Prime Suspects in a Quantum Ladder

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In this Letter we set up a suggestive number theory interpretation of a quantum ladder system made of $\mathcal{N}$ coupled chains of spin $1/2$. Using the hard-core boson representation and a leg-Hamiltonian made of a magnetic field and a hopping term, we can associate to the spins σ_a the prime numbers p_a so that the chains become quantum registers for square-free integers. The rung Hamiltonian involves permutation terms between next-neighbor chains and a coprime repulsive interaction. The system has various phases; in particular, there is one whose ground state is a coherent superposition of the first $\mathcal{N}$ prime numbers. We also discuss the realization of such a model in terms of an open quantum system with a dissipative Lindblad dynamics.

Introduction.—The aim of this Letter is to point out some interesting connections between quantum many-body systems and number theory, in particular, prime numbers. Prime numbers are the building blocks of arithmetics and, arguably, one of the pillars of the entire mathematics [1,2]. Their nature has two fascinating but opposite features [3]: If their appearance in the sequence of natural numbers is rather unpredictable, their coarse-graining properties [e.g., their total number $\pi(x)$ less than x] can be captured instead rather efficiently by simple statistical considerations [4–8]. In particular, the scaling of the kth prime is particularly plain:

$$p_k \simeq k \log k. \tag{1}$$

Equally fascinating is the connection between prime numbers and quantum mechanics: Prime numbers, for instance, were the main concern of Shor's algorithm, one of the first quantum computing algorithms [9]. Moreover, the scaling behavior (1) permits one to show the existence of a single-particle one-dimensional quantum mechanical potential $V(x)$ with eigenvalues given just by the prime numbers and, therefore, permits one to address the primality test of a natural number in terms of a quantum scattering [10]. Such a potential $V(x)$ can be determined either semiclassically [10] or exactly, using in this case methods of supersymmetric quantum mechanics [11,12]. In experimental setups of cold atom systems, $V(x)$ could be realized using a holographic trap [13].

Turning now our attention to quantum many-body systems, for the dense nature of their spectra it is obviously impossible to have energy levels given by prime numbers, but we can have instead many-body ground state wave functions expressed in terms of prime numbers. This is what we are going to present below, where we consider a quantum ladder system with a suggestive number theoretic interpretation. We will see that such a system has a rich spectrum of ground states and, in particular, there is one whose wave function is given in terms of a highly coherent superposition of prime number occupations. To the best of our knowledge, this is the first time where a ground state of this type has been constructed.

Quantum ladder systems, made of coupled one-dimensional chains, have attracted considerable interest in recent years as truly interpolating between one- and two-dimensional systems [14–20]. In our case, we have $\mathcal{N}$ coupled half-infinite chains of spins $1/2$ subjected to a magnetic field and a hopping term. As discussed below, properly tuning these two interactions, we can put in correspondence the spins with the prime numbers and reformulate the spin-spin rung interaction in terms of coprimality conditions (two integers are coprime if they do not share common factors other than 1).

Degrees of freedom.—As it is well known, spin $1/2$ can be described by hard-core bosons: The mapping between the Pauli matrices σ_a and the hard-core annihilation and creation operators f and $f^\dagger$ [$f^2 = (f^\dagger)^2 = 0$] is provided by $\sigma_z = f^\dagger f - 1/2$; $\sigma_+ = f^\dagger$; $\sigma_- = f$ [21]. Hence, instead of the spins, we can equivalently take as degrees of freedom the hard-core boson operators $f_i(a)$ and $f_i^\dagger(a)$, where the index i refers to the ith chain ($i = 1, 2, \ldots, \mathcal{N}$), while $a = 1, 2, \ldots$ to the vertical position along the half-infinite chain (see Fig. 1). Since $[f_i(a)]^2 = [f_i^\dagger(a)]^2 = 0$, the occupation number of each vertical site in the ladder can take only values $\{0, 1\}$. Let $|\mathrm{vac}\rangle$ be the vacuum state, i.e.,

