Sustainable Management of Tourist Flow Networks: A Mean Field Model

Joseph Andria[1] · **Rosario Maggistro**[2] · **Raffaele Pesenti**[3]

Abstract
In this article, we propose a mean field game approach for modeling the flows of excursionists within a network of tourist attractions. We prove the existence of an equilibrium within the network using a balance ordinary differential equation together with optimality conditions in terms of the value function. We also propose a bi-level formulation of the problem where we aim at achieving a sustainable-oriented control strategy in the upper level and at maximizing excursionists' satisfaction in the lower level. Our proposed model may provide an effective management tool for local authorities who deal with the challenging problem of finding an optimal control policy to the often conflicting objectives of ensuring the maximum excursionists' satisfaction while pursuing the highest sustainability benefits.

Keywords Network flow optimal control · Mean field game · Bi-level optimization · Sustainability

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✉ Rosario Maggistro
rosario.maggistro@deams.units.it

Joseph Andria
joseph.andria@unipa.it

Raffaele Pesenti
pesenti@unive.it

1 Dipartimento di Scienze Economiche, Aziendali e Statistiche, Università di Palermo, 90128 Palermo, Italy

2 Dipartimento di Scienze Economiche, Aziendali, Matematiche e Statistiche "Bruno de Finetti", Università di Trieste, 34123 Trieste, Italy

3 Dipartimento di Management, Università Ca' Foscari Venezia, 30121 Venezia, Italy

1 Introduction

In this paper, we tackle the problem of managing the excursionists' tours in an art city with the aim of defining sustainable-oriented control strategies while attaining the maximum excursionists' satisfaction.

The success of tours depends on their ability to allow both excursionists and local authorities to achieve experiential, operational, and political targets [43]. In general, a *touristic area* is identified by a set of icon attractions, the so-called *primary attractions*, around which tours are conducted. Different kinds of excursionists may follow different routes within an area depending on their needs and their preferences over the attractions attributes [13]. The more the needs of excursionists are met along the tour, the more effective and experiential-based the tour is.

Operational factors can influence the success of a tour. They are concerned with pragmatic objectives such as safety, timing and route selection, parking, proximity to transportation lines, and linkages or commercial areas. Traffic congestion and on-site crowding threaten the success of a tour. Indeed, overcrowded sites or routes have a negative impact on excursionists' satisfaction, especially in places with narrow streets or walkways, since excursionists may be forced to curtail their tours to meet their planned itinerary time [21].

The literature emphasizes also the importance for authorities of developing and adopting effective governance strategies for tourism management. The adoption of itineraries management policies constitutes a strategic factor in the representation of a tourist area and in influencing the quality of the excursionist experience, the length of stay, and the resulting economic benefits for a local community. Moreover, governments may use tourism strategically to address issues of national significance [17]. Providing a visiting experience of high quality is of extreme importance in today's increasingly competitive marketplace [41, 42]. Directing excursionists to the right places at the right times not only can lead to higher quality visits but may also shield the excursionists from the bad aspects of a city from less pleasant areas. Furthermore, an effective itinerary management policy may also attain the right combination of commercial needs, excursionists' experiential desires and residents well-being. Indeed, excursionists' satisfaction can have also a positive impact on local commercial and industrial activities by stimulating repeat visits, positive word of mouth recommendations and consequently new customers, reputation enhancement, higher acceptance of price increases and higher profitability [9, 19, 23, 32].

In this study, we apply the tools of the mean field game theory to support local authorities to deal with the challenging problem of attaining both the excursionists' experiential satisfaction and the maximum sustainability benefits. To this end, moving from the problem in [8], we introduce two theoretical models. The first one describes the excursionists' flows in a network that depicts an area of tourist attractions. We show that this problem can be solved within a mean field scheme and the existence of an equilibrium of flows is proven. The second one is a bi-level model to define sustainable-oriented control strategies while attaining the maximum excursionists' satisfaction.

Mean field game (MFG) theory was introduced in the seminal papers by Lasry and Lions [29, 30], as well as by Huang et al. [24, 25]. It is a branch of dynamic games aiming at modeling complex decision processes involving an infinite number of agents. Each agent has a small influence on the overall system but is influenced by the average behavior of other agents, hence the use of mean field terminology.

An MFG model is generally described by a system coupling a transport equation for the distribution of the agents over time with a Hamilton–Jacobi–Bellman (HJB) equation governing the value function of a single agent. The solution of the model is an equilibrium configuration (a *mean field equilibrium*) in which no agent is interested in deviating from his/her current route. We herein simplify the evolution of the agents' distribution by using a balance ordinary differential equations and considering optimality conditions in terms of the value function due the presence of discontinuous exit costs.

The concept of mean field equilibrium recall the one of Wardrop equilibrium widely used in the domain of transportation. The Wardrop equilibrium is a configuration in which the perceived cost associated to any source-destination path chosen by a nonzero fraction of agents does not exceed the perceived cost associated to any other path. The two concepts of equilibrium, however, differ among themselves: in MFG models on networks, the cost is more comprehensive than in the Wardrop model where the cost is a function of the edge, and each edge is treated as an aggregate entity. Specifically, in a Wardrop equilibrium, the cost incurred by an agent on an edge is the travel time which depends on the agents' flow. On the other hand, MFG models consider more general travel costs that depend not only on the agents' flow but also on their strategies which are solutions of an optimal control problem.
Very recently, in [35], the reformulation of the MFG problem into a Wardrop one has been considered and how to recover the MFG solution from the corresponding Wardrop equilibrium has been showed.

Applications of mean field games are several and cover different fields such as economics, physics, biology, and network engineering (see, e.g., [1, 5, 12, 22, 28]). In particular, models for crowd and population dynamics on networks were investigated by Camilli et al. [10, 11], Cristiani et al. [16], Lachapelle and Wolfram [27] and Bagagiolo et al. [6–8].

The bi-level model that we propose addresses the often conflicting objectives of local authorities and excursionists. The former ones aim at defining sustainable-oriented policy while the latter ones at maximizing their satisfaction [2–4]. We deal with the problem of selecting an optimal sustainable oriented control strategy at the upper level of the model, while we describe the excursionists' search for maximum satisfaction at the lower level of the model.

The herein adopted definition of sustainability is that declared by the World Tourism Organization (WTO). The WTO states that sustainable tourism can be defined as "tourism that takes full account of its current and future economic, social, and environmental impacts, addressing the needs of excursionists, the industry, the environment, and the host communities" [40].
The reminder of this paper is organized as follows. In Sect. 2, we introduce the main assumptions of the MFG model which is described in Sect. 3. In Sect. 4, we derive the value function and the corresponding optimal control for the problem at hand. In Sect. 5, we prove the existence of a mean field equilibrium while, in Sect. 6, we

consider the bi-level model from a leader/followers perspective. Finally, in Sect. 7, we draw some conclusions and suggest future research.

2 Assumptions

In this section, we introduce the assumptions underlying our model.
As observed in Introduction, we aim at modeling and controlling the excursionists' movement within an art city during the daytime. Accordingly, the excursionists' visiting time is formally defined as the closed interval $[0, T]$, where $T > 0$ is the final horizon. Here, let us recall that: an *excursionist*, also indicated as *day-tripper* in the literature, is a visitor who does not stay overnight and a *route* is a path chosen by an excursionist for his/her journey inside the city. We refer to the entrance, the exit, and the attractions of the city as *points of interest* POI. The excursionists have to traverse the *entrance POI*, respectively, the *exit POI*, to enter, respectively, to leave, the city. They have to traverse the remaining POIS to visit the attractions.

We make the following assumptions.

Assumption 1 (*Alternative means of transport*) Excursionists may follow their route by a priori choosing at most between two means of transport. The first of the two means (walking) is always available and its velocity is computed minimizing a suitable cost functional. The second mean may not always be available and its velocity is given constant.

Assumption 2 (*Excursionist predefined routes and orientation*)

1. Each excursionist visits the city following a predefined route using a set of predefined means of transport.
2. The excursionists never backtrack along a route.
3. Each excursionist has a predefined orientation toward the means of transport.

In Assumption 2(3), orientation means that each excursionist may be either process- or outcome-oriented. The former kind of excursionists prefers to explore a destination more widely by taking indirect routes and walk slowly to explore the area of attractions more freely and deeply. Differently, the latter minimizes the transit times between POIs by using a means of transport such as buses or cars [31].

3 The Model

In this section, we first model the city as a set $\mathcal{A}$ of connected POIs using a directed network. Then, we describe the excursionists' movements in terms of the dynamics of agents that traverse the network edges.

We recall that Assumption 2(3) states that we have two kinds of excursionists that choose two different types of routes. Hereinafter, we denote by $\Xi = \{1, 2\}$ the set of the different kinds of excursionists and, with a little abuse, we use the same of notation to identify the different types of routes. Accordingly, we write that the excursionists of type $\xi \in \Xi$ (*ξ-excursionists* for short) follow routes of type ξ.

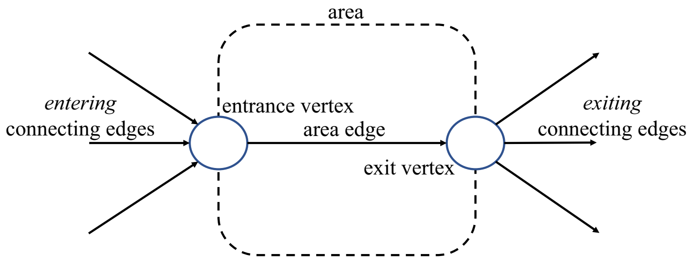

Fig. 1 Underlying subnetwork induced by a point of interest

3.1 The Underlying Network

Each POI in $\mathcal{A}$ has an entrance and an exit point. Then, we represent the city as a directed network $G = (V, E)$, where the edgeset V includes a vertex for each entrance point and each exit point of the POIs in $\mathcal{A}$. We call *entrance*, respectively, *exit*, *vertices* the vertices corresponding to entrance, respectively, exit, points. The edgeset E includes, for each POI, a directed edge that joins its entrance vertex with its exit vertex. The edgeset E includes also, for each ordered pair of POIs and each means of transport between the two POIs, a directed edge that connects the exit vertex of the first POI with the entrance vertex of the second POI. We call *POI edges* the former kind of edges, *connecting edges* the second kind of edges. We denote by l_e the length of edge e, for $e \in E$. Figure 1 represents the underlying subnetwork induced by a POI.

In light of the above definitions, we observe that the edgeset E is partitioned into two subsets: the subset E^p of POI edges and the subset E^c of connecting edges. In turn, the subset E^c is partitioned into two subsets E^ξ, $\xi \in \Xi$. Each edge in subset E^p models the path followed by an excursionist visiting a POI. Each edge in a subset E^ξ models the path followed by an ξ-excursionist moving from a POI to another POI using the mean of transport of choice, i.e., by walking if $\xi = 1$, by bus or car if $\xi = 2$. We remark that, for the exit vertex of a POI and the entrance vertex of its subsequent POI, there always exists an edge in E^1 and may exist an edge E^2. We say that two edges are of different *type* if they do not belong to the same subset E^p or E^1 or E^2. Hereinafter, we assume that the network G contains no cycles. We denote by o the entrance vertex of the POI corresponding to the entrance of the city, by d the exit vertex of the POI corresponding to the exit of the city. We refer to vertices o and d as to the *origin* and *destination vertices*, respectively, of the network G and assume that any vertex in V can be reached from the origin vertex and the destination vertex is reachable from any vertex in V. A route r is a path of G that joins the origin vertex with the destination one, i.e., an ordered subset of consecutive edges in E from o to d. We denote by Γ the set of all routes and by Γ^ξ the subset of routes of type ξ in Γ.

3.2 Excursionists' Dynamics and Objective Function

We initially introduce the necessary notation. Then, we formalize the excursionists' dynamics.
Hereinafter, we use the term *mass* to refer to the number of excursionists as we will make use of the mass conservation law equation.

