Minimal lengths in 3D via the generalized uncertainty principle

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A R T I C L E   I N F O

A B S T R A C T

We investigate an extension of the Generalized Uncertainty Principle (GUP) in three dimensions by modifying the three dimensional position and momentum operators in a manner that remains coordinate-independent and retains as much of the standard position-momentum commutators as possible. Moreover, we bound the physical momentum which leads to an effective minimal length in every coordinate direction. The physical consequences of these modified operators are explored in two scenarios: (i) when a spherically-symmetric wave function is 'compressed' into the smallest possible volume; (ii) when the momentum is directed in a single direction. In case (ii), we find that the three dimensional GUP exhibits interesting phenomena that do not occur in one dimension: the minimal distance in the direction parallel to a particle's momentum is different from the minimal distance in the orthogonal directions.

1. Introduction

The Generalized Uncertainty Principle (GUP) and its associated minimal length scale, or maximum energy-momentum scale, are considered fruitful phenomenological approaches to quantum gravity [1–9]. The GUP approach to quantum gravity has led to testable predictions and observations, *e.g.* explanations for photon dispersion in vacuum [10] and in various table-top laboratory experiments [11,12]. While many works mention the three dimensional generalization of the position-momentum commutators and the associated three dimensional uncertainty relationships, they generally restrict their GUPs to one spatial dimension. There are a few exceptions including [7] and more recently the work in [13], which looked at a fully relativistic GUP in four space-time dimensions.

In this work, we want to explore the GUP in three spatial dimensions and note some new features which are present in three spatial dimensions, but are absent in a single spatial dimension. This is similar to the theory of special relativity where boosts were introduced in only one spatial dimension and later expanded to three dimensions. Looking at special relativity in three spatial dimensions leads not only to a more complex form for the boosts, but also gives rise to the mixing

of boost transformations, rotations, and related effects such as Thomson precession. In a similar way to Lorentz boosts, we find that certain variations of three dimensional GUPs lead to an interplay between the different coordinate directions, resulting in different minimal distances in the directions orthogonal to the direction of momentum.

It is interesting to notice that a similar behavior has already been observed in the context of string inspired noncommutative geometry [14–16]. Noncommutative relations determine a foliation of the space-time in noncommutative planes. If, on the one hand, the foliation is an invariant character, on the other hand, on each of such planes the minimal length transforms like a antisymmetric second rank tensor, *i.e.*, the Kalb-Ramond field – see *e.g.* [17].

2. Minimal lengths in one dimension

In this section we will review the general construction of a one dimensional GUP, and introduce a specific form of the GUP that we will focus on in this work. The specific GUP studied here will modify the position and momentum operators, while *trying* to retain the standard commutator between conventional position and momentum.

Most works on GUP start with a modified commutator of the form

$$[\hat{X}, \hat{P}] = i\hbar F\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right),\qquad(1)$$

where $F\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right)$ is a function of the canonical position and momentum operators which satisfy $[\hat{x}, \hat{p}] = i\hbar$; and the capitalization on $\hat{X}$ and $\hat{P}$ is used to denote the modified operators. The constants x_m (p_M) set the small distance (large momentum) cut-off scale. When $\Delta x \gg x_m$ and/or $\Delta p \ll p_M$, the ansatz function has the limit $\langle F\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right)\rangle \to 1$, and the modified commutator becomes the standard one for large distances and/or small momenta. A common example of such a modified commutator is one from [7] where $F\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right) = 1 + \beta \hat{p}^2 = 1 + \frac{\hat{p}^2}{p_M^2}$, with $\beta \equiv p_M^{-2}$.

With a modified commutator defined, it is necessary to define corresponding modified quantum operators that lead to that modified commutator. An acceptable modification would be to modify the operators in momentum space to have the form

$$\hat{X} = i\hbar G\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right)\partial_p \quad \text{and} \quad \hat{P} = H\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right)\hat{p},\qquad(2)$$

and choose G and H so as to yield (1).

A concrete example of the above comes from reference [7], where the various ansatz functions are $F\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right) = G\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right) = 1 + \beta \hat{p}^2$, and $H\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right) = 1$. This leads to the following modified operators: $\hat{X} = i\hbar(1 + \beta p^2)\partial_p$ and $\hat{P} = \hat{p} = p$.[1] The position operator is modified while the momentum operator is not.

