How Do Personal Preferences Influence the Flow Dynamics in Networks?

Rosario Maggistro and Raffaele Pesenti

1 Introduction

Networks are a widely used paradigm to describe many kinds of systems, for example, communications, logistics, social, and data. In recent years, the technological evolution has favored that the network literature intermingled with the ones devoted to describing the behavior of very large amounts of agents.

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R. Maggistro (✉)
Network Organization, Innovation, and Strategy Research Center (NOIS), Department of Management, Ca' Foscari University of Venice, Venezia, Italy

Department of Economics, Business, Mathematics and Statistics 'Bruno de Finetti', University of Trieste, Trieste, Italy
e-mail: rosario.maggistro@deams.units.it

R. Pesenti
Network Organization, Innovation, and Strategy Research Center (NOIS), Department of Management, Ca' Foscari University of Venice, Venezia, Italy
e-mail: pesenti@unive.it

Within this framework, network scholars debate on defining network performance as the sum of individual members' performance versus the assessment of whole networks' outcome (see, e.g., [1]). From this latter perspective, some confusion still exists about two possible ways of defining network performance: as the ability to reach the collective goal, or as the effectiveness in coordinating members. Moreover, as discussed in [2], network literature suffered from a static/structuralist approach. In particular, the network outcome would be incomplete and partially flawed if we do not look at how individual network members act in their strategic behavior aimed at modifying networks' systems of benefits and constraints for their interests.

From another perspective, we may distinguish between two main lines of studies in the literature. The first one describes agents' behavior in terms of the dynamics of the network. In this case, the network topology evolves, for example, new connections or new vertices may be built or destroyed over time (see, e.g., [3] for a review and research agenda on dynamics of organizational networks). This is the situation, for example, of interfirm networks defined as an institutional arrangement among distinct but related for-profit organizations which are characterized by a special kind of (network) relationship, a certain degree of reflexivity, and a logic of exchange that operates differently from that of markets. Such interfirm networks have been analyzed from different economic and social perspectives, while their inter-organizational effectiveness was first introduced in [1] and subsequently revised under a structurationist perspective in [4] (see also [5]).

The second line of studies describes agents' behavior in terms of dynamics in the network. In this latter case, the network topology remains fixed or, in any case, changes slowly; differently, agents flow dynamically through the network.

Our work falls within the scope of the second line of studies on the networks and provides a small contribution to the literature describing how past experience and available information may influence the behavior of bounded rational agents (see, e.g., Hainer's seminal paper [6]). From a performance perspective, our contribution is in showing how network performance could be defined in terms of effectiveness in coordinating flow dynamics on a fixed network structure. This idea could be assimilated to information flows within intra-organizational networks, to flows of goods and resources in sparse supply chains. From an agency perspective, our contribution is in providing a model that describes how individual agents' preferences and behaviors may lead to different network performance.

Our approach is based on the mean-field games (MFG) to model the flow of many (theoretically infinite) agents over a network. Specifically, we study how agents moving on a network reach a dynamic equilibrium which is a function of the network congestion. Our results can be exploited by network managers interested in controlling network congestion by making available relevant information.

Mean-field game theory is the study of strategic decision-making by small interacting agents in very large populations, where by small agent we mean an agent who has very little influence on the overall system. More precisely, the idea underlying the introduction of this theory is that, in the case of a large number of agents, interactions are such that each agent only considers the statistical distribution of the others to make his decisions (see [7] and [8]). Several application domains such as economics, physics, biology, and network engineering accommodate MFG theoretical models (see, e.g., [9, 10, 11]). In particular, models to study dynamics on networks and/or pedestrian movement can be found, for example, in [12, 13, 14].

In the present work, we consider the agent's path preferences dynamics in addition to the usual framing of mean-field games (typically defined by the pair made of Hamilton-Jacobi-Bellman and mass conservation equations).