Let A^ξ be the $|E| \times |\Gamma^\xi|$ edge-route incidence matrix with entries

$$A^\xi_{e,r} = \begin{cases} 1 & \text{if} \quad e \in r,\ r \in \Gamma^\xi, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

for each $\xi \in \Xi$. In addition, denote the total number of times that an edge belongs to a route by

$$\Theta = \sum_{\xi \in \Xi} \sum_{e \in E} \sum_{r \in \Gamma^\xi} A^\xi_{e,r}, \quad \text{with} \quad |E| \le \Theta \le |\Xi|(|E| \times |\Gamma|).$$

Finally, let:

- $\lambda^\xi : [0, T] \to \mathbb{R}_+$ be a function denoting the rate of arrival of ξ-excursionists, $\xi \in \Xi$, entering the network G in the origin vertex o at time t.
- $\lambda(t) = \sum_{\xi \in \Xi} \lambda^\xi(t)$ be the throughput of the excursionists at time t, i.e., the total flow of excursionists entering G in o at time t.
- $\rho^{\xi,e}_r : [0, T] \to \mathbb{R}_+$ be a function indicating the mass of ξ-excursionists, $\xi \in \Xi$, present on edge e that chose route r when entered G in o.
- $\rho^e_r(t) = \sum_{\xi \in \Xi} \rho^{\xi,e}_r(t)$ be the total mass of excursionists present on edge e at time t that chose route r when entered G in o.
- $\rho(t)$ be the vector $\{\rho^{\xi,e}_r(t), \xi \in \Xi, e \in r, r \in \Gamma\} \in \mathbb{R}^\Theta$.
- $f^{\xi,e}_r : [0, T] \to \mathbb{R}_+$ be a function indicating the flow of ξ-excursionists, $\xi \in \Xi$, leaving edge e that chose route r when entered G in o.
- $f^e_r(t) = \sum_{\xi \in \Xi} f^{\xi,e}_r(t)$ be the total flow of excursionists leaving edge e at time t that chose route r when entered G in o.
- $f(t)$ be the vector $\{f^{\xi,e}_r(t), \xi \in \Xi, e \in r, r \in \Gamma\} \in \mathbb{R}^\Theta$.
- $u^{\xi,e}$ the velocity at which the ξ-excursionists traverse edge $e \in E$, for $\xi \in \Xi$.
- $\widetilde{u}^{\xi,e} > 0$ be the desired velocity at which the ξ-excursionists would like to traverse edge $e \in E^p \cup E^1$, for $\xi \in \Xi$.
- v_e the velocity at which the ξ-excursionists traverse edge $e \in E^2$, for $\xi = 2$.

We recall that Assumption 2(3) states that an excursionist decides the velocity $u^{\xi,e}$ at which he/she traverses an edge $e \in E^p \cup E^1$, $\xi \in \Xi$. Differently, he/she cannot decide the velocity at which he/she traverses an edge $e \in E^\xi$, if $\xi = 2$. In the former case, the velocity $u^{\xi,e}$ is the excursionist control. It is a function $u^{\xi,e} : [0, T] \to \mathbb{R}_+$ for $e \in E^p \cup E^1$, $\xi \in \Xi$ which is measurable and integrable, namely $u^{\xi,e} \in L^1(0, T)$ for $t \in [0, T]$. In the latter case, $u^{\xi,e} = v_e = const > 0$, that may be considered as a degenerate control. In both cases, the non-negativity of $u^{\xi,e}$ is imposed by Assumption 2.

The following conditions describe the dynamics of each ξ-excursionist, $\xi \in \Xi$, on each edge $e \in E$.
Let $x^{\xi,e} \in [0, l_e]$ be the state of each ξ-excursionist, $\xi \in \Xi$, over an edge $e = (\nu_e, \kappa_e) \in E$. Specifically, the value $x^{\xi,e}(s)$ is the position of the ξ-excursionist at time s from the tail vertex ν_e of e. Then, the controlled dynamics of an excursionist who entered the edge $e \in E$ at time $t \in [0, T]$ is:

$$\dot{x}^{\xi,e}(s) = u^{\xi,e}(s) \quad \forall s \in]t, T], \tag{2a}$$

$$x^{\xi,e}(t) = 0, \tag{2b}$$

where, in particular, $u^{\xi,e} = v_e > 0$ if $e \in E^2$. Condition (2b) trivially states that a ξ-excursionist that reaches edge $e = (\nu_e, \kappa_e)$ at time t enters e through its tail vertex ν_e. The excursionist is inside the edge e as long as $0 \leq x^{\xi,e}(s) \leq l_e$. He/she reaches the head vertex κ_e and exits from e through κ_e at time $\tau = \min\{s > t : x^{\xi,e}(s) = l_e\}$. We also note that (2a) describes the state evolution of a hypothetical ξ-excursionist assumed to be in ν_e at time t, independently of the fact whether there is actually someone present at ν_e at that time.

We refer to the principle by McDowall [33] to formulate the excursionists' objective function. It says: "the excursionists form their judgment [on their visiting experiences] by comparing their actual experiences with their expectations. If their actual experiences exceed their expectations, they will become satisfied excursionists. If not, they will be dissatisfied or unhappy".

We assume that every excursionist wants to maximize the satisfaction that he/she acquires while visiting each POI or equivalently to minimize the cost that he/she pays at each POI. According to the literature, the satisfaction gained at visiting a POI depends on the excursionist's preference for the POI itself [13, 38] and the time spent in it [20]. Moreover, the overall cost also depends on a congestion cost defined as in [7] as the pain for an excursionist of being entrapped in a highly congested edge.

With regard to the time spent at visiting the POIs, we consider the difference between the *actual time spent* and the *desired time spent* for visiting a POI.

We assume that aim of a ξ-excursionist is to minimize the whole cost for traversing the route $r \in \Gamma$ that he/she has a priori chosen. The cost of traversing each edge e of the route r depends on the edge type and it can be defined as follows:

$$J^{\xi,e}(t, u^{\xi,e}) = \chi_{\{e \in E^1\}} \Bigg\{ \int_t^T \chi_{\{0 \leq x^{\xi,e}(s) \leq l_e\}} \left(\frac{(u^{\xi,e}(s))^2}{2} + \varphi_e\Bigg(\sum_{\hat{r} : e \in \hat{r}} \rho_{\hat{r}}^{\xi,e}(s) \Bigg) \right) \mathrm{d}s$$

$$+ \chi_{\{0 \leq x^{\xi,e}(T) < l_e\}} \alpha_1 \sum_{j \in r_e^{\xi}} l_j \Bigg\} \tag{3a}$$

$$+ \chi_{\{e \in E^2\}} \left\{ \int_t^T \varphi_e \left(\sum_{\hat{r} : e \in \hat{r}} \rho_{\hat{r}}^{\xi,e}(s) \right) \mathrm{d}s + \chi_{\{0 \leq x^{\xi,e}(T) < l_e\}} \alpha_2 \sum_{j \in r_e^{\xi}} l_j \right\} \tag{3b}$$

$$
\begin{aligned}
&+ \chi_{\{e\in E^p\}}\Bigg\{\int_t^T \Bigg(c_{\xi,1}\frac{(u^{\xi,e}(s))^2}{2} + c_{\xi,2}\widetilde{\varphi}_e\Bigg(\sum_{\xi\in\Xi}\sum_{\hat{r}\,:\,e\in\hat{r}}\rho_{\hat{r}}^{\xi,e}(s)\Bigg)\\
&- \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_e} + \frac{c_{\xi,4}(\widetilde{u}^{\xi,e}-u^{\xi,e}(s))^2}{2}\Bigg)\mathrm{d}s\\
&+ \chi_{\{0\le x^{\xi,e}(T)<l_e\}}\Delta\Bigg(q_e + \sum_{j\in r^{\xi}_{succ(e)}} l_j\Bigg)\Bigg\},
\end{aligned} \tag{3c}
$$

where χ is the characteristic function

$$
\chi_{\{condition\}} = \begin{cases} 1 \text{ if condition holds true,}\\ 0 \text{ otherwise,}\end{cases}
$$

$\alpha_1 > 0, \alpha_2 > 0$ and $\Delta > 0$ are constant parameters representing costs per unit of length; r_e^{ξ} is the shortest route for ξ-excursionists from the tail v_e to the destination d ; $q_e > 0$ is the intangible cost for not having experienced a complete visit of the attraction, i.e., for not reaching κ_e; $r^{\xi}{}_{succ(e)}$ is the shortest route for ξ-excursionists from the tail vertex $v_{e'}$ of the edge e' following e still in the same route r to reach the destination d; the constants $c_{\xi,j} > 0$, $j \in \{1\ldots,4\}$, express the weights that the excursionists assign to the different components of the cost depending on their orientation as well as on other behavioral characteristics, such as if they are more likely to adapt to an unforeseen/undesirable situation.

The cost of traversing each edge e is made of three parts: (3a)–(3c). The component (3a) is paid if $e \in E^1$. It takes into account: (i.a) the possible hassle of running in the edge to reach d on time; (ii.a) the pain of being entrapped in a highly congested edge; (iii.a) the disappointment of not being able to reach d by the final horizon T. The component (3b) is paid if $e \in E^2$. It takes into account: (i.b) the pain of being entrapped on a crowded mean of transport; (ii.b) the disappointment of not being able to reach d by T. The component (3c) is paid if $e \in E^p$. It takes into account: (i.c) the hassle of running through the POI; (ii.c) the pain of being entrapped in a highly crowded POI (highly aggregated tourist crowds have, in fact, a negative effect not only on the visit experience but also on safety issues); (iii.c) the sightseeing value of the POI which, for reasons that will be detailed below, we model by using a hyperbolic tangential function as a saturation function; (iv.c) the difference between the desired time spent and the actual time spent for the visit of the POI and, finally, (v.c) the penalty for not leaving the POI within the scheduled time and not having reached d at time T. We remark that the excursionist's preference for a POI is implicitly defined in (iii.c) where different values of the coefficient $c_{\xi,3}$ imply a different level of attraction assigned by ξ-type excursionists toward a particular POI.

The first term inside the integral of (3a) stands for the cost component (i.a). The second term stands for the congestion cost component (ii.a) and it is characterized by the congestion function

$$
\varphi_e : [0, +\infty[\to [0, +\infty[. \tag{4}
$$

The last term outside the integral in (3a) stands for the cost component iii.a). Such a term is equal to the length of the shortest path on the network G from tail ν_e of edge e to the destination d, if at the final time T the excursionist is still inside the edge, i.e., he/she has not reached the head κ_e yet; it is equal to zero otherwise. We remark that an excursionist who is on the head κ_e of an edge e is considered also on the tail ν'_e, and hence inside, of a successive edge e'. Then, the final cost paid by an excursionist, still inside the network G at the final time T, is always equal to the distance from the tail of his/her current edge to d.
The term inside the integral of (3b) is equivalent to the second term of (3a). The term outside the integral in (3b) is similar to the corresponding one in (3a).
The first term inside the integral of (3c) stands for the cost component i.c). The second term, expressed by the congestion function

$$\widetilde{\varphi}_e : [0, +\infty[\to [0, +\infty[, \tag{5}$$

defines the cost component (ii.c); the third term which stands for the cost component (iii.c), measures the tourists' emotional experience while visiting the POI and, as shown in literature, it changes over time. In particular, it has been observed that tourists' marginal satisfaction starts to decrease at some point in time. This phenomenon is referred to as *aesthetic fatigue* or *accumulated satisfaction*, and it can be related to the standard economic assumption of decreasing marginal profit [37]. We model the above mentioned concept by the hyperbolic tangent function as it naturally models tourists' emotional saturation over the duration time of their visit at the same POI. Tourist start their visit with full of enthusiasm (increasing marginal satisfaction), but as the time passes, their marginal satisfaction starts to decrease as a consequence of a prolonged exposition to the same stimuli. Moreover, in our work the above quantity is scaled down by a factor equal to $1/m_e$, where m_e is the ticket price that tourists pay for visiting the POI: the higher the price they pay, the less satisfaction they acquire from the visit. This latter assumption is justified from evidence in literature which suggest a negative relationship between the touristic price and the degree of tourists' satisfaction.

The last term inside the integral in (3c) stands for the cost component (iv.c). It is proportional to the squared difference between the desired velocity $\widetilde{u}^{\xi,e}$ and the actual velocity $u^{\xi,e}$ of the excursionist that traverses the POI. Finally, the last term stands for the cost component (v.c).

Given (3), we define the cost that the ξ-excursionists would pay along of each route r at time $t \in [0, T]$ as:

$$J_r^{\xi}(t) = \sum_{e \in r} J^{\xi,e}(t_r^{\xi,e}(t), u_r^{\xi,e}). \tag{6}$$

In (6), $u_r^{\xi,e} \in L^1(0, T)$, for every $e \in r$, is the optimal control implemented along the edges by ξ-excursionists who are in the route r. We will prove that this control is constant and we will show how to compute it in the next section. Moreover, $t_r^{\xi,e}(t)$ is the time instant at which the ξ-excursionist, entering G in the origin o at time t and following the route r, reaches v_e using the controls $u_r^{\xi,e}$. We take $t_r^{\xi,e}(t) = \infty$ if an excursionist does not reach e within T and we define $J^{\xi,e}(\infty, u_r^{\xi,e}) = 0$.