One subtle issue is that one can obtain the same $F\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right)$ with different $G\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right)$ and $H\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right)$. For example, the modified operators - $\hat{X} = \hat{x} = i\hbar\partial_p$ and $\hat{P} = p\left(1 + \frac{1}{3}\beta p^2\right)$ from [18], give the same right hand side of the modified commutator in (1) as [7]. However, the momentum uncertainty will be different in these two cases. It will be Δp, with unmodified momentum, for the case from [7], while for the case from [18] it will be ΔP, with modified momentum. Thus the way the generalized uncertainty principle plays out will be different for [7] versus [18]. There are still other examples where the momentum operator is modified in a way that leads to the modified commutator being in terms of the modified momentum operator. Reference [19] proposed the modified operators $\hat{X} = \hat{x} = i\hbar\partial_p$ and $\hat{P} = \frac{1}{\sqrt{\beta}}\tan(\sqrt{\beta}p)$. This led to $F\left(\frac{\hat{x}}{x_m}, \frac{\hat{p}}{p_M}\right) = 1 + \beta\hat{P}^2$ i.e. the modified commutator is in terms of the modified momentum.

To summarize the previous paragraph – for general operators $\hat{A}$ and $\hat{B}$ the uncertainty relationship is $\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A}, \hat{B}]\rangle|$. Applying this to the GUPs from [7] versus [18] we see that the right hand side of (1) is $1 + \beta\hat{p}^2$ with the standard momentum. However, the uncertainty in the momentum is different – for the GUP from [7] it is Δp (i.e. in terms of the standard momentum) while for the GUP from [18] it is ΔP (i.e. in terms of the modified momentum).

In this work we follow reference [20] and take the ansatz functions G and H in (2) such that $\hat{X}$ and $\hat{P}$ satisfied the standard commutator relationship

$$[\hat{X}, \hat{P}] = i\hbar.\qquad(3)$$

H was chosen such that when $p \ll p_M$, $\hat{P} = p\,H\left(\frac{p}{p_M}\right) \approx p$; and such that when $p \gg p_M$, $\hat{P} = p\,H\left(\frac{p}{p_M}\right) \approx p_M$, to cap momentum at p_M.

For the ansatz function $G\left(\frac{p}{p_M}\right)$, we require $G\left(\frac{p}{p_M}\right) \to 1$ for $\Delta p \ll p_M$. The limit of $G\left(\frac{p}{p_M}\right)$ for $p \gg p_M$ is fixed below by requiring that

the modified position and momentum operators still satisfy the standard commutator (3). These choices for the ansatz functions still give a minimum in position because of the following:

(i) The general form of the ansatz function H gives $\hat{P} \leq p_M$.
(ii) Since $G\left(\frac{p}{p_M}\right)$ and $H\left(\frac{p}{p_M}\right)$ are fixed to maintain $[\hat{X}, \hat{P}] = i\hbar$, this yields the standard uncertainty principle $\Delta X \Delta P \geq \frac{\hbar}{2}$.
(iii) Finally, this leads to a minimum uncertainty in position, $\Delta X \geq \frac{\hbar}{2p_M}$.

Now, one obtains a condition on G and H,

$$G\left(\frac{p}{p_M}\right)\partial_p\left[H\left(\frac{p}{p_M}\right)p\right] = 1.\qquad(4)$$

To fix the functions $G\left(\frac{p}{p_M}\right)$ and $H\left(\frac{p}{p_M}\right)$, using the above constraints, we follow reference [20], and start by determining the momentum function $H\left(\frac{p}{p_M}\right)$. Once the momentum ansatz function is fixed the condition (4) fixes the position ansatz function $G\left(\frac{p}{p_M}\right)$. For the modified momentum function we want something for which $H(p) \sim const.$ for $p \ll p_M$ and $H(p) \sim \frac{1}{p}$ for $p \gg p_M$. Recall, that the modified momentum is $\hat{P} \sim pH(p)$, so for the modified momentum this gives $\hat{P} \sim p$ for $p \ll p_M$ and $\hat{P} \sim const.$ for $p \gg p_M$. This behavior for the modified momentum can be obtained by choosing $H(p) \sim \frac{1}{p}\tanh(p)$ or $H(p) \sim \frac{1}{p}\arctan(p)$. These conditions can be satisfied by the following, non-unique two sets of ansatz functions

$$G\left(\frac{p}{p_M}\right) = \cosh^2\left(\frac{p}{p_M}\right) \quad \text{and} \quad H\left(\frac{p}{p_M}\right) = \frac{p_M}{p}\tanh\left(\frac{p}{p_M}\right),\qquad(5)$$

and

$$G\left(\frac{p}{p_M}\right) = \left[1 + \left(\frac{\pi p}{2p_M}\right)^2\right] \quad \text{and} \quad H\left(\frac{p}{p_M}\right) = \frac{2p_M}{\pi p}\arctan\left(\frac{\pi p}{2p_M}\right).\qquad(6)$$

The functions $G\left(\frac{p}{p_M}\right)$ in (5) and (6) were fixed via the condition in (4).

