Subharmonic solutions of weakly coupled Hamiltonian systems

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Abstract. We prove the existence of an arbitrarily large number of subharmonic solutions for a class of weakly coupled Hamiltonian systems which includes the case when the Hamiltonian function is periodic in all of its variables and its critical points are non-degenerate. Our results are obtained through a careful analysis of the dynamics of the planar components, combined with an application of a generalized version of the Poincaré–Birkhoff Theorem.

1 Introduction

We consider a Hamiltonian system of the type

$$J\dot{z} = \nabla H(z) + \varepsilon \nabla P(t,z)\,, \tag{1}$$

where $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ denotes the standard $2N \times 2N$ symplectic matrix, the Hamiltonian function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is twice continuously differentiable, $P : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ is a continuous function, T-periodic in its first variable and continuously differentiable with respect to its second variable (with $\nabla P(t,z)$ denoting the gradient with respect to z), and ε is a small real parameter.

Writing $z = (x,y)$, with $x = (x_1, \dots, x_N) \in \mathbb{R}^N$ and $y = (y_1, \dots, y_N) \in \mathbb{R}^N$, we use the notation $z_k = (x_k, y_k) \in \mathbb{R}^2$, and we assume that there are N functions $H_k : \mathbb{R}^2 \to \mathbb{R}$ such that

$$H(z) = \sum_{k=1}^{N} H_k(z_k)\,. \tag{2}$$

Under this assumption, system (1) is said to be *weakly coupled*, since ε is supposed to be a small parameter.

We are looking for subharmonic solutions, i.e., periodic solutions $z : \mathbb{R} \to \mathbb{R}^{2N}$ whose minimal period is mT, for some integer $m \geq 2$. Clearly enough, under the sole above assumptions the existence of these solutions is not guaranteed, as the simple example $H = P = 0$ shows; this is why, for this kind of problem, some further assumptions are necessary.

In [3], Conley and Zehnder consider a general Hamiltonian system of the type

$$J\dot{z} = \nabla H(t,z)\,, \tag{3}$$

assuming the Hamiltonian function H to be T-periodic in t and also periodic in all its remaining variables, a setting already adopted by the same authors in [2]. They ask all the iterates of the possible T-periodic solutions of the Hamiltonian system to be *non-degenerate*; in other words, if $z(t)$ is a T-periodic solution of (3), denoting by $Z(t)$ the $2N \times 2N$ matrix solution of

$$J\dot{Z} = H_z''(t, z(t))Z\,, \qquad Z(0) = \mathrm{Id}\,,$$

the number 1 cannot be an eigenvalue of $Z(mT)$, for any integer $m \geq 1$. Under these assumptions, it is proved in [3] that for any sufficiently large *prime* number m there is a periodic solution having minimal period mT. (See also [9, 11, 13, 17] for similar results.) Although their non-degeneracy assumption seems to be generically satisfied (see e.g. [14]), it is very difficult to verify it in concrete examples.

We are looking here for some conditions which can be more easily checked in practice. For instance, we will prove that if the function H of the form (2) is periodic in all its variables and has only non-degenerate critical points, then for ε small enough system (1) has a large number of subharmonic solutions, whose planar components perform a prescribed number of rotations in their period time. (We recall that Q is a *non-degenerate* critical point of the Hamiltonian function H if $\det H''(Q) \neq 0$.)

In order to precisely state our results, let us first recall the definition of rotation number associated with a planar curve, around a point Q. For $\tau_1 < \tau_2$, let $\zeta : [\tau_1, \tau_2] \to \mathbb{R}^2$ be continuously differentiable and such that $\zeta(t) \neq Q$, for every $t \in [\tau_1, \tau_2]$. Writing $\zeta(t) = Q + \rho(t)(\cos\theta(t), \sin\theta(t))$, with $\rho(t) > 0$ and $\theta(t)$ continuous functions, one has

$$\mathrm{Rot}\,(\zeta; Q; [\tau_1, \tau_2]) = -\frac{\theta(\tau_2) - \theta(\tau_1)}{2\pi}\,.$$

If Q is the origin, we will simply write $\mathrm{Rot}\,(\zeta; [\tau_1, \tau_2])$.

We can now present our first result.

Theorem 1.1. *Assume that the Hamiltonian function H, of the form* (2)*, has at least two critical points, one of which is a non-degenerate local minimum point $\mathcal{Q} = (\mathcal{Q}_1, \ldots, \mathcal{Q}_N) \in \mathbb{R}^{2N}$. In addition assume that, for $\varepsilon = 0$, the solutions of* (1) *are globally defined.*

Let $M_1, \ldots, M_N$ be arbitrary positive integers. Then, there is a positive integer $\overline{m}$ with the following property: for every integer $m \geq \overline{m}$, there exists $\varepsilon_m > 0$ such that, if $|\varepsilon| \leq \varepsilon_m$, system (1) *has at least $N+1$* distinct *mT-periodic solutions $z(t)$, whose components satisfy*

$$\mathrm{Rot}(z_k; \mathcal{Q}_k; [0, mT]) = M_k\,, \quad \textit{for every } k = 1, \ldots, N\,.$$

Taking into account (2), in Theorem 1.1 we are assuming that all the functions H_k have at least two critical points, one of which is a non-degenerate local minimum point. A similar statement holds in the case when there is a non-degenerate local maximum point, taking $M_1, \dots, M_N$ to be negative integers.

Let us clarify what we mean by *distinct* subharmonic solutions. Being the nonlinearities T-periodic in t, once an mT-periodic solution $z(t)$ has been found, many others appear by just making a shift in time, thus giving rise to the periodicity class

$$z(t),\, z(t+T),\, z(t+2T)\,,\, \dots\,,\, z(t+(m-1)T)\,.$$

We say that two mT-periodic solutions are *distinct* if they are not related to each other in such a way.

As already noticed in [7], if at least one of the components of a solution $z(t)$ makes exactly one rotation in its period time mT, then necessarily this solution has *minimal period* equal to mT. As a consequence, if $N \geq 2$, there will be a myriad of periodic solutions having minimal period mT: when one of the components performs exactly one rotation, the others rotate an arbitrary number of times. We thus have the following direct consequence of Theorem 1.1.

Corollary 1.2. *Under the assumptions of Theorem 1.1, let $N \geq 2$ and fix an arbitrary positive integer $\aleph$. Then, there is a positive integer $\overline{m}$ with the following property: for every integer $m \geq \overline{m}$, there exists $\varepsilon_m > 0$ such that, if $|\varepsilon| \leq \varepsilon_m$, system* (1) *has at least $\aleph$ distinct periodic solutions with minimal period mT.*

We then easily deduce the following result.

Corollary 1.3. *Let $N \geq 2$, and assume that the Hamiltonian function H is of the form* (2)*, it is periodic with respect to all variables x_k and y_k, and all its critical points are non-degenerate. Then, the same conclusion of Corollary 1.2 holds.*

Indeed, by Weierstrass Theorem, there surely is a minimum point for H, and it is non-degenerate, by assumption. Moreover, being ∇H periodic in all its variables, it is bounded, so the solutions of (1) with $\varepsilon = 0$ are globally defined. Corollary 1.2 thus applies.

