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On some multilinear type integral systems

SSD MAT/05 Analisi Matematica

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Introduction

In recent years, the study of Euler-Lagrange equations associated to multilinear fractional inequalities received great attention (see [3, 5, 11, 23]). In particular, the following Hardy-Littlewood-Sobolev system has been widely studied:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{v^q(y)}{|x-y|^{N-\alpha}}\,dy, \quad x \in \mathbb{R}^N, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\alpha}}\,dy, \quad x \in \mathbb{R}^N, \\ u, v \geq 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $0 < \alpha < N$ and $p, q > 0$. This system is associated to the well known Hardy-Littlewood-Sobolev inequality:

$$\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}}\,dx\,dy \right| \leq H\|f\|_{s_1}\,\|g\|_{s_2}, \quad \text{for all } f \in L^{s_1}(\mathbb{R}^N), g \in L^{s_2}(\mathbb{R}^N), \tag{2}$$

where $1 < s_1, s_2 < +\infty$ and $\frac{1}{s_1} + \frac{1}{s_2} + \frac{N-\alpha}{N} = 2$ (see [3, 12] for details).

In the same spirit, in this thesis we study the following problem:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^q(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz, \quad x \in \mathbb{R}^N, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^p(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\gamma}|z-x|^{N-\beta}}\,dy\,dz, \quad x \in \mathbb{R}^N, \\ w(x) = \displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^p(y)v^q(z)}{|x-y|^{N-\gamma}|y-z|^{N-\alpha}|z-x|^{N-\beta}}\,dy\,dz, \quad x \in \mathbb{R}^N, \\ u, v \geq 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{3}$$

where $p, q, r > 0$ and $0 < \alpha, \beta, \gamma < N$. Indeed, (3) is the system of Euler-Lagrange equations associated to the following generalization of (2), proved by Beckner [1]:

Theorem 0.0.1 (Beckner inequality)**.** *Let $1 < s_1, \dots, s_k < \infty$, $\sum_{j=1}^{k} s_j^{-1} > 1$ and $0 < h_{ij} = h_{ji} < N$ be real numbers satisfying*

$$\sum_{\substack{1 \leq j \leq k \\ j \neq m}} \frac{N - h_{jm}}{2N} = \frac{s_m - 1}{s_m} \quad \text{for all } m \in \{1, \dots, k\},$$

$$\sum_{1 \leq i < j \leq k} \frac{N - h_{ij}}{N} + \sum_{j=1}^{k} \frac{1}{s_j} = k.$$

If $f_i \in L^{s_i}(\mathbb{R}^N)$ for all $1 \leq i \leq k$, then

$$\left| \int_{\mathbb{R}^{Nk}} \frac{\prod_{j=1}^{k} f_j(x_j)}{\prod\limits_{1 \leq i < j \leq k} |x_i - x_j|^{N - h_{ij}}} \, dx_1 \dots dx_k \right| \leq b_k \prod_{j=1} \|f_j\|_{s_j}, \tag{4}$$

where $b_k = b_k(h_{ij}, N)$.

Moreover, the best constant b_k in (4) is attained for the estremal functions

$$f_j(x) = C(1 + |x|^2)^{-\frac{N}{s_j}},$$

up to a conformal automorphism.

Fractional integration arises in the context of Green's functions and potential theory, restriction phenomena for Fourier transform, intertwining operators for representations of the Lorentz groups and correlation functions in conformal field theory and statistical mechanics (see [1, 4, 7]).

In particular, multilinear fractional integral inequalities of the type (4) can be used to investigate the endpoint estimates for restriction theorems of Fourier transform (see [4]). Said results establish conditions on the exponents $a, b > 0$ and the curve $\psi : [-1, 1] \to \mathbb{R}^N$, such that the following estimate holds:

$$\int_{-\delta}^{\delta} |\hat{f}(\psi(t))|^b \, dt \leq C \|f\|_{L^a(\mathbb{R}^N)}^b, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^N),$$

where $\delta > 0$, $\hat{f}$ is the Fourier transform of f and $\mathcal{S}(\mathbb{R}^N)$ is the set of Schwartz function.

Moreover, Beckner inequality is connected also with Selberg integrals, indeed the best constant b_k is given by the following formula computed in [1]:

$$b_k = b_k(h_{ij}, N) = |S^N|^{-k+\sum_{1\le i<j\le k}\frac{N-h_{ij}}{N}} \int\limits_{(S^N)^k} \prod_{1\le i<j\le k} |\xi_i - \xi_j|^{h_{ij}-N}\, d\xi_1 \dots d\xi_k.$$

For the sake of clarity, in what follows we consider the inequality (4) in the particular case $k=3$:

$$\left| \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{f_1(x)f_2(y)f_3(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dx\, dy\, dz \right| \le b_3 \|f_1\|_{s_1} \|f_2\|_{s_2} \|f_3\|_{s_3}, \quad (5)$$

where, for simplicity, we recall $\alpha := h_{12}$, $\beta := h_{23}$ and $\gamma := h_{13}$.

As for Hardy-Littlewood-Sobolev inequality, the optimizers of (5) are related to the solutions to (3). More precisely, if $(u, v, w) \in L^{p+1}(\mathbb{R}^N) \times L^{q+1}(\mathbb{R}^N) \times L^{r+1}(\mathbb{R}^N)$ is a solution to (3),with $p = \frac{1}{s_1-1}$, $q = \frac{1}{s_2-1}$, $r = \frac{1}{s_3-1}$, then the optimizers of (5) are given by:

$$u = \lambda_1 f_1^{s_1-1}, \quad v = \lambda_2 f_2^{s_2-1}, \quad \text{and} \quad w = \lambda_3 f_3^{s_3-1},$$

where $\lambda_1, \lambda_2, \lambda_3$ are suitable positive constants.

The corresponding value of the best constant b_3 was computed in an explicit form by Grafakos and Morpurgo [8]:

$$b_3 = b_3(\alpha, \beta, \gamma) = (2\pi)^N |S^N|^{-1} \frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}{\Gamma\left(N-\frac{\alpha}{2}\right)\Gamma\left(N-\frac{\beta}{2}\right)\Gamma\left(N-\frac{\gamma}{2}\right)},$$

in the case $\alpha + \beta + \gamma = 2N$.

Unfortunately, for $k \ge 4$ the Grafakos and Morpurgo formula does not hold [20] and the value of b_k is still unknown.

To the best of our knowledge, there is no study of the system (3) in the literature. Our goal is to introduce the problem (3) and to prove related non-existence results. In particular we give a proof of the following theorem:

Theorem 0.0.2. *Let $0 < \alpha, \beta, \gamma < N$, $\alpha + \beta + \gamma > 3N - 1$ and $p, q, r > 0$.*

If

$$\frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} \neq \frac{3N - \alpha - \beta - \gamma}{N}, \quad (6)$$

then the problem (3) has no nontrivial globally Lipschitz [1] *solution*

$$(u,v,w) \in \left(C^1(\mathbb{R}^N) \cap L^{s_1}(\mathbb{R}^N)\right) \times \left(C^1(\mathbb{R}^N) \cap L^{s_2}(\mathbb{R}^N)\right) \times \left(C^1(\mathbb{R}^N) \cap L^{s_3}(\mathbb{R}^N)\right),$$

for all $1 < s_1, s_2, s_3 < +\infty$.

The proof of Theorem 0.0.2 is based on the identity contained in [3, Teorema 5.1]. In [3] Caristi, D'Ambrosio and Mitidieri proved the following non-existence result for system (1):

Theorem 0.0.3 (Caristi - D'Ambrosio - Mitidieri)**.** *Let* $2 \le \alpha < N$ *and* $p, q > 0$. *If*

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-\alpha}{N}, \tag{7}$$

then the problem (1) has no nontrivial radial solution $(u,v) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$.

We obtain Theorem 0.0.2 combining the ideas contained in the proof of Theorem 0.0.3 with the identity (2.29).

This thesis is organized as follow.

In the next chapter we consider some preliminary results: we prove some original theorems to differentiate under the integral sign; moreover, we introduce the system (3) and some properties of the kernel

$$\frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}, \quad x, y, z \in \mathbb{R}^N,$$

with $0 < \alpha, \beta, \gamma < N$.

In Chapter 2 we prove non-existence results for finite energy solutions, i.e. solutions to the system (3) such that $(u,v,w) \in L^{p+1}(\mathbb{R}^N) \times L^{q+1}(\mathbb{R}^N) \times L^{r+1}(\mathbb{R}^N)$. Whereas, solutions to (3) that are not necessarily finite energy solutions are considered in Chapter 3 and 4.

Results stated in this thesis are contained in [15, 16], unless otherwise specified.

[1] Observe that $u \in C^1(\mathbb{R}^N)$ is a globally Lipschitz function if and only if the gradient is bounded: $|\nabla u| \le C$.

Chapter 1

Preliminary results

Let us consider the Beckner inequality

$$\left| \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{f_1(x) f_2(y) f_3(z)}{|x-y|^{N-\alpha} |y-z|^{N-\beta} |z-x|^{N-\gamma}} \, dx \, dy \, dz \right| \leq b_3 \|f_1\|_{s_1} \|f_2\|_{s_2} \|f_3\|_{s_3}, \quad (1.1)$$

with s_1, s_2, s_3 and $\alpha := h_{12}$, $\beta := h_{23}$, $\gamma := h_{13}$ as in Theorem 0.0.1.

The best constant b_3 is given by

$$b_3 = \max_{\|f_1\|_{s_1} = \|f_2\|_{s_2} = \|f_3\|_{s_3} = 1} \left\{ \left| \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{f_1(x) f_2(y) f_3(z)}{|x-y|^{N-\alpha} |y-z|^{N-\beta} |z-x|^{N-\gamma}} \, dx \, dy \, dz \right| \right\}.$$

In order to obtain b_3, applying the Lagrange multiplier theorem, we have to maximize the following functional

$$\begin{aligned} E[f_1, f_2, f_3] := & \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{f_1(x) f_2(y) f_3(z)}{|x-y|^{N-\alpha} |y-z|^{N-\beta} |z-x|^{N-\gamma}} \, dx \, dy \, dz + \\ & -\mu_1 \int\limits_{\mathbb{R}^N} f_1^{s_1}(x) \, dx - \mu_2 \int\limits_{\mathbb{R}^N} f_2^{s_2}(x) \, dx - \mu_3 \int\limits_{\mathbb{R}^N} f_3^{s_3}(x) \, dx, \end{aligned} \quad (1.2)$$

where $f_1, f_2, f_3 \geq 0$ and μ_1, μ_2, μ_3 are positive constants.

The optimizing problem above leads us to a system of integral equations on f_1, f_2 and f_3. Letting $p = \frac{1}{s_1 - 1}$, $q = \frac{1}{s_2 - 1}$, $r = \frac{1}{s_3 - 1}$, $u = \lambda_1 f_1^{s_1 - 1}$,$v = \lambda_2 f_2^{s_2 - 1}$, $w = \lambda_3 f_3^{s_3 - 1}$ and choosing suitable positive constants $\lambda_1, \lambda_2, \lambda_3$, we obtain the following system of

Euler-Lagrange equations for the inequality (1.1):

$$\begin{cases} u(x) = \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{v^q(y) w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz, \quad x \in \mathbb{R}^N, \\ v(x) = \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{u^p(y) w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\gamma}|z-x|^{N-\beta}}\, dy\, dz, \quad x \in \mathbb{R}^N, \\ w(x) = \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{u^p(y) v^q(z)}{|x-y|^{N-\gamma}|y-z|^{N-\alpha}|z-x|^{N-\beta}}\, dy\, dz, \quad x \in \mathbb{R}^N, \\ u, v, w \geq 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

where $p, q, r > 0$ and $0 < \alpha, \beta, \gamma < N$.

In what follows we shall refer to (1.3) as **Beckner system**.

Definition 1.0.1. *A solution* (u, v, w) *to (1.3) is a* **finite energy solution** *(or a* **variational solution***) if* $(u, v, w) \in L^{p+1}(\mathbb{R}^N) \times L^{q+1}(\mathbb{R}^N) \times L^{r+1}(\mathbb{R}^N)$*.*

1.1 Selberg integrals and formulae

In this section we prove some properties of the kernel

$$\frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}, \quad x, y, z \in \mathbb{R}^N, \tag{1.4}$$

with $0 < \alpha, \beta, \gamma < N$.

Lemma 1.1.1. *Let* $0 < \alpha, \beta, \gamma < N$ *such that* $\alpha + \beta + \gamma > N$*. Then, the following integral is finite:*

$$\int\limits_{B_R} \int\limits_{B_R} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz < +\infty, \quad R > 0, \tag{1.5}$$

where $B_R := \{x \in \mathbb{R}^N \,:\, |x| < R\}$*. Moreover,*

$$\int\limits_{\mathbb{R}^N \setminus B_1} \int\limits_{\mathbb{R}^N \setminus B_1} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz = +\infty. \tag{1.6}$$

Proof. First, we prove (1.5). Without loss of generality we assume that $x = 0$. Considering $z \neq 0$, we denote

$$A_0 = B_R \cap B_{\frac{|z|}{2}},$$
$$A_1 = B_R \cap \left(B_{\frac{|z|}{2}}(z) \setminus B_{\frac{|z|}{2}} \right),$$
$$A_2 = B_R \setminus \left(B_{\frac{|z|}{2}}(z) \cup B_{\frac{|z|}{2}} \right).$$

We have that $B_R = A_0 \cup A_1 \cup A_2$. Hence,

$$\int\limits_{B_R} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}}\, dy = \int\limits_{A_0} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}}\, dy + \int\limits_{A_1} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}}\, dy + \\ + \int\limits_{A_2} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}}\, dy. \tag{1.7}$$

We estimate every integral of (1.7) separately:

$$\int\limits_{A_0} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}}\, dy \leq C\frac{1}{|z|^{N-\beta}} \int\limits_{B_{\frac{|z|}{2}}} \frac{1}{|y|^{N-\alpha}}\, dy = C\frac{1}{|z|^{N-\beta}} \int\limits_0^{\frac{|z|}{2}} \frac{r^{N-1}}{r^{N-\alpha}}\, dr = C\frac{1}{|z|^{N-\beta-\alpha}},$$

$$\int\limits_{A_1} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}}\, dy \leq C\frac{1}{|z|^{N-\alpha}} \int\limits_{A_1} \frac{1}{|y-z|^{N-\beta}}\, dy \leq C\frac{1}{|z|^{N-\alpha}} \int\limits_{B_{\frac{|z|}{2}}(z)} \frac{1}{|y-z|^{N-\beta}}\, dy \\ = C\frac{1}{|z|^{N-\beta-\alpha}}.$$

Finally,

$$\int\limits_{A_2} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}}\, dy \leq C \int\limits_{A_2} \frac{1}{|z|^{N-\beta-\alpha}}\, dy = C\frac{1}{|z|^{N-\beta-\alpha}}.$$

Then,

$$\int\limits_{B_R}\int\limits_{B_R} \frac{1}{|y|^{N-\alpha}|y-z|^{N-\beta}|z|^{N-\gamma}}\, dy\, dz \leq C \int\limits_{B_R} \frac{1}{|z|^{2N-\alpha-\beta-\gamma}}\, dz \\ = C\int\limits_0^R \frac{r^{N-1}}{r^{2N-\alpha-\beta-\gamma}}\, dr = CR^{\alpha+\beta+\gamma-N}.$$

This implies (1.5).

Next, we prove (1.6). Consider $R > |x|$ and $R > 1$, we have that

$$\int_{B_R\setminus B_1}\int_{B_R\setminus B_1} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \geq CR^{\alpha+\beta+\gamma-3N}R^{2N} = CR^{\alpha+\beta+\gamma-N},$$

letting $R \to +\infty$ we obtain (1.6). □

The kernel (1.4) is a generalization of the kernel

$$\frac{1}{|x-y|^{N-\alpha}}, \quad x, y \in \mathbb{R}^N, \tag{1.8}$$

which appears on Hardy-Littlewood-Sobolev system. This kernel is related to the Beta integral by the following formula (see [8]):

Theorem 1.1.2 (Beta integral formula)**.** *Let $0 < \alpha, \beta < N$ with $\alpha + \beta < N$. Then, the following formula holds*

$$\int_{\mathbb{R}^N} \frac{1}{|x-t|^{N-\alpha}|y-t|^{N-\beta}}\,dt = C\frac{1}{|x-y|^{N-\alpha-\beta}},$$

where $C = C(\alpha, \beta, N)$.

Grafakos and Morpurgo [8] proved a similar result for the kernel (1.4) thus establishing a relation with Selberg integral. More precisely, they proved the following theorem:

Theorem 1.1.3 (Selberg integral formula)**.** *Let $0 < \alpha, \beta, \gamma < N$ with $\alpha+\beta+\gamma = 2N$. Then, the following formula holds*

$$\int_{\mathbb{R}^N} \frac{1}{|x-t|^{\beta}|y-t|^{\gamma}|z-t|^{\alpha}}\,dt = C\frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}},$$

where $C = C(N, \alpha, \beta, \gamma)$.