In both cases, the modified operators, $\hat{X}$ and $\hat{P}$, formulated by using the G's and H's in (2) satisfy the standard commutator $[\hat{X}, \hat{P}] = i\hbar$, and the related standard uncertainty $\Delta X \Delta P \geq \frac{\hbar}{2}$. One might think recovering the standard commutator (3) would not lead to a minimum length, but due to the modified behavior of $\hat{P}$, and therefore ΔP, one still finds a non-zero minimum distance. In conclusion, the crux of the question is if operators are capped or not. Physical operators must display ultraviolet convergence/finiteness. Uncapped operators have only a mathematical meaning, although can be used as auxiliary variables in calculations. The algebra of operators is per se irrelevant to discriminate the short distance nature of observables.

3. Generalizing to three dimensions

In this section, we generalize the above construction from one spatial dimension to three spatial dimensions. The modified operators must now become three-vector operators – $\hat{X} \to \hat{X}_i$ and $\hat{P} \to \hat{P}_i$, where $i = 1, 2, 3$. This can be done by applying different conditions to the ansatz functions, G and H as we now discuss in detail.

For the hyperbolic ansatz functions of (5), one could try letting $H\left(\frac{p}{p_M}\right) = \frac{p_M}{p_i}\tanh\left(\frac{p_i}{p_M}\right)$, which gives a modified momentum of the form $\hat{P}_i = p_i H\left(\frac{p}{p_M}\right) = p_M \tanh\left(\frac{p_i}{p_M}\right)$. This would have a low momentum limit of $\hat{P}_i \approx p_i$, and a high momentum limit of p_M in the direction of p_i. The vector character of the modified position operator arises through $\partial_p \to \partial_{p_i}$, and the ansatz function becomes $G\left(\frac{p_i}{p_M}\right) = \cosh^2\left(\frac{p_i}{p_M}\right)$. In order to satisfy (4), the $\cosh^2$ term for the modified position operator has to have the same functional dependence as the $\tanh$

term in the modified momentum operator. Note that condition (4) becomes $G\left(\frac{p_i}{p_M}\right)\partial_{p_i}\left[p_j H\left(\frac{p_j}{p_M}\right)\right] = \delta_{ij}$ in this three dimensional case.

This approach has a fundamental problem, though. G and H are functions of p_i, which spoils the rotational symmetry of the modified position and momentum operators, as discussed in section 6.2 of [7].

In order to preserve rotational symmetry and still retain the feature that the momentum operators have upper bounds of the form given in section 2, we need ansatz functions of the form

$$G\left(\frac{|\vec{p}|}{p_M}\right) = \cosh^2\left(\frac{|\vec{p}|}{p_M}\right) \quad \text{and} \quad H\left(\frac{|\vec{p}|}{p_M}\right) = \frac{p_M}{|\vec{p}|}\tanh\left(\frac{|\vec{p}|}{p_M}\right), \tag{7}$$

or for the arctan modification

$$G\left(\frac{|\vec{p}|}{p_M}\right) = \left[1 + \left(\frac{\pi|\vec{p}|}{2p_M}\right)^2\right] \quad \text{and} \quad H\left(\frac{|\vec{p}|}{p_M}\right) = \frac{2p_M}{\pi|\vec{p}|}\arctan\left(\frac{\pi|\vec{p}|}{2p_M}\right). \tag{8}$$

These ansatz functions in (7) and (8) lead to modified three dimensional position and momentum operators of the form

$$\hat{X}_i = i\hbar\cosh^2\left(\frac{|\vec{p}|}{p_M}\right)\partial_{p_i} \quad \text{and} \quad \hat{P}_i = p_i H\left(\frac{|\vec{p}|}{p_M}\right) = p_i\frac{p_M}{|\vec{p}|}\tanh\left(\frac{|\vec{p}|}{p_M}\right), \tag{9}$$

or for the arctan modification

$$\hat{X}_i = i\hbar\left[1 + \left(\frac{\pi|\vec{p}|}{2p_M}\right)^2\right]\partial_{p_i} \quad \text{and}$$
$$\hat{P}_i = p_i H\left(\frac{|\vec{p}|}{p_M}\right) = p_i\frac{2p_M}{\pi|\vec{p}|}\arctan\left(\frac{\pi|\vec{p}|}{2p_M}\right). \tag{10}$$

The modified position and momentum operators in (9) and (10) are the three dimensional generalizations of (5) and (6), which satisfy rotational symmetry and still have capped momentum.