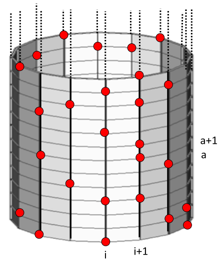

FIG. 1. Ladder system of $\mathcal{N}$ coupled half-infinite quantum chains of hard-core bosons. Red circles refer to occupied sites.

the state which is annihilated by all the $f_i(a)$'s. For each chain, we can then define the state

$$|n_i\rangle = \left(\prod_{a=1}^{k}[f_i^\dagger(a)]^{\alpha_a}\right)|\mathrm{vac}\rangle, \qquad \alpha_a = \{0,1\}. \quad (2)$$

We will show below [see, in particular, Eqs. (9)–(11)] that it is possible to associate to the ath hard-core boson along each chain the ath prime number: This allows us to use the notation $|p_a\rangle_i = f_i^\dagger(a)|\mathrm{vac}\rangle$ and to define for each chain a set of integers whose general form is

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}, \qquad \alpha_a = \{0,1\}. \quad (3)$$

These are the so-called *square-free numbers*, i.e., those integers whose prime factors do not divide them more than once. Their first representatives are $n = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, \ldots$. Remarkably, these numbers are a finite fraction (i.e., $6/\pi^2$) of all the integers [7]: Indeed, assuming $1/p$ to be the probability that a generic integer is divisible by a prime p, the probability that it is not divisible more than once by a prime is given by $\prod_p (1 - 1/p^2) = 1/\zeta(2) = 6/\pi^2$, where $\zeta(x)$ is the Riemann-zeta function. Any occupation number configuration of the vertical chains can be associated to a square-free number and vice versa, so each chain plays the role of a quantum register for the square-free numbers.

It is useful to define $F_i(a) = f_i^\dagger(a) f_i(a)$ (the number operators of the ath hard-core boson on the ith chain), $F_i = \sum_{a=1} F_i(a)$ (the total number of hard-core bosons of the ith chain), and $F^{(a)} = \sum_i F_i(a)$ (the total number of the a hard-core bosons on the entire ladder lattice). It is also convenient to introduce the numbers operators $\hat{N}_i$ for each chain, such that

$$\hat{N}_i |n_j\rangle = \delta_{i,j} n_j |n_j\rangle. \quad (4)$$

It is worth stressing that the n_i's are *not* the true occupation numbers (the actual occupation numbers at each leg are given by the F_i's). The n_i's may be just regarded as useful labels of the hard-core boson degrees of freedom present at each chain. Notice that there is a one-to-one correspondence between configurations of hard-core bosons and the n_i's since, after all, a decomposition as Eq. (3) not only exists but is also unique.

The Hamiltonian of our system consists of two terms, relative to rung (R) and leg (L) interactions:

$$H = \sum_{i=1}^{\mathcal{N}} [H_{i,+1} + H_i] \equiv \mathcal{H}_R + \mathcal{H}_L. \quad (5)$$

Rung Hamiltonian.—Let us discuss first the rung Hamiltonian $\mathcal{H}_R$ which acts on a bosonic Hilbert space given by occupation numbers n_i's given by the square-free numbers. How to realize such occupation numbers will be discussed later in relation with $\mathcal{H}_L$. Taking for granted this Hilbert space, the explicit expression of $\mathcal{H}_R(\lambda)$ is given by ($\lambda \geq 0$)

$$\mathcal{H}_R(\lambda) = \sum_{i=1}^{\mathcal{N}} H_{i,i+1} = \sum_{i=1}^{\mathcal{N}} [P_{i,i+1} + \lambda \mathcal{C}_{i,i+1}]. \quad (6)$$

$P_{i,i+1}$ is the permutation operator between the two next-neighbor occupation numbers n_i, while $\mathcal{C}_{i,i+1}$ is the *coprimality operator* which counts how many common prime factors are shared between the square-free numbers n_i and n_{i+1}. In terms of the hard-core boson operators, the coprimality operators can be expressed as $\mathcal{C}_{i,i+1} \equiv \sum_{a=1}^{\Lambda} F_i(a) F_{i+1}(a)$, where Λ is a convenient cutoff in the length of the vertical chain (with $\Lambda \to \infty$ taken first and independently of $\mathcal{N}$).

