Zero sets and Nullstellensatz type theorems for slice regular quaternionic polynomials

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A B S T R A C T

We study the vanishing sets of slice regular polynomials in several quaternionic variables. We obtain a geometric description of the vanishing sets in two variables, which leads to a new version of the *Strong* Hilbert Nullstellensatz in the quaternionic setting.

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1. Introduction

The so called *Weak* Hilbert Nullstellensatz can be regarded as a generalization of the Fundamental Theorem of Algebra to the case of polynomials in several complex variables, as pointed out in [9]. The Hilbert Nullstellensatz has also a *Strong* formulation which provides a correspondence between radical ideals and vanishing sets of polynomials in several complex variables. This theorem represents a central tool in many active research fields in Mathematics, and especially in Algebraic Geometry. In the complex setting these two versions of the Hilbert Nullstellensatz, are actually equivalent thanks to the fact that point-evaluation of a polynomial is a ring homomorphism of complex polynomials.

The Hilbert Nullstellensatz is generally stated in the framework of algebraically closed fields, but in recent times some new interest has been addressed to a formalization of the Nullstellensatz in a noncommutative setting (see, e.g., [1–3]). In particular, Alon and Paran in [1] proved both a Weak and a Strong version of the Nullstellensatz in the ring $\mathbb{H}[x_1, \ldots, x_n]$ of quaternionic polynomials with *central* variables, i.e. such that $x_\ell x_m = x_m x_\ell$ for all $\ell, m = 1 \ldots n$. We remark here that, in this framework, the equivalence of the two versions of the Nullstellensatz cannot be shown using the point-evaluation of polynomials since it is not a ring homomorphism in general.

In the last decade the theory of slice regular quaternionic functions has proved to be central for the development of the study of quaternionic maps which resemble the main properties of holomorphic functions in the complex setting (see, e.g., [4,5]). Furthermore, even though an analog of the Fundamental Theorem of Algebra does not hold in general for polynomials over the quaternions, in [6] a positive result in this sense has been obtained for slice regular polynomials over quaternions and octonions. Thus it is quite natural to look for a version of the Nullstellensatz in the framework of slice regular polynomials in n quaternionic variables, i.e. polynomial functions $P : \mathbb{H}^n \to \mathbb{H}$ of the form

$$(q_1, \ldots, q_n) \mapsto P(q_1, \ldots, q_n) = \sum_{\substack{\ell_1=0,\ldots,L_1 \\ \cdots \\ \ell_n=0,\ldots,L_n}} {q_1}^{\ell_1} \cdots {q_n}^{\ell_n} a_{\ell_1,\ldots,\ell_n}$$

with $a_{\ell_1,\ldots,\ell_n} \in \mathbb{H}$, where $\deg_{q_{\ell_j}} P := L_j$. Slice regular polynomial functions of several variables can be endowed with an appropriate notion of product, the so called *slice product*, that will be denoted by the symbol $*$. Let us recall here how it works for slice regular polynomials in two variables. If $P(q_1, q_2) = \sum_{\substack{n=0,\ldots,N_1 \\ m=0,\ldots,N_2}} q_1^n q_2^m a_{n,m}$ and $Q(q) = \sum_{\substack{n=0,\ldots,L_1 \\ m=0,\ldots,L_2}} q_1^n q_2^m b_{n,m}$ are two slice regular polynomials, then the $*$-*product* of P and Q is the slice regular polynomial defined by

$$P * Q(q_1, q_2) := \sum_{\substack{n=0,\ldots,N_1+L_1 \\ m=0,\ldots,N_2+L_2}} q_1^n q_2^m \sum_{\substack{r=0,\ldots,n \\ s=0,\ldots,m}} a_{r,s} b_{n-r,m-s}$$

It is possible to establish an isomorphism φ between the ring of slice regular polynomials in n quaternionic variables $\mathbb{H}[q_1, \dots, q_n]$ equipped with the $*$-product and the ring of polynomials in several central variables $\mathbb{H}[x_1, \dots, x_n]$ equipped with the standard pointwise product, considered in [1]. Thanks to the isomorphism φ, it is immediate to rephrase in our setting the weak version of the quaternionic Hilbert Nullstellensatz proved in [1]. However, the established isomorphism cannot be directly used to study the vanishing sets of polynomials. Indeed, an element in $\mathbb{H}[x_1, \dots, x_n]$ can be evaluated as a function only on n-tuples $(a_1, \dots, a_n) \in \mathbb{H}^n$ with commuting components $a_l a_m = a_m a_l$ for any $l, m = 1, \dots, n$ (a nowhere dense subset of $\mathbb{H}^n$) whereas the corresponding regular polynomial function is well defined in $\mathbb{H}^n$.

In order to study the zero set of slice regular polynomials, we begin by focusing our attention to their factorization properties in terms of the $*$-product. While polynomials vanishing at points with commuting components are studied in [1], we characterize slice regular polynomials which vanish at any given point in $\mathbb{H}^n$.

Let I be a right ideal in $\mathbb{H}[q_1, \dots, q_n]$; we define $\mathcal{V}(I)$ to be the set of common zeros of polynomials in I. Let Z be a subset of $\mathbb{H}^n$, we denote by $\mathcal{I}(Z)$ the right ideal given by the intersection, for $(a_1, \dots, a_n) \in Z$, of the right ideals $\mathcal{I}_{(a_1,\dots,a_n)}$ generated, via the $*$-product, by $q_1 - a_1, q_2 - a_2, \dots, q_n - a_n$ in $\mathbb{H}[q_1, \dots, q_n]$.
We point out that, in the quaternionic setting, $\mathcal{I}(Z)$ does not always coincide with the set of polynomials vanishing on Z. Indeed the set of polynomials whose zero locus contains Z is not an ideal, in general. So it is natural to consider also the ideal $\mathcal{J}(Z)$ generated by polynomials vanishing on Z; we then investigate the relations of $\mathcal{J}(Z)$ with $\mathcal{I}(Z)$. The two sets $\mathcal{I}(Z)$ and $\mathcal{J}(Z)$ coincide, for instance, when Z consists only of points with commuting components (this is the case considered in [1]). In the general case, we can show the inclusion $\mathcal{J}(Z) \subseteq \mathcal{I}(Z)$.
After introducing the radical of an ideal I as the intersection of all *completely prime* ideals containing I, we can rephrase the Strong version of the Nullstellensatz proved in [1] in the setting of slice regular polynomials.

In the two variable case, we show some relevant geometric properties of the vanishing set of an ideal in $\mathbb{H}[q_1, q_2]$. We say that a subset $D \subseteq \mathbb{H}^2$ is q_1-*symmetric* if, for any $(a, b) \in D$, such that $ab \neq ba$, the set $\mathbb{S}_a \times \{b\}$ is contained in D. With this notation, we show that, given a right ideal $I \subseteq \mathbb{H}[q_1, q_2]$, then $\mathcal{V}(I)$ is q_1-symmetric. This is the key ingredient to show that $\mathcal{J}(\mathcal{V}(I))$ coincides with $\mathcal{I}(\mathcal{V}(I))$ in $\mathbb{H}[q_1, q_2]$ and thus to have a more geometric interpretation of the Strong Nullstellensatz in this framework.

Theorem *(Strong Nullstellensatz in $\mathbb{H}^2$). Let I be a right ideal in $\mathbb{H}[q_1, q_2]$. Then*

$$\mathcal{J}(\mathcal{V}(I)) = \sqrt{I}.$$

Moreover, $\sqrt{I}$ coincides with the ideal of polynomials vanishing on $\mathcal{V}(I)$.

