Fast rotation and inviscid limits for the SQG equation with general ill-prepared initial data

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Abstract. In the present paper, we study the fast rotation and inviscid limits for the 2-D dissipative surface quasi-geostrophic equation with a dispersive forcing term, in the domain $\Omega = \mathbb{T}^1 \times \mathbb{R}$. In the case when we perform the fast rotation limit (keeping the viscosity fixed), in the context of general ill-prepared initial data, we prove that the limit dynamics is described by a linear equation with parabolic structure. Conversely, performing the combined fast rotation and inviscid limits, we show that the means of the target initial datum $\overline{\vartheta}_0$ are conserved along the motion. The proof of the convergence is based on a compensated compactness argument which allows, on the one hand, to get compactness properties for suitable quantities hidden in the wave system and, on the other hand, to exclude the oscillatory part of waves at the limit.

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1. Introduction

In this paper, we are interested in the description of the surface temperature ϑ on the ocean. The importance of this model is justified from the geophysical point of view in Chapter 19 of [30] and in Chapter 20 of [30] where the appearing of nonlinear phenomena is discussed. We refer also to Chapter 16 of [13] for the description of the quasi-geostrophic dynamics (in particular Section 16.7 is devoted to the quasi-geostrophic ocean model) and to Chapter 20 of [13] for physical insights on the oceanic circulation. We consider the two-dimensional surface quasi-geostrophic (SQG) equation with dissipation determined by a fractional Laplacian and a dispersive forcing term in the domain $\Omega = \mathbb{T}^1 \times \mathbb{R}$ given by:

$$\begin{cases} \partial_t \vartheta + \operatorname{div}(\vartheta \boldsymbol{u}) + \nu\Lambda\vartheta + A\mathcal{R}_1\vartheta = 0 \\ \boldsymbol{u} = \mathcal{R}^{\perp}\vartheta := (-\mathcal{R}_2\vartheta, \mathcal{R}_1\vartheta) \\ \vartheta_{|t=0} = \vartheta_0, \end{cases} \tag{1.1}$$

where ϑ is a real-valued scalar function representing the ocean's surface temperature, $\boldsymbol{u}$ is the divergence-free velocity field of the fluid and A represents the amplitude parameter for the dispersive forcing term $\mathcal{R}_1\vartheta$. To define the fractional Laplacian operator $\Lambda := \sqrt{-\Delta}$ of the dissipative term $\nu\Lambda\vartheta$, we can adapt the Fourier transform for functions $f(x_1, x_2) \in L^1(\Omega)$ with $(n_1, \xi_2) \in \mathbb{Z} \times \mathbb{R}$, so that

$$\hat{f}(n_1, \xi_2) = \int_{\mathbb{Z}\times\mathbb{R}} e^{-i(n_1 x_1 + \xi_2 x_2)} f(x_1, x_2)\, dx_1\, dx_2,$$

and Λ is then defined by

$$\widehat{\Lambda f}(n_1, \xi_2) = (n_1^2 + \xi_2^2)^{1/2}\, \widehat{f}(n_1, \xi_2) := |\tilde{\xi}|\widehat{f}(n_1, \xi_2).$$

Further, we have $\nu > 0$ and $\mathcal{R}_i := \partial_i \Lambda^{-1}$, for $i = 1, 2$, are the usual Riesz transforms. The dissipative term $\nu\Lambda\vartheta$ comes from the Ekman pumping mechanism (we refer to [22, 25, 27] for details); while the presence of an environmental horizontal gradient $\mathcal{R}_1\vartheta = \partial_1\Lambda^{-1}\vartheta$ represents the advection of a large-scale buoyancy coming from the meridional variation of the Coriolis force. The parameter $A > 0$ is the analogue of the Rossby number which determines the typically large weight of the Coriolis term in geophysical fluid-dynamics. The mathematical analysis of (1.1) started with the work [23] by Kiselev and Nazarov, where the existence of smooth solutions in the torus was proved.

It is worth noting that, due to technical reasons, we need to work with the domain $\Omega = \mathbb{T}^1 \times \mathbb{R}$ so that non-trivial test functions are still allowed by the compact support condition (see also Remark 4.1 below).

In the case without the presence of the dispersive forcing term $A\mathcal{R}_1\vartheta$, the non-dissipative SQG equations ($\nu = 0$) are the two-dimensional analogue of the 3D Euler equations in vorticity form (see e.g. [11]), while the SQG equations (with $\nu \neq 0$) are analogous to the 3D Navier–Stokes system (see [24]). Due to this analogy, the global regularity of the SQG equations has been intensively studied in recent decades, and for an overview of these equations we mention [12] and references therein.

Moreover, the presence of the dispersive forcing term makes equations (1.1) analogous to the Navier–Stokes-Coriolis (NSC) system, which is a fundamental geophysical model dealing with large-scale phenomena. We refer the interested reader to [1–3] for the pioneering studies on the Navier–Stokes-Coriolis equations and to [10] for a more in-depth survey of fundamental results concerning rotating fluids. The main advantage of the NSC system is that in the limit of vanishing Rossby number, the fast rotating term "produces" a stabilization effect which ensures the global well-posedness of strong solutions with large initial data, unlike the case of the Navier–Stokes equations. In particular, in [10] this was proved by establishing Strichartz estimates, which show how the dispersive phenomena weaken the non-linearity and stabilize NSC towards a 2D Navier–Stokes type system. Following the analogy between the Navier–Stokes-Coriolis and SQG systems, a similar result for the supercritical dispersive SQG equation was shown in [9], where in (1.1) the dissipative term is represented by $\nu\Lambda^{\alpha}\vartheta$, $\alpha < 1$. In this context, we mention also the work [24] (in the context of well-prepared initial data), where the main tool employed is the relative energy inequality, in order to treat the inviscid incompressible limit of the SQG system with large rotation (see [8,18]). Furthermore, the method used in [24] allows to obtain the inviscid limit with fixed or no dispersion, providing an alternative to previous results [5,32] for the SQG equation without the dispersive forcing term $A\mathcal{R}_1\vartheta$.

Motivated by the previous discussion, our goal is to perform the fast rotation and inviscid limits in the more general framework of *ill-prepared* initial data (see e.g. Section 4.6 of [19]).

The so-called *ill-prepared* initial data are general data which do *not* require any strong convergence to the initial data of the target systems, or do *not* require any *a priori* structural conditions. Instead, initial data which satisfy the previous conditions are called *well-prepared* initial data (e.g. as in [9] where the initial data have a special structure, or as in [24] where the initial data of the primitive system strongly converge to the initial data of the target system).

In this direction, we remark that, in the domain $\Omega = \mathbb{T}^1 \times \mathbb{R}$, the stability of the SQG system without dispersive forcing and with horizontal dissipation, was recently proved in [31]. Thus, this result shows that periodic boundary conditions allow for a decomposition into the mean flow and its oscillations, in which global existence, and indeed, stability, can be obtained in the Sobolev space $H^2(\Omega)$, without explicit recourse to dispersive effects.

In our work, first of all, we study the regime when the rotational effects are predominant in the dynamics, keeping fixed the viscous coefficient $\nu > 0$, i.e.

$$A = \frac{1}{\varepsilon}, \tag{1.2}$$

for a given $\varepsilon \in]0,1]$. Next, we analyse the combined fast rotation and inviscid limits where

$$A = \frac{1}{\varepsilon} \qquad \text{and} \qquad \nu = \nu(\varepsilon) = \varepsilon^{\beta}, \quad \beta > 0. \tag{1.3}$$

The scaling for ν in (1.3) is motivated by the physical Stommel boundary layer model for the western intensification of oceanic currents, where the Ekman pumping dissipation must be taken into account. In the inviscid limit case, the exponent of ε would typically belong to the interval $]-1,-2/3[$ (see Section 4 in [16]), but making a suitable rescaling of the parameter ε in the equations leads to the positive exponent β introduced in (1.3). Note also that we disregard the frictional sublayer needed to adjust the no-slip boundary conditions (for more details, we refer to [16,21]).