Hereinafter, the following assumption on the excursionists' behavior holds true.

Assumption 3

1. The throughput λ is $C^1([0,T])$ and $\lambda(t) > 0$ for all $t \in [0,T]$. In particular this implies that there exist $0 < \underline{\lambda} \le \overline{\lambda} < +\infty$ such that $\underline{\lambda} \le \lambda(t) \le \overline{\lambda}$ for all $t \in [0,T]$.
2. The initial mass of excursionists is null, i.e., $\rho(0) = 0$.
3. For every $e \in E$, the congestion cost functions φ_e and $\widetilde{\varphi}_e$ are continuous, bounded and they only depend on the masses $\rho_r^{\xi,e}$ and not on the state variable $x^{\xi,e}$.
4. When more than one optimal control is available, excursionists choose the smallest one.

Assumption 3.2 means that no one is around the network at $t = 0$, while Assumption 3.3 implies that all excursionists on the same edge at the same time pay the same congestion cost. Assumption 3.4 implies that excursionists prefer to move slower than faster when they must choose.

We describe the mass' evolution of the ξ-excursionists as a conservation law for every non-destination vertex and outward-directed edge $e \in r$, $r \in \Gamma$:

$$\dot{\rho}(t) = \mathcal{H}(f(t)), \quad \rho(0) = \rho_0, \tag{7}$$

where the flow $t \mapsto f(t)$ is described next and $\mathcal{H} : \mathbb{R}^\Theta \to \mathbb{R}^\Theta$ is defined for every $t \in [0,T]$, by

$$H_r^{\xi,e}(f(t)) = \left(\lambda^\xi(t)\Pi_r^{\xi,e} + f_r^{\xi,prec_r(e)}(t)\right) - f_r^{\xi,e}(t), \quad \forall \xi \in \Xi,\ \forall r \in \Gamma,\ \forall e \in r, \tag{8}$$

where the function $prec_r(e)$ returns the edge that precedes e on the route r, if it exists; $f_r^{\xi,e}(t)$ is the component of the flow vector $f(t)$ that represents the outgoing flow of ξ-excursionists from the edge $e \in r$ at time t; finally, $\Pi_r^{\xi,e}$ is the percentage of ξ-excursionists entering G in o that choose route $r \in \Gamma$, if $e = (o, \kappa_e) \in E^p$ is the edge that traverses the entrance POI, and, hence, it is the first edge of every route; differently, $\Pi_r^{\xi,e} = 0$ for all the other edges $e \in E$. Note that $\Pi \in \mathbb{R}^\Theta$ is a constant vector because of Assumption 2.

Following Bagagiolo et al. [8] excursionists assess the outgoing flow out of an edge e at time t assuming a constant traverse time which depends on the type of edge. In particular, the traverse time is fixed and equal to l_e/v_e if $e \in E^2$.

We denote by ω the traverse time in $]0,T]$ of an edge $e \in E^p$ which justifies the choice of a null optimal control $u_r^{\xi,e}(t)$ by a ξ-excursionist. It is the maximum time such that it is not convenient to traverse the edge e for a ξ-excursionist reaching the tail vertex of e at any $t \in [T-\omega, T]$, as the cost of running through the edge $e \in E^p$ to reach d at T is for sure greater than the cost of the disappointment of not being able to reach d.

Then, we define the outgoing flow for the edge traversing the entrance POI as:

$$f_r^{\xi,e}(t) = \begin{cases} 0 & t \in [0, \omega], \\ \lambda^{\xi}(t-\omega)\Pi_r^{\xi,e} sign(u_r^{\xi,e}[t-\omega]) & t \in [\omega, T]. \end{cases} \tag{9a}$$

Differently, the outgoing flow for all the other edges $e \in E^p$ is:

$$f_r^{\xi,e}(t) = \begin{cases} 0 & t \in [0, \omega], \\ f_r^{\xi,prec_r(e)}(t-\omega) sign(u_r^{\xi,e}[t-\omega]) & t \in [\omega, T], \end{cases} \tag{9b}$$

where $u_r^{\xi,e}[t-\omega] \geq 0$ is the constant optimal control implemented by a ξ-excursionist who, following route r, enters the edge $e \in E^p$ at time $t-\omega$, and $sign(\sigma) = 1$ if $\sigma > 0$ and $sign(\sigma) = 0$ if $\sigma = 0$.
We now denote by $k \in]0, T]$ the traverse time for every $e \in E^1$ for which the same considerations given above for the traverse time ω of each $e \in E^p$ hold. Then, the outgoing flow for each $e \in E^1$ is:

$$f_r^{\xi,e}(t) = \begin{cases} 0 & \text{if } t \in [0, k], \\ f_r^{\xi,prec_r(e)}(t-k) sign(u_r^{\xi,e}[t-k]) & \text{if } t \in [k, T], \end{cases} \tag{10}$$

where $u_r^{\xi,e}[t-k] \geq 0$ is the constant optimal control implemented by a ξ-excursionist who, following route r, enters the edge $\in E^1$ at time $t-k$. Finally, for $e \in E^2$ the outgoing flow is:

$$f_r^{\xi,e}(t) = \begin{cases} 0 & \text{if } t \in [0, l_e/v_e], \\ f_r^{\xi,prec_r(e)}\left(t - \frac{l_e}{v_e}\right) & \text{if } t \in [l_e/v_e, T]. \end{cases} \tag{11}$$

Remark 3.1 By imposing conditions (9), one gets that an excursionist entering $e \in E^p$ at time $t-\omega$ estimates the outgoing flow $f_r^{\xi,e}(t)$ by assuming that all the other excursionists who are currently on e and are following the same route r, are implementing the same controls $u_r^{\xi,e}[t-\omega]$ as him/her-self. Similar considerations also apply to (10) and (11). Of course, a more precise formulation of the outgoing flow should consider the actual value of the control [and not only its sign as in (9)–(10)] and estimate the actual traverse time (something similar in this direction is made in [6]). Similarly, the mass ρ that satisfies (7) may be more precisely defined to represent the actual dynamics of the excursionists.
However, the estimation of flows and masses proposed in this paper may be seen as an approximation used by a network manager that must elaborate in real time the information to distribute to the excursionists for strategically controlling the flows.
The analysis of the discrepancy between the approximated flows and masses and the actual ones will be considered in future research. Here, we emphasize that the estimated flows $f_r^{\xi,e}$, when implemented in (7), ensure that the principle of mass conservation

is satisfied. For example when $\rho_0 \equiv 0$, the actual total mass present in the network is the mass entered through the origin:

$$\sum_{\xi \in \Xi} \sum_{e \in E} \sum_{r\,:\,e \in r} \rho_r^{\xi,e}(t) = \int_0^t \lambda(s)\,\mathrm{d}s \quad \forall\, t \in [0, T].$$

Remark 3.2 Note that for all $t \in [0, T]$ all the components $\rho_r^{\xi,e}$ of ρ are uniformly bounded by

$$0 \leq \rho_r^{\xi,e}(t) \leq K \equiv \int_0^T \lambda^{\xi}(s)\,\mathrm{d}s. \tag{12}$$

Moreover, by (7)–(11), and by Assumption 3.1, we have that any solution ρ of (7) is Lipschitz continuous with Lipschitz constant $L = 3\overline{\lambda}$, independently of the optimal control and of the initial value ρ_0.

4 Optimization and Control

Given a vector mass concentration $\rho(t)$, $t \in [0, T]$, we define the following *value function*, representing the optimum cost that ξ-excursionists, entering edge e of route r at time t, must pay for traversing e:

$$\begin{aligned}
\mathcal{V}_r^{\xi,e}(t) = \inf_{u_r^{\xi,e} \in L^1} \Bigg\{ & \chi_{\{e \in E^1\}} \Bigg\{ \int_t^{T \wedge \tau} \left(\frac{(u^{\xi,e}(s))^2}{2} + \varphi_e \left(\sum_{\hat{r}\,:\,e \in \hat{r}} \rho_{\hat{r}}^{\xi,e}(s) \right) \right) \mathrm{d}s + \Phi_r^{\xi,e}(T \wedge \tau) \Bigg\} \\
& + \chi_{\{e \in E^2\}} \Bigg\{ \int_t^{T \wedge \tau} \varphi_e \left(\sum_{\hat{r}\,:\,e \in \hat{r}} \rho_{\hat{r}}^{\xi,e}(s) \right) \mathrm{d}s + \mathcal{F}_r^{\xi,e}(T \wedge \tau) \Bigg\} \\
& + \chi_{\{e \in E^p : e \in r \setminus \{last(r)\}\}} \Bigg\{ \int_t^{T \wedge \tau} \Bigg(c_{\xi,1} \frac{(u^{\xi,e}(s))^2}{2} + c_{\xi,2} \widetilde{\varphi}_e \left(\sum_{\xi \in \Xi} \sum_{\hat{r}\,:\,e \in \hat{r}} \rho_{\hat{r}}^{\xi,e}(s) \right) \\
& - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_e} + \frac{c_{\xi,4}(\widetilde{u}^{\xi,e} - u^{\xi,e}(s))^2}{2} \Bigg) \mathrm{d}s \\
& + \Psi_r^{\xi,e}(T \wedge \tau) \Bigg\} + \chi_{\{e \in E^p : e = last(r)\}} \left\{ J^{\xi,e}(t, u_r^{\xi,e}) \right\} \Bigg\}.
\end{aligned} \tag{13}$$

In (13), τ is the first exit time from the edge $e \in E$, i.e., $\tau = \min\{s \in]t, T]$: (2) holds and $x^{\xi,e}(s) = l_e\}$ is the time at which the excursionists reach the head of e and exit from e through κ_e. In particular, when $e \in E^2$ the exit time τ is $\tau = t + (l_e/v_e)$ due to the constant velocity v_e to traverse the edge e. Also in (13), function $last(r)$ returns the last edge of a route r used by ξ-excursionists while the exit costs $\Phi_r^{\xi,e}(T \wedge \tau)$, $\mathcal{F}_r^{\xi,e}(T \wedge \tau)$ and $\Psi_r^{\xi,e}(T \wedge \tau)$ are given by

$$
\Phi_r^{\xi,e}(T \wedge \tau) = \begin{cases} \mathcal{V}_r^{\xi,succ_r(e)}(\tau) & \text{if } \tau < T, \\ \alpha_1 \sum_{j \in r_e^{\xi}} l_j & \text{if } \tau > T, \\ \min\left\{ \alpha_1 \sum_{j \in r_e^{\xi}} l_j, \mathcal{V}_r^{\xi,succ_r(e)}(\tau) \right\} & \text{if } \tau = T, \end{cases} \tag{14}
$$

$$
\mathcal{F}_r^{\xi,e}(T \wedge \tau) = \begin{cases} \mathcal{V}_r^{\xi,succ_r(e)}(\tau) & \text{if } \tau < T, \\ \alpha_2 \sum_{j \in r_e^{\xi}} l_j & \text{if } \tau > T, \\ \min\left\{ \alpha_2 \sum_{j \in r_e^{\xi}} l_j, \mathcal{V}_r^{\xi,succ_r(e)}(\tau) \right\} & \text{if } \tau = T, \end{cases} \tag{15}
$$

$$
\Psi_r^{\xi,e}(T \wedge \tau) = \begin{cases} \mathcal{V}_r^{\xi,succ_r(e)}(\tau) & \text{if } \tau < T, \\ \Delta\left(q_e + \sum_{j \in r^{\xi}_{succ(e)}} l_j \right) & \text{if } \tau > T, \\ \min\left\{ \Delta\left(q_e + \sum_{j \in r^{\xi}_{succ(e)}} l_j \right), \mathcal{V}_r^{\xi,succ_r(e)}(\tau) \right\} & \text{if } \tau = T, \end{cases} \tag{16}
$$

where $succ_r(e)$ is the function which returns the edge that follows e on route r followed by ξ-excursionists.

The value function (13) is recursively and backwardly defined, starting from the traversing edge of the exit POI, i.e., the edge e such that $\kappa_e = d$. We remark that this definition is valid as we assumed that network G has no cycles and hence self-referring is prevented.