These promising results and their potential generalizations in several quaternionic variables are in the direction of developing a theory of quaternionic algebraic varieties in

$\mathbb{H}^n$, without any assumption on the commutativity of the components of the considered points.
The present paper is organized as follows: in Section 2 we shortly recall the main definitions and results from the theory of slice regular polynomial functions which will be used in the sequel. The factorization of slice regular polynomials and application to the study of ideals in $\mathbb{H}[q_1, \ldots, q_n]$ is treated in Section 3, where moreover one can find several properties of the vanishing sets of slice regular polynomials in two quaternionic variables. The Nullstellensatz type theorems for slice regular polynomials are investigated in Section 4; in particular, a new version of the Strong Nullstellensatz Theorem is proved for ideals in $\mathbb{H}[q_1, q_2]$. Finally we provide examples in several quaternionic variables that enforce the evidence that this new version of the Strong Nullstellensatz should hold in $\mathbb{H}[q_1, \ldots, q_n]$.

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2. Introduction to quaternionic slice regular polynomials

Let $\mathbb{H}= \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ denote the skew field of quaternions and let $\mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\}$ be the two dimensional sphere of quaternionic imaginary units. Then

$$\mathbb{H} = \bigcup_{J\in\mathbb{S}} (\mathbb{R} + \mathbb{R}J),$$

where the "slice" $\mathbb{C}_J := \mathbb{R} + \mathbb{R}J$ can be identified with the complex plane $\mathbb{C}$ for any $J \in \mathbb{S}$. In this way, any $q \in \mathbb{H}$ can be expressed as $q = x + yJ$ with $x, y \in \mathbb{R}$ and $J \in \mathbb{S}$. The *real part* of q is $\mathrm{Re}(q) = x$ and its *imaginary part* is $\mathrm{Im}(q) = yJ$; the *conjugate* of q is $\bar{q} := \mathrm{Re}(q) - \mathrm{Im}(q)$. For any non-real quaternion $a \in \mathbb{H} \setminus \mathbb{R}$ we will denote by $J_a := \frac{\mathrm{Im}(a)}{|\mathrm{Im}(a)|} \in \mathbb{S}$ and by $\mathbb{S}_a := \{\mathrm{Re}(a) + J|\mathrm{Im}(a)| \ : \ J \in \mathbb{S}\}$. If $a \in \mathbb{R}$, then J_a is any imaginary unit.

The central object of the present paper is the class of slice regular quaternionic polynomial functions $P : \mathbb{H}^n \to \mathbb{H}$,

$$(q_1, \ldots, q_n) \mapsto P(q_1, \ldots, q_n) = \sum_{\substack{\ell_1=0,\ldots,L_1 \\ \cdots \\ \ell_n=0,\ldots,L_n}} {q_1}^{\ell_1} \cdots {q_n}^{\ell_n} a_{\ell_1,\ldots,\ell_n}$$

with $a_{\ell_1,\ldots,\ell_n} \in \mathbb{H}$, where $\deg_{q_{\ell_j}} P := L_j$.
These polynomial functions are examples of *slice regular functions* on $\mathbb{H}^n$. When considering functions of several quaternionic variables, the definition of slice regularity relies on the notion of *stem functions*. The formulation of the theory in several quaternionic variables (in the more general setting of real alternative $*$-algebras) can be found in [7], whereas for an updated survey on the theory in one quaternionic variable we refer to the book [4].

Slice regular polynomial functions of several variables can be endowed with an appropriate notion of product, the so called *slice product*, that will be denoted by the symbol $*$. Let us recall here how it works for slice regular polynomials in two variables.

Definition 2.1. If $P(q_1, q_2) = \sum_{\substack{n=0,\dots,N_1\\ m=0,\dots,N_2}} q_1^n q_2^m a_{n,m}$ and $Q(q) = \sum_{\substack{n=0,\dots,L_1\\ m=0,\dots,L_2}} q_1^n q_2^m b_{n,m}$ are two slice regular polynomials, then the $*$-*product* of P and Q is the slice regular polynomial defined by

$$P * Q(q_1, q_2) := \sum_{\substack{n=0,\dots,N_1+L_1\\ m=0,\dots,N_2+L_2}} q_1^n q_2^m \sum_{\substack{r=0,\dots,n\\ s=0,\dots,m}} a_{r,s} b_{n-r,m-s}$$

For example, if $a, b \in \mathbb{H}$, then

- $q_1 * q_2 = q_2 * q_1 = q_1 q_2$;
- $a * (q_1 q_2) = (q_1 q_2) * a = q_1 q_2 a$;
- $(q_1^n q_2^m a) * (q_1^r q_2^s b) = q_1^{n+r} q_2^{m+s} ab$.

Moreover we point out that, if P or Q have real coefficients, then $P * Q = Q * P$.
For slice regular polynomials of one variable, the relation of the $*$-product with the usual pointwise product is the following (see [4, Theorem 3.4]):

$$P * Q(q) = \begin{cases} 0 & \text{if } P(q) = 0 \\ P(q) \cdot Q(P(q)^{-1} \cdot q \cdot P(q)) & \text{if } P(q) \neq 0\,, \end{cases} \tag{2.1}$$

for any P, Q slice regular polynomials. Notice that $P(q)^{-1} \cdot q \cdot P(q)$ belongs to the sphere $\mathbb{S}_q$. Hence each zero of $P * Q$ in $\mathbb{S}_q$ is given either by a zero of P or by a point which is a conjugate of a zero of Q in the same sphere.

A peculiar aspect of slice regular polynomials (more in general slice regular functions) in one quaternionic variable is the structure of their zero sets. In fact, besides isolated zeros, they can also vanish on two dimensional spheres. As an example, the polynomial $q^2 + 1$ vanishes on the entire sphere of imaginary units $\mathbb{S}$. It can be proven that also spherical zeros cannot accumulate.

Theorem 2.2. *[4, Theorem 3.13] Let P be a slice regular polynomial in one quaternionic variable. If P does not vanish identically, then its zero set consists of isolated points or isolated 2-spheres of the form $x + y\mathbb{S}$ with $x, y \in \mathbb{R}$, $y \neq 0$.*

Remark 2.3. Observe that the evaluation of slice regular power series is not a multiplicative homomorphism. Therefore, the zeros of the $*$-product of two slice regular polynomials are not in general the union of the zeros of each of the factors. For instance, $q_1 - i$ vanishes on $\{i\} \times \mathbb{H}$, while $(q_1 - i) * (q_2 - j) = q_1 q_2 - q_1 j - q_2 i + k$, when $q_1 = i$, vanishes only for $q_2 \in \mathbb{C}_i$.

In the sequel we will denote by $\mathbb{H}[q_1,\dots,q_n]$ the set of slice regular polynomials in n quaternionic variables. Since the $*$-product is associative but not commutative, $(\mathbb{H}[q_1,\dots,q_n],+,*)$ is a noncommutative ring (without zero divisors).

Definition 2.4. A subset I of $\mathbb{H}[q_1,\dots,q_n]$, closed under addition, is called

- a *left ideal* if for any $P \in \mathbb{H}[q_1,\dots,q_n]$, $P * I = \{P * Q \ : \ Q \in I\} \subseteq I$;
- a *right ideal* if for any $P \in \mathbb{H}[q_1,\dots,q_n]$, $I * P = \{Q * P \ : \ Q \in I\} \subseteq I$;
- a *two-sided ideal* if I is both a left and a right ideal.