Let us compare this result with the above quoted one by Conley and Zehnder. Clearly, we have a strong restriction in dealing here only with weakly coupled systems, while in [3] the general Hamiltonian system (3) was studied. However, the non-degeneracy is now assumed only on the critical points of H, a condition which is rather easy to verify in practice. Moreover, we now have, *for any* sufficiently large integer m, an *arbitrarily large* number of periodic solutions having minimal period mT, provided that ε is chosen sufficiently small. Notice also that we do not require the function $P(t,z)$ to be periodic in the space variables.

We also recall that a perturbation theory has been developed in the literature for nearly integrable Hamiltonian systems (see, e.g. [1, 4]). However, we emphasize that our results are not of perturbative type, since the periodic solutions we find do not bifurcate from some particular solution of the uncoupled system corresponding to $\varepsilon = 0$.

The paper is organized as follows. The next three sections are devoted to the study of *planar* Hamiltonian systems, providing some new existence results in this setting. In Section 2 we develop some preliminaries on autonomous Hamiltonian systems, which will be used in the subsequent existence theorems. In Section 3 we prove our main result for planar Hamiltonian systems, followed by some useful corollaries. In Section 4 we analyse in detail some more specific systems, and propose more explicit conditions so to get the existence of subharmonic solutions. In particular, we obtain an existence result for scalar second order equations which turns out to be optimal. Finally, in Section 5 we provide a generalization of Theorem 1.1 along the lines developed in the previous sections.

2 Some preliminaries on autonomous planar Hamiltonian systems

In this section we concentrate on autonomous planar Hamiltonian systems of the type

$$J\dot{\zeta} = \nabla\mathscr{H}(\zeta)\,, \tag{4}$$

where $\mathscr{H} : \mathbb{R}^2 \to \mathbb{R}$ is a twice continuously differentiable function. We are first interested in the dynamics near a nonconstant periodic solution. We recall that the orbit Γ of such a solution is a Jordan curve, so that $\mathbb{R}^2 \setminus \Gamma$ is the disjoint union of two open connected sets, $\mathrm{int}(\Gamma)$, the "interior" set, which is bounded, and $\mathrm{ext}(\Gamma)$, the "exterior" set, which is unbounded. As we now recall, such a solution "generates" a period annulus, i.e., a connected set in the plane (which may be unbounded) covered by orbits of nonconstant periodic solutions.

Proposition 2.1. *Any nonconstant periodic orbit of* (4) *is contained in the interior of a period annulus.*

Proof. Let $\zeta^*(t)$ be a nonconstant periodic solution of (4), with minimal period $\tau^* > 0$, and set $h^* = \mathscr{H}(\zeta^*(t))$ (recall that the Hamiltonian is constant along the orbits). We know that $\nabla\mathscr{H}(\zeta^*(t)) \neq 0$ for every $t \in [0, \tau^*]$, hence, by continuity and compactness, there is an open neighborhood U^* of the orbit of ζ^* on which $\nabla\mathscr{H}$ remains away from 0.

Let us consider in U^* the system

$$\dot{z} = \frac{\nabla\mathscr{H}(z)}{|\nabla\mathscr{H}(z)|}\,, \tag{5}$$

and for every $s \in \mathbb{R}$ let $z_s(t)$ be the solution of (5) satisfying $z_s(0) = \zeta^*(s)$. We can then fix $\delta > 0$ such that the set

$$U^{**} = \{z_s(t) : t \in [-\delta, \delta] \text{ and } s \in [0, \tau^*]\}$$

is contained in U^*. Let $f, g : [0, \tau^*] \to \mathbb{R}$ be the continuous functions defined by

$$f(s) = \min\{\mathscr{H}(z_s(t)) : t \in [-\delta, \delta]\}, \quad g(s) = \max\{\mathscr{H}(z_s(t)) : t \in [-\delta, \delta]\}.$$

Since $\nabla\mathscr{H} \neq 0$ on U^*, the function $t \mapsto \mathscr{H}(z_s(t))$ is strictly increasing on $[-\delta, \delta]$, for every $s \in [0, \tau^*]$, hence $f(s) < h^* < g(s)$, for every $s \in [0, \tau^*]$. Then, defining

$$h^*_- = \max\{f(s) : s \in [0, \tau^*]\}, \quad h^*_+ = \min\{g(s) : s \in [0, \tau^*]\},$$

we have that $h^*_- < h^* < h^*_+$, and

$$\mathscr{H}(z_s(-\delta)) \leq h^*_-, \quad \mathscr{H}(z_s(\delta)) \geq h^*_+, \qquad \text{for every } s \in [0, \tau^*].$$

The set $U^{**} \cap \mathscr{H}^{-1}(\,]h^*_-, h^*_+[\,)$ contains the orbit of ζ^*, it is open and arcwise connected; to conclude the proof, we need to show that every solution of (4) starting from this set is periodic.

Fix any $h^\sharp \in \,]h^*_-, h^*_+[\,$, any $P \in U^{**} \cap \mathscr{H}^{-1}(h^\sharp)$, and let $\zeta(t)$ be the solution of (4) such that $\zeta(0) = P$. Since $\mathscr{H}(\zeta(t)) = h^\sharp$ for every $t \in \mathbb{R}$ and $\zeta(0) \in U^{**}$, by the choice of h^*_- and h^*_+, it has to be that $\zeta(t) \in U^{**}$ for every $t \in \mathbb{R}$, hence $\zeta(t)$ is bounded. Therefore, by the Poincaré–Bendixson Theorem, the ω-limit of ζ is a periodic orbit. Let us show that $\omega(\zeta)$ coincides with the orbit of ζ itself. First notice that $\omega(\zeta) \subseteq \mathscr{H}^{-1}(h^\sharp)$. Next, let $Q \in \omega(\zeta)$. By the Implicit Function Theorem, there is an $r > 0$ such that the set $\mathscr{H}^{-1}(h^\sharp) \cap B(Q, r)$ is the graph of a C^1-function, hence it is an arc of the periodic orbit $\omega(\zeta)$. On the other hand, being Q in $\omega(\zeta)$, by definition there is a sequence $(Q_n)_n$ in the orbit of ζ which belongs to $\mathscr{H}^{-1}(h^\sharp) \cap B(Q, r)$. Then Q_n belongs both to the orbit of ζ and to $\omega(\zeta)$, which is also an orbit of (4). Therefore, the orbit of ζ must coincide with $\omega(\zeta)$. We have thus shown that $\zeta(t)$ is periodic. □

In the following, we will denote by $\mathcal{A}(\Gamma)$ the *maximal* period annulus determined by a nonconstant periodic orbit Γ, i.e., the maximal connected set covered by the orbits of nonconstant periodic solutions containing Γ. By Proposition 2.1, $\mathcal{A}(\Gamma)$ is an open set. Clearly, it may be unbounded, but it cannot coincide with the whole space $\mathbb{R}^2$, since it is well-known that, when there is a nonconstant periodic orbit Γ, there must exist an equilibrium point in $\mathrm{int}(\Gamma)$. By Hopf's Theorem (the *Umlaufsatz* [12]), we can associate a direction of rotation to the nonconstant periodic solution $\zeta^*(t)$, according to whether the degree of its derivative is $+1$ or -1; in the first case, we say that the solution rotates clockwise, while in the second case it rotates counter-clockwise. Since the period annulus is connected, all the periodic solutions contained in it have the same direction of rotation.

