Distributed Dynamic Pricing of Multiscale Transportation Networks

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***Abstract*—We study transportation networks controlled by dynamic feedback tolls. We focus on a multiscale model, whereby the dynamics of the traffic flows are intertwined with those of the routing choices. The latter are influenced by the current traffic state of the network as well as by dynamic tolls controlled in feedback by the system planner. We prove that a class of decentralized monotone flow-dependent tolls allows for globally stabilizing the transportation network around a generalized Wardrop equilibrium. In particular, our results imply that using decentralized marginal cost tolls, stability of the dynamic transportation network is guaranteed around the social optimum traffic assignment. This is particularly remarkable as such dynamic feedback tolls can be computed in a fully local way without the need for any global information about the network structure, its state, or the exogenous network loads. Through numerical simulations, we also compare the performance of such decentralized dynamic feedback marginal cost tolls with constant offline (and centrally) optimized tolls both in the asymptotic and in the transient regime, and we investigate their robustness to information delays.**

***Index Terms*—Congestion pricing, distributed control, dynamical flow networks, marginal cost tolls, robust control, social optimum, transportation networks, user equilibrium.**

I. Introduction

OVER the past years, there has been an increasing interest in the control analysis and synthesis of dynamical transportation networks. This is especially motivated by the wide-spreading sensing, communication, information, and actuation technologies that are dramatically changing the transportation system dynamics and affecting the users' decision making and behavior. There is a growing awareness that the new opportunities and risks created by these technologies can be fully understood only within a dynamical network framework.

Dynamics and control of traffic flows over networks have received a great deal of research attention, motivated by applications both to communication networks [2]–[6] and to road transportation systems [8]–[12]. Special emphasis in this literature has been put on mathematical properties of the dynamical system model—e.g., convexity, monotonicity, contractivity, Lyapunov functions' separability—that allow for scalable control architectures, such as, e.g., distributed or decentralized control policies [14]–[17].

A central aspect of dynamical flow networks is related to the routing decisions. In classical approaches to road traffic networks, the routing is considered static (see, e.g., the cell transmission model [18]), possibly determined by a network flow optimization problem such as a system or user optimum traffic assignment problem (see [19] and [20]). In fact, it is widely recognized that when drivers make their routing decisions by choosing the paths that minimize their own experienced delays, network congestion can increase significantly with respect to a hypothetical scenario, where a central planner was able to directly impose an optimized routing, a phenomenon known as the price of anarchy [21], [22]. On the other hand, the impact of dynamic routing on the stability and resilience of traffic flow networks has been recently analyzed [23]–[25] and there has been also a significant research effort to understand the drivers' answer to external communications from intelligent traveller information devices [26]–[29]. Charging tolls or providing signalling schemes subject to a nontrivial amount of uncertainty are, therefore, two potential strategies to influence drivers to make routing choices that result in globally optimal routing (see [30]–[38]).

In this article, we study multiscale dynamical flow networks whereby the physical dynamics of the traffic flows are intertwined with those of the routing choices. In particular, we extend the model and results of [25] by introducing decentralized flow-dependent tolls in order to influence the route choice behavior. Specifically, we consider a multiscale dynamical model of the transportation network whereby the traffic dynamics describing the real-time evolution of the local traffic level are coupled with those of the path preferences. We assume that the latter evolve

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following a perturbed best response to global information about the traffic status of the whole network and to decentralized flow-dependent tolls.

Our main result shows that by using monotone decentralized flow-dependent tolls and in the limit of small update rate of the aggregate path preferences, the transportation network globally stabilizes around a generalized Wardrop equilibrium [39]. The latter is a configuration in which the perceived cost associated to any source–destination path chosen by a nonzero fraction of users does not exceed the perceived cost associated to any other path. As in [25], we assume that the path preferences evolve at a slower time scale than the physical traffic flows and adopt a singular perturbation approach [40] to the stability analysis of the ensuing multiscale closed-loop traffic dynamics. In fact, classical results from evolutionary game theory and population dynamics [41]–[42] cannot be directly applied to our framework since they assume that information is accessed at a single temporal and spatial scale while the traffic dynamics are neglected as they are assumed to be instantaneously equilibrated.

The introduction of tolls has long been studied as a way to influence the rational and selfish behavior of drivers so that the associated user equilibrium can be aligned with the system optimum network flow. A particular taxation mechanism that guarantees this alignment is marginal-cost pricing, see, e.g., [43] and [44]. Marginal-cost tolls do not require any global information about the network structure or traffic state, nor of the exogenous user demands, and can be computed in a fully local way. We prove that using marginal-cost tolls our multiscale dynamical flow network stabilizes around the social optimum traffic assignment. We observe that our results go well beyond the traditional setting [43] where only static frameworks are considered as well as the evolutionary game theoretic approaches [44], where only path preference dynamics are considered, as the physical ones are assumed equilibrated. In fact, our analysis is performed in a fully dynamical flow network setting. In this respect, the global optimality guarantees obtained in this article should be compared with recent results on global performance and resilience of robust distributed control of dynamical flow networks [23], [13].

In the last part of this article, we present numerical simulations comparing the asymptotic and transient performance of the system with dynamic distributed feedback marginal cost tolls and constant marginal cost tolls. While it is known that the latter can be computed to enforce the social optimum equilibrium—provided that the system planner has a complete knowledge of the network topology, user demand profile, and delay functions—we show that not only do the former achieve the same optimal asymptotic performance, but they also guarantee faster convergence and are strongly robust to variation of network topology and exogenous traffic load. It is worth pointing out that robustness of the marginal cost tolls was recently investigated also in the case of static models [22], [45]. Finally, we study the effect of time-delays in the global information of the routing decision dynamics and analyze their influence on the evolution of the multiscale dynamical system. For different values of such time delays, one observes different behaviors of the system depending on whether dynamic feedback marginal cost tolls are used instead of constant marginal cost ones. With the latter, the system remains stable and converges to the equilibrium, instead with the former a phase transition and an oscillatory behavior may emerge for large enough delays.

The rest of this article is organized as follows. In Section II, we describe the multiscale model of network traffic flow dynamics and introduce distributed dynamics tolls. In Section III, we present the main technical results of this article, whose proofs are then presented in Section IV. In Section V, we discuss possible extensions of the results presented in the previous sections. In Section VI, we provide a numerical study of the transient and asymptotic performance of both dynamic feedback and constant tolls and we analyze their robustness with respect to information delays. Section VII concludes this article.

A. Notation

For two finite sets $\mathcal{A}$ and $\mathcal{B}$, $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$, $\mathbb{R}^{\mathcal{A}}$ the space of real-valued vectors whose entries are indexed by elements of $\mathcal{A}$, and $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ the space of real-valued matrices whose entries are indexed by pairs in $\mathcal{A}\times\mathcal{B}$. The transpose of a matrix Q in $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ is denoted by Q' in $\mathbb{R}^{\mathcal{B}\times\mathcal{A}}$, I is an identity matrix and $\mathbf{1}$ an all-one vector whose size depends on the context. For, i in $\mathcal{A}$, $\delta^{(i)}$ in $\mathbb{R}^{\mathcal{A}}$ denotes the vector with all entries equal to 0 except for the ith that is equal to 1. We use the notation $\Phi := I - |\mathcal{A}|^{-1}\mathbf{1}\mathbf{1}'$ in $\mathbb{R}^{\mathcal{A}\times\mathcal{A}}$ to denote the projection matrix of the space orthogonal to $\mathbf{1}$. The simplex of a probability vector over $\mathcal{A}$ is denoted by $S(\mathcal{A}) = \{x \in \mathbb{R}^{\mathcal{A}}_{+} : \mathbf{1}'x = 1\}$. Let $\|\cdot\|_p$ be the class of p-norms for p in $[1,\infty]$, and by default, let $\|\cdot\| := \|\cdot\|_2$. Let now sgn : $\mathbb{R} \to \{-1, 0, 1\}$ be the sign function, defined by $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = -1$ if $x < 0$ and $\text{sgn}(x) = 0$ if $x = 0$. By convention, we will assume the identity $d|x|/dx = \text{sgn}(x)$ to be valid for every x in $\mathbb{R}$, including $x = 0$. Finally, given the gradient ∇f of a function $f : D \to \mathbb{R}$ with $D \subseteq \mathbb{R}^{\mathcal{A}}$, we denote with $\tilde{\nabla} f = \Phi\nabla f$ the projected gradient on $S(\mathcal{A})$.

II. Model Description

A. Transportation Network

We model the topology of the transportation network as a directed multigraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set of nodes and $\mathcal{E}$ is a finite set of directed links. Each link i in $\mathcal{E}$ is directed from its tail node θ_i to its head node $\kappa_i \neq \theta_i$. We shall allow for parallel links, i.e., links $i \neq j$ such that $\theta_i = \theta_j$ and $\kappa_i = \kappa_j$. On the other hand, we shall assume that there are no self-loops, i.e., that $\theta_i \neq \kappa_i$ for every link i in $\mathcal{E}$. We shall denote by B in $\{-1, 0, 1\}^{\mathcal{V}\times\mathcal{E}}$ the node-link incidence matrix of a multigraph $\mathcal{G}$, whose entries are given by

$$B_{vi} = \begin{cases} +1 & \text{if} \quad v = \theta_i \\ -1 & \text{if} \quad v = \kappa_i \\ 0 & \text{if} \quad v \neq \theta_i, \kappa_i. \end{cases}$$

A length-l path from a node v_0 to a node v_l is an ordered l-tuple of links $\gamma = (i_1, i_2, \ldots, i_l)$ such that the tail node of the first link is $\theta_{i_1} = v_0$, the head node of the last link is $\kappa_{i_l} = v_l$, the tail node of the next link coincides with the head node of the previous link, i.e., $v_s = \kappa_{i_s} = \theta_{i_{s+1}}$ for $1 \leq s \leq l-1$, and no node is visited twice, i.e., $v_r \neq v_s$ for all $0 \leq r < s \leq l$, except possibly for $v_0 = v_l$, in which case the path is referred to a cycle. A node d is said to be reachable from another node o if there exists at least a path from o to d. Observe that, in contrast to [25], where the transportation network was assumed to be cycle-free, in this article, we allow for the possible presence of cycles.

Throughout this article, we will consider a given origin node o and a destination node $d \neq o$ that is reachable from o and let Γ be the set of paths from o to d of any length $l \geq 1$. We shall denote the corresponding link-path incidence matrix by A in $\{0,1\}^{\mathcal{E}\times\Gamma}$ with entries

$$A_{i\gamma} = \begin{cases} 1 & \text{if} \quad i \in \gamma \\ 0 & \text{if} \quad i \notin \gamma. \end{cases}$$

We shall assume that every link i lies on some path from o to d so that A has no all-zero rows. We shall refer to nonnegative vectors y in $\mathbb{R}_+^{\mathcal{E}}$ generally as flow vectors. Upon recalling that $\delta^{(o)}$ ($\delta^{(d)}$) is the vector with all entries equal to 0 except for the one in the origin (destination) node that is equal to 1, we shall refer to a flow vector y such that

$$By = \lambda\left(\delta^{(o)} - \delta^{(d)}\right) \tag{1}$$

for some $\lambda \geq 0$ as an o-d equilibrium flow vector of throughput λ. For $\lambda \geq 0$, let us consider the simplex

$$\mathcal{S}_\lambda = \left\{z \in \mathbb{R}_+^{\Gamma} : \mathbb{1}'z = \lambda\right\} . \tag{2}$$

Observe that, for every z in $\mathcal{S}_\lambda$, one has $BAz = \lambda(\delta^{(o)} - \delta^{(d)})$, so that

$$y^z := Az \tag{3}$$

is an o-d equilibrium flow vector of throughput λ. Throughout, we shall refer to any z in $\mathcal{S}_\lambda$ as a *path preference* vector and to y^z defined as in (3) as the *associated equilibrium flow* vector.