In order to prove our results and get the improvement to the more general ill-prepared data with respect to [9, 24] where the initial data are well-prepared, we employ an approach based on a compensated compactness argument. This technique was first exploited to analyze the barotropic Navier-Stokes equations in [26] and later developed in the fast rotation, incompressible and homogeneous framework in [20]. We mention also the works [14, 15] and references therein, in which the compensated compactness method has been recently applied to NSC-type systems (see in addition [17] and references therein).

In this paper, we specifically employ the algebraic structure hidden behind the system (recasted as a wave system) to find compactness properties for the means of temperature ϑ_ε with respect to the x_1-variable. Nonetheless, the dispersive forcing term $A\mathcal{R}_1\vartheta$ is responsible for strong in time oscillations of solutions which may prevent the convergence of the non-linearities. To overcome this issue, we inspect the structure of the wave system and the special form of the test functions which allow us to avoid the interaction of wave oscillations in the limit and then provide us with the description of the target dynamics.

The strategy employed is a standard matter in the context of singular perturbations problems and it consists of the following steps:

(i) to develop an existence theory, which holds for any value of $\varepsilon > 0$ fixed;
(ii) to state uniform bounds for the family of solutions in order to extract weak limit points;
(iii) to find the constraints that the limit points have to satisfy;
(iv) to pass to the limit for test functions in the kernel of the singular perturbation operator.

In this way, on one hand, in the fast rotation regime we will obtain convergence of the primitive system towards a linear equation with a parabolic structure; while on the other hand, in the combined fast rotation and inviscid limits, we will show that the target dynamics is trivial in the sense that the dissipative and dispersive terms will not contribute to the limit and the means of the target initial temperature $\overline{\vartheta}_0$ are conserved along the motion.

Remark 1.1. The general case, where one considers $\nu\Lambda^\alpha\vartheta$ with $\alpha \in]0,1[$ and ν fixed or $\nu = \nu(\varepsilon) = \varepsilon^\beta$ with $\beta > 0$, can be easily treated adapting the analysis developed in this paper with no essential troubles (see also Remark 2.7 below).

Let us now give an overview of the paper. In Sect. 2 we collect our assumptions and we state our main results. In Sect. 3 we study the singular perturbation part of the equations, recalling the uniform bounds on our family of weak solutions and establishing constraints that the limit points have to satisfy. Section 4 is devoted to the proof of the convergence results for the fast rotation limit and the combined fast rotation and inviscid limits.

Some notation and conventions. Let $B \subset \mathbb{R}^2$. The symbol $C_c^\infty(B)$ denotes the space of ∞-times continuously differentiable functions on $\mathbb{R}^2$ and having compact support in B. The dual space $\mathcal{D}'(B)$ is the space of distributions on B. Given $p \in [1, +\infty]$, by $L^p(B)$ we mean the classical space of Lebesgue measurable functions g, where $|g|^p$ is integrable over the set B (with the usual modifications for the case $p = +\infty$). We use also the notation $L^p_T(L^q)$ to indicate the space $L^p\big([0,T];L^q(B)\big)$, with $T>0$. Given $k\geq 0$, we denote by $W^{k,p}(B)$ the Sobolev space of functions which belongs to $L^p(B)$ together with all their derivatives up to order k. When $p=2$, we alternately use the notation $W^{k,2}(B)$ and $H^k(B)$. We denote by $\dot W^{k,p}(B)$ the corresponding homogeneous Sobolev spaces, i.e. $\dot W^{k,p}(B) = \{g\in L^1_{\rm loc}(B) \,:\, D^j g\in L^p(B),\ |j|=k\}$. Recall that $\dot W^{k,p}$ is the completion of $C^\infty_c(\overline{B})$ with respect to the L^p norm of the k-th order derivatives. For the sake of simplicity, we will omit from the notation the set B, that we will explicitly point out if needed.

Moreover, since we will deal with a periodic problem in the x_1-variable, we also introduce the following decomposition: for a vector-field $\boldsymbol{V}$, we write

$$\boldsymbol{V}(\boldsymbol{x}) = \langle \boldsymbol{V}\rangle(x_2) + \widetilde{\boldsymbol{V}}(\boldsymbol{x}) \qquad \text{with} \quad \langle \boldsymbol{V}\rangle(x_2) := \frac{1}{|\mathbb{T}^1|}\int_{\mathbb{T}^1} \boldsymbol{V}(x_1,x_2)\,{\rm d}x_1, \tag{1.4}$$

where $\mathbb{T}^1 := [-1,1]/\sim$ is the one-dimensional flat torus (here $\sim$ denotes the equivalence relation which identifies -1 and 1) and $\big|\mathbb{T}^1\big|$ denotes its Lebesgue measure.

In the whole paper, the symbols c and C will denote generic multiplicative constants, which may change from line to line, and which do not depend on the small parameter ε. Sometimes, we will explicitly point out the quantities that these constants depend on, by putting them inside brackets.

Let $\big(f_\varepsilon\big)_{0<\varepsilon\leq 1}$ be a sequence of functions in a normed space X. If this sequence is bounded in X, we use the notation $\big(f_\varepsilon\big)_\varepsilon\subset X$.

2. Setting of the SQG problem and main statements

In this section, we formulate our working hypotheses (see Sect. 2.1) and we state our main results (in Sect. 2.2).

2.1. Formulation of the problem

In this subsection, we present the rescaled SQG system with the dispersive forcing term, which we are going to consider in our study, and we formulate the main working hypotheses. The material of this part is mostly classical: unless otherwise specified, we refer to [24] for details.

2.1.1. Primitive system. To begin with, let us introduce the "primitive system", i.e. the rescaled SQG system, supplemented with the scaling (1.2) presented in the introduction, where $\varepsilon\in\,]0,1]$ is a small parameter. Thus, the system consists of the momentum equation and the quasi-geostrophic balance: respectively,

$$\partial_t\vartheta_\varepsilon + {\rm div}\,(\vartheta_\varepsilon\boldsymbol{u}_\varepsilon) + \nu\Lambda\vartheta_\varepsilon + \frac{1}{\varepsilon}\mathcal{R}_1\vartheta_\varepsilon = 0 \tag{2.1}$$

$$\boldsymbol{u}_\varepsilon = \mathcal{R}^\perp\vartheta_\varepsilon := \left(-\mathcal{R}_2\vartheta_\varepsilon, \mathcal{R}_1\vartheta_\varepsilon\right). \tag{2.2}$$

The unknown is the fluid surface temperature $\vartheta_\varepsilon = \vartheta_\varepsilon(t,x)$, with $t \in \mathbb{R}_+$ and $x \in \Omega$. The velocity vector field in $\mathbb{R}^2$ is indicated by $\boldsymbol{u}_\varepsilon = \boldsymbol{u}_\varepsilon(t,x)$.

Remark 2.1. In the case of scaling (1.3), we replace ν with ε^β, $\beta > 0$.

2.1.2. Initial data and finite energy weak solutions. We address the singular perturbation problem described in Sect. 2.1.1, with scaling (1.2), for general *ill prepared initial data*, in the framework of *finite energy weak solutions* (see e.g. [9]). Since we work with weak solutions based on dissipation estimates, we need to assume that the initial data satisfy the following bound

$$\sup_{\varepsilon \in \,]0,1]} \|\vartheta_{0,\varepsilon}\|_{L^2(\Omega)} \leq C. \tag{2.3}$$

Thanks to the previous uniform estimate, up to extraction, we can argue that

$$\overline{\vartheta}_0 := \lim_{\varepsilon\to 0} \vartheta_{0,\varepsilon}\ , \tag{2.4}$$

where we agree that the previous limit is taken in the weak topology of $L^2(\Omega)$.

Let us specify better what we mean for *finite energy weak solution* (see [10] for details).