The value function (13) will be reintroduced in Sect. 5, where we study the existence of a mean field equilibrium.

Note that (14)–(16) are non usual exit costs of (13) and they may be discontinuous in τ. This fact implies the discontinuity of the Hamiltonian associated to the value function and/or of the boundary conditions. In this paper, instead of considering discontinuous HJB equations, following Bagagiolo et al. [8], we will write optimality conditions in terms of the value function for the exit time/exit cost problem on each edge.

The value function (13), which includes the term $(u^{\xi,e})^2$, is upper bounded as an example, by the cost of deciding to spend all the time interval $[0, T]$ at the entrance POI. This fact implies the optimal value of the control $u^{\xi,e}$ is upper bounded. Moreover, (13) does not depend on the position $x^{\xi,e}$ of the ξ-excursionists on the edge $e \in r$, because, as we will prove in the following, the optimal control of the excursionists traversing an edge e is a constant control $u_r^{\xi,e} \geq 0$, whose value is chosen when entering the edges in $E^p \cup E^1$, or fixed to v_e when $e \in E^2$. This result is a consequence of the structure of the congestion functions φ_e and $\widetilde{\varphi}_e$ that do not depend on the state position of the single excursionist.

We start our analysis by considering an edge $e \in E^p$. We consider a ξ-excursionist leaving the tail ν_e of e at time t' and reaching the head κ_e of e at time t''. Moreover, we suppose the mass ρ as given, as it will be done in Sect. 5. Under the above hypotheses, the component

$$\int_{t'}^{t''} c_{\xi,2}\widetilde{\varphi}_e \left(\sum_{\xi \in \Xi} \sum_{\hat{r}\,:\,e\in\hat{r}} \rho_{\hat{r}}^{\xi,e}(s) \right) - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_e} + \frac{c_{\xi,4}}{2}(\widetilde{u}^{\xi,e})^2 \, \mathrm{d}s, \quad (17)$$

of the cost (3c) is given.

We here emphasize some points which follow from the assumptions we imposed and related to the optimal behavior of the excursionists. The following discussion moves from the study in [8] with the difference that here the optimal control chosen by the excursionists is a function of the edge type:

(i) When t'' is chosen, Problem (13) reduces to determining the optimal control that optimize the quantity $\int_{t'}^{t''} \left(\frac{c_{\xi,1}+c_{\xi,4}}{2}(u^{\xi,e}(s))^2 - c_{\xi,4}\widetilde{u}^{\xi,e}u^{\xi,e}(s) \right) \mathrm{d}s$ of the cost $J^{\xi,e}$. The above expression rules out the possibility that an optimal control for a ξ-excursionist is to remain at ν_e (i.e, to choose $u_r^{\xi,e} = 0$) for a non-null time interval and then move later or, similarly, to stop and stay still in an intermediate point of the edge for a non-null time interval; or to go back and forth along edge e (as required by Assumption 2).

(ii) The solution to the problem considered in the previous point i) is an optimal control constant and equal to $u_r^{\xi,e} = \frac{l_e}{t''-t'}$. With such a choice of the control, in (13), it is $\tau = t''$ which changes according to the type ξ of excursionist that we are considering. Indeed, to different values of $\xi \in \Xi$ correspond different values of the constants $c_{\xi,j}$, $j \in \{1, \dots, 4\}$ and hence a different arrival time τ to the head vertex κ_e.

(iii) Points (i)–(ii) imply that ξ-excursionists implementing their optimal controls can neither accumulate on points strictly internal to an edge nor overtake each other along the edge. As a consequence, if at time t the optimal control is $u_r^{\xi,e} = 0$, and hence the arrival time is $+\infty$, then $u_r^{\xi,e} = 0$ will be the unique optimal control from t onward.

(iv) if $e \in E^p$ is the edge of the exit POI, a control that allows an excursionist to reach d before T and wait there for a non-null time interval is certainly not optimal. Indeed, the excursionist would pay the congestion costs in d as well as the sightseeing value of the POI while waiting in d, in addition to the cost of running [see the cost (17)].

(v) It is possible that a ξ-excursionist moving from ν_e at time t' may have multiple alternative optimal controls $u_r^{\xi,e}$ and then he/she may choose whether to reach κ_e at t_1'' or at time $t_2'' > t_1''$. In this case, only ξ-excursionists entering the edge e at time t' may reach κ_e at a time $t'' \in [t_1'', t_2'']$, since optimally behaving ξ-excursionists cannot get over each other along the edge e. In this contest, for every $t \in [0, T]$, let us define

$$\tau^*_{\xi,e,r}(t) = \max\left\{\tau \in]t, T] : u^{\xi,e}_r \equiv \frac{l_e}{\tau - t} \text{ is optimal}\right\}, \tag{18}$$

that is, $\tau^*_{\xi,e,r}(t)$ is the last time in $]t, T]$ at which the ξ-excursionist in ν_e may reach κ_e implementing an optimal control. Assumption 3.4 forces the ξ-excursionists to choose the smallest optimal velocity. Hence, the ξ-excursionists actually arrive in κ_e at time $\tau^*_{\xi,e,r}(t)$, if $\tau^*_{\xi,e,r}(t)$ exists, otherwise he/she stops in ν_e indefinitely. The optimal control implemented by the ξ-excursionist is

$$u^{\xi,e}_r \equiv \begin{cases} \frac{l_e}{\tau^*_{\xi,e,r}(t)-t} & \text{if } \tau^*_{\xi,e,r}(t')\text{exists,} \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Consider now a ξ-excursionist that, in an edge $e \in E^1$, moves from ν_e at time t' and reaches κ_e at time t'' and, as before, we suppose the mass concentration ρ as given. The component $\int_{t'}^{t''} \varphi_e\left(\sum_{\hat{r}\,:\,e\in\hat{r}} \rho^{\xi,e}_{\hat{r}}(s)\right) \mathrm{d}s$ of the cost in (13) can be then assumed as given, whenever the ξ-excursionist in ν_e at time t' decides to reach κ_e at time t''. The facts (i)–(iii) and (v) considered before for a ξ-excursionist inside every edge $e \in E^p$, continue to apply also in this case with simple appropriate changes. For example, in the point (i) when t'' is chosen, the only quantity to minimize by the ξ-excursionist is $\frac{1}{2}\int_{t'}^{t''} (u^{\xi,e}(s))^2\,\mathrm{d}s$ and hence by (ii) the constant optimal control is $u^{\xi,e}_r = \frac{l_e}{t''-t'}$. Using that control, in (13), it is $\tau = t''$. Points (iii) and (v) do not require any change unless to consider $e \in E^1$, while point iv) does not apply since the exit POI does not include any edge $e \in E^1$.

Instead, for a ξ-excursionist moving on an edge $e \in E^2$, since the optimal control is fixed, $u^{\xi,e} = v_e$, only some of the above considerations apply. Point (i) continues to hold. Points (ii) and (v) does not hold since the ξ-excursionist cannot choose his/her arrival time t'' which, in this case, is equal to $t' + (l_e/v_e)$. Point (iii) is only partially satisfied since $u_{\xi,e} = 0$ cannot be an optimal control. Point iv) does not hold since the exit POI does not include any edge $e \in E^2$.

Remark 4.1 Function $t \mapsto \tau^*_{\xi,e,r}(t)$ defined in (18), relative to ξ-excursionists moving on $e \in E^p \cup E^1$, is an increasing function, whenever it exists. Hence it is continuous almost everywhere and defines the unique optimal control (19).

To simplify notations and statements, hereinafter, we consider a graph G on which every ξ-excursionist has only four possible routes to reach d starting from o (see Fig. 2).

Accordingly, for every $\xi \in \Xi$, the corresponding set of routes is $\Gamma = \{r_1, r_2, r_3, r_4\}$ where $r_1 = (e_0, e_1, e_3, e_8, e_{11})$, $r_2 = (e_0, e_2, e_5, e_7, e_9, e_{10}, e_{11})$, $r_3 = (e_0, e_1, e_3, e_4, e_5, e_7, e_9, e_{10}, e_{11})$, $r_4 = (e_0, e_1, e_3, e_6, e_9, e_{10}, e_{11})$.
However, all the results proved in the following continue to be valid for any directed acyclic networks such that any vertex can be reached from the origin o and the desti-

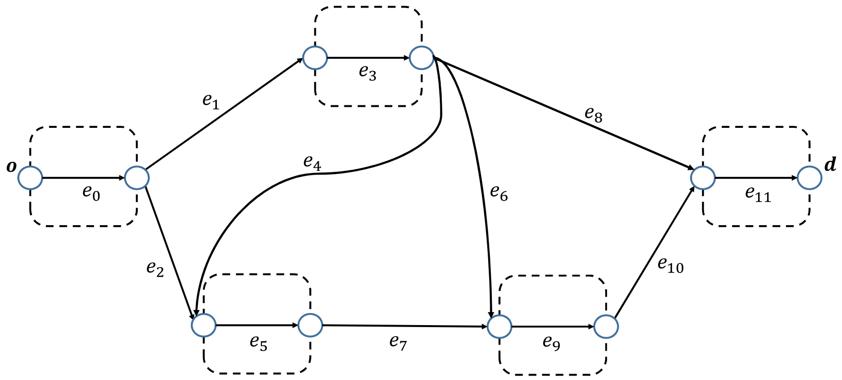

Fig. 2 The graph topology used in the paper

nation d is reachable from any vertex. Moreover, from now on we denote by

$$u = \{u_r^{\xi,e}[\cdot] : \xi \in \varXi, e \in r,\ r \in \varGamma,\ u_r^{\xi,e}[\cdot] \geq 0\},$$

the controls' vector, and we do not report the arguments $\sum_{\hat{r}\,:\,e\in\hat{r}} \rho_{\hat{r}}^{\xi,e}$ and $\sum_{\xi\in\varXi} \sum_{\hat{r}\,:\,e\in\hat{r}} \rho_{\hat{r}}^{\xi,e}$ whenever they are not strictly necessary. Finally, we characterize the positions of excursionists by the *edge-route pair* $(e, r) \in E \times \varGamma$ to mean that the excursionists are on edge e following route r.

A ξ-excursionist standing at $\nu_{e_{11}}$ at time $t \in [0, T[$, for the edge-route pairs $(e_{11}, r), \forall r \in \varGamma$, has two possible choices: either staying at $\nu_{e_{11}}$ indefinitely or moving to reach the destination d exactly at time T. Accordingly, the candidate constant optimal controls to be chosen at time t are:

$$u_{r,1}^{\xi,e_{11}}[t] \equiv 0, \qquad u_{r,2}^{\xi,e_{11}}[t] \equiv l_{e_{11}}/(T-t). \tag{20}$$

Hence, given the cost functional (3), we obtain the following structure for the value function (13):

$$\begin{aligned} \mathcal{V}_r^{\xi,e_{11}}(t) = \min\Bigg\{ & \Delta q_{e_{11}} + \frac{c_{\xi,4}}{2}(\widetilde{u}^{\xi,e_{11}})^2\,,\ \frac{c_{\xi,1}}{2}\frac{(l_{e_{11}})^2}{T-t} \\ & + \frac{c_{\xi,4}}{2}\left(\widetilde{u}^{\xi,e_{11}} - \frac{l_{e_{11}}}{T-t}\right)^2 (T-t)\Bigg\} \\ & + \int_t^T \left(c_{\xi,2}\widetilde{\varphi}_{e_{11}} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_{11}}}\right) \mathrm{d}s. \end{aligned} \tag{21}$$

Consider a ξ-excursionist at ν_e for the pairs $(e, r) \in \{(e_8, r_1), (e_{10}, r_2), (e_{10}, r_3), (e_{10}, r_4)\}$ at time $t \in [0, T]$. If $e \in E^1$ then the excursionist has two possible choices: either staying at ν_e indefinitely or moving to reach κ_e at a certain time $\tau \in]t, T]$. Consequently, he/she has to choose between the following two kinds of candidate constant optimal controls:

$$u_{r,1}^{\xi,e}[t] \equiv 0, \qquad u_{r,2}^{\xi,e}[t] \equiv l_e/(\tau - t). \tag{22}$$