$\mathcal{H}_R$ clearly conserves the total numbers of each n_i's. The Hilbert space is then partitioned in sectors $\mathcal{S}_{\omega_1 \ldots \omega_k}(u_1, \ldots, u_k)$, identified by a set of square-free numbers $(u_1, u_2, \ldots, u_k)$, with $k \leq \mathcal{N}$ and multiplicities $(\omega_1, \omega_2, \ldots, \omega_k)$ such that $\sum_{i=1}^{k} \omega_i = \mathcal{N}$. The dimensions of these sectors are $d(\omega_1, \ldots, \omega_k) = (\mathcal{N}!/\omega_1! \omega_2! \ldots \omega_k!)$. Even though the number $\mathcal{N}$ of legs may be finite, there are nevertheless infinite sectors, which are obtained by varying both the set of the numbers u_i and their multiplicity ω_i.

Notice that the symmetry of the system in each of these sectors is $S_{\omega_1} \otimes S_{\omega_2} \cdots \otimes S_{\omega_k}$ rather than S_N (S_A denotes the permutation group of A objects). Indeed, the permutation operators $P_{i,i+1}$ of S_N enter directly the Hamiltonian, and, therefore, they do not implement a symmetry of the system, since $P_{i,j} \mathcal{H}_R P_{i,j}^{-1} \neq H$, where $P_{i,j}$ is a generic operator of S_N which interchanges n_i with n_j. So, in light of the actual symmetries of the system, it is natural to consider different n_i's as different species of bosons and require the validity of spin statistics only for particles of the same species (for more details, see Supplemental Material [22]).

Manifold of the ground states of $\mathcal{H}_R$.—Let us consider the ground states of $\mathcal{H}_R(\lambda)$ by varying the coupling λ.

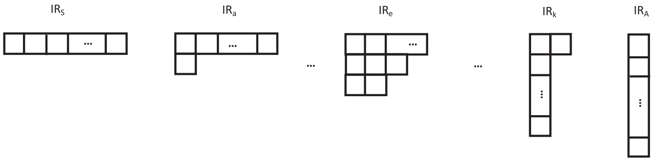

FIG. 2. Young tableaux of the symmetrix group $\mathcal{S}_{\mathcal{N}}$.

$\lambda = 0$ case.—When $\lambda = 0$, the Hamiltonian $\mathcal{H}_R(0)$ can be decomposed in block forms according to the irreducible representation (IR) of the symmetric group $\mathcal{S}_{\mathcal{N}}$ given by Young tableaux [23] and then each block diagonalized separately [24]. While this diagonalization procedure is, in general, highly elaborate, it is instead quite easy to identify the two IRs which give rise to the *highest* and *lowest* energy states $E = \pm\mathcal{N}$: These are given, respectively, by the first and the last Young tableaux in Fig. 2, relative to the fully symmetric and antisymmetric one-dimensional IR_S and IR_A. Note that, while IR_S always appears in the decomposition of any sector, IR_A on the contrary appears only in the decomposition of those sectors where all the n_i's are different numbers. The minimum energy $E^* = -\mathcal{N}$ is obtained when each permutation operator $P_{i,i+1}$ can simultaneously take value -1: Hence, the relative state corresponds to the totally antisymmetric IR (corresponding to the vertically longest Young tableau) and can be written as a Slater determinant built in terms of any set of $\mathcal{N}$ different square-free numbers $|n_a\rangle_i$:

$$|n^{(1)}, \ldots, n^{(\mathcal{N})}\rangle_- = \frac{1}{\sqrt{\mathcal{N}!}} \begin{vmatrix} |n^{(1)}\rangle_1 & \cdots & |n^{(\mathcal{N})}\rangle_1 \\ |n^{(1)}\rangle_2 & \cdots & |n^{(\mathcal{N})}\rangle_2 \\ \cdots & \cdots & \cdots \\ |n^{(1)}\rangle_{\mathcal{N}} & \cdots & |n^{(\mathcal{N})}\rangle_{\mathcal{N}} \end{vmatrix}. \quad (7)$$

One may be surprised that a (hard-)boson wave function is expressed by a Slater determinant, but there is nothing strange with this result, since different occupation numbers (as is the case here) correspond to different particle species (i.e., S_N is not a symmetry), and, therefore, there is no violation of spin statistics. Given that we can freely change the square-free numbers involved in Eq. (7), the ground state of the Hamiltonian $\mathcal{H}_R(0)$ is then infinitely degenerate; i.e., for any finite number of legs $\mathcal{N}$, the density of these ground states is as much as the density of all states of the Hilbert space. Indeed, let Δ be a cutoff for the number of square-free integers: On a lattice of $\mathcal{N}$ sites, the dimension of the Hilbert space is $D = \Delta^{\mathcal{N}}$, while the ground states are given by $\mathcal{N}$ different square-free integers, whose number is then $\tilde{d} = \Delta(\Delta-1)(\Delta-2)\ldots(\Delta-\mathcal{N}+1)$. Hence, $P = \tilde{d}/D$, and, taking the limit $\Delta \to \infty$, we see that $P \to 1$, independently on the number $\mathcal{N}$ of chains.

$\lambda > 0$ *case.*—When we switch on $\lambda > 0$, the coprimality term lifts the degeneracy of many of the previous ground states, but it leaves several of them untouched. The new set of ground states of $\mathcal{H}_R(\lambda)$ still has $E = -\mathcal{N}$ and corresponds to square-free numbers n_i which have to be not only different but this time also coprime each other. It is indeed the only way to minimize both the permutation and the coprime operators, because in this case the matrix elements of the coprime operator simply vanish. As shown in Supplemental Material [22], for $\lambda > 0$ the fraction of the ground states of the Hamiltonian (6) with respect to the total number of states of the Hilbert space changes radically and is given by

$$\mathrm{P} = \prod_{p_a}\left[\left(1 + \frac{\mathcal{N}-1}{p_a+1}\right)\left(1 - \frac{1}{p_a+1}\right)^{\mathcal{N}-1}\right]. \quad (8)$$

This quantity now depends on $\mathcal{N}$ and rapidly decreases to 0 by increasing the number $\mathcal{N}$ of chains; i.e., in the presence of the coprimality interaction, the number of ground states of $\mathcal{H}_R(\lambda)$ for $\mathcal{N} \gg 1$ is infinitesimally small with respect to the dimension of the Hilbert space.

Leg Hamiltonian.—Let us now go back to the issue of how one can realize occupation numbers in terms of square-free integers. As mentioned before, the key is to use hard-core bosons, but we have to assign them the right energies for putting in correspondence with the prime numbers. This can be achieved by a leg Hamiltonian $\mathcal{H}_L$:

$$\mathcal{H}_L = \sum_{i=1}^{\mathcal{N}} H_i(h,\nu) = \sum_{i=1}^{\mathcal{N}}\left[\sum_{a=1}^{\Lambda} h(a)F_i(a) - \frac{1}{2}\sum_{a\neq b} J_{ab}[f_i^\dagger(a)f_i(b) + f_i^\dagger(b)f_i(a)]\right], \quad (9)$$

which simply involves a magnetic field $h(a)$ and a hopping term J_{ab} properly tuned:

$$h(a) = h\log p_a, \qquad J_{ab} = J_{|a-b|} = \nu/|a-b| \quad (10)$$

($h, \nu > 0$ are two coupling constants). As shown in detail in Supplemental Material [22], the increasing values of the magnetic field $h(a)$ along the chain according to the logarithm of the primes dictate the long-range nature of the hopping term J_{ab}. It is precisely thanks to this long-range dependence of J_{ab} that the ratio h/ν of the two couplings truly captures the competition present in $\mathcal{H}_i$: Indeed, when $h/\nu \to \infty$, the eigenstates of $\mathcal{H}_i$ are given by the $|p_a\rangle_i$'s (in this case, they are "localized" on the primes). The magnetic contribution of a state as Eq. (2) in the leg Hamiltonian is given by

$$M_{n_i} = h\sum_{a=1}^{k}\alpha_a \log(p_a)_i = h\log n_i, \quad (11)$$

and these values are *never* degenerate for the unique decomposition in terms of primes of any number n_i [see Eq. (3)]. When $h/\nu \to 0$, the eigenstates are instead "delocalized" along the whole chain: With a periodic boundary condition (J_{ab} is a circulant matrix in this case), the ground state of H_i is the so-called prime state [26]

$$|\mathbb{P}_0\rangle_\Lambda = \frac{1}{\sqrt{\Lambda}}(|p_1\rangle + |p_2\rangle + \cdots |p_\Lambda\rangle), \qquad (12)$$

which is completely delocalized in the space of the primes, as are also delocalized the excited states ($\omega = e^{2\pi i/\Lambda}$):

$$|\mathbb{P}_k\rangle = \frac{1}{\sqrt{\Lambda}}(|p_1\rangle + \omega^k|p_2\rangle + \cdots \omega^{(\Lambda-1)k}|p_\Lambda\rangle). \qquad (13)$$

Phases of the system.—The phase diagram of the full Hamiltonian (5) is quite rich. Let us briefly discuss some of its cases. It is easy to see that, taking $\nu \to \infty$ (keeping all other coupling constants fixed), the system goes into a "stripe phase" described by the factorized ground state made of the prime states (12):

$$|\Psi\rangle_0 \simeq \otimes_i |\mathbb{P}\rangle_i. \qquad (14)$$

Expanding each $|\mathbb{P}\rangle_i$ in the prime basis, this ground state is made of equally weighted vectors of all possible sectors of the theory, whose degeneracy will be eventually solved by taking into account the coprimality interaction and the magnetic field (λ and h both small compared to ν). In this phase, factorized expressions also hold for the excited states, $|\Psi\rangle_k \simeq \otimes_i |\mathbb{P}_k\rangle_i$. Assuming periodic boundary conditions and a cutoff Λ along each chain, for the ground and excited state energies of this phase, we get $E_k \simeq -\nu\mathcal{N} e_k$, where, for $\Lambda \to \infty$,

$$e_0 \simeq -2\left(\log\frac{\Lambda}{2} + \gamma_E + \frac{1}{\Lambda} + \cdots\right),$$
$$e_k \simeq \log\left[1 - \cos\left(\frac{2\pi k}{\Lambda}\right)\right] + \log 2,$$

which can be made finite by subtracting the leading divergent term $\log \Lambda/2$.

Taking instead $h \to \infty$ (and neglecting for simplicity the hopping term in $\mathcal{H}_L$), the system goes into its "ordered phase," characterized by an occupation number at each chain given by the lowest prime $p_1 = 2$:

$$|\Psi\rangle_0 \simeq \bigotimes_{i=1}^{\mathcal{N}} |2\rangle_i \qquad (15)$$

and ground state energy $E_0^{(\mathrm{ord})} \simeq \mathcal{N}(h\log 2 + 1 + \lambda)$. For small h, the ground state (15) is, however, unstable with respect to the proliferation of other numbers n_i (which may replace some of the 2's present). Indeed, when such numbers $n_i \neq 2$ exist in some of the chains, the ground state energy tends to decrease for (i) the presence of other IRs in the decomposition of the permutation term of the Hamiltonian (in addition to IR_S, the only IR present in the ordered phase); typically, these IRs have lower energy than $E = \mathcal{N}$ (the rule of thumb being the longer the Young tableau in the vertical direction, the lower the corresponding minimum energy in that IR); (ii) a lower contribution coming from the coprimality term, since there are fewer pairs of equal particles. Imagine, for instance, replacing one of the 2's in the ordered phase with a 3: The new IR needed in this case is the second Young tableau (from left) in Fig. 2 which, with periodic boundary conditions, has dimension $\mathcal{N}$ and is spanned by the $\mathcal{N}$ vectors ($m = 1, 2, \ldots, \mathcal{N}$)

$$|m\rangle \equiv |2, 2, 2, \ldots, \underset{\substack{\uparrow \\ m\text{th chain}}}{3}, \ldots, 2, 2, \ldots\rangle. \qquad (16)$$

The number 3 plays a role of a defect with respect to the ordered ground state. On the space spanned by these $\mathcal{N}$ vectors, the term $\sum_i^{\mathcal{N}} P_{i,i+1}$ in the Hamiltonian has $|v_k\rangle = 1/\sqrt{\mathcal{N}} e^{ikm}|m\rangle$ as eigenvectors and a spectrum given by $\hat{E}_k = \mathcal{N} - 2 + 2\cos 2\pi k/\mathcal{N}$, whose minimum is $\hat{E}_{\min} = (\mathcal{N} - 4)$. The expectation value of the coprimality operators on the $|v_k\rangle$ eigenvectors is simply $(\mathcal{N} - 2)\lambda$. So, putting together the two terms, the minimum energy in the defect sector is $E_{\min}^{(\mathrm{def})} = \mathcal{N} - 4 + \lambda(\mathcal{N} - 2) + \mathcal{N} h \log 2 + h\log(3/2)$ (neglecting for simplicity the hopping term in $\mathcal{H}_L$). Comparing now $E_0^{(\mathrm{ord})}$ with $E_{\min}^{(\mathrm{def})}$, we can determine the minimum value of h, i.e., h_c, for which the ordered phase is indeed stable:

$$E_0^{(\mathrm{ord})} \le E_{\min}^{(\mathrm{def})} \quad \text{if } h_c \log(3/2) \ge 4 + 2\lambda. \qquad (17)$$

Let us finally consider the limit in which $\lambda \to \infty$ and $h \to 0$ (in a way determined below), also imposing the extra condition $\nu \ll h$. In this case, the system goes into a "prime-number phase," which consists in minimizing simultaneously the permutation and the coprimality operators, adjusting accordingly the magnetization operators. A state which satisfies all these requirements consists of the Slater determinant of the first $\mathcal{N}$ primes. As for a fermionic system, also for hard-core bosons this condition defines a "Fermi energy" given by filling the first $\mathcal{N}$ levels, and its value is

$$E_F = \sum_{a=1}^{\mathcal{N}} \log p_a = \log\left(\prod_{a=1}^{\mathcal{N}} p_a\right) = \log \tilde{P}(\mathcal{N}), \qquad (18)$$

where $\tilde{P}(\mathcal{N})$ is the *primorial*, i.e., the product of the first $\mathcal{N}$ consecutive prime numbers. Since this quantity goes asymptotically as $\tilde{P}(\mathcal{N}) \simeq e^{p_{\mathcal{N}}} \simeq e^{\mathcal{N}\log\mathcal{N}}$ [27], we have $E_F \simeq \mathcal{N}\log\mathcal{N}$. Hence, the ground state energy in the "prime phase" is given by

$$E_0^{(\mathrm{prime})} = -\mathcal{N} + h\mathcal{N}\log\mathcal{N}. \tag{19}$$