From the algebraic point of view, the ring $(\mathbb{H}[q_1,\dots,q_n],+,*)$ is isomorphic to the ring $(\mathbb{H}[x_1,\dots,x_n],+,\cdot)$ of quaternionic polynomials in several central variables with left coefficients considered in [1], via the map defined on monomials as

$$\begin{aligned}\varphi : (\mathbb{H}[q_1,\dots,q_n],+,*) &\longrightarrow (\mathbb{H}[x_1,\dots,x_n],+,\cdot)\\ \varphi : q_1^{\ell_1}\cdots q_n^{\ell_n} a &\longmapsto \overline{a} x_1^{\ell_1}\cdots x_n^{\ell_n},\end{aligned} \tag{2.2}$$

and then extended by additivity to polynomials. The Identity Principle for Polynomials guarantees that φ is a bijection. Moreover it satisfies the equality

$$\begin{aligned}&\varphi(q_1^{\ell_1}\cdots q_n^{\ell_n} a * q_1^{k_1}\cdots q_n^{k_n} b) = \varphi(q_1^{\ell_1+k_1}\cdots q_n^{\ell_n+k_n} ab) = \overline{ab} x_1^{\ell_1+k_1}\cdots x_n^{\ell_n+k_n} =\\ =&\overline{b}\overline{a} x_1^{\ell_1+k_1}\cdots x_n^{\ell_n+k_n} = \overline{b} x_1^{\ell_1}\cdots x_n^{\ell_n}\cdot \overline{a} x_1^{k_1}\cdots x_n^{k_n} = \varphi(q_1^{k_1}\cdots q_n^{k_n} b)\cdot\varphi(q_1^{\ell_1}\cdots q_n^{\ell_n} a).\end{aligned}$$

The isomorphism φ inverts the order of the factors, this is in accordance with the fact that ideals considered in [1] are *left* ideals, while in our context it is natural to consider *right* ideals.

Remark 2.5. Observe that any slice regular polynomial function $P \in \mathbb{H}[q_1,\dots,q_n],+,*)$ can be evaluated at any point in $\mathbb{H}^n$, whereas any polynomial with central variables $\varphi(P) \in (\mathbb{H}[x_1,\dots,x_n],+,\cdot)$ can be evaluated only at points in $\bigcup_{J\in\mathbb{S}}(\mathbb{C}_J)^n \Subset \mathbb{H}^n$. In [1], the set $\bigcup_{J\in\mathbb{S}}(\mathbb{C}_J)^n$ is denoted by $\mathbb{H}^n_c$.

As a consequence of the previous remark, the isomorphism φ cannot be used to have information on the evaluations of the isomorphic polynomials in the two polynomial rings. However, in $\bigcup_{J\in\mathbb{S}}(\mathbb{C}_J)^n \Subset \mathbb{H}^n$ it is possible to establish a correspondence between values of slice regular polynomials in $\mathbb{H}[q_1,\dots,q_n]$ and those of their images in $\mathbb{H}[x_1,\dots,x_n]$ through φ. Indeed, let $(a_1,\dots,a_n) \in \mathbb{H}^n$ be such that $a_\ell a_m = a_m a_\ell$ for any ℓ, m, and let $P \in \mathbb{H}[q_1,\dots,q_n]$. Then, it is immediate to see that

$$\varphi(P(a_1,\dots,a_n)) = (\varphi(P))(\overline{a_1},\dots,\overline{a_n}). \tag{2.3}$$

As in the one-variable case (see [4, Definition 1.46]), it is possible to introduce the conjugation and the symmetrization operators for slice regular polynomials in several quaternionic variables. For the sake of simplicity, we define them only in the case of two quaternionic variables.

Definition 2.6. Let $P(q_1, q_2) = \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} q_1^n q_2^m a_{n,m}$ be a slice regular polynomial. Then the *regular conjugate* of P is the slice regular polynomial

$$P^c(q_1, q_2) = \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} q_1^n q_2^m \overline{a_{n,m}},$$

and the *symmetrization* of P is the slice regular polynomial

$$P^s = P * P^c = P^c * P.$$

A very useful result, that will be used in the sequel, is the following version of *Identity Principle* for slice regular polynomial functions. See [7, Corollary 2.13] for the details.

Theorem 2.7 *(Identity Principle). Let P, Q be two slice regular polynomials in n quaternionic variables. If there exist $J_1, \dots, J_n \in \mathbb{S}$ such that $P \equiv Q$ on $\mathbb{C}_{J_1} \times \cdots \times \mathbb{C}_{J_n}$, then $P \equiv Q$ on $\mathbb{H}^n$.*

3. Some remarks on factorization of slice regular polynomials

In this paper we will only consider slice regular polynomials in quaternionic variables. For brevity, we will refer to them simply as *regular polynomials*. In the one-variable setting it is possible to perform both left and right Euclidean $*$-division between regular polynomials, see [4, Proposition 3.42]. For regular polynomials in several quaternionic variables, as in the complex case, we need to restrict our setting to the case of division of regular polynomials by a *monic* regular polynomial.

Definition 3.1. A regular polynomial $P(q_1, \dots, q_2)$ is *monic* of degree d in the variable q_j if it can be written as

$$P(q_1, \dots, q_n) = q_j^d + \sum_{k=1}^{d-1} q_j^k * P_k(q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_n)$$

with $P_1, \dots, P_{d-1} \in \mathbb{H}[q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_n]$.

Proposition 3.2. *Let $M \in \mathbb{H}[q_1, \dots, q_n]$ be a monic regular polynomial of degree d in q_j, with $1 \leq j \leq n$. Then, for any $P \in \mathbb{H}[q_1, \dots, q_n]$, there exist, and are unique, regular polynomials $Q \in \mathbb{H}[q_1, \dots, q_n]$ and $R_0, \dots, R_{d-1} \in \mathbb{H}[q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_n]$ such that*

$$P = M * Q + \sum_{k=0}^{d-1} q_j^k * R_k.$$

Proof. Let $P_0, \dots, P_s \in \mathbb{H}[q_1, \dots, q_{j-1}, q_{j+1}, \dots, q_n]$ be such that

$$P = \sum_{k=0}^{s} q_j^k * P_k,$$

and proceed by induction on the degree s of P in q_j.
If $s < d$, then we set $Q \equiv 0$ and $R_k = P_k$ and we immediately prove the statement.
Otherwise, consider $\hat{P} = P - M * q_j^{s-d} * P_s$. Since M is monic of degree d in q_j, we get that $\deg_{q_j} \hat{P} < s$. We can therefore use the induction hypothesis to write $\hat{P} = M * \hat{Q} + \sum_{k=0}^{d-1} q_j^k \hat{R}_k$, so that

$$P = \hat{P} + M * q_j^{s-d} * P_s = M * (\hat{Q} + q_j^{s-d} * P_s) + \sum_{k=0}^{d-1} q_j^k \hat{R}_k.$$

Thus, setting $Q = \hat{Q} + q_j^{s-d} * P_s$ and $R_k = \hat{R}_k$, we conclude. $\square$

Here we begin the study of zeros of regular polynomials. Our interest is twofold: on the one hand we aim to better understand the structure of the vanishing sets of regular polynomials, on the other hand we want to investigate the relation between such vanishing sets and ideals in $\mathbb{H}[q_1, \dots, q_n]$.

Definition 3.3. If $a \in \mathbb{H}$, let C_a be the set of $q \in \mathbb{H}$ such that $aq = qa$, namely

$$C_a = \begin{cases} \mathbb{C}_{J_a} & \text{if} \quad a \in \mathbb{H} \setminus \mathbb{R} \\ \mathbb{H} & \text{if} \quad a \in \mathbb{R} \end{cases}$$

Let us first consider a simple example that enlights how the lack of commutativity of the product affects the structure of the zero locus of a regular polynomial in two quaternionic variables and shows that these two variables are not interchangeable.