Each link i in $\mathcal{E}$ of the transportation network topology $\mathcal{G}$ represents a cell. We shall denote the density on and the outflow from cell i in $\mathcal{E}$ by x_i and y_i, respectively. We shall assume that density and outflow of each cell are related by a functional dependence

$$y_i = \varphi_i(x_i), \qquad i \in \mathcal{E} \tag{4}$$

satisfying the following property.

Assumption 1: For every link i in $\mathcal{E}$ the flow-density function $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable, strictly increasing, strictly concave, and such that

$$\varphi_i(0) = 0, \qquad \varphi_i'(0) < +\infty .$$

For every link i in $\mathcal{E}$, let

$$C_i := \sup\{\varphi_i(x_i) : x_i \geq 0\}$$

be its maximum flow capacity.

Remark 1: Notice that in road traffic networks the assumption that the flow-density functions are strictly increasing remains valid provided that we confine ourselves to so-called the free-flow region, as is done in [25]. In Section V, we will discuss how the framework of this article could possibly be extended to more accurate dynamical models for road traffic flow networks, such as the cell transmission model [18].

Let us denote cell i's latency function by $\tau_i : \mathbb{R}_+ \to [0, +\infty]$, returning the delay incurred in traversing link i in $\mathcal{E}$ as a function of the current flow out of it. This is defined as

$$\tau_i(y_i) := \begin{cases} 1/\varphi_i'(0) & \text{if } y_i = 0 \\ \varphi_i^{-1}(y_i)/y_i & \text{if } 0 < y_i < C_i \\ +\infty & \text{if } y_i \geq C_i . \end{cases} \tag{5}$$

Notice that the third line in (5) is merely a convenient mathematical convention allowing us to formally extend the range of the flow variable y_i to values above the cell i's capacity, albeit such values of flow remain not physically achievable. The following simple useful result is proven in Appendix A.

Lemma 1: Let $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ be a flow-density function satisfying Assumption 1. Then, the corresponding latency function τ_i defined in (5) is twice continuously differentiable, strictly increasing on the interval $[0, C_i)$, and such that $\tau_i(0) > 0$. Moreover, its first derivative is given by

$$\tau_i'(y) = \frac{y - x\varphi_i'(x)}{\varphi_i'(x)y^2}, \qquad x = \varphi_i^{-1}(y) \tag{6}$$

and the function $y \mapsto y\tau_i(y)$ is strictly convex on $[0, C_i)$.

Let us now define the set of *feasible flow* vectors as

$$\mathcal{F} := \left\{y \in \mathbb{R}_+^{\mathcal{E}} : y_i < C_i , \ i \in \mathcal{E}\right\}$$

and the set of *feasible path preferences* as

$$\mathcal{Z} := \{z \in \mathcal{S}_\lambda : y^z \in \mathcal{F}\}.$$

Moreover, let the *total latency* associated to a nonnegative vector y in $\mathbb{R}_+^{\mathcal{E}}$ be

$$L(y) = \sum_{i\in\mathcal{E}} y_i\tau_i(y_i) . \tag{7}$$

Observe that the total latency $L(y)$ is finite if and only if the flow vector y is feasible. In fact, as a consequence of Lemma 1, we have that the total latency function $L(y)$ is a strictly convex function of y in $\mathcal{F}$. Notice that, by the max-flow min-cut theorem (see [19], Th. 4.1), the set of feasible flows $\mathcal{F}$ contains equilibrium o-d flows if and only if the throughput $\lambda < C_{o,d}^{\text{min cut}}$, where

$$C_{o,d}^{\text{min cut}} = \min_{\substack{\mathcal{U}\subseteq\mathcal{V}: \\ o\in\mathcal{U},\, d\notin\mathcal{U}}} \sum_{\substack{i\in\mathcal{E}: \\ \theta_i\in\mathcal{U},\, \kappa_i\notin\mathcal{U}}} C_i$$

is the min-cut capacity. It then follows that, for every λ in $[0, C_{o,d}^{\text{min cut}})$, the total latency $L(y)$ admits a unique minimizer $y^*(\lambda)$ in the set of feasible equilibrium o-d flows of throughput λ. We shall refer to such unique minimizer

$$y^*(\lambda) := \operatorname*{argmin}_{\substack{y\in\mathbb{R}_+^{\mathcal{E}} \\ By=\lambda(\delta^{(o)}-\delta^{(d)})}} L(y) \tag{8}$$

as the *social optimum* equilibrium flow.

Example 1: Consider the network in Fig. 1 with node set $\mathcal{V} = \{o, a, b, d\}$ and link set $\mathcal{E} = \{i_1, i_2, i_3, i_4, i_5, i_6\}$. It contains four distinct paths from o to d. In fact, we may write $\Gamma = \{\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)}, \gamma^{(4)}\}$, where $\gamma^{(1)} = (i_1, i_5)$, $\gamma^{(2)} = (i_2, i_6)$, $\gamma^{(3)} = (i_1, i_3, i_6)$, and $\gamma^{(4)} = (i_2, i_4, i_5)$. Notice that there is a cycle $\gamma^{(o)} = (i_3, i_4)$. For every link i in $\mathcal{E}$, let the flow-density functions be given by

$$\varphi_i(x_i) = C_i(1 - e^{-x_i}) , \qquad x_i \in \mathbb{R}_+ \tag{9}$$

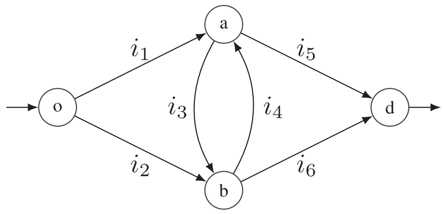
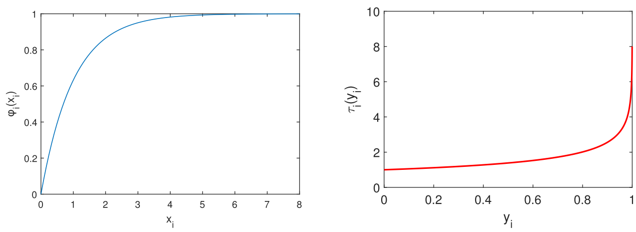
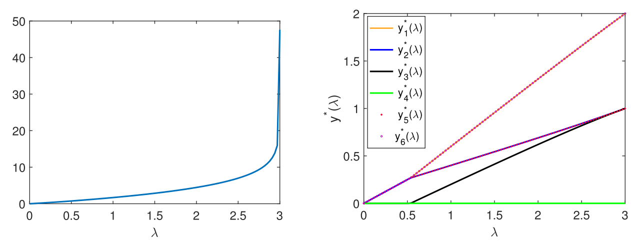


Fig. 1. Example of network with cycle.



(a) (b)

Fig. 2. Plots of (a) flow-density function (9) and (b) latency function (10) in the special case of capacity $C_i = 1$.



(a) (b)

Fig. 3. (a) Plot of the minimum total latency as a function of the throughput λ for a transportation network with topology as in Fig. 1, flow-density functions as in (9), and link capacities as in (11). (b) Plots of the corresponding social optimum flow vector $y^*(\lambda)$. In particular $y^*_6(\lambda)$ is overlapped to $y^*_1(\lambda)$, while $y^*_5(\lambda)$ is overlapped to $y^*_2(\lambda)$.

where $C_i > 0$ is link i's capacity. Then, the corresponding latency functions are given by

$$\tau_i(y_i) = \begin{cases} 1/C_i & \text{if } y_i = 0 \\ \dfrac{1}{y_i} \log\left(\dfrac{C_i}{C_i - y_i}\right) & \text{if } 0 < y_i < C_i \\ +\infty & \text{if } y_i \geq C_i\,. \end{cases} \tag{10}$$

Plots of the flow-density function (9) and of the latency function (10) are reported in Fig. 2 . In the special case, when the link capacities are

$$C_{i_1} = 3\,,\ C_{i_2} = 1\,,\ C_{i_3} = 1\,,\ C_{i_4} = 1\,,\ C_{i_5} = 1\,,\ C_{i_6} = 3\,, \tag{11}$$

the min-cut capacity is $C^{\text{min cut}}_{o,d} = 3$ and the minimum total latency and social optimum flow are plotted in Fig. 3 as a function of the throughput λ in $[0, C^{\text{min cut}}_{o,d})$.

B. Multiscale Model of Network Traffic Flow Dynamics

We shall consider a physical traffic flow entering the network from the origin node o at a constant rate λ, travelling on the different paths and finally exiting the network from the destination node d. Conservation of mass implies that the density on every link i in $\mathcal{E}$ at time $t \geq 0$ evolves as

$$\dot{x}_i(t) = \lambda \delta^{(o)}_{\theta_i} R_{oi} + \sum_{j\in\mathcal{E}} R_{ji}(t) y_j(t) - y_i(t) \tag{12}$$

where

$$y_i(t) = \varphi_i(x_i(t)) \tag{13}$$

is the total outflow from link i, the terms $R_{ji}(t)$ and $R_{oi}(t)$ stand for the fractions of outflow from link j and, respectively, from the origin node o, that moves directly toward link i, and the term $\lambda\delta^{(o)}_{\theta_i}$ accounts for the constant exogenous inflow in the origin node o. Topological constraints and mass conservation imply that: 1) $R_{ij}(t) = 0$ whenever $\kappa_i \neq \theta_j$, i.e., whenever link j is not immediately downstream of link i; 2) that $R_{oj}(t) = 0$ whenever $\theta_j \neq o$; and 3) that $\sum_{j\in\mathcal{E}} R_{ij}(t) = 1$ for $i = o$ and for every i in $\mathcal{E}$ such that $\theta_i \neq d$. The square matrix $R(t) = (R_{ij}(t))_{i,j\in\mathcal{E}}$ will be referred to as the *routing matrix*.