We denote by $\mathcal{T}(\zeta_0)$ the period of the solution with initial position $\zeta_0 = \zeta(0)$ in $\mathcal{A}(\Gamma)$ (in the following, the "period" of a solution of an autonomous system is always meant to be its *minimal* period). It is well-known that, since $\mathscr{H}$ is a C^2-function, the function $\zeta_0 \mapsto \mathcal{T}(\zeta_0)$ is continuously differentiable on $\mathcal{A}(\Gamma)$. The periods of all the orbits in $\mathcal{A}(\Gamma)$ thus determine what we will call the *associated period interval* $\mathcal{I}(\Gamma)$.

Proposition 2.2. *Let Γ be a nonconstant periodic orbit of* (4)*, and assume that $\mathcal{A}(\Gamma) \cup \mathrm{int}(\Gamma) \neq \mathbb{R}^2$. If the solutions of* (4) *are globally defined, then $\mathcal{I}(\Gamma)$ is unbounded. More precisely, the periods of the orbits of* (4) *in $\mathcal{A}(\Gamma) \cap \mathrm{ext}(\Gamma)$ cover an unbounded interval.*

Proof. Let P be a point belonging to Γ, and v be a vector such that $P + v \notin \mathcal{A}(\Gamma) \cup \mathrm{int}(\Gamma)$. Consider the set of points $p_\lambda = P + \lambda v$, with $\lambda \in [0, 1]$, i.e., the segment joining P with $P + v$, and set

$$\alpha = \max\{\lambda \in [0, 1] : p_\lambda \in \Gamma\}\,, \quad \bar{\lambda} = \sup\{\lambda \in [\alpha, 1] : p_\lambda \in \mathcal{A}(\Gamma)\}\,.$$

By Proposition 2.1, we know that $\bar{\lambda} > \alpha$.

For $\lambda \in [\alpha, 1]$, let $\zeta_\lambda(t)$ be the solution of (4) satisfying the initial condition $\zeta_\lambda(0) = p_\lambda$. Denoting by $\tau(\lambda)$ the period of $\zeta_\lambda(t)$, with $\lambda \in [\alpha, \bar{\lambda}[$, we want to prove that

$$\lim_{\lambda \to \bar{\lambda}^-} \tau(\lambda) = +\infty\,.$$

By contradiction, assume that there is an increasing sequence $(\lambda_n)_n$ such that $\lambda_n \to \bar{\lambda}$ and $(\tau(\lambda_n))_n$ remains bounded. Then, for a subsequence, keeping the same notation, $\tau(\lambda_n) \to \bar{\tau}$, for some $\bar{\tau} \in [0, +\infty[$. Since the solutions of (4) are globally defined, the set

$$K = \{\zeta_\lambda(t) : \lambda \in [\alpha, 1],\, t \in [0, \bar{\tau} + 1]\}$$

is compact in $\mathbb{R}^2$. For n large enough, the orbit of $\zeta_{\lambda_n}(t)$ is contained in the set K. Moreover, there is a constant $\bar{c} > 0$ such that

$$|\nabla \mathscr{H}(\zeta)| \leq \bar{c}\,, \qquad \text{for every } \zeta \in K\,. \tag{6}$$

Let $Q \neq P$ be another point belonging to Γ. Observing that, when λ varies in $]\alpha, \bar{\lambda}[$, the orbit of $\zeta_\lambda(t)$ belongs to the period annulus associated to Γ and is contained in $\mathrm{ext}(\Gamma)$, it is easy to see, using (6), that

$$\bar{\tau} \geq \frac{2|Q - P|}{\bar{c}}\,.$$

So, $\bar{\tau}$ cannot be equal to zero.

By (6), the sequence $(\zeta_{\lambda_n})_n$ is equi-uniformly continuous on $[0, \bar{\tau} + 1]$, and we know that it is uniformly bounded. By the Ascoli–Arzelà Theorem, there is a subsequence, for which we maintain the same notation, and a continuous function $\bar{\zeta} : [0, \bar{\tau} + 1] \to \mathbb{R}^2$ such that $\zeta_{\lambda_n}(t) \to \bar{\zeta}(t)$, uniformly on $[0, \bar{\tau} + 1]$.

By a standard argument, $\bar{\zeta}(t)$ is a $\bar{\tau}$-periodic solution of (4), and $\bar{\zeta}(0) = p_{\bar{\lambda}}$. By Proposition 2.1, the orbit of $\bar{\zeta}(t)$ is contained in the interior of a period annulus, contradicting the definition of $\bar{\lambda}$. □

Remark 2.3. The assumption $\mathcal{A}(\Gamma) \cup \mathrm{int}(\Gamma) \neq \mathbb{R}^2$ is verified if, e.g., (4) has an equilibrium point in $\mathrm{ext}(\Gamma)$. This situation surely occurs if $\nabla \mathscr{H} : \mathbb{R}^2 \to \mathbb{R}^2$ is periodic along some vector $v \in \mathbb{R}^2 \setminus \{0\}$, i.e.,

$$\nabla \mathscr{H}(\zeta + v) = \nabla \mathscr{H}(\zeta)\,, \quad \text{for every } \zeta \in \mathbb{R}^2. \tag{7}$$

Indeed, in this case $\Gamma + v$ is a periodic orbit of (4), with $\mathrm{int}(\Gamma + v) \subseteq \mathrm{ext}(\Gamma)$, and we know that there is an equilibrium point in $\mathrm{int}(\Gamma + v)$.

In order to provide the existence of a nonconstant periodic solution of (4), we will need the following result.

Proposition 2.4. *Let Ω be a bounded connected open subset of $\mathbb{R}^2$ such that, either*

$$\min_{\partial\Omega} \mathscr{H} > \min_{\overline{\Omega}} \mathscr{H}\,, \tag{8}$$

or

$$\max_{\partial\Omega} \mathscr{H} < \max_{\overline{\Omega}} \mathscr{H}\,. \tag{9}$$

Then, there exists a nonconstant periodic solution of (4) *whose orbit is contained in Ω.*

Proof. Let us assume (8). By Sard's Lemma, there exists a $c \in \mathbb{R}$ such that

$$\min_{\overline{\Omega}} \mathscr{H} < c < \min_{\partial\Omega} \mathscr{H}\,, \tag{10}$$

and

$$\nabla \mathscr{H}(\zeta) \neq 0\,, \quad \text{for every } \zeta \in \mathscr{H}^{-1}(c)\,. \tag{11}$$

Fix $\zeta_0 \in \Omega \cap \mathscr{H}^{-1}(c)$ and let $\zeta(t)$ be the solution of (4) such that $\zeta(0) = \zeta_0$. Since $\mathscr{H}(\zeta(t)) = c$ for every $t \in \mathbb{R}$, by (10) it has to be that $\zeta(t) \in \Omega$, for every $t \in \mathbb{R}$. By the Poincaré–Bendixson Theorem, the ω-limit of ζ is either a nonconstant periodic orbit, or contains an equilibrium point. But the second possibility is excluded, because of (11). Hence, it is a nonconstant periodic orbit which, by (10), is contained in Ω.

The case when (9) holds can be treated in a similar way. □

Remark 2.5. Condition (8) is surely verified if $\partial\Omega$ is smooth and

$$\langle \nabla \mathscr{H}(\zeta), \nu(\zeta) \rangle > 0\,, \qquad \text{for every } \zeta \in \partial\Omega\,,$$

where $\nu(\zeta)$ denotes the outward unit normal to $\partial\Omega$ at ζ. A similar observation holds for (9), reversing the inequality.

