BRANCHES OF POSITIVE SOLUTIONS OF A SUPERLINEAR INDEFINITE PROBLEM DRIVEN BY THE ONE-DIMENSIONAL CURVATURE OPERATOR*

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ABSTRACT. This paper aims at proving the existence and the localization of an unbounded connected set of positive regular solutions (λ, u) of the quasilinear Neumann problem

$$-\big(u'/\sqrt{1+(u')^2}\,\big)' = \lambda a(x) f(u),\ 0 < x < 1, \quad u'(0) = u'(1) = 0,$$

bifurcating from $u = 0$ as $\lambda \to +\infty$. Here, $\big(u'/\sqrt{1+(u')^2}\,\big)'$ is the one-dimensional curvature operator, $\lambda \in \mathbb{R}$ is a parameter, the weight a changes sign, and the function f is superlinear at 0. A novel approach is introduced based on the explicit construction of non-ordered sub and supersolutions.

1. Results

The aim of this paper is analyzing the set of positive regular solutions of the quasilinear Neumann problem

$$\begin{cases} -\Big(u'/\sqrt{1+(u')^2}\Big)' = \lambda a(x) f(u), \qquad 0 < x < 1, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a parameter and the functions a and f satisfy:

(a_1) *$a \in L^\infty(0,1)$, $\int_0^1 a(x)\,dx < 0$, and there is $z \in (0,1)$ such that $a(x) > 0$ almost everywhere in $(0, z)$ and $a(x) < 0$ almost everywhere in $(z, 1)$.*

(f_1) *$f \in \mathcal{C}^0[0, +\infty)$, $f(s) > 0$ if $s > 0$, and, for some constant $p > 1$, $\lim_{s\to 0^+} \frac{f(s)}{s^p} = 1$.*

As a is sign indefinite and f is superlinear at zero, (1.1) is a *superlinear indefinite* elliptic problem. These problems have attracted a huge amount of attention during the last few decades.

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The problem (1.1) can be regarded as a simple prototype of the more sophisticated multidimensional problem

$$\begin{cases} -\text{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) = F(x,u,\nabla u), & \text{in } \Omega, \\ -\nabla u \cdot \nu/\sqrt{1+|\nabla u|^2} = \sigma, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $: \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. The problem (1.2) plays a central role in the mathematical analysis of a number of important geometrical and physical issues, ranging from prescribed mean curvature problems for cartesian surfaces in the Euclidean space, to the study of capillarity phenomena for compressible or incompressible fluids, as well as to the analysis of reaction-diffusion processes where the flux features saturation at high regimes. An extensive discussion on these and other closely related issues is presented in our previous papers [9–14], where rather complete lists of relevant references can be found.

Although the study of (1.1), or (1.2), is often settled in the space of *bounded variation* functions, here we will be instead concerned with the regular solutions of (1.1). Namely, by a *regular* solution of (1.1), we mean a function $u \in W^{2,1}(0,1)$ which fulfills the differential equation almost everywhere in $(0,1)$, as well as the boundary conditions. It is straightforward to see that u is a regular solution of (1.1) if, and only if, it satisfies

$$\begin{cases} -u'' = \lambda a(x) f(u)(1+(u')^2)^{\frac{3}{2}}, & 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.3}$$

A function $u \in \mathcal{C}^0[0,1]$ is said to be *positive* if $\min_{[0,1]} u \geq 0$ and $\max_{[0,1]} u > 0$, whereas it is said *strictly positive* if $\min_{[0,1]} u > 0$. Throughout this paper, the positive solutions of (1.1) are regarded as couples (λ, u) as, eventually, we will adopt the point of view of 'bifurcation theory' in our analysis. Naturally, for any given $\lambda \geq 0$, a couple (λ, u) is said to be a positive, or strictly positive, solution of (1.1) if u is a positive, or strictly positive, solution of (1.1), respectively.

The first, preliminary, result of this paper is related to the Vázquez strong maximum principle [15] and yields, under conditions (a_1) and (f_1), the strict positivity of any positive regular solution of (1.3), or, equivalently, (1.1). Indeed, setting $g(x,s,\xi) = \lambda a(x) f(s)(1+\xi^2)^{\frac{3}{2}}$, it is apparent that the condition (g_1) below holds true.