Combining Lemma 1.1.1 and Beta integral formula we obtain the following result:

Theorem 1.1.4. *Let $0 < \alpha, \beta, \gamma < N$ and $\chi = \alpha + \beta + \gamma$. We have the following cases:*

i) If $0 < \chi < N$ *then*

$$\int\limits_{\mathbb{R}^N \setminus B_R(x)} \int\limits_{\mathbb{R}^N \setminus B_R(x)} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz < +\infty \tag{1.9}$$

and

$$\int\limits_{B_R(x)} \int\limits_{B_R(x)} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz = +\infty. \tag{1.10}$$

ii) If $N < \chi < 3N$ *then*

$$\int\limits_{\mathbb{R}^N \setminus B_R(x)} \int\limits_{\mathbb{R}^N \setminus B_R(x)} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz = +\infty$$

and

$$\int\limits_{B_R(x)} \int\limits_{B_R(x)} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz < +\infty.$$

Proof. Proceeding as in Lemma 1.1.1 we obtain the proof of ii).
Next, we prove i). If $\chi < N$, by Theorem 1.1.2, we get

$$\int\limits_{\mathbb{R}^N} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy = \frac{C}{|x-z|^{2N-\chi}}. \tag{1.11}$$

Hence,

$$\begin{aligned} \int\limits_{\mathbb{R}^N \setminus B_R(x)} \int\limits_{\mathbb{R}^N} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dy\, dz &= \int\limits_{\mathbb{R}^N \setminus B_R(x)} \frac{C}{|x-z|^{2N-\chi}}\, dz \\ &= \int\limits_{R}^{+\infty} \frac{C}{r^{N-\chi+1}}\, dr = C < +\infty. \end{aligned} \tag{1.12}$$

This conclude the proof of (1.9).

We proceed to prove (1.10). First, we observe that, (1.11) implies

$$\begin{aligned} &\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \geq \\ &\geq \int\limits_{B_R}\int\limits_{\mathbb{R}^N} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz = \int\limits_0^R \frac{C}{r^{N-\chi+1}}\,dr = +\infty. \end{aligned} \tag{1.13}$$

On the other hand, applying De Morgan Laws we have

$$\begin{aligned} \mathbb{R}^N \times \mathbb{R}^N &= \Big(B_R(x)\times B_R(x)\Big) \cup \Big((\mathbb{R}^N\times\mathbb{R}^N)\setminus(B_R(x)\times B_R(x))\Big) \\ &= \Big(B_R(x)\times B_R(x)\Big) \cup \Big(\big(\mathbb{R}^N\setminus B_R(x)\big)\times\mathbb{R}^N\Big) \cup \Big(\mathbb{R}^N\times\big(\mathbb{R}^N\setminus B_R(x)\big)\Big). \end{aligned} \tag{1.14}$$

Combining the decomposition (1.14) with (1.13) and (1.12) we obtain the claim. □

Corollary 1.1.5. *Let $p, q, r > 0$, $0 < \alpha, \beta, \gamma < N$ and $\chi = \alpha + \beta + \gamma$.*
If $\chi < N$, then the problem (1.3) has no nontrivial solution

$$(u, v, w) \in \left(L^p_{loc}(\mathbb{R}^N)\cap L^1_{loc}(\mathbb{R}^N)\right)\times\left(L^q_{loc}(\mathbb{R}^N)\cap L^1_{loc}(\mathbb{R}^N)\right)\times\left(L^r_{loc}(\mathbb{R}^N)\cap L^1_{loc}(\mathbb{R}^N)\right).$$

Proof. The idea is to prove that

$$\max\{u(x)v(x), v(x)w(x), u(x)w(x)\} = 0, \tag{1.15}$$

for almost every $x \in \mathbb{R}^N$. The identity (1.15) implies that the problem (1.3) has no nontrivial solution.

First, we prove (1.15) assuming the additional hypothesis

$$(u, v, w) \in C^0(\mathbb{R}^N)\times C^0(\mathbb{R}^N)\times C^0(\mathbb{R}^N).$$

Without loss of generality, we consider the product vw. If there exists x_0 such that $v(x_0)w(x_0) \neq 0$, then there exist $C > 0$ and a radius $R > 0$ such that

$$v^q(y)w^r(z) > C \qquad \text{for all } (y, z) \in B_{2R}(x_0)\times B_{2R}(x_0).$$

Let $x \in B_R(x_0)$. By (1.3) we have

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^q(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \\ &\geq \int_{B_R(x)}\int_{B_R(x)} \frac{v^q(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \\ &\geq C\int_{B_R(x)}\int_{B_R(x)} \frac{1}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz. \end{aligned}$$

Hence, applying Theorem 1.1.4, we obtain the contradiction: $u(x) = +\infty$ for all $x \in B_R(x_0)$. Then, (1.15) follows.

Next, we consider a solution

$$(u,v,w) \in \left(L^p_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^q_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^r_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right),$$

to the problem (1.3) and we prove that (1.15) holds.

Let $K \subset \mathbb{R}^N$ a compact set and u_n, v_n, w_n a sequence of continuous functions such that $u_n^p \leq u^p$, $v_n^q \leq v^q$, $w_n^r \leq w^r$ and $u_n(x) \to u(x)$, $v_n(x) \to v(x)$ and $w_n(x) \to w(x)$ for almost every $x \in K$.

Since,

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^q(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \\ &\geq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v_n^q(y)w_n^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz, \qquad n \in \mathbb{N},\ x \in K. \end{aligned}$$

Then, $v_n^q(x)w_n^r(x) = 0$ for all $x \in K$. Letting $n \to +\infty$, we obtain the claim. □

1.2 Differentiation under the integral sign

In this section we consider the following problem: let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a function. When does the following identity hold?

$$\frac{d}{d\lambda}\int_{\mathbb{R}^N} f(x,\lambda)\,dx = \int_{\mathbb{R}^N} \frac{\partial f}{\partial \lambda}(x,\lambda)\,dx, \quad \lambda \in \mathbb{R}. \tag{1.16}$$

A first answer to this problem is given by Lebesgue theorem (see [24, Proposition 23.37]). Unfortunately, it is not possible to use the Lebesgue theorem in order to apply (1.16) to the function defined by (2.8). Therefore, we prove some original results in order to differentiate integral functions using the identity (1.16): Theorem 1.2.2 and Theorem 1.2.3 are new, whereas Theorem 1.2.1 is a particular case of [22, Theorem 4]. Said results are fundamental tools to prove the theorems contained in Chapter 2.

Theorem 1.2.1. *Let $a < b$ and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a function such that*
i) $f(\cdot, \lambda) \in L^1(\mathbb{R}^N)$ for all $\lambda \in [a, b]$.
ii) $f(x, \cdot)$ is differentiable in (a, b) for all $x \in \mathbb{R}^N$.
iii) The following integral is finite:

$$\int_a^b \int_{\mathbb{R}^N} \left| \frac{\partial f}{\partial \lambda}(x, \lambda) \right| dx \, d\lambda < +\infty.$$

Then, $\int_{\mathbb{R}^N} f(x, \lambda) \, dx$ is differentiable and

$$\frac{d}{d\lambda} \int_{\mathbb{R}^N} f(x, \lambda) \, dx = \int_{\mathbb{R}^N} \frac{\partial f}{\partial \lambda}(x, \lambda) \, dx, \quad \lambda \in (a, b).$$

Proof. By the fundamental theorem of Lebesgue integral calculus and the Fubini theorem we have

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{\partial f}{\partial \lambda}(x, \lambda) \, dx &= \frac{d}{d\lambda} \int_a^\lambda \int_{\mathbb{R}^N} \frac{\partial f}{\partial t}(x, t) \, dx \, dt = \frac{d}{d\lambda} \int_{\mathbb{R}^N} \int_a^\lambda \frac{\partial f}{\partial t}(x, t) \, dt \, dx \\ &= \frac{d}{d\lambda} \int_{\mathbb{R}^N} [f(x, \lambda) - f(x, a)] \, dx = \frac{d}{d\lambda} \int_{\mathbb{R}^N} f(x, \lambda) \, dx. \end{aligned}$$

□

Theorem 1.2.2. *Let $a < b$ and $f : \mathbb{R}^N \times (a, b) \to \mathbb{R}$ be a function such that*
i) $f(\cdot, \lambda) \in L^1(\mathbb{R}^N)$ for all $\lambda \in (a, b)$.
ii) $f(x, \cdot) \in C^1(a, b)$ for all $x \in \mathbb{R}^N$.

iii) The derivative $\frac{\partial f}{\partial \lambda} \in L^1_{loc}(\mathbb{R}^N \times (a,b))$ *and there exist a sequence* $R_n \to +\infty$ *such that the following improper integral exists:*

$$G(\lambda) := \lim_{n \to +\infty} \int\limits_{B_{R_n}} \frac{\partial f}{\partial \lambda}(x,\lambda)\, dx < +\infty, \quad \text{for all } \lambda \in (a,b).$$

iv) There exists a function $I \in L^1(a,b)$ *such that*

$$\left| \int\limits_{B_{R_n}} \frac{\partial f}{\partial \lambda}(x,\lambda)\, dx \right| \leq I(\lambda), \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda \in (a,b).$$

Then, $\int\limits_{\mathbb{R}^N} f(x,\lambda)\, dx$ *is differentiable and*

$$\frac{d}{d\lambda} \int\limits_{\mathbb{R}^N} f(x,\lambda)\, dx = \lim_{n \to +\infty} \int\limits_{B_{R_n}} \frac{\partial f}{\partial \lambda}(x,\lambda)\, dx = G(\lambda), \quad \lambda \in (a,b).$$

Proof. Proceeding as in Theorem 1.2.1, by dominated convergence theorem we have

$$\begin{aligned} G(\lambda) = \frac{d}{d\lambda} \int_a^b \left(\lim_{n \to +\infty} \int\limits_{B_{R_n}} \frac{\partial f}{\partial \lambda}(x,\lambda)\, dx \right) d\lambda &= \frac{d}{d\lambda} \left(\lim_{n \to +\infty} \int\limits_{B_{R_n}} \int_a^b \frac{\partial f}{\partial \lambda}(x,\lambda)\, d\lambda\, dx \right) \\ &= \frac{d}{d\lambda} \int\limits_{\mathbb{R}^N} f(x,\lambda)\, dx\, d\lambda. \end{aligned}$$

□

Theorem 1.2.3. *Let* $a < b$ *and* $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be a function such that*

i) $f(\cdot,\lambda) \in L^1(\mathbb{R}^N)$ *for all* $\lambda \in [a,b]$.

ii) $f(x,\cdot) \in C^1(a,b)$ *for all* $x \in \mathbb{R}^N$.

iii) There exists a function $h \in L^1(\mathbb{R}^N \times (a,b))$ *such that:*

$$\frac{\partial f}{\partial \lambda}(x,\lambda) + h(x,\lambda) \geq 0 \quad \text{for all } x \in \mathbb{R}^N,\ \lambda \in (a,b). \tag{1.17}$$

Then, $\int\limits_{\mathbb{R}^N} f(x,\lambda)\,dx$ *is differentiable and*

$$\frac{d}{d\lambda}\int\limits_{\mathbb{R}^N} f(x,\lambda)\,dx = \int\limits_{\mathbb{R}^N} \frac{\partial f}{\partial \lambda}(x,\lambda)\,dx, \quad \lambda \in (a,b).$$

Proof. By the fundamental theorem of calculus

$$f(x,\lambda) = f(x,a) + \int\limits_a^\lambda \frac{\partial f}{\partial t}(x,t)\,dt, \quad \text{for all } x \in \mathbb{R}^N,\ \lambda \in (a,b),$$

and integrating we have

$$\int\limits_{\mathbb{R}^N} f(x,\lambda)\,dx = \int\limits_{\mathbb{R}^N}\int\limits_a^\lambda \frac{\partial f}{\partial t}(x,t)\,dt\,dx + \int\limits_{\mathbb{R}^N} f(x,a)\,dx.$$

Adding $\int\limits_{\mathbb{R}^N}\int\limits_a^\lambda h(x,t)\,dt\,dx$, we obtain

$$\int\limits_{\mathbb{R}^N} f(x,\lambda)\,dx + \int\limits_{\mathbb{R}^N}\int\limits_a^\lambda h(x,t)\,dt\,dx = \int\limits_{\mathbb{R}^N}\int\limits_a^\lambda \frac{\partial f}{\partial t}(x,t) + h(x,t)\,dt\,dx + \int\limits_{\mathbb{R}^N} f(x,a)\,dx.$$

By assumption (1.17) and Tonelli theorem we have

$$\int\limits_{\mathbb{R}^N} f(x,\lambda)\,dx + \int\limits_a^\lambda\int\limits_{\mathbb{R}^N} h(x,t)\,dx\,dt = \int\limits_a^\lambda\int\limits_{\mathbb{R}^N} \frac{\partial f}{\partial t}(x,t) + h(x,t)\,dx\,dt + \int\limits_{\mathbb{R}^N} f(x,a)\,dx.$$

Differentiating with respect to λ we obtain the claim. $\square$

Chapter 2

Non existence of finite energy solutions

In this chapter we prove some results of non-existence of solution to the Euler-Lagrangian equations associated to the general Beckner inequality (see Theorem 0.0.1):

$$\begin{cases} u_s(x_s) = \displaystyle\int\limits_{\mathbb{R}^{N(k-1)}} \frac{\prod_{j\neq s} u_j^{p_j}(x_j)}{\prod_{1\leq i<j\leq k} |x_i - x_j|^{N-h_{ij}}}\, dX_{\hat{s}}, \\ u_s \geq 0 \text{ in } \mathbb{R}^N, \quad s = 1,\ldots,k, \end{cases} \tag{2.1}$$

where $p_j > 0$, $0 < h_{ij} < N$ for all $i, j \in \{1, \ldots, k\}$. We have used the notation $X_{\hat{s}} := (x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_k) \in \mathbb{R}^{N(k-1)}$ on the variables of integration $x_s \in \mathbb{R}^N$.

In particular, in this chapter we assume that the possible solutions of (2.1) are globally integrable: $u_s \in L^{p_s}(\mathbb{R}^N)$ $s = 1, \ldots, k$.

2.1 Beckner system: $k = 3$

First, we address the problem (2.1) in the case $k = 3$:

Theorem 2.1.1. *Let* $0 < \alpha, \beta, \gamma < N$, $\chi := \alpha + \beta + \gamma$ *and* $p, q, r > 0$.

If

$$\frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} \neq \frac{3N - \chi}{N}, \tag{2.2}$$

then the problem (1.3) has no nontrivial solution

$$(u,v,w) \in \left(C^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)\right) \times \left(C^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)\right) \times \left(C^1(\mathbb{R}^N) \cap L^{r+1}(\mathbb{R}^N)\right),$$

satisfying

$$\begin{aligned}
&\int_a^b \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^{q-1}(\lambda y) w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} |\nabla v(\lambda y) \cdot y| \, dy \, dz \, d\lambda < +\infty, \\
&\int_a^b \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^{q}(\lambda y) w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} |\nabla w(\lambda z) \cdot z| \, dy \, dz \, d\lambda < +\infty,
\end{aligned} \tag{2.3}$$

for some $a < 1 < b$.

In order to prove Theorem 2.1.1 we apply the following lemma, which is a direct consequence of mean value theorem.

Lemma 2.1.2. *Let* $f : \mathbb{R} \to \mathbb{R}^N$ *be an integrable nonnegative function* [1]. *Then there exists a sequence* (ξ_n) *such that* $\xi_n \to +\infty$ *and*

$$\xi_n f(\xi_n) \to 0.$$

Proof. The mean value theorem implies that there exists $\xi_n \in [n, 2n]$ such that

$$\left| \int_n^{2n} f(x) \, dx \right| = n|f(\xi_n)| \geq \frac{\xi_n}{2} |f(\xi_n)|,$$

hence,

$$\lim_{n \to +\infty} \xi_n f(\xi_n) = 0.$$

□

Proof. of Theorem 2.1.1 By divergence theorem we have

$$\int_{B_R} \operatorname{div}\left(x u^{p+1}(x)\right) dx = \int_{\partial B_R} u^{p+1}(x) \, x \cdot \hat{n} \, dS(x) = R I(R), \tag{2.4}$$

[1] We say that a vector function $f = (f_1, \ldots, f_N) : \mathbb{R} \to \mathbb{R}^N$ is integrable and nonnegative, if f_i is integrable and nonnegative for all $i = 1, \ldots, N$.

where,

$$I(R) := \int\limits_{\partial B_R} u^{p+1}(x)\, dS(x),$$

and $\hat{n}$ is the unit normal vector to ∂B_R. We have used the notation $x \cdot n$ in order to indicate the Euclidean scalar product.