In the following, we say that a critical point $\zeta_0 \in \mathbb{R}^2$ of $\mathscr{H}$ is *non-degenerate* if $\det \mathscr{H}''(\zeta_0) \neq 0$.

Proposition 2.6. *Let ζ_0 be a non-degenerate local minimum point of $\mathscr{H}$. Then, in any neighbourhood of ζ_0 there exists a nonconstant periodic solution of* (4)*, rotating in clockwise sense. Similarly if ζ_0 is a non-degenerate local maximum point of $\mathscr{H}$, the nonconstant periodic solutions rotating in counter-clockwise sense in this case.*

Proof. Being $\nabla\mathscr{H}(\zeta_0) = 0$, we can write

$$\mathscr{H}(\zeta) = \mathscr{H}(\zeta_0) + \langle \mathscr{H}''(\zeta_0 + \xi(\zeta - \zeta_0))(\zeta - \zeta_0), \zeta - \zeta_0 \rangle \,,$$

for some $\xi \in]0, 1[$. The conclusion easily follows from Proposition 2.4, taking $\Omega = B(\zeta_0, \rho)$, with $\rho > 0$ sufficiently small. □

3 Subharmonic solutions in the plane

We consider the time-dependent planar Hamiltonian system

$$J\dot{\zeta} = \nabla\mathscr{H}(\zeta) + \varepsilon\nabla\mathscr{P}(t, \zeta) \,. \tag{12}$$

Here, the Hamiltonian function $\mathscr{H} : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable, $\mathscr{P} : \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ is a continuous function, T-periodic in its first variable and continuously differentiable with respect to $\zeta = (x, y)$, and ε is a small real parameter. We denote by $\nabla\mathscr{P}(t, \zeta)$ the gradient with respect to ζ.

Theorem 3.1. *Let the solutions of* (4) *be globally defined, and assume that there exists a nonconstant periodic orbit Γ of* (4) *such that $\mathcal{A}(\Gamma)\cup\mathrm{int}(\Gamma) \neq \mathbb{R}^2$. Then, there exists an integer $\overline{m} \geq 2$ such that, for every integer $m \geq \overline{m}$, there is a $\bar{\varepsilon}_m > 0$ with the following property: if $|\varepsilon| \leq \bar{\varepsilon}_m$, then system* (12) *has at least two periodic solutions having minimal period mT.*

Proof. We first concentrate on the autonomous system (4), corresponding to (12) with $\varepsilon = 0$. To fix the ideas, assume for instance that the orbit Γ rotates clockwise. By Propositions 2.1 and 2.2, Γ generates a period annulus $\mathcal{A}(\Gamma)$, and the periods of the orbits of (4) in $\mathcal{A}(\Gamma) \cap \mathrm{ext}(\Gamma)$ cover an interval $[\hat{\tau}, +\infty[$. Let $\widehat{\Gamma}$ be an orbit in $\mathcal{A}(\Gamma)\cap\mathrm{ext}(\Gamma)$ with period $\hat{\tau}$. Let $\overline{m}$ be the minimal positive integer such that $\overline{m}T > \hat{\tau}$, and take $m \geq \overline{m}$. Let Γ_m be an orbit in $\mathcal{A}(\Gamma)\cap\mathrm{ext}(\widehat{\Gamma})$ with period mT. Finally, let $\widetilde{\Gamma}$ be an orbit in $\mathcal{A}(\Gamma)\cap\mathrm{ext}(\Gamma_m)$ with period $\tilde{\tau} > mT$. The orbits $\widehat{\Gamma}$ and $\widetilde{\Gamma}$ determine a bounded open annulus $\mathscr{A}$ in the plane.

We can now proceed as in [6, Lemma 4.3] and construct a symplectic diffeomorphism $\Lambda : \mathscr{A} \to \mathscr{B}$, where $\mathscr{B}$ is an open annulus of the type $\{v \in \mathbb{R}^2 : r_i < |v| < r_e\}$, such that the change of variables $v(t) = \Lambda(\zeta(t))$ transforms the orbits of the autonomous system (4) in $\mathscr{A}$ into the orbits in $\mathscr{B}$ of a Hamiltonian system with Hamiltonian function $\mathcal{L}(v) = \mathscr{H}(\Lambda^{-1}(v))$, with

$\nabla \mathcal{L}(v) = \frac{2\pi}{\mathcal{T}(\Lambda^{-1}(v))}\, v$, while preserving their periods. Notice that $\mathcal{L} : \mathscr{B} \to \mathbb{R}$ is a C^2-function. We thus have the new system

$$J\dot{v} = \frac{2\pi}{\mathcal{T}(\Lambda^{-1}(v))}\, v\,. \tag{13}$$

All the orbits of this system are circular, and the periods vary in an interval containing $]\hat{\tau}, \tilde{\tau}[$. In particular, the periods of those lying near the circle of radius r_i are close to $\hat{\tau}$, while the periods of those lying near the circle of radius r_e are close to $\tilde{\tau}$. The orbit Γ_m is transformed into a circle of radius $r_m \in]r_i, r_e[$, whose period is still equal to mT.

The same change of variables translates the solutions of the perturbed system (12) lying in $\mathscr{A}$ into solutions of a Hamiltonian system with Hamiltonian function

$$\mathcal{L}_\varepsilon(t, v) = \mathcal{L}(v) + \varepsilon \mathscr{P}(t, \Lambda^{-1}(v))\,,$$

defined on $\mathbb{R} \times \mathscr{B}$. We now modify and extend from $\mathscr{B}$ to the whole plane $\mathbb{R}^2$ this Hamiltonian function. We fix some numbers r_i', r_e', r_i'', r_e'', with

$$r_i < r_i' < r_i'' < r_m < r_e'' < r_e' < r_e\,,$$

and in such a way that, denoting by $\tau(r_i'')$, $\tau(r_e'')$ the periods of the circular orbits of (13) with radius r_i'', r_e'', respectively, we have that

$$\tau(r_i'') < mT < \tau(r_e'')\,. \tag{14}$$

Consider a C^∞-function $\chi : [0, +\infty[\to [0,1]$, whose support is contained in $]r_i, r_e[$, such that $\chi(r) = 1$ when $r \in [r_i', r_e']$, and let $\widetilde{\mathcal{L}}_\varepsilon : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$\widetilde{\mathcal{L}}_\varepsilon(t, v) = \begin{cases} \chi(|v|)\mathcal{L}_\varepsilon(t, v) & \text{if } v \in \mathscr{B}\,, \\[2ex] 0 & \text{if } v \in \mathbb{R}^2 \setminus \mathscr{B}\,. \end{cases}$$

This is a continuous function, T-periodic in its first variable, and continuously differentiable in its second variable. We can then consider the Hamiltonian system

$$J\dot{v} = \nabla \widetilde{\mathcal{L}}_\varepsilon(t, v)\,. \tag{15}$$

Notice that all the points in $\mathbb{R}^2 \setminus \mathscr{B}$ are equilibria for (15). Moreover, if $\varepsilon = 0$, system (15) coincides with (13) on

$$\overline{\mathscr{B}}\,' = \{v \in \mathbb{R}^2 : r_i' \le |v| \le r_e'\}\,.$$