Theorem 1.1. *Assume that*

(g_1) $g\colon [0,1]\times[0,+\infty)\times\mathbb{R} \to \mathbb{R}$ *satisfy the Carathéodory conditions and, for every compact subset K of $\mathbb{R}$,*

$$\lim_{s\to 0^+} \frac{g(x,s,\xi)}{s} = 0, \quad \textit{uniformly for almost every } x \in [0,1] \textit{ and every } \xi \in K.$$

Then, any positive solution $u \in W^{2,1}(0,1)$ of

$$\begin{cases} -u'' = g(x,u,u'), & 0 < x < 1, \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.4}$$

is strictly positive.

Subsequently, we denote by $\mathscr{S}^+$ the set of all couples $(\lambda, u) \in [0,\infty) \times C^1[0,1]$ such that (λ, u) is a positive, and hence strictly positive, regular solution of (1.1).

The second, and main, goal of this paper is establishing the existence of an unbounded closed connected subset $\mathscr{C}^+$ of $\mathscr{S}^+$, bifurcating from $u = 0$ as $\lambda \to +\infty$, and providing simultaneously some sharp information on its localization. The existence of unstable solutions, however not necessarily belonging to $\mathscr{C}^+$, is also detected.

Theorem 1.2. *Assume* (a_1) *and* (f_1). *Then, there exists an unbounded closed connected subset* $\mathscr{C}^+$ *of* $\mathscr{S}^+$ *for which the following properties hold:*

(i) *there is* $\lambda^* > 0$ *such that* $[\lambda^*, \infty) \subseteq \mathrm{proj}_{\mathbb{R}}(\mathscr{C}^+)$;

(ii) *there are functions* α *and* β, *explicitly defined by* (3.8) *and* (3.12) *respectively, such that, for every* $(\lambda, u_\lambda) \in \mathscr{C}^+$, *one has that:*
- $u_\lambda(x_\lambda) < \lambda^{\frac{1}{1-p}} \alpha(x_\lambda)$ *for some* $x_\lambda \in [0, z)$,
- $u_\lambda(y_\lambda) > \lambda^{\frac{1}{1-p}} \beta(y_\lambda)$ *for some* $y_\lambda \in [0, 1]$;

(iii) *there is* $C > 0$ *such that, for every* $(\lambda, u_\lambda) \in \mathscr{C}^+$,
$$\|u'_\lambda\|_{L^\infty(0,1)} < C\lambda^{\frac{1}{1-p}}. \tag{1.5}$$

Moreover, for every $\lambda \in [\lambda_*, \infty)$, *there exists at least one (Lyapunov) unstable solution* $u \in \mathscr{S}^+$ *of* (1.1) *satisfying the conditions expressed by properties* (ii) *and* (iii).

The existence of positive solutions of (1.1) for large $\lambda > 0$ has been previously established, under (a_1) and (f_1), in [9] by variational methods and in [10] by topological degree techniques, however without getting any information on the global structure of the solution set. The existence of a component of $\mathscr{S}^+$ bifurcating from 0 as $\lambda \to +\infty$ was proven in [14] through some global bifurcation techniques inspired by [1]. Theorem 1.2 is a substantial sharpening of all these previous results. In this occasion we are exploiting an alternative method based on the construction of some non-ordered sub and supersolutions and on the use of the Leray-Schauder degree. This new approach, which appears of interest in its own, yields, in addition, the localization and the instability information established by Theorem 1.2, which is a novel result in the context of the problem (1.1).

This paper is organized as follows: Section 2 delivers the proof of Theorem 1.1 and Section 3 consists of the proof of Theorem 1.2.

2. Proof of Theorem 1.1

Suppose $u \in W^{2,1}(0,1)$ is a positive solution of (1.4). Let us introduce the auxiliary function $q(x,s,\xi)$ defined, for almost every $x \in [0,1]$ and every $(s,\xi) \in \mathbb{R} \times \mathbb{R}$, by
$$q(x,s,\xi) = \begin{cases} \frac{g(x,s,\xi)}{s} & \text{if } s > 0, \\ 0 & \text{if } s \le 0. \end{cases}$$

Assumption (g_1) implies that, for every compact subset K of $\mathbb{R}$, $\lim_{s\to 0} q(x,s,\xi) = 0$, uniformly for almost every $x \in [0,1]$ and every $\xi \in K$. It is easy to check that q satisfies the Carathéodory conditions (see [6, p. 28]). Consequently, the function V, defined, for almost every $x \in [0,1]$, by $V(x) = q(x, u(x), u'(x))$, belongs to $L^1(0,1)$ and u solves the linear equation
$$-u'' = Vu. \tag{2.1}$$

Suppose that u is such that $u(x_0) = 0$ for some $x_0 \in [0,1]$. Also due to the boundary conditions $u'(0) = u'(1) = 0$, it follows that necessarily $u'(x_0) = 0$. Thus, the uniqueness of solutions for the Cauchy problem associated with (2.1), which is a consequence of, e.g., [6, Ch. I, Thm. 5.3], yields $u = 0$ in $[0,1]$, which is impossible as u was assumed positive. This contradiction ends the proof of Theorem 1.1.