Throughout, we shall assume that the routing matrix is determined by the path preferences that are continuously updated in response to available current traffic information and dynamic tolls. Formally, the relative appeal of the different paths to the users is modelled by a time-varying nonnegative vector $z(t)$ in the simplex $\mathcal{S}_\lambda$, to be referred to as the current *aggregate path preference*.[1] We shall assume that such aggregate path preferences determine the routing matrix as

$$R_{ij}(t) = \begin{cases} G_j(z(t)) & \text{if } \theta_j = \kappa_i \\ 0 & \text{if } \theta_j \neq \kappa_i \end{cases} \tag{14}$$

for i, j in $\mathcal{E}$ and $t \geq 0$, where $G : \mathcal{Z} \to \mathbb{R}^{\mathcal{E}}_+$ is given by

$$G_j(z) = \begin{cases} \dfrac{y^z_j}{\sum_{i\in\mathcal{E}:\theta_i=\theta_j} y^z_i} & \text{if } \sum_{i\in\mathcal{E}:\theta_i=\theta_j} y^z_i > 0 \\ \dfrac{1}{|\{i \in \mathcal{E} : \theta_i = \theta_j\}|} & \text{if } \sum_{i\in\mathcal{E}:\theta_i=\theta_j} y^z_i = 0 \end{cases} \tag{15}$$

for each cell j in $\mathcal{E}$. Equations (14) and (15) state that at every junction, represented by a node v in $\mathcal{V}$, the outflow from every incoming cell i such that $\kappa_i = v$ gets split among the cells j immediately downstream (i.e., such that $\theta_j = v$) according to the proportion associated to the equilibrium flow vector y^z corresponding to the path preference z, provided that y^z is such there is flow passing through node v, and otherwise the split is uniform among the immediately downstream cells. Notice that $G(z)$ as defined in (15) is continuously differentiable on the interior of the set $\mathcal{Z}$, to be denoted as

$$\mathcal{Z}^\circ := \{z \in \mathcal{Z} :\ z_\gamma > 0\, \forall \gamma \in \Gamma\}\,.$$

In the considered dynamical network traffic model, the aggregate path preference vector $z(t)$ is continuously updated as route

[1] Recall that $\mathcal{S}_\lambda$ stands for the simplex over the set of o-d-paths Γ, as defined in (2).

decision makers access global information about the current traffic state of the whole network embodied by the vector

$$l(t) = (l_i(t))_{i\in\mathcal{E}}\,, \qquad l_i(t) = \tau_i(y_i(t)) \tag{16}$$

of current latencies on the different links. The aggregate path preference vector is also influenced by a vector $w(t) = (w_i(t))_{i\in\mathcal{E}}$ of *dynamic tolls*, that are to be determined by the transportation system operator. Specifically, let the cost perceived by each user, crossing a link i in $\mathcal{E}$, be given by the sum of the latency $l_i(t)$ and the toll $w_i(t)$ so that the perceived total cost that is expected to incur on a path γ in Γ assuming that the traffic levels on that path won't change during the journey is $\sum_i A_{i\gamma}(l_i(t) + w_i(t))$. We shall then assume that the path preferences are updated at some rate $\eta > 0$, according to a noisy best response (a.k.a. logit) dynamics

$$\dot{z}(t) = \eta\left(F^{(\beta)}(l(t), w(t)) - z(t)\right) \tag{17}$$

where for every fixed uncertainty parameter $\beta > 0$ the function $F^{(\beta)} : \mathbb{R}^{\mathcal{E}}_+ \times \mathbb{R}^{\mathcal{E}}_+ \to \mathcal{Z}$ is the perturbed best response defined as follows:

$$F^{(\beta)}(l, w) = \frac{\lambda \exp(-\beta(A'(l+w)))}{\mathbf{1}' \exp(-\beta(A'(l+w)))}. \tag{18}$$

We shall compactly rewrite the coupled dynamics of the physical flow and the path preferences defined in (12)–(18) as

$$\begin{cases} \dot{x}(t) = & H(y(t), z(t))\,, \qquad y(t) = \varphi(x(t)) \\ \dot{z}(t) = & \eta\left(F^{(\beta)}\left(l(t), w(t)\right) - z(t)\right) \end{cases} \tag{19}$$

where $H : \mathcal{F} \times \mathcal{Z} \to \mathbb{R}^{\mathcal{E}}$ is defined as

$$H_i(y, z) := G_i(z)\left(\lambda\delta^{(o)}_{\theta_i} + \sum_{j:\kappa_j=\theta_i} y_j\right) - y_i\,, \qquad i \in \mathcal{E}\,. \tag{20}$$

III. Problem Statement and Main Results

The goal of this article is to design robust scalable feedback pricing policies

$$\omega : \mathcal{F} \to \mathbb{R}^{\mathcal{E}}_+ \tag{21}$$

determining in real time the dynamic tolls

$$w(t) = \omega(y(t)) \tag{22}$$

with the objective of guaranteeing stability and achieving social optimality for the closed-loop network traffic flow dynamics (19)–(22).

Observe that, for any given fixed inflow vector $\lambda\delta^{(o)}$ and constant toll vector w, and in the special case of cycle-free network topology, stability and convergence to the corresponding Wardrop equilibrium—as defined later in this section—follow from the results in [25]. In fact, given full knowledge of the exogenous inflow $\lambda\delta^{(o)}$ and of the whole transportation network characteristics, one could use classical results in order to precompute static tolls that would align such Wardrop equilibrium with the social optimum. However, even for cycle-free networks, such an approach would result in an inadequate solution as it would lack robustness with respect to the value of the exogenous inflow $\lambda\delta^{(o)}$, as well as to changes of the network characteristics in response, e.g., to accidents and other disruptions.

In contrast, we seek to design feedback pricing policies that are universal with respect to values of the exogenous inflow and robustly adapt in real time to changes of the network characteristics. We shall particularly focus on the class of *decentralized monotone feedback pricing policies*, defined as follows.

Definition 1: In a transportation network with topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a feedback pricing policy $\omega : \mathcal{F} \to \mathbb{R}^{\mathcal{E}}_+$ is said to be:

i) *Monotone* if $\omega(y) \geq \omega(y')$ for every y, y' in $\mathcal{F}$ such that $y \geq y'$, where inequalities are meant to hold true entrywise.

ii) *Decentralized* if, for every i in $\mathcal{E}$, the toll $w_i = \omega_i(y)$ is a function of the flow y_i on link i only.

Throughout the rest of this article, we shall emphasize the local structure of decentralized pricing policies by writing $w_i = \omega_i(y_i)$, with a slight abuse of notation. As shown in the following, such robust fully local feedback pricing policies can be designed with global guarantees on stability and optimality. Before stating our main results, we introduce the notion of generalized Wardrop equilibrium with feedback pricing.

Definition 2: (Generalized Wardrop equilibrium with feedback pricing). For a transportation network with topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and latency functions τ_i, let o and d in $\mathcal{V}$, with $d \neq o$ reachable from o, be an origin and a destination, respectively. Let Γ the set of o–d paths and A the link-path incidence matrix. Then, for a feedback pricing policy $\omega : \mathcal{F} \to \mathbb{R}^{\mathcal{E}}_+$, an o–d equilibrium flow vector y in $\mathcal{F}$ of throughput λ is a *generalized Wardrop equilibrium* if $y = Az$ for some path preference vector z in $\mathcal{S}_\lambda$ such that for every path γ in Γ with $z_\gamma > 0$, we have

$$\left(A'\left(\tau(y) + \omega(y)\right)\right)_\gamma \leq \left(A'\left(\tau(y) + \omega(y)\right)\right)_{\tilde{\gamma}} \quad \forall\tilde{\gamma} \in \Gamma. \tag{23}$$

Equation (23) states that the sum of the total delay and the total toll associated to an o–d path γ at the equilibrium flow y are less than or equal to the sum of the total delay and the total toll associated to any other o–d path $\tilde{\gamma}$. Hence, a generalized Wardrop equilibrium with feedback pricing is characterized as being the flow associated to a path preference vector supported on the subset of paths with minimal sum of total latency plus total toll. In the special case with no tolls, i.e., when the feedback pricing policy $\omega(y) \equiv 0$, this reduces to the classical notion of Wardrop equilibrium [39]. More in general, for constant tolls $\omega(y) \equiv w$, we get the standard notion of Wardrop equilibrium with tolls. For general decentralized monotone feedback pricing policies, existence and uniqueness of a generalized Wardrop equilibrium are guaranteed by the following result, proven in Appendix B.

Proposition 1: Consider a transportation network with topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and strictly increasing latency functions. Let o and d in $\mathcal{V}$, with $d \neq o$ reachable from o, be an origin and a destination, respectively. Then, for every throughput λ in $[0, C^{\text{min cut}}_{o,d})$ and every decentralized monotone feedback pricing policy $\omega : \mathcal{F} \to \mathbb{R}^{\mathcal{E}}_+$, there exists a unique generalized Wardrop equilibrium $y^{(\omega)}$ and it can be characterized as the solution of the convex optimization problem

$$y^{(\omega)} = \underset{\substack{y\in\mathbb{R}^{\mathcal{E}}_+ \\ By=\lambda(\delta^{(o)}-\delta^{(d)})}}{\arg\min} \sum_{i\in\mathcal{E}} D_i(y_i) \tag{24}$$

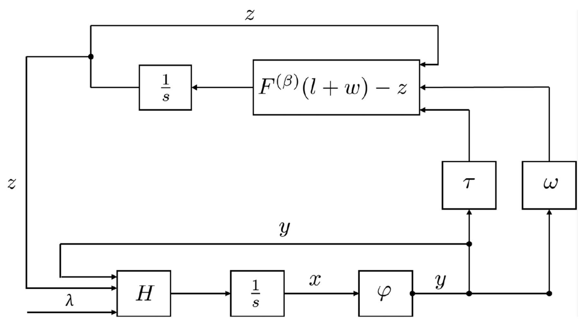


Fig. 4. Block diagram of the problem.

where for each link i in $\mathcal{E}$

$$D_i(y_i) = \int_0^{y_i} (\tau_i(s) + \omega_i(s))\, \mathrm{d}s \tag{25}$$

is the primitive of the perceived cost $\tau_i(y_i) + \omega_i(y_i)$.

Remark 2: It is possible to modify the definition of perceived cost by weighing τ_i differently from ω_i. This modification would cause no restriction on the validity of our results.

In the following, we shall prove that for small values of η and large values of β, the long-time behavior of the system (19) is approximately at the corresponding generalized Wardrop equilibrium, which—under proper distributed feedback pricing policies—coincides with the social optimum equilibrium. The following is the main result of this article. It will be proved in the next section using a singular perturbation approach.

Theorem 1: Consider a transportation network with topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and flow-density functions satisfying Assumption 1. Let λ in $[0, C_{o,d}^{\text{min cut}})$ be the throughput and $\omega : \mathcal{F} \to \mathbb{R}_+^{\mathcal{E}}$ be a Lipschitz-continuous monotone decentralized feedback pricing policy. Then, there exists a perturbed equilibrium flow $y^{(\omega,\beta)}$ in $\mathcal{F}$ such that, for every initial condition $(z(0), x(0))$ in $\mathcal{Z}^\circ \times \mathbb{R}_+^{\mathcal{E}}$, the solution of the closed-loop network traffic flow dynamics (19)–(22) satisfies

$$\limsup_{t\to\infty} \|y(t) - y^{(\omega,\beta)}\| \le \bar{\delta}(\eta), \qquad \eta > 0 \tag{26}$$

where $\bar{\delta}(\eta)$ is a nonnegative-real-valued, nondecreasing function such that $\lim_{\eta\to 0} \bar{\delta}(\eta) = 0$. Moreover

$$\lim_{\beta\to\infty} y^{(\omega,\beta)} = y^{(\omega)}. \tag{27}$$

Theorem 1 states that the system planner globally stabilizes the transportation network around the Wardrop equilibrium using nondecreasing decentralized flow-dependent tolls. Notice that the case $\lambda \ge C_{o,d}^{\text{min cut}}$ is not covered by Theorem 1 and in fact in that case one can show that the transportation system would become unstable as time grows large (see e.g., [23]).