If $e \in E^2$ the ξ-excursionist can not sit still at ν_e nor decide the velocity to reach κ_e. In fact, he/she uses a mean of transport and the travel time is (l_e/v_e). In particular, we get a value function whose value depends on whether the arrival time $t + (l_e/v_e)$ is less than or greater than the final horizon T:

$$\begin{aligned}
\mathcal{V}_r^{\xi,e}(t) = {} & \chi_{\{e \in E^1\}} \min \Bigg\{ \alpha_1(l_e + l_{e_{11}}) + \int_t^T \varphi_e \, \mathrm{d}s, \\
& \inf_{\tau \in]t,T]} \left\{ \frac{1}{2} \frac{(l_e)^2}{\tau - t} + \int_t^\tau \varphi_e \, \mathrm{d}s + \mathcal{V}_r^{\xi,e_{11}}(\tau) \right\} \Bigg\} \\
& + \chi_{\{e \in E^2\}} \Bigg\{ \int_t^{T \wedge \left(t + \frac{l_e}{v_e}\right)} \varphi_e \, \mathrm{d}s + \chi_{\{t + \frac{l_e}{v_e} < T\}} \mathcal{V}_r^{\xi,e_{11}} \left(t + \frac{l_e}{v_e} \right) \\
& + \chi_{\left\{t + \frac{l_e}{v_e} > T\right\}} \alpha_2(l_e + l_{e_{11}}) \\
& + \chi_{\left\{t + \frac{l_e}{v_e} = T\right\}} \min \left\{ \mathcal{V}_r^{\xi,e_{11}} \left(t + \frac{l_e}{v_e} \right), \alpha_2(l_e + l_{e_{11}}) \right\} \Bigg\}.
\end{aligned} \tag{23}$$

A ξ-excursionist standing at ν_{e_9} at time t for the pairs (e_9, r), $r \in \{r_2, r_3, r_4\}$, may choose between staying in ν_{e_9} or reaching κ_{e_9} at time $\tau \in]t, T]$. Accordingly, the candidate constant optimal controls are:

$$u_{r,1}^{\xi,e_9}[t] \equiv 0, \qquad u_{r,2}^{\xi,e_9}[t] \equiv l_{e_9}/(\tau - t); \tag{24}$$

the associated value function is:

$$\begin{aligned}
\mathcal{V}_r^{\xi,e_9}(t) = \min \Bigg\{ & \Delta(q_{e_9} + l_{e_{10}} + l_{e_{11}}) + \frac{c_{\xi,4}}{2} (\widetilde{u}^{\xi,e_9})^2 \\
& + \int_t^T \left(c_{\xi,2} \widetilde{\varphi}_{e_9} - \frac{c_{\xi,3}(\tanh(s-1) - 1)}{m_{e_9}} \right) \mathrm{d}s \, , \\
& \inf_{\tau \in]t,T]} \Bigg\{ \frac{c_{\xi,1}}{2} \frac{(l_{e_9})^2}{\tau - t} + \frac{c_{\xi,4}}{2} \left(\widetilde{u}^{\xi,e_9} - \frac{l_{e_9}}{\tau - t} \right)^2 (\tau - t) \\
& + \int_t^\tau \left(c_{\xi,2} \widetilde{\varphi}_{e_9} - \frac{c_{\xi,3}(\tanh(s-1) - 1)}{m_{e_9}} \right) \mathrm{d}s + \mathcal{V}_r^{\xi,e_{10}}(\tau) \Bigg\} \Bigg\}.
\end{aligned} \tag{25}$$

A ξ-excursionist at ν_e at time t for the pairs $(e, r) \in \{(e_6, r_4), (e_7, r_2), (e_7, r_3)\}$, has two possible choices when the considered edge belongs to E^1: either staying at ν_e indefinitely or moving to reach κ_e at a certain time $\tau \in]t, T]$. Accordingly, the candidate constant optimal controls are, respectively:

$$u_{r,1}^{\xi,e}[t] \equiv 0, \qquad u_{r,2}^{\xi,e}[t] \equiv l_e/(\tau - t). \tag{26}$$

If instead $e \in E^2$, the same arguments as in (23) hold (and, of course, they also holds for the other value functions associated to edges in E^2). The associated value function is:

$$
\begin{aligned}
\mathcal{V}_r^{\xi,e}(t) = \chi_{\{e\in E^1\}} \min \Bigg\{ & \alpha_1\Big(l_e + \sum_{j=9}^{11} l_{e_j}\Big) + \int_t^T \varphi_e \, \mathrm{d}s, \\
& \inf_{\tau\in]t,T]} \left\{ \frac{1}{2}\frac{(l_e)^2}{\tau - t} + \int_t^\tau \varphi_e \, \mathrm{d}s + \mathcal{V}_r^{\xi,e_9}(\tau) \right\} \Bigg\} \\
& + \chi_{\{e\in E^2\}} \Bigg\{ \int_t^{T\wedge\left(t+\frac{l_e}{v_e}\right)} \varphi_e \, \mathrm{d}s + \chi_{\left\{t+\frac{l_e}{v_e}<T\right\}} \mathcal{V}_r^{\xi,e_9}\left(t + \frac{l_e}{v_e}\right) \\
& + \chi_{\left\{t+\frac{l_e}{v_e}>T\right\}} \alpha_2 \Big(l_e + \sum_{j=9}^{11} l_{e_j}\Big) \\
& + \chi_{\left\{t+\frac{l_e}{v_e}=T\right\}} \min\left\{ \mathcal{V}_r^{\xi,e_9}\left(t + \frac{l_e}{v_e}\right), \alpha_2\Big(l_e + \sum_{j=9}^{11} l_{e_j}\Big) \right\} \Bigg\}.
\end{aligned}
\tag{27}
$$

A ξ-excursionist standing at ν_{e_5} at time t for the pairs (e_5, r), $r \in \{r_2, r_3\}$, may choose between staying in ν_{e_5} or reaching κ_{e_5} at time $\tau \in]t, T]$. Accordingly, the candidate constant optimal controls are:

$$
u_{r,1}^{\xi,e_5}[t] \equiv 0, \qquad u_{r,2}^{\xi,e_5}[t] \equiv l_{e_5}/(\tau - t); \tag{28}
$$

the associated value function is:

$$
\begin{aligned}
\mathcal{V}_r^{\xi,e_5}(t) = \min \Bigg\{ & \Delta\Big(q_{e_5} + l_{e_7} + \sum_{j=9}^{11} l_{e_j}\Big) + \frac{c_{\xi,4}}{2}(\widetilde{u}^{\xi,e_5})^2 \\
& + \int_t^T \left(c_{\xi,2}\widetilde{\varphi}_{e_5} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_5}} \right) \mathrm{d}s\,, \\
& \inf_{\tau\in]t,T]} \Bigg\{ \frac{c_{\xi,1}}{2}\frac{(l_{e_5})^2}{\tau - t} + \frac{c_{\xi,4}}{2}\left(\widetilde{u}^{\xi,e_5} - \frac{l_{e_5}}{\tau - t}\right)^2 (\tau - t) \\
& + \int_t^\tau \left(c_{\xi,2}\widetilde{\varphi}_{e_5} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_5}} \right) \mathrm{d}s + \mathcal{V}_r^{\xi,e_7}(\tau) \Bigg\} \Bigg\}.
\end{aligned}
\tag{29}
$$

Analogous arguments to (27) hold for computing $\mathcal{V}_{r_3}^{\xi,e_4}(t)$ when a ξ-excursionist is standing at ν_{e_4}. The candidate constant optimal controls are:

$$
u_{r_3,1}^{\xi,e_4}[t] \equiv 0, \qquad u_{r_3,2}^{\xi,e_4}[t] \equiv l_{e_4}/(\tau - t); \tag{30}
$$

the value function is:

$$
\begin{aligned}
\mathcal{V}_{r_3}^{\xi,e_4}(t) = \chi_{\{e_4\in E^1\}} \min\Bigg\{&\alpha_1\left(\sum_{e\in r_3\setminus\{e_0,e_1,e_3\}} l_e\right) + \int_t^T \varphi_{e_4}\,\mathrm{d}s,\\
&\inf_{\tau\in]t,T]}\left\{\frac{1}{2}\frac{(l_{e_4})^2}{\tau-t} + \int_t^\tau \varphi_{e_4}\,\mathrm{d}s + \mathcal{V}_{r_3}^{\xi,e_5}(\tau)\right\}\Bigg\}\\
&+\chi_{\{e_4\in E^2\}}\Bigg\{\int_t^{T\wedge\left(t+\frac{l_{e_4}}{v_{e_4}}\right)} \varphi_e\,\mathrm{d}s + \chi_{\left\{t+\frac{l_{e_4}}{v_{e_4}}<T\right\}}\mathcal{V}_{r_3}^{\xi,e_5}\left(t+\frac{l_{e_4}}{v_{e_4}}\right)\\
&+\chi_{\left\{t+\frac{l_{e_4}}{v_{e_4}}>T\right\}}\alpha_2\Bigg(\sum_{e\in r_3\setminus\{e_0,e_1,e_3\}} l_e\Bigg)\\
&+\chi_{\left\{t+\frac{l_{e_4}}{v_{e_4}}=T\right\}}\min\left\{\mathcal{V}_{r_3}^{\xi,e_5}\left(t+\frac{l_{e_4}}{v_{e_4}}\right), \alpha_2\Bigg(\sum_{e\in r_3\setminus\{e_0,e_1,e_3\}} l_e\Bigg)\right\}\Bigg\}. \quad (31)
\end{aligned}
$$

A ξ-excursionist standing at v_{e_3} at time t and following a route $r\in\{r_1,r_3,r_4\}$ may choose between staying in v_{e_3} or reaching κ_{e_3} at a certain $\tau\in]t,T]$. Hence, the candidate constant optimal controls are:

$$
u_{r,1}^{\xi,e_3}[t]\equiv 0, \qquad u_{r,2}^{\xi,e_3}[t]\equiv l_{e_3}/(\tau-t); \quad (32)
$$

the associated value function is:

$$
\begin{aligned}
\mathcal{V}_r^{\xi,e_3}(t) = \min\Bigg\{&\Delta\Bigg(q_{e_3} + \sum_{j\in r^\xi_{succ(e_3)}} l_j\Bigg) + \frac{c_{\xi,4}}{2}(\widetilde{u}^{\xi,e_3})^2\\
&+\int_t^T\left(c_{\xi,2}\widetilde{\varphi}_{e_3} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_3}}\right)\mathrm{d}s\,,\\
&\inf_{\tau\in]t,T]}\Bigg\{\frac{c_{\xi,1}}{2}\frac{(l_{e_3})^2}{\tau-t} + \frac{c_{\xi,4}}{2}\left(\widetilde{u}^{\xi,e_3} - \frac{l_{e_3}}{\tau-t}\right)^2(\tau-t)\\
&+\int_t^\tau\left(c_{\xi,2}\widetilde{\varphi}_{e_3} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_3}}\right)\mathrm{d}s + \mathcal{V}_r^{\xi,e}(\tau)\Bigg\}\Bigg\}, \quad (33)
\end{aligned}
$$

where:
(i) if $r=r_1$ then $\mathcal{V}_r^{\xi,e}(\tau)=\mathcal{V}_{r_1}^{\xi,e_8}(\tau)$; ii) if $r=r_3$ then $\mathcal{V}_r^{\xi,e}(\tau)=\mathcal{V}_{r_3}^{\xi,e_4}(\tau)$; iii) if $r=r_4$ then $\mathcal{V}_r^{\xi,e}(\tau)=\mathcal{V}_{r_4}^{\xi,e_6}(\tau)$.
Similarly to (31) we consider a ξ-excursionist standing at v_{e_2} to compute $\mathcal{V}_{r_2}^{\xi,e_2}(t)$. The candidate constant optimal controls for $e_2\in E^1$ are:

$$
u_{r_2,1}^{\xi,e_2}[t]\equiv 0, \qquad u_{r_2,2}^{\xi,e_2}[t]\equiv l_{e_2}/(\tau-t); \quad (34)
$$

the associated value function is:

$$\begin{aligned}\mathcal{V}_{r_2}^{\xi,e_2}(t) = \chi_{\{e_2\in E^1\}} \min\Bigg\{&\alpha_1\Bigg(\sum_{e\in r_2\backslash\{e_0\}} l_e\Bigg) + \int_t^T \varphi_{e_2}\,\mathrm{d}s,\\ &\inf_{\tau\in]t,T]}\left\{\frac{1}{2}\frac{(l_{e_2})^2}{\tau-t} + \int_t^\tau \varphi_{e_2}\,\mathrm{d}s + \mathcal{V}_{r_2}^{\xi,e_5}(\tau)\right\}\Bigg\}\\ &+ \chi_{\{e_2\in E^2\}}\Bigg\{\int_t^{T\wedge\left(t+\frac{l_{e_2}}{v_{e_2}}\right)} \varphi_{e_2}\,\mathrm{d}s + \chi_{\left\{t+\frac{l_{e_2}}{v_{e_2}}<T\right\}}\mathcal{V}_{r_2}^{\xi,e_5}\left(t+\frac{l_{e_2}}{v_{e_2}}\right)\\ &+ \chi_{\left\{t+\frac{l_{e_2}}{v_{e_2}}>T\right\}}\alpha_2\Bigg(\sum_{e\in r_2\backslash\{e_0\}} l_e\Bigg)\\ &+ \chi_{\left\{t+\frac{l_{e_2}}{v_{e_2}}=T\right\}}\min\left\{\mathcal{V}_{r_2}^{\xi,e_5}\left(t+\frac{l_{e_2}}{v_{e_2}}\right), \alpha_2\Bigg(\sum_{e\in r_2\backslash\{e_0\}} l_e\Bigg)\right\}\Bigg\}. \end{aligned} \tag{35}$$

A ξ-excursionist standing at ν_{e_1} at time t and following a route $r\in\{r_1, r_3, r_4\}$ may choose between staying in ν_{e_1} or reaching κ_{e_1} at a certain $\tau\in]t,T]$.