3. Proof of Theorem 1.2

Since $f(0) = 0$ and we are focusing attention on the positive solutions of (1.1), or (1.3), without loss of generality we can extend f to the whole of $\mathbb{R}$ as an even function. By performing the change of variable

$$u = \varepsilon v, \qquad \varepsilon = \lambda^{\frac{1}{1-p}}, \tag{3.1}$$

and setting

$$h(s) = \begin{cases} \frac{f(s)}{|s|^p} & \text{if } s \neq 0, \\ 1 & \text{if } s = 0, \end{cases} \tag{3.2}$$

the problem (1.3) can be equivalently written in the form

$$\begin{cases} -v'' = a(x)|v|^p \, h(\varepsilon v) \, (1 + (\varepsilon v')^2)^{\frac{3}{2}}, & 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{3.3}$$

Note that $h(-s) = h(s)$ for all $s > 0$, like f. Throughout the rest of this proof, for every $r > 0$, we consider the auxiliary function

$$\ell_r(x,s) = \begin{cases} |s|^p & \text{if } s \leq 0, \\ a(x)\, s^p & \text{if } 0 < s \leq r, \\ a(x)\, s^p \,(r+1-s) & \text{if } r < s \leq r+1, \\ -s + r + 1 & \text{if } s > r+1, \end{cases} \tag{3.4}$$

as well as the associated problem

$$\begin{cases} -v'' = \ell_r(x,v) \, h(\varepsilon v) \, (1 + (\varepsilon v')^2)^{\frac{3}{2}}, & 0 < x < 1, \\ v'(0) = v'(1) = 0. \end{cases} \tag{3.5}$$

It is obvious that any solution v of (3.5), with $0 \leq v \leq r$ in $[0,1]$, solves (3.3). Moreover, due to (3.2), the problem (3.5) perturbs from the semilinear problem

$$\begin{cases} -v'' = \ell_r(x,v), & 0 < x < 1, \\ v'(0) = v'(1) = 0, \end{cases} \tag{3.6}$$

as $\varepsilon > 0$ separates away from 0.

3.1. **Existence of a couple of non-ordered strict sub and supersolutions for** (3.6)**.** The next result, of technical nature, holds.

Proposition 3.1. *There exists a constant $r_0 > 0$ such that, for all $r \geq r_0$, the problem* (3.6) *admits a subsolution α and a supersolution β such that:*

(i) *$\beta - \alpha$ changes sign in $[0,1]$;*
(ii) *any solution v of* (3.6) *such that $\alpha \leq v$ in $[0,1]$, satisfies $\alpha(x) < v(x)$ for all $x \in [0,1]$;*
(iii) *any solution v of* (3.6) *such that $v \leq \beta$ in $[0,1]$, satisfies $v(x) < \beta(x)$ for all $x \in [0,1]$.*

In other words, according to [3, Ch. III], α and β are strict sub and supersolutions of (3.6). In particular, they cannot be solutions. By Property (i), they are not ordered.

Proof. The proof can be divided in two steps.

Step 1: Construction of α. Let denote by $\mu_1 > 0$ the unique positive eigenvalue of the linear weighted eigenvalue problem

$$\begin{cases} -\varphi'' = \mu\, a(x)\, \varphi, & 0 < x < \frac{z}{2}, \\ \varphi'(0) = 0, \;\; \varphi(\frac{z}{2}) = 0, \end{cases}$$

whose existence and uniqueness follow, e.g., from [8, Thm. 9.2] by taking into account of the strict positivity of the principal eigenvalue $\sigma_1 = (\frac{\pi}{z})^2$ of the problem

$$\begin{cases} -\psi'' = \sigma\, \psi, & 0 < x < \frac{z}{2}, \\ \psi'(0) = 0, \;\; \psi(\frac{z}{2}) = 0. \end{cases}$$