Definition 2.2. We say that ϑ_ε is a *weak solution* to the dispersive SQG system (2.1)–(2.2) in $[0,T[\,\times\Omega$ (for some time $T>0$) with the initial condition $\vartheta_{0,\varepsilon}$, if:

(i) $\vartheta_\varepsilon \in L^\infty([0,T[\,;L^2(\Omega)) \cap L^2([0,T[\,;\dot H^{1/2}(\Omega))$;

(ii) the momentum equation is satisfied in a weak sense: for any $\varphi \in C_c^\infty([0,T[\,\times\Omega)$, one has

$$\begin{aligned} &-\int_0^T\int_\Omega \left(\vartheta_\varepsilon\,\partial_t\varphi + \vartheta_\varepsilon\boldsymbol{u}_\varepsilon\cdot\nabla\varphi + \nu\Lambda^{1/2}\vartheta_\varepsilon\,\Lambda^{1/2}\varphi - \frac{1}{\varepsilon}\mathcal{R}_1\vartheta_\varepsilon\,\varphi\right)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_\Omega \theta_{0,\varepsilon}\,\varphi(0)\,\mathrm{d}x\,; \end{aligned} \tag{2.5}$$

(iii) the quasi-geostrophic balance is satisfied in $\mathcal{D}'(]0,T[\,\times\Omega)$;

(iv) the solution satisfies the following energy estimate

$$\|\vartheta_\varepsilon(T)\|_{L^2}^2 + 2\nu\int_0^T \|\Lambda^{1/2}\vartheta_\varepsilon(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leq \|\vartheta_{0,\varepsilon}\|_{L^2}^2 \quad \text{for all} \quad T>0. \tag{2.6}$$

The solution is *global* if the previous conditions are satisfied for all $T>0$.

Under the previous assumptions (2.3) and (2.4), at any *fixed* value of the parameter $\varepsilon \in]0,1]$, the existence of a global in time finite energy weak solution ϑ_ε to system SQG, related to the initial data $\vartheta_{0,\varepsilon}$, has been proved in e.g. [28] (see also [9] in this respect).

Remark 2.3. The term involving the Riesz transform $\mathcal{R}_1$ does not contribute to the energy estimate (2.6), because for any $s \in \mathbb{R}$, it holds

$$\begin{aligned} \langle\Lambda^s\mathcal{R}_1\theta,\ \Lambda^s\theta\rangle_{L^2} &= \int_\Omega \Lambda^s\mathcal{R}_1\theta\,\overline{\Lambda^s\theta}\,\mathrm{d}x = -\int_\Omega i\boldsymbol{n}_1|\tilde\xi|^{s-1}\hat\theta\,\overline{|\tilde\xi|^s\hat\theta}\,\mathrm{d}\tilde\xi \\ &= -\langle\Lambda^s\theta,\ \Lambda^s\mathcal{R}_1\theta\rangle_{L^2}. \end{aligned} \tag{2.7}$$

2.2. Main results

We can now state our main results. The first statement concerns the case when the rotational and viscosity effects, with fixed ν, are predominant in the dynamics.

Theorem 2.4. *For any fixed value of $\varepsilon \in]0,1]$, assume the initial data $\vartheta_{0,\varepsilon}$ verifies the hypothesis in Sect. 2.1.2 and let ϑ_ε be a corresponding weak solution to system (2.1)–(2.2). Then, one has the following convergence property, for any $T > 0$*

$$\vartheta_\varepsilon \stackrel{*}{\rightharpoonup} \overline{\vartheta} \qquad \text{weakly-}* \ \text{ in } \qquad L^\infty_T(L^2(\Omega)) \cap L^2_T(\dot{H}^{1/2}(\Omega)), \tag{2.8}$$

with the property that $\partial_1 \overline{\vartheta} = 0$ a.e. in $\mathbb{R}_+ \times \Omega$.
In addition, $\overline{\vartheta}$ is a weak solution to the following linear equation in $\mathbb{R}_+ \times \Omega$

$$\partial_t \overline{\vartheta} + \nu \Lambda_2 \overline{\vartheta} = 0, \tag{2.9}$$

where $\Lambda_2 := \sqrt{-\partial_2^2}$ and (2.9) is supplemented with the initial condition $\overline{\vartheta}_{|t=0} = \overline{\vartheta}_0$, where $\overline{\vartheta}_0$ has been identified in (2.4).

The previous theorem characterizes the limit dynamics of system (2.1)–(2.2) when one consider a fast rotation regime. In contrast with [9,24], in Theorem 2.4, we are able to consider data which are *ill-prepared.*

Instead, in the fast rotation and inviscid limits, we need more regularity on ϑ_ε since we have to control $\Lambda\vartheta_\varepsilon$. In this case, the limit dynamics is trivial and tells us that the means of the limit temperature $\overline{\vartheta}$ are constant (in time). We recall that the notation $\langle f \rangle$ introduced in (1.4) represents the mean of the quantity f with respect to the x_1-variable.

Theorem 2.5. *For any fixed value of $\varepsilon \in]0,1]$, let the initial data $\vartheta_{0,\varepsilon} \subset H^s(\Omega)$ for $s > 2$. Let ϑ_ε be a corresponding solution to system (2.1)–(2.2) with scaling (1.3), that means $\nu = \nu(\varepsilon) = \varepsilon^\beta$ and $\beta > 0$. Then, there exists $T > 0$ such that one has the following convergence property, for $s > 2$*

$$\vartheta_\varepsilon \stackrel{*}{\rightharpoonup} \overline{\vartheta} \qquad \text{weakly-}* \text{ in } \qquad L^\infty_T(H^s(\Omega)). \tag{2.10}$$

Moreover, one deduces the relation

$$\partial_t \langle \overline{\vartheta} \rangle = 0,$$

with the initial condition $\langle \overline{\vartheta} \rangle_{|t=0} = \langle \overline{\vartheta}_0 \rangle$, where $\overline{\vartheta}_0$ has been identified in (2.4).

Remark 2.6. We point out that the condition $s > 2$ is necessary to have the embedding of Sobolev spaces H^s in the space $W^{1,\infty}$ of globally Lipschitz functions (see the Appendix A for more details in this respect).

Remark 2.7. With minor modifications the results of the previous theorems can be extended to the supercritical case, where one has $\nu\Lambda^\alpha\vartheta_\varepsilon$ with $0 < \alpha < 1$ in (2.1) (see [9] for connected results in the whole space).

3. Inspection of the singular perturbation

The purpose of this section is twofold. First of all, in Sect. 3.1 we recall the uniform bounds and further properties for our family of weak solutions. Then, we study the singular operator underlying to the primitive SQG equations, and determine constraints that the limit points of our family of weak solutions have to satisfy (see Sect. 3.2). We recall that from now on the viscous parameter $\nu > 0$ is kept fixed.

3.1. Uniform bounds

In this section we will state the uniform bounds on the sequence $(\vartheta_\varepsilon)_\varepsilon$.

Indeed, with the energy estimate (2.6) at hand, we can derive uniform bounds for our family of weak solutions. To begin with, we point out that, owing to the assumption (2.3), the right-hand side of (2.6) is *uniformly bounded* for all $\varepsilon \in]0,1]$. Then, one has

$$\sup_{T\in\mathbb{R}_+} \|\vartheta_\varepsilon(T)\|_{L^2(\Omega)} \leq c \tag{3.1}$$

$$\int_0^T \|\Lambda^{1/2}\vartheta_\varepsilon(\tau)\|^2_{L^2(\Omega)}\, \mathrm{d}\tau \leq c \quad \text{for all } T > 0\,. \tag{3.2}$$

Due to the relation (2.2) and the fact that the Riesz operators $\mathcal{R}_j$ are 0-th order operators, one gets also

$$\sup_{T\in\mathbb{R}_+} \|\boldsymbol{u}_\varepsilon(T)\|_{L^2} + \int_0^T \|\Lambda^{1/2}\boldsymbol{u}_\varepsilon(\tau)\|^2_{L^2}\, \mathrm{d}\tau \leq c, \tag{3.3}$$

where the generic constant c is independent of $\varepsilon > 0$.

In view of the previous properties, there exist $\overline{\vartheta}$, $\overline{\boldsymbol{u}} \in L^\infty_T(L^2)\cap L^2_T(H^{1/2})$ such that (up to the extraction of a suitable subsequence) one has

$$\vartheta_\varepsilon \overset{*}{\rightharpoonup} \overline{\vartheta} \qquad\qquad \text{and} \qquad\qquad \boldsymbol{u}_\varepsilon \overset{*}{\rightharpoonup} \overline{\boldsymbol{u}}. \tag{3.4}$$

3.2. Constraints on the limit dynamics

In this subsection, we establish some properties that the limit points of the family $\left(\vartheta_\varepsilon\right)_\varepsilon$ have to satisfy. These are static relations, which do not characterise the limit dynamics yet.