In particular, we propose a model in which the agents choose their path having access to global information about the network congestion, but also being influenced by the decision of agents that has already made their decisions. We assume that an agent that enters the network at the time t first estimates how the congestion of the network will evolve. Then, it individuates the "least expensive" path to reach its destination by evaluating the optimal control (the velocity) that it should implement edge by edge along each possible path. Finally, it makes its choice of the followed path being influenced also by its a priori path preference. The agents' behavior just described makes the evolution of actual network congestion depend on the congestion estimated by the agents when entering the network. We say that the system has reached an equilibrium when the actual congestion and the estimated one coincide. One possible physical interpretation of our model is to consider the agents as pedestrians traversing possible paths within a city described as a network. However, it may also be seen as well suited to describe car traffic flow in highway networks; and possibly adopted to explain information flows in organizations.

Besides the novelty of the model introduced, the present work aims to introduce the reader to the assumptions that are sufficient to guarantee the existence of a mean-field equilibrium and to forward-backward approach that is generally used to prove it.

The rest of this work is structured as follows. Section 2 describes the model and presents the used hypotheses. Moreover, it analyzes all the agents' dynamics which constitute the transportation system. Section 3 introduces the approach that can be used to prove the existence of a mean-field equilibrium and Sect. 4 presents concluding remarks and highlights directions for future research.

2 The Model

In this section, we describe the flow dynamics over a network of possible paths that the agents can decide to traverse within a time interval $[0, T]$, where $T>0$ is the final horizon.

2.1 Network Features

We consider a directed network $\mathcal{G}=(\mathcal{V},\mathcal{E})$, where: $\mathcal{V}$ is a finite set of vertices, and $\mathcal{E}$ is a finite set of directed edges $e=(v_e, \kappa_e)$ being v_e the tail vertex of e and $\kappa_e \neq v_e$ the head vertex. The set $\mathcal{V}$ includes the *origin* o and the *destination* d, where the agents enter and leave the network, respectively. Each edge $e \in \mathcal{E}$ is characterized by three finite parameters: its *length* ℓ_e; its flow *capacity* C_e, expressing the maximum number of agents that can enter in e per unit of time; and *maximum mass* $\rho_{\max}$ denoting the maximum mass/number of agents that can be present at the same time in e. We assume $\rho_{\max}$ be the same for each $e \in \mathcal{E}$.

An (oriented) *path* from a vertex v_0 to a vertex v_r is an ordered set of r adjacent edges $p = (e_1, e_2, \ldots, e_r)$ such that $\nu_{e_1} = v_0$, $\kappa_{e_r} = v_r$, $v_s = \kappa_{e_s} = \nu_{e_{s+1}}$ for $1 \le s \le r-1$, and no vertex is visited twice, that is, $v_l \ne v_s$ for all $0 \le l < s \le r$, except possibly for $v_0 = v_r$, in which case the path is referred to as a *cycle*. A vertex v_j is said to be *reachable* from another vertex v_k if there exists at least a path from v_k to v_j. In particular, we hold the following assumptions on the network $\mathcal{G}$: i) $\mathcal{G}$ contains no cycles; ii) any vertex in $\mathcal{V}$ can be reached from the origin vertex o and the destination vertex d is reachable from any vertex in $\mathcal{V}$.

We denote by Γ the set of all the paths p from o to d, by A the $|\mathcal{E}| \times |\Gamma|$ *edge-path incidence matrix* with entries

$$A_{ep} = \begin{cases} 1 & \text{if} \quad e \in p, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

and by

$$\Xi = \sum_{e \in \mathcal{E}} \sum_{p \in \Gamma} A_{ep}, \quad \text{with} \quad |\mathcal{E}| \le \Xi \le |\mathcal{E}| \times |\Gamma|,$$

the number of the elements equal to 1 of the matrix A, that is, the number of pairs edge-path $(e, p) \in \mathcal{E} \times \Gamma$ such that $e \in p$.

For every path $p \in \Gamma$ and edge $e \in p$, we define the functions $\rho_p^e : [0, T] \to [0, \rho_{\max}]$, $f_p^e : [0, T] \to [0, C_e]$, which denote the current mass and current flow of agents following path p, respectively, present and leaving the edge e at each time instant $t \in [0, T]$. We let

$$\rho(t) = \{\rho_p^e(t) : e \in p, p \in \Gamma\} \in \mathbb{R}^{\Xi}, \quad f(t) = \{f_p^e(t) : e \in p, p \in \Gamma\} \in \mathbb{R}^{\Xi}, \tag{2}$$

be the vectors of masses and flows, respectively.