Let φ_1 be any positive eigenfunction associated to μ_1. Since $\varphi_1(x) > 0$ for all $x \in [0, \frac{z}{2})$ and $\varphi_1'(\frac{z}{2}) < 0$, by the mean value theorem, there exists $\overline{x} \in (0, \frac{z}{2})$ such that

$$\varphi_1(\overline{x}) + \varphi_1'(\overline{x})(z - \overline{x}) = 0.$$

Indeed, the function

$$\Phi(x) = \varphi_1(x) + \varphi_1'(x)(z - x)$$

is continuous in $[0, \frac{z}{2}]$ and it satisfies $\Phi(0) = \varphi_1(0) > 0$ and $\Phi(\frac{z}{2}) = \varphi_1'(\frac{z}{2})\frac{z}{2} < 0$, whence the existence of $\overline{x}$. Since φ_1 is decreasing and $p > 1$, we can also find $c > 0$ such that

$$[c\varphi_1(x)]^{p-1} \geq [c\varphi_1(\overline{x})]^{p-1} > \mu_1, \quad \text{for all } x \in [0, \overline{x}]. \tag{3.7}$$

Next, we define

$$\alpha(x) = \begin{cases} c\varphi_1(x) & \text{if } 0 \leq x < \overline{x}, \\ c\varphi_1(\overline{x}) + c\varphi_1'(\overline{x})(x - \overline{x}) & \text{if } \overline{x} \leq x < z, \\ 0 & \text{if } z \leq x \leq 1. \end{cases} \tag{3.8}$$

It is clear that $\alpha \in \mathcal{C}^0[0,1] \cap W^{2,\infty}(0,z) \cap W^{2,\infty}(z,1)$ and, moreover,

$$\alpha'(z^-) = c\varphi_1'(\overline{x}) = -\frac{c\varphi_1(\overline{x})}{z - \overline{x}} < 0 = \alpha'(z^+). \tag{3.9}$$

Further, it follows from (3.8) and (3.7) that, for almost every $x \in (0, \overline{x})$,

$$-\alpha''(x) = -c\varphi_1''(x) = \mu_1 a(x) c\varphi_1(x) < [c\varphi_1(x)]^{p-1} a(x) c\varphi_1(x) = a(x)\alpha^p(x).$$

Similarly, we have that, for almost every $x \in (\overline{x}, z)$, $-\alpha''(x) = 0 < a(x)\alpha^p(x)$, and, for almost every $x \in (z,1)$, $-\alpha''(x) = 0 = a(x)\alpha^p(x)$. Thus, setting $r_0 = 1 + \max_{[0,1]} \alpha$, from (3.4) we infer that

$$-\alpha''(x) \leq a(x)\alpha^p(x) = \ell_r(x, \alpha(x)), \quad \text{for almost every } x \in (0,1),$$

provided that $r \geq r_0$. In addition, α is such that $\alpha'(0) = 0$ and $\alpha'(1) = 0$. Therefore, for every $r \geq r_0$, α is a subsolution of (3.6) as discussed in [3, Def. II-2.1] (see also [2]).

Finally, we will check that α satisfies the assertion (ii) of Proposition 3.1. Let v be a solution of (3.6), with $v \geq \alpha$ in $[0,1]$, and set $w = v - \alpha$, with $w \geq 0$ in $[0,1]$. We want to show that $w(x) > 0$ for all $x \in [0,1]$. Indeed, suppose, by contradiction, that there is some $x_0 \in [0,1]$ such that $w(x_0) = \min_{[0,1]} w = 0$. Since $v'(z^-) = v'(z^+)$, it follows from (3.9) that

$$w'(z^-) = v'(z^-) - \alpha'(z^-) > v'(z^+) - \alpha'(z^+) = w'(z^+).$$

As this is impossible at an interior minimum point, $x_0 \neq z$. Therefore, $w'(x_0) = 0$, as w is differentiable for $x \neq z$ and it satisfies $w'(0) = w'(1) = 0$. Assume that $x_0 \in [0,z)$. Then, since $v(x_0) = \alpha(x_0) < r_0 \leq r$, there exists an interval $J \subseteq [0,z)$, with $x_0 \in J$, such that $v(x) < r$ for all $x \in J$. Thus, for almost every $x \in J$, we have that

$$-w''(x) = -v''(x) + \alpha''(x) > \ell_r(x, v(x)) - \ell_r(x, \alpha(x)) = a(x)(v^p(x) - \alpha^p(x)) \geq 0,$$

as $\alpha''(x) > -\ell_r(x,\alpha(x)) = -a(x)\alpha^p(x)$ for almost every $x \in [0,z)$. Hence, w is strictly concave in J. Since $w(x_0) = \min_J w = 0$ and $w'(x_0) = 0$, the strong maximum principle (see, e.g., [8, Thm. 7.11]) implies that $w = 0$ in J. As this contradicts the strict concavity of w in J, necessarily $x_0 \in (z,1]$. As $\alpha = 0$ in $[z,1]$, we have that $w = v$ in $[z,1]$. Thus, $v(x_0) = w(x_0) = 0$ and $v'(x_0) = w'(x_0) = 0$, because $v = w \geq 0$ and $v'(1) = 0$. Consequently, v is a local solution of the Cauchy problem