Since $u \in L^{p+1}(\mathbb{R}^N)$, then $I \in L^1(\mathbb{R})$. Hence, by Lemma 2.1.2, there exists a sequence $R_n \to +\infty$ such that

$$R_n I(R_n) \to 0.$$

Therefore, by (2.4) we have

$$\lim_{n\to+\infty} \int\limits_{B_{R_n}} \operatorname{div}\left(x u^{p+1}(x)\right) dx = 0,$$

i.e.

$$\lim_{n\to+\infty} \int\limits_{B_{R_n}} u^p(x) x \cdot \nabla u(x)\, dx = -\frac{N}{p+1} \int\limits_{\mathbb{R}^N} u^{p+1}(x)\, dx. \tag{2.5}$$

Similarly, we get

$$\lim_{n\to+\infty} \int\limits_{B_{R_n}} v^q(x) x \cdot \nabla v(x)\, dx = -\frac{N}{q+1} \int\limits_{\mathbb{R}^N} v^{q+1}(x)\, dx, \tag{2.6}$$

and

$$\lim_{n\to+\infty} \int\limits_{B_{R_n}} w^r(x) x \cdot \nabla w(x)\, dx = -\frac{N}{r+1} \int\limits_{\mathbb{R}^N} w^{r+1}(x)\, dx. \tag{2.7}$$

Next, let λ close to 1. Using the change variables rule, we have

$$\begin{aligned} u(\lambda x) &= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{v^q(y) w^r(z)}{|\lambda x - y|^{N-\alpha} |y-z|^{N-\beta} |z - \lambda x|^{N-\gamma}}\, dy\, dz \\ &= \lambda^{\chi - N} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{v^q(\lambda y) w^r(\lambda z)}{|x-y|^{N-\alpha} |y-z|^{N-\beta} |z-x|^{N-\gamma}}\, dy\, dz. \end{aligned} \tag{2.8}$$

Differentiating with respect to λ this identity we obtain

$$\begin{aligned}\frac{\partial}{\partial\lambda}u(\lambda x) = \nabla u(\lambda x)\cdot x =&(\chi-N)\lambda^{\chi-N-1}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^q(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz+\\&+\lambda^{\chi-N}q\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\cdot y\,dy\,dz+\\&+\lambda^{\chi-N}r\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^q(\lambda y)w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(\lambda z)\cdot z\,dy\,dz.\end{aligned}$$

The differentiation under the integral sign is justified by Theorem 1.2.1 and the assumption (2.3). In particular, for $\lambda = 1$ we have

$$\begin{aligned}\nabla u(x)\cdot x =&(\chi-N)u(x)+q\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^{q-1}(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(y)\cdot y\,dy\,dz+\\&+r\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^q(y)w^{r-1}(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(z)\cdot z\,dy\,dz.\end{aligned}$$

Multiplying with u^p and integrating over B_{R_n} with respect to x we obtain

$$\begin{aligned}\int\limits_{B_{R_n}} u^p(x)\nabla u(x)\cdot x\,dx &= (\chi-N)\int\limits_{B_{R_n}} u^{p+1}(x)\,dx+\\&+q\int\limits_{B_{R_n}}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{u^p(x)v^{q-1}(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(y)\cdot y\,dy\,dz\,dx+\\&+r\int\limits_{B_{R_n}}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{u^p(x)v^q(y)w^{r-1}(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(z)\cdot z\,dy\,dz\,dx,\end{aligned}$$

i.e.

$$\int\limits_{B_{R_n}} u^p(x)\nabla u(x)\cdot x\,dx = (\chi - N)\int\limits_{B_{R_n}} u^{p+1}(x)\,dx+$$

$$+\,q\int\limits_{\mathbb{R}^N} v^{q-1}(y)\nabla v(y)\cdot y\int\limits_{\mathbb{R}^N}\int\limits_{B_{R_n}} \frac{u^p(x)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dx\,dz\,dy+$$

$$+\,r\int\limits_{\mathbb{R}^N} w^{r-1}(z)\nabla w(z)\cdot z\int\limits_{\mathbb{R}^N}\int\limits_{B_{R_n}} \frac{u^p(x)v^q(y)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dx\,dy\,dz.$$

Letting $n\to+\infty$ we have

$$\begin{aligned}\lim_{n\to+\infty}\int\limits_{B_{R_n}} u^p(x)\nabla u(x)\cdot x\,dx =&(\chi-N)\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx+\\ &+q\lim_{n\to+\infty}\int\limits_{B_{R_n}} v^q(y)\nabla v(y)\cdot y\,dy+\\ &+r\lim_{n\to+\infty}\int\limits_{B_{R_n}} w^r(z)\nabla w(z)\cdot z\,dz.\end{aligned}\tag{2.9}$$

Combining (2.9) with (2.5), (2.6) and (2.7) we obtain

$$\begin{aligned}-\frac{N}{p+1}\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx =&(\chi-N)\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx-\frac{qN}{q+1}\int\limits_{\mathbb{R}^N} v^{q+1}(x)\,dx\\ &-\frac{rN}{r+1}\int\limits_{\mathbb{R}^N} w^{r+1}(x)\,dx.\end{aligned}\tag{2.10}$$

Next, we claim

$$\begin{aligned}\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx &= \int\limits_{\mathbb{R}^N} v^{q+1}(x)\,dx = \int\limits_{\mathbb{R}^N} w^{r+1}(x)\,dx\\ &=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{u^p(x)v^q(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dx\,dz\,dy.\end{aligned}\tag{2.11}$$

Indeed, by Tonelli theorem we have

$$\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx = \int\limits_{\mathbb{R}^N} u^p(x) \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{v^q(y)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz\,dx,$$
$$= \int\limits_{\mathbb{R}^N} v^q(y) \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{u^p(x)w^r(z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dx\,dz\,dy = \int\limits_{\mathbb{R}^N} v^{q+1}(y)\,dy.$$

Similarly,

$$\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx = \int\limits_{\mathbb{R}^N} w^{r+1}(x)\,dx.$$

Combining (2.10) with (2.11) we obtain

$$\left(\chi - N + \frac{N}{p+1} - \frac{qN}{q+1} - \frac{rN}{r+1}\right) \int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx = 0,$$

Hence, (2.2) implies $u \equiv v \equiv w \equiv 0$. □

In order to remove the assumption (2.3) in Theorem 2.1.1, first we consider the simpler problem

$$\begin{cases} u(x) = \displaystyle\int\limits_{\mathbb{R}^N} \frac{v^q(y)}{|x-y|^{N-\alpha}}\,dy, & x \in \mathbb{R}^N, \\ v(x) = \displaystyle\int\limits_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\alpha}}\,dy, & x \in \mathbb{R}^N, \\ u, v \geq 0 \quad \text{in } \mathbb{R}^N, \end{cases} \tag{2.12}$$

where $0 < \alpha < N$ and $p, q > 0$. Then, we generalize the obtained results to the system (1.3).

Proceeding as in Theorem 2.1.1 we get the following result:

Theorem 2.1.3. *Let $0 < \alpha < N$ and $p, q > 0$.*

If

$$\frac{1}{p+1} + \frac{1}{q+1} \neq \frac{N-\alpha}{N}, \tag{2.13}$$

then the problem (2.12) has no nontrivial solution

$$(u, v) \in \left(C^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)\right) \times \left(C^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)\right),$$

satisfying

$$\int_a^b \int_{\mathbb{R}^N} \frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} |\nabla v(\lambda y) \cdot y| \, dy \, d\lambda < +\infty, \tag{2.14}$$

for some $a < 1 < b$.

By Theorem 1.2.2, we can weaken the hypothesis of Theorem 2.1.3 by replacing the condition (2.14) with: there exist a sequence $R_n \to +\infty$ and a function $I : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that the following improper integral exists

$$\lim_{n\to+\infty} \int_{B_{R_n}(x)} q \frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \cdot y \, dy < +\infty, \tag{2.15}$$

and

$$\left| \int_{B_{R_n}(x)} q \frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \cdot y \, dy \right| \leq I(\lambda, x), , \tag{2.16}$$

with $I(\cdot, x) \in L^1(a, b)$ for some $a < 1 < b$.

Proposition 2.1.4. *Let $0 < \alpha < N$ and $p, q > 0$. If*

$$(u, v) \in \left(C^1(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)\right) \times \left(C^1(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)\right)$$

is a solution to (2.12). Then, there exists a sequence $R_n \to +\infty$, such that the following improper integral exist and

$$\lim_{n\to+\infty} \int_{B_{R_n}(x)} q \frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \cdot (y - x) \, dy = -\frac{\alpha}{\lambda} u(\lambda x), \quad \textit{for all } \lambda > 0.$$

Proof. Let $0 < \varepsilon < R$. By divergence theorem we have

$$\begin{aligned} \int_{B_R(x)\setminus B_\varepsilon(x)} \operatorname{div}_y \left(\frac{v^q(\lambda y)}{|x-y|^{N-\alpha}} (y-x) \right) dy &= \int_{\partial(B_R(x)\setminus B_\varepsilon(x))} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}} (y-x) \cdot \hat{n}(y) \, dS(y), \\ &= R \int_{\partial B_R(x)} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}} \, dS(y) - \varepsilon \int_{\partial B_\varepsilon(x)} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}} \, dS(y), \end{aligned} \tag{2.17}$$

where $\hat{n}$ is the unit normal vector to $\partial\left(B_R(x) \setminus B_\varepsilon(x)\right)$. Since,

$$\varepsilon \int\limits_{\partial B_\varepsilon(x)} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}}\, dS(y) \leq C\varepsilon^\alpha,$$

then, by the absolute continuity of Lebesgue integral, letting $\varepsilon \to 0$, (2.17) becomes

$$\int\limits_{B_R(x)} \operatorname{div}_y \left(\frac{v^q(\lambda y)}{|x-y|^{N-\alpha}}(y-x) \right) dy = R \int\limits_{\partial B_R(x)} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}}\, dS(y). \tag{2.18}$$

By Lemma 2.1.2, there exists a sequence $R_n \to +\infty$ such that

$$\lim_{n\to+\infty} R_n \int\limits_{\partial B_{R_n}(x)} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}}\, dS(y) = 0,$$

hence,

$$\lim_{n\to+\infty} \int\limits_{B_{R_n}(x)} \operatorname{div}_y \left(\frac{v^q(\lambda y)}{|x-y|^{N-\alpha}}(y-x) \right) dy = 0.$$

On the other hand,

$$\operatorname{div}_y \left(\frac{v^q(\lambda y)}{|y-x|^{N-\alpha}}(y-x) \right) = \lambda q \frac{v^{q-1}(\lambda y)\nabla v(\lambda y)\cdot(y-x)}{|y-x|^{N-\alpha}} + v^q(\lambda y)\operatorname{div}\left(\frac{y-x}{|y-x|^{N-\alpha}} \right) =$$
$$= \lambda q \frac{v^q(\lambda y)\nabla v(\lambda y)\cdot(y-x)}{|y-x|^{N-\alpha}} + N\frac{v^q(\lambda y)}{|y-x|^{N-\alpha}} + v^q(\lambda y)\nabla_y\left(\frac{1}{|y-x|^{N-\alpha}} \right)\cdot(y-x) =$$
$$= \lambda q \frac{v^q(\lambda y)\nabla v(\lambda y)\cdot(y-x)}{|y-x|^{N-\alpha}} + N\frac{v^q(\lambda y)}{|y-x|^{N-\alpha}} + (\alpha-N)\frac{v^q(\lambda y)}{|y-x|^{N-\alpha+1}}\frac{y-x}{|y-x|}\cdot(y-x) =$$
$$= \lambda q \frac{v^q(\lambda y)\nabla v(\lambda y)\cdot(y-x)}{|y-x|^{N-\alpha}} + \alpha\frac{v^q(\lambda y)}{|y-x|^{N-\alpha}}.$$

Therefore,

$$\lim_{n\to+\infty} \int\limits_{B_{R_n}(x)} q\frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}}\nabla v(\lambda y)\cdot(y-x)\, dy = -\frac{\alpha}{\lambda}\lim_{n\to+\infty} \int\limits_{B_{R_n}(x)} \frac{v^q(\lambda y)}{|y-x|^{N-\alpha}}\, dy =$$
$$= -\frac{\alpha}{\lambda}\int\limits_{\mathbb{R}^N} \frac{v^q(\lambda y)}{|y-x|^{N-\alpha}}\, dy = -\frac{\alpha}{\lambda}u(\lambda x).$$

□

Corollary 2.1.5. *Let $0 < \alpha < N$ and $p, q > 0$. If (2.13) is satisfied, then the problem (2.12) has no nontrivial, globally Lipschitz solution*

$$(u, v) \in \left(L^{p+1}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)\right) \times \left(L^{q+1}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)\right).$$

Proof. It is sufficient to prove that (2.15) and (2.16) are satisfied. We have that

$$\int_{B_{R_n}(x)} q\frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \cdot y \, dy =$$

$$= x \cdot \int_{B_{R_n}(x)} q\frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \, dy + \int_{B_{R_n}(x)} q\frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \cdot (y - x) \, dy,$$

for all $x \in \mathbb{R}^N$ and λ close to 1.

Let

$$I(\lambda, x) := \int_{\mathbb{R}^N} q\frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \, dy,$$

we get

$$\left| \int_{\mathbb{R}^N} q\frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \, dy \right| \leq \int_{\mathbb{R}^N} q\frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} |\nabla v(\lambda y)| \, dy \leq C I(\lambda, x).$$

Let $\varphi \in C_0^\infty(\mathbb{R}^N)$, applying the Hardy-Littlewood-Sobolev inequality with exponents $s_1 = \frac{q+1}{q-1}$ and $s_2 = \frac{N(q+1)}{2N+\alpha(q+1)}$, we obtain

$$\int_{\mathbb{R}^N} I(\lambda, x)\varphi(x) \, dx \leq \frac{B_2}{\lambda^{\frac{N(q-1)}{q+1}}} \|v\|_{q+1}^{q-1} \|\varphi\|_{s_2}.$$

Hence, integrating with respect to λ over a neighborhood $[a, b]$ of 1, we get

$$\int_a^b \int_{\mathbb{R}^N} I(\lambda, x)\varphi(x) \, dx \, d\lambda = \int_{\mathbb{R}^N} \left(\int_a^b I(\lambda, x) \, d\lambda \right) \varphi(x) \, dx < +\infty, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Then $I(\cdot, x) \in L^1(a, b)$. Finally, by Proposition 2.1.4, the condition (2.15) is satisfied. Moreover, by Lemma 2.1.2 and (2.18) we have

$$\left| \int_{B_{R_n}(x)} q \frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \cdot (y-x)\, dy \right| \le \frac{\alpha}{\lambda} \int_{B_{R_n}(x)} \frac{v^q(\lambda y)}{|y-x|^{N-\alpha}}\, dy +$$

$$+ R_n \int_{\partial B_{R_n}(x)} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}}\, dS(y) = \frac{\alpha}{\lambda} \int_{B_{R_n}(x)} \frac{v^q(\lambda y)}{|y-x|^{N-\alpha}}\, dy +$$

$$+ 2 \int_n^{2n} \int_{\partial B_R(x)} \frac{v^q(\lambda y)}{|x-y|^{N-\alpha}}\, dS(y)\, dR,$$

hence,

$$\left| \int_{B_{R_n}(x)} q \frac{v^{q-1}(\lambda y)}{|x-y|^{N-\alpha}} \nabla v(\lambda y) \cdot (y-x)\, dy \right| \le \left(\frac{\alpha}{\lambda} + 2 \right) u(\lambda x), \quad \lambda > 0.$$

Therefore, (2.16) is also satisfied. □

The regularity of the solutions to system (2.12) assumed in Corollary 2.1.9 is motivated by the following result, proved by Li, Chen and Ma [10]:

Theorem 2.1.6 (Li - Chen - Ma). *Let $p, q > 1$ and $0 < \alpha < N$ satisfying*

$$\frac{1}{p+1} + \frac{1}{q+1} = \frac{N-\alpha}{N}. \tag{2.19}$$

If $(u, v) \in L^{p+1}(\mathbb{R}^N) \times L^{q+1}(\mathbb{R}^N)$ is a positive solution of (2.12), then u, v are bounded and globally Lipschitz continuous.

In the particular case $\alpha = 2$, the problem (2.12) corresponds to the system of differential equations of Lane-Emden:

$$\begin{cases} -\Delta u = v^q, & \text{in } \mathbb{R}^N, \\ -\Delta v = u^p, & \text{in } \mathbb{R}^N, \\ u, v \ge 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.20}$$

In the following non-existence result for the system (2.20), we assume the boundness of $|\nabla u|$ and $|\nabla v|$ with respect to the L^2-norm, instead of L^∞-norm as in Corollary 2.1.5.

Theorem 2.1.7. *Let $p, q > 0$. If*

$$\frac{1}{p+1} + \frac{1}{q+1} \neq \frac{N-2}{N}, \tag{2.21}$$

then the problem (2.20) has no nontrivial solution

$$(u, v) \in \left(C^2(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N) \cap W^{1,2}(\mathbb{R}^N)\right) \times \left(C^2(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N) \cap W^{1,2}(\mathbb{R}^N)\right).$$

Proof. We apply the following Rellich type identity (see [13])

$$\int_{B_R} \Delta u(x) \nabla v(x) \cdot x + \Delta v(x) \nabla u(x) \cdot x \, dx - (N-2) \int_{B_R} \nabla u(x) \cdot \nabla v(x) \, dx =$$
$$= \int_{\partial B_R} 2 \frac{(\nabla u(x) \cdot x)(\nabla v(x) \cdot x)}{|x|} - \nabla u(x) \cdot \nabla v(x) |x| \, dS(x). \tag{2.22}$$

We have that

$$\left| \int_{\partial B_R} 2 \frac{(\nabla u(x) \cdot x)(\nabla v(x) \cdot x)}{|x|} - \nabla u(x) \cdot \nabla v(x) |x| \, dS(x) \right| \leq 3R \int_{\partial B_R} |\nabla u(x)| |\nabla v(x)| \, dS(x).$$

Since

$$\int_{\mathbb{R}^N} |\nabla u(x)| |\nabla v(x)| \, dx \leq \|\nabla u\|_2 \|\nabla v\|_2 < +\infty,$$

by Lemma 2.1.2, there exists a sequence $R_n \to +\infty$ such that

$$R_n \int_{\partial B_{R_n}} |\nabla u(x)| |\nabla v(x)| \, dS(x) \to 0. \tag{2.23}$$

Proceeding as in Theorem 2.1.1, by the divergence theorem, we obtain

$$\lim_{n \to +\infty} \int_{B_{R_n}} v^q(x) \nabla v(x) \cdot x \, dx = -\frac{N}{q+1} \int_{B_{R_n}} v^{q+1}(x) \, dx, \tag{2.24}$$

$$\lim_{n \to +\infty} \int_{B_{R_n}} u^p(x) \nabla u(x) \cdot x \, dx = -\frac{N}{p+1} \int_{B_{R_n}} u^{p+1}(x) \, dx. \tag{2.25}$$

On the other hand,

$$\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx = \int\limits_{\mathbb{R}^N} v^{q+1}(x)\,dx = \int\limits_{\mathbb{R}^N} \nabla u(x)\cdot\nabla v(x)\,dx. \tag{2.26}$$

Indeed, multiplying by v the first equation of the system (2.20) and integrating over B_{R_n} we have

$$\int\limits_{B_{R_n}} v^{q+1}(x)\,dx = \int\limits_{B_{R_n}} (-\Delta u(x))v(x)\,dx =$$

$$= \int\limits_{B_{R_n}} \nabla u(x)\cdot\nabla v(x)\,dx + \int\limits_{\partial B_{R_n}} v(x)\nabla u(x)\cdot\frac{x}{|x|}\,dS(x).$$

Since

$$\int\limits_{\mathbb{R}^N} v(x)|\nabla u(x)|\,dx \leq \|v\|_2\|\nabla u\|_2,$$

letting $n\to+\infty$, we get

$$\int\limits_{\partial B_R} v(x)\nabla u(x)\cdot\frac{x}{|x|}\,dS(x)\to 0,$$

and the identity (2.26) follows.