Let us see how the solutions of (15) behave when starting from the set

$$\overline{\mathscr{B}}\,'' = \{v \in \mathbb{R}^2 : r_i'' \le |v| \le r_e''\}\,.$$

If $\varepsilon = 0$, we know from (14) that the solutions $v(t)$ of (13) starting with $|v(0)| = r_i''$ rotate clockwise more than once in the time interval $[0, mT]$, while those starting with $|v(0)| = r''_e$ make less than one clockwise rotation in the same time interval. In symbols we thus have, for the solutions of (15),

$$\begin{aligned} \big[\, \varepsilon = 0 \quad \text{and} \quad |v(0)| = r''_i \,\big] \quad &\Rightarrow \quad \mathrm{Rot}(v; [0, mT]) > 1\,, \\ \big[\, \varepsilon = 0 \quad \text{and} \quad |v(0)| = r''_e \,\big] \quad &\Rightarrow \quad \mathrm{Rot}(v; [0, mT]) < 1\,. \end{aligned} \tag{16}$$

We claim that, for ε small enough, the solutions of (15) starting with $v(0) \in \overline{\mathscr{B}}''$ will be such that

$$v(t) \in \overline{\mathscr{B}}', \qquad \text{for every } t \in [0, mT]\,. \tag{17}$$

Indeed, since $\mathcal{L}$ is twice continuously differentiable on $\mathscr{B}$, and $\overline{\mathscr{B}}'$ is a compact subset of $\mathscr{B}$, there are two constants $C > 0$ and $L > 0$ such that

$$|\nabla \mathcal{L}(v_1) - \nabla \mathcal{L}(v_2)| \le L|v_1 - v_2|\,, \qquad \text{for every } v_1, v_2 \in \overline{\mathscr{B}}', \tag{18}$$

and

$$|\nabla(\mathscr{P}(t, \cdot) \circ \Lambda^{-1})(v)| \le C\,, \qquad \text{for every } (t, v) \in \mathbb{R} \times \overline{\mathscr{B}}'. \tag{19}$$

Fix $\tilde{\varepsilon} > 0$ such that

$$\tilde{\varepsilon} < \frac{1}{CmTe^{LmT}} \min\{r''_i - r'_i, r'_e - r''_e\}\,, \tag{20}$$

and assume that $|\varepsilon| \le \tilde{\varepsilon}$. Let $v(t)$ be a solution of (15) with $v(0) \in \overline{\mathscr{B}}''$, let $w(t)$ be a solution of (13) with $w(0) = v(0)$, and let $t \in [0, mT]$ be such that $v(s) \in \overline{\mathscr{B}}'$ for every $s \in [0, t]$. Then, by (18) and (19),

$$\begin{aligned} |v(t) - w(t)| &= \left| \int_0^t J\nabla \widetilde{\mathcal{L}}_\varepsilon(s, v(s)) - J\nabla \mathcal{L}(w(s))\, ds \right| \\ &\le \int_0^t |\nabla \mathcal{L}(v(s)) - \nabla \mathcal{L}(w(s))|\, ds + \\ &\qquad + \varepsilon \int_0^t |\nabla(\mathscr{P}(s, \cdot) \circ \Lambda^{-1})(v(s))|\, ds \\ &\le L \int_0^t |v(s) - w(s)|\, ds + \varepsilon CmT\,. \end{aligned}$$

By Gronwall Lemma,

$$|v(t) - w(t)| \le \varepsilon CmTe^{Lt} \le \tilde{\varepsilon} CmTe^{LmT},$$

showing that $v(t) \notin \partial \overline{\mathscr{B}}'$, by (20). This proves that, if $|\varepsilon| \le \tilde{\varepsilon}$, the solution $v(t)$ remains in $\overline{\mathscr{B}}'$ for every $t \in [0, mT]$.

Then, by (16) and (19), there exists $\bar{\varepsilon} \in]0, \tilde{\varepsilon}\,]$ such that

$$\begin{aligned} \big[\, |\varepsilon| \le \bar{\varepsilon} \quad \text{and} \quad |v(0)| = r''_i \,\big] \quad &\Rightarrow \quad \mathrm{Rot}(v; [0, mT]) > 1\,, \\ \big[\, |\varepsilon| \le \bar{\varepsilon} \quad \text{and} \quad |v(0)| = r''_e \,\big] \quad &\Rightarrow \quad \mathrm{Rot}(v; [0, mT]) < 1\,. \end{aligned}$$

Hence, we can apply the generalized version of the Poincaré–Birkhoff Theorem in [8, Theorem 1.2] (which does not require the uniqueness for initial value problems), providing the existence of two distinct mT-periodic solutions $v^1(t), v^2(t)$ of (15), with $v^1(0), v^2(0) \in \overline{\mathscr{B}}''$, such that

$$\operatorname{Rot}(v^j; [0, mT]) = 1\,, \quad \text{for every } j = 1, 2\,.$$

The minimal period of these solutions is mT and, by the above considerations,

$$v^j(t) \in \overline{\mathscr{B}}', \qquad \text{for every } t \in [0, mT]\,.$$

Hence, since $\widetilde{\mathcal{L}}_\varepsilon(t, v) = \mathcal{L}_\varepsilon(t, v)$ when $v \in \overline{\mathscr{B}}'$, by the inverse change of variables we obtain two distinct periodic solutions of the original system (12),

$$\zeta^j(t) = \Lambda^{-1}(v^j(t))\,, \qquad \text{with } j = 1, 2\,,$$

both having minimal period mT. □

Remark 3.2. In the above proof, assuming that the orbit Γ rotates clockwise, we could fix an arbitrary positive integer M and choose $\overline{m}$ to be the minimal positive integer such that $\overline{m}T > M\hat{\tau}$. Then, taking $m \geq \overline{m}$, it is possible to find $r_i'' < r_m < r_e''$ such that the orbit of (13) with radius r_i'' has a smaller period than mT/M, while the period of the orbit with radius r_e'' is greater than mT. The corresponding annulus $\overline{\mathscr{B}}''$ is such that the solutions $v(t)$ starting with $|v(0)| = r_i''$ rotate clockwise more than M times in the time interval $[0, mT]$, while those starting with $|v(0)| = r_e''$ make less than one clockwise rotation in the same time interval. We thus eventually find two mT-periodic solutions of (15) such that

$$\operatorname{Rot}(v^j; [0, mT]) = M\,, \quad \text{for every } j = 1, 2\,.$$

A similar argument holds when the orbit Γ rotates counter-clockwise, provided that M is negative. This remark will be useful in the proof of Theorem 1.1.

We now provide some useful corollaries of Theorem 3.1.