Example 3.4. Let $a, b \in \mathbb{H}$ be such that $ab \neq ba$ and consider the regular polynomials

$$P(q_1, q_2) = (q_1 - a) * (q_2 - b) = q_1 q_2 - q_1 b - q_2 a + ab$$

and

$$Q(q_1, q_2) = (q_2 - b) * (q_1 - a) = q_1 q_2 - q_1 b - q_2 a + ba.$$

For $q_1 = a$ we have

$$P(a, q_2) = aq_2 - q_2 a \quad \text{and} \quad Q(a, q_2) = aq_2 - ab - q_2 a + ba.$$

Hence P vanishes on $\{a\} \times C_a$, while Q does not.
On the other hand, for $q_2 = b$,

$$P(q_1, b) = -ba + ab \quad \text{and} \quad Q(q_1, b) \equiv 0.$$

Hence Q vanishes on $\mathbb{H} \times \{b\}$, while P is never zero when $q_2 = b$.

Remark 3.5. In addition to Remark 2.5, observe that there is not a direct correspondence between the zero locus of regular polynomials in $(\mathbb{H}[q_1, q_2], +, *)$ with the zero locus of polynomials in $(\mathbb{H}[x_1, x_2], +, \cdot)$. In fact, the zero locus of the regular polynomial $P(q_1, q_2) = (q_2 - i) * (q_1 - j)$ in $\mathbb{H}^2$ contains $\mathbb{H} \times \{i\}$, while its isomorphic image via φ in $(\mathbb{H}[x_1, x_2], +, \cdot)$, namely $(x_1 + j) \cdot (x_2 + i)$, for $x_2 = -i$ vanishes only when $x_1 \in \mathbb{C}_i$ (since it cannot be evaluated at points with noncommutative coordinates).

The next proposition gives a first geometrical description of the zero set of regular polynomials in several quaternionic variables.

Proposition 3.6. *Let $P \in \mathbb{H}[q_1, \dots, q_n]$ be a regular polynomial in n variables and let $1 \leq m \leq n$. Then P vanishes on $\mathbb{H}^{m-1} \times \{a\} \times (C_a)^{n-m}$ if and only if there exists $P_m \in \mathbb{H}[q_1, \dots, q_n]$ such that*

$$P(q_1, \dots, q_n) = (q_m - a) * P_m(q_1, \dots, q_n).$$

Proof. Let $P_m(q_1, \dots, q_n) := \sum_{\substack{k_j=0,\dots,N_j \\ j=1,\dots,n}} q_1^{k_1} \cdots q_n^{k_n} b_{k_1 \dots k_n}$ be a regular polynomial in $\mathbb{H}[q_1, \dots, q_n]$ and consider $P = (q_m - a) * P_m$. Then

$$\begin{aligned} P(q_1, \dots, q_n) &= q_m * P_m(q_1, \dots, q_n) - a * P_m(q_1, \dots, q_n) \\ &= \sum_{\substack{k_j=0,\dots,N_j \\ j=1,\dots,n}} q_1^{k_1} \cdots q_m^{k_m+1} \cdots q_n^{k_n} b_{k_1 \dots k_n} \\ &\quad - \sum_{\substack{k_j=0,\dots,N_j \\ j=1,\dots,n}} q_1^{k_1} \cdots q_m^{k_m} \cdots q_n^{k_n} \cdot a \cdot b_{k_1 \dots k_n}, \end{aligned}$$

and hence, for any $(q_1, \dots, q_{m-1}, a, u_{m+1}, \dots, u_n) \in \mathbb{H}^{m-1} \times \{a\} \times (C_a)^{n-m}$,

$$\begin{aligned} &P(q_1, \dots, q_{m-1}, a, u_{m+1}, \dots, u_n) \\ &= \sum_{\substack{k_j=0,\dots,N_j \\ j=1,\dots,n}} q_1^{k_1} \cdots a^{k_m+1} \cdots u_n^{k_n} \cdot b_{k_1 \cdots k_n} - \sum_{\substack{k_j=0,\dots,N_j \\ j=1,\dots,n}} q_1^{k_1} \cdots a^{k_m} \cdots u_n^{k_n} \cdot a \cdot b_{k_1 \cdots k_n} \\ &= 0. \end{aligned}$$

On the other hand, suppose that a regular polynomial P vanishes on $\mathbb{H}^{m-1} \times \{a\} \times (C_a)^{n-m}$. Then, performing the $*$-division of P by $q_m - a$ as in Proposition 3.2, we obtain

$$P(q_1, \dots, q_n) = (q_m - a) * P_m(q_1, \dots, q_n) + R(q_1, q_2, \dots, q_{m-1}, q_{m+1}, \dots, q_n).$$

Now, for any $(q_1, \dots, q_{m-1}, a, u_{m+1}, \dots, u_n) \in \mathbb{H}^{m-1} \times \{a\} \times (C_a)^{n-m}$ we have that $P(q_1, \dots, q_{m-1}, a, u_{m+1}, \dots, u_n) = 0$ and, as in the previous considerations, also

$$[(q_m - a) * P_m(q_1, \dots, q_n)]|_{(q_1, \dots, q_{m-1}, a, u_{m+1}, \dots, u_n)} = 0.$$

Hence R vanishes identically on $\mathbb{H}^{m-1} \times (C_a)^{n-m}$. Thanks to the Identity Principle 2.7, we get that R is identically zero and we conclude the proof. □

We now want to establish a relation between zeros of regular polynomials and ideals in $\mathbb{H}[q_1, \dots, q_n]$. If $(a_1, \dots, a_n) \in \mathbb{H}^n$, then denote by

$$\mathcal{I}_{(a_1, \dots, a_n)} = \{(q_1 - a_1) * P_1 + \dots + (q_n - a_n) * P_n \mid P_1, \dots, P_n \in \mathbb{H}[q_1, \dots, q_n]\}.$$

Thanks to the properties of the $*$-product, it is not difficult to prove that $\mathcal{I}_{(a_1, \dots, a_n)}$ is a right ideal, generated (via the $*$-product) by $q_1 - a_1, q_2 - a_2, \dots, q_n - a_n$ in $\mathbb{H}[q_1, \dots, q_n]$.

Remark 3.7. The isomorphism φ introduced in (2.2) maps the ideal $\mathcal{I}_{(a_1, \dots, a_n)} \subseteq \mathbb{H}[q_1, \dots, q_n]$ to the ideal $I_{\overline{a}}$ in $\mathbb{H}[x_1, \dots, x_n]$, where $\overline{a} = (\overline{a_1}, \dots, \overline{a_n})$ (as in [1]).

From Lemma 2.1 and Proposition 2.2 in [1], taking into account Remark 3.7, we directly get:

Proposition 3.8 *([1, Lemma 2.1 and Proposition 2.2]). Let $(a_1, \dots, a_n) \in \mathbb{H}^n$.*

(1) *If $a_l a_m = a_m a_l$ for any $1 \leq l, m \leq n$, then $\mathcal{I}_{(a_1, \dots, a_n)}$ is a maximal ideal in $\mathbb{H}[q_1, \dots, q_n]$;*
(2) *if there exist $l, m \in \{1, \dots, n\}$ such that $a_l a_m \neq a_m a_l$, then $\mathcal{I}_{(a_1, \dots, a_n)} = \mathbb{H}[q_1, \dots, q_n]$.*

Moreover, recalling Formula (2.3) and Remark 3.7, we immediately obtain:

Proposition 3.9 *([1, Proposition 2.2]). Let $(a_1, \dots, a_n) \in \mathbb{H}^n$ with $a_l a_m = a_m a_l$ for any $1 \leq l, m \leq n$, and let $\mathcal{I}_{(a_1, \dots, a_n)}$ be the right ideal generated by $q_1 - a_1, \dots, q_n - a_n$. Then a regular polynomial $P \in \mathbb{H}[q_1, \dots, q_n]$ belongs to $\mathcal{I}_{(a_1, \dots, a_n)}$ if and only if $P(a_1, \dots, a_n) = 0$.*

Let us now investigate what kind of information we obtain on the $*$–factorization of a regular polynomial which vanishes at a generic point $(a_1, \dots, a_n)$, without any assumption on the commutativity of its components.