Combining (2.22) with (2.23),(2.24),(2.25) and (2.26) we get

$$\left(\frac{N}{p+1}+\frac{N}{q+1}-(N-2)\right)\int\limits_{\mathbb{R}^N} u^{p+1}(x)\,dx = 0.$$

This conclude the proof. □

Next, we want to generalize the ideas that we have used to prove Corollay 2.1.5 and Proposition 2.1.4 to obtain a non-existence result for the problem (1.3). Proceeding as in the Corollary 2.1.5, by Theorem 1.2.2, we can weaken the hypothesis of Theorem 2.1.1 by replacing the assumption (2.3) with the following conditions:

i) There exists a sequence $R_n \to +\infty$ such that the following improper integral exists

$$\begin{aligned}\lim_{n\to+\infty} \int\limits_{B^{2N}_{R_n}(x,x)} & q\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\cdot y\, d(y,z)+ \\ + \int\limits_{B^{2N}_{R_n}(x,x)} & r\frac{v^{q}(\lambda y)w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(\lambda z)\cdot z\, d(y,z) < +\infty,\end{aligned} \tag{2.27}$$

where $B^{2N}_{R_n}(x,x) = \{(y,z) \in \mathbb{R}^{2N} \ : \ |y-x|^2 + |z-x|^2 < R_n^2\}$.

ii) There exists a function $I : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that

$$\begin{aligned}&\left| \int\limits_{B^{2N}_{R_n}(x,x)} q\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\cdot y\, d(y,z)+ \right. \\ &\left. + \int\limits_{B^{2N}_{R_n}(x,x)} r\frac{v^{q}(\lambda y)w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(\lambda z)\cdot z\, d(y,z) \right| \le I(\lambda,x)\end{aligned} \tag{2.28}$$

and $I(\cdot,x) \in L^1(a,b)$ for some $a < 1 < b$.

Proposition 2.1.8. *Let $0 < \alpha, \beta, \gamma < N$, $\chi = \alpha+\beta+\gamma > 3N-1$ and $p, q, r > 0$. If*

$$(u,v,w) \in \big(C^1(\mathbb{R}^N)\cap L^{p+1}(\mathbb{R}^N)\big) \times \big(C^1(\mathbb{R}^N)\cap L^{q+1}(\mathbb{R}^N)\big) \times \big(C^1(\mathbb{R}^N)\cap L^{r+1}(\mathbb{R}^N)\big)$$

is a solution to (1.3). Then, there exists a sequence of radius $R_n \to +\infty$, such that the following improper integral exist and

$$\begin{aligned}\lim_{n\to+\infty} \int\limits_{B^{2N}_{R_n}(x,x)} & q\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\cdot (y-x)+ \\ & + r\frac{v^{q}(\lambda y)w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(\lambda z)\cdot (z-x)\, d(y,z) = \\ = - & \frac{\chi-2N}{\lambda}u(\lambda x), \quad \text{for all } \lambda > 0.\end{aligned} \tag{2.29}$$

Proof. Let $0 < \varepsilon < R$ and

$$\begin{aligned} C_1 &:= \{(y,z) \in \mathbb{R}^{2N} \ : \ |y-z| < \varepsilon\}, \\ C_2 &:= \{(y,z) \in \mathbb{R}^{2N} \ : \ |x-z| < \varepsilon\}, \\ C_3 &:= \{(y,z) \in \mathbb{R}^{2N} \ : \ |x-y| < \varepsilon\}, \\ I_\varepsilon &:= (C_1 \cup C_2 \cup C_3) \cap B_R^{2N}(x,x). \end{aligned}$$

By divergence theorem, we get

$$\begin{aligned} &\int\limits_{B_R^{2N}(x,x)\setminus I_\varepsilon} \operatorname{div}_{(y,z)} \left(\frac{v^q(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} (y-x, z-x) \right) d(y,z) = \\ &\quad = \int\limits_{\partial\left(B_R^{2N}(x,x)\setminus I_\varepsilon\right)} \frac{v^q(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} (y-x, z-x) \cdot \hat{n}(y,z)\, dS(y,z) \\ &\quad = R \int\limits_{\partial B_R^{2N}(x,x)} \frac{v^q(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dS(y,z) - \qquad (2.30) \\ &\qquad - \int\limits_{\partial I_\varepsilon} \frac{v^q(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} (y-x, z-x) \cdot \hat{n}(y,z)\, dS(y,z), \end{aligned}$$

where $\hat{n}$ is the unit normal vector to $\partial\left(B_R^{2N}(x,x) \setminus I_\varepsilon\right)$.

As in Proposition 2.1.4, we need to estimate the following surface integral:

$$\begin{aligned} &\int\limits_{\partial I_\varepsilon} \frac{v^q(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} |(y-x, z-x) \cdot \hat{n}(y,z)|\, dS(y,z) \leq \\ &\quad \leq C\varepsilon \int\limits_{\partial I_\varepsilon} \frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\, dS(y,z) = C\varepsilon^{\chi-3N+1}. \end{aligned}$$

Indeed, considering the boundary $\partial C_1 = \{(y,z) \ : \ |y-z|^2 - \varepsilon^2 = 0\}$ we have that

$$\hat{n}(y,z) = \frac{\nabla(|y-z|^2-\varepsilon^2)}{|\nabla(|y-z|^2-\varepsilon^2)|} = \frac{1}{\sqrt{2}|z-y|}(y-z, z-y),$$

therefore on ∂C_1 we get

$$\hat{n} \cdot (y-x, z-x) = \frac{|y-z|}{\sqrt{2}} = \frac{\varepsilon}{\sqrt{2}}. \qquad (2.31)$$

The same argument prove that (2.31) holds also on ∂C_2 and ∂C_3.

By (1.3) we have that the function defined by $\frac{v^q(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}$ is integrable. Then, by absolute continuity of Lebesgue integral and Lemma 1.1.1, letting $\varepsilon \to 0$ in (2.30) we obtain

$$\int_{B_R^{2N}(x,x)} \operatorname{div}_{(y,z)} \left(\frac{v^q(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}(y-x, z-x) \right) d(y,z) = \tag{2.32}$$
$$= R \int_{\partial B_R^{2N}(x,x)} \frac{v^q(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \, dS(y,z)$$

By Lemma 2.1.2, there exists a sequence $R_n \to +\infty$ such that

$$\lim_{n\to+\infty} R_n \int_{\partial B_{R_n}^{2N}(x,x)} \frac{v^q(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \, dS(y,z) = 0.$$

Hence,

$$\lim_{n\to+\infty} \int_{B_{R_n}^{2N}(x,x)} \operatorname{div}_{(y,z)} \left(\frac{v^q(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}(y-x, z-x) \right) d(y,z) = 0.$$

On the other hand, we have

$$\begin{aligned}
\operatorname{div}_{\binom{y}{z}} \left(\frac{v^q(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}(y-x, z-x) \right) &= \\
\lambda q \frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \nabla v(\lambda y) \cdot (y-x+ \\
+\lambda r \frac{v^q(\lambda y)w^{r-1}(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \nabla w(\lambda z) \cdot (z-x)+ \\
+N \frac{v^q(\lambda y)w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}+ \\
+v^q(\lambda y)w^r(\lambda z) \nabla_y \left(\frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \right) \cdot (y-x)+ \\
+v^q(\lambda y)w^r(\lambda z) \nabla_z \left(\frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \right) \cdot (z-x).
\end{aligned}$$

Since,

$$
\begin{aligned}
&\nabla_y \left(\frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \right) \cdot (y-x) + \\
&+ \nabla_z \left(\frac{1}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \right) \cdot (z-x) = \\
=& \frac{\alpha - N}{|y-x|^{N-\alpha+1}|y-z|^{N-\beta}|z-x|^{N-\gamma}} |y-x| \\
&+ \frac{\beta - N}{|y-x|^{N-\alpha}|y-z|^{N-\beta+1}|z-x|^{N-\gamma}} \frac{(y-z)\cdot(y-x)}{|y-z|} + \\
&+ \frac{\gamma - N}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma+1}} |z-x| \\
&+ \frac{N - \beta}{|y-x|^{N-\alpha}|y-z|^{N-\beta+1}|z-x|^{N-\gamma}} \frac{(y-z)\cdot(z-x)}{|y-z|} = \\
=& \frac{\alpha - N}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} + \\
&+ \frac{\gamma - N}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} + \frac{\beta - N}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\operatorname{div}_{(y,z)} \left(\frac{v^q(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} (y-x, z-x) \right) &= \\
\lambda q \frac{v^{q-1}(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \nabla v(\lambda y) \cdot (y-x) &+ \\
+ \lambda r \frac{v^q(\lambda y) w^{r-1}(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \nabla w(\lambda z) \cdot (z-x) &+ \\
+ (\chi - 2N) \frac{v^q(\lambda y) w^r(\lambda z)}{|y-x|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}&.
\end{aligned}
$$

This conclude the proof. □

Corollary 2.1.9. *Let $p, q, r > 0$ and $0 < \alpha, \beta, \gamma < N$ with $\alpha + \beta + \gamma > 3N - 1$. If (2.2) is satisfied, then the problem (1.3) has no nontrivial, globally Lipschitz solution*

$$
(u, v, w) \in \left(L^{s_1}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)\right) \times \left(L^{s_2}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)\right) \times \left(L^{s_3}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)\right),
$$

for every $1 < s_1, s_2, s_3 < +\infty$.

Proof. It is sufficient to prove that the assumptions (2.27) and (2.28) are satisfied. We have that

$$\int\limits_{B^{2N}_{R_n}(x,x)} q\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\cdot y\, d(y,z)+$$

$$+\int\limits_{B^{2N}_{R_n}(x,x)} r\frac{v^{q}(\lambda y)w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(\lambda z)\cdot z\, d(y,z)=$$

$$=x\cdot\int\limits_{B^{2N}_{R_n}(x,x)} q\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\, d(y,z)+$$

$$+\,x\cdot\int\limits_{B^{2N}_{R_n}(x,x)} r\frac{v^{q}(\lambda y)w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(\lambda z)\, d(y,z)+$$

$$\int\limits_{B^{2N}_{R_n}(x,x)} q\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\cdot (y-x)\, d(y,z)+$$

$$+\int\limits_{B^{2N}_{R_n}(x,x)} r\frac{v^{q}(\lambda y)w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla w(\lambda z)\cdot (z-x)\, d(y,z).$$

Let

$$I_1(\lambda,x):=q\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz.$$

We have that

$$\begin{aligned}&\left|\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\nabla v(\lambda y)\,dy\,dz\right|\le\\ &\le\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{v^{q-1}(\lambda y)w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}|\nabla v(\lambda y)|\,dy\,dz\le CI_1(\lambda,x).\end{aligned}\tag{2.33}$$

Let $\varphi\in C_0^\infty(\mathbb{R}^N)$, by Beckner inequality, there exist $s_1,s_2,s_3>0$ such that

$$\int\limits_{\mathbb{R}^N} I(\lambda,x)\varphi(x)\,dx\le\frac{b_3}{\lambda^{\frac{N}{s_1}+\frac{N}{s_2}}}\|v\|_{s_1}\|w\|_{s_2}\|\varphi\|_{s_3}.$$

Integrating with respect to λ over a neighborhood $[a,b]$ of 1 we obtain that

$$\int_a^b \int_{\mathbb{R}^N} I(\lambda,x)\varphi(x)\,dx\,d\lambda = \int_{\mathbb{R}^N} \left(\int_a^b I(\lambda,x)\,d\lambda \right) \varphi(x)\,dx < +\infty, \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^N).$$

Then $I_1(\cdot,x) \in L^1(a,b)$. Similarly, there exists $I_2 : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ such that $I_2(\cdot,x) \in L^1(a,b)$ and

$$\begin{aligned} &\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^q(\lambda y) w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \nabla w(\lambda y)\,dy\,dz \right| \leq \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^q(\lambda y) w^{r-1}(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} |\nabla w(\lambda y)|\,dy\,dz \leq C I_2(\lambda,x). \end{aligned} \tag{2.34}$$

Combining Proposition 2.1.8 with (2.33) and (2.34), we have that the condition (2.27) is satisfied.

Moreover, proceeding as in Corollary 2.1.5 we obtain that also the condition (2.28) holds. □

2.2 Radial solutions to Beckner system

We consider the possible radial solutions

$$(u,v,w) \in \left(C^2(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)\right) \times \left(C^2(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)\right) \times \left(C^2(\mathbb{R}^N) \cap L^{r+1}(\mathbb{R}^N)\right),$$

to (1.3) and we apply a different strategy to justify the differentiation under integral sign in (2.8). The idea is to prove that v and w are superharmonic functions in order to obtain estimates for $\nabla v(x) \cdot x$ and $\nabla w(x) \cdot x$. More precisely, we apply the following lemma proved by Mitidieri [13]:

Lemma 2.2.1. *Let $N \geq 3$. If $\varphi \in C^2(\mathbb{R}^N \setminus \{0\})$ is a radial and non-negative solution to*

$$-\Delta\varphi \geq 0 \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

then

$$\nabla\varphi(x) \cdot x + (N-2)\varphi(x) \geq 0, \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.$$

Proposition 2.2.2. *Let $N \geq 3$, $p, q, r > 0$ and $0 < \alpha, \beta, \gamma < N$ such that $\alpha + \beta = N + 2$ and $\beta + \gamma = N + 2$. If $(u, v, w) \in C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N) \times C^2(\mathbb{R}^N)$ is a solution to (1.3), then v and w are superharmonic.*

Proof. We prove that v is superharmonic, the same argument can be used to prove that w is superharmonic.