To simplify notations and statements, hereinafter we consider a network $\mathcal{G}$ on which agents have only three possible paths to reach d starting from o (see Fig. 1). Accordingly, the set of paths is $\Gamma = \{p_1, p_2, p_3\}$, where $p_1 = (e_1, e_4)$, $p_2 = (e_2, e_5)$, $p_3 = (e_1, e_3, e_5)$. However, all the results obtained in the next sections can be proved for more general networks, still satisfying the assumptions (i) and (ii) above.

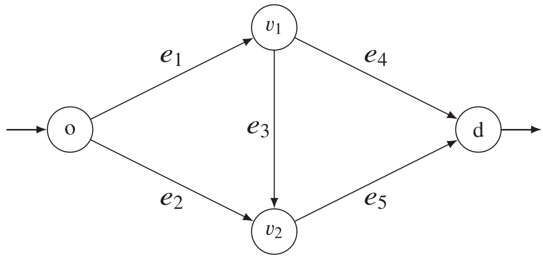

Fig. 1 The network topology used in the chapter

2.2 Agents' Dynamics and Costs

We assume that the agents are indistinguishable and that every agent enters the network $\mathcal{G}$ by the origin o, chooses a path $p \in \Gamma$, travels through $\mathcal{G}$ along p, and finally leaves the network from the destination d. Let $\lambda : [0, T] \to [0, +\infty[$ be a given function describing the *throughput* of the agents, that is, $\lambda(t)$ is the total flow of agents entering the network in the origin o at time t. In addition, we let $\theta_e \in [0, \ell_e]$ be the state of the generic agent over an edge $e \in \mathcal{E}$. The value $\theta_e(s)$ describes the position of the agent at time s from the tail of e, that is, $\theta_e(s) = 0$ means that the agent is in ν_e, while $\theta_e(s) = \ell_e$ means that the agent is in κ_e and hence it is inside the edge e as long as $0 \le \theta_e(s) \le \ell_e$. Note that $\theta_e(s)$ describes the state of an hypothetical agent assumed to be in ν_e at time t, independently of the fact whether there is actually someone present at ν_e at that time. The controlled dynamics in any edge $e \in \mathcal{E}$ of an agent who entered the edge at time $t \in [0, T]$ is:

$$\begin{cases} \dot{\theta}_e(s) = u^e(s), & s \in]t, T], \\ \theta_e(t) = 0, \end{cases} \tag{3}$$

where the control, $s \mapsto u^e(s)$, is measurable and integrable, namely $u^e \in L^1(0, T)$.

Each agent traversing an edge e at a given time t, aims at minimizing a cost that takes into account: (i) the possible hassle of running in the edge to reach d on time; (ii) the pain of being entrapped in a highly congested edge; (iii) the disappointment of not being able to reach d by the final horizon T. We model this cost analytically as

$$J_e(t,u^e) = \int_t^T \chi_{\{0\le\theta_e(s)\le\ell_e\}} \left(\frac{(u^e(s))^2}{2} + \varphi_e\left(\sum_{\hat{p}\in\Gamma | e\in\hat{p}} \rho^e_{\hat{p}}(s) \right) \right) ds \\ + \chi_{\{0\le\theta_e(T)<\ell_e\}} \alpha \sum_{j\in p_e} \ell_j , \tag{4}$$

where χ is the characteristic function

$$\chi_{\{0\le\theta_e(s)\le\ell_e\}} = \begin{cases} 1 \text{ if } 0 \le \theta_e(s) \le \ell_e, \\ 0 \text{ otherwise,} \end{cases}$$

and similarly for $\chi_{\{0\le\theta_e(T)<\ell_e\}}$; $\alpha > 0$ is a constant parameter representing a cost per unit of length, and p_e is the shortest path from the tail v_e to d. The quadratic term inside the integral in (4) stands for the cost component i), while the other term, characterized by the congestion function $\varphi_e : [0, \rho_{\max}] \to [0, +\infty]$ stands for the congestion cost component. Finally, the last addendum in (4) stands for cost component iii). In particular, note that, due to the presence of the characteristic functions, the integral part is paid as long as the agent stays on the edge e. The cost outside the integral acts as follows: (1) if at the final horizon T the agent is still in between the edge (not reached the head κ_e yet), then the final paid cost is the minimum distance in the network from the tail v_e of the actual edge to the destination d; (2) if at the final horizon T the agent is at the head of the edge κ_e (i.e., it has already traversed the whole edge), then the corresponding paid cost with respect to the actual edge e is zero. Anyway it will be paid as the minimum distance in the network from the head vertex κ_e to the destination d just by interpreting that head as the tail $v_{e'}$ of any other subsequent edge e' hypothetically entered by the agent at time T.