Remark 3: Even in the cycle-free case, Theorem 1 does not follow from [25, Th. 2.5] if the tolls are not constant. Indeed, although the functions τ and ω both depend on the flow y, it is not always possible consider an auxiliary function $\bar{\tau} = \tau + \omega$ and directly apply the result from [25] due to the specific structure imposed on τ in (5). The feedback structure of the considered closed-loop multiscale transportation network dynamics is illustrated in Fig. 4 .

Now, we focus on the special case of decentralized feedback tolls, namely, when

$$w_i(t) = \omega_i(y_i(t)) = y_i(t)\tau_i'(y_i(t)), \qquad i \in \mathcal{E}. \tag{28}$$

Due the properties of the delay function τ_i, the marginal cost tolls $\omega_i(y_i(t))$ defined in (28) are increasing functions of the flow $y_i(t)$, so that Theorem 1 applies in this case. Moreover, the following additional result holds true.

Corollary 1: Consider a transportation network with topology $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and flow-density functions satisfying Assumption 1. Let λ in $[0, C_{o,d}^{\text{min cut}})$ be the throughput and $\omega : \mathcal{F} \to \mathbb{R}_+^{\mathcal{E}}$ be the dynamic feedback marginal cost tolls defined in (28). Then, the transportation network globally stabilizes around the social optimum traffic assignment $y^*(\lambda)$, i.e., for every initial condition $(z(0), x(0))$ in $\mathcal{Z}^\circ \times \mathbb{R}_+^{\mathcal{E}}$, the solution of the closed-loop network traffic flow dynamics (19)–(22) satisfies

$$\lim_{\beta\to\infty} y^{(\omega,\beta)} = y^*(\lambda)\,. \tag{29}$$

Proof: First, notice that with feedback marginal cost tolls $\omega_i(y_i) = y_i\tau_i'(y_i)$, the perceived cost $\tau_i(y_i) + \omega_i(y_i)$ on each link i in $\mathcal{E}$ has primitive

$$D_i(y_i) = \int_0^{y_i} \Big(\tau_i(s) + s\tau_i'(s)\Big)\mathrm{d}s = y_i\tau_i(y_i)$$

so that

$$\sum_{i\in\mathcal{E}} D_i(y_i) = L(y)$$

coincides with the total latency. It then follows from the characterization (24) of Proposition 1 that

$$y^{(\omega)} = \underset{\substack{y\in\mathbb{R}_+^{\mathcal{E}} \\ By=\lambda(\delta^{(o)}-\delta^{(d)})}}{\arg\min} \sum_{i\in\mathcal{E}} D_i(y_i) = \underset{\substack{y\in\mathbb{R}_+^{\mathcal{E}} \\ By=\lambda(\delta^{(o)}-\delta^{(d)})}}{\arg\min} L(y) = y^*(\lambda).$$

The claim then follows by directly applying Theorem 1. ■

Remark 4: Corollary 1 holds true also if the dynamic feedback marginal cost tolls (28) are replaced by the constant tolls

$$w_i^* = y_i^*(\lambda)\tau_i'(y_i^*(\lambda))\,, \qquad i \in \mathcal{E}\,. \tag{30}$$

However, in contrast to the dynamic feedback marginal cost tolls (28), such constant marginal cost tolls (30) require knowledge both of the social optimum flow and the exogenous inflow $\lambda\delta^{(o)}$ and lack robustness with respect to changes of the value of λ, as well as to changes of the network.

Remark 5: In order to implement the dynamic feedback marginal cost tolls (28), each local controller is required to compute the product $y_i\tau_i'(y_i)$ of the link's current flow and its latency function's derivative. Notice that, using (5), we get

$$\omega_i(y_i) = y_i\tau_i'(y_i) = \frac{1}{\varphi_i'(x_i)} - \frac{x_i}{y_i}\,.$$

Hence, for the computation of $\omega_i(y_i)$ is is sufficient to measure the current link density x_i, the flow $y_i = \varphi_i(x_i)$ and the derivative of the flow-density function $\varphi_i(x_i)$.

IV. Proof of Theorem 1

In this section, we prove Theorem 1. First of all, notice that since the functions $F^{(\beta)}$, G, and φ are differentiable, standard results imply the existence and uniqueness of a solution of the initial value problem associated to (19), with initial condition

$(z(0), x(0))$ in $\mathcal{Z}^\circ \times \mathbb{R}^{\mathcal{E}}_+$. In order to prove the stability result, we shall adopt a singular perturbation approach. Our strategy consists in thinking of the path preference vector z as quasi-static when we analyse the fast-scale dynamics (12), and considering the flow vector y almost equilibrated (i.e., close to y^z) when study the slow-scale dynamics (17). Below, we will derive a series of intermediate results that will then be combined to prove Theorem 1.

Before proceeding, we introduce some notation to be used throughout the section. Similar to (16) and (22) let

$$l^z(t) = (l^z_i(t))_{i\in\mathcal{E}}\,, \qquad l^z_i(t) = \tau_i(y^z_i(t))$$

and

$$w^z(t) = (w^z_i(t))_{i\in\mathcal{E}}\,, \qquad w^z_i(t) = \omega_i(y^z_i(t))$$

be respectively the vector of current latencies and the one of dynamic tolls both corresponding to the flow y^z associated to the path preference z.

Furthermore, observe that the perturbed best response function (18) satisfies

$$F^{(\beta)}(l,w) := \arg\min_{\alpha\in\mathcal{Z}_h}\{\alpha' A'(l+w) + h(\alpha)\} \tag{31}$$

where $h : \mathcal{Z} \to \mathbb{R}$ is the negative entropy function defined as

$$h(z) := \beta^{-1}\sum_{\gamma\in\Gamma} z_\gamma \log z_\gamma \tag{32}$$

using the standard convention that $0\log 0 = 0$. In fact, all our analysis and results apply to a more general setting where the perturbed best response function is defined as

$$F^{(h)}(l,w) := \arg\min_{\alpha\in\mathcal{Z}_h}\{\alpha' A'(l+w) + h(\alpha)\} \tag{33}$$

for some *admissible perturbation h:* $\mathcal{Z}_h \to \mathbb{R}$ such that $\mathcal{Z}_h \subseteq \mathcal{Z}$ is a closed convex set, $h(\cdot)$ is strictly convex, twice differentiable in the interior $\mathcal{Z}^\circ_h$ of $\mathcal{Z}_h$, and $\lim_{z\to\partial\mathcal{Z}_h}\|\tilde{\nabla} h(z)\| = \infty$. These conditions on h imply that $F^h(l,w)$ belongs to $\mathcal{Z}^\circ_h$ and that it is continuously differentiable on $\mathbb{R}^{\mathcal{E}}_+ \times \mathbb{R}^{\mathcal{E}}_+$. Notice that clearly the negative entropy function (32) is an admissible perturbation as defined above. We shall then proceed to proving Theorem 1 in this more general setting.

Now, let

$$x^z_i := \varphi_i^{-1}(y^z_i), \qquad \sigma_i := \mathrm{sgn}(x_i - x^z_i) = \mathrm{sgn}(y_i - y^z_i)$$

denote, respectively, the density corresponding to the flow associated to the path preference z and the sign of the difference between it and the actual density x_i. Then, we define the functions

$$V(y,z) = \|y - y^z\|_1, \quad \text{and} \quad W(x,z) = \|x - x^z\|_1. \tag{34}$$

The following technical results aim at showing that (34) are Lyapunov functions for the fast-scale dynamics (12) with stationary path preference z.

Lemma 2: Let $\overline{\mathcal{E}} \subseteq \mathcal{E}$ be a nonempty set of cells. Then

$$\max_{j\in\overline{\mathcal{E}}}\left\{1 - \sum_{\substack{i\in\overline{\mathcal{E}}:\\ \theta_i=\kappa_j}} G_i(z)\right\} \geq \frac{1}{|\mathcal{V}|} \tag{35}$$

Proof: Let $\overline{\mathcal{V}} = \{v\in\mathcal{V} : v = \kappa_i, i\in\overline{\mathcal{E}}\}$. Observe that

$$\sum_{\substack{i\in\overline{\mathcal{E}}\\ \theta_i=d}} G_i(z) = 0$$

so that, if d in $\overline{\mathcal{V}}$ then

$$\max_{j\in\overline{\mathcal{E}}}\left\{1 - \sum_{\substack{i\in\overline{\mathcal{E}}:\\ \theta_i=\kappa_j}} G_i(z)\right\} = 1$$

and the claim follows immediately.

We can then focus on the case when $d\notin\overline{\mathcal{V}}$. Let

$$\alpha = \sum_{\substack{i:\kappa_i\in\overline{\mathcal{V}}\\ \theta_i\notin\overline{\mathcal{V}}}} y^z_i + \lambda\delta^{(o)}_i \tag{36}$$

be the total inflow in $\overline{\mathcal{V}}$ which is also equal to the total outflow from $\overline{\mathcal{V}}$. Indeed α in (36) can be also written as

$$\alpha = \sum_{\substack{i:\kappa_i\notin\overline{\mathcal{V}}\\ \theta_i\in\overline{\mathcal{V}}}} y^z_i = \sum_{v\in\overline{\mathcal{V}}}\sum_{\substack{i:\kappa_i\notin\overline{\mathcal{V}}\\ \theta_i=v}} y^z_i \leq \sum_{v\in\overline{\mathcal{V}}}\sum_{\substack{i\notin\overline{\mathcal{E}}\\ \theta_i=v}} y^z_i \tag{37}$$

Now, let

$$f_v = \sum_{i:\kappa_i=v} y^z_i$$

be outflow from a single node v and observe that $f_v \leq \alpha$ for every node v. Using this and (37), we get

$$\alpha \leq \sum_{v\in\overline{\mathcal{V}}}\sum_{\substack{i\notin\overline{\mathcal{E}}\\ \theta_i=v}} y^z_i = \sum_{v\in\overline{\mathcal{V}}} f_v \sum_{\substack{i\notin\overline{\mathcal{E}}\\ \theta_i=v}} G_i(z) \leq \alpha\sum_{v\in\overline{\mathcal{V}}}\sum_{\substack{i\notin\overline{\mathcal{E}}\\ \theta_i=v}} G_i(z). \tag{38}$$

Hence

$$\frac{1}{|\mathcal{V}|} \leq \frac{1}{|\overline{\mathcal{V}}|} \leq \frac{1}{|\overline{\mathcal{V}}|}\sum_{v\in\overline{\mathcal{V}}}\sum_{\substack{i\notin\overline{\mathcal{E}}\\ \theta_i=v}} G_i(z) \leq \max_{v\in\overline{\mathcal{V}}}\sum_{\substack{i\notin\overline{\mathcal{E}}\\ \theta_i=v}} G_i(z) \tag{39}$$

so that

$$\max_{j\in\overline{\mathcal{E}}}\left(1 - \sum_{\substack{i\in\overline{\mathcal{E}}\\ \theta_i=\kappa_j}} G_i(z)\right) = \max_{v\in\overline{\mathcal{V}}}\sum_{\substack{i\notin\overline{\mathcal{E}}\\ \theta_i=v}} G_i(z) \geq \frac{1}{|\mathcal{V}|}$$

hence proving the claim. ∎

Lemma 3: For every $y = \varphi(x)$ in $\mathcal{F}$ and z in $\mathcal{Z}$

$$\nabla_x W(x,z)' H(y,z) \leq -\varsigma V(y,z)$$

where $\varsigma = 1/|\mathcal{V}||\mathcal{E}|$.