Corollary 3.3. *Let the solutions of* (4) *be globally defined, assume that there exists a bounded connected open subset Ω of $\mathbb{R}^2$ such that either* (8) *or* (9) *is satisfied, and that there is a ζ_0 in the unbounded connected component of $\mathbb{R}^2 \setminus \Omega$ such that $\nabla\mathscr{H}(\zeta_0) = 0$. Then, the same conclusion of Theorem 3.1 holds.*

Proof. In this case, by Proposition 2.4, there is a nonconstant periodic orbit Γ of (4) contained in Ω. Since ζ_0 is an equilibrium point which belongs to the unbounded connected component of $\mathbb{R}^2 \setminus \Omega$, it surely does not belong to $\mathcal{A}(\Gamma) \cup \operatorname{int}(\Gamma)$, so that Theorem 3.1 applies. □

Corollary 3.4. *Let the solutions of* (4) *be globally defined, and assume that $\nabla\mathscr{H}$ is periodic along some vector $v \in \mathbb{R}^2 \setminus \{0\}$ (i.e., that* (7) *holds). If there exists a bounded connected open subset Ω of $\mathbb{R}^2$ such that either* (8) *or* (9) *is satisfied, then the same conclusion of Theorem 3.1 holds.*

Proof. By Proposition 2.4 there is a nonconstant periodic orbit Γ of (4) contained in Ω, and hence there is an equilibrium point in $\zeta^* \in \mathrm{int}(\Gamma)$. By the periodicity assumption, all points $\zeta^* + kv$, with $k \in \mathbb{Z}$, are still equilibria, hence the conclusion follows from Corollary 3.3. □

Corollary 3.5. *Let the solutions of* (4) *be globally defined, and assume that* $\mathscr{H}$ *has at least two critical points, one of which is a non-degenerate local minimum or maximum point. Then, the same conclusion of Theorem 3.1 holds.*

Proof. We use Proposition 2.6 to find a sufficiently small nonconstant orbit Γ surrounding the non-degenerate local minimum or maximum point, so that the second critical point belongs to $\mathrm{ext}(\Gamma)$. Hence, Theorem 3.1 applies, in view of Remark 2.3. □

Corollary 3.6. *Let the solutions of* (4) *be globally defined, and assume that* $\nabla\mathscr{H}$ *is periodic along some vector* $v \in \mathbb{R}^2 \setminus \{0\}$*. If there exists a non-degenerate local minimum or maximum point* ζ_0 *of* $\mathscr{H}$*, then the same conclusion of Theorem 3.1 holds.*

Proof. It is a direct consequence of Corollary 3.5. □

We now consider the case when the Hamiltonian function $\mathscr{H}$ is periodic in two different directions.

Corollary 3.7. *Let* $\mathscr{H}$ *be periodic along two linearly independent vectors* $v, w \in \mathbb{R}^2 \setminus \{0\}$*, and assume that all its critical points are non-degenerate. Then, the same conclusion of Theorem 3.1 holds.*

Proof. By Weierstrass Theorem, there surely are a minimum and a maximum point for $\mathscr{H}$, and they are non-degenerate, by assumption. Moreover, $\nabla\mathscr{H}$ is periodic along the same vectors v, w, hence bounded, so the solutions of (4) are globally defined. Corollary 3.6 thus applies. □

We end this section dealing with the case when the gradient of $\mathscr{H}$ is periodic in two different directions.

Corollary 3.8. *Let* $\nabla\mathscr{H}$ *be periodic along two linearly independent vectors* $v, w \in \mathbb{R}^2 \setminus \{0\}$*. Assume that there exist* $\alpha, \beta \in \mathbb{R}$ *such that*

$$\frac{\partial\mathscr{H}}{\partial v}(\alpha v + \lambda w) \cdot \frac{\partial\mathscr{H}}{\partial v}(\beta v + \lambda w) < 0\,, \quad \textit{for every } \lambda \in \mathbb{R}\,,$$

and that there exist $a, b \in \mathbb{R}$ *such that*

$$\frac{\partial\mathscr{H}}{\partial w}(\lambda v + a w) \cdot \frac{\partial\mathscr{H}}{\partial w}(\lambda v + b w) < 0\,, \quad \textit{for every } \lambda \in \mathbb{R}\,.$$

Then, the same conclusion of Theorem 3.1 holds.

Proof. By the periodicity assumptions, $\nabla\mathscr{H}$ is bounded, so the solutions of (4) are globally defined. Moreover we can always assume, without loss of generality, that $\alpha < \beta$, $a < b$,

$$\frac{\partial\mathscr{H}}{\partial v}(\alpha v + \lambda w) < 0 < \frac{\partial\mathscr{H}}{\partial v}(\beta v + \lambda w)\,,$$

and

$$\frac{\partial\mathscr{H}}{\partial w}(\lambda v + a w) < 0 < \frac{\partial\mathscr{H}}{\partial w}(\lambda v + b w)\,,$$

for every $\lambda \in \mathbb{R}$. Defining the bounded connected open set

$$\Omega = \{\gamma_1 v + \gamma_2 w : \alpha < \gamma_1 < \beta,\, a < \gamma_2 < b\}\,,$$

we see that (8) holds, and Corollary 3.4 applies. □

4 Further existence results

The aim of this section is to provide the existence of subharmonic solutions in a more specific setting, which includes as a special case the planar systems generated by scalar second order differential equations. The final result of the section will indeed be specifically stated for such type of equations, involving a periodic nonlinearity, thus proving the existence of subharmonic solutions for a periodically perturbed pendulum-type equation.

As in the previous section, we consider the planar system (12), with the same regularity assumptions on the Hamiltonian function $\mathscr{H} : \mathbb{R}^2 \to \mathbb{R}$ and on $\mathscr{P} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$.

Theorem 4.1. *Let the solutions of* (4) *be globally defined, and assume that there exist three real constants $\alpha < \beta < \gamma$ with the following properties:*
A1 *either*

$$\max\left\{\frac{\partial\mathscr{H}}{\partial x}(\alpha, y), \frac{\partial\mathscr{H}}{\partial x}(\gamma, y)\right\} < 0 < \frac{\partial\mathscr{H}}{\partial x}(\beta, y)\,, \quad \textit{for every } y \in \mathbb{R}\,,$$

or

$$\frac{\partial\mathscr{H}}{\partial x}(\beta, y) < 0 < \min\left\{\frac{\partial\mathscr{H}}{\partial x}(\alpha, y), \frac{\partial\mathscr{H}}{\partial x}(\gamma, y)\right\}\,, \quad \textit{for every } y \in \mathbb{R}\,;$$

A2 *for $x \in [\alpha, \gamma]$, the function $y \mapsto \mathscr{H}(x, y)$ is convex;*
A3 *there exist two real constants $a < b$ such that*

$$\frac{\partial\mathscr{H}}{\partial y}(x, a) < 0 < \frac{\partial\mathscr{H}}{\partial y}(x, b)\,, \quad \textit{for every } x \in [\alpha, \gamma]\,.$$

Then the same conclusion of Theorem 3.1 holds.

Proof. We first notice that, if A2 holds, then A3 is equivalent to

$$\lim_{|y|\to\infty} \mathscr{H}(x,y) = +\infty\,, \quad \text{uniformly for } x \in [\alpha,\gamma]\,.$$

Assume the first of the two conditions in A1 holds. We want to construct a bounded connected open set $\Omega \subset]\alpha,\beta[\times\mathbb{R}$ satisfying condition (8). Let us introduce the multivalued function μ which associates to every $x \in [\alpha,\beta]$ the compact interval

$$\mu(x) = \{y \in \mathbb{R} : \mathscr{H}(x,y) = \min \mathscr{H}(x,\cdot)\}\,.$$

There exists a $\rho > 0$ such that $\mu(x) \subseteq [-\rho,\rho]$, for every $x \in [\alpha,\beta]$. Indeed, assuming the contrary, for every positive integer n there would exist a $x_n \in [\alpha,\beta]$ and a $y_n \in \mu(x_n)$ with $|y_n| > n$. But, since $\frac{\partial\mathscr{H}}{\partial y}(x_n,y_n) = 0$, we would find a contradiction with A2 and A3.