$$\begin{cases} -v'' = \ell_r(x,v) \\ v(x_0) = 0, \ \ v'(x_0) = 0. \end{cases}$$

As the function $\ell_r(x,s)$ is locally Lipschitz with respect to s, because $p > 1$ and $a \in L^\infty(0,1)$, by uniqueness, we conclude that $v = 0$ in $[0,1]$. This is impossible, since we already proved that $v(x) > \alpha(x) > 0$ for all $x \in [0,z)$. Therefore, the property (ii) of Proposition 3.1 holds for all $r \geq r_0$.

Step 2: Construction of β. Fix $r \geq r_0$. Subsequently, for every $\kappa > 0$, we denote by z_κ the unique solution of the linear problem

$$\begin{cases} -z'' = \Big(a(x) - \displaystyle\int_0^1 a(t)\,dt\Big)\kappa^p, \qquad 0 < x < 1, \\ z'(0) = z'(1) = 0, \ \ \displaystyle\int_0^1 z(t)\,dt = 0. \end{cases} \tag{3.10}$$

Combining the Poincaré–Wirtinger inequality with (3.10) yields

$$\|z_\kappa\|_{L^\infty(0,1)} \leq \|z'_\kappa\|_{L^1(0,1)} \leq \|z'_\kappa\|_{L^\infty(0,1)} \leq \|z''_\kappa\|_{L^1(0,1)} \leq 2\|a\|_{L^1(0,1)}\,\kappa^p. \tag{3.11}$$

Consequently, since $p > 1$ in the estimate (3.11), the function $\beta \in W^{2,1}(0,1)$ defined by

$$\beta = z_\kappa + \kappa \tag{3.12}$$

satisfies, for sufficiently small $\kappa > 0$,

$$0 < \min_{[0,1]} \beta \le \max_{[0,1]} \beta < \max_{[0,1]} \alpha \le r_0. \tag{3.13}$$

Moreover, for almost every $x \in [0,1]$, we find from (3.10) that

$$-\beta''(x) = -z''_\kappa = a(x)\kappa^p - \kappa^p \int_0^1 a(t)\,dt = a(x)\beta^p(x) + a(x)[\kappa^p - \beta^p(x)] - \kappa^p \int_0^1 a(t)\,dt,$$

and thus, by rearranging terms and using (3.12),

$$-\beta''(x) = a\beta^p(x) + \kappa^p \left[a(x)\left(1 - \left(1 + \tfrac{z_\kappa(x)}{\kappa}\right)^p\right) - \int_0^1 a(t)\,dt \right]. \tag{3.14}$$

Using (3.11) and the condition $p > 1$ again, it is easily seen that

$$\lim_{\kappa \to 0} \left[a(x)\left(1 - \left(1 + \tfrac{z_\kappa(x)}{\kappa}\right)^p\right)\right] = 0, \quad \text{uniformly almost everywhere in } [0,1].$$

Consequently, since $\int_0^1 a(t)\,dt < 0$, we can conclude from (3.14) that, for sufficiently small $\kappa > 0$,

$$-\beta''(x) \ge a(x)\beta^p(x) - \tfrac{1}{2}\kappa^p \int_0^1 a(t)\,dt > \ell_r(x, \beta(x)), \quad \text{for almost every } x \in [0,1]. \tag{3.15}$$

Therefore, for every $r \ge r_0$, the function β provides us with a supersolution of (3.6) fulfilling the boundary conditions.

To complete the proof of Proposition 3.1 it remains to show the property (iii). Indeed, let v be a solution of (3.6) such that $v \le \beta$ in $[0,1]$ and consider the function $w = \beta - v$, where $w \ge 0$. Suppose, by contradiction, that $\min_{[0,1]} w = 0$. Then, there exists $x_0 \in [0,1]$ such that $w(x_0) = 0$, i.e., $\beta(x_0) = v(x_0)$. Note that, owing to (3.13), $0 < v(x_0) < r_0 \le r$. Thus, there is an interval $J \subseteq [0,1]$, with $x_0 \in J$, such that $0 < v(x) < r$, for all $x \in J$, and

$$|a(x)(\beta^p(x) - v^p(x))| < -\tfrac{1}{2}\kappa^p \int_0^1 a(t)\,dt, \quad \text{for almost every } x \in J.$$

