, Nonlinear Anal. 42 (2000), 1351–1363.

[6] P. Drábek, Landesman-Lazer condition for nonlinear problems with jumping nonlinearities, J. Differential Equations 85 (1990), 186–199.

[7] P. Drábek and S. Invernizzi, On the periodic BVP for the forced Duffing equation with jumping nonlinearity, Nonlinear Anal. 10 (1986), 643–650.

[8] C. Fabry, Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities, J. Differential Equations 116 (1995), 405–418.

[9] C. Fabry and A. Fonda, Periodic solutions of nonlinear differential equations with double resonance, Ann. Mat. Pura Appl. 157 (1990), 99–116.

[10] C. Fabry and A. Fonda, Nonlinear equations at resonance and generalized eigenvalue problems, Nonlinear Anal. 18 (1992), 427–444.

[11] C. Fabry and A. Fonda, Periodic solutions of perturbed isochronous Hamiltonian systems at resonance, J. Differential Equations 116 (2005), 299–325

[12] C. Fabry and P. Habets, Periodic solutions of second order differential equations with superlinear asymmetric nonlinearities, Arch. Math. 60 (1993), 266–276.

[13] A. Fonda, Positively homogeneous Hamiltonian systems in the plane, J. Differential Equations 200 (2004), 162–184.

[14] A. Fonda, Playing around resonance: an invitation to the search of periodic solutions for second order ordinary differential equations, Birkhäuser Advanced Texts Basler Lehrbücher, Springer International Publishing, 2016.

[15] A. Fonda and M. Garrione, Double resonance with Landesman-Lazer conditions for planar systems of ordinary differential equations, J. Differential Equations 250 (2011), 1052–1082.

[16] A. Fonda and M. Garrione, Generalized Sturm-Liouville boundary conditions for first order differential systems in the plane, Topological Methods Nonlinear Anal. 42 (2013), 293–325.

[17] A. Fonda and J. Mawhin, Planar differential systems at resonance, Adv. Differential Equations 11 (2006), 1111–1133.

[18] A. Fonda and A. Sfecci, Periodic solutions of a system of coupled oscillators with one-sided superlinear retraction forces, Differential Integral Equations 25 (2012) , 993–1010.

[19] A. Fonda and A. Sfecci, Multiple periodic solutions of Hamiltonian systems confined in a box, Discrete Contin. Dynam. Syst. 37 (2017) 1425–1436.

[20] S. Fučík, Boundary value problems with jumping nonlinearities, Časopis Pěst. Mat. 101 (1976), 69–87.

[21] M. Henrard, Optimal integral criterion of nonresonance for asymptotically positively homogeneous equations with Sturm-Liouville boundary conditions, Acad. Roy. Belg. Cl. Sci. Mém. Collect. 8 18 (2000), 1-51.

[22] B.P. Rynne, The Fučík spectrum of general Sturm-Liouville problems, J. Differential Equations 161 (2000), 87–109.

[23] B.P. Rynne, Non-resonance conditions for semilinear Sturm-Liouville problems with jumping nonlinearities, J. Differential Equations 170 (2001), 215–227.

[24] A. Sfecci, Positive periodic solutions for planar differential systems with repulsive singularities on the axes, J. Math. Anal. Appl. 415 (2014), 110–120.

[25] A. Sfecci, Double resonance for one-sided superlinear or singular nonlinearities, Ann. Mat. Pura Appl. 195 (2016), 2007–2025.

[26] A. Sfecci, Periodic impact motions at resonance of a particle bouncing on spheres and cylinders, Adv. Nonlinear Studies 17 (2017), 481–496.

[27] P. Tomiczek, Potential Landesman-Lazer type conditions and the Fučík spectrum, Electron. J. Differential Equations 2005 (2005), No. 94, pp. 1–12.

Andrea Sfecci
Dipartimento di Matematica e Geoscienze
Università degli Studi di Trieste
Via Valerio 12/1, 34127 Trieste, Italy.
    E-mail address: asfecci@units.it