Hence, the candidate constant optimal controls for $e_1\in E^1$ are

$$u_{r,1}^{\xi,e_1}[t]\equiv 0, \qquad u_{r,2}^{\xi,e_1}[t]\equiv l_{e_1}/(\tau-t); \tag{36}$$

the associated value functions are:

$$\begin{aligned}\mathcal{V}_{r}^{\xi,e_1}(t) = \chi_{\{e_1\in E^1\}} \min\Bigg\{&\alpha_1\Bigg(\sum_{e\in r\backslash\{e_0\}} l_e\Bigg) + \int_t^T \varphi_{e_1}\,\mathrm{d}s,\\ &\inf_{\tau\in]t,T]}\left\{\frac{1}{2}\frac{(l_{e_1})^2}{\tau-t} + \int_t^\tau \varphi_{e_2}\,\mathrm{d}s + \mathcal{V}_{r}^{\xi,e_3}(\tau)\right\}\Bigg\}\\ &+ \chi_{\{e_1\in E^2\}}\Bigg\{\int_t^{T\wedge\left(t+\frac{l_{e_1}}{v_{e_1}}\right)} \varphi_{e_1}\,\mathrm{d}s + \chi_{\left\{t+\frac{l_{e_1}}{v_{e_1}}<T\right\}}\mathcal{V}_{r}^{\xi,e_3}\left(t+\frac{l_{e_1}}{v_{e_1}}\right)\\ &+ \chi_{\left\{t+\frac{l_{e_1}}{v_{e_1}}>T\right\}}\alpha_2\Bigg(\sum_{e\in r\backslash\{e_0\}} l_e\Bigg)\\ &+ \chi_{\left\{t+\frac{l_{e_1}}{v_{e_1}}=T\right\}}\min\left\{\mathcal{V}_{r}^{\xi,e_3}\left(t+\frac{l_{e_1}}{v_{e_1}}\right), \alpha_2\Bigg(\sum_{e\in r\backslash\{e_0\}} l_e\Bigg)\right\}\Bigg\}. \end{aligned} \tag{37}$$

A ξ-excursionist standing at ν_{e_0} at time t for the pairs (e_0, r), $\forall r\in\Gamma$, may choose between staying in ν_{e_0} or reaching κ_{e_0} at time $\tau\in]t,T]$. Accordingly, the candidate constant optimal controls are:

$$u_{r,1}^{\xi,e_0}[t]\equiv 0, \qquad u_{r,2}^{\xi,e_0}[t]\equiv l_{e_0}/(\tau-t); \tag{38}$$

the associated value function is:

$$\begin{aligned}\mathcal{V}_r^{\xi,e_0}(t) = \min\Bigg\{ &\Delta\Bigg(q_{e_0} + \sum_{j\in r^{\xi}_{succ(e_0)}} l_j\Bigg) + \frac{c_{\xi,4}}{2}(\widetilde{u}^{\xi,e_0})^2 \\ &+ \int_t^T \left(c_{\xi,2}\widetilde{\varphi}_{e_0} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_0}}\right) \mathrm{d}s\,, \\ &\inf_{\tau\in]t,T]}\Bigg\{ \frac{c_{\xi,1}}{2}\frac{(l_{e_0})^2}{\tau-t} + \frac{c_{\xi,4}}{2}\left(\widetilde{u}^{\xi,e_0} - \frac{l_{e_0}}{\tau-t}\right)^2(\tau-t) \\ &+ \int_t^{\tau}\left(c_{\xi,2}\widetilde{\varphi}_{e_0} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_0}}\right)\mathrm{d}s + \mathcal{V}_r^{\xi,e}(\tau)\Bigg\}\Bigg\}. \end{aligned} \tag{39}$$

In (39) if $r = r_1$ then $\mathcal{V}_r^{\xi,e}(\tau) = \mathcal{V}_{r_1}^{\xi,e_1}(\tau)$; if $r = r_2$ then $\mathcal{V}_r^{\xi,e}(\tau) = \mathcal{V}_{r_2}^{\xi,e_2}(\tau)$; if $r = r_3$ then $\mathcal{V}_r^{\xi,e}(\tau) = \mathcal{V}_{r_3}^{\xi,e_1}(\tau)$; finally, if $r = r_4$ then $\mathcal{V}_r^{\xi,e}(\tau) = \mathcal{V}_{r_4}^{\xi,e_1}(\tau)$.
Before ending this section, we prove the Lipschitz continuity of the above defined value functions. This is a result that will turn useful in the next section.

Proposition 4.1 *Let the mass vector ρ be given continuous and Assumption 3 hold. Then, every value function $V_r^{\xi,e} : [0,T] \to \mathbb{R}$, for all $\xi \in \Xi$ and for all $e \in r$, $r \in \Gamma$, defined by (21)–(39) is Lipschitz continuous, with Lipschitz constant independent of ρ.*

Proof Assumption 3.3 implies that there exist two positive constants k_1, k_2 such that, for every $0 \le t_1 \le t_2 \le T$, it always holds:

$$\begin{aligned}\left|\int_{t_1}^{t_2} \varphi_e\left(\sum_{\hat{r}\in\Gamma\,:\,e\in\hat{r}} \rho_{\hat{r}}^{\xi,e}(s)\right)\mathrm{d}s\right| &\le \left\|\varphi_e\left(\sum_{\hat{r}\in\Gamma\,:\,e\in\hat{r}} \rho_{\hat{r}}^{\xi,e}(\cdot)\right)\right\|_{\infty} |t_2-t_1| \\ &\le k_1\,|t_2-t_1| \le k_1 T. \end{aligned} \tag{40}$$

$$\begin{aligned}\left|\int_{t_1}^{t_2} \widetilde{\varphi}_e\left(\sum_{\xi\in\Xi}\sum_{\hat{r}\in\Gamma\,:\,e\in\hat{r}} \rho_{\hat{r}}^{\xi,e}(s)\right)\mathrm{d}s\right| &\le \left\|\widetilde{\varphi}_e\left(\sum_{\xi\in\Xi}\sum_{\hat{r}\in\Gamma\,:\,e\in\hat{r}} \rho_{\hat{r}}^{\xi,e}(\cdot)\right)\right\|_{\infty} |t_2-t_1| \\ &\le k_2\,|t_2-t_1| \le k_2 T. \end{aligned} \tag{41}$$

Moreover, by the formulation of the sightseeing value as a hyperbolic tangent, there exists a positive constant k_3 such that, for every $0 \le t_1 \le t_2 \le T$, it always holds

$$\left|\int_{t_1}^{t_2}(\tanh(s-1)-1)\,\mathrm{d}s\right| \le \|\tanh(\cdot-1)-1\|_{\infty}\,|t_2-t_1| \le k_3\,|t_2-t_1| \le k_3 T. \tag{42}$$

Now, take the edge e_{11} for every $r \in \Gamma$ and consider $\mathcal{V}_r^{\xi,e_{11}}$ as defined in (21). It appears that it is of the form

$$\mathcal{V}_r^{\xi,e_{11}}(t) = \frac{c_{\xi,1}}{2}\frac{(l_{e_{11}})^2}{T-t} + \frac{c_{\xi,4}}{2}\left(\widetilde{u}^{\xi,e_{11}} - \frac{l_{e_{11}}}{T-t}\right)^2 (T-t) \\ + \int_t^T \left(c_{\xi,2}\widetilde{\varphi}_{e_{11}} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_{11}}}\right) \mathrm{d}s,$$

only if $T - \max\{h', h''\} \le t \le T - \min\{h', h''\}$ with $h', h'' > 0$ independently of r, of ρ and of the control (see Appendix 1). By making use of conditions (41) and (42), we obtain that all the components of the value functions $\mathcal{V}_r^{\xi,e_{11}}$ in (21) are Lipschitz continuous and equi-bounded.

Proceeding backwards, let us consider $\mathcal{V}_r^{\xi,e}(t)$ given by (23). We focus on the first part of (23) which involves edges in E^1 and consider the term minimized with respect to $\tau \in]t, T]$. As before there exists $h > 0$ independent of ρ, of controls and of $t \in [0, T]$ such that, for any t, whenever the value of $\mathcal{V}_r^{\xi,e}(t)$ is defined by the value of

$$\inf_{\tau\in]t,T]}\left\{\frac{1}{2}\frac{(l_e)^2}{\tau - t} + \int_t^\tau \varphi_e \,\mathrm{d}s + \mathcal{V}_r^{\xi,e_{11}}(\tau)\right\},$$

then the minimizing values τ belongs to $[t+h, T]$. Differently, the value of $\mathcal{V}_r^{\xi,e}(t)$ is certainly defined by the value of

$$\alpha_1(l_e + l_{e_{11}}) + \int_t^T \varphi_e \,\mathrm{d}s,$$

when $t + h > T$. Hence, for every t, we can consider the function:

$$\psi^t : [t+h, T] \to \mathbb{R},\ \tau \mapsto \frac{1}{2}\frac{(l_e)^2}{\tau - t} + \int_t^\tau \varphi_e \,\mathrm{d}s + \mathcal{V}_r^{\xi,e_{11}}(\tau).$$

By (40) and the Lipschitz continuity of $\mathcal{V}_r^{\xi,e_{11}}$ follows that ψ^t is Lipschitz continuous for every t, with Lipschitz constant $M > 0$ independent of t and ρ. Analogously, due to (40), we have that the following inequalities hold for $0 \le t_1 < t_2 \le T$, and for $\tau \in [t_2 + h, T]$, with $M > 0$ and again independent of t and of ρ:

$$\left|\psi^{t_1}(\tau) - \psi^{t_2}(\tau)\right| \le \frac{1}{2}\left|\frac{(\ell_e)^2}{\tau - t_1} - \frac{(\ell_e)^2}{\tau - t_2}\right| + \int_{t_1}^{t_2} \varphi_e \,\mathrm{d}s \\ \le \frac{1}{2}\frac{(\ell_e)^2}{h^2}|t_1 - t_2| + k_1|t_1 - t_2| = M|t_1 - t_2|.$$

Let τ_1, τ_2 be two points of minimum for ψ^{t_1} and ψ^{t_2}, respectively. We get

$$\psi^{t_1}(\tau_1) - \psi^{t_2}(\tau_2) \le \psi^{t_1}(\tau_2) - \psi^{t_2}(\tau_2) \le M|t_1 - t_2|.$$

Fig. 3 Fixed point scheme

$$\rho \longrightarrow u \longrightarrow f \longrightarrow \mathcal{H}(f) \longrightarrow \rho'$$

If $\tau_1 \geq t_2 + h$, we then get

$$\psi^{t_2}(\tau_2) - \psi^{t_1}(\tau_1) \leq \psi^{t_2}(\tau_1) - \psi^{t_1}(\tau_1) \leq M|t_1 - t_2|.$$

If instead, $t_1 + h \leq \tau_1 < t_2 + h$, then we get

$$\psi^{t_2}(\tau_2) - \psi^{t_1}(\tau_1) = \psi^{t_2}(\tau_2) \pm \psi^{t_2}(t_2 + h) \pm \psi^{t_1}(t_2 + h) - \psi^{t_1}(\tau_1) \leq 2M|t_1 - t_2|.$$