Let us show that $\mu(x)$ is an upper semicontinuous multivalued function, i.e., that for every $x \in [\alpha,\beta]$ and every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\xi - x| < \delta$ implies $\mu(\xi) \subseteq B_\varepsilon(\mu(x))$. (Here $B_\varepsilon(A)$ denotes the open ε-neighbourhood of the set A.)

Indeed, assume by contradiction that there are $x \in [\alpha,\beta]$, $\varepsilon > 0$, a sequence $(x_n)_n$ in $[\alpha,\beta]$ and a sequence $(y_n)_n$ in $[-\rho,\rho]$ such that $x_n \to x$, $y_n \in \mu(x_n)$ and $\mathrm{dist}(y_n,\mu(x)) \geq \varepsilon$. Then, up to a subsequence, $y_n \to y$, for some $y \in [-\rho,\rho]$, and $\mathrm{dist}(y,\mu(x)) \geq \varepsilon$. As a consequence,

$$\eta := \mathscr{H}(x,y) - \min \mathscr{H}(x,\cdot) > 0\,.$$

Fix $\bar{y} \in \mu(x)$; then, since

$$\mathscr{H}(x_n,\bar{y}) \to \mathscr{H}(x,\bar{y}) = \min \mathscr{H}(x,\cdot) \qquad \text{and} \qquad \mathscr{H}(x_n,y_n) \to \mathscr{H}(x,y)\,,$$

for n large enough,

$$\mathscr{H}(x_n,\bar{y}) \leq \min \mathscr{H}(x,\cdot) + \frac{\eta}{2} = \mathscr{H}(x,y) - \frac{\eta}{2} < \mathscr{H}(x_n,y_n)\,,$$

contradicting the fact that $\mathscr{H}(x_n,y_n) = \min \mathscr{H}(x_n,\cdot)$.

By a compactness argument, for every $\varepsilon > 0$ there are a finite number of points x_k in $[\alpha,\beta]$ and corresponding constants $\delta_k > 0$ such that the open intervals $]x_k - \delta_k, x_k + \delta_k[$, with $k = 1,\ldots,n$, cover $[\alpha,\beta]$ and

$$x \in]x_k - \delta_k, x_k + \delta_k[\quad \Rightarrow \quad \mu(x) \subseteq B_\varepsilon(\mu(x_k))\,. \tag{21}$$

Define

$$\Omega = \Big(\bigcup_{k=1}^{n}]x_k - \delta_k, x_k + \delta_k[\times B_\varepsilon(\mu(x_k))\Big) \cap \big(]\alpha,\beta[\times\mathbb{R}\big)\,.$$

This is a bounded connected open subset of $\mathbb{R}^2$ and, by A1 and (21), condition (8) is satisfied. By A1 and A3, the Poincaré–Miranda Theorem (see, e.g., [5]) ensures the existence of a point (x_0,y_0) in $]\beta,\gamma[\times]a,b[$ such that $\nabla\mathscr{H}(x_0,y_0) = (0,0)$. The conclusion follows from Corollary 3.3.

If the second condition in A1 holds, one proceeds similarly, defining the multivalued function $\mu(x)$ on $[\beta, \gamma]$ and finding (x_0, y_0) in $]\alpha, \beta[\times]a, b[$. $\square$

As a particular case, let $\mathscr{H}(x, y) = F(x) + G(y)$, so that system (12) becomes

$$\dot{x} = g(y) + \varepsilon \frac{\partial \mathscr{P}}{\partial y}(t, x, y), \quad -\dot{y} = f(x) + \varepsilon \frac{\partial \mathscr{P}}{\partial x}(t, x, y), \tag{22}$$

where $f(x) = F'(x)$ and $g(y) = G'(y)$.

We first assume both functions f, g to be periodic.

Corollary 4.2. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be periodic, and assume that*

$$\textit{there exist } \alpha, \beta \in \mathbb{R} \textit{ such that } f(\alpha)f(\beta) < 0,$$

and that

$$\textit{there exist } a, b \in \mathbb{R} \textit{ such that } g(a)g(b) < 0.$$

Then, the same conclusion of Theorem 3.1 holds for system (22).

Proof. It is an immediate consequence of Corollary 3.8. $\square$

We now enter into the framework of Theorem 4.1. We have the following two corollaries.

Corollary 4.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be increasing. Assume the existence of a constant $C > 0$ such that*

$$|f(x)| + |g(y)| \leq C(1 + |x| + |y|), \quad \textit{for every } x \in \mathbb{R} \textit{ and } y \in \mathbb{R}.$$

If there exist $\alpha < \beta < \gamma$ and $a < b$ such that

$$\textit{either} \quad \max\{f(\alpha), f(\gamma)\} < 0 < f(\beta) \quad \textit{or} \quad f(\beta) < 0 < \min\{f(\alpha), f(\gamma)\},$$

and

$$g(a) < 0 < g(b),$$

then the same conclusion of Theorem 3.1 holds for system (22).

Proof. It is an immediate consequence of Theorem 4.1. $\square$

Corollary 4.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic, and assume that*

$$\textit{there exist } \alpha, \beta \in \mathbb{R} \textit{ such that } f(\alpha)f(\beta) < 0.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be increasing, assume that

$$\limsup_{|y| \to \infty} \frac{g(y)}{y} < +\infty,$$

and that

$$\textit{there exist } a < b \textit{ such that } g(a) < 0 < g(b).$$

Then, the same conclusion of Theorem 3.1 holds for system (22)*.*

Proof. If f is τ-periodic, we can assume without loss of generality that $\alpha < \beta < \alpha + \tau$ and $f(\alpha) < 0 < f(\beta)$. Moreover, the global existence is guaranteed by the fact that f is bounded and g has an at most linear growth. Taking $\gamma = \alpha + \tau$, the conclusion follows from Corollary 4.3. □

We end this section with an application of the above two corollaries to a scalar differential equation of the type

$$\ddot{x} + f(x) = \varepsilon p(t, x) . \tag{23}$$

Here, $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $p : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, and T-periodic in its first variable.

Corollary 4.5. *Assume the existence of a constant $C > 0$ such that*

$$|f(x)| \le C(1 + |x|) , \quad \textit{for every } x \in \mathbb{R} .$$

If there exist $\alpha < \beta < \gamma$ such that

$$\textit{either} \quad \max\{f(\alpha), f(\gamma)\} < 0 < f(\beta) \quad \textit{or} \quad f(\beta) < 0 < \min\{f(\alpha), f(\gamma)\} ,$$

then the same conclusion of Theorem 3.1 holds for system (22)*.*

Proof. Equation (23) can be written into the equivalent Hamiltonian system

$$\dot{x} = y , \quad -\dot{y} = f(x) - \varepsilon p(t, x) ,$$

which is of the type (22), with $g(y) = y$. Corollary 4.3 then applies, yielding to the conclusion. □

Corollary 4.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic, and assume that*

$$\textit{there exist } \alpha, \beta \in \mathbb{R} \textit{ such that } f(\alpha)f(\beta) < 0 .$$

Then, the same conclusion of Theorem 3.1 holds for equation (23)*.*

Proof. It is an immediate consequence of Corollary 4.5, following the argument in the proof of Corollary 4.4. □

Notice that the above result is optimal since, in the case when $p(t, x)$ is identically equal to zero, if $f(x)$ does not change sign, the only possible periodic solutions of (23) are constant.