So, letting h vanish as $h \simeq \tilde{h}/\log\mathcal{N}$ for $\mathcal{N} \to \infty$, we have a ground state energy of the prime phase which scales linearly with the number $\mathcal{N}$ of chains. Hence, this state gives rise to a "Fermi surface" in terms of the first $\mathcal{N}$ prime numbers. These primes are simultaneously present on each chain, being spread on the entire ladder system, although quantum coherently assembled by a Slater determinant; see Eq. (7). As for other Fermi surfaces, there are soft modes above this ground state: Indeed, if we replace one prime number $p_c = p_{\mathcal{N}-\epsilon}$ (inside and close to the Fermi surface) with other one $p_e = p_{\mathcal{N}+\delta}$ (placed outside and close to it), the variation of the energy of the corresponding wave functions is simply

$$\Delta E = \tilde{h}\log\frac{p_e}{p_c} \simeq \tilde{h}\frac{\delta+\epsilon}{\mathcal{N}}, \tag{20}$$

where we have used the scaling law (1). So, for a finite $\mathcal{N}$, the system has a gap which, however, scales to zero as $1/\mathcal{N}$ if we send the number of chains to infinity.

Lindbladian dynamics.—A natural question is how the system is able to reach one of its ground states, say, the "prime ground state" given by the coherent superposition of the first $\mathcal{N}$ primes. One way is to set up a dissipative dynamics able to efficiently "filter" such a ground state starting from an initial configuration made of an arbitrary mixture of excited states. This procedure can be implemented by choosing a suitable and optimized set of Lindblad operators (see, e.g., [28]) which induce a dissipative dynamics for the density matrix ρ of our system ruled by the master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \mathcal{L}[\rho], \tag{21}$$

where H is the ladder Hamiltonian (5) while $\mathcal{L}[\rho]$ is the Lindbladian term describing spontaneous emission processes. Of course, one has to specify the Lindbladian operator, a goal achieved by posing

$$\mathcal{L}[\rho] = \sum_{i=1}^{\mathcal{N}} [\gamma_i L_i \rho L_i^\dagger - (\gamma_i/2)\{L_i^\dagger L_i, \rho\}]$$

and identifying a suitable set of the L_i's operators.

For our purposes, notice that the quantum superposition is induced by the rung Hamiltonian $\mathcal{H}_R$ while the intraleg term $\mathcal{H}_L$ permits the hopping, i.e., the reshuffling, of the particles among the $|p_a\rangle$ levels inside each of the chains. Hence, if our aim is to target the ground state made of the first $\mathcal{N}$ primes, it is sufficient to choose for L_i the following operators:

$$L_i = \sum_{a>\mathcal{N}}\sum_{b\leq\mathcal{N}} f_i(a) f_i^{\dagger(b)} \tag{22}$$

(in principle, one could also consider level-dependent coefficients $\gamma_i^{(ab)}$). The rationale behind this choice is that the dissipative term does not act in the space of the first $\mathcal{N}$ levels while at the same time does favor the occupation of such a subspace. We expect an interplay between the term $\propto \nu$ present in $\mathcal{H}_L$ and the Lindbladian term $\mathcal{L}$, in the sense that a nonvanishing ν tends to decrease the characteristic time in which the system reaches our target subspace.

Conclusions.—Number theory is the paradigmatic example of pure mathematics. Yet the theory of integers can appear totally unexpected in quantum mechanics systems, providing new perspectives on their dynamics. In this Letter, we have considered a many-body quantum ladder system, made of $\mathcal{N}$ coupled quantum chains, whose degrees of freedom and interactions have a very direct interpretation in terms of prime numbers and basic properties thereof. We have shown that such a system has many different phases. Among the major capabilities of this system, there is the possibility of realizing a ground state made of a coherent superposition of the first $\mathcal{N}$ primes.

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