Proof: Observe that by (15), we get

$$y^z_i = G_i(z)\Bigg(\lambda\delta^{(o)}_{\theta_i} + \sum_{j:\kappa_j=\theta_i} y^z_j\Bigg).$$

We will use the above in the second equality of the computation below. Indeed we have

$$
\begin{aligned}
&\nabla_x W(x,z)' H(y,z) \\
&= \sum_{i\in\mathcal{E}} \sigma_i \left(G_i(z) \Big(\lambda \delta^{(o)}_{\theta_i} + \sum_{j:\kappa_j=\theta_i} y_j \Big) - y_i \right) \\
&= \sum_{i\in\mathcal{E}} \sigma_i \Big(G_i(z) \Big(\lambda \delta^{(o)}_{\theta_i} + \sum_{j:\kappa_j=\theta_i} y_j \Big) \\
&- G_i(z) \Big(\lambda \delta^{(o)}_{\theta_i} + \sum_{j:\kappa_j=\theta_i} y^z_j \Big)\Big) + \sum_{i\in\mathcal{E}} \sigma_i (y^z_i - y_i) \\
&= \sum_{i\in\mathcal{E}} \sigma_i \left(G_i(z) \sum_{j:\kappa_j=\theta_i} (y_j - y^z_j) \right) - \sum_{i\in\mathcal{E}} \sigma_i (y_i - y^z_i).
\end{aligned} \tag{40}
$$

Now, define

$$\overline{\mathcal{E}} = \{ i \in \mathcal{E} : \sigma_i \neq 0 \}$$

and put

$$\delta_i = |y_i - y^z_i|\,, \qquad i \in \mathcal{E}\,.$$

We have that

$$\delta_i \geq \min_{k\in\overline{\mathcal{E}}} \delta_k \geq \frac{\|\delta\|_1}{|\mathcal{E}|}, \quad \forall\, i \in \overline{\mathcal{E}}.$$

Then by (40)

$$
\begin{aligned}
&\sum_{i\in\mathcal{E}} \sigma_i \left(G_i(z) \sum_{j:\kappa_j=\theta_i} (y_j - y^z_j) \right) - \sum_{i\in\mathcal{E}} \sigma_i (y_i - y^z_i) \\
&\leq \sum_{i\in\overline{\mathcal{E}}} \left(G_i(z) \sum_{j\in\overline{\mathcal{E}}:\kappa_j=\theta_i} \delta_j \right) - \sum_{i\in\overline{\mathcal{E}}} \delta_i \\
&= - \sum_{j\in\overline{\mathcal{E}}} \delta_j \left(1 - \sum_{i\in\overline{\mathcal{E}}:\theta_i=\kappa_j} G_i(z) \right) \\
&\leq - \frac{\|\delta\|_1}{|\mathcal{E}|} \max_{j\in\overline{\mathcal{E}}} \left(1 - \sum_{i\in\overline{\mathcal{E}}:\theta_i=\kappa_j} G_i(z) \right) \\
&\leq - \frac{||\delta||_1}{|\mathcal{V}||\mathcal{E}|} = -\varsigma V(y,z)
\end{aligned} \tag{41}
$$

by using Lemma 2 ■

The following two results show that both $y^z_i(t)$ and $y_i(t)$ stay bounded away from the maximum flow capacity C_i.

Lemma 4: Given the admissible perturbation (32), there exists t_0 in $\mathbb{R}_+$ and, for every link i in $\mathcal{E}$, a finite positive constant $\overline{C}_i$, dependent on h, but not on η, such that for every initial condition $(z(0), x(0))$ in $\mathcal{Z}^\circ \times \mathbb{R}^{\mathcal{E}}_+$

$$y^z_i(t) \leq \overline{C}_i < C_i \qquad \forall t \geq t_0,\ \forall i \in \mathcal{E}.$$

Proof: The fact that $y^z_i(t) \leq \lambda$ for all i in $\mathcal{E}$ follows from the fact that the arrival rate at the origin is unitary. Hence, for all i in $\mathcal{E}$ with $C_i > \lambda$ (and therefore also for $C_i = \infty$) the claim follow with $\overline{C}_i = \lambda$ and $t_0 = 0$. We now consider the case when $C_i < \lambda$ for all i in $\mathcal{E}$. Recall that by the definition of admissible perturbation, the domain of (32) is a closed set $\mathcal{Z}_\beta \subseteq \mathcal{Z}^\circ$. This implies that

$$\xi_i := C_i - \sup\{(A\alpha)_i : \alpha \in \mathcal{Z}_\beta\} > 0.$$

It follows from (18) that

$$C_i - \xi_i = \sup\{(A\alpha)_i : \alpha \in \mathcal{Z}_\beta\} \geq \sup\{(AF^{(\beta)}(l,w))_i\}.$$

Hence, one gets

$$\frac{d}{dt} y^z_i(t) = \eta (A(F^{(\beta)}(l(t), w(t)) - z(t)))_i \leq \eta (C_i - \xi_i - y^z_i).$$

This implies that

$$y^z_i(t) - C_i + \xi_i \leq (y^z_i(0) - C_i + \xi_i) e^{-\eta t} \leq \lambda e^{-\eta t},\ t \geq 0 \tag{42}$$

where the last inequality comes from the fact that $y^z_i(0) \leq \lambda$ and $C_i \geq \xi_i$. For i in $\mathcal{E}$ with $C_i < \lambda$ the claim now follows from (42) by choosing, for example, $\overline{C}_i := C_i - \xi/2$ with $\xi := \min\{\xi_i : i \in \mathcal{E}$ s.t. $C_i < \lambda\}$ and $t_0 := -\eta^{-1} \log(\xi/2\lambda)$. ■

Lemma 5: Given the admissible perturbation (32), there exist some $\eta^* > 0$ and $\tilde{C}_i > 0$ for i in $\mathcal{E}$, such that for every $\eta < \eta^*$ and every initial condition $(z(0), x(0))$ in $\mathcal{Z}^\circ \times \mathbb{R}^{\mathcal{E}}_+$

$$y_i(t) \leq \tilde{C}_i < C_i \qquad \forall t \geq 0,\ \forall i \in \mathcal{E}.$$

Proof: For $t \geq 0$, let us define

$$\zeta(t) := W(x(t), z(t)), \qquad \chi(t) := V(y(t), z(t))$$

where V and W are defined in (34). By the Lemma 4 there exists $t_0 \geq 0$ and a positive constant $\overline{C}_i$ for every i in $\mathcal{E}$, such that for every $t \geq t_0$ and applying the inverse of the function φ_i we get

$$x^z_i(t) \leq x^*_i, \qquad x^*_i := \varphi_i^{-1}(\overline{C}_i) \quad \forall i \in \mathcal{E}. \tag{43}$$

Since $x^z_i(t) \geq 0$, (43) implies that if $|x_i(t) - x^z_i(t)| \geq 2x^*_i$ for some $t \geq t_0$, then $x_i(t) \geq 2x^*_i$ for $t \geq t_0$. Hence $y_i(t) - y^z_i(t) \geq \chi^*_i$ for all $t \geq t_0$, where $\chi^*_i = \varphi_i(2x^*_i) - \overline{C}_i$. Since $\varphi_i(x_i)$ is a strictly increasing function, one has that

$$\chi^*_i = \varphi_i(2x^*_i) - \overline{C}_i > \varphi_i(x^*_i) - \overline{C}_i = 0.$$

Now, let

$$\zeta^* := 2|\mathcal{E}| \max\{x^*_i : i \in \mathcal{E}\}, \quad \chi^* := \min\{\chi^*_i : i \in \mathcal{E}\}.$$

and observe that

$$
\begin{aligned}
&W(x,z) \leq |\mathcal{E}| \max\{|x_i - x^z_i| : i \in \mathcal{E}\} \\
&V(y,z) \geq |y_i - y^z_i| \quad \forall i \in \mathcal{E}.
\end{aligned}
$$

Therefore, it follows that for any $t \geq t_0$, if $\zeta(t) \geq \zeta^*$, then for some i' in $\mathcal{E}$ we have that $|x_{i'}(t) - x^z_{i'}(t)| \geq 2x^*_{i'}$ for $t \geq t_0$. This in turn implies that $\chi(t) \geq \chi^*_{i'} \geq \chi^*$. Hence

$$\zeta(t) \geq \zeta^* \implies \chi(t) \geq \chi^* > 0 \quad \forall t \geq t_0. \tag{44}$$

Moreover by (43) follows that there exists some $\mu > 0$ such that:

$$\sum_{i\in\mathcal{E}} \frac{1}{\varphi'_i(x^z_i(t))} \leq \mu \qquad \forall t \geq t_0.$$

By combining the above with Lemma 3 one finds that for every $u, t \geq t_0$

$$
\begin{aligned}
&\zeta(t) - \zeta(u) = \int_u^t \sum_{i\in\mathcal{E}} \sigma_i \left(\frac{d}{ds} x_i - \frac{d}{ds} x^z_i \right) ds \\
&\leq \int_u^t \nabla_x W(x,z)' H(y,z) ds
\end{aligned}
$$

$$+\int_u^t \sum_{i\in\mathcal{E}} \frac{\eta}{\varphi_i'(x_i^z(t))} |(AF^{(\beta)}(l^z, w^z))_i - (Az)_i| ds$$

$$\leq \int_u^t (-\varsigma\,\chi(s) + 2\lambda\eta\mu)\; ds. \tag{45}$$

Now, by contradiction, let us assume that $\limsup_{t\to\infty} y_i(t) \geq C_i$ for some i in $\mathcal{E}$. Since $y_i(t) = \varphi_i(x_i(t)) < C_i$ for every $t \geq 0$, this would imply that $\limsup_{t\to\infty} x_i(t) = \infty$. From this follows that the $\limsup_{t\to\infty} \zeta(t) = \infty$. Then, in particular, the set $\mathcal{T} := \{t > 0 : \zeta(t) > \zeta(s)\ \forall\ s < t\}$ is an unbounded union of open intervals with $\lim_{t\in\mathcal{T}, t\to\infty} \zeta(t) = \infty$. This and (44) imply that there exists a nonnegative constant $t^* \geq t_0$ such that

$$\chi(t) \geq \chi^* \quad \forall t \in \mathcal{T} \cap [t^*, \infty). \tag{46}$$

Defining $\eta^* := \varsigma\,\chi^*/2\lambda\mu$, for every $\eta < \eta^*$, (45) and (46) give

$$\zeta(t) - \zeta(u) \leq \int_u^t (-\varsigma\,\chi(s) + 2\lambda\eta\mu)\; ds$$

$$\leq \int_u^t (-\varsigma\,\chi^* + 2\lambda\eta\mu)\; ds < 0$$

for every $t > u \geq t^*$ such that t and u belong to the same connected component of $\mathcal{T}$. But this contradicts the definition of $\mathcal{T}$. Hence, if $\eta < \eta^*$ then $\limsup_{t\to\infty} y_i(t) < C_i$ for any i in $\mathcal{E}$. Since $\sup_{t\in\mathcal{I}} y_i(t) = y_i(\hat{t}) < C_i$ for some $\hat{t}$ on every compact time interval $\mathcal{I} \subseteq \mathbb{R}_+$, the claim follows. ■

Lemma 6: There exist constants $K > 0$ and $t_1 \geq 0$ such that for every initial condition $(z(0), x(0))$ in $\mathcal{Z}^\circ \times \mathbb{R}_+^{\mathcal{E}}$, $\|\tilde{\nabla}_z h(z(t))\| \leq K$ for all $t \geq t_1$.