Throughout this chapter we will assume the following basic assumptions to hold on the agents' behavior:

Assumptions

1. *The throughput* λ *is* $C^1([0, T])$ *and* $\lambda(t) > 0$ *for all* $t \in [0, T]$. *In particular, this implies that there exist* $0 < \underline{\lambda} \leq \overline{\lambda} < +\infty$ *such that* $\underline{\lambda} \leq \lambda(t) \leq \overline{\lambda}$ *for all* $t \in [0, T]$.
2. *The initial mass of agents is null, that is,* $\rho\ (0) = 0$.
3. *For every* $e \in \mathcal{E}$, *the congestion cost function* Φ_e *is Lipschitz continuous. Moreover, it only depends on the masses* ρ_p^e *and not on the state variable* θ_e.
4. *The network edges' maximum mass is such that* $\rho_{\max} > \overline{\lambda} T \geq \int_0^T \lambda(s)\, ds$ *and the flow capacity* $C_e > \overline{\lambda}, \forall e \in \mathcal{E}$, *i.e., neither the mass capacity nor the flow capacity of the edges can impede the agents' movements even in the worst-case scenario.*
5. *When more than one optimal control is available, agents choose the smallest one.*
6. *Agents have bounded rationality in the sense that, even when they access the full available information, the cognitive limitations of their minds, and the finite amount of time they have to prevent them from using the pieces of information to their full extent when making their decisions.*

We now assume that agents entering the network have access to global information about the current congestion status of the network through the knowledge of the actual mass vector ρ. Then, they choose the path to follow on the basis of their appraisal of the costs of the different paths and on the observation of the decision of the agents that have preceded. The relative appeal of the different paths to the agents is modeled by a time-varying non-negative *(aggregate) path preferences* vector $z : [0,T] \rightarrow \mathbb{R}_+^{|\Gamma|}$, whose generic element $z_p(t)$ represents the flow's density of agents entering path p at the origin o at time t. The vector z varies within the simplex

$$\mathcal{S}_{\lambda(t)} = \left\{ z \in \mathbb{R}_+^{|\Gamma|} : \sum_{p \in \Gamma} z_p(t) = \lambda(t) \right\}, \tag{5}$$

where we recall that by $\lambda(t)$ we denote the agents' throughput at time t.

The path preferences vector $z(t)$ evolves over time as a function of the appraisal of the costs that the agents would pay along the different paths. The agents assess these costs in terms of the optimal controls that they would implement and assuming known the congestion level described by ρ. Specifically, the assessed cost for each path $p \in \Gamma$ at time t is:

$$J^p(t) = \sum_{e \in \mathcal{E}: e \in p} J_e(t_e^p(t), u_p^e), \tag{6}$$

where, for every $e \in p$, $u_p^e \in L^1(0,T)$ is the optimal control implemented along the edges by an agent who is in the path p (these controls are discussed in the following subsection); $t_e^p(t)$ is the time instant in which an agent, arriving at t in the origin o and following the path p, reaches v_e using the controls u_p^e. We write $t_e^p(t) = \infty$ if an agent does not reach e within T and we define $J_e(\infty, u_p^e) = 0$. This last definition is justified by the fact that the sum (6) must involve non-null costs only for the edges that an agent actually reaches.