Consequently, by (3.15), (3.4) and (3.6), for almost every $x \in J$, we have that

$$\begin{aligned} -w''(x) = -\beta''(x) + v''(x) &\ge a(x)\beta^p(x) - \tfrac{1}{2}\kappa^p \int_0^1 a(t)\,dt - \ell_r(x, v(x)) \\ &= a(x)(\beta^p(x) - v^p(x)) - \tfrac{1}{2}\kappa^p \int_0^1 a(t)\,dt > 0. \end{aligned}$$

Hence, w is strictly concave in J. Since $w(x_0) = \min_J w = 0$ and $w'(0) = w'(1) = 0$, it follows that $w'(x_0) = 0$. The strong maximum principle then implies that $w = 0$ in J. As this is impossible, by the strict concavity of w in J, we find that indeed $v(x) < \beta(x)$ for all $x \in [0,1]$. The proof is complete. □

3.2. **Positivity. A priori bounds.** The next result is the main positivity result of this section.

Proposition 3.2. *Fix any $r > 0$. Then, the following assertions hold:*

(i) *every solution of* (3.6) *is non-negative;*
(ii) *every positive solution of* (3.6) *is strictly positive.*

Proof. Let v be a solution of (3.6). Suppose, by contradiction, that $\min_{[0,1]} v < 0$ and let $x_0 \in [0,1]$ be such that $v(x_0) = \min_{[0,1]} v$. As $v'(0) = v'(1) = 0$, we have that $v'(x_0) = 0$. From (3.6) and (3.4) we infer that $v''(x_0) = -\ell_r(x_0, v(x_0)) = -|v(x_0)|^p < 0$, which is impossible at a minimum critical point. This yields assertion (i).

To prove (ii), it is enough to observe that, for any given $r > 0$, the function ℓ_r satisfies assumption (g_1). Theorem 1.1 then guarantees that any positive solution of (3.6) is indeed strictly positive. This ends the proof. □

The following result provides us with a priori bounds for the positive solutions of (3.6).

Proposition 3.3. *The following assertions hold:*

(i) *for every $r > 0$, any solution v of* (3.6) *satisfies*

$$0 \leq v(x) \leq r + 1, \quad \textit{for all } \; x \in [0,1], \tag{3.16}$$

and

$$\|v'\|_{L^\infty(0,1)} < C = \|a\|_{L^1(0,1)}(r+1)^{p+1}; \tag{3.17}$$

(ii) *for every $r \geq r_0$, any solution v of* (3.6)*, with $v(x_0) \leq a(x_0)$ for some $x_0 \in [0,1]$, satisfies*

$$\max_{[0,1]} v < R = \|\alpha\|_{L^\infty(0,1)} + \|\alpha'\|_{L^\infty(0,1)}. \tag{3.18}$$

Proof. Let v be a solution of (3.6). Proposition 3.2 implies that $\min_{[0,1]} v \geq 0$. Suppose, by contradiction, that $\max_{[0,1]} v > r+1$ and let $x_0 \in [0,1]$ be such that $v(x_0) = \max_{[0,1]} v$. As $v'(0) = v'(1) = 0$, we have that $v'(x_0) = 0$. From (3.6) and (3.4) we get $v''(x_0) = v(x_0) - r - 1 > 0$, which is impossible at a maximum critical point. Hence, (3.16) follows. To prove (3.17), we integrate (3.6) in $(0,x)$ and thus we obtain $v'(x) = -\int_0^x \ell_r(s, v(s))\, ds$. Hence, we have, from (3.16) and (3.4), that

$$|v'(x)| \leq \int_0^x |\ell_r(s, v(s))|\, ds < \|a\|_{L^1(0,1)}(r+1)^{p+1}, \qquad \text{for all } x \in [0,1].$$

Setting $C = \|a\|_{L^1(0,1)}(r+1)^{p+1}$, the proof of assertion (i) is completed.