We will now show that the position and momentum operators' dependence on $|\vec{p}|$ causes the standard commutator (i.e., $[\hat{X}_i, \hat{P}_j] = i\hbar\delta_{ij}$) to no longer necessarily be maintained in all directions. The standard commutator can be maintained in one direction but at the cost of altering the commutator in the two orthogonal directions, resulting in there being different minimal distances in different directions. In one dimension, the condition for the commutator to retain its standard form was given by (4). To generalize (4) to three dimensions one would like to require

$$G\left(\frac{|\vec{p}|}{p_M}\right)\partial_{p_i}\left[p_j H\left(\frac{|\vec{p}|}{p_M}\right)\right] \overset{?}{=} \delta_{ij}, \tag{11}$$

where the question mark indicates we are asking if it is possible for the left hand side to equal the right hand side.

The complication that arises with (11), that is not found in (4), is that the momentum derivative operates on all the components of momentum "hidden" in the function in square brackets due to the $|\vec{p}|$ dependence of H. This leads to p_i and p_j terms that are not found in the one dimensional case.

For instance, for the tanh modification of the momentum, condition (11) becomes

$$G\left(\frac{|\vec{p}|}{p_M}\right)\partial_{p_i}\left[p_j H\left(\frac{|\vec{p}|}{p_M}\right)\right]$$
$$= \left[\frac{p_M}{|\vec{p}|}\sinh\left(\frac{|\vec{p}|}{p_M}\right)\cosh\left(\frac{|\vec{p}|}{p_M}\right)\left(\delta_{ij} - \frac{p_j p_i}{|\vec{p}|^2}\right) + \frac{p_j p_i}{|\vec{p}|^2}\right]$$
$$= \delta_{ij} + \mathcal{O}\left(\frac{|\vec{p}|^2}{p_M^2}\right), \tag{12}$$

and the arctan modification of the momentum, condition (11) becomes

$$G\left(\frac{|\vec{p}|}{p_M}\right)\partial_{p_i}\left[p_j H\left(\frac{|\vec{p}|}{p_M}\right)\right]$$

$$= \left[\left(1 + \left(\frac{\pi|\vec{p}|}{2p_M}\right)^2\right)\frac{2p_M}{\pi|\vec{p}|}\arctan\left(\frac{\pi|\vec{p}|}{2p_M}\right)\left(\delta_{ij} - \frac{p_j p_i}{|\vec{p}|^2}\right) + \frac{p_j p_i}{|\vec{p}|^2}\right]$$
$$= \delta_{ij} + \mathcal{O}\left(\frac{|\vec{p}|^2}{p_M^2}\right). \tag{13}$$

In the second lines of (12) and (13), we expanded the functions in square brackets on the right hand side to first order in $|\vec{p}|/p_M$. To first order, one can recover the standard commutators in three dimensions. However, to second order, there is a difference with the one dimensional case, and the expressions in (12) and (13) are no longer equal to solely δ_{ij}.

Using the modified position and momentum from (9) and (10), and the results of (12) and (13), we find that the modified commutator for the tanh modification, to second order, is

$$[\hat{X}_i, \hat{P}_j] \approx i\hbar\left[\delta_{ij} + \frac{|\vec{p}|^2}{2p_M^2}\left(\delta_{ij} - \frac{p_j p_i}{|\vec{p}|^2}\right)\right]. \tag{14}$$

For the arctan modification there is a bit if a subtlety. From (13), the term we want to expand is of the form $(1 + x^2)\frac{\arctan(x)}{x}$, with $x \equiv \frac{\pi|\vec{p}|}{2p_M}$. Due to the x in the denominator of this expression, one needs to expand arctan to $\mathcal{O}(x^3)$ namely $\arctan(x) \approx x - \frac{x^3}{3}$. In this way (13) to $\mathcal{O}(x^2)$ is

$$[\hat{X}_i, \hat{P}_j] \approx i\hbar\left[\delta_{ij} + \frac{2}{3}\left(\frac{\pi|\vec{p}|}{2p_M}\right)^2\left(\delta_{ij} - \frac{p_j p_i}{|\vec{p}|^2}\right)\right]. \tag{15}$$

The second order approximations in (14) and (15), are similar to the modified commutators in [7] that the modifications are quadratic in $|\vec{p}|$. However, unlike the one dimensional case, one can no longer recover the standard commutator (3) with the modified operators from (9) and (10).