All the above inequalities imply the Lipschitz continuity of $\mathcal{V}_r^{\xi,e}$ in (23) for $e \in E^1$, with Lipschitz constant independent of ρ. Let us now focus on the second part of (23) that holds for edges $e \in E^2$. It is again Lipschitz continuous with Lipschitz constant independent of ρ because of (40) and the Lipschitz continuity of $\mathcal{V}_r^{\xi,e_{11}}$. In conclusion, $\mathcal{V}_r^{\xi,e}$ in (23) is Lipschitz continuous for every $e \in E^1 \cup E^2$. The Lipschitz continuity of the value functions in (25)– (39), with Lipschitz constant independent of ρ, is achieved proceeding as before in a backward way. □

5 Existence of a Mean Field Equilibrium

In this section, we prove the existence of a mean field equilibrium for ρ over the network G depicted in Fig. 2. We proceed in two steps. As a first step, we let $L(w)$ be the Lipschitz constant of a function w and consider as a space to look for a fixed point:

$$\mathcal{S} = \left\{ w : [0, T] \to \mathbb{R}_+ : L(w) \leq \tilde{L},\ |w| \leq K \right\}^{\Theta}, \tag{43}$$

the Cartesian product Θ times of the space of Lipschitzian functions with Lipschitz constant not greater than $\tilde{L}$ and overall bounded by K, where $\tilde{L}$ is a constant and K is defined in Remark 3.2. Space $\mathcal{S}$ is convex and compact with respect to the uniform topology.
As a second step, we look for a fixed point of the function $\psi : \mathcal{S} \to \mathcal{S}$, with $\rho \mapsto \rho' = \psi(\rho)$ where ρ' is obtained carrying out the next steps (see diagram in Fig. 3):

1. given the mass ρ the optimal control u is derived through (21)–(39);
2. the optimal control u is used to compute the flow vector f through (9)–(11);
3. the mass vector ρ' is derived from f through (7) by first computing H through (8).

Note that a suitable constant $\tilde{L}$ exists such that the function ψ maps $\mathcal{S}$ into itself. Indeed, by construction, $\psi(\rho)$ must satisfy (7) and hence, by Remark 3.2 we can take as Lipschitz constant $\tilde{L} = 3\overline{\lambda}$.

Definition 1 Let ψ be the function described above. Then a mean field equilibrium is a total mass $\rho \in \mathcal{S}$ such that $\rho = \psi(\rho)$.

Now we prove the continuity of the function ψ so that Brouwer fixed-point theorem can be applied and a mean field equilibrium exists. As a convenience to the readers, we report the statement of the Brouwer fixed-point theorem: *Every continuous function from a convex compact subset S of a Euclidean space to S itself has a fixed point.*

Lemma 5.1 *The function $\psi : \mathcal{S} \to \mathcal{S}$ is continuous.*

Proof We show that for every sequence $\{\rho^n\} \subset \mathcal{S}$ and every $\rho \in \mathcal{S}$ such that $\rho^n \to \rho$ uniformly, we get $\psi(\rho^n) \to \psi(\rho)$ uniformly. We divide the proof into two steps.

Step 1) Consider the value functions $V_{r,n}^{\xi,e}$ and $V_r^{\xi,e}$, for every $\xi \in \Xi$ and every $e \in r$, $r \in \Gamma$ defined by (21)–(39). These two value functions are associated, respectively, to the choices of masses ρ^n and ρ in the congestion cost vectors $\varphi = \{\varphi_e : e \in E^1 \cup E^2\}$ and $\widetilde{\varphi} = \{\widetilde{\varphi}_e : e \in E^p\}$, where each entry of φ_e and $\widetilde{\varphi}_e$ has as an argument the sum of the suitable entries of ρ^n and ρ, respectively. The value functions $V_{r,n}^{e,\xi}$ and $V_r^{e,\xi}$ are equi-bounded and equi-Lipschitz in time and, by Proposition 4.1, they continuously depend on the relative components of ρ^n, ρ, respectively. Since $\rho^n \to \rho$ uniformly, then $V_{r,n}^{\xi,e} \to V_r^{\xi,e}$ uniformly in $[0, T]$, for all ξ and for all $e \in r$.
For every fixed t, let $u^n[t]$ and $u[t]$ be the corresponding constant optimal controls used by the ξ-excursionists for traversing at time t a given edge e in a given route r (these last indexes are not displayed for the easy of notation), with the corresponding optimal arrival time $\tau_n^*(t)$, $\tau^*(t)$ [see (19)]. Recall that the optimal arrival time is defined only for edges $e \in E^p \cup E^1$, since for $e \in E^2$ the optimal control is a priori fixed as well as the arrival time. By compactness, there exists a real number u^t such that, at least for a subsequence, $u^n[t] \to u^t$. By the convergence of the value functions and, consequently, of the minimizing expressions in (21)–(39) (when $e \in E^p \cup E^1$),

we have that the constant u^t is an optimal constant control used by the ξ-excursionists for traversing, at time t, the edge e, as part of the route r, with the given limit mass ρ. By Remark 4.1, if t is a continuity point of $\tau^*(\cdot)$, then the only optimal control for the limit problem when $e \in E^p \cup E^1$ is $u[t] \equiv \frac{l_e}{\tau^*(t)-t}$, and hence the limit is independent of the subsequence. If $e \in E^2$ the only optimal control for the limit problem is $u[t] \equiv v_e$ which is also independent of the subsequence. Again by Remark 4.1 relative to edges $e \in E^p \cup E^1$, and by definition of the optimal control for $e \in E^2$, we then get that the sequence of optimal control functions $u^n[\cdot]$ almost everywhere converges to the limit optimal control $u[\cdot]$. By the dominated convergence theorem it then converge in $L^1(0, T)$.

Step (2) Consider the optimal controls $u^n[\cdot]$ and $u[\cdot]$ introduced in Step (1). Given the throughput λ for every $t \in [0, T]$ and the constant vector Π, we can compute the corresponding flows f^n and f as in (9)–(11).
We now want to prove that $f^n \to f$ in $L^1(0, T)$. To this end, is enough to show that $sign(u^n[\cdot]) \to sign(u[\cdot])$ in $L^1(0, T)$ for $e \in E^p \cup E^1$, while for $e \in E^2$ there is nothing to prove due to the a priori fixed optimal control and (11).
By the optimization procedure (20)–(39) follows that each ξ-excursionist when enters an edge $e = (\nu_e, \kappa_e) \in E^p \cup E^1$ decides either to stop or to keep a constant control strictly greater than zero, which allows the excursionist to reach the head vertex κ_e within time T. Then, any control $u[\cdot] > 0$ is lower bounded by a constant $\frac{l_e}{T} > 0$ (for

every $e \in E^p \cup E^1$ in a given route r). As a consequence if $u^n[\cdot] \to u[\cdot] > 0$, we have $u[\cdot] \geq \frac{l_e}{T} > 0$. Hence, $sign(u^n[\cdot]) \to sign(u[\cdot]) = 1$.
Differently, if $u^n[\cdot] \to u[\cdot] = 0$, by the limit definition follows that from a certain n onward $u^n[\cdot] < \frac{l_e}{T}$ and hence, by its optimality, $u^n[\cdot] = 0$ which in turn implies that $sign(u^n[\cdot]) \to sign(u[\cdot]) = 0$. Therefore we have proven the almost everywhere convergence of signs from which, by the dominated convergence theorem, follows the convergence in $L^1(0, T)$. Then we can compute (edge by edge) $\psi(\rho^n)$ and $\psi(\rho)$ integrating the mass conservation (7):

$$\psi(\rho^n(t)) = \rho^n(0) + \int_0^t \big(\lambda(s)\Pi + f^{prec,n}(s)\big)\,\mathrm{d}s - \int_0^t f^n(s)\,\mathrm{d}s, \tag{44a}$$

$$\psi(\rho(t)) = \rho(0) + \int_0^t \big(\lambda(s)\Pi + f^{prec}(s)\big)\,\mathrm{d}s - \int_0^t f(s)\,\mathrm{d}s. \tag{44b}$$

Using all the previous arguments in the points (1) and (2) we have proven that the right hand side of (44a) converges to the right hand side of (44b), and hence that $\psi(\rho^n(t)) \to \psi(\rho(t))$ for every $t \in [0, T]$ and also uniformly, being them equi-bounded and equi-Lipschitz because belonging to $\mathcal{S}$. Hence, by Brouwer fixed point theorem, the map $\rho \to \psi(\rho)$ has a fixed point which is the mean field equilibrium. □

Remark 5.1 Note that the existence of a mean field equilibrium continues to be valid for any directed acyclic network, since both the value function (13) and the subsequent discussion on the optimal controls were given in general without considering any particular network structure. At the end of page 16, we decided to consider a specific network only to simplify notations and to have an explicit expression of the optimality conditions in term of the value function for the exit cost problem on each edge. Therefore, the result of Lipschitz continuity, Proposition 4.1, holds for every recursively and backwardly defined value function under the assumption of no cycle in the network and such that any vertex can be reached from the origin o and the destination d is reachable from any vertex. Consequently also Lemma 5.1 is still verified.

Remark 5.2 The presented results can be easily generalized to a directed acyclic network with multiple origins and multiple destinations. Suppose we are given a network $\mathcal{G}$, a set of origins $\mathcal{O}$ and a set of destinations $\mathcal{D}$ and recall that we assume that each excursionist follows a predefined route (see Assumption 2.1). Let $r_i \in \Gamma_i$ be the generic (o_i, d_i)-route with origin $o_i \in \mathcal{O}$ and destination $d_i \in \mathcal{D}$.

The following standard trick reduces the problem with multiple origins and destinations to the our problem with a single origin and a single destination.

Initially, we add a new "super-origin" $\bar{o}$ and a new "super-destination"$\bar{d}$ nodes to $\mathcal{G}$. Then, we create an edge $(\bar{o}, o_i)$ from the super-origin to every $o_i \in \mathcal{O}$, and an edge $(d_i, \bar{d})$ from every $d_i \in \mathcal{D}$ to the super-destination. These edges have zeros costs and lengths. In this way, we have transformed network $\mathcal{G}$ in a new single origin - single destination network. Finally, we extend every route r_i with an initial edge $(\bar{o}, o_i)$ from the super-origin to the origin of r_i and a final edge $(d_i, \bar{d})$ from the destination of r_i to the super destination. In this way, each agent excursionist is requested to follow a new route from a same super-origin to the same super-destination. However, the new route in practice overlaps the excursionist's original route r_i as the excursionist traverses the added edge in no time and paying no cost.

6 Bi-level Optimization

In this section, we propose a bi-level optimization problem where the upper level addresses the problem of selecting an optimal *sustainable oriented* control strategy, while the lower level describes the excursionist flows in the assumption that the excursionists optimize their satisfaction within the visiting experience. The optimal *sustainable* control strategy for the whole set $\mathcal{A}$ of connected POI is taken following tourism sustainability criteria at destination level [2–4]. The World Tourism Organization states that the sustainable tourism can be defined as "tourism that takes full account of its current and future economic, social and environmental impacts, addressing the needs of excursionists, the industry, the environment and host communities" [40]. In our model, we refer to the European Tourism System of Indicators for Sustainable Management at Destination Level (ETIS). This is actually also consistent with what results from Sardianou et al. [36] where the authors, through a summary of the literature, assert that sustainable tourism development must focus on the four pillars: the economic, the environmental, the social, and the cultural ones. This framework seem also to validate the perspectives at a local level on the main benefits and costs of tourism on residents' subjective well-being [14, 34]. Then, we propose the following bi-level optimization problem:

$$\begin{aligned} &\underset{\lambda(\cdot)}{\text{maximize}} && \mathcal{Q}(O^{ec}, O^{env}, O^{so}) \\ &\text{subject to} && \arg\min_{u} J(\cdot), \end{aligned} \tag{45}$$

where $\mathcal{Q}$ is the overall sustainability function which depends on three sub-sustainability objectives, i.e., respectively, the economic, the environmental, and the sociocultural, and J is the vector of costs paid by the excursionists on all the routes $r \in \Gamma$, i.e, $J(\cdot) = \{J_r^{\xi}(\cdot) : \xi \in \Xi, r \in \Gamma\}$ with $J_r^{\xi}(\cdot)$ as in (6).
In particular, we consider the problem in (45) from a leader/followers perspective. The leader, in our case the local authorities or the network manager, would like to control the throughput $\lambda(t)$, for every $t \in [0, T]$, to obtain the maximum gain from the visit of the POIs and, at same time, to ensure the best possible experience both at the socio-cultural and the environmental level. The optimal value of $\lambda(t)$ indicates to the local authorities which is the best throughput that they should encourage through, e.g., advertisement but possibly also coordinating the schedules of the different means of transport, such as trains, coaches, cruise ships, that take the excursionists to the city.