Proposition 3.10. *A regular polynomial in $P \in \mathbb{H}[q_1, \dots, q_n]$ vanishes at $(a_1, \dots, a_n) \in \mathbb{H}^n$ if and only if there exist $P_k \in \mathbb{H}[q_1, \dots, q_k]$ for any $k = 1, \dots, n$ such that*

$$P(q_1, \dots, q_n) = \sum_{k=1}^{n} (q_k - a_k) * P_k(q_1, \dots, q_k).$$

Proof. We start performing the $*$-division of P by $(q_n - a_n)$ as in Proposition 3.2; we thus obtain

$$P(q_1, \dots, q_n) = (q_n - a_n) * P_n(q_1, \dots, q_n) + R_n(q_1, \dots, q_{n-1}).$$

Notice that $R_n(q_1, \dots, q_{n-1})$ does not depend on q_n since $\deg_{q_n} R < 1$. If we now divide R_n by $(q_{n-1} - a_{n-1})$, we obtain

$$\begin{aligned} P(q_1, \dots, q_n) = &(q_n - a_n) * P_n(q_1, \dots, q_n) + (q_{n-1} - a_{n-1}) * P_{n-1}(q_1, \dots, q_{n-1}) \\ &+ R_{n-1}(q_1, \dots, q_{n-2}). \end{aligned}$$

Iterating this process, at the $(n-2)$-th step we get

$$\begin{aligned} P(q_1, \dots, q_n) = &(q_n - a_n) * P_n(q_1, \dots, q_n) + (q_{n-1} - a_{n-1}) * P_{n-1}(q_1, \dots, q_{n-1}) + \\ &+ \dots + (q_2 - a_2) * P_2(q_1, q_2) + R_2(q_1) \end{aligned}$$

where R_2 is a one-variable regular polynomial in q_1. Thanks to Proposition 3.6, evaluating at $(a_1, \dots, a_n)$ the previous equality gives

$$0 = P(a_1, \dots, a_n) = R_2(a_1)$$

which, recalling the one-variable theory (see [4, Proposition 3.18]), implies that there exists $P_1 \in \mathbb{H}[q_1]$ such that $R_2(q_1) = (q_1 - a_1) * P_1(q_1)$. Therefore we prove the statement. □

Let $(a_1, \dots, a_n) \in \mathbb{H}^n$, and let us denote by $E_{(a_1, \dots, a_n)}$ the set of regular polynomials in $\mathbb{H}[q_1, \dots, q_n]$ which vanish at $(a_1, \dots, a_n)$, namely

$$E_{(a_1, \dots, a_n)} := \left\{ \sum_{k=1}^{n} (q_k - a_k) * P_k(q_1, \dots, q_k) \ : P_k \in \mathbb{H}[q_1, \dots, q_k] \text{ for any } k = 1, \dots, n \right\}.$$

Proposition 3.11. *Let $(a_1, \dots, a_n) \in \mathbb{H}^n$. The set $E_{(a_1, \dots, a_n)}$ of regular polynomials vanishing at $(a_1, \dots, a_n)$ is an ideal if and only if $a_l a_m = a_m a_l$ for any $l, m = 1, \dots, n$.*

Proof. If $(a_1, \dots, a_n) \in \mathbb{H}^n$ is such that $a_l a_m = a_m a_l$ for any $l, m = 1, \dots, n$, then $E_{(a_1, \dots, a_n)}$ coincides with the ideal $\mathcal{I}_{(a_1, \dots, a_n)}$. Indeed $E_{(a_1, \dots, a_n)} \subseteq \mathcal{I}_{(a_1, \dots, a_n)}$ by definition. On the other hand, by Proposition 3.9 all regular polynomials in $\mathcal{I}_{(a_1, \dots, a_n)}$ vanish at $(a_1, \dots, a_n)$ and hence $\mathcal{I}_{(a_1, \dots, a_n)} \subseteq E_{(a_1, \dots, a_n)}$.

Let now $(a_1, \dots, a_n) \in \mathbb{H}^n$ be such that $a_l a_m \neq a_m a_l$ for some $l, m \in \{1, \dots, n\}$ and suppose that $E_{(a_1,\dots,a_n)}$ is an ideal. Since the regular polynomials $q_1 - a_1, \dots, q_n - a_n$ belong to $E_{(a_1,\dots,a_n)}$, the same holds for the ideal generated by them $\mathcal{I}_{(a_1,\dots,a_n)} = \langle q_k - a_k \ : \ k = 1, \dots, n\rangle$. Recalling Proposition 3.8, $\mathcal{I}_{(a_1,\dots,a_n)} = \mathbb{H}[q_1, \dots, q_n] \subseteq E_{(a_1,\dots,a_n)}$, a contradiction. □

Remark 3.12. The previous Proposition can be read as follows: if a regular polynomial P vanishes at $(a_1, \dots, a_n)$ with $a_l a_m = a_m a_l$ for any $l, m = 1, \dots, n$, then $P * Q$ still vanishes at $(a_1, \dots, a_n)$ for any Q in $\mathbb{H}[q_1, \dots, q_n]$. The same is not in general true if a zero does not have commuting components.

The previous remark is also relevant when considering the zeros of the symmetrization of a regular polynomial. Recall that if a slice regular function in one quaternionic variable vanishes at $a \in \mathbb{H}$, then also its symmetrization does. In several quaternionic variables, we can show that the situation is the same only assuming the commutativity of the components of the assigned zero.

Corollary 3.13. *If $P \in \mathbb{H}[q_1, \dots, q_n]$ vanishes at $(a_1, \dots, a_n)$, with $a_l a_m = a_m a_l$ for any $l, m = 1, \dots, n$, then also its symmetrization does since $P^s = P * P^c$.*

This is not necessarily true if $a_l a_m \neq a_m a_l$ for some $l, m \in \{1, \dots, n\}$. As an example, consider $P(q_1, q_2) = q_1 q_2 - k$, whose symmetrization is $P^s(q_1, q_2) = q_1^2 q_2^2 + 1$. We have $P(i, j) = 0$, while $P^s(i, j) = 2$.

3.1. The two variables case

Using the properties of regular polynomials in one quaternionic variable, we are able to prove several results concerning the vanishing sets of regular polynomials in two quaternionic variables which enlighten different interesting phenomena passing from one to several quaternionic variables.

We begin by assigning the value of the first variable. The next proposition reminds the geometric origin of spherical zeros for slice regular functions in one quaternionic variable.