Proposition 3.1. *Let $(\vartheta_\varepsilon)_\varepsilon$ be a family of weak solutions, related to initial data $(\vartheta_{0,\varepsilon})_\varepsilon$ verifying hypothesis of Sect. 2.1.2. Let $\overline{\vartheta}$ be a limit point of the previous sequence as identified in Sect. 3.1. Then, one deduces the relation*

$$\partial_1\overline{\vartheta} = 0 \qquad\qquad a.e.\ in\ \mathbb{R}_+\times\Omega. \tag{3.5}$$

Proof. Let us consider the weak formulation of (2.1). We test it against $\varepsilon\varphi$ where φ is a compactly supported test function in $C^\infty_c(\mathbb{R}_+\times\Omega)$. Denoting $[0,T]\times K = \operatorname{supp}\varphi$ with $\varphi(T,\cdot) = 0$ (and $T > 0$), we have

$$-\varepsilon\int_0^T\int_K \left(\vartheta_\varepsilon\partial_t\varphi + \vartheta_\varepsilon\boldsymbol{u}_\varepsilon\cdot\nabla\varphi + \nu\Lambda^{1/2}\vartheta_\varepsilon\,\Lambda^{1/2}\varphi\right)\,\mathrm{d}x\mathrm{d}t + \int_0^T\int_K \mathcal{R}_1\vartheta_\varepsilon\,\varphi\,\mathrm{d}x\mathrm{d}t$$
$$= \varepsilon\int_K \vartheta_{0,\varepsilon}\varphi(0)\,\mathrm{d}x. \tag{3.6}$$

By uniform bounds (3.1) and (3.2), the first and third integrals on the left-hand side converge to 0, when tested against any smooth compactly supported φ. Thanks to the weak convergence of ϑ_ε and $\boldsymbol{u}_\varepsilon$ in $L^\infty_T(L^2) \cap L^2_T(L^4)$ by the uniform bounds in Sect. 3.1 and Sobolev embeddings, we get the convergence to 0 of the second integral on the left-hand side when tested against φ.

Analogously, thanks to the bound (2.3) on the initial data, also the term on the right-hand side of (3.6) vanishes.

Then, passing to the limit for $\varepsilon \to 0$, we find

$$\int_0^T \int_K \mathcal{R}_1 \overline{\vartheta}\, \varphi \, \mathrm{d}x\mathrm{d}t = 0,$$

for any test function $\varphi \in C^\infty_c(\mathbb{R}_+ \times \Omega)$, which in particular implies

$$\mathcal{R}_1 \overline{\vartheta} = 0 \qquad\qquad \text{a.e. in } \mathbb{R}_+ \times \Omega. \tag{3.7}$$

At this point, employing the Fourier transform the function needs to have the support localised at the point 0. This implies that it is a multiple of the Dirac delta distribution δ_0. However, in view of the L^2 regularity, the previous situation cannot happen and one can deduce that

$$\partial_1 \overline{\vartheta} = 0 \qquad\qquad \text{a.e. in } \mathbb{R}_+ \times \Omega. \tag{3.8}$$

This completes the proof of the proposition. □

4. Limit dynamics

4.1. Fast rotation regime

In this section, we will show the convergence of momentum equation (2.1) towards the linear equation (2.9) depicted in Theorem 2.4.

The uniform bounds of Sect. 3.1 are not enough for proving convergence in the weak formulation of the momentum equation: the main problem relies on identifying the weak limit of the convective term $\mathrm{div}\,(\vartheta_\varepsilon \boldsymbol{u}_\varepsilon)$.

In view of Proposition 3.1, we define the test-function ψ lying in the kernel of the singular perturbation operator, namely

$$\psi \in C^\infty_c([0,T[\times \Omega;\mathbb{R}) \quad \text{ such that } \quad \partial_1 \psi = 0. \tag{4.1}$$

Notice that, in order to pass to the limit in the weak formulation of the momentum equation and derive the limit system, it is enough to use test functions ψ as above.

Remark 4.1. Observe that the choice of $\Omega = \mathbb{T}^1 \times \mathbb{R}$ is fundamental for the fact that the compact support of test functions does not force them to be 0 everywhere.

At this point, we rewrite the convective term in the weak formulation

$$\begin{aligned} -\int_0^T \int_\Omega \vartheta_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla \psi \, \mathrm{d}x\,\mathrm{d}t &= -\int_0^T \int_{\mathbb{T}^1\times\mathbb{R}} \vartheta_\varepsilon (u_\varepsilon)_2 \, \partial_2 \psi \, \mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_{\mathbb{R}} \langle \vartheta_\varepsilon (u_\varepsilon)_2 \rangle \, \partial_2 \psi \, \mathrm{d}x_2\,\mathrm{d}t, \end{aligned} \tag{4.2}$$

where the test-function ψ has been introduced in (4.1) and the notation $\langle \vartheta_\varepsilon (u_\varepsilon)_2\rangle$ represents the mean of $\vartheta_\varepsilon (u_\varepsilon)_2$ with respect to the x_1-variable (see (1.4) in this respect).

In addition, to avoid the appearing of the (irrelevant) multiplicative constant in the equation (4.2), we have supposed that the torus $\mathbb{T}^1$ has been renormalised so that its Lebesgue measure is equal to 1.

The previous relation (4.2) points our attention to the term

$$\langle \vartheta_\varepsilon (u_\varepsilon)_2\rangle = \langle \vartheta_\varepsilon\rangle\langle (u_\varepsilon)_2\rangle + \langle \widetilde{\vartheta}_\varepsilon (\widetilde{u}_\varepsilon)_2\rangle, \tag{4.3}$$

where we have recast the notation presented in (1.4).

4.1.1. Analysis of the means. We start by finding compactness properties for $\langle \vartheta_\varepsilon\rangle$. Taking the mean in Eq. (2.1) and thanks to the fact that the mean of the Riesz term $\langle \mathcal{R}_1\vartheta_\varepsilon\rangle$ is zero, we get

$$\partial_t\langle \vartheta_\varepsilon\rangle = -\mathrm{div}\,\langle \vartheta_\varepsilon \boldsymbol{u}_\varepsilon\rangle - \nu\Lambda\langle \vartheta_\varepsilon\rangle. \tag{4.4}$$

To analyse the former term in the right-hand side of (4.4), we employ the paradifferential calculus (see Appendix B and Chapter 2 of [4]). Due to previous estimates, one can consider e.g. that $\vartheta_\varepsilon \in L^\infty_T L^2$ and $\boldsymbol{u}_\varepsilon \in L^2_T H^{1/2}$. Then, the product $\vartheta_\varepsilon \boldsymbol{u}_\varepsilon$ is L^2 in time. The regularity in space instead follows from Corollary B.2 in Appendix B. Indeed, one can find that the term $\vartheta_\varepsilon \boldsymbol{u}_\varepsilon$ is in $B^{-1/2}_{2,1}$ in the space variables and due to the embedding $B^{-1/2}_{2,1} \hookrightarrow H^{-1/2}$ (see Proposition A.5) we finally get $\vartheta_\varepsilon \boldsymbol{u}_\varepsilon$ is in $L^2_T H^{-1/2}$. Therefore, one can immediately argue that $\mathrm{div}\,\langle \vartheta_\varepsilon \boldsymbol{u}_\varepsilon\rangle$ belongs to $L^2_T H^{-3/2}$. Since all the computations have to be understood in the sense of distributions, this means that when we test $\mathrm{div}\,\langle \vartheta_\varepsilon \boldsymbol{u}_\varepsilon\rangle$ against a test function $\psi = \psi(x_2) \in C^\infty_T H^{3/2}$, the resulting integral is well-defined.