We also assume that information on the congestion of the network provided to the agents may be inexact, so that they assess a path p having a minimum cost with probability $e^{-\beta J^p(t)} / \sum_{\hat{p} \in \Gamma} e^{-\beta J^{\hat{p}}(t)}$, where $\beta > 0$ is a fixed noise parameter. Hence, the fraction of agents entering the network at time t that would consider a path p having minimum cost is

$$F_\beta^p(t) = \lambda(t) \frac{e^{-\beta J^p(t)}}{\sum_{\hat{p} \in \Gamma} e^{-\beta J^{\hat{p}}(t)}}.$$

Note that when β tends to 0, then $F_\beta^p(t)$ tends to $\lambda(t) / |\Gamma|$, that is, agents consider all the paths equivalent. Differently, when β tends to infinite the agents have the possibility of surely determining the exact costs of the paths and indeed $F_\beta^p(t)$ tends to 0 for all p, except for the path minimum cost, for which it tends to $\lambda(t)$.

Hereinafter, we denote by $F_\beta(t)$ the vector $\{F_\beta^p(t) : p \in \Gamma\}$ and by $J(t) = \{J^p(t) : p \in \Gamma\}$ the vector of costs on all the paths $p \in \Gamma$. Agents make their final decision on the path to choose comparing the value of $F_\beta(t)$ with the choice of the agents that have preceded them. Specifically, we assume that they correct the difference $z(t) - F_\beta(t)$ with a proportional control, as described by the following equation:

$$\dot{z}(t) - \dot{F}_\beta(t) = -\eta(z(t) - F_\beta(t)),\ z(0) = z_0, \tag{7}$$

where, the parameter $\eta > 0$ can be interpreted as the rate at which the path preferences are updated. In other words, Eq. (7) says that the bounded rationality of the agents makes them, on the one side, like the idea to split as indicated by F_β; on the other side, prefer not to stray from previous agents' decisions. We remark that the dynamics described by (7) makes $z(t)$ satisfies constraint (5) for all $t \in]0, T]$, whenever the same happens for z_0.

The path preferences vector z turns then useful, to define, for every $t \in [0, T]$ the *local decision function* $G[t] : \mathcal{S}_{\lambda(t)} \to \mathbb{R}_+^\Xi$, which characterizes the fractions of agents choosing each outward-directed edge $e \in p$, $p \in \Gamma$ when traversing a non-destination vertex v. Actually, in this chapter, we are interesting only on the first three components of this functions, (e_1, p_1), (e_1, p_3), (e_2, p_2), which are relative to the two edges e_1, e_2 outgoing from the origin o (see Fig. 1). We restrict our attention to these three components since once the path is chosen in the origin, in the following non-destination vertices the agents get split according such a choice.

Hence, we define the first three component of $G[t]$ and fix the others equal to zero as follows:

$$G[t]_p^e(z) = \begin{cases} \dfrac{z_p}{\sum\limits_{\hat{p}\in\Gamma} z_{\hat{p}}} & \text{for} \quad e \in \{e_1, e_2\},\ p \ni e, \\ 0 & \text{for} \quad e \in \{e_3, e_4, e_5\},\ p \ni e. \end{cases} \tag{8}$$

Note that in (8), for every $t \in [0, T]$ and for every $z \in \mathcal{S}_{\lambda(t)}$, it is $\sum_{\hat{p}\in\Gamma} z_{\hat{p}} = \lambda(t) \geq \underline{\lambda} > 0$, because of (5) and Assumption 1.1. Hence, for every $t \in [0, T]$, $G[t]$ is a continuous function defined over the compact set $S_{\lambda(t)}$, and so uniformly continuous. Definition (8) allows to write the equation that describes mass conservation, for every vertex $v \neq d$ and outward-directed edge $e \in p$, $p \in \Gamma$, as:

$$\dot{\rho}(t) = H(f(t), z(t); t), \qquad \rho(0) = \rho_0, \tag{9}$$

where the flow $t \mapsto f(t) = (f_p^e(t))_p^e \in \left(\prod_{e \in p}[0, C_e]\right)_p$ is defined next, $t \mapsto z(t) = (z_p(t))_p \in \mathcal{S}_{\lambda(t)}$ is the solution of (7), and $H : \prod_{e \in p}[0, C_e] \times \mathcal{S}_{\lambda(t)} \to \mathbb{R}^{\Xi}$ is defined, for every $t \in [0, T]$, by