Proposition 3.14. *Let $a \in \mathbb{H} \setminus \mathbb{R}$. If $P(q_1, q_2)$ vanishes on $\{a\} \times C_a$ and also at (a, b), with $b \notin C_a$, then P vanishes on $\{a\} \times \mathbb{S}_b$.*

Proof. Thanks to Proposition 3.6,

$$P(q_1, q_2) = (q_1 - a) * Q(q_1, q_2)$$

with $Q(q_1, q_2) = \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} q_1^n q_2^m a_{nm}$. Notice that

$$(q_1 - a) * Q(q_1, q_2) = q_1 Q(q_1, q_2) - aQ(a^{-1} q_1 a, a^{-1} q_2 a);$$

hence

$$\begin{aligned} P(a, q_2) &= a \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} a^n q_2^m a_{nm} - a \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} a^{n-1} q_2^m a a_{nm} \\ &= a \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} a^{n-1} [a q_2^m - q_2^m a] a_{nm}. \end{aligned}$$

If $q_2 = x + Ky$, put $q_2^m := \alpha_m + K\beta_m$, with $x,\ y,\ \alpha_m, \beta_m$ real numbers and $K \in \mathbb{S}$. Similarly, write $a^n := u_n + J v_n$ if $a = a_o + J a_1$ with $a_0,\ a_1,\ u_n,\ v_n$ real numbers and $J \in \mathbb{S}$. Hence

$$\begin{aligned} P(a, q_2) &= a \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} a^{n-1} \underbrace{[aK - Ka]}_{:=T} \beta_m a_{nm} \\ &= a \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} (u_{n-1} + J v_{n-1}) T \beta_m a_{nm}. \end{aligned}$$

Now

$$T = aK - Ka = (a_o + J a_1) K - K(a_o + J a_1) = (JK - KJ) a_1,$$

which implies that $JT = -TJ$. Therefore

$$\begin{aligned} P(a, q_2) &= aT \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} (u_{n-1} - J v_{n-1}) \beta_m a_{nm} \\ &= aT \sum_{\substack{n=0,\dots,N \\ m=0,\dots,M}} \overline{a^{n-1}} \beta_m a_{nm}. \end{aligned} \tag{3.1}$$

If P vanishes at (a, b) with $b \notin C_a$, then $T \neq 0$ and hence P vanishes on $\{a\} \times \mathbb{S}_b$. Indeed (3.1) is a function of β_m which only depends on $x = \mathrm{Re}(q_2)$ and $y = |\mathrm{Im}(q_2)|$ and not on the imaginary unit of q_2. $\square$

We want to describe ideals contained in the set $E_{(a,b)}$ of regular polynomials vanishing at a point $(a, b) \in \mathbb{H}^2$. First we need the following

Proposition 3.15. *A regular polynomial in two variables $P \in \mathbb{H}[q_1, q_2]$ vanishes on $\mathbb{S}_a \times \{b\}$ if and only if there exist $P_1 \in \mathbb{H}[q_1]$ and $P_2 \in \mathbb{H}[q_1, q_2]$ such that*

$$P(q_1, q_2) = (q_1^2 - 2\mathrm{Re}(a) q_1 + |a|^2) * P_1(q_1) + (q_2 - b) * P_2(q_1, q_2)$$

Proof. Dividing P by $(q_2 - b)$ as in the proof of Proposition 3.10, we can write

$$P(q_1, q_2) = (q_2 - b) * P_2(q_1, q_2) + R(q_1),$$

and hence $R(q_1) = P(q_1, q_2) - (q_2 - b) * P_2(q_1, q_2)$ vanishes on $\mathbb{S}_a \times \{b\}$. Thus, since R is a slice regular polynomial in one quaternionic variable, thanks to Proposition 3.18 in [4],

$$R(q_1) = (q_1^2 - 2\mathrm{Re}(a)q_1 + |a|^2) * P_1(q_1). \quad \square$$

Let us denote by $E_{\mathbb{S}_a \times \{b\}}$ the set of regular polynomials vanishing on $\mathbb{S}_a \times \{b\}$, namely

$$E_{\mathbb{S}_a \times \{b\}} = \{(q_1^2 - 2\mathrm{Re}(a)q_1 + |a|^2) * P_1(q_1) + (q_2 - b) * P_2(q_1, q_2) \ : \ P_1 \in \mathbb{H}[q_1], P_2 \in \mathbb{H}[q_1, q_2]\}.$$

Proposition 3.16. *The set $E_{\mathbb{S}_a \times \{b\}}$ is an ideal contained in $E_{(a,b)}$.*

Proof. The fact that $(q_1^2 - 2\mathrm{Re}(a)q_1 + |a|^2)$ is a regular polynomial only in the first variable, with real coefficients, vanishing identically on $\mathbb{S}_a$, guarantees that

$$(q_1^2 - 2\mathrm{Re}(a)q_1 + |a|^2) * Q = (q_1^2 - 2\mathrm{Re}(a)q_1 + |a|^2) \cdot Q$$

vanishes on $\mathbb{S}_a \times \mathbb{H}$ for any $Q \in \mathbb{H}[q_1, q_2]$. Hence any regular polynomial of the form

$$(q_1^2 - 2\mathrm{Re}(a)q_1 + |a|^2) * Q_1 + (q_2 - b) * Q_2,$$

with $Q_1, Q_2 \in \mathbb{H}[q_1, q_2]$, vanishes on $\mathbb{S}_a \times \{b\}$. Therefore $E_{\mathbb{S}_a \times \{b\}}$ is an ideal. Since $a \in \mathbb{S}_a$, $E_{\mathbb{S}_a \times \{b\}} \subset E_{(a,b)}$. $\square$

Let us now show the following

Proposition 3.17. *Let $a, b \in \mathbb{H}$ be such that $ab \neq ba$ and let I be an ideal of $\mathbb{H}[q_1, q_2]$ contained in $E_{(a,b)}$. Then $I \subseteq E_{\mathbb{S}_a \times \{b\}}$.*

Proof. Let $P \in I$. Then $P(a, b) = 0$. Consider now $P * q_2$ which still belongs to I. Hence

$$0 = P(q_1, q_2) * q_2{}_{|_{(a,b)}} = q_2 * P(q_1, q_2)_{|_{(a,b)}} = q_2 P(q_2^{-1} q_1 q_2, q_2)_{|_{(a,b)}} = bP(b^{-1}ab, b).$$

Thus the regular polynomial $P(\cdot, b) \in \mathbb{H}[q_1]$ vanishes at $q_1 = a$ and at $q_1 = b^{-1}ab$ which are two different points on the two sphere $\mathbb{S}_a$. Thanks to Theorem 3.1 in [4]), $P(\cdot, b)$ vanishes on the entire sphere $\mathbb{S}_a$. Hence $P \in E_{\mathbb{S}_a \times \{b\}}$. $\square$

Propositions 3.15 and 3.16, and 3.17 will be used in the next section in order to study the vanishing set of regular polynomials in $\mathbb{H}[q_1, q_2]$; the first two results can be easily generalized to regular polynomials in n variables vanishing on sets of the form $\mathbb{S}_{a_1} \times \{a_2\} \times \cdots \times \{a_n\}$, where $a_l a_m = a_m a_l$ for all l, m such that $2 \leq l, m \leq n$.

4. Nullstellensatz type theorems for regular polynomials

In this Section, we further investigate the relations between zero sets of regular polynomials and ideals in $\mathbb{H}[q_1, \dots, q_n]$. In the complex setting such correspondence is established by the Hilbert Nullstellensatz. This result admits two equivalent formulations. The first one, known as "Weak Nullstellensatz", states that I is a proper ideal of $\mathbb{C}[z_1, \dots, z_n]$ if and only if there exists a common zero of all regular polynomials in I. The second one, the "Strong Nullstellensatz", is more abstract and involves the notion of *radical* of an ideal. The equivalence of the two statements deeply relies on the fact that point-evaluation is a homomorphism (see, e.g., [8]). As already pointed out in Remark 2.3, this is not the case in the quaternionic setting. Using a proper strategy, Alon and Paran in [1] prove a Weak and a Strong Nullstellensatz for quaternionic polynomials with central variables. Thanks to the isomorphism φ introduced in (2.2) it is immediate to rephrase in our setting the Weak Nullstellensatz proven by Alon and Paran [1, Theorem 1.1].

Theorem 4.1 *(Weak Nullstellensatz for regular polynomials). Let I be a proper right ideal of $\mathbb{H}[q_1, \dots, q_n]$. Then there exists a point $(a_1, \dots, a_n) \in (\mathbb{C}_J)^n$, for some $J \in \mathbb{S}$, such that every regular polynomial in I vanishes at $(a_1, \dots, a_n)$.*

To discuss a Strong version of the Nullstellensatz for regular polynomials we need to introduce some additional notation.