At this point, recalling Eq. (4.4), we deduce that $(\partial_t\langle \vartheta_\varepsilon\rangle)_\varepsilon$ is uniformly bounded in $L^2_T H^{-3/2}$, which implies $(\langle \vartheta_\varepsilon\rangle)_\varepsilon \subset W^{1,2}_T H^{-3/2}$. Moreover, we already know that $(\langle \vartheta_\varepsilon\rangle)_\varepsilon \subset L^\infty_T L^2 \cap L^2_T H^{1/2}$. Therefore, the Aubin-Lions lemma gives compactness of $(\langle \vartheta_\varepsilon\rangle)_\varepsilon$ in e.g. $L^2_T L^2_{\mathrm{loc}}$. Thus, we deduce the strong convergence (up to extraction) for $\varepsilon \to 0$:

$$\langle \vartheta_\varepsilon\rangle \to \langle \overline{\vartheta}\rangle \quad \text{in} \quad L^2_T L^2_{\mathrm{loc}}.$$

Now, we have that $\langle \vartheta_\varepsilon\rangle$ converges strongly to $\langle \overline{\vartheta}\rangle$ in $L^2_T(L^2_{\mathrm{loc}})$ and $\langle \boldsymbol{u}_\varepsilon\rangle$ converges weakly to $\langle \overline{\boldsymbol{u}}\rangle$ in $L^2_T(L^2_{\mathrm{loc}})$. Then, we deduce that

$$\langle \vartheta_\varepsilon\rangle\langle \boldsymbol{u}_\varepsilon\rangle \to \langle \overline{\vartheta}\rangle\langle \overline{\boldsymbol{u}}\rangle \qquad \text{in} \qquad \mathcal{D}'(\mathbb{R}_+ \times \Omega). \tag{4.5}$$

4.1.2. Analysis of the oscillations. This subsection is devoted to the analysis of the term $\langle \widetilde{\vartheta}_\varepsilon (\widetilde{u}_\varepsilon)_2\rangle$ introduced in (4.3). In order to take advantage of the algebraic structure of the system, we rewrite (2.1) as a wave equation in the following way

$$\varepsilon\,\partial_t\vartheta_\varepsilon + \mathcal{R}_1\vartheta_\varepsilon = \varepsilon\, f_\varepsilon, \tag{4.6}$$

where $f_\varepsilon := -\mathrm{div}\,(\vartheta_\varepsilon \boldsymbol{u}_\varepsilon) - \nu\Lambda\vartheta_\varepsilon$. Further, one can observe that $\mathcal{R}_1\theta_\varepsilon$ is in fact equal to $(u_\varepsilon)_2$.

Now, the aim is to work with approximate smooth quantities in order to give sense to all the computations in the sequel. For this reason, for any $M \in \mathbb{N}$ we consider the low-frequency cut-off operator S_M of a Littlewood-Paley decomposition, as introduced in (A.1). Then, we define

$$\widetilde{\vartheta}_{\varepsilon,M} = S_M \widetilde{\vartheta}_{\varepsilon} \qquad \text{and} \qquad \widetilde{\boldsymbol{u}}_{\varepsilon,M} = S_M \widetilde{\boldsymbol{u}}_{\varepsilon}. \tag{4.7}$$

The previous regularised quantities satisfy the following properties.

Proposition 4.2. *For any $T > 0$, we have the following convergence properties in the limit $M \to +\infty$*

$$\begin{aligned} \sup_{0<\varepsilon\leq 1} \left\| \widetilde{\vartheta}_{\varepsilon} - \widetilde{\vartheta}_{\varepsilon,M} \right\|_{L^2([0,T];H^{1/2-s})} &\longrightarrow 0 \\ \sup_{0<\varepsilon\leq 1} \left\| \widetilde{\boldsymbol{u}}_{\varepsilon} - \widetilde{\boldsymbol{u}}_{\varepsilon,M} \right\|_{L^2([0,T];H^{1/2-s})} &\longrightarrow 0, \end{aligned} \tag{4.8}$$

for any $s > 0$. Moreover, for any $M > 0$, the couple $(\widetilde{\vartheta}_{\varepsilon,M}, \widetilde{\boldsymbol{u}}_{\varepsilon,M})$ satisfies the approximate wave equation

$$\varepsilon\, \partial_t \widetilde{\vartheta}_{\varepsilon,M} + \left(\widetilde{u}_{\varepsilon,M}\right)_2 = \varepsilon\, \widetilde{f}_{\varepsilon,M}, \tag{4.9}$$

where $(\widetilde{f}_{\varepsilon,M})_{\varepsilon}$ is a family of smooth (in the space variables) functions satisfying, for any $s \geq 0$, the uniform bound

$$\sup_{0<\varepsilon\leq 1} \left\| \widetilde{f}_{\varepsilon,M} \right\|_{L^2([0,T];H^s)} \leq C(s,M), \tag{4.10}$$

with the constant $C(s,M)$ depending on the fixed values of $s \geq 0$ and $M > 0$, but not on $\varepsilon > 0$.

Proof. Thanks to characterization (A.2) of H^s and Lemma A.4, properties (4.8) follow straightforward from the uniform bounds established in Sect. 3.1.

Next, applying the operator S_M to (4.6) we get Eq. (4.9), where we have defined

$$\widetilde{f}_{\varepsilon,M} := S_M \left(-\mathrm{div}\left(\widetilde{\vartheta}_{\varepsilon} \widetilde{\boldsymbol{u}}_{\varepsilon} \right) - \nu \Lambda \widetilde{\vartheta}_{\varepsilon} \right).$$

Again, due to (A.2) and Lemma A.4, it is easy to prove inequality (4.10). $\square$

As a corollary of Proposition 4.2, one can state the following result.

Proposition 4.3. *Let $T > 0$. For any $\varphi \in C_c^\infty\big([0,T[\times\Omega;\mathbb{R}\big)$, we have*

$$\lim_{M\to+\infty} \limsup_{\varepsilon\to 0} \left| \int_0^T \int_\Omega \widetilde{\vartheta}_{\varepsilon} (\widetilde{u}_{\varepsilon})_2\, \partial_2\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \widetilde{\vartheta}_{\varepsilon,M} (\widetilde{u}_{\varepsilon,M})_2\, \partial_2\varphi \,\mathrm{d}x\,\mathrm{d}t \right| = 0.$$

Proof. Let $\varphi \in C_c^\infty([0,T[\times\Omega;\mathbb{R})$ with $\mathrm{supp}\,\varphi \subset [0,T] \times K$ for some compact $K \subset \Omega$. Therefore, we can write

$$\begin{aligned} \int_0^T \int_K \widetilde{\vartheta}_{\varepsilon} (\widetilde{u}_{\varepsilon})_2\, \partial_2\varphi \,\mathrm{d}x\,\mathrm{d}t = &\int_0^T \int_K \widetilde{\vartheta}_{\varepsilon,M} (\widetilde{u}_{\varepsilon})_2\, \partial_2\varphi \,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_K (\mathrm{Id} - S_M) \widetilde{\vartheta}_{\varepsilon}\, (\widetilde{u}_{\varepsilon})_2\, \partial_2\varphi \,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

As a consequence of the estimate (4.8) for $\widetilde{\vartheta}_\varepsilon$, we gather that

$$\lim_{M\to+\infty}\ \limsup_{\varepsilon\to0}\left|\int_0^T\int_K (\mathrm{Id}-S_M)\widetilde{\vartheta}_\varepsilon\,(\widetilde{u}_\varepsilon)_2\,\partial_2\varphi\,\mathrm{d}x\,\mathrm{d}t\right|=0.$$

At this point, we take

$$\begin{aligned}\int_0^T\int_K \widetilde{\vartheta}_{\varepsilon,M}(\widetilde{u}_\varepsilon)_2\,\partial_2\varphi\,\mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_K \widetilde{\vartheta}_{\varepsilon,M}(\widetilde{u}_{\varepsilon,M})_2\,\partial_2\varphi\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_0^T\int_K \widetilde{\vartheta}_{\varepsilon,M}\,(\mathrm{Id}-S_M)(\widetilde{u}_\varepsilon)_2\,\partial_2\varphi\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and again thanks to (4.8) for $\widetilde{\boldsymbol{u}}_\varepsilon$, we obtain

$$\lim_{M\to+\infty}\ \limsup_{\varepsilon\to0}\left|\int_0^T\int_K \widetilde{\vartheta}_{\varepsilon,M}\,(\mathrm{Id}-S_M)(\widetilde{u}_\varepsilon)_2\,\partial_2\varphi\,\mathrm{d}x\,\mathrm{d}t\right|=0.$$

This concludes the proof of the proposition. □

We notice that, in order to analyse the oscillatory part of the term in (4.3), it is enough to reduce the study to the case of smooth quantities $\widetilde{\vartheta}_{\varepsilon,M}$ and $\widetilde{\boldsymbol{u}}_{\varepsilon,M}$.