Pick now a solution v of (3.6), with $\max_{[0,1]} v > 0$ and $v(x_0) \leq \alpha(x_0)$ for some $x_0 \in [0,1]$. By Proposition 3.2, we know that $\min_{[0,1]} v > 0$. As $\alpha(x) > 0$ if and only if $x \in [0,z)$, we necessarily have that $x_0 \in [0,z)$. Thus, since $v(z) > 0 = \alpha(z)$, there exists $x_1 \in [x_0, z)$ such that $v(x_1) = \alpha(x_1)$ and $v'(x_1) \geq \alpha'(x_1)$. From (a_1) and (3.16) it follows that v is concave in $[0,z)$, convex in $(z,1]$, and, as $v'(0) = v'(1) = 0$, strictly decreasing in $[0,1]$. By concavity, we get $v(x) \leq v(x_1) + v'(x_1)(x - x_1)$, for all $x \in [0, x_1]$. Hence, as v is decreasing, we find that

$$\max_{[0,1]} v = v(0) \leq v(x_1) - v'(x_1)x_1 \leq \alpha(x_1) - \alpha'(x_1)x_1 \leq \|\alpha\|_{L^\infty(0,1)} + \|\alpha'\|_{L^\infty(0,1)}.$$

Setting $R = \|\alpha\|_{L^\infty(0,1)} + \|\alpha'\|_{L^\infty(0,1)}$, the conclusion (ii) is achieved. This ends the proof. □

3.3. **Existence of a couple of ordered strict sub and supersolutions for** (3.6)**.** Fix any $r \geq r_0$. Due to the definition of the function ℓ_r, all negative costants are subsolutions of (3.6), while all constants larger than $r+1$ are supersolutions of (3.6); accordingly, we set $\alpha_1 = -1$ and $\beta_1 = r+2$. By our choice of r, it follows that α, β both satisfy

$$\alpha_1 < 0 \leq \alpha(x), \beta(x) \leq r_0 < \beta_1, \quad \text{for all } x \in [0,1]. \tag{3.19}$$

Moreover, Proposition 3.3 implies that every solution v of (3.6) is such that $\alpha_1 = -1 < v(x) < r+2 = \beta_1$, for all $x \in [0,1]$. Thus, α_1 and β_1 form a couple of ordered strict sub and supersolutions for (3.6).

3.4. **Degree computation.** Fix any $r \geq \max\{r_0, R\}$, where R is the constant defined in (3.18). Subsequently, C being the constant introduced in (3.17), we consider the open bounded subsets of $C^1[0,1]$ defined by

$$\begin{aligned}
&\Omega_1 = \{v \in \mathcal{C}^1[0,1] \ : \alpha_1 < v(x) < \beta_1 \text{ for all } x \in [0,1], \ \|v'\|_\infty < C\},\\
&\Omega_2 = \{v \in \mathcal{C}^1[0,1] \ : \alpha_1 < v(x) < \beta(x) \text{ for all } x \in [0,1], \ \|v'\|_\infty < C\},\\
&\Omega_3 = \{v \in \mathcal{C}^1[0,1] \ : \alpha(x) < v(x) < \beta_1 \text{ for all } x \in [0,1], \ \|v'\|_\infty < C\},\\
&\Omega = \Omega_1 \setminus \overline{\Omega_2 \cup \Omega_3} = \{v \in \Omega_1 \ : v(x_0) < \alpha(x_0) \text{ and } \beta(y_0) < v(y_0) \text{ for some } x_0, y_0 \in [0,1]\}.
\end{aligned}$$

From (3.19), it follows that $\Omega_2 \cup \Omega_3 \subset \Omega_1$. Moreover, we have that $\Omega_2 \cap \Omega_3 = \emptyset$, because $\beta - \alpha$ changes sign in $[0,1]$ by Proposition 3.1.

Let denote by $\mathcal{T} : [0,\infty) \times \mathcal{C}^1[0,1] \to \mathcal{C}^1[0,1]$ the operator sending each $(\varepsilon, v) \in [0,\infty) \times \mathcal{C}^1[0,1]$ to the unique solution $w \in W^{2,\infty}(0,1)$ of the linear problem

$$\begin{cases} -w'' + w = \ell_r(x,v)\, h(\varepsilon v)\, (1 + (\varepsilon v')^2)^{\frac{3}{2}} + v, & 0 < x < 1, \\ w'(0) = w'(1) = 0. \end{cases}$$

It is clear that $\mathcal{T}$ is completely continuous and that its fixed points are precisely the solutions of the problem (3.5). Moreover, by Propositions 3.1 and 3.3 and our choice of C, the operator $T(0,\cdot)$ cannot have fixed points on $\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$. Thus, by the additivity property of the degree,

$$\begin{aligned}
\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega) &= \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_1 \setminus \overline{\Omega_2 \cup \Omega_3})\\
&= \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_1) - \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_2 \cup \Omega_3)\\
&= \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_1) - \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_2) - \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_3).
\end{aligned}$$