4. Physical consequences of three dimensional GUP

We now want to investigate the physical consequences of the modified commutators of the previous section on the resulting uncertainty principle. The left hand side of the uncertainty principle will be the product of the variation of the modified position and momentum operators, $\Delta X_i \Delta P_j$, where for a general operator $\hat{A}$ the variation is $\Delta A = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$. The right hand side of the uncertainty relationship involves taking the expectation of the right hand side of (12) or (13). However, the presence of the hyperbolic functions in (12) or the inverse tangent function in (13) makes it hard to take the expectation values and get an analytical form. In order to more easily calculate the expectation values, we will instead use the second order commutators given in (14) and (15). Even after this, we will need to make another approximation to obtain the final form of the uncertainty principle.

We will look at two different cases for evaluating the physical consequences of the 3D GUP: (a) We assume a spherically symmetric wave function so that the average of the modified momentum and position operators is zero so that $\Delta X_i = \sqrt{\langle \hat{X}_i^2 \rangle}$, and $\Delta P_j = \sqrt{\langle \hat{P}_j^2 \rangle}$. (b) We assume that the system has a non-zero, average momentum in one direction (e.g. the $i = 1$ direction) so that $\langle \hat{P}_i \rangle \neq 0$.

For both cases, the minimum distance will be proportional to the inverse of the cut-off scale $\Delta X^{min} \propto \frac{1}{p_M}$, up to numerical factors of $\mathcal{O}(1)$. For the second case, with an average momentum in one direction, we find that the minimum position uncertainty in the direction of the momentum is smaller than in the two perpendicular directions.

4.1. Spherically symmetric case

In this subsection we study the case where the wave function is spherically symmetric in momentum space. We use this to explore how focused/concentrated a stationary particle can get in position space in these theories. We utilize the spherical symmetry to make the substitution $\langle p_j p_i \rangle = \delta_{ij}\langle p_i^2 \rangle$.

Starting with the hyperbolic case (14), the modified uncertainty relationship is

$$\Delta X_i \Delta P_j \geq \frac{\hbar}{2}\left(\delta_{ij} + \frac{\delta_{ij}}{2p_M^2}\langle |\vec{p}|^2 - p_i^2\rangle\right) = \frac{\hbar}{2}\delta_{ij}\left(1 + \frac{\langle p_\perp^2\rangle}{2p_M^2}\right), \tag{16}$$

where $\langle p_\perp^2\rangle = \langle |\vec{p}|^2 - p_i^2\rangle$ is the expectation of the square of the perpendicular part of the momentum. We can use the symmetry to write $\langle p_\perp\rangle^2 = (d-1)\langle p_i^2\rangle$, where d is the number of spatial dimensions. In this case, $d = 3$ and $|p_\perp|^2 = 2|p_i|^2$ so (16) becomes

$$\Delta X_i \Delta P_j \geq \frac{\hbar}{2}\delta_{ij}\left(1 + \frac{\langle |p_i|^2\rangle}{p_M^2}\right). \tag{17}$$

Next, we need to estimate ΔP_j,

$$\Delta P_j = \sqrt{\langle (\hat{P}_j)^2\rangle - \langle \hat{P}_j\rangle^2} \leq \sqrt{\langle (\hat{P}_j)^2\rangle} = \sqrt{\left\langle \frac{\hat{p}_j^2 p_M^2}{|\vec{p}|^2}\tanh^2\left(\frac{|\vec{p}|}{p_M}\right)\right\rangle} \tag{18}$$

where we take $\langle \hat{P}_j\rangle = 0$; and bounding $\tanh^2(x) \leq x^2$, equation (18) yields

$$\Delta P_j \leq \sqrt{\left\langle \hat{p}_j^2\right\rangle} = \Delta p_j, \tag{19}$$

where the uncertainty of the modified momentum is bound by the uncertainty of the canonical momentum. Combining (17) and (19) gives a ΔX_i of

$$\Delta X_i \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p_i} + \frac{\Delta p_i}{p_M^2}\right). \tag{20}$$

The minimum of (20) occurs at $\Delta p_i = p_M$ which then gives

$$\Delta X_i^{min} = \frac{\hbar}{p_M}. \tag{21}$$

The structure for this 3D GUP given by (17) (20) and (21) is very similar to, but not exactly identical to, the original KMM model [7].