At this point, taking Eq. (4.9) and multiplying by $\widetilde{\vartheta}_{\varepsilon,M}$ we obtain

$$\widetilde{\vartheta}_{\varepsilon,M}\,(\widetilde{u}_{\varepsilon,M})_2=\varepsilon\,\widetilde{\vartheta}_{\varepsilon,M}\widetilde{f}_{\varepsilon,M}-\frac{\varepsilon}{2}\,\partial_t\widetilde{\vartheta}^2_{\varepsilon,M}.$$

The former term in the right-hand side is of order ε as a consequence of the uniform bounds for $\widetilde{\vartheta}_{\varepsilon,M}$ and $\widetilde{f}_{\varepsilon,M}$. The latter one is a remainder due to the particular structure of the test function in (4.1).

This implies that, for any $T>0$ and any test-function ψ as in (4.1), one has the convergence (at any $M\in\mathbb{N}$ fixed) when $\varepsilon\to0$

$$\int_0^T\int_{\mathbb{R}}\langle\widetilde{\vartheta}_{\varepsilon,M}(\widetilde{u}_{\varepsilon,M})_2\rangle\,\partial_2\psi\,\mathrm{d}x_2\,\mathrm{d}t\longrightarrow0. \tag{4.11}$$

4.1.3. Description of the limit system. With the relations established in the previous subsections, we can pass to the limit in the Eq. (2.1).

To begin with, we take a test-function ψ (for $T>0$) as in (4.1). For such a ψ, the Riesz term $\mathcal{R}_1\vartheta_\varepsilon$ vanishes identically. Hence, the momentum equation in its weak formulation reads

$$-\int_0^T\int_\Omega\left(\vartheta_\varepsilon\,\partial_t\psi+\vartheta_\varepsilon\boldsymbol{u}_\varepsilon\cdot\nabla\psi+\nu\Lambda^{1/2}\vartheta_\varepsilon\,\Lambda^{1/2}\psi\right)\,\mathrm{d}x\,\mathrm{d}t=\int_\Omega\theta_{0,\varepsilon}\,\psi(0)\,\mathrm{d}x. \tag{4.12}$$

Making use of the uniform bounds of Sect. 3.1, we can pass to the limit in the ∂_t term and in the viscosity term $\nu\Lambda\vartheta_\varepsilon$. Moreover, our assumptions imply that $\vartheta_{0,\varepsilon}\rightharpoonup\overline{\vartheta}_0$ in e. g. L^2. It is worth noticing that, thanks to relations (4.2), (4.5) and (4.11), for the convective term $\mathrm{div}\,(\vartheta_\varepsilon\boldsymbol{u}_\varepsilon)$ we get (when $\varepsilon\to0$)

$$-\int_0^T\int_{\mathbb{R}}\langle\vartheta_\varepsilon(u_\varepsilon)_2\rangle\,\partial_2\psi\,\mathrm{d}x_2\,\mathrm{d}t\longrightarrow-\int_0^T\int_{\mathbb{R}}\langle\overline{\vartheta}\rangle\langle\overline{u}_2\rangle\,\partial_2\psi\,\mathrm{d}x_2\,\mathrm{d}t,$$

which vanishes identically due to relation (3.7), recalling that $\mathcal{R}_1 \overline{\vartheta} = \overline{u}_2$.

Finally, letting $\varepsilon \to 0$, we may infer that

$$-\int_0^T \int_\Omega \left(\overline{\vartheta}\, \partial_t \psi + \nu \Lambda^{1/2} \overline{\vartheta}\, \Lambda^{1/2} \psi \right)\, \mathrm{d}x\, \mathrm{d}t = \int_\Omega \overline{\theta}_0\, \psi(0)\, \mathrm{d}x. \tag{4.13}$$

This concludes the proof of Theorem 2.4.

4.2. Proof of the convergence in the inviscid and fast rotation case

In this subsection, we consider the case when $\nu = \nu(\varepsilon) = \varepsilon^\beta$, $\beta > 0$. In that scaling, the energy estimate reads

$$\|\vartheta_\varepsilon(T)\|_{L^2}^2 + 2\varepsilon^\beta \int_0^T \|\Lambda^{1/2} \vartheta_\varepsilon(\tau)\|_{L^2}^2\, \mathrm{d}\tau \le \|\vartheta_{0,\varepsilon}\|_{L^2}^2 \quad \text{for all} \quad T > 0. \tag{4.14}$$

Therefore, one completely loses the uniform control on $\Lambda^{1/2}\vartheta_\varepsilon$. In order to gain space compactness for ϑ_ε, we have to consider strong solutions with the following regularity properties

$$\vartheta_\varepsilon \in L^\infty([0,T[\,; H^s(\Omega)) \quad \text{and} \quad s > 2.$$

The aim now is to show higher order estimates for the temperature. We have already presented in (4.14) the L^2 estimate. In order to obtain the H^s control, we employ the Littlewood-Paley decomposition (see Appendix A). Applying the dyadic blocks Δ_j to Eq. (2.1) (see e.g. [29]), we obtain

$$\partial_t \Delta_j \vartheta_\varepsilon + \boldsymbol{u}_\varepsilon \cdot \nabla \Delta_j \vartheta_\varepsilon + \varepsilon^\beta \Lambda \Delta_j \vartheta_\varepsilon + \frac{1}{\varepsilon} \mathcal{R}_1 \Delta_j \vartheta_\varepsilon = [\boldsymbol{u}_\varepsilon \cdot \nabla, \Delta_j] \vartheta_\varepsilon.$$

Multiplying by $\Delta_j \vartheta_\varepsilon$ and using the orthogonality of the Riesz term, it follows that

$$\partial_t \|\Delta_j \vartheta_\varepsilon\|_{L^2}^2 + 2\varepsilon^\beta \|\Lambda^{1/2} \Delta_j \vartheta_\varepsilon\|_{L^2}^2 \le 2 \|[\boldsymbol{u}_\varepsilon \cdot \nabla, \Delta_j] \vartheta_\varepsilon\|_{L^2} \|\Delta_j \vartheta_\varepsilon\|_{L^2}.$$

In particular, we get

$$\|\Delta_j \vartheta_\varepsilon\|_{L^2} \le \|\Delta_j \vartheta_{0,\varepsilon}\|_{L^2} + C \int_0^t \|[\boldsymbol{u}_\varepsilon \cdot \nabla, \Delta_j] \vartheta_\varepsilon\|_{L^2}\, \mathrm{d}\tau.$$

At this point, thanks to the commutator estimates (we refer to Lemma B.3 and also to [29]) we have

$$\begin{aligned} 2^{js} \|[\boldsymbol{u}_\varepsilon \cdot \nabla, \Delta_j] \vartheta_\varepsilon\|_{L^2} &\le C c_j(t) \left(\|\nabla \vartheta_\varepsilon\|_{L^\infty} \|\boldsymbol{u}_\varepsilon\|_{H^s} + \|\vartheta_\varepsilon\|_{H^s} \|\nabla \boldsymbol{u}_\varepsilon\|_{L^\infty} \right) \\ &\le C c_j(t) \|\vartheta_\varepsilon\|_{H^s}^2, \end{aligned}$$

where $(c_j(t))_{j \ge -1}$ is a sequence in the unit ball of ℓ^2.