As an illustrative example, we see that Corollary 4.6 directly applies to the perturbed pendulum equation

$$\ddot{x} + \sqrt{\frac{g}{\ell}} \sin x = \lambda + \varepsilon p(t, x) ,$$

when $\lambda^2 \ell < g$, yielding the existence of an arbitrarily large number of subharmonic solutions when $|\varepsilon|$ is small enough. We remark that the existence of subharmonic solutions for such kind of equations has been already considered, e.g., in [4, 9, 10, 14, 15, 18].

The results of these last two sections can be easily extended to weakly coupled systems in $\mathbb{R}^{2N}$, with $N \geq 2$. For example, for a system of the type

$$\begin{cases} \ddot{x}_1 + f_1(x_1) = \varepsilon \dfrac{\partial P}{\partial x_1}(t, x_1, \dots, x_N) \\ \dots \\ \ddot{x}_N + f_N(x_N) = \varepsilon \dfrac{\partial P}{\partial x_N}(t, x_1, \dots, x_N)\,, \end{cases}$$

assuming that the functions $f_i : \mathbb{R} \to \mathbb{R}$ are periodic and change sign, we have an arbitrarily large number of subharmonic solutions, as in the statement of Corollary 1.2.

5 The main result

We are in the position of considering the general system (1) in $\mathbb{R}^{2N}$. We recall that the Hamiltonian function $H : \mathbb{R}^{2N} \to \mathbb{R}$ is twice continuously differentiable and satisfies (2), i.e., $H(z) = \sum_{k=1}^N H_k(z_k)$. Hence, if $\varepsilon = 0$, we have the uncoupled planar systems

$$J\dot{\zeta} = \nabla H_k(\zeta)\,, \tag{24}$$

with $k = 1, \dots, N$.

Let us state the main result of this paper.

Theorem 5.1. *Assume that the Hamiltonian function H is of the form* (2) *and that, for $\varepsilon = 0$, the solutions of* (1) *are globally defined. For every $k = 1, \dots, N$, assume that there exists a nonconstant periodic orbit Γ_k of* (24) *such that $\mathcal{A}(\Gamma_k) \cup \mathrm{int}(\Gamma_k) \neq \mathbb{R}^2$, and let $\mathcal{Q}_k \in \mathrm{int}(\Gamma_k)$ be such that $\nabla H_k(\mathcal{Q}_k) = 0$.*

Let $M_1, \dots, M_N$ be arbitrary positive integers. Then there is a positive integer $\overline{m}$ with the following property: for every integer $m \geq \overline{m}$, there exists $\varepsilon_m > 0$ such that, if $|\varepsilon| \leq \varepsilon_m$, system (1) *has at least $N+1$* distinct *mT-periodic solutions $z(t)$, whose components satisfy*

$$\big|\mathrm{Rot}(z_k; \mathcal{Q}_k; [0, mT])\big| = M_k\,, \quad \textit{for every } k = 1, \dots, N\,.$$

Proof. Let us first assume that all the orbits Γ_k rotate clockwise. As seen in the proof of Theorem 3.1, for each $k = 1, \dots, N$ there is a bounded open annulus $\mathscr{A}_k$ for (24) and a symplectic diffeomorphism $\Lambda_k : \mathscr{A}_k \to \mathscr{B}_k$, where $\mathscr{B}_k$ is an open annulus of the type $\{v \in \mathbb{R}^2 : r_k^i < |v| < r_k^e\}$, transforming the orbits of (24) in $\mathscr{A}_k$ into circular orbits in $\mathscr{B}_k$, without changing their periods. By the argument in Remark 3.2, we can then find a positive integer $\overline{m}_k$ with the property that for every $m \geq \overline{m}_k$ there exists a smaller annulus $\mathscr{B}_{k,m} = \{v \in \mathbb{R}^2 : r_{k,m} \leq |v| \leq R_{k,m}\}$ such that the solutions of the transformed planar system starting from the interior boundary circle rotate clockwise more than M_k times in the time interval $[0, mT]$, while those starting from the exterior boundary circle make less than one clockwise rotation in the same time interval. Setting

$$\overline{m} = \max\{\overline{m}_1, \dots, \overline{m}_N\},$$

for every $m \geq \overline{m}$, we choose

$$\mathscr{B} = \mathscr{B}_{1,m} \times \cdots \times \mathscr{B}_{N,m}.$$

We now consider system (1) with arbitrary ε, and apply the change of variables $v(t) = \Lambda(z(t))$ to the solutions lying in $\mathscr{A}_1 \times \cdots \times \mathscr{A}_N$, where

$$\Lambda(z) = (\Lambda_1(z_1), \dots, \Lambda_N(z_N)).$$

Arguing as in the proof of Theorem 3.1, we can modify the transformed Hamiltonian function and extend it to the whole space $\mathbb{R}^{2N}$, so that the twist properties of each component of the solutions are preserved when $|\varepsilon|$ is small enough. We can thus apply [8, Theorem 1.2] to obtain the existence of $N+1$ *distinct* mT-periodic solutions $v^0(t), \dots, v^N(t)$ of the transformed system, whose components satisfy

$$\mathrm{Rot}(v_k^j; [0, mT]) = M_k, \quad \text{for every } k = 1, \dots, N \text{ and } j = 0, \dots, N.$$

Moreover, if $|\varepsilon|$ is small enough, the orbits of these solutions lie in the region where the transformed Hamiltonian function has not been modified.

These solutions are *distinct*, according to the definition given in the Introduction, since they are obtained as critical points of a suitable functional $\varphi : \mathbb{T}^N \times \mathcal{H} \to \mathbb{R}$, using a generalized Lusternik–Schnirelmann Theorem (see the proof of [8, Theorem 1.2]). Here, $\mathbb{T}^N$ is the N-dimensional torus, and $\mathcal{H}$ is a suitable Hilbert space. Hence, either all the corresponding $N+1$ critical levels are different, or the set of critical points is not contractible in $\mathbb{T}^N \times \mathcal{H}$. The claim then follows, since if two solutions $v^i(t)$ and $v^j(t)$ are not *distinct* according to the definition given in the Introduction, then $\varphi(v^i) = \varphi(v^j)$.

Going back to the original system with the inverse change of variables $z^i(t) = \Lambda^{-1}(v^i(t))$, we obtain $N+1$ distinct mT-periodic solutions of (1), whose components satisfy

$$\mathrm{Rot}(z_k^i; \mathcal{Q}_k; [0, mT]) = M_k, \quad \text{for every } k = 1, \dots, N \text{ and } i = 0, \dots, N.$$

In the case when some of the orbits Γ_k rotate counter-clockwise the argument is similar, the only difference being that the corresponding components of the solutions $z^i(t)$ satisfy $\mathrm{Rot}(z_k^i; \mathcal{Q}_k; [0, mT]) = -M_k$.

The proof is thus completed. □

Clearly enough, Theorem 1.1 follows directly from Theorem 5.1, by the same argument in the proof of Corollary 3.5.

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