One could also do a rougher, but simpler, approximation and get close to the same result as (21): instead of (17), simply take $\Delta X_i \Delta P_j \geq \frac{\hbar}{2}\delta_{ij}$; and instead of (19) one could take $\Delta P_j \leq p_M$. This yields $\Delta X_i^{min} = \frac{\hbar}{2p_M}$ which is of the same order as (21). There is not a big difference between this rough estimate for ΔX_i^{min} and the more refined one in (21).

We now repeat this analysis for the arctan modification given by (13). At first glance, it seems this is straightforward, but there is a subtle issue as compared to the previous tanh case. For the tanh modification, in going from (12) to the approximation in (14), we note that the right hand side of (14) is a lower bound to (12), thus in (16) we are justified in using $\geq$. However since the expansion for arctan has alternating signs, the approximation in (15) is not always a lower bound on (13). This makes getting the uncertainty relationship coming from (13) more complicated.

One could make a similar approximation to the rough approximations mentioned after (21). Note that the $(1+x^2)\frac{\arctan(x)}{x}$ term from (13) (here $x = \frac{\pi|\vec{p}|}{2p_M}$) has a lower bound of 1, i.e. $(1+x^2)\frac{\arctan(x)}{x} \geq 1$. Replacing this term in (13) by 1 gives $[X_i, P_j] = i\hbar\delta_{ij}$, which then gives $\Delta X_i \Delta P_j \geq \delta_{ij}\frac{\hbar}{2}$. As was for the tanh modification, the modified momentum uncertainty for the arctan modification has $\Delta P_j \leq p_M$, which then gives the same minimum as for the tanh modification. Namely, $\Delta X_i^{min} = \frac{\hbar}{2p_M}$. The fact that this rough estimate for ΔX_i^{min} is the same for both tanh and arctan modification is not surprising. To this level of approximation, the right hand side of the commutator is the standard one, $i\hbar\delta_{ij}$. ΔX_i^{min} ultimately comes from the cut-off in the momentum uncertainty, which is the same in both cases, namely $\Delta P_j \leq p_M$.

To get a better estimate for ΔX_i^{min}, we need a finer lower bound on $(1+x^2)\frac{\arctan(x)}{x}$. This is given by the function $(1+x^2)^{1/2}$, i.e. $(1 +$

$x^2)\frac{\arctan(x)}{x} \geq (1+x^2)^{1/2}$. The fact that $(1+x^2)^{1/2}$ provides a lower bound to $(1+x^2)\frac{\arctan(x)}{x}$ is not obvious (there are other possible functions that may provide a tighter lower bound), but can be verified by expanding both functions, or by graphing them.

In any case, replacing $(1+x^2)\frac{\arctan(x)}{x}$ by $(1+x^2)^{1/2}$ in (13) and expanding to second order in $x = \frac{\pi|\vec{p}|}{2p_M}$ gives

$$[\hat{X}_i, \hat{P}_j] \approx i\hbar\left[\delta_{ij} + \frac{1}{2}\left(\frac{\pi|\vec{p}|}{2p_M}\right)^2\left(\delta_{ij} - \frac{p_j p_i}{|\vec{p}|^2}\right)\right]. \tag{22}$$

The commutator in (22) yields the following uncertainty relationship

$$\Delta X_i \Delta P_j \geq \frac{\hbar}{2}\left(\delta_{ij} + \frac{\delta_{ij}\pi^2}{8p_M^2}\langle |\vec{p}|^2 - p_i^2\rangle\right) = \frac{\hbar}{2}\delta_{ij}\left(1 + \frac{\pi^2\langle p_\perp^2\rangle}{8p_M^2}\right). \tag{23}$$

In going from the (22) to (23) we have again used spherical symmetry to write the expectation of the last term in (22) as $\frac{\pi^2\langle p_j p_i\rangle}{8p_M^2} = \delta_{ij}\frac{\pi^2\langle p_i^2\rangle}{8p_M^2}$, such that the expectation is zero unless $i = j$; and as before, we define $\langle p_\perp^2\rangle = \langle |\vec{p}|^2 - p_i^2\rangle$. From (23) we now repeat all the steps going from (17) to (19) to obtain the following expressions for the arctan modification

$$\Delta X_i \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p_i} + \frac{\pi^2\Delta p_i}{8p_M^2}\right), \tag{24}$$

The minimum of (20) occurs at $\Delta p_i = p_M$ which then gives

$$\Delta X_i^{min} = \frac{\pi\hbar}{2\sqrt{2}p_M}. \tag{25}$$

Equations (24) and (25) are the arctan modification analogs of (20) and (21), only they differ by factors of order 1. Notice, there is not that big a difference between the minima from (21) and (25), and the rougher estimate obtained when the minimum in position uncertainty which came entirely from ΔP_i being bounded by p_M, where $\Delta X_i^{min} = \frac{\hbar}{2p_M}$.