As, from, e.g., [3, Ch. III], we already know that

$$\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_1) = \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_2) = \deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega_3) = 1,$$

we can conclude that

$$\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \Omega) = -1.$$

Therefore, by the existence property of the degree, the problem (3.6) possesses a solution $v \in \Omega$, where necessarily $x_0 \in [0,z)$, because $\alpha(x_0) > v(x_0) > 0$ and $\alpha = 0$ on $[z,1]$. In addition, having chosen $r > R$, Proposition 3.3 guarantees that $v(x) < r$ for all $x \in [0,1]$ and therefore v is a solution of the problem (3.3) for $\varepsilon = 0$. Hence, if we define an open subset $\mathcal{O}$ of Ω by

$$\mathcal{O} = \{v \in \Omega : \min_{[0,1]} v > 0, \max_{[0,1]} v < r\},$$

then every solution $v \in \Omega$ must belong to $\mathcal{O}$. Thus, the excision property of the degree yields

$$\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\cdot), \mathcal{O}) = -1.$$

3.5. **Existence of a continuum. Conclusion of the proof.** The boundedness of $\partial\mathcal{O}$ in $\mathcal{C}^1[0,1]$ and the complete continuity of the operator $\mathcal{T}$ guarantee the existence of some $\varepsilon^* > 0$ such that $\mathcal{T}(\varepsilon,\cdot)$ has no fixed points on $\partial\mathcal{O}$ for all $\varepsilon \in [0,\varepsilon^*]$. Consequently, the homotopy property of the degree implies that

$$\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\varepsilon), \mathcal{O}) = -1, \qquad \text{for all } \varepsilon \in [0,\varepsilon^*],$$

and hence the existence of at least one solution $v = v_\varepsilon \in \mathcal{O}$ of the problem (3.3) for all $\varepsilon \in [0,\varepsilon^*]$. Actually, the Leray–Schauder continuation theorem [7, p. 63] provides us with a continuum $\mathscr{K}^+$ of solutions $(\varepsilon, v_\varepsilon)$ of (3.3) with $\varepsilon \in [0,\varepsilon^*]$ and $v_\varepsilon \in \mathcal{O}$.

The change of variables (3.1) then implies the existence of a closed connected set $\mathscr{C}^+$ of solutions (λ, u_λ) of (1.1), where $\lambda = \varepsilon^{1-p} \in [\lambda_*, \infty)$, with $\lambda_* = (\varepsilon^*)^{1-p}$, and $u_\lambda = \varepsilon v_\varepsilon = \lambda^{\frac{1}{1-p}} v_\varepsilon$. Since $v_\varepsilon \in \mathcal{O}$, there exist $x_\varepsilon \in [0,z)$ such that $v(x_\varepsilon) < \alpha(x_\varepsilon)$ and $y_\varepsilon \in [0,1]$ such that $v(y_\varepsilon) < \alpha(y_\varepsilon)$. This in turn implies that every $(\lambda, u_\lambda) \in \mathscr{C}^+$ satisfies

$$u_\lambda(x_\lambda) < \lambda^{\frac{1}{1-p}} \alpha(x_\lambda), \qquad u_\lambda(y_\lambda) > \lambda^{\frac{1}{1-p}} \beta(y_\lambda), \quad \text{for some } x_\lambda \in [0,z) \text{ and } y_\lambda \in [0,1]. \tag{3.20}$$

Moreover, u_l is strictly positive and satisfies (1.5).

Finally, adapting the results in [4, 5], we can prove the existence, for each $\varepsilon \in [0,\varepsilon^*]$, of an extremal solution $v \in \mathcal{O}$ of (3.5), satisfying the following condition: there is a sequence $(v_n)_n$, either of subsolutions of (3.5) if v is maximal, or of supersolutions of (3.5) if v is minimal, which converges to v in $\mathcal{C}^1[0,1]$ and is such that, for every n, a strong solution w_n of the parabolic initial value problem

$$\begin{cases} w_t - w_{xx} = \ell_r(x,w)\, h(\varepsilon w)\, (1 + (\varepsilon w_x)^2)^{\frac{3}{2}}, & 0 < x < 1,\ t > 0, \\ w_x(0,t) = w_x(1,t) = 0, & t > 0, \\ w(x,0) = v_n, & 0 < x < 1, \end{cases}$$

exists and either it blows up in finite time, or otherwise it satisfies

$$\limsup_{t \to +\infty} \|w_n(\cdot,t) - v(\cdot)\|_{\mathcal{C}^1[0,1]} \geq \eta,$$

for some $\eta > 0$ independent of n. Hence, for every $\lambda \in [\lambda_*, \infty)$, we can infer the existence an unstable solution u of (3.3) which is strictly positive and satisfies (3.20) and (1.5). This ends the proof of Theorem 1.2.

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