Testing the second equation of (1.3) with $-\Delta\varphi$, where $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $\varphi \geq 0$, we obtain

$$\int_{\mathbb{R}^N} -\Delta\varphi(x)v(x)\,dx = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^p(y)w^r(z)}{|y-z|^{N-\gamma}} \left(\int_{\mathbb{R}^N} \frac{-\Delta\varphi(x)}{|x-y|^{N-\alpha}|z-x|^{N-\beta}}\,dx \right) dy\,dz.$$

Let $\sigma = 2N - \alpha - \beta = N - 2$. By Selberg integral formula we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{-\Delta\varphi(x)}{|x-y|^{N-\alpha}|z-x|^{N-\beta}}\,dx &= |y-z|^{N-\sigma} \int_{\mathbb{R}^N} \frac{-\Delta\varphi(x)}{|x-y|^{N-\alpha}|y-z|^{N-\sigma}|z-x|^{N-\beta}}\,dx \\
&= C|y-z|^2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{-\Delta\varphi(x)}{|x-t|^{N-2}|y-t|^{\beta}|z-t|^{\alpha}}\,dx\,dt \\
&= C \int_{\mathbb{R}^N} \frac{\varphi(t)|y-z|^2}{|y-t|^{\beta}|z-t|^{\alpha}}\,dt \geq 0.
\end{aligned}$$

□

Theorem 2.2.3. *Let $N \geq 3$, $p, q, r > 0$, $0 < \alpha, \beta, \gamma < N$ and $\chi = \alpha + \beta + \gamma$. Suppose that $\alpha + \beta = N + 2$ and $\beta + \gamma = N + 2$. If (2.2) is satisfied, then the problem (1.3) has no nontrivial radial solution*

$$(u, v, w) \in \left(C^2(\mathbb{R}^N) \cap L^{p+1}(\mathbb{R}^N)\right) \times \left(C^2(\mathbb{R}^N) \cap L^{q+1}(\mathbb{R}^N)\right) \times \left(C^2(\mathbb{R}^N) \cap L^{r+1}(\mathbb{R}^N)\right).$$

Proof. We proceed as in the proof of Theorem 2.1.1, the only difference is the application of Theorem 1.2.3 to differentiate under the integral sign with respect to λ the following function

$$u(\lambda x) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(\lambda, x, y, z)dy\,dz,$$

where

$$f(\lambda, x, y, z) = \frac{v^q(\lambda y) w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}.$$

Let $0 < a < 1 < b$ and

$$h(x, y, z, \lambda) = C\lambda^{\chi - N} \frac{v^q(\lambda y) w^r(\lambda z)}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}},$$

where $C = \frac{N-2}{a^{\chi-N+1}}(q+r)$. We have

$$\int_a^b \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |h(x, y, z, \lambda)| \, dy \, dz \, d\lambda = C \int_a^b u(\lambda x) \, d\lambda < +\infty, \quad \text{for all } x \in \mathbb{R}^N.$$

Moreover, by Lemma 2.2.1 we obtain

$$\begin{aligned}\partial_\lambda f(\lambda, y, z) &= q \frac{v^{q-1}(\lambda y) w^r(\lambda z) \nabla v(\lambda y) \cdot y}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} + r \frac{v^q(\lambda y) w^{r-1}(\lambda z) \nabla w(\lambda z) \cdot z}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}} \\ &\geq -\frac{N-2}{a}(q+r) f(\lambda, y, z) \geq -\frac{N-2}{a^{\chi-N+1}}(q+r)\lambda^{\chi-N} f(\lambda, y, z) \geq h(\lambda, y, z),\end{aligned}$$

hence, (1.17) is satisfied. □

2.3 Beckner and Stein-Weiss system: general case

Let us consider the following generalization of the Beckner system:

$$\begin{cases} u_s(x_s) = \displaystyle\int_{\mathbb{R}^{N(k-1)}} \frac{\prod_{j \neq s} u_j^{p_j}(x_j)}{\prod_{1 \leq i < j \leq k} |x_i|^{\sigma_{ij}} |x_i - x_j|^{N-h_{ij}} |x_j|^{\sigma_{ji}}} \, dX_{\hat{s}}, \quad x_s \in \mathbb{R}^N \setminus \{0\}, \\ u_s \geq 0 \text{ in } \mathbb{R}^N, \quad s = 1, \ldots, k, \end{cases} \tag{2.35}$$

where $N > h_{ij} = h_{ji} \geq 2$, $\sigma_{ij} \in \mathbb{R}$ and $p_s > 0$.

This problem has been studied in the particular case $k = 2$ (see [5, 23]), i.e.:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^N} \frac{v^q(y)}{|x|^\zeta |x-y|^{N-\alpha}|y|^\eta} \, dy, \quad x \in \mathbb{R}^N \setminus \{0\}, \\ v(x) = \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x|^\eta |x-y|^{N-\alpha}|y|^\zeta} \, dy, \quad x \in \mathbb{R}^N \setminus \{0\}, \\ u, v \geq 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{2.36}$$

where $0 < \alpha, \beta < N$, $\zeta, \eta \in \mathbb{R}$ and $p, q > 0$.

(2.36) is the system of Euler-Lagrange equations associated to the following inequality, proved by Stein and Weiss [21].

Theorem 2.3.1 (Stein-Weiss inequality)**.** *Let $0 < s_1, s_2 < +\infty$, $\zeta, \eta \in \mathbb{R}$, $\zeta + \eta \geq 0$ and $0 < \alpha < N$ be real number satisfying*

$$\frac{1}{s_1} + \frac{1}{s_2} = \frac{N + \alpha - \zeta - \eta}{N}.$$

If $f \in L^{s_1}(\mathbb{R}^N)$ and $g \in L^{s_2}(\mathbb{R}^N)$, then

$$\left| \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{f(x)g(y)}{|x|^\zeta |x-y|^{N-\alpha} |y|^\eta} \, dx \, dy \right| \leq S \|f\|_{s_1} \|g\|_{s_2},$$

where $S = S(s_1, s_2, \alpha, \zeta, \eta)$.

Motivated by the above-mentioned description, in this section we study the general system (2.35).

Theorem 2.3.2. *Let $N > h_{ij} = h_{ji} \geq 2$, $\sigma_{ij} \in \mathbb{R}$, $p_s > 0$ and*

$$\chi = \sum_{1 \leq i < j \leq k} h_{ij}, \quad \rho = \sum_{i \neq j} \sigma_{ij}.$$

If

$$\sum_{s=1}^{k} \frac{1}{p_s + 1} \neq \frac{(k-1)k}{2} - \frac{\chi - \rho}{N}, \tag{2.37}$$

then the problem (2.35) has no nontrivial solution $u_s \in C^1(\mathbb{R}^N) \cap L^{p_s+1}(\mathbb{R}^N)$ $s = 1, \ldots, k$ satisfying the following condition: there exists $a < 1 < b$ such that the following integral is finite

$$\int\limits_a^b \int\limits_{\mathbb{R}^{N(k-1)}} \frac{u_r^{p_r-1}(\lambda x_r) |\nabla u_r(\lambda x_r) \cdot x_r| \prod_{j \neq r,1} u_j^{p_j}(\lambda x_j)}{\prod_{1 \leq i < j \leq k} |x_i|^{\sigma_{ij}} |x_i - x_j|^{N - h_{ij}} |x_j|^{\sigma_{ji}}} \, dX_{\hat{1}} \, d\lambda < +\infty, \quad \text{for all } r \in \{2, \ldots, k\},$$

where the variables of integration are $X_{\hat{1}} = (x_2, \ldots, x_k) \in \mathbb{R}^{N(k-1)}$.

Proof. We proceed as in the Theorem 2.1.1: let λ close to 1, we have that

$$\begin{aligned} u_1(\lambda x_1) &= \frac{\lambda^{N(k-1)}}{\lambda^{\binom{k}{2}N-\chi+\rho}} \int\limits_{\mathbb{R}^{N(k-1)}} \frac{\prod_{j=2}^{k} u_j^{p_j}(\lambda x_j)}{\prod_{1\le i<j\le k} |x_i|^{\sigma_{ij}}|x_i-x_j|^{N-h_{ij}}|x_j|^{\sigma_{ji}}}\, dX_{\hat{1}} \\ &= \lambda^{\chi-\rho-N\frac{(k-1)(k-2)}{2}} \int\limits_{\mathbb{R}^{N(k-1)}} \frac{\prod_{j=2}^{k} u_j^{p_j}(\lambda x_j)}{\prod_{1\le i<j\le k} |x_i|^{\sigma_{ij}}|x_i-x_j|^{N-h_{ij}}|x_j|^{\sigma_{ji}}}\, dX_{\hat{1}}. \end{aligned} \tag{2.38}$$

Therefore,

$$\begin{aligned} \nabla u_1(x_1)\cdot x_1 = \frac{\partial u_1}{\partial \lambda}(\lambda x_1)\Big|_{\lambda=1} &= \left(\chi-\rho-N\frac{(k-1)(k-2)}{2}\right) u_1(x_1)+ \\ &+ \sum_{r=2}^{k} \int\limits_{\mathbb{R}^{N(k-1)}} p_r \frac{u_r^{p_r-1}(x_r)\nabla u_r(x_r)\cdot x_r \prod_{j\neq r,1} u_j^{p_j}(x_j)}{\prod_{1\le i<j\le k} |x_i|^{\sigma_{ij}}|x_i-x_j|^{N-h_{ij}}|x_j|^{\sigma_{ji}}}\, dX_{\hat{1}}. \end{aligned} \tag{2.39}$$

Hence, multiplying by $u_1^{p_1}$ and integrating with respect to x_1 the previous identity, we have

$$\begin{aligned} \int\limits_{\mathbb{R}^N} u_1^{p_1}(x_1)\nabla u_1(x_1)\cdot x_1\, dx_1 &= \left(\chi-\rho-N\frac{(k-1)(k-2)}{2}\right) \int\limits_{\mathbb{R}^N} u_1^{p_1+1}(x_1)\, dx_1+ \\ &+ \sum_{r=2}^{k} \int\limits_{\mathbb{R}^N} p_r u_r^{p_r}(x_r)\nabla u_r(x_r)\cdot x_r\, dx_r. \end{aligned} \tag{2.40}$$

By divergence theorem there exists a sequence $R_n \to +\infty$, such that

$$\lim_{n\to+\infty} \int\limits_{B_{R_n}} u_r^{p_r}(x_r)\nabla u_r(x_r)\cdot x_r\, dx_r = -\frac{N}{p_r+1}\int\limits_{\mathbb{R}^N} u_r^{p_r+1}(x_r)\, dx_r, \quad \text{for all } r\in\{1,\dots,k\}.$$

Therefore, by (2.40) we obtain

$$\begin{aligned} -\frac{N}{p_1+1}\int\limits_{\mathbb{R}^N} u_1^{p_1+1}(x_1)\, dx_1 &= \left(\chi-\rho-N\frac{(k-1)(k-2)}{2}\right) \int\limits_{\mathbb{R}^N} u_1^{p_1+1}(x_1)\, dx_1- \\ &- \sum_{r=2}^{k} \frac{Np_r}{p_r+1}\int\limits_{\mathbb{R}^N} u_r^{p_r+1}(x_r)\, dx_r. \end{aligned} \tag{2.41}$$

By Tonelli Theorem we have that

$$\int_{\mathbb{R}^N} u_1^{p_1+1}(x_1)\,dx_1 = \int_{\mathbb{R}^N} u_r^{p_r+1}(x_r)\,dx_r, \quad \text{for all } r \in \{1, \dots, k\}.$$

Hence,

$$\left(\chi - \rho - N\frac{(k-1)(k-2)}{2} + \frac{N}{p_1+1} - \sum_{r=2}^{k} \frac{Np_r}{p_r+1}\right) \int_{\mathbb{R}^N} u_1^{p_1+1}(x_1)\,dx_1 = 0.$$

This conclude the proof. □

In the particular case $\sigma_{ij} = 0$ for all $i, j = 1, \dots, k$, as a consequence of Theorem 2.3.2 we obtain the following result about the system (2.1).

Theorem 2.3.3. *Let $N > h_{ij} \geq 2$, and $p_s > 0$.*

If

$$\sum_{s=1}^{k} \frac{1}{p_s+1} \neq \sum_{1\leq i<j\leq k} \frac{N-h_{ij}}{N}, \tag{2.42}$$

then the problem (2.1) has no nontrivial solution $u_s \in C^1(\mathbb{R}^N) \cap L^{p_s+1}(\mathbb{R}^N)$ $s = 1, \dots, k$ satisfying the following condition: there exists $a < 1 < b$ such that the following integral is finite

$$\int_a^b \int_{\mathbb{R}^{N(k-1)}} \frac{u_r^{p_r-1}(\lambda x_r)|\nabla u_r(\lambda x_r)\cdot x_r| \prod_{j\neq r,1} u_j^{p_j}(\lambda x_j)}{\prod_{1\leq i<j\leq k} |x_i - x_j|^{N-h_{ij}}}\,dX_{\hat{1}}\,d\lambda < +\infty, \quad \textit{for all } r \in \{2, \dots, k\},$$

where the variables of integration are $X_{\hat{1}} = (x_2, \dots, x_k) \in \mathbb{R}^{N(k-1)}$.

Chapter 3

Non existence results without finite energy condition

Let us consider solutions $u_s \in L^{p_s}_{loc}(\mathbb{R}^N)$ $s = 1, \dots, k$ to the system (2.1) (not necessarily $u_s \in L^{p_s+1}(\mathbb{R}^N)$ $s = 1, \dots, k$). If $s = 2$ the problem (2.1) coincides with (2.12) and it is known that the Sobolev hyperbola (2.19) plays a key role in the non-existence criteria: the HLS conjecture [3] states that if

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-\alpha}{N}, \quad 0 < \alpha < N, \tag{3.1}$$

then (2.12) has no nontrivial solutions.

Even in the case where $\alpha = 2$, the conjecture, better known as the Lane-Emden conjecture, remains an open problem. However, partial results has been proved: for $n \leq 4$ the Lane-Emden conjecture has been verified (see [17, 18, 19]). For $n \geq 5$ other contributions with stronger assumptions than (3.1) on exponents are known (see [2, 6, 14, 19]).

For α not necessarily $\alpha = 2$, Caristi, D'Ambrosio and Mitidieri [3] prove a Rellich type identity for solutions to (2.12) obtaining the proof of the Hardy-Littlewood-Sobolev conjecture for radial solutions to (2.12) (see Theorem 0.0.3). Another contribution to the Hardy-Littlewood-Sobolev conjecture contained in [3] is the following result, where no conditions on symmetry or energy of solutions are assumed:

Theorem 3.0.1 (Caristi - D'Ambrosio - Mitidieri)**.** *Let $p, q > 0$ and $0 < \alpha < N$. If $pq \leq 1$ or*

$$pq > 1 \quad \textit{and} \quad N - \alpha \leq \alpha \max\left\{\frac{q+1}{pq-1}, \frac{p+1}{pq-1}\right\}, \tag{3.2}$$

then the system

$$\begin{cases} u(x) \geq \displaystyle\int_{\mathbb{R}^N} \frac{v^q(y)}{|x-y|^{N-\alpha}}\, dy, \quad x \in \mathbb{R}^N, \\ v(x) \geq \displaystyle\int_{\mathbb{R}^N} \frac{u^p(y)}{|x-y|^{N-\alpha}}\, dy, \quad x \in \mathbb{R}^N, \\ u, v \geq 0 \quad in\ \mathbb{R}^N. \end{cases} \tag{3.3}$$

has no nontrivial solution

$$(u,v) \in \left(L^1_{loc}(\mathbb{R}^N) \cap L^p_{loc}(\mathbb{R}^N)\right) \times \left(L^1_{loc}(\mathbb{R}^N) \cap L^q_{loc}(\mathbb{R}^N)\right).$$

In view of the description above, it is then natural to formulate the following conjecture:

Conjecture. Let $0 < h_{ij} < N$, $p_j > 0$ for all $i, j \in \{1, \dots, k\}$. If

$$\sum_{j=1}^{k} \frac{1}{p_j + 1} > \sum_{1 \leq i < j \leq k} \frac{N - h_{ij}}{N}, \tag{3.4}$$

then (2.1) has no nontrivial solutions.

In this chapter we prove some contributions to this conjecture. More precisely, we obtain some non-existence results for solutions $u_s \in L^{p_s}_{loc}(\mathbb{R}^N)$ $s = 1, \dots, k$ to the system

$$\begin{cases} u_s(x_s) \geq \displaystyle\int_{\mathbb{R}^{N(k-1)}} \frac{\prod_{j \neq s} u_j^{p_j}(x_j)}{\prod_{1 \leq i < j \leq k} |x_i - x_j|^{N-h_{ij}}}\, dX_{\hat{s}}, \\ u_s \geq 0 \text{ in } \mathbb{R}^N, \quad s = 1, \dots, k. \end{cases} \tag{3.5}$$

3.1 Beckner system of inequalities: $k = 3$

Lemma 3.1.1. *Let* $0 < \alpha, \beta, \gamma < N$, $\chi = \alpha + \beta + \gamma$, *and* $p, q, r > 0$.