$$H_p^e(f(t), z(t); t) = \left(\lambda(t) G[t]_p^e(z(t)) + f_p^{prec_p(e)}(t)\right) - f_p^e(t), \ \forall \ p \in \Gamma, e \in p, \tag{10}$$

with $prec_p(e)$ the function that returns the edge that precedes e on the path p. Each component $f_p^e(t)$ of the flow $f(t)$ represents the outgoing flow from the edge e at time t. Given Assumption 1.6, agents assess the outgoing flow assuming a minimal length of the traverse time interval, $k \in]0, T]$, for each edge $e \in \mathcal{E}$. Specifically, k is assessed as the minimal length of a time interval such that to cross the edge in less time is certainly non-optimal, as the traversing cost would be for sure greater than the cost of non-traversing, given by the disappointment of not being able to reach the destination d at time T. Actually, such a value $k > 0$ can be a priori evaluated by the data of the problem. Then, we write the outgoing flows as:

$$f_p^e(t) \quad = \begin{cases} 0 & \text{if } t \in [0, k], \\ \lambda(t-k) G[t-k]_p^e(z(t-k)) sign(u_p^e[t-k]) & \text{if } t \in [k, T], \\ \text{for } e \in \{e_1, e_2\},\ p \ni e, & \end{cases} \tag{11a}$$

$$f_p^e(t) \quad = \begin{cases} 0 & \text{if } t \in [0, k], \\ f_p^{prec_p(e)}(t-k) sign(u_p^e[t-k]) & \text{if } t \in [k, T], \\ \text{for } e \in \{e_3, e_4, e_5\},\ p \ni e, & \end{cases} \tag{11b}$$

where $u_p^e[t-k] \geq 0$ is the constant optimal control implemented by an agent that, following path *p*, enters the edge *e* at time $t-k$, and $sign(\xi)=1$ if $\xi>0$ and $sign(\xi)=0$ if $\xi=0$.

Remark 1 Conditions (11), coherently with Assumption 1.6, model the outgoing flows $f_p^e(t)$ as possibly estimated by an agent entering *e* at time $t-k$ that assumes that all the other agents that are currently present on *e* and that are following the same path *p*, are implementing the same controls $u_p^e[t-k]$, as itself. Hereinafter, the flows (11) are sometimes called "estimated flows." Of course, a more precise formulation of them should consider the actual value of the control (and not only its sign) and estimate the real traverse time (something similar in this direction is made in [14]). Similarly, the mass ρ that satisfies (9) may be more precisely defined to represent the real dynamics of the agents. Anyway, such estimated flows and mass evolution may be also seen as an approximation for the elaboration in real time of the information that a possible network manager has to implement and to send them to the agents. The study of the real discrepancy of such estimated flows and mass evolution from the actual ones may be the subject of future works. However, note that the estimated flows f_p^e (11), when implemented in (9), make the principle of mass conservation satisfied. Finally, let us observe that (9)–(11) do not preclude the possibility that agents accumulate at the beginning of an edge *e*, that is, on the vertex ν_e. This situation may occur, when the optimal control is $u_p^e=0$, since the corresponding outflow $f_p^e=0$.

3 Value Functions and Optimal Controls

Given a vector mass concentration $\rho(\cdot)$, for each $p \in \Gamma$, $e \in p$ and $t \in [0, T]$, the following functions represents the optimum cost that an agent, entering edge *e* of a path *p* at time *t*, must pay:

$$
V_p^e(t) = \begin{cases} \inf_{u_p^e \in L^1(0,T)} \left\{ \int_t^{T\wedge\tau} \left(\frac{(u_p^e(s))^2}{2} + \varphi_e\left(\sum_{\hat{p}\in\Gamma | e\in\hat{p}} \rho_{\hat{p}}^e(s) \right) \right) ds + \mathcal{F}_p^e(T\wedge\tau) \right\} \\ \text{if } e \in p \setminus \{last(p)\}, \\ \inf_{u_p^e \in L^1(0,T)} \{ J_e(t, u_p^e) \} \quad \text{if } e = last(p), \end{cases} \tag{12}
$$

where τ is the first exit time from the closed interval $[0, \ell_e]$, $last(p)$ is a function that returns the last edge of a path p and $\mathcal{F}_p^e(T\wedge\tau)$ is given by