Definition 4.2. Given a right ideal I in $\mathbb{H}[q_1, \dots, q_n]$, we define $\mathcal{V}(I)$ to be the set of common zeros of $P \in I$, i.e., if $Z_P \subset \mathbb{H}^n$ represents the zero set of a regular polynomial $P \in I$, then

$$\mathcal{V}(I) := \bigcap_{P \in I} Z_P.$$

Furthermore, we set

$$\mathcal{V}_c(I) := \mathcal{V}(I) \cap \bigcup_{J \in \mathbb{S}} (\mathbb{C}_J)^n.$$

Notice that $\mathcal{V}_c(I)$ is contained in $\mathcal{V}(I)$ and coincides with the set $Z(I)$ introduced in [1]. Recalling the notation introduced in Section 3, we have that a generic point $(a_1, \dots, a_n) \in \mathbb{H}^n$ belongs to $\mathcal{V}(I)$ if and only if $I \subseteq E_{(a_1, \dots, a_n)}$.

Let us now investigate some properties of the two sets $\mathcal{V}(I)$ and $\mathcal{V}_c(I)$ when I is a principal ideal.

Proposition 4.3. *Let $\langle P \rangle$ be the principal right ideal generated by $P \in \mathbb{H}[q_1, \dots, q_n]$. Then*

$$\mathcal{V}_c(\langle P \rangle) = Z_P \cap \bigcup_{J \in \mathbb{S}} (\mathbb{C}_J)^n.$$

Proof. By definition $\mathcal{V}_c(\langle P\rangle) \subseteq \bigcup_{J\in\mathbb{S}}(\mathbb{C}_J)^n$ and $\mathcal{V}_c(\langle P\rangle) \subseteq \mathcal{V}(\langle P\rangle) \subseteq Z_P$. On the other hand, recalling Remark 3.12, $Z_P \cap \bigcup_{J\in\mathbb{S}}(\mathbb{C}_J)^n \subseteq \mathcal{V}_c(\langle P\rangle)$ and hence they coincide. □

Theorem 4.1 guarantees that every regular polynomial has at least a zero with commuting components, i.e., $\mathcal{V}_c(\langle P\rangle) \neq \varnothing$ for any $P \in \mathbb{H}[q_1,\dots,q_n]$.
We give an example of a regular polynomial in $\mathbb{H}[q_1,q_2]$ with all the zeros with commuting components, that is such that $\mathcal{V}_c(\langle P\rangle) = \mathcal{V}(\langle P\rangle)$.

Example 4.4. The regular polynomial

$$P(q_1,q_2) = q_1q_2 - 1$$

is such that, if $J \neq K$, then $Z_P \cap (\mathbb{C}_J \times \mathbb{C}_K) = \varnothing$; indeed $Z_P = \{(q,q^{-1}) \ : \ q \in \mathbb{H}\setminus\{0\}\}$, which implies that if $P(a,b) = 0$ there exists $J \in \mathbb{S}$ such that $(a,b) \in \mathbb{C}_J \times \mathbb{C}_J$.

However, not all regular polynomials have all the zeros with commuting components, not even if they have real coefficients

Example 4.5. The regular polynomial $Q(q_1,q_2) = q_1^2 + q_2^2 + 2$ vanishes at (i,j) (and actually at any pair (J_1,J_2) with $J_1,J_2 \in \mathbb{S}$), which implies that $Z_Q \supsetneq \mathcal{V}_c(\langle Q\rangle)$. Moreover, since

$$Q(q_1,q_2) * q_1^n q_2^m a = q_1^n Q(q_1,q_2) q_2^m a$$

for any monomial $q_1^n q_2^m a$, any regular polynomial in $\langle Q\rangle$ vanishes on Z_Q and hence $\mathcal{V}(\langle Q\rangle) = Z_Q$.

This gives us an example in which $\mathcal{V}(I) \supsetneq \mathcal{V}_c(I)$.

Since the set of regular polynomials vanishing on a given subset Z of $\mathbb{H}^n$ is not in general an ideal (even if Z consists of a single point, as noted in Proposition 3.11), it becomes natural to associate two different ideals with Z.

Definition 4.6. Let Z be a non-empty subset of $\mathbb{H}^n$.
We denote by $\mathcal{J}(Z)$ the right ideal generated in $\mathbb{H}[q_1,\dots,q_n]$ by regular polynomials which vanish on Z,

$$\mathcal{J}(Z) := \left\{ \sum_{k=1}^{N} P_k * Q_k \ : \ P_k, Q_k \in \mathbb{H}[q_1,\dots,q_n] \text{ with } P_{k|_Z} \equiv 0 \right\}.$$

We denote by $\mathcal{I}(Z)$ the right ideal

$$\mathcal{I}(Z) := \bigcap_{(a_1,\dots,a_n)\in Z} \mathcal{I}_{(a_1,\dots,a_n)}.$$

Recalling Proposition 3.11, note that, in general, neither $\mathcal{J}(Z)$ nor $\mathcal{I}(Z)$ coincide with the set of regular polynomials vanishing on Z. Let us give an example in two variables.

Example 4.7. Consider the case $Z = \{(i, j)\}$, from Proposition 3.8 it follows that

$$\mathcal{I}(Z) = \mathcal{I}_{(i,j)} = \mathbb{H}[q_1, q_2] \neq E_{(i,j)}.$$

Moreover, since $q_1 - i, q_2 - j \in \mathcal{J}(Z)$, we get that $\mathcal{J}(Z) = I_{(i,j)}$ as well.

Again by Proposition 3.11, given a right ideal I in $\mathbb{H}[q_1, \dots, q_n]$, both the ideals $\mathcal{J}(\mathcal{V}_c(I))$ and $\mathcal{I}(\mathcal{V}_c(I))$ coincide with the set of regular polynomials vanishing on $\mathcal{V}_c(I)$. In particular this yields that we always have the equality

$$\mathcal{J}(\mathcal{V}_c(I)) = \mathcal{I}(\mathcal{V}_c(I)).$$

Observe that

$$\mathcal{I}(\mathcal{V}_c(I)) = \bigcap_{(a_1,\dots,a_n)\in\mathcal{V}_c(I)} \mathcal{I}_{(a_1,\dots,a_n)} = \bigcap_{(a_1,\dots,a_n)\in\mathcal{V}(I)} \mathcal{I}_{(a_1,\dots,a_n)} = \mathcal{I}(\mathcal{V}(I)). \tag{4.1}$$

Moreover, since $\mathcal{V}_c(I) \subseteq \mathcal{V}(I)$, it turns out that

$$\mathcal{J}(\mathcal{V}(I)) \subseteq \mathcal{J}(\mathcal{V}_c(I)) = \mathcal{I}(\mathcal{V}_c(I)) = \mathcal{I}(\mathcal{V}(I)). \tag{4.2}$$

To state a version of the Strong Nullstellensatz for regular polynomials, first we recall the following counterpart of prime ideals in the noncommutative setting (see [10]).

Definition 4.8. A right ideal I in $\mathbb{H}[q_1, \dots, q_n]$ is *completely prime* if for any $P, Q \in \mathbb{H}[q_1, \dots, q_n]$ such that $P * Q \in I$ and $P * I \subseteq I$ we have that $P \in I$ or $Q \in I$.

The notion of radical of an ideal introduced in [1] can be defined also in the setting of regular polynomials.

Definition 4.9. Let I be a right ideal in $\mathbb{H}[q_1, \dots, q_n]$. The *right radical* $\sqrt{I}$ of I is the intersection of all completely prime right ideals that contain I.