If

$$(u,v,w) \in \left(L^p_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^q_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^r_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right)$$

is a solution to

$$\begin{cases} u(x) \geq \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{v^q(y) w^r(z)}{|x-y|^{N-\alpha} |y-z|^{N-\beta} |z-x|^{N-\gamma}} \, dy \, dz, \quad x \in \mathbb{R}^N, \\ v(x) \geq \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{u^p(y) w^r(z)}{|x-y|^{N-\alpha} |y-z|^{N-\gamma} |z-x|^{N-\beta}} \, dy \, dz, \quad x \in \mathbb{R}^N, \\ w(x) \geq \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} \frac{u^p(y) v^q(z)}{|x-y|^{N-\gamma} |y-z|^{N-\alpha} |z-x|^{N-\beta}} \, dy \, dz, \quad x \in \mathbb{R}^N, \\ u, v, w \geq 0 \ \text{in } \mathbb{R}^N, \end{cases} \tag{3.6}$$

then

$$\int\limits_{B_R} u^p(x) \, dx \geq \frac{C}{R^{p(3N-\chi)-N}} \left(\int\limits_{B_R} v^q(y) \, dy \right)^p \left(\int\limits_{B_R} w^r(z) \, dz \right)^p, \tag{3.7}$$

$$\int\limits_{B_R} v^q(x) \, dx \geq \frac{C}{R^{q(3N-\chi)-N}} \left(\int\limits_{B_R} u^p(y) \, dy \right)^q \left(\int\limits_{B_R} w^r(z) \, dz \right)^q, \tag{3.8}$$

$$\int\limits_{B_R} w^r(x) \, dx \geq \frac{C}{R^{r(3N-\chi)-N}} \left(\int\limits_{B_R} u^p(y) \, dy \right)^r \left(\int\limits_{B_R} v^q(z) \, dz \right)^r, \tag{3.9}$$

for all $R > 0$*.*

Proof. By the system (3.6), we have

$$\begin{aligned} w(x) &\geq \int\limits_{B_R} \int\limits_{B_R} \frac{u^p(y) v^q(z)}{|x-y|^{N-\gamma} |y-z|^{N-\alpha} |z-x|^{N-\beta}} \, dy \, dz \\ &\geq \frac{C}{R^{N-\alpha} (|x|+R)^{2N-\beta-\gamma}} \left(\int\limits_{B_R} u^p(y) \, dy \right) \left(\int\limits_{B_R} v^q(z) \, dz \right). \end{aligned}$$

Raising to r and integrating over B_R with respect to x we obtain

$$\int\limits_{B_R} w^r(x) \, dx \geq \frac{C}{R^{r(3N-\chi)-N}} \left(\int\limits_{B_R} u^p(y) \, dy \right)^r \left(\int\limits_{B_R} v^q(z) \, dz \right)^r.$$

In a similar way, we can prove (3.7) and (3.8). $\square$

Theorem 3.1.2. *Let $p, q, r > 0$, $0 < \alpha, \beta, \gamma < N$ and $\chi = \alpha + \beta + \gamma = 2N + \sigma$ with $\sigma > 0$. If*

$$\min\{pq, qr, pr\} \le 1,$$

then the problem (3.6) has no nontrivial solution

$$(u, v, w) \in \left(L^p_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^q_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^r_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right).$$

Proof. We may assume that $pq \le 1$ without losing of generality. If $w \not\equiv 0$, i.e. $w \ne 0$ almost everywhere in $\mathbb{R}^N$, then

$$\int\limits_{B_R} w^r(x)\, dx > C > 0, \quad R \ge R_0.$$

Consequently, by Lemma 3.1.1, we have

$$\int\limits_{B_R} u^p(x)\, dx \ge \frac{C}{R^{p(3N-\chi)-N}} \left(\int\limits_{B_R} v^q(y)\, dy \right)^p, \tag{3.10}$$

$$\int\limits_{B_R} v^q(x)\, dx \ge \frac{C}{R^{q(3N-\chi)-N}} \left(\int\limits_{B_R} u^p(y)\, dy \right)^q. \tag{3.11}$$

Combining (3.10) and (3.11), we obtain

$$\int\limits_{B_R} u^p(x)\, dx \ge C R^t \left(\int\limits_{B_R} u^p(y)\, dy \right)^{pq}. \tag{3.12}$$

Since $pq \le 1$ and $\chi > 2N$, than the exponent of R in (3.12) is

$$t := -\left(p(3N-\chi) - N + pq(3N-\chi) - pN\right) = N(1-pq) + \sigma(p+pq) > 0.$$

If $pq = 1$ then (3.12) implies $R \le C$. Hence, letting $R \to +\infty$, we have a contradiction. If $pq < 1$ then

$$\int\limits_{B_R} u^p(x)\, dx \ge C R^{\frac{t}{1-pq}}. \tag{3.13}$$

On the other hand, taking $R \geq |x|$ and proceeding as in the proof of Lemma 3.1.1 we have that

$$v(x) \geq \frac{C}{R^{3N-\chi}} \left(\int\limits_{B_R} u^p(y)\,dy \right). \tag{3.14}$$

Combining (3.13) with (3.14) we obtain

$$v(x) \geq CR^{\sigma\left(1+p\frac{q+1}{pq-1}\right)} \geq C|x|^{\sigma\left(1+p\frac{q+1}{pq-1}\right)}. \tag{3.15}$$

Similarly we get

$$u(x) \geq CR^{\sigma\left(1+q\frac{p+1}{pq-1}\right)} \geq C|x|^{\sigma\left(1+q\frac{p+1}{pq-1}\right)}. \tag{3.16}$$

The estimates (3.15) and (3.16) imply that u and v are bounded from below by a positive constant on $\mathbb{R}^N \setminus B_1$. Hence, applying Lemma 1.1.1, the integral on the right hand side of the third inequality (3.6) is not finite, and we have a contradiction. □

Theorem 3.1.3. *Let* $0 < \alpha, \beta, \gamma < N$, $\chi = \alpha + \beta + \gamma = 2N + \sigma$ *with* $\sigma > 0$ *and* $p, q, r > 0$ *such that* $pqr > 1$. *If*

$$N \leq \sigma \max\left\{ p\frac{1+q+qr}{pqr-1}, q\frac{1+r+rp}{pqr-1}, r\frac{1+p+pq}{pqr-1} \right\},$$

then the problem (3.6) has no nontrivial solution

$$(u,v,w) \in \left(L^p_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^q_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^r_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right).$$

Proof. Without losing of generality, we may assume that

$$N \leq \sigma p \frac{1+q+qr}{pqr-1}.$$

By Lemma 3.1.1, if $u \not\equiv 0$, $v \not\equiv 0$ and $w \not\equiv 0$, then (3.10) holds and

$$\int\limits_{B_R} v^q(x)\,dx \geq \frac{C}{R^{q(3N-\chi)-N}} \left(\int\limits_{B_R} w^r(z)\,dz \right)^q, \tag{3.17}$$

$$\int\limits_{B_R} w^r(x)\,dx \geq \frac{C}{R^{r(3N-\chi)-N}} \left(\int\limits_{B_R} u^p(y)\,dy \right)^r. \tag{3.18}$$

Hence,

$$\int_{B_R} u^p(x)\,dx \geq \frac{C}{R^{p(3N-\chi)-N}} \frac{C}{R^{p[q(3N-\chi)-N]}} \left(\int_{B_R} w^r(z)\,dz \right)^{pq}$$
$$\geq \frac{C}{R^{p(3N-\chi)-N}} \frac{C}{R^{p[q(3N-\chi)-N]}} \frac{C}{R^{pq[r(3N-\chi)-N]}} \left(\int_{B_R} u^p(y)\,dy \right)^{rpq}$$
$$= CR^a \left(\int_{B_R} u^p(y)\,dy \right)^{rpq},$$

where

$$\begin{aligned} a :=& -\{p(3N-\chi) - N + p[q(3N-\chi)-N] + pq[r(3N-\chi)-N]\} \\ =& \sigma p(1+q+qr) - N(pqr-1) \geq 0. \end{aligned}$$

Therefore,

$$\int_{B_R} u^p(x)\,dx \leq CR^{-\frac{a}{pqr-1}}.$$

Letting $R \to +\infty$, if $a > 0$ then $u \equiv 0$. Hence, we may assume that $a = 0$ and $u \in L^p(\mathbb{R}^N)$. Next, we proceed as in the proof of Lemma 3.1.1 and we obtain that

$$u(x) \geq \frac{C}{R^{N-\beta}(|x|+R)^{2N-\alpha-\gamma}} \left(\int_{B_R} w^r(y)\,dy \right) \left(\int_{B_R} v^q(z)\,dz \right),$$

raising to the power p and integrating over $A_R = B_{2R} \setminus B_R$ both hand-side we have

$$\begin{aligned} \int_{A_R} u^p(x)\,dx &\geq \frac{C}{R^{p(3N-\chi)-N}} \left(\int_{B_R} w^r(y)\,dy \right)^p \left(\int_{B_R} v^q(y)\,dy \right)^p \\ &\geq \frac{C}{R^{p(3N-\chi)-N}} \left(\int_{B_R} v^q(y)\,dy \right)^p. \end{aligned} \tag{3.19}$$

Since $a = 0$, by (3.17) and (3.18), the estimate (3.19) becomes

$$\int_{A_R} u^p(x)\,dx \geq C \left(\int_{B_R} u^p(x)\,dx \right)^{pqr}.$$

Letting $R \to +\infty$, since $u \in L^p(\mathbb{R}^N)$ therefore

$$\lim_{R\to+\infty} \int\limits_{A_R} u^p(x)\,dx = 0,$$

and we have the claim. □

Theorem 3.1.4. *Let* $0 < \alpha, \beta, \gamma < N$, $\chi = \alpha + \beta + \gamma = 2N + \sigma$ *with* $\sigma > 0$ *and* $p, q, r > 0$. *If one of the following conditions holds*

$$pq > 1 \quad \text{and} \quad N \leq \sigma \max\left\{\frac{p(q+1)}{pq-1}, \frac{q(p+1)}{pq-1}\right\}, \tag{3.20}$$

$$qr > 1 \quad \text{and} \quad N \leq \sigma \max\left\{\frac{r(q+1)}{rq-1}, \frac{q(r+1)}{rq-1}\right\}, \tag{3.21}$$

$$pr > 1 \quad \text{and} \quad N \leq \sigma \max\left\{\frac{p(r+1)}{pr-1}, \frac{r(p+1)}{pr-1}\right\}, \tag{3.22}$$

then the problem (3.6) has no nontrivial solution

$$(u, v, w) \in \left(L^p_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^q_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^r_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right).$$

Proof. Without losing of generality, we may assume that (3.20) holds and

$$\max\left\{\frac{p(q+1)}{pq-1}, \frac{q(p+1)}{pq-1}\right\} = \frac{p(q+1)}{pq-1}.$$

Proceeding as in the proof of Theorem 3.1.2, by (3.12) we have

$$\int\limits_{B_R} u^p(x)\,dx \leq CR^{-\frac{b}{pq-1}},$$

where

$$b = -p(3N - \chi) + N - pq(3N - \chi) + pN = \sigma p(q+1) - N(pq-1) \geq 0.$$

Letting $R \to +\infty$, if $b > 0$ then $u \equiv 0$. Hence, we may assume that $b = 0$ and $u \in L^p(\mathbb{R}^N)$. Next, we proceed as in the proof of Lemma 3.1.1 and we obtain that

$$u(x) \geq \frac{C}{R^{N-\beta}(|x| + R)^{2N-\alpha-\gamma}} \left(\int\limits_{B_R} w^r(y)\,dy\right) \left(\int\limits_{B_R} v^q(z)\,dz\right),$$

raising to the power p and integrating over $A_R = B_{2R} \setminus B_R$ both hand-side we have

$$\begin{aligned}\int_{A_R} u^p(x)\,dx &\geq \frac{C}{R^{p(3N-\chi)-N}} \left(\int_{B_R} w^r(y)\,dy \right)^p \left(\int_{B_R} v^q(y)\,dy \right)^p \\ &\geq \frac{C}{R^{p(3N-\chi)-N}} \left(\int_{B_R} v^q(y)\,dy \right)^p .\end{aligned} \tag{3.23}$$

Since $b = 0$, combining this estimates with (3.11), we get

$$\int_{A_R} u^p(x)\,dx \geq C \left(\int_{B_R} u^p(y)\,dy \right)^{pq} .$$

Letting $R \to +\infty$, since $u \in L^p(\mathbb{R}^N)$ therefore

$$\lim_{R\to+\infty} \int_{A_R} u^p(x)\,dx = 0,$$

and we have the claim. □

Theorem 3.1.5. *Let $0 < \alpha, \beta, \gamma < N$, $\chi = \alpha + \beta + \gamma > N$ and $p, q, r > 0$ such that*

$$\min\{p+q,\, q+r,\, p+r\} > 1.$$

If

$$p + q + r \leq \frac{3N}{3N-\chi}, \tag{3.24}$$

then the problem (3.6) has no nontrivial solution

$(u, v, w) \in \left(L^p_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^q_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right) \times \left(L^r_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N)\right).$

Proof. By Lemma 3.1.1, we have

$$\begin{aligned}\left(\int_{B_R} u^p(x)\,dx \right) \left(\int_{B_R} v^q(x)\,dx \right) \left(\int_{B_R} w^r(x)\,dx \right) &\geq \frac{C}{R^{(p+q+r)(3N-\chi)-3N}} \cdot \\ \cdot \left(\int_{B_R} u^p(x)\,dx \right)^{q+r} &\left(\int_{B_R} v^q(x)\,dx \right)^{p+r} \left(\int_{B_R} w^r(x)\,dx \right)^{p+q} ,\end{aligned}$$

i. e.

$$\left(\int_{B_R} u^p(x)\,dx\right)^{q+r-1}\left(\int_{B_R} v^q(x)\,dx\right)^{p+r-1}\left(\int_{B_R} w^r(x)\,dx\right)^{p+q-1}\leq CR^b, \quad (3.25)$$

where $b=(p+q+r)(3N-\chi)-3N$.

If $p+q+r<\frac{3N}{3N-\chi}$ then $b<0$. Hence, letting $R\to+\infty$ in the inequality (3.25), we have that $u,v,w\equiv 0$.

On the other hand, if $p+q+r=\frac{3N}{3N-\chi}$ then $b=0$. Hence, one of the following possibilities occur: $u\in L^p(\mathbb{R}^N)$, $v\in L^q(\mathbb{R}^N)$ or $w\in L^r(\mathbb{R}^N)$.

Without loss of generality, we suppose $u\in L^p(\mathbb{R}^N)$, therefore

$$\lim_{R\to+\infty}\int_{A_R} u^p(x)\,dx=0.$$

Proceeding as in the proof of Lemma 3.1.1, we obtain

$$\int_{A_R} u^p(x)\,dx\geq \frac{C}{R^{(3N-\chi)p-N}}\left(\int_{B_R} v^q(y)\,dy\right)^p\left(\int_{B_R} w^r(z)\,dz\right)^p. \quad (3.26)$$

Combining the inequalities (3.8) and (3.9) with (3.26) we get

$$\left(\int_{A_R} u^p(x)\,dx\right)\left(\int_{B_R} v^q(x)\,dx\right)\left(\int_{B_R} w^r(x)\,dx\right)\geq \frac{C}{R^{(p+q+r)(3N-\chi)-3N}}\cdot$$
$$\cdot\left(\int_{B_R} u^p(x)\,dx\right)^{q+r}\left(\int_{B_R} v^q(x)\,dx\right)^{p+r}\left(\int_{B_R} w^r(x)\,dx\right)^{p+q},$$

i. e.

$$\left(\int_{A_R} u^p(x)\,dx\right)\geq \frac{C}{R^b}\left(\int_{B_R} u^p(x)\,dx\right)^{q+r}\left(\int_{B_R} v^q(x)\,dx\right)^{p+r-1}\cdot$$
$$\cdot\left(\int_{B_R} w^r(x)\,dx\right)^{p+q-1}\geq C\left(\int_{B_R} u^p(x)\,dx\right)^{q+r}, \qquad R\geq R_0. \quad (3.27)$$

Finally, letting $R \to +\infty$ in (3.27), we have that $u \equiv 0$, therefore we obtain the claim.

$\square$

3.2 Beckner system of inequalities: general case

Theorem 3.2.1. *Let $k \geq 3$, $0 < h_{ij} = h_{ji} < N$ and $p_s > 0$ for all $s = 1, \dots, k$.*

Suppose that both the following conditions are satisfied:

$$\prod_{i=1}^{k-1} p_i = \min_{j \in \{1,\dots,k\}} \prod_{\substack{i=1,\\ i\neq j}}^{k} p_i, \tag{3.28}$$

$$\chi = \sum_{1 \leq i < j \leq k} h_{ij} = (C_k - 1)N + \sigma, \tag{3.29}$$

where $\sigma > 0$ and $C_k := \binom{k}{2}$. If

$$\prod_{i=1}^{k-1} p_i \leq 1, \tag{3.30}$$

then the problem (3.5) has no nontrivial solution

$$u_s \in L^{p_s}_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N) \quad s = 1, \dots, k.$$

Proof. Let $R > 0$, by system (3.5), we have that

$$\begin{aligned} u_s(x_s) \geq & \frac{C}{\displaystyle\prod_{1 \leq i < j \leq k, i,j \neq s} R^{N-h_{ij}} \prod_{1 \leq i < s} (R + |x_s|)^{N-h_{is}} \prod_{s < j \leq k} (R + |x_s|)^{N-h_{sj}}} \\ & \cdot \prod_{1 \leq i \leq k, i \neq s} \int_{B_R} u_j^{p_j}(x_j)\, dx_j, \end{aligned} \tag{3.31}$$

for all $s = 1, \dots, k-1$.

Supposing $u_s \not\equiv 0$ almost everywhere in $\mathbb{R}^N$ for all $s = 1, \dots, k$, we prove that we obtain a contradiction. Since,

$$\int_{B_R} u_j^{p_j}(x_j)\, dx_j > C > 0, \quad R \geq R_0, \text{ for all } j = 1, \dots k,$$

then by system (3.31) we get

$$
\begin{aligned}
u_s(x_s) \geq & \frac{C}{\prod\limits_{1\leq i<j\leq k, i,j\neq s} R^{N-h_{ij}} \prod\limits_{1\leq i<s} (R+|x_s|)^{N-h_{is}} \prod\limits_{s<j\leq k} (R+|x_s|)^{N-h_{sj}}} \cdot \\
& \cdot \int\limits_{B_R} u_{s+1}^{p_{s+1}}(x_{s+1})\, dx_{s+1}, \quad \text{for all } s=1,\dots,k-2,
\end{aligned}
\tag{3.32}
$$

$$
\begin{aligned}
u_{k-1}(x_{k-1}) \geq & \frac{C}{\prod\limits_{1\leq i<j\leq k, i,j\neq s} R^{N-h_{ij}} \prod\limits_{1\leq i<s} (R+|x_s|)^{N-h_{is}} \prod\limits_{s<j\leq k} (R+|x_s|)^{N-h_{sj}}} \cdot \\
& \cdot \int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1.
\end{aligned}
\tag{3.33}
$$

Rising the inequality (3.32) to the power p_s and integrating it over B_R we obtain

$$
\int\limits_{B_R} u_s^{p_s}(x_s)\, dx_s \geq CR^{-(C_kN-\chi)p_s+N} \cdot \left(\int\limits_{B_R} u_{s+1}^{p_{s+1}}(x_{s+1})\, dx_{s+1} \right)^{p_s}, \text{ for all } s=1,\dots,k-2,
\tag{3.34}
$$

Similarly, rising the inequality (3.33) to the power p_{k-1} and integrating it over B_R we have

$$
\int\limits_{B_R} u_{k-1}^{p_{k-1}}(x_{k-1})\, dx_{k-1} \geq CR^{-(C_kN-\chi)p_{k-1}+N} \cdot \left(\int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1 \right)^{p_{k-1}}.
\tag{3.35}
$$

Combining (3.34) with (3.35) we get

$$
\int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1 = CR^t \left(\int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1 \right)^{\prod_{j=1}^{k-1} p_j},
\tag{3.36}
$$

where

$$t = (N - (C_k N - \chi)p_1) + \sum_{i=2}^{k-1} \left((N - (C_k N - \chi)p_i) \prod_{j=1}^{i-1} p_j \right) =$$

$$= (N - (N - \sigma)p_1) + \sum_{i=2}^{k-1} \left(N \left(\prod_{j=1}^{i-1} p_j - \prod_{j=1}^{i} p_j \right) - \sigma \prod_{j=1}^{i} p_j \right) =$$

$$= N \left(1 - \prod_{j=1}^{k-1} p_j \right) + \sigma \sum_{i=1}^{k-1} \prod_{j=1}^{i} p_j.$$

If $\prod_{j=1}^{k-1} p_j = 1$, then we conclude that $R \leq C$ and we have a contradiction.