4.2. Case with momentum in one direction

We now look at the physical consequences on the modified uncertainty relationship when the system has some overall average momentum in one direction, e.g. $\vec{p} = (p_1, 0, 0)$. For this choice in momentum, the right hand side of (12) and (13) simply becomes 1 when $i = j = 1$, which gives the standard commutator $[X_1, P_1] = i\hbar$ for both cases. In the 1 direction, the uncertainty relationship becomes $\Delta X_1 \Delta P_1 \geq \frac{\hbar}{2}$, which is just the standard uncertainty but constructed with modified operators.

For both tanh and arctan modifications, one has $\Delta P_1 \leq p_M$; the same position uncertainty minimum in the 1 direction of $\Delta X_1^{min} \geq \frac{\hbar}{2p_M}$. Note that this minimum in the 1-direction is the same for both types of modified operators when the right hand side is bounded by the standard commutator.

We now turn to the uncertainty in the two orthogonal directions 2 and 3. For the modified operators $\hat{X}_j$ and $\hat{P}_j$ with $j = 2$ or 3 for the tanh modification, the commutator becomes

$$[\hat{X}_j, \hat{P}_j] = i\hbar\left[\frac{p_M}{|\vec{p}|}\sinh\left(\frac{|\vec{p}|}{p_M}\right)\cosh\left(\frac{|\vec{p}|}{p_M}\right)\right]$$
$$= i\hbar\left[\cosh^2\left(\frac{|\vec{p}|}{p_M}\right)\frac{p_M}{|\vec{p}|}\tanh\left(\frac{|\vec{p}|}{p_M}\right)\right]. \tag{26}$$

The associated uncertainty is then

$$\Delta X_j \Delta P_j \geq \frac{\hbar}{2}\left\langle \frac{p_M}{|\vec{p}|}\sinh\left(\frac{|\vec{p}|}{p_M}\right)\cosh\left(\frac{|\vec{p}|}{p_M}\right)\right\rangle \geq \frac{\hbar}{2}\left(1 + \frac{\langle |\vec{p}|^2\rangle}{2p_M^2}\right), \tag{27}$$

where in the last step we expanded the hyperbolic functions to second order. For the 1-direction modified momentum uncertainty, we take the rough estimate that the momentum uncertainties in the 2,3-directions

are bounded by p_M, *i.e.* $\Delta P_j \leq p_M$. Again, we assume $\langle p_j \rangle = 0$ and $\langle P_j \rangle = 0$. With this, the modified position uncertainty minimum is

$$\Delta X_j^{min} = \frac{\hbar}{2 p_M}\left(1 + \frac{\langle |\vec{p}|^2 \rangle}{2 p_M^2}\right) \geq \frac{\hbar}{2 p_M}\left(1 + \frac{p_1^2}{2 p_M^2}\right). \qquad (28)$$

In the last expression in (28), we used $\langle |\vec{p}|^2 \rangle \geq p_1^2$ since we assume that the momentum is all in the 1-direction. This leads to the conclusion that the minimum uncertainty for the modified position in the 1-direction is smaller than in the orthogonal, 2, 3-directions; such that $\Delta X_{2,3}^{min} > \Delta X_1^{min}$.

We now repeat the above calculations for the arctan modification. In the 1-direction one has as before $\Delta X_1^{min} = \frac{\hbar}{2 p_M}$, by taking the rough bound $\Delta P_1 \leq p_M$. Now for the two orthogonal directions, $j = 2, 3$ the commutator becomes

$$[\hat{X}_j, \hat{P}_j] = i\hbar \left[\left(1 + \left(\frac{\pi |\vec{p}|}{2 p_M}\right)^2\right)\frac{2 p_M}{\pi |\vec{p}|} \arctan\left(\frac{\pi |\vec{p}|}{2 p_M}\right)\right]. \qquad (29)$$