After summing on indices $j \ge -1$, we finally derive for all $t \ge 0$

$$\|\vartheta_\varepsilon(t)\|_{H^s} + C\varepsilon^\beta \int_0^t \|\vartheta_\varepsilon(\tau)\|_{H^{s+1/2}}^2\, \mathrm{d}\tau \le \|\vartheta_{0,\varepsilon}\|_{H^s} + C \int_0^t \|\vartheta_\varepsilon(\tau)\|_{H^s}^2\, \mathrm{d}\tau. \tag{4.15}$$

The scope now is finding a time $T^* > 0$ for which the solutions are uniformly bounded (in ε) in the interval $[0, T^*]$.

We define $T^*_\varepsilon > 0$ such that

$$T^*_\varepsilon := \sup \left\{ t > 0 : \int_0^t \|\vartheta_\varepsilon\|^2_{H^s} \leq \|\vartheta_{0,\varepsilon}\|_{H^s} \right\}. \tag{4.16}$$

Then, we deduce $\|\vartheta_\varepsilon(t)\|_{H^s} \leq C\|\vartheta_{0,\varepsilon}\|_{H^s}$ for all times $t \in [0, T^*_\varepsilon]$ and for some positive constant $C > 0$. Therefore, for all $t \in [0, T^*_\varepsilon]$ we gather

$$\int_0^t \|\vartheta_\varepsilon(\tau)\|^2_{H^s} \, \mathrm{d}\tau \leq Ct\|\vartheta_{0,\varepsilon}\|^2_{H^s}.$$

By using the definition (4.16) of T^*_ε, we finally argue that

$$T^*_\varepsilon \geq \frac{C}{\|\vartheta_{0,\varepsilon}\|_{H^s}}. \tag{4.17}$$

We recall also that $(\vartheta_{0,\varepsilon})_\varepsilon$ is uniformly bounded in H^s. In particular, relation (4.17) implies that there exists a time $T^* > 0$ such that

$$T^* := \sup_{\varepsilon > 0} T^*_\varepsilon > 0.$$

From the definition of $T^* > 0$ and relation (4.15), one can obtain that

$$\sup_{\varepsilon \in]0,1]} \|\vartheta_\varepsilon\|_{L^\infty_{T^*}(H^s)} \leq C. \tag{4.18}$$

At this point, since those strong solutions are in particular weak solutions, one can adapt the arguments developed in Sect. 4.1 to treat the convective term $\mathrm{div}\,(\vartheta_\varepsilon \boldsymbol{u}_\varepsilon)$. However, in this case, the regularisation procedure presented in Sect. 4.1.2 is no longer necessary due to the H^s regularity of the solutions. *Mutatis mutandis* also in this instance the convective term vanishes identically at the limit for $\varepsilon \to 0$.

Furthermore, since $(\Lambda\vartheta_\varepsilon)_\varepsilon$ is uniformly bounded in $L^\infty_{T^*}(H^{s-1})$, the presence of the prefactor ε^β provides the convergence to zero for the dissipative term.

Finally, letting $\varepsilon \to 0$, one has

$$-\int_0^{T^*} \int_\Omega \overline{\vartheta} \, \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t = \int_\Omega \overline{\theta}_0 \, \psi(0) \, \mathrm{d}x, \tag{4.19}$$

for all test functions ψ defined as in (4.1). This proves the convergence to the limit dynamics claimed in Theorem 2.5.

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A. Appendix—Littlewood–Paley theory

In this appendix, we present some tools from Littlewood-Paley theory, which we have exploited in our analysis. We refer e.g. to Chapter 2 of [4] for details. For simplicity of exposition, we deal with the $\mathbb{R}^d$ case, with $d \geq 1$; however, the whole construction can be adapted also to the d-dimensional torus $\mathbb{T}^d$, and to the "hybrid" case $\mathbb{T}^{d_2} \times \mathbb{R}^{d_1}$.

First of all, we introduce the *Littlewood-Paley decomposition*. For this, we fix a smooth radial function χ such that $\operatorname{supp}\chi \subset B(0,2)$, $\chi \equiv 1$ in a neighborhood of $B(0,1)$ and the map $r \mapsto \chi(r\, e)$ is non-increasing over $\mathbb{R}_+$ for all unitary vectors $e \in \mathbb{R}^d$. Set $\varphi\left(\xi\right) = \chi\left(\xi\right) - \chi\left(2\xi\right)$ and $\varphi_j(\xi) := \varphi(2^{-j}\xi)$ for all $j \geq 0$. The dyadic blocks $(\Delta_j)_{j\in\mathbb{Z}}$ are defined by[1]

$$\Delta_j := 0 \quad \text{if } j \leq -2, \qquad \Delta_{-1} := \chi(D) \qquad \text{and} \qquad \Delta_j := \varphi(2^{-j}D) \quad \text{if } j \geq 0.$$

For any $j \geq 0$ fixed, we also introduce the *low frequency cut-off operator*

$$S_j := \chi(2^{-j}D) = \sum_{k \leq j-1} \Delta_k. \tag{A.1}$$

Note that S_j is a convolution operator. More precisely, after defining

$$K_0 := \mathcal{F}^{-1}\chi \qquad \text{and} \qquad K_j(x) := \mathcal{F}^{-1}[\chi(2^{-j}\cdot)](x) = 2^{jd}K_0(2^jx),$$

for all $j \in \mathbb{N}$ and all tempered distributions $u \in \mathcal{S}'$, we have that $S_ju = K_j * u$. Thus, the L^1 norm of K_j is independent of $j \geq 0$, hence S_j maps continuously L^p into itself, for any $1 \leq p \leq +\infty$.

Moreover, the following property states the usefulness of such a decomposition.

Lemma A.1. *For any $u \in \mathcal{S}'$, then one has the equality $u = \sum_j \Delta_j u$ in the sense of $\mathcal{S}'$.*

Let us also recall the so-called *Bernstein inequalities*, which describe the way derivatives take effect on the spectrally localized functions.

[1] We agree that $f(D)$ stands for the pseudo-differential operator $u \mapsto \mathcal{F}^{-1}[f(\xi)\,\widehat{u}(\xi)]$.

Lemma A.2. *Let $0<r<R$. A constant C exists so that, for any non-negative integer k, any couple (p,q) in $[1,+\infty]^2$, with $p\leq q$, and any function $u\in L^p$, we have, for all $\lambda>0$,*

$$\begin{aligned}
&\mathit{supp}\ \widehat{u}\subset B(0,\lambda R) \quad \Longrightarrow \quad \|\nabla^k u\|_{L^q}\ \leq\ C^{k+1}\,\lambda^{k+d\left(\frac{1}{p}-\frac{1}{q}\right)}\,\|u\|_{L^p};\\
&\mathit{supp}\ \widehat{u}\subset \{\xi\in\mathbb{R}^d\,:\,\lambda r\leq|\xi|\leq\lambda R\}\Longrightarrow C^{-k-1}\,\lambda^k\|u\|_{L^p}\ \leq\ \|\nabla^k u\|_{L^p}\ \leq\ C^{k+1}\,\lambda^k\|u\|_{L^p}.
\end{aligned}$$

By use of Littlewood-Paley decomposition, we can define the class of Besov spaces.

Definition A.3. Let $s\in\mathbb{R}$ and $1\leq p,r\leq+\infty$. The *non-homogeneous Besov space* $B^s_{p,r}$ is defined as the subset of tempered distributions u for which

$$\|u\|_{B^s_{p,r}}:=\left\|\left(2^{js}\,\|\Delta_j u\|_{L^p}\right)_{j\geq-1}\right\|_{\ell^r}\ <\ +\infty.$$

Besov spaces are interpolation spaces between Sobolev spaces. In fact, for any $k\in\mathbb{N}$ and $p\in[1,+\infty]$ we have the chain of continuous embeddings $B^k_{p,1}\hookrightarrow W^{k,p}\hookrightarrow B^k_{p,\infty}$, which when $1<p<+\infty$ can be refined to $B^k_{p,\min(p,2)}\hookrightarrow W^{k,p}\hookrightarrow B^k_{p,\max(p,2)}$. In particular, for all $s\in\mathbb{R}$ we deduce that $B^s_{2,2}\equiv H^s$, with equivalence of norms

$$\|f\|_{H^s}\ \sim\ \left(\sum_{j\geq-1}2^{2js}\,\|\Delta_j f\|^2_{L^2}\right)^{1/2}. \tag{A.2}$$

Observe that, from that equivalence, we easily get the following property whose proof is presented for the sake of completeness.