Proof: From Lemma 5, there exists $T^*, \upsilon^* > 0$ such that $\|l(t)\| \leq T^*$ and $\|w(t)\| \leq \upsilon^*$ for all $t \geq 0$. This fact together with the definition of $F^{(\beta)}(l, w)$ (18) implies that $F^{(\beta)}(l(t), w(t))$ belongs to $\mathcal{Z}_\beta^\circ$ and $\tilde{\nabla}_z h(F^{(\beta)}(l(t), w(t))) = -\Phi A'(l(t) + w(t))$. Hence $\|\tilde{\nabla}_z h(F^{(\beta)}(l(t), w(t)))\| \leq \|\Phi\|\|A'\|S^*$, with $S^* = T^* + \upsilon^*$. This implies the existence of a convex compact $\mathcal{K} \subset \mathcal{Z}_\beta^\circ$ such that $F^{(\beta)}(l(t), w(t))$ belongs to $\mathcal{K}$ for all $t \geq 0$. Define

$$\Delta(t) := \frac{\eta}{1 - e^{-\eta t}} \int_0^t e^{-\eta(t-s)} F^{(\beta)}(l(s), w(s))\, ds.$$

Since $\Delta(t)$ is an average of elements of the convex set $\mathcal{K}$, then $\Delta(t) \in \mathcal{K}\ \forall t \geq 0$. Moreover, $z(t) = e^{-\eta t} z(0) + (1 - e^{-\eta t})\Delta(t)$ approaches $\mathcal{K}$, which implies that for large enough t, $z(t)$ belongs to a closed subset $\mathcal{K}_1$ of $\mathcal{Z}_\beta^\circ$ that contains $\mathcal{K}$. Hence, after large enough t, say, t_1, $\tilde{\nabla}_z h(z(t))$ stays bounded. ■

Lemma 7: There exist $\ell > 0$ and $t_0 \geq 0$ such that for every initial condition $(z(0), x(0))$ in $\mathcal{Z}^\circ \times \mathbb{R}_+^{\mathcal{E}}$,

$$\tilde{\nabla}_z W(x(t), z(t))'(F^{(\beta)}(l(t), w(t)) - z(t)) \leq 2\lambda\ell|\mathcal{E}| \quad \forall t \geq t_0.$$

Proof: Observe that thanks to Lemma 4 there exists $t_0 \geq 0$ such that $\ell_i := \sup\{1/\varphi_i'(x_i^z(t)) : t \geq t_0\} < +\infty$. Put $\ell := \max\{\ell_i : i \in \mathcal{E}\}$. Then, for every path γ in Γ and for every $t \geq t_0$, one has

$$\left|\frac{\partial W(x,z)}{\partial z_\gamma}\right| = \left|-\sum_{i\in\mathcal{E}} \sigma_i \frac{\partial}{\partial z_\gamma} x_i^z\right|$$

$$= \left|\sum_{i\in\mathcal{E}} \sigma_i \frac{\partial}{\partial z_\gamma} \varphi_i^{-1}\left(\sum_\gamma A_{i\gamma} z_\gamma\right)\right|$$

$$\leq \sum_{i\in\mathcal{E}} A_{i\gamma} \frac{1}{\varphi_i'(x_i^z)} \leq \sum_{i\in\mathcal{E}} A_{i\gamma}\ell_i \leq \ell|\mathcal{E}|.$$

Therefore

$$2\lambda\ell|\mathcal{E}| \geq \sum_\gamma F_\gamma^{(\beta)}(l, w) \left|\frac{\partial W(x,z)}{\partial z_\gamma}\right| + \sum_\gamma z_\gamma \left|\frac{\partial W(x,z)}{\partial z_\gamma}\right|$$

$$\geq \sum_\gamma F_\gamma^{(\beta)}(l, w) \frac{\partial W(x,z)}{\partial z_\gamma} - \sum_\gamma z_\gamma \frac{\partial W(x,z)}{\partial z_\gamma}$$

$$= \tilde{\nabla}_z W(x,z)'(F^{(\beta)}(l, w) - z)$$

thus proving the claim. ■

We now combine Lemmas 3 and 7 in order to estimate the behavior in time of $W(x(t), z(t))$.

Lemma 8: There exist $\ell, L, \eta^* > 0$ and $t_0 \geq 0$ such that for every initial condition $z(0)$ in $\mathcal{Z}$, $x(0)$ in $[0, +\infty)^{\mathcal{E}}$

$$W(x(t), z(t)) \leq$$

$$\frac{2\lambda\ell L\eta|\mathcal{E}|}{\varsigma} + e^{-\varsigma(t-t_0)/L}\left(W(x(t_0), z(t_0)) - \frac{2\lambda\ell L\eta|\mathcal{E}|}{\varsigma}\right)$$

for $t \geq t_0$ and $\eta < \eta^*$.

Proof: Define $\zeta(t) := W(x(t), z(t))$. Note that thanks to Lemmas 4 and 5, there exist $L > 0$, $\eta^* > 0$ and $t_0 \geq 0$ such that for any $\eta < \eta^*$

$$|x_i(t) - x_i^z(t)| \leq L|y_i(t) - y_i^z(t)| \quad \forall i \in \mathcal{E}, t \geq t_0.$$

This involves that

$$V(y(t), z(t)) \geq \frac{1}{L} W(x(t), z(t)) = \frac{1}{L}\zeta(t) \quad \forall \eta < \eta^*, t \geq t_0.$$

Moreover $W(x, z)$ is a Lipschitz function of x and z, while both $x(t)$ and $z(t)$ are Lipschitz on every compact time interval. Therefore $\zeta(t)$ is Lipschitz on every compact time interval and hence absolutely continuous. Thus, $d\zeta(t)/dt$ exists for almost every $t \geq 0$, and, thanks to Lemmas 3 and 7 it satisfies

$$\frac{d\zeta(t)}{dt} = \frac{dW(x(t), z(t))}{dt}$$

$$= \nabla_x W(x,z)' H(y,z) + \eta\tilde{\nabla}_z W(x,z)'(F^{(\beta)}(l,w) - z)$$

$$\leq -\varsigma V(y,z) + 2\lambda\ell\eta|\mathcal{E}| \leq -\frac{\varsigma\,\zeta(t)}{L} + 2\lambda\ell\eta|\mathcal{E}|.$$

Then, integrating both sides we get the claim. ■

A. Proof of Theorem 1

We are now in a position to prove Theorem 1. Let us consider the function

$$\Theta : \mathcal{Z} \to \mathbb{R}_+, \quad \Theta(z) := \sum_{i\in\mathcal{E}} \int_0^{y_i^z} (\tau_i(s) + \omega_i(s))\; ds \tag{47}$$

and observe that

$$\tilde{\nabla}\Theta(z) = \Phi A'(l^z + w^z) \qquad \forall z \in \mathcal{Z}^\circ. \tag{48}$$

Note that since $\tau_i(y_i) + \omega_i(y_i)$ is increasing, then the map $y_i \mapsto \int_0^{y_i^z} (\tau_i(y_i) + \omega_i(y_i))\, dy_i$ is convex. Hence, the composition with the linear map $z \mapsto y_i^z = \sum_\gamma A_{i\gamma} z_\gamma$ is convex in z, which in turn implies convexity of Θ over $\mathcal{Z}$. Since $h(z)$ defined in (32) is strictly convex, we obtain strict convexity of $\Theta(z) + h(z)$ on $\mathcal{Z}_\beta$. Then, since $\mathcal{Z}_\beta$ is a compact and convex set, there exists a unique minimizer

$$z^\beta := \arg\min\{\Theta(z) + h(z) : z \in \mathcal{Z}_\beta\}. \tag{49}$$

Let now $y^{(\omega,\beta)} := y^{z^\beta}$. Then, the following result holds true.

Lemma 9: The perturbed equilibrium flow $y^{(\omega,\beta)}$ in $\mathcal{F}$ is such that

$$\lim_{\beta\to+\infty} y^{(\omega,\beta)} = y^{(w)}.$$

Proof: Since $\{Az^\beta\} \subseteq A\mathcal{Z}$, and $A\overline{\mathcal{Z}}$ is compact, there exists a converging subsequence $\{Az^{\beta_k} : k \in \mathbb{N}\}$. Let us denote by $\hat{y} := \lim_k Az^{\beta_k}$ in $A\overline{\mathcal{Z}}$ its limit and choose some $\hat{z}$ in $\overline{\mathcal{Z}}$ such that $\hat{y} = A\hat{z}$. Notice that since

$$\sup\{\tau_i(y_i^z) + \omega_i(y_i^z) : z \in \mathcal{Z}_\beta\} < +\infty\,, \qquad \forall i \in \mathcal{E}$$

the differentiability of h in the interior set $\mathcal{Z}_\beta^\circ$ of $\mathcal{Z}_\beta$ implies that the minimizer in (49) belongs to $\mathcal{Z}_\beta^\circ$. As a consequence, one finds that necessarily

$$\tilde{\nabla}_z h(z^{\beta_k}) = -\Phi A'(\tau(Az^{\beta_k}) + \omega(Az^{\beta_k}))$$

which successively implies that $F^{\beta_k}(\tau(Az^{\beta_k}), \omega(Az^{\beta_k})) = z^{\beta_k}$. Then, using (33), one finds that

$$\begin{aligned} &(Az^{\beta_k})'(\tau(Az^{\beta_k}) + \omega(Az^{\beta_k})) + h_{\beta_k}(z^{\beta_k}) \\ &\leq (Az^{\beta_k})'(\tau(Az^{\beta_k}) + \omega(Az^{\beta_k})) + h_{\beta_k}(\alpha) \end{aligned} \tag{50}$$

for all α in $\mathcal{Z}_{\beta_k}$. Now, fix any z in $\mathcal{Z}$. Since $\mathcal{Z}_\beta \to \overline{\mathcal{Z}}$ as $\beta \to +\infty$,[2] then there exists a sequence $\{\tilde{z}^k\}$ such that $\tilde{z}^k$ belongs to $\mathcal{Z}_{\beta_k}$ for all k and $\lim_k \tilde{z}^k = z$. Hence, taking $\alpha = \tilde{z}^k$ in (50) and passing to the limit as k grows large, one finds that

$$\hat{z}' A'(\tau(\hat{y}) + \omega(\hat{y})) \leq z' A'(\tau(\hat{y}) + \omega(\hat{y})) \quad \forall\, z \in \mathcal{Z}.$$

In turn, the above can be easily shown to be equivalent to the characterization (23) of Wardrop equilibria. From the uniqueness of the Wardrop equilibrium, it follows that necessarily $\hat{y} = y^{(w)}$. Then, the claim follows from the arbitrariness of the accumulation point $\hat{y}$, hence $y^{(\omega,\beta)} \to y^{(w)}$. ■

We now estimate the time derivative of $\Theta(z) + h(z)$ along trajectories of our dynamical system. Toward this goal, define

$$\begin{aligned} \Psi(t) &:= \Theta(z(t)) + h(z(t)) \\ \psi(t) &:= \Phi A'(l^z(t) + w^z(t)) + \tilde{\nabla}_z h(z(t))\,. \end{aligned}$$

Then, using (48), we get

$$\begin{aligned} \dot{\Psi}(t) &= \left(\tilde{\nabla}_z \Theta + \tilde{\nabla} h(z)\right) \dot{z} \\ &= \eta\psi(t)'(F^{(\beta)}(l(t), w(t)) - z(t)) \\ &= \eta\psi(t)'(F^{(\beta)}(l^z(t), w^z(t)) - z(t)) \\ &\quad + \eta\psi(t)'(F^{(\beta)}(l(t), w(t)) - F^{(\beta)}(l^z(t), w^z(t)))\,. \end{aligned} \tag{51}$$

[2]Here, $\overline{\mathcal{Z}}$ stands for the closure of $\mathcal{Z}$ and the convergence $\mathcal{Z}_\beta \to \overline{\mathcal{Z}}$ is meant to hold true with respect to the Hausdorff metric.