$$
\mathcal{F}_p^e(T\wedge\tau) = \begin{cases} V_p^{succ_p(e)}(\tau) & \text{if } \tau < T, \\ \alpha \sum_{j\in p_e} \ell_j & \text{if } \tau > T, \\ \min\left\{ \alpha \sum_{j\in p_e} \ell_j, V_p^{succ_p(e)}(\tau) \right\} & \text{if } \tau = T, \end{cases}
$$

with $succ_p(e)$ the function which returns the edge that follows e on path p, for $e \in p \setminus \{last(p)\}$. Functions $V_p^e(t)$ can be recursively and backwardly computed, starting from the ones corresponding to the last edges ending in the destination vertex d. We call them, with a little abuse of terminology, *value functions*. Note that such a recursive definition is valid as the absence of oriented cycles in the network $\mathcal{G}$ prevents self-referring. Note also that $\mathcal{F}_p^e(T\wedge\tau)$ may be discontinuous in τ. This fact implies the possible discontinuity of the Hamiltonian associated with the value function and/or of the boundary data. Hence, we will write, as in [14], optimality conditions in terms of the value functions for the exit-time/exit cost problem on each edge. The value functions do not take into consideration the position θ_e of the agents on the edges due to the hypothesis that the congestion functions φ_e depend on the total mass actually present on the edge and not on the state position of the single agent.

The considered value functions imply that, for each $p \in \Gamma$, $e \in p$, the optimal control implemented by an agent that, at time $t \in [0, T]$, starts to traverse the edge e as part of the path p is either

$$u_p^e \equiv \frac{\ell_e}{\tau - t} \quad \text{or} \quad u_p^e \equiv 0. \tag{13}$$

Once t is fixed, the control in (13) is constant.

Hereinafter, we denote by $u_p^e[t]$ the optimal control chosen by an agent that stands in v_e at time t when following the path p and by $u[\cdot] = \{u_p^e[\cdot] : e \in p, p \in \Gamma, u_p^e[\cdot] \geq 0\}$ the vector of these controls.

Consider now the network as Fig. 1 we can determine the optimal controls on each edge and the corresponding value functions using backward dynamic programming.

An agent standing at v_e at time $t < T$, and hence at $\theta_e(t) = 0$, where $\kappa_e = d$, that is, for the pairs $(e, p) \in \{(e_4, p_1), (e_5, p_2), (e_5, p_3)\}$ has two possible choices: either staying at v_e indefinitely or moving to reach $\kappa_e = d$ exactly at time T. Accordingly, the candidate constant optimal controls to be chosen at the time t are

$$u_{p,1}^e[t] \equiv 0, \qquad u_{p,2}^e[t] \equiv \frac{\ell_e}{T - t}. \tag{14}$$

Hence, given the cost functional (4), we derive

$$V_p^e(t) = \min\left\{\alpha \ell_e, \frac{1}{2}\frac{(\ell_e)^2}{T - t}\right\} + \int_t^T \varphi_e \, ds. \tag{15}$$

Given the above value functions, we can proceed backward. As an example, we consider $V_{p_3}^{e_5}(\tau)$ to determine the optimal control of an agent standing in v_{e_3} at time $t \in [0, T]$. It has two possible choices: staying in v_{e_3} or moving to reach κ_{e_3} at some (optimal) instants $\tau \in]t, T]$. Hence, The agent has to choose between the following two candidate constant optimal controls:

$$u_{p_3,1}^{e_3}[t] \equiv 0, \qquad u_{p_3,2}^{e_3}[t] \equiv \frac{\ell_{e_3}}{\tau - t}, \tag{16}$$

whose associated value function is:

$$V_{p_3}^{e_3}(t) = \min\left\{\alpha\left(\ell_{e_3} + \ell_{e_5}\right) + \int_t^\tau \varphi_{e_3}\, ds, \inf_{\tau \in]t,T]}\left\{\frac{1}{2}\frac{(\ell_{e_3})^2}{\tau - t} + \int_t^\tau \varphi_{e_3}\, ds + V_{p_3}^{e_5}(\tau)\right\}\right\}. \quad (17)$$

Iterating a similar argument, we can determine the optimal controls and the value functions of all the agents that are entering an arc e of a path p at time t.