Lemma A.4. *For any $f\in H^s$ and any $j\in\mathbb{N}$, one has*

$$\left\|\left(\mathrm{Id}-S_j\right)f\right\|_{H^\sigma}\ \leq\ C\,\|\nabla f\|_{H^{s-1}}\,2^{-j(s-\sigma)}\qquad \textit{for all}\qquad \sigma\leq s, \tag{A.3}$$

where $C>0$ is a "universal" constant, independent of f, j, s and σ.

Proof. We make use of the characterization (A.2) of H^σ and we write

$$\begin{aligned}
\|(\mathrm{Id}-S_j)f\|^2_{H^\sigma}&\leq C\sum_{k\geq j}2^{2k\sigma}\,\|\Delta_k f\|^2_{L^2}\,2^{2ks}\,2^{-2ks}\\
&\leq C\sum_{k\geq j}2^{-2k(s-\sigma)}\,2^{2ks}\,\|\Delta_k f\|^2_{L^2}\\
&\leq C\sum_{k\geq j}2^{-2k(s-\sigma)}\,2^{2k(s-1)}\,\|\Delta_k\nabla f\|^2_{L^2}\\
&\leq C\sum_{k\geq j}2^{-2k(s-\sigma)}\,\|\nabla f\|^2_{H^{s-1}}\\
&\leq C\,2^{-2j(s-\sigma)}\|\nabla f\|^2_{H^{s-1}},
\end{aligned}$$

for all $\sigma\leq s$. Then, we obtain (A.3), concluding the proof of the lemma. □

As an immediate consequence of the first Bernstein inequality (see Lemma A.2), one gets the following embedding result, which generalises Sobolev embeddings.

Proposition A.5. *The space $B^{s_1}_{p_1,r_1}$ is continuously embedded in the space $B^{s_2}_{p_2,r_2}$ for all indices satisfying $p_1 \leq p_2$ and either $s_2 < s_1 - d\big(1/p_1 - 1/p_2\big)$, or $s_2 = s_1 - d\big(1/p_1 - 1/p_2\big)$ and $r_1 \leq r_2$.*

In particular, we get the following chain of continuous embeddings

$$B^s_{p,r} \hookrightarrow W^{1,\infty},$$

whenever the triplet $(s,p,r) \in \mathbb{R} \times [1,+\infty]^2$ satisfies

$$s > 1 + \frac{d}{p} \qquad \text{or} \qquad s = 1 + \frac{d}{p} \quad \text{and} \quad r = 1. \tag{A.4}$$

B. Appendix—Paradifferential calculus

Let us introduce the Bony decomposition (see [6]). Formally, the product of two tempered distributions u and v can be decomposed into

$$u\,v = \mathcal{T}_u v + \mathcal{T}_v u + \mathcal{R}(u,v),$$

where the *paraproduct* $\mathcal{T}$ and the *remainder* $\mathcal{R}$ are defined as follows

$$\mathcal{T}_u v = \sum_j S_{j-1} u\,\Delta_j v \quad \text{and} \quad \mathcal{R}(u,v) := \sum_j \sum_{|k-j|\leq 1} \Delta_j u\,\Delta_k v.$$

The paraproduct and remainder operators have the following nice continuity properties.

Proposition B.1. *For any $(s,t) \in \mathbb{R}\times\,]0,+\infty[$ and for any $(p,r_1,r_2) \in [1,+\infty]^3$, the paraproduct operator $\mathcal{T}$ maps continuously $L^\infty \times B^s_{p,r}$ into $B^s_{p,r}$ and $B^{-t}_{\infty,r_1} \times B^s_{p,r_2}$ into $B^{s-t}_{p,r}$ with $1/r := \min\{1,\, 1/r_1 + 1/r_2\}$. Moreover, we have the following estimates*

$$\|\mathcal{T}_u v\|_{B^s_{p,r}} \leq C\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} \quad \text{and} \quad \|\mathcal{T}_u v\|_{B^{s-t}_{p,r}} \leq C\|u\|_{B^{-t}_{\infty,r_1}}\|v\|_{B^s_{p,r_2}}.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R} \times [1,+\infty]^2$ such that $s_1 + s_2 > 0$, $1/p := 1/p_1 + 1/p_2 \leq 1$ and $1/r := 1/r_1 + 1/r_2 \leq 1$, the remainder operator $\mathcal{R}$ maps continuously $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ into $B^{s_1+s_2}_{p,r}$. In the case when $s_1 + s_2 = 0$ and $1/r_1 + 1/r_2 = 1$, the operator $\mathcal{R}$ is continuous from $B^{s_1}_{p_1,r_1} \times B^{s_2}_{p_2,r_2}$ to $B^0_{p,\infty}$.

Corollary B.2. *Assume that u is in L^2 and v belongs to the space $H^{1/2}$. Then, the product uv belongs to $B^{-1/2}_{2,1}$ and one has the following estimate*

$$\|uv\|_{B^{-1/2}_{2,1}} \leq C\|u\|_{L^2}\|v\|_{H^{1/2}}.$$

Proof. Taking advantage of Bony decomposition, one can write

$$uv = \mathcal{T}_u v + \mathcal{T}_v u + \mathcal{R}(u,v)$$

and employing Proposition B.1, we get

$$\|\mathcal{T}_u v\|_{B^{-1/2}_{2,1}} \leq C\|u\|_{B^{-1}_{\infty,2}}\|v\|_{B^{1/2}_{2,2}} \leq C\|u\|_{B^{-1}_{\infty,2}}\|v\|_{H^{1/2}}$$

$$\|\mathcal{T}_v\, u\|_{B^{-1/2}_{2,1}} \leq C\|v\|_{B^{-1/2}_{\infty,2}}\|u\|_{B^0_{2,2}} \leq C\|v\|_{B^{-1/2}_{\infty,2}}\|u\|_{L^2}$$

$$\|\mathcal{R}(u,v)\|_{B^{1/2}_{1,1}} \leq C\|u\|_{B^0_{2,2}}\|v\|_{B^{1/2}_{2,2}} \leq C\|u\|_{L^2}\|v\|_{H^{1/2}}.$$

This completes the proof of the corollary, recalling the embeddings $L^2 \hookrightarrow B^{-1}_{\infty,2}$ and $H^{1/2} \hookrightarrow B^{-1/2}_{\infty,2}$ for the paraproducts and the embedding $B^{1/2}_{1,1} \hookrightarrow B^{-1/2}_{2,1}$ for the remainder. $\square$

To conclude this appendix, we present a standard commutator estimate between the transport operator and the frequency localisation operator.

Lemma B.3. *Assume that $v \in B^s_{p,r}$ with (s,p,r) satisfying the Lipschitz condition (A.4). Denote by $[v\cdot\nabla, \Delta_j]f = (v\cdot\nabla)\Delta_j f - \Delta_j(v\cdot\nabla)f$ the commutator between the transport operator $v\cdot\nabla$ and the frequency localisation operator Δ_j. Then, for every $f \in B^s_{p,r}$,*

$$\left\| \left(2^{js}\|[v\cdot\nabla,\Delta_j]f\|_{L^p}\right)_j \right\|_{\ell^r} \leq C\left(\|\nabla v\|_{L^\infty}\|f\|_{B^s_{p,r}} + \|\nabla v\|_{B^{s-1}_{p,r}}\|\nabla f\|_{L^\infty}\right)$$

and also, for every $f \in B^{s-1}_{p,r}$,

$$\left\| \left(2^{j(s-1)}\|[v\cdot\nabla,\Delta_j]f\|_{L^p}\right)_j \right\|_{\ell^r} \leq C\left(\|\nabla v\|_{L^\infty}\|f\|_{B^{s-1}_{p,r}} + \|\nabla v\|_{B^{s-1}_{p,r}}\|f\|_{L^\infty}\right),$$

for some constant $C = C(s,p,d) > 0$.

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