By Lemma 8, there exist $t_2 \geq 0, \eta^* > 0$ and $M_1 > 0$ such that $W(x(t), z(t)) \leq \eta M_1$ for all $\eta < \eta^*$ and $t \geq t_2$. From the definition of W it follows that $W(x, z) \geq \|x - x^z\|_1/|\mathcal{E}|$ for all x, z. Moreover, the properties of φ imply that $\|y - y^z\|_1 \leq \overline{L}\|x - x^z\|_1$ for all y, z, and $\overline{L} := \max\{\varphi_i'(0) : i \in \mathcal{E}\}$. Combining all these relationships, we get that there exists $M > 0$ such that, for every $\eta < \eta^*$

$$\|y(t) - y^z(t)\| \leq \eta M \qquad \forall t \geq t_2 \tag{52}$$

where $M = |\mathcal{E}| M_1 \overline{L}$. Thanks to the differentiability of $F^{(\beta)}$ on $\mathbb{R}_+^{\mathcal{E}} \times \mathbb{R}_+^{\mathcal{E}}$ and the boundedness of both $y^z(t)$ and $y(t)$ one gets that

$$\|F^{(\beta)}(l(t), w(t)) - F^{(\beta)}(l^z(t), w^z(t))\| \leq K_1 \eta$$

for some positive constant $K_1, \eta < \eta^*$ and large enough t. Since Lemmas 4 and 6 guarantee that that $l^z(t)$, $w^z(t)$ and $\tilde{\nabla}_z h(z(t))$ are eventually bounded, there exists a positive constant K_2 such that $\|\psi(t)\| \leq K_2$ for t large enough. This implies that the second addend in the last line of (51) can be bounded as

$$\eta\psi(t)'(F^{(\beta)}(l(t), w(t)) - F^{(\beta)}(l^z(t), w^z(t))) \leq K\eta^2 \tag{53}$$

where $K = K_1 K_2$, for all $\eta < \eta^*$ and $t \geq t_3$ for some sufficiently large but finite value of t_3. Now, observe that

$$\Phi A'(l^z(t) + w^z(t))) = -\tilde{\nabla}_z h(F^{(\beta)}(l^z(t), w^z(t)))$$

for every z in $\mathcal{Z}$, so that the first addend in the last line of (51) may be rewritten as

$$\psi(t)'(F^{(\beta)}(l^z(t), w^z(t)) - z(t)) = -\Upsilon(z(t)) \tag{54}$$

where

$$\begin{aligned} \Upsilon(z(t)) = &\left(\tilde{\nabla}_z h(F^{(\beta)}(l^z(t), w^z(t))) - \tilde{\nabla}_z h(z(t))\right)' \\ &\cdot (F^{(\beta)}(l^z(t), w^z(t)) - z(t)). \end{aligned}$$

It follows from (51), (53), and (54) that for $\eta < \eta^*$ and $t \geq t_3$

$$\dot{\Psi}(t) \leq -\eta\Upsilon(z(t)) + K\eta^2. \tag{55}$$

From the strict convexity of $h(z)$ on $\mathcal{Z}_\beta$, $\Upsilon(z(t)) \geq 0$ for every z, with equality if and only if $z = z^\beta$. Now, put

$$\bar{\delta}(r) = \begin{cases} \sup\{\|y^z - y^{(\omega,\beta)}\| : \Upsilon(z) \leq Kr\} + Kr & \text{if} \quad 0 \leq r < \eta^* \\ \tilde{C}\sqrt{|\mathcal{E}|} & \text{if} \quad r \geq \eta^* \end{cases}$$

where $\tilde{C} := \max\{1, \tilde{C}_i : i \in \mathcal{E}\}$, with $\tilde{C}_i$ as defined in Lemma 5. It can be proved that $\bar{\delta}(r)$ is nondecreasing, right-continuous, and such that $\lim_{\eta\to 0} \bar{\delta}(\eta) = \bar{\delta}(0) = 0$. Then, (52) and (55) imply that for $\eta < \eta^*$

$$\limsup_{t\to\infty} \|y(t) - y^{(\omega,\beta)}\| \leq \bar{\delta}(\eta). \tag{56}$$

Note that since $y(t)$ in $[0, \tilde{C}]^{\mathcal{E}}$ and $y^{(\omega,\beta)}$ in $AZ \subseteq [0,1]^{\mathcal{E}}$ then $|y_i(t) - y_i^{(\beta)}| \leq \max\{\tilde{C}_i, 1\} \leq \tilde{C}$ for all i in $\mathcal{E}$ and hence $\|y(t) - y^{(\omega,\beta)}\|^2 \leq |\mathcal{E}|\tilde{C}^2$. Then (56) also holds for $\eta \geq \eta^*$, since in that range $\bar{\delta}(r) = \tilde{C}\sqrt{|\mathcal{E}|}$. Together with Lemma 9, this concludes the proof of Theorem 1. □

V. Possible Extensions of the Results

As discussed, the framework and results presented in the previous sections have arguably two major limitations: the assumption that there is a single origin/destination pair and the assumption that the link flow-density functions are strictly increasing. In this section, we briefly discuss possible extensions of our results that include relaxations of these two assumptions.

First, it is possible to extend our results to the case of multiple origin-destination pairs as follows. Let $\{(o_k, d_k)\}_{k\in\mathcal{K}}$ be a set of origin-destination pairs, where $o_k \neq d_k$ in $\mathcal{V}$ for each k in $\mathcal{K}$. Let λ in $\mathbb{R}_+^{\mathcal{K}}$ be a vector of associated throughputs

$$\nu = \sum_{k\in\mathcal{K}} \lambda_k \left(\delta^{(\theta_{o_k})} - \delta^{(\kappa_{d_k})}\right), \qquad \nu^+ = [\nu] \qquad \nu^- = [\nu]_- .$$

Let Γ_k be the set of (o_k, d_k)-paths and $A^{(k)}$ in $\{0,1\}^{\mathcal{E}\times\Gamma_k}$ the link-path incidence matrix. Let $\Gamma = \cup_{k\in\mathcal{K}}\Gamma_k$ and A in $\{0,1\}^{\mathcal{E}\times\Gamma}$ be the link-path incidence matrix. Let

$$\mathcal{S}_\lambda = \left\{ z \in \mathbb{R}_+^\Gamma : \sum\nolimits_{\gamma\in\Gamma_k} z_\gamma = \lambda_k \right\}.$$

For every z in $\mathcal{Z}_\lambda$, $y^z = Az$ is an equilibrium flow vector satisfying $By^z = \nu$. Define $G(z)$ as in (14) and extend (12) and (20) as

$$\dot{x}_i(t) = \nu_i^+ + \sum_{j\in\mathcal{E}} R_{ji}(t) y_j(t) - y_i(t) \tag{57}$$

and

$$H_i(y,z) := G_i(z)\left(\nu_i^+ + \sum_{j:\kappa_j=\theta_i} y_j\right) - y_i\,, \qquad i \in \mathcal{E}\,. \tag{58}$$

respectively. Then, all the results carry over with the notion of Wardrop equilibrium defined as in [20, Sec. 2.1] and the min-cut feasibility condition (cf. [15])

$$\sum_{i\in\mathcal{U}} \nu_i < \sum_{\substack{i\in\mathcal{E}:\\ \theta_i\in\mathcal{U},\, \kappa_i\notin\mathcal{U}}} C_i\,, \qquad \forall \mathcal{U}\subseteq\mathcal{V}\,.$$

Notice that the extension illustrated above allows one for considering multiple origin-destination pairs. However, it considers physical dynamics of the traffic flows with a single aggregate commodity, while it keeps the commodities separated as far as the route decision dynamics are concerned. An alternative approach could entail a multicommodity model also of the physical dynamics of the traffic flows. However, such multicommodity dynamical flow networks would lose fundamental monotonicity properties (cf. [46]) that enable, in particular, the proof of Lemma 2 as presented in this article. This means that, in order to generalize the results of this article with a multicommodity physical dynamics of the traffic flows, one should be able to find different ways to guarantee their global exponential stability.

Finally, as mentioned in Remark 1, the fact that the flow-density functions are strictly increasing limits the applicability of the results in this article in road traffic network applications to the so-called free-flow region. One possible approach to extend the setting outside such free-flow region consists in modeling the physical dynamics of the traffic flows with monotone non-FIFO versions of the cell transmission model [18] as proposed and analyzed, e.g., in [47], thus keeping monotonicity and contractivity properties of the physical flow dynamics. The difficulty in this case comes from the fact that the outflow from and the latency on a cell would depend on the densities both on that cell and on the ones immediately downstream, thus making one lose separability of the latency functions. Such an approach may possibly be pursued using techniques developed in the context of traffic assignment problems with nonseparable cost functions, see, e.g., [48]–[50] and [20, Sec. 2.5].

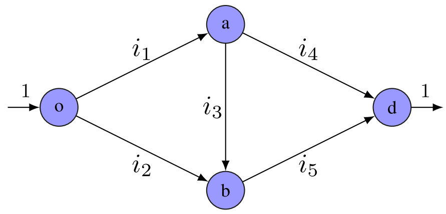


Fig. 5. Graph topology used for the simulations.

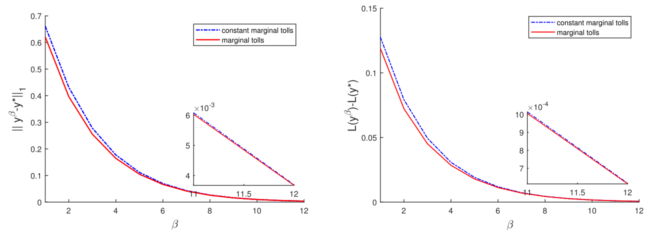


Fig. 6. Plot of $\|y^{(\omega,\beta)}(T) - y^*\|_1$ and $\mathcal{L}(y^{(\omega,\beta)}(T)) - \mathcal{L}(y^*)$ for decentralised marginal and constant marginal tolls.

VI. Numerical Simulations

In this section, we present a numerical study comparing the asymptotic and transient performance of multiscale transportation networks controlled by dynamic feedback marginal cost tolls (28) and precomputed constant marginal cost tolls (30).