The literature on tourism has examined the relationship between consumer satisfaction and expenditure. Satisfaction appears to produce both direct and indirect positive impacts on expenditure. As regards the direct impacts, a positive relationship between customer satisfaction and its willingness to pay is proven in [23] and, more recently in [19]. The influence on the expenditure on ancillary services, such as museums, shows,

entertainment, guided excursions, exerted by the overall satisfaction is shown in [18]. Satisfaction with the landscape also positively affects excursionists' expenditures on accommodation, internal transport, food and beverage. This last fact suggests that landscaping maintenance policies may impact positively the economic variable.
As regards the indirect impact, the literature reveals that satisfaction stimulates repeat visits, positive recommendations and thereby new customers, reputation enhancement, higher acceptance of price increase, and consequently overall higher profitability [26]. In light of the above considerations, let us now introduce possible structures for the components of $\mathcal{Q}$ in (45). We can formulate the economic component as:

$$O^{ec} = \int_t^T \lambda(s) \sum_{\xi \in \Xi} \sum_{r \in \Gamma} J_r^{\xi}(s)\, \mathrm{d}s, \tag{46}$$

where J_r^{ξ} is the actual cost faced by the ξ-excursionists on r.
To define the two other dimensions in (45), i.e., environmental and sociocultural, we refer to the concept of *Tourism Carrying Capacity* (TCC). Although there are several definitions in literature on the subject, they all refer to the capacity of a system to endure despite adverse tourism impacts. More in detail, in [15], TCC is defined as a "…certain threshold level of tourism activity beyond which there will occur damage to the environment, including natural habitats."
The World Tourism Organization (UNWTO) defines three levels at which TCC can be addressed [39]: *ecological*, *psychological*, and *sociocultural*. Each one of them permits to estimate the allowable level of utilization of a tourist area while attaining the primary objective of preserving, respectively, the environment, the quality of tourism experiences, and the way of life of local people, their culture and traditions. Following the above considerations and regarding the environmental objective in (45), we introduce the following index:

$$\widetilde{\pi}^{\xi,e} = \frac{\widetilde{\alpha}^e S^e}{B^{\xi,e}}, \ \forall\, e \in E^p, \tag{47}$$

where $\widetilde{\pi}^{\xi,e}$ denotes the estimated allowed ξ-capacity (i.e., the number of ξ-type excursionists) of the POI, S^e is the total POI surface area, $\widetilde{\alpha}^e$ is a correction coefficient whose value range depends on the characteristics and on the level of vulnerability of the given POI and, finally, $B^{\xi,e}$ denotes the normative/benchmark area per ξ-type excursionist. The value $B^{\xi,e}$ corresponds to the best target value for attaining either the desired excursionists' experiential satisfaction or the area preservation. Concerning (47), excursionist stakeholders and network managers would want to set different thresholds of $B^{\xi,e}$, for every $e \in E^p$, with respect to the different types of excursionists in Ξ. For example, one would want to set a lower level of $B^{\xi,e}$ for the first type of excursionists, i.e., the ones who prefer walking, with the effect of reducing the environmental impacts of tourism transport/mobility. The objective of the network manager would be that of minimizing the distances between the estimated capacities of the POIs and the actual number of excursionists visiting them, i.e.,

$$O^{env} = \eta_{env} \left(\sum_{\xi \in \Xi} \sum_{e \in E^p} \int_0^T \left(\widetilde{\pi}^{\xi,e} - \sum_{r : e \in r} \rho_r^{\xi,e}(s) \right)^2 \mathrm{d}s \right), \tag{48}$$

where every distance is expressed as a squared difference between the desired and the actual value and η_{env} is a penalty parameter.
Concerning the third objective in (45), we apply the same methodology as in (48), i.e., we minimize the difference between the *sociocultural capacity* of a given route $r \in \Gamma$ and the actual number of excursionists following it. Here the attainment target is that of controlling the social pressure, i.e., the proportion of excursionists out of the total number of residents. The network manager is then interested in knowing the carrying capacity on a social level as:

$$\widetilde{\varrho}_r^{\xi} = \frac{\beta_r R_r}{F_r^{\xi}}, \tag{49}$$

where $\widetilde{\varrho}_r^{\xi}$ is the allowed sociocultural capacity for route $r \in \Gamma$ with respect to ξ-type excursionists, β_r is a correction coefficient whose value range might depend, for example, on the residents feeling towards excursionists, R_r is the number of residents along r and F_r^{ξ} is the normative/benchmark number of residents per ξ-type of excursionists. Moreover, from (49), as it holds the condition $\widetilde{\varrho} = \sum_{\xi \in \Xi} \sum_{r \in \Gamma} \widetilde{\varrho}_r^{\xi}$, the network manager can estimate the carrying capacity for the whole network. Then, he/she can monitor if the actual mass of excursionists is greater than the allowed one. In this framework, the objective would be that of minimizing the differences between the ideal social capacity of routes and the actual number of excursionists following them, i.e.:

$$O^{so} = \eta_s \left(\sum_{\xi \in \Xi} \sum_{r \in \Gamma} \int_0^T \left(\widetilde{\varrho}_r^{\xi} - \sum_{e \in r} \rho_r^{\xi,e}(s) \right)^2 \mathrm{d}s \right), \tag{50}$$

where again, we use the squared differences and η_s is a penalty parameter.
In the view of (48) and (50), it is worth stressing that if on one hand tourism development contributes to economic growth, on the other hand, it also leads to massive problems such as overcrowding and over-tourism. A way to cope with these phenomena is to implement efficient and effective strategies for managing tourist attractive sites within a sustainable development framework. In this context, a quantitative approach based on the TCC principle might be used not only as a control policy to limit excursionists' flows but also as a dynamic tool to improve the quality of the excursionists' tours.
According to all stated above, the optimization problem faced by either the local authorities or network managers can be formulated as

$$\begin{aligned} &\text{maximize} \quad \mathcal{Q} = \zeta_1 O^{ec} - \zeta_2 O^{env} - \zeta_3 O^{so} \\ &\text{subject to} \quad \arg\min J(\cdot), \end{aligned} \tag{51}$$

where $\zeta_{i=1,2,3}$ are weights applicable depending on which objective the authorities or the network managers consider more important for achieving their target.

7 Conclusions and Future Developments

In this paper, we have presented some theoretical results that can justify a mean field game approach to model the flows of excursionists visiting the POIs of the historical centers of art cities. We prove the existence of an equilibrium of flows within the network and we also propose and formalize a bi-level optimal control model which addresses the often conflicting objectives of defining a sustainable-oriented policy by the local authorities while excursionists aim at maximizing their satisfaction. Specifically, the model upper level addresses the problem of selecting an optimal sustainable oriented control strategy, while its lower level describes the excursionist flows in the assumption that the excursionists' satisfaction can be expressed in terms of the minimization of an appropriate cost function. The contribution of this study is to propose a theoretical framework for developing new models and methods based on the mean field theory that can support local authorities to deal with the challenging problem of finding the total excursionists' experiential satisfaction while attaining the maximum sustainability benefits.

Future research must be focused on some issues that have to be addressed before being able to apply the theoretical approach proposed in this paper in the practice.

A first issue is that the number of routes increases exponentially with the number of POIs. However, in the practice, this exponential increase may not occur. Indeed, historical centers usually present very few main attractions, e.g., the cathedral or the main museum, and the excursionists' routes typically pivot around the main attractions and a few minor ones. As an example, it is the authors' experience that: excursionists of the city of Venice very rarely visit more than 5 or 6 POIs; the number of tickets sold each day to visit the main two attractions is usually much greater than the number of tickets sold to visit all the other POIs.

A second issue is that we assume that the network G has no oriented cycles and its "physical" entrance and exit points are different. In the practice, there may be oriented cycles, although it is reasonable to assume that no route visits the same POI twice. Moreover, physical entrance and exit points of the network may coincide, e.g., with the main gate of the town medieval walls.

A final issue is related to a numerical algorithm that implements the approach presented in Sect. 5 to determine a fixed point for the excursionists' mass ρ. Indeed, we proved the existence of a fixed point by Brouwer Theorem. However, this theorem does not provide information about the speed of convergence to the fixed point of an algorithm based on the fixed point scheme described in Fig. 3. Such a question requires further investigation and experimentation.

In addition, a numerical algorithm cannot deal with continuous time functions. As an example, it can assess only approximately the value of the continuous time vectorial function $\rho(t)$, by sampling the time and, e.g., assuming fixed the values of $\rho(t)$ within the sampling period. Obviously, on the one hand, the higher the sampling frequency, the better the approximation of ρ. On the other hand, the higher the sampling frequency, the greater is the computational burden which adds up to a possible slow convergence of the algorithm.

Data Availability 'Not applicable' for that paper.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

The function $\mathcal{V}_r^{\xi,e_{11}}$ as defined in (21) is of the form

$$\mathcal{V}_r^{\xi,e_{11}}(t) = \frac{c_{\xi,1}}{2}\frac{(l_{e_{11}})^2}{T-t} + \frac{c_{\xi,4}}{2}\left(\widetilde{u}^{\xi,e_{11}} - \frac{l_{e_{11}}}{T-t}\right)^2 (T-t) + \int_t^T \left(c_{\xi,2}\widetilde{\varphi}_{e_{11}} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_{11}}}\right) \mathrm{d}s, \tag{52}$$

only if

$$\Delta q_{e_{11}} + \frac{c_{\xi,4}}{2}(\widetilde{u}^{\xi,e_{11}})^2 \geq \frac{c_{\xi,1}}{2}\frac{(l_{e_{11}})^2}{T-t} + \frac{c_{\xi,4}}{2}\left(\widetilde{u}^{\xi,e_{11}} - \frac{l_{e_{11}}}{T-t}\right)^2 (T-t),$$

from which we get

$$(T-t)\left(1 - \frac{c_{\xi,4}(\widetilde{u}^{\xi,e_{11}})^2(T-t) - 2c_{\xi,4}\widetilde{u}^{\xi,e_{11}}l_{e_{11}}}{2\Delta q_{e_{11}} + c_{\xi,4}(\widetilde{u}^{\xi,e_{11}})^2}\right) \geq \frac{(c_{\xi,1}+c_{\xi,4})l_{e_{11}}^2}{2\Delta q_{e_{11}} + c_{\xi,4}(\widetilde{u}^{\xi,e_{11}})^2}. \tag{53}$$

Considering the first factor of the left hand side of (53), we get that

$$t \leq T - \frac{(c_{\xi,1}+c_{\xi,4})l_{e_{11}}^2}{2\Delta q_{e_{11}} + c_{\xi,4}(\widetilde{u}^{\xi,e_{11}})^2} \leq T - h',$$

with $h' > 0$ independent of r, on ρ and of the control. If instead, we analyze the second factor of the left hand side of (53) we have

$$t \geq T - \frac{l_{e_{11}}(-l_{e_{11}}(c_{\xi,1}+c_{\xi,4}) + 2c_{\xi,4}\widetilde{u}^{\xi,e_{11}}) + c_{\xi,4}(\widetilde{u}^{\xi,e_{11}})^2 + 2\Delta q_{e_{11}}}{c_{\xi,4}(\widetilde{u}^{\xi,e_{11}})^2} \geq T - h'', \tag{54}$$

with h'' independent of r, on ρ and on the control. If in (54) $h'' > 0$ then $\mathcal{V}_r^{e_{11},\xi}$ has the shape of (52) only if $T - \max\{h', h''\} \leq t \leq T - \min\{h', h''\}$. If, instead, $h'' \leq 0$

then $\mathcal{V}_r^{e_{11},\xi}$ is not defined as in (52) but rather by (see (21)):

$$\mathcal{V}_r^{\xi,e_{11}}(t) = \Delta q_{e_{11}} + \frac{c_{\xi,4}}{2}(\widetilde{u}^{\xi,e_{11}})^2 + \int_t^T \left(c_{\xi,2}\widetilde{\varphi}_{e_{11}} - \frac{c_{\xi,3}(\tanh(s-1)-1)}{m_{e_{11}}} \right) \mathrm{d}s.$$

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