On the other hand, if $\prod_{j=1}^{k-1} p_j < 1$, then

$$\int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1 \geq C R^{\frac{t}{1-\prod_{j=1}^{k-1} p_j}}. \tag{3.37}$$

Next, we consider $|x_s| \leq R$ for all $s = 1, \ldots, k-1$. Combining (3.37) with (3.33) and (3.35), we obtain

$$u_{k-1}(x_{k-1}) \geq C R^{a_{k-1}}, \quad \text{and} \quad \int\limits_{B_R} u_{k-1}^{p_{k-1}}(x_{k-1})\, dx_{k-1} \geq C R^{N+b_{k-1}},$$

where

$$a_{k-1} = \frac{t}{1 - \prod_{j=1}^{k-1} p_j} - (C_k N - \chi) = \sigma \left(1 + \frac{\sum_{i=1}^{k-1} \prod_{j=1}^{i} p_j}{1 - \prod_{j=1}^{k-1} p_j} \right) > 0,$$

and

$$b_{k-1} = \frac{t p_{k-1}}{1 - \prod_{j=1}^{k-1} p_j} - (C_k N - \chi) p_{k-1} = p_{k-1} a_{k-1} > 0$$

Similarly, assuming by induction that

$$u_{s+1}(x_{s+1}) \geq C R^{a_{s+1}}, \quad \text{and} \quad \int\limits_{B_R} u_{s+1}^{p_{s+1}}(x_{s+1})\, dx_{s+1} \geq C R^{N+b_{s+1}},$$

with $a_{s+1}, b_{s+1} > 0$. Then, by (3.32) and (3.34) we have

$$u_s(x_s) \geq C R^{a_s}, \quad \text{and} \quad \int\limits_{B_R} u_s^{p_s}(x_s)\, dx_s \geq C R^{N+b_s},$$

where

$$a_s = b_{s+1} + \sigma > 0 \quad \text{and} \quad b_s = \sigma p_s + b_{s+1} p_s = p_s a_s > 0.$$

Therefore,

$$u_s(x_s) \geq C R^{a_s} \geq C|x_s|^{a_s}, \quad \text{for all } s = 1, \dots, k-1. \tag{3.38}$$

The estimates (3.38) imply that u_s is bounded from below by a positive constant on $\mathbb{R}^N \setminus B_1$ for all $s = 1, \dots, k-1$. Hence, the integrals on right hand side of (3.5) are not finite and we have a contradiction. □

Theorem 3.2.2. *Let $k \geq 3$, $\sigma > 0$, $0 < h_{ij} = h_{ji} < N$ and $p_s > 0$ for all $s = 1 \dots, k$. Suppose that both conditions (3.29) and (3.28) are satisfied.*
Consider

$$\mathcal{S}_{k-1} := \{\pi : \{1, \dots, k-1\} \to \{1, \dots, k-1\} \text{ bijective}\},$$

the set of permutations of $\{1, \dots, k-1\}$.
If

$$\prod_{j=1}^{k-1} p_j > 1 \quad \text{and} \quad N \leq \sigma \max_{\pi \in \mathcal{S}_{k-1}} \left\{ \frac{\sum_{i=1}^{k-1} \prod_{j=1}^{i} p_{\pi(j)}}{\prod_{j=1}^{k-1} p_j - 1} \right\},$$

then the problem (3.5) has no nontrivial solution

$$u_s \in L^{p_s}_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N) \quad s = 1, \dots, k.$$

Proof. Without loss of generality we may assume that

$$\max_{\pi \in \mathcal{S}_{k-1}} \left\{ \frac{\sum_{i=1}^{k-1} \prod_{j=1}^{i} p_{\pi(j)}}{\prod_{j=1}^{k-1} p_j - 1} \right\} = \frac{\sum_{i=1}^{k-1} \prod_{j=1}^{i} p_i}{\prod_{j=1}^{k-1} p_j - 1}.$$

Let $R > 0$ such that $|x_s| \leq R$ for all $s = 1, \dots, k-1$. Proceeding as in Theorem 3.2.1, by (3.36) we obtain

$$\left(\int_{B_R} u_1^{p_1}(x_1)\, dx_1 \right)^{\prod_{j=1}^{k-1} p_j - 1} \leq C R^{-t}, \tag{3.39}$$

where

$$t = N \left(1 - \prod_{j=1}^{k-1} p_j \right) + \sigma \sum_{i=1}^{k-1} \prod_{j=1}^{i} p_j.$$

If

$$\sigma \frac{\sum_{i=1}^{k-1} \prod_{j=1}^{i} p_j}{\prod_{j=1}^{k-1} p_j - 1} > N,$$

then $t > 0$. Therefore, letting $R \to +\infty$, (3.39) implies $u_1 \equiv 0$.

On the other hand, if

$$\sigma \frac{\sum_{i=1}^{k-1} \prod_{j=1}^{i} p_j}{\prod_{j=1}^{k-1} p_j - 1} = N,$$

then $t = 0$ and $u_1 \in L^{p_1}(\mathbb{R}^N)$. Since $u_1 \in L^{p_1}(\mathbb{R}^N)$, we have

$$\lim_{R \to +\infty} \int\limits_{A_R} u_1^{p_1}(x_1)\, dx_1 = 0,$$

where $A_R := \{x \in \mathbb{R}^N \,:\, R < |x| < 2R\}$. In order to obtain the contradiction, we rise to the power p_1 and integrate over A_R the inequality of (3.32):

$$\int\limits_{A_R} u_1^{p_1}(x_1)\, dx_1 \geq C R^{-(C_k N - \chi) p_1 + N} \cdot \left(\int\limits_{B_R} u_2^{p_2}(x_2)\, dx_2 \right)^{p_1}. \tag{3.40}$$

Next, combining (3.40) with the estimates (3.34) and (3.35), we have

$$\int\limits_{A_R} u_1^{p_1}(x_1)\, dx_1 \geq C R^t \left(\int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1 \right)^{\prod_{j=1}^{k-1} p_j} = C \left(\int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1 \right)^{\prod_{j=1}^{k-1} p_j}.$$

Finally, letting $R \to +\infty$ we get $u_1 \equiv 0$. □

Theorem 3.2.3. *Let $k \geq 2$, $0 < h_{ij} = h_{ji} < N$ and $p_s > 0$*

Suppose that the condition(3.29) is satisfied.

If

$$\prod_{j=1}^{k} p_j \leq 1,$$

or

$$\prod_{j=1}^{k} p_j > 1 \quad \text{and} \quad N \leq \sigma \frac{\sum_{i=1}^{k} \prod_{j=1}^{i} p_j}{\prod_{j=1}^{k} p_j - 1},$$

then the problem (3.5) has no nontrivial solution

$$u_s \in L^{p_s}_{loc}(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N) \quad s = 1, \ldots, k.$$

Proof. The case $k = 2$ of Theorem 3.2.3 was proved in [3, Theorem 5.7]. Hence, we may assume $k \geq 3$.

Moreover, we may suppose $\prod_{j=1}^{k} p_j > 1$, or else Theorem 3.2.1 implies the thesis. In fact, if (3.30) does not hold:

$$\min_{j\in\{1,\dots,k\}} \prod_{\substack{i=1,\\ i\neq j}}^{k} p_i > 1,$$

then

$$1 < \prod_{j=1}^{k} \prod_{\substack{i=1,\\ i\neq j}}^{k} p_i = \left(\prod_{j=1}^{k} p_j\right)^{k-1},$$

i.e. $\prod_{j=1}^{k} p_j > 1$.

Next, we suppose $\prod_{j=1}^{k} p_j > 1$. Proceeding as in Theorem 3.2.1, we obtain that the inequalities (3.32) hold and

$$\begin{aligned} u_{k-1}(x_{k-1}) \geq & \frac{C}{\prod\limits_{1\leq i<j\leq k, i,j\neq s} R^{N-h_{ij}} \prod\limits_{1\leq i<s} (R+|x_s|)^{N-h_{is}} \prod\limits_{s<j\leq k} (R+|x_s|)^{N-h_{sj}}} \cdot \\ & \cdot \int\limits_{B_R} u_k^{p_k}(x_k)\, dx_k, \end{aligned} \tag{3.41}$$

$$\begin{aligned} u_{k}(x_{k}) \geq & \frac{C}{\prod\limits_{1\leq i<j\leq k, i,j\neq s} R^{N-h_{ij}} \prod\limits_{1\leq i<s} (R+|x_s|)^{N-h_{is}} \prod\limits_{s<j\leq k} (R+|x_s|)^{N-h_{sj}}} \cdot \\ & \cdot \int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1. \end{aligned} \tag{3.42}$$

By estimates (3.32), (3.41) and (3.42), we get

$$\left(\int\limits_{B_R} u_1^{p_1}(x_1)\, dx_1\right)^{\prod_{j=1}^{k} p_j - 1} \leq CR^{-t}, \tag{3.43}$$

where

$$t = N\left(1 - \prod_{j=1}^{k} p_j\right) + \sigma \sum_{i=1}^{k} \prod_{j=1}^{i} p_j.$$

Finally, it is sufficient to proceed as in Theorem 3.2.2 in order to obtain that $u_1 \equiv 0$.
□

Chapter 4

Existence of solutions to Beckner system of inequalities

The problem to determine necessary and sufficient conditions for the existence of solutions to (3.5) is still open for $k \geq 3$. However, in the case $k = 2$, Theorem 3.0.1 is optimal, since Lei and Li [9] proved that if $pq > 1$ and (3.2) is not satisfied, then there exists a solution (u, v) to (3.3) given by

$$u(x) = \frac{A_1}{(1+|x|^2)^{\vartheta_1}}, \quad v(x) = \frac{A_2}{(1+|x|^2)^{\vartheta_2}},$$

where $A_1, A_2, \vartheta_1, \vartheta_2 > 0$ are suitable constants.

The idea of the proof of Lei and Li is to estimate the integrals on the right hand side of (3.3), using the following decomposition of the domain of integration:

$$\mathbb{R}^N = B_R \cup B_{|x|/2}(x) \cup \left(\mathbb{R}^N \setminus \left(B_R \cup B_{|x|/2}(x)\right)\right), \qquad R > 0. \tag{4.1}$$

On the other hand, considering

$$u_j(x_j) = \frac{A_j}{(1+|x|^2)^{\vartheta_j}}, \qquad A_j, \vartheta_j > 0,$$

although the decomposition (4.1) of $\mathbb{R}^N$ does not allow us to estimate the integrals on the right hand side of system (3.5), we apply a different type of decomposition of of $\mathbb{R}^N$ (see Lemma 4.1.2), in order to prove the following results:

Theorem 4.0.1. *Let $p, q, r > 0$, $0 < \alpha, \beta, \gamma < N$ and $\chi := \alpha + \beta + \gamma$ such that $N < \chi < 2N$. Suppose that two of the following conditions hold:*

$$pq > 1 \quad \textit{and} \quad N > (\chi - N) \max \left\{ \frac{p(q+1)}{pq-1}, \frac{q(p+1)}{pq-1} \right\}, \tag{4.2}$$

$$rq > 1 \quad \textit{and} \quad N > (\chi - N) \max \left\{ \frac{r(q+1)}{rq-1}, \frac{q(r+1)}{rq-1} \right\}, \tag{4.3}$$

$$pr > 1 \quad \textit{and} \quad N > (\chi - N) \max \left\{ \frac{p(r+1)}{pr-1}, \frac{r(p+1)}{pr-1} \right\}. \tag{4.4}$$

Then there exist $\vartheta_1, \vartheta_2, \vartheta_3 > 0$ and an infinite number of trios of positive constants (A_1, A_2, A_3) such that

$$u(x) = \frac{A_1}{(1+|x|^2)^{\vartheta_1}}, \quad v(x) = \frac{A_2}{(1+|x|^2)^{\vartheta_2}}, \quad w(x) = \frac{A_3}{(1+|x|^2)^{\vartheta_3}}, \tag{4.5}$$

is a solution to the system (3.6).

Theorem 4.0.2. *Let $p, q, r > 0$, $0 < \alpha, \beta, \gamma < N$ and $\chi := \alpha + \beta + \gamma$ such that $N < \chi < 2N$. Suppose that the following condition hold:*

$$pqr > 1 \quad \textit{and} \quad N > (\chi - N) \max \left\{ p\frac{1+q+qr}{pqr-1}, q\frac{1+r+rp}{pqr-1}, r\frac{1+p+pq}{pqr-1} \right\}. \tag{4.6}$$

Then there exist $\vartheta_1, \vartheta_2, \vartheta_3 > 0$ and an infinite number of trios of positive constants (A_1, A_2, A_3) such that (u,v,w) given by (4.5) is a solution to the system (3.6).

4.1 Decompositions of $\mathbb{R}^N$

In order to prove Theorems 4.0.1 and 4.0.2 we apply the following lemmas.

Lemma 4.1.1. *Let $s, t > 0$ such that $t < 2s < N$. Then there exists a constant $C > 0$ such that*

$$\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\, dy \leq \frac{C}{(1+|x|^2)^{s-t/2}}, \quad x \in \mathbb{R}^N. \tag{4.7}$$

We recall the following inequality between real numbers that we need in the proof: let $a_1, \dots, a_k \geq 0$, $s_1, \dots, s_k > 0$ and $S = \sum_{i=1}^k s_i$. Then

$$\prod_{i=1}^{k} a_i^{s_i} \leq \left(\max_{i \in \{1,\dots,k\}} a_i \right)^S \leq \sum_{i=1}^{k} a_i^S. \tag{4.8}$$

Proof of Lemma 4.1.1. First, we consider $x \leq 1$. We use the following decomposition of $\mathbb{R}^N$:

$$\mathbb{R}^N = A_0 \cup A_1 \cup A_2,$$

where

$$\begin{aligned} A_0 &= B_1(x), \\ A_1 &= B_1 \setminus B_1(x), \\ A_2 &= \mathbb{R}^N \setminus (B_1 \cup B_1(x)). \end{aligned}$$

We have that

$$\begin{aligned} \int\limits_{A_0} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\, dy &\leq \int\limits_{B_1(x)} \frac{1}{|x-y|^{N-t}}\, dy \leq C, \\ \int\limits_{A_1} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\, dy &\leq \int\limits_{B_1} 1\, dy \leq C. \end{aligned} \tag{4.9}$$

Applying the inequality (4.8), we obtain

$$\begin{aligned} \int\limits_{A_2} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\, dy &\leq \int\limits_{A_2} \frac{1}{|y|^{N-t+2s}}\, dy + \int\limits_{A_2} \frac{1}{|x-y|^{N-t+2s}}\, dy \leq \\ \leq \int\limits_{\mathbb{R}^N \setminus B_1} \frac{1}{|y|^{2s+N-t}}\, dy &+ \int\limits_{\mathbb{R}^N \setminus B_1(x)} \frac{1}{|y-x|^{2s+N-t}}\, dy = C \int\limits_1^{+\infty} \frac{1}{r^{2s+N-t}} r^{N-1} dr \leq C. \end{aligned} \tag{4.10}$$

Using (4.9) and (4.10) we have

$$\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\, dy \leq C \leq \frac{C}{(1+|x|^2)^{s-t/2}}, \quad x \leq 1.$$

Next, we suppose $|x| > 1$. Now, we consider the following decomposition of $\mathbb{R}^N$:

$$\mathbb{R}^N = D_0 \cup D_1 \cup D_2,$$

where

$$D_0 = B_{|x|/2}(x),$$
$$D_1 = B_{1/2} \setminus B_{|x|/2}(x),$$
$$D_2 = \mathbb{R}^N \setminus (B_{|x|/2}(x) \cup B_{1/2}).$$

If $y \in D_0$ then $|y| \geq |x| - |y - x| \geq |x|/2$. Hence,

$$\begin{aligned} \int\limits_{D_0} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\, dy &\leq \frac{C}{(1+|x|^2)^s} \int\limits_{B_{|x|/2}(x)} \frac{1}{|x-y|^{N-t}}\, dy = \\ = \frac{C}{(1+|x|^2)^s} \int\limits_0^{|x|/2} \frac{1}{r^{N-t}} r^{N-1}\, dr &= \frac{C}{(1+|x|^2)^s}|x|^t \leq \frac{C}{(1+|x|^2)^{s-t/2}}. \end{aligned} \tag{4.11}$$