For the network topology of Fig. 5 and for several values of the parameter η, we found that dynamic feedback marginal cost tolls outperform the constant marginal ones. Specifically

1) Concerning the transient convergence, it appears that the time needed to reach the perturbed equilibrium associated to the dynamic feedback marginal cost tolls is lower than the time to reach the perturbed equilibrium associated to the constant marginal cost ones.
2) As the uncertainty parameter β of the route choice goes to infinity the perturbed equilibrium associated to dynamic feedback marginal cost tolls asymptotically converges to the social optimum flow faster than the one associated to the constant marginal cost tolls.

We illustrate these findings in the following simple case:

- Network topology $\mathcal{G}$ as in Fig. 5.
- Flow-density function as in (9) and corresponding latency function as in (10), with capacity $C_i = 2$ for every link i.
- $F^{(\beta)}$ as in (18), $\eta = 0.1$, G as in (15) and $\lambda = 1$.
- Initial conditions: $z_{\gamma^{(1)}}(0) = 1/2$, $z_{\gamma^{(2)}}(0) = 1/6$, $z_{\gamma^{(3)}}(0) = 1/3$ $x_{i_1}(0) = 4$, $x_{i_2}(0) = 2$, $x_{i_3}(0) = 3$, $x_{i_4}(0) = 1$, $x_{i_5}(0) = 5$.

Having settled a time horizon $T = 350$, Fig. 6 displays the l_1-distance and the latency loss of $y^{(\omega,\beta)}(T)$ from the system optimum $y^* = (1/2, 1/2, 0, 1/2, 1/2)$, for different values of the uncertainty parameter β. This is done both considering (28) and the constant marginal tolls (30). Note that while our theoretical

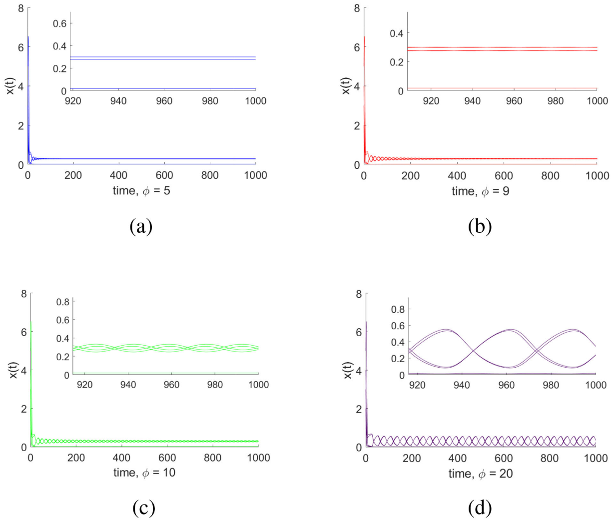


Fig. 7. The density vector trajectory $x(t)$ for different values of the information delay, ϕ.

results guarantee that $y^{(\omega,\beta)}(T)$ converges to y^* only in the double limit of large T (asymptotically in time) and large β (vanishing noise), in our numerical examples convergence is practically observed already for relatively small values of β. Our simulations also suggest that convergence of $y^{(\omega,\beta)}(T)$ to the system optimum is faster for the feedback marginal cost tolls (28) than for the fixed marginal cost (30). Hence, in addition to variations of network's parameters and exogenous loads, feedback marginal cost tolls appear to be more robust than their constant counterparts also when it comes to noise.

A. Effect of Information Delays

In this section, we study the effects of delays in the global information of the slow scale dynamics (17) on the system (19). Considering at first the case of marginal cost tolls, we fix a time-delay ϕ so that the cost perceived by each user crossing a link i in $\mathcal{E}$ is $l_i(t-\phi)+w_i(t-\phi)$. Fixing the uncertainty parameter β and varying ϕ, we observe how the time-evolution of the density $x(t)$ is changed and how the corresponding flow y approximates the social optimum flow $y^*(\lambda)$ with $\lambda=1$. For that, we consider the graph topology as in Fig. 5 and the same parameters as before. Then, fixing $\beta=5$, we numerically compute the trajectory $x(t)$ for different values of the delay ϕ as shown in Fig. 7. In Fig. 7(a) and (b) we can note that the density vector $x(t)$ converges to an equilibrium. By numerical simulations, one gets that $\phi=9$ is the largest value for which one has convergence [see Fig. 7(b)]. In fact, for $\phi>9$ one witnesses a phase transition of the system, with the emergence of an oscillatory behavior. We can also note in Fig. 7(c) and (d) that the larger ϕ is, the larger the oscillation amplitude and phase are. A similar situation can be observed in the plot of the l_1-distance of y from y^* in Fig. 8, for the same value of ϕ used in Fig. 7.

Consider now the case of constant marginal cost tolls (30). Let ϕ be the time delay as before and $\tau_i(y_i(t-\phi))+w_i^*$ the cost perceived by each user crossing a link i in $\mathcal{E}$. Still using

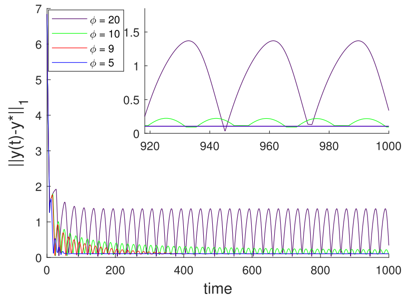


Fig. 8. Plot of $\|y(t)-y^*\|_1$ for different values of the delay ϕ.

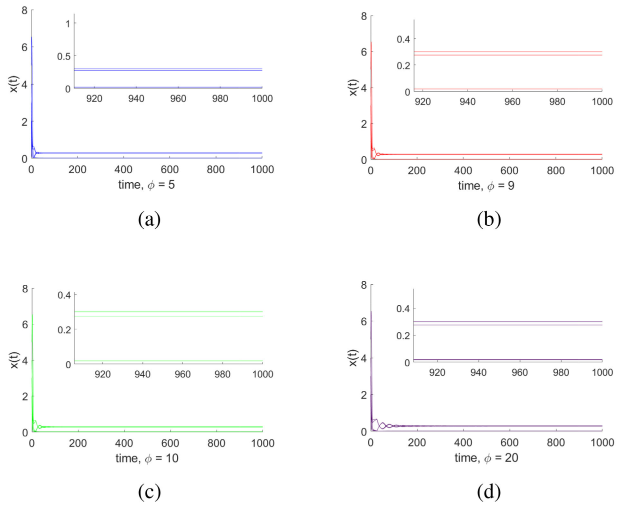


Fig. 9. Trajectories with constant marginal tolls, for different values of the delay ϕ.

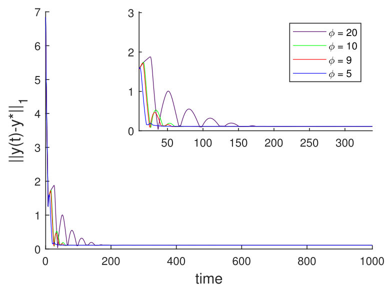


Fig. 10. Plot of $\|y(t)-y^*\|_1$ for different values of ϕ.

the graph topology as in Fig. 5 and fixing $\beta=5$ we numerically compute the trajectory of the density vector $x(t)$ and the l_1-distance of the corresponding flow vector $y(t)$ from the social optimal y^*. We perform this for the same values of time delay ϕ used before. From Fig. 9, we can note that for all considered values of ϕ the trajectory x converges to the equilibrium. This differs from what happens using the marginal cost tolls (see Fig. 7) and highlights how time-delays affect marginal cost tolls more than their constant counterpart. The plot of the 1-norm, Fig. 10, confirms the same trend, indeed after some initial oscillations, the 1-norm is the same for the different values of ϕ.

VII. Conclusion

We have studied the stability of multiscale dynamical transportation networks with distributed dynamic feedback pricing. We have proved that, if the frequency of path preferences updates is sufficiently low, monotone decentralized flow-dependent dynamical tolls make the network asymptotically approach a neighborhood of a generalized Wardrop equilibrium. For a particular class of dynamic feedback tolls, i.e., the marginal cost ones, we have proved that the stability is guaranteed to be around the social optimum equilibrium.

Through numerical experiments, both asymptotic and transient performance have been shown to be better with dynamic feedback marginal cost tolls than with constant ones. Finally, the impact of information delays has been investigated through numerical simulations, showing how such delays influence the stability and convergence of the network flow dynamics. In particular, feedback marginal cost tolls appear to be more fragile to information delays that constant tolls.

These findings motivate future research aimed at providing analytical estimates of the different convergence rates. It would also be worth analytically investigating the robustness of feedback tolls to information delays and to consider anticipatory learning dynamics incorporating derivative actions (c.f., [51]).

Appendix A
Proof of Lemma 1

The fact that the latency function $\tau_i(y)$ is twice continuously differentiable on $[0, C_i)$, strictly increasing, and such that $\tau_i(0) > 0$ directly follows from Assumption 1.

For a given y in $[0, C_i)$, let $x = \varphi_i^{-1}(y)$, $a = \varphi_i'(x)$, and $b = \varphi_i''(x)$. Then

$$\tau_i'(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\varphi_i^{-1}(y)}{y}\right) = \frac{y/a - x}{y^2} = \frac{y - ax}{ay^2}$$

thus proving (6).

We now prove that $y \mapsto y\tau_i(y)$ is strictly convex by computing its second derivative. For that, first notice that

$$\frac{\mathrm{d}a}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}\varphi_i'(\varphi^{-1}(y)) = \frac{\varphi_i''(x)}{\varphi_i'(x)} = \frac{b}{a}$$
$$\frac{\mathrm{d}}{\mathrm{d}y}(y - ax) = 1 - \frac{b}{a}x - a\frac{1}{a} = -\frac{b}{a}x$$

and

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(ay^2\right) = \frac{b}{a}y^2 + 2ya\,.$$

Then

$$\begin{aligned}(y\tau_i(y))'' &= 2\tau_i'(y) + y\tau_i''(y)\\ &= \frac{2(y - xa)}{ay^2} + y\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y - xa}{ay^2}\right)\\ &= \frac{2(y - xa)}{ay^2} + \frac{-bxy^2 - (y - xa)\left(y^2\frac{b}{a} + 2ya\right)}{a^2y^3}\\ &= -\frac{b}{a^3}\,.\end{aligned}$$

Now, observe that Assumption 1 guarantees that $a > 0$ and $b < 0$. Hence, $(y\tau_i(y))'' > 0$ and therefore $y\tau_i(y)$ is strictly convex, thus completing the proof. □

Appendix B
Proof of Proposition 1

From Assumption 1 and the fact that the toll on a link is a nondecreasing function of the flow on that link only, it follows that the perceived cost function $\tau_i(y_i) + \omega_i(y_i)$ on link i is continuous, strictly increasing, and grater than zero when $y_i = 0$. The claim then follows as a direct application of [20, Th. 2.4 and 2.5]. □

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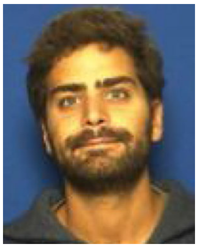

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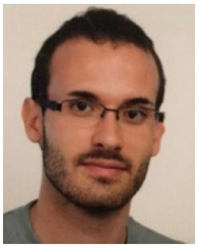

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