We remark that the minimization processes in τ are admissible because of the coercivity of the minimizing term when $\tau \to t^+$.

4 Existence of a Mean-Field Equilibrium

The procedure to prove the existence of a mean-field equilibrium for ρ over the considered network $\mathcal{G}$ is discussed in this section. We proceed as follows: first, we choose the space of functions to search for a fixed point, that is, for a function ρ which describes the desired mean-field equilibrium. We choose

$$X = \left\{w : [0,T] \to [0, \rho_{\max}] : L(w) \le \tilde{L}, |w| \le \rho_{\max}\right\}^{\Xi}, \quad (18)$$

that is, the Cartesian product Ξ times of the space of Lipschitzian functions w with Lipschitz constant $L(w)$ not greater than a constant value $\tilde{L}$ and overall bounded by $\rho_{\max}$. Space X is convex and compact with respect to the uniform topology.

Then, fixed the noisy parameter $\beta > 0$, we search for a fixed point of the function ψ: $X \to X$, with $\rho' = \psi(\rho)$ where ρ' is obtained performing the following steps (see diagram in Fig. 2):

(i) Given the mass ρ the optimal control u is derived as described in the previous section;

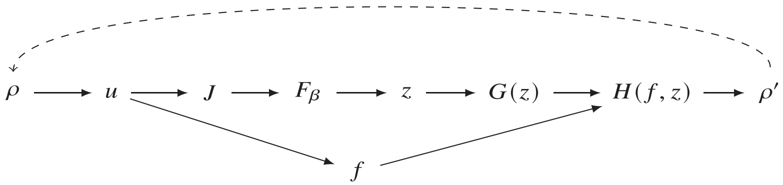

Fig. 2 Fixed-point scheme

(ii) The optimal control u is used both to compute the flow vector f through (11) and to obtain the path preferences vector z through (7) by first computing the vector of costs J and thus the vector F_β;

(iii) The mass vector ρ' is derived from f and z through (9) by first computing the vectors G through (8) and H through (10);

(iv) We set $\rho = \rho'$ and we iterate steps (i)–(iii) until we converge to a value ρ which satisfies $\rho = \psi(\rho)$.

Note that a suitable constant $\tilde{L}$ exists such that the function ψ maps X into itself. Indeed, note that, by construction, $\psi(\rho)$ must satisfy (9) and hence, by Remark 1 and Assumption 1.4, the bound $|\rho| \leq \rho_{\max}$ is satisfied and, as Lipschitz constant we can take $\tilde{L} = 3\bar{\lambda}$.

Finally, the following theorem guarantees that the above procedure converges.

Theorem 1 *Given Assumptions 1, a mean-field equilibrium, that is a total mass $\rho \in X$ that satisfies $\rho = \psi(\rho)$, exists.*

The proof of the above theorem is out of the scope of the current work. The proof would show that if ρ is continuous and Assumptions 1 hold, then every value function defined in the previous section is: Lipschitz continuous, with Lipschitz constant independent of ρ; bounded independently on ρ; continuous with respect to the mass density ρ. Then, it would exploit this fact to prove that ψ is continuous (see [15]) so that we can apply the Brouwer fixed-point theorem which in turn guarantees the existence of a mean-field equilibrium.

5 Conclusions

In view of the link among the network literature and the one aiming to describe the crowd's behavior, in this work we have introduced a novel mean-field game model to represent the agents' flows over network. Our model takes into account the agents' preferences about the path choices and, in particular, the fact that the agents choose their path on the basis of both the network congestion state and the observation of the decision of whom have preceded them.

We have also introduced a set of conditions that are sufficient to ensure the existence of a mean-field equilibrium. Possibly, the strongest condition is the one assuming the absence of (oriented) cycles in the network. This assumption may limit the application of our work to the description of information flows only for particular organizational networks, for example, the ones characterized by a hierarchical structure.

In light of the above considerations, our model can be framed in the literature on network performance and on how individual agents' preferences and behaviors may influence it.

Our future research will look at how far it is possible to relax the conditions introduced in this work and still ensure the existence of a mean-field equilibrium. In addition, it will investigate deeper the connection between network equilibrium and network effectiveness.

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