If $y \in D_1$ then

$$|x-y| \geq \frac{|x|}{2} = \frac{1}{2}\left(\frac{|x|^2}{2} + \frac{|x|^2}{2}\right)^{1/2} \geq C(1+|x|^2)^{1/2}.$$

Hence,

$$\int\limits_{D_1} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\, dy \leq \frac{C}{(1+|x|^2)^{\frac{N-t}{2}}} \int\limits_{B_{1/2}} \frac{C}{(1+|y|^2)^s}\, dy = \frac{C}{(1+|x|^2)^{\frac{N-t}{2}}}. \tag{4.12}$$

Next, we consider the following decomposition of the set D_2:

$$D_2 = H_1 \cup H_2,$$

where

$$H_1 = D_2 \setminus B_{|x|/2} = \left(\mathbb{R}^N \setminus (B_{|x|/2}(x) \cup B_{|x|/2})\right), \qquad H_2 = D_2 \cap B_{|x|/2}.$$

By straightforward calculation, we have

$$\begin{aligned} \int\limits_{H_1} \frac{1}{|x-y|^{N-t+2s}}\, dy &= \int\limits_{\mathbb{R}^N \setminus B_{|x|/2}(x)} \frac{1}{|x-y|^{N-t+2s}}\, dy \leq \\ \leq C \int\limits_{|x|/2}^{+\infty} \frac{1}{r^{N-t+s}} r^{N-1}\, dr &= \frac{C}{|x|^{2s-t}} \leq \frac{C}{(1+|x|^2)^{s-t/2}}, \end{aligned} \tag{4.13}$$

and

$$\int\limits_{H_1} \frac{1}{|y|^{N-t+2s}}\,dy \le \int\limits_{\mathbb{R}^N\setminus B_{|x|/2}} \frac{1}{|y|^{N-t+2s}}\,dy \le \frac{C}{(1+|x|^2)^{s-t/2}}. \tag{4.14}$$

Combining the inequality (4.8) with (4.13) and (4.14) we obtain

$$\begin{aligned}\int\limits_{H_1} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\,dy \le \int\limits_{H_1} \frac{1}{|x-y|^{N-t+2s}}\,dy + \int\limits_{H_1} \frac{1}{|y|^{N-t+2s}}\,dy \le \\ \le \frac{C}{(1+|x|^2)^{s-t/2}}.\end{aligned} \tag{4.15}$$

On the other hand, if $y \in H_2$ then $|x-y| \ge |y|$ and $|x-y| \ge |x|-|y| \ge |x|/2$. Hence,

$$\frac{1}{|x-y|^{N-t}} \le \frac{C}{|x|^{N-t-(N-2s-1)}}\frac{1}{|y|^{N-2s-1}} = \frac{C}{|x|^{2s-t+1}}\frac{1}{|y|^{N-2s-1}}. \tag{4.16}$$

Using the estimate (4.16), we get

$$\begin{aligned}\int\limits_{H_2} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\,dy \le \frac{C}{|x|^{2s-t+1}} \int\limits_{B_{|x|/2}\setminus B_{1/2}} \frac{1}{|y|^{N-1}}\,dy = \\ = \frac{C}{|x|^{2s-t+1}}(|x|-1) \le \frac{C}{(1+|x|^2)^{s-t/2}}.\end{aligned} \tag{4.17}$$

Therefore, by (4.15) and (4.17) we have that

$$\int\limits_{D_2} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\,dy \le \frac{C}{|x|^{2s-t+1}}(|x|-1) \le \frac{C}{(1+|x|^2)^{s-t/2}}. \tag{4.18}$$

Finally, combining (4.11), (4.12) and (4.18) we obtain

$$\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-t}(1+|y|^2)^s}\,dy \le \frac{C}{(1+|x|^2)^{s-t/2}} + \frac{C}{(1+|x|^2)^{\frac{N-t}{2}}}. \tag{4.19}$$

Moreover, since $t < 2s < N$, (4.19) leads to our desidered result. □

Lemma 4.1.2. *Let $0 < \alpha, \beta, \gamma < N$ and $\chi := \alpha + \beta + \gamma$ such that $N < \chi < 2N$. If $s_1, s_2 > 0$ satisfy the following condition:*

$$\chi - N < 2s_1, 2s_2 < N, \tag{4.20}$$

then there exist $C > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{s_1}(1+|z|^2)^{s_2}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \leq \\ &\leq \min\left\{\frac{C}{(1+|x|^2)^{s_1-\frac{\chi-N}{2}}}, \frac{C}{(1+|x|^2)^{s_2-\frac{\chi-N}{2}}}\right\}, \end{aligned} \tag{4.21}$$

for every $x \in \mathbb{R}^N$.

Proof. First, we prove that.

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{s_1}(1+|z|^2)^{s_2}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \leq \frac{C}{(1+|x|^2)^{s_1-\frac{\chi-N}{2}}}. \tag{4.22}$$

We consider the following decomposition of $\mathbb{R}^N$:

$$\mathbb{R}^N = D_0 \cup D_1 \cup D_2 \cup D_3,$$

where

$$\begin{aligned} D_0 &:= B_{|x-z|/2}(x), \\ D_1 &:= B_{|x-z|/2}(z), \\ D_2 &:= B_{|x-z|/2} \setminus (D_0 \cup D_1), \\ D_3 &:= \mathbb{R}^N \setminus (D_0 \cup D_1 \cup D_2). \end{aligned}$$

If $y \in D_0$, then $|y-z| \geq |x-z| - |y-x| \geq |x-z|/2$. Hence,

$$\begin{aligned} \int_{D_0} \frac{1}{|x-y|^{N-\alpha}|y-z|^{N-\beta}(1+|y|^2)^{s_1}}\,dy &\leq \frac{2}{|x-z|^{N-\beta}} \int_{B_{|x-z|/2}(x)} \frac{1}{|x-y|^{N-\alpha}}\,dy = \\ &= \frac{C}{|x-z|^{N-\beta}} \int_0^{|x-z|/2} \frac{1}{r^{N-\alpha}} r^{N-1}\,dr = \frac{C}{|x-z|^{N-\alpha-\beta}}. \end{aligned} \tag{4.23}$$

If $y \in D_1$ then $|x-y| \geq |x-z| - |y-z| \geq |x-z|/2$. Hence,

$$\int\limits_{D_1} \frac{1}{|x-y|^{N-\alpha}|y-z|^{N-\beta}(1+|y|^2)^{s_1}}\, dy \leq \frac{C}{|x-z|^{N-\alpha-\beta}}. \tag{4.24}$$

If $y \in D_2$, then $|y-z| \geq |x-z|/2$ and $|y-x| \geq |x-z|/2$. It follows that

$$\begin{aligned} \int\limits_{D_2} \frac{1}{|x-y|^{N-\alpha}|y-z|^{N-\beta}(1+|y|^2)^{s_1}}\, dy &\leq \frac{C}{|x-z|^{2N-\alpha-\beta}} \int\limits_{B_{|x-z|/2}} 1\, dy = \\ &= \frac{C}{|x-z|^{N-\alpha-\beta}}. \end{aligned} \tag{4.25}$$

By the inequality (4.8), as well as $2N - \chi + 2s_1 > 0$, we have

$$\begin{aligned} &\int\limits_{D_3} \frac{1}{|x-y|^{N-\alpha}|y-z|^{N-\beta}(1+|y|^2)^{s_1}}\, dy \leq \int\limits_{D_3} \frac{1}{|x-y|^{N-\alpha}|y-z|^{N-\beta}|y|^{2s_1}}\, dy \leq \\ &\leq \int\limits_{D_3} \frac{1}{|x-y|^{2N-\alpha-\beta+2s_1}}\, dy + \int\limits_{D_3} \frac{1}{|y-z|^{2N-\alpha-\beta+2s_1}}\, dy + \int\limits_{D_3} \frac{1}{|y|^{2N-\alpha-\beta+2s_1}}\, dy \leq \\ &\leq \int\limits_{\mathbb{R}^N \setminus B_{|x-z|/2}(x)} \frac{1}{|x-y|^{2N-\alpha-\beta+2s_1}}\, dy + \int\limits_{\mathbb{R}^N \setminus B_{|x-z|/2}(z)} \frac{1}{|y-z|^{2N-\alpha-\beta+2s_1}}\, dy + \\ &+ \int\limits_{\mathbb{R}^N \setminus B_{|x-z|/2}} \frac{1}{|y|^{2N-\alpha-\beta+2s_1}}\, dy = C \int\limits_{|x-z|/2}^{+\infty} \frac{1}{r^{2N-\alpha-\beta+2s_1}} r^{N-1}\, dr = \frac{C}{|x-z|^{N-\alpha-\beta+2s_1}}. \end{aligned} \tag{4.26}$$

Using inequalities (4.23),(4.24), (4.25) and (4.26) we get

$$\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|^{N-\alpha}|y-z|^{N-\beta}(1+|y|^2)^{s_1}}\, dy \leq \frac{C}{|x-z|^{N-\alpha-\beta+2s_1}} + \frac{C}{|x-z|^{N-\alpha-\beta}}. \tag{4.27}$$

By Lemma 4.1.1 as well as (4.27), we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{s_1}(1+|z|^2)^{s_2}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \le$$
$$\le \int_{\mathbb{R}^N} \frac{C}{(1+|z|^2)^{s_2}|x-z|^{2N-\chi+2s_1}}\,dz + \int_{\mathbb{R}^N} \frac{C}{(1+|z|^2)^{s_2}|x-z|^{2N-\chi}}\,dz \le \tag{4.28}$$
$$\le \frac{C}{(1+|x|^2)^{s_1+s_2-\frac{(\chi-N)}{2}}} + \frac{C}{(1+|x|^2)^{s_2-\frac{(\chi-N)}{2}}} \le \frac{C}{(1+|x|^2)^{s_2-\frac{(\chi-N)}{2}}}.$$

Then, the proof of (4.22) is completed. Changing the names of the variables in the inequality (4.22) we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{s_1}(1+|z|^2)^{s_2}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \le \frac{C}{(1+|x|^2)^{s_2-\frac{\chi-N}{2}}},$$

and this completes the proof. □

Proof of Theorem 4.0.1. Without loss of generality, we suppose that (4.2) and (4.3) are satisfied. Let $\vartheta_1, \vartheta_2, \vartheta_3$ such that

$$\chi - N < 2\vartheta_1 p, 2\vartheta_2 q, 2\vartheta_3 r < N, \tag{4.29}$$

and $A_1, A_2, A_3 > 0$. By Lemma 4.1.2, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_2^q A_3^r}{(1+|y|^2)^{\vartheta_2 q}(1+|z|^2)^{\vartheta_3 r}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \le \frac{CA_2^q A_3^r}{(1+|x|^2)^{\vartheta_2 q-\frac{\chi-N}{2}}},$$
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_1^p A_3^r}{(1+|y|^2)^{\vartheta_1 p}(1+|z|^2)^{\vartheta_3 r}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \le \frac{CA_1^p A_3^r}{(1+|x|^2)^{\vartheta_1 p-\frac{\chi-N}{2}}},$$
$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_1^p A_2^q}{(1+|y|^2)^{\vartheta_1 p}(1+|z|^2)^{\vartheta_2 q}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz \le \frac{CA_1^p A_2^q}{(1+|x|^2)^{\vartheta_1 p-\frac{\chi-N}{2}}}. \tag{4.30}$$

In order to complete the proof we need to find A_1, A_2, A_3 such that

$$A_1 \ge CA_2^q A_3^r, \quad A_2 \ge CA_1^p A_3^r, \quad A_3 \ge CA_1^p A_2^q, \tag{4.31}$$

and $\vartheta_1, \vartheta_2, \vartheta_3$ such that both (4.29) and the following conditions

$$\vartheta_1 \le \vartheta_2 q - \frac{\chi - N}{2}, \quad \vartheta_2 \le \vartheta_1 p - \frac{\chi - N}{2}, \quad \vartheta_3 \le \vartheta_1 p - \frac{\chi - N}{2}. \tag{4.32}$$

We observe that $(\vartheta_1, \vartheta_2, \vartheta_3)$ given by

$$\vartheta_1 = \frac{\chi - N}{2}\frac{q+1}{pq-1}, \quad \vartheta_2 = \vartheta_3 = \frac{\chi - N}{2}\frac{p+1}{pq-1}, \tag{4.33}$$

satisfy both (4.29) and (4.32).

Finally, it is easy to check that there exists a constant $\delta > 0$ such that the trios

$$(A_1, A_2, A_3) := (C^{-\frac{q+1}{pq-1}} A_3^{-\frac{r(q+1)}{pq-1}}, C^{-\frac{p+1}{pq-1}} A_3^{-\frac{r(p+1)}{pq-1}}, A_3), \quad A_3 \in [\delta, +\infty), \tag{4.34}$$

are solutions to

$$A_1 = C A_2^q A_3^r, \quad A_2 = C A_1^p A_3^r, \quad A_3 \ge C A_1^p A_2^q.$$

For the details see Appendix.

This leads us to the desired result. □

Proof of Theorem 4.0.2. Since $pqr > 1$, then one of the following conditions holds:

$$pq > 1, \quad pr > 1, \quad qr > 1.$$

Without loss of generality, we suppose $pq > 1$, then we proceed as in the proof of Theorem 4.0.1: we consider $\vartheta_1, \vartheta_2, \vartheta_3$ satisfying (4.29) and $A_1, A_2, A_3 > 0$. Then, applying Lemma 4.1.2, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_2^q A_3^r}{(1+|y|^2)^{\vartheta_2 q}(1+|z|^2)^{\vartheta_3 r}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz &\le \frac{C A_2^q A_3^r}{(1+|x|^2)^{\vartheta_2 q - \frac{\chi-N}{2}}},\\
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_1^p A_3^r}{(1+|y|^2)^{\vartheta_1 p}(1+|z|^2)^{\vartheta_3 r}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz &\le \frac{C A_1^p A_3^r}{(1+|x|^2)^{\vartheta_3 r - \frac{\chi-N}{2}}},\\
\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{A_1^p A_2^q}{(1+|y|^2)^{\vartheta_1 p}(1+|z|^2)^{\vartheta_2 q}|x-y|^{N-\alpha}|y-z|^{N-\beta}|z-x|^{N-\gamma}}\,dy\,dz &\le \frac{C A_1^p A_2^q}{(1+|x|^2)^{\vartheta_1 p - \frac{\chi-N}{2}}}.
\end{aligned} \tag{4.35}$$

Choosing $(\vartheta_1, \vartheta_2, \vartheta_3)$ given by

$$\vartheta_1 = \frac{\chi - N}{2}\frac{qr+q+1}{pq-1}, \quad \vartheta_2 = \frac{\chi - N}{2}\frac{pr+r+1}{pq-1}, \quad \vartheta_3 = \frac{\chi - N}{2}\frac{pq+p+1}{pq-1}, \tag{4.36}$$

both (4.29) and the following conditions are satisfied

$$\vartheta_1 \le \vartheta_2 q - \frac{\chi - N}{2}, \quad \vartheta_2 \le \vartheta_3 r - \frac{\chi - N}{2}, \quad \vartheta_3 \le \vartheta_1 p - \frac{\chi - N}{2}.$$

Hence, by (4.35), (u, v, w) defined by (4.5), with A_1, A_2, A_3 and $\vartheta_1, \vartheta_2, \vartheta_3$ given by (4.34) and (4.36) respectively, is a solution to (3.6). □

4.2 Appendix

In this section we check that there exists an infinite number of trios (A_1, A_2, A_3) of positive real numbers such that

$$A_1 = C A_2^q A_3^r, \tag{4.37}$$

$$A_2 = C A_1^p A_3^r, \tag{4.38}$$

$$A_3 \ge C A_1^p A_2^q. \tag{4.39}$$

Substituting (4.37) in (4.38) we have

$$C^{p+1} A_2^{pq-1} A_3^{r(p+1)} = 1,$$

that is,

$$A_2 = C^{-\frac{p+1}{pq-1}} A_3^{-\frac{r(p+1}{pq-1})}. \tag{4.40}$$

Similarly, substituting (4.40) in (4.37) we obtain

$$A_1 = C^{1-\frac{q(p+1)}{pq-1}} A_3^{r-\frac{qr(p+1}{pq-1})} = C^{-\frac{q+1}{pq-1}} A_3^{-\frac{r(q+1}{pq-1})}. \tag{4.41}$$

Combining (4.40) and (4.41) with (4.39) we get

$$A_3^{\frac{pr(q+1)}{pq-1}+\frac{qr(p+1)}{pq-1}+1} \ge C^{1-\frac{p(q+1)}{pq-1}-\frac{p(q+1)}{pq-1}} = C^{-\frac{(p+1)(q+1)}{pq-1}}, \tag{4.42}$$

i.e.

$$A_3 \ge C^{-b}, \tag{4.43}$$

where

$$b = \frac{\frac{(p+1)(q+1)}{pq-1}}{\frac{pr(q+1)}{pq-1} + \frac{qr(p+1)}{pq-1} + 1} = \frac{(p+1)(q+1)}{2pqr + pq + qr + pr - 1} > 0.$$

The trios

$$(C^{-\frac{q+1}{pq-1}} A_3^{-\frac{r(q+1)}{pq-1}}, C^{-\frac{p+1}{pq-1}} A_3^{-\frac{r(p+1)}{pq-1}}, A_3), \quad A_3 \in [C^{-b}, +\infty),$$

satisfy conditions (4.37), (4.38) and (4.39).

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