From the isomorphism φ introduced in (2.2) and equality (4.1), Theorem 4.6 in [1], directly leads to the following version of the Strong Nullstellensatz for regular polynomials

Theorem 4.10. *Let I be a right ideal in $\mathbb{H}[q_1, \dots, q_n]$. Then*

$$\mathcal{I}(\mathcal{V}(I)) = \sqrt{I}.$$

In the two-variable case, thanks to the better understanding of the correspondence between zeros of regular polynomials and ideals, we are able to give a more concrete version of the Strong Nullstellensatz, stated in terms of the ideal operator $\mathcal{J}$ instead of $\mathcal{I}$.

Definition 4.11. A subset $D \subseteq \mathbb{H}^2$ is called q_1*-symmetric* if for any $(a,b) \in D$ such that $ab \neq ba$, the set $\mathbb{S}_a \times \{b\}$ is contained in D.

Observe that if D contains only points with commuting components, then D is clearly q_1-symmetric.

Proposition 4.12. *Let $I \in \mathbb{H}[q_1, q_2]$ be a right ideal such that $\mathcal{V}(I)$ is q_1-symmetric. Then $\mathcal{J}(\mathcal{V}_c(I))$ coincides with the set of regular polynomials vanishing on $\mathcal{V}(I)$, which thus is a right ideal in $\mathbb{H}[q_1, q_2]$.*

Proof. If $\mathcal{V}(I) = \mathcal{V}_c(I)$ there is nothing to prove. Let $(a,b) \in \mathcal{V}(I) \setminus \mathcal{V}_c(I)$. Since $\mathcal{V}(I)$ is q_1-symmetric, the set $\mathcal{V}_c(I)$ contains two points (a_1, b) and $(\bar{a_1}, b)$ in $\mathbb{S}_a \times \{b\}$ with $a_1 b = b a_1$. Let $P \in \mathcal{J}(\mathcal{V}_c(I))$. Then P vanishes on $\mathcal{V}_c(I)$ and hence $P(a_1, b) = P(\bar{a_1}, b) = 0$. Since $P(\cdot, b)$ is a regular polynomial in q_1, thanks to Theorem 3.1 in [4], we get that $P(\cdot, b)$ vanishes on the entire sphere $\mathbb{S}_a$. Therefore $P(q_1, q_2)$ vanishes on $\mathbb{S}_a \times \{b\}$, thus on (a, b). Since (a,b) is a generic point in $\mathcal{V}(I)$, we have that P vanishes on $\mathcal{V}(I)$. □

Corollary 4.13. *Let $I \in \mathbb{H}[q_1, q_2]$ be a right ideal such that $\mathcal{V}(I)$ is q_1-symmetric. Then $\mathcal{J}(\mathcal{V}(I)) = \mathcal{J}(\mathcal{V}_c(I))$.*

The crucial geometric property of $\mathcal{V}(I)$ is shown in the following

Proposition 4.14. *Given a right ideal $I \subseteq \mathbb{H}[q_1, q_2]$, then $\mathcal{V}(I)$ is q_1-symmetric.*

Proof. Suppose there exists a point $(a,b) \in \mathcal{V}(I) \setminus \mathcal{V}_c(I)$. Since $I \subseteq E_{(a,b)}$ and $ab \neq ba$, thanks to Proposition 3.17, we get $I \subseteq E_{\mathbb{S}_a \times \{b\}}$ and thus $\mathbb{S}_a \times \{b\} \subseteq \mathcal{V}(I)$. □

The case $\mathcal{V}_c(I) = \mathcal{V}(I)$, for example, occurs when $I = \langle P \rangle$, with $P = q_1 q_2 - 1$. Since $Z_P \cap (\mathbb{C}_J \times \mathbb{C}_K) = \varnothing$ for imaginary units $J \neq K$, then every regular polynomial in $\langle P \rangle$ vanishes on Z_P. Hence $\mathcal{V}(\langle P \rangle) = Z_P = \mathcal{V}_c(\langle P \rangle)$.
Combining Proposition 4.12, Corollary 4.13 and Proposition 4.14, we obtain

Theorem 4.15. *Let I be an ideal in $\mathbb{H}[q_1, q_2]$. Then $\mathcal{J}(\mathcal{V}(I)) = \mathcal{J}(\mathcal{V}_c(I))$; thus $\mathcal{J}(\mathcal{V}(I)) = \{P \in \mathbb{H}[q_1, q_2] \ : \ P(q_1, q_2) = 0, \ (q_1, q_2) \in \mathcal{V}(I)\}$.*

Recalling Equation (4.2) we then have the following

Theorem 4.16 *(Strong Nullstellensatz in $\mathbb{H}^2$). Let I be a right ideal in $\mathbb{H}[q_1, q_2]$. Then*

$$\mathcal{J}(\mathcal{V}(I)) = \sqrt{I}.$$

This formulation of the Strong Nullstellensatz has a relevant geometric interpretation since, combining Proposition 4.12 with Theorem 4.15, we obtain that $\sqrt{I}$ coincides with the ideal of regular polynomials vanishing on $\mathcal{V}(I)$.

There are several examples suggesting that the equality $\mathcal{J}(\mathcal{V}(I)) = \mathcal{J}(\mathcal{V}_c(I))$, holds also in $\mathbb{H}[q_1, \dots, q_n]$, with $n > 2$. This has been the key ingredient to give a more geometric interpretation of the Strong version of the Nullstellensatz for regular polynomials in two quaternionic variables and is expected to be crucial also for regular polynomials in several quaternionic variables.

Example 4.17. Here we list some examples in which $\mathcal{J}(\mathcal{V}(I)) = \mathcal{J}(\mathcal{V}_c(I))$.

(1) Any ideal I in $\mathbb{H}[q_1, \dots q_n]$ such that $\mathcal{V}_c(I) = \mathcal{V}(I)$.
(2) $I = \langle q_1 - a_1 \rangle$; indeed, $P \in \langle q_1 - a_1 \rangle$ if and only if P vanishes on $\{a_1\} \times (C_{a_1})^{n-1} = \mathcal{V}_c(\langle q_1 - a_1 \rangle)$. Hence $\langle q_1 - a_1 \rangle = \mathcal{J}(\mathcal{V}_c(\langle q_1 - a_1 \rangle)) \supseteq \mathcal{J}(\mathcal{V}(\langle q_1 - a_1 \rangle)) \supseteq \langle q_1 - a_1 \rangle$.
(3) $I = \mathcal{I}_{(a_1, \dots, a_n)}$:
if $a_l a_m \neq a_m a_l$ for some $l, m \in \{1, \dots, n\}$, then $\mathcal{I}_{(a_1, \dots, a_n)} = \mathbb{H}[q_1, \dots, q_n]$ so $\mathcal{V}(\mathcal{I}_{(a_1, \dots, a_n)}) = \mathcal{V}_c(\mathcal{I}_{(a_1, \dots, a_n)}) = \varnothing$.
If $a_l a_m = a_m a_l$ for any $l, m = 1, \dots, n$, then $P \in \mathcal{I}_{(a_1, \dots, a_n)}$ if and only if P vanishes at $(a_1, \dots, a_n)$. Therefore
$\mathcal{I}_{(a_1, \dots, a_n)} \supseteq \mathcal{J}(\mathcal{V}_c(\mathcal{I}_{(a_1, \dots, a_n)})) \supseteq \mathcal{J}(\mathcal{V}(\mathcal{I}_{(a_1, \dots, a_n)})) \supseteq \mathcal{I}_{(a_1, \dots, a_n)}$.

The authors have in mind to investigate the general statement which might be behind these examples in a forthcoming paper.

Declaration of competing interest

The authors have no competing interests to declare that are relevant to the content of this article.

Data availability

No data was used for the research described in the article.

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