This commutator in (29) now gives the modified uncertainty relationship for $j = 2, 3$

$$\Delta X_j \Delta P_j \geq \frac{\hbar}{2}\left\langle \left(1 + \left(\frac{\pi |\vec{p}|}{2 p_M}\right)^2\right)\left(\frac{2 p_M}{\pi |\vec{p}|}\right)\arctan\left(\frac{\pi |\vec{p}|}{2 p_M}\right)\right\rangle$$
$$\geq \frac{\hbar}{2}\left\langle \sqrt{\left(1 + \left(\frac{\pi |\vec{p}|}{2 p_M}\right)^2\right)}\right\rangle \geq \frac{\hbar}{2}\left(1 + \frac{\pi^2 \langle |\vec{p}|^2 \rangle}{8 p_M}\right). \qquad (30)$$

In going from the first line to the second line in (30), we have employed the previously used non-obvious lower bound $(1 + x^2)\frac{\arctan(x)}{x} \geq \sqrt{1 + x^2}$, and taken a Taylor expansion $\sqrt{1 + x^2} \approx 1 + \frac{1}{2}x^2$. If we again take the rough bound of $\Delta P_j \leq p_M$ for the modified momentum in the two orthogonal directions we obtain a minimum in the modified position uncertainty of

$$\Delta X_j^{min} = \frac{\hbar}{2 p_M}\left(1 + \frac{\pi^2 \langle |\vec{p}|^2 \rangle}{8 p_M}\right) \geq \frac{\hbar}{2 p_M}\left(1 + \frac{\pi^2 p_1^2}{8 p_M}\right). \qquad (31)$$

Again, we have used $\langle |\vec{p}|^2 \rangle \geq p_1^2$ in arriving at the last expression in (31).

As for the tanh modification, the term in parentheses in (31) is greater than 1. So, as before, the uncertainty in the orthogonal 2, 3-directions is larger than in the 1-direction - the direction of the momentum of the system. This can be compared to when the electric field of a boosted, charged particle is stronger in the directions orthogonal to the momentum of the particle.

5. Summary and conclusions

In this paper we formulated 3D GUP models based on earlier 1D GUP models studied in [18,20,21]. Our aim in 1D was to find a GUP model which gave a minimal length which only modified the position and momentum operators, while leaving the commutator of these modified operators unmodified. In section 2, we summarized the 1D GUP models. In section 3, we extended these 1D models to 3D GUP models so that they were rotationally invariant (coordinate-independent) by requiring the modifications to be functions of the total momentum $|\vec{p}|$. Consequently, the 3D commutator must be modified in at least one direction. This can be seen explicitly in (14) and (15) via the second order corrections to the position-momentum commutators which had a form $\propto \delta_{ij} - \frac{p_j p_i}{|\vec{p}|^2}$. In 1D GUP models, the term $\delta_{ij} - \frac{p_j p_i}{|\vec{p}|^2}$ is always zero. We then investigated the uncertainty relationships produced by the modified 3D operators and modified commutators. In general, we found that the minimum uncertainty in position was approximately $\Delta X_i^{min} \sim \frac{\hbar}{p_M}$ - essentially the same result as in 1D.

We also considered the minimum distance in two cases. In the first case, we used spherically-symmetric wave functions to analyze how small of a volume the wave function could be compressed; we found that it was the minimal distance cubed, up to a factor of order one.

In the second case, we considered a system with a large, non-zero momentum in some particular direction. Here we found that the minimum distance in the direction of the average momentum was different from the minimum distance in the orthogonal directions. For either the tanh and arctan GUPs, the positional uncertainty in the orthogonal directions was larger than in the direction of the non-zero momentum. A similar result occurs in the GUP model in reference [7] *i.e.* $\Delta X_{2,3}^{min} > \Delta X_1^{min}$. This difference in the uncertainties in different directions could have interesting consequences on the momentum-position phase space, which in turn would impact on the thermodynamic properties of a system of particles. This modification of the phase space could also potentially impact the calculation of the cosmological constant. There has been some preliminary work in this direction [22], but we hope to explore the effect of the anisotropic position uncertainties on the thermodynamics of systems of particle more in future work.

The difference between the minimum position uncertainties in the second case above hints that there may be a violation of Lorentz symmetry. Moreover, the existence of minimal distances would naively imply violation of Lorentz symmetry. However in [20], a GUP model was formulated which gave a minimal length and preserved the special relativistic energy and momentum relationship, $E^2 - |\vec{p}|^2 = m^2$, and thus preserved some aspects of the Lorentz symmetry. A complementary issue is the connection of our work to a breaking of the isotropy of space, as observed for other GUP models [23]. In future work, we plan to study the issue of the fate of the Lorentz symmetry of these modified position and momentum operators.

The standard Lorentz algebra is generated by the angular momentum operators and boost operators in terms of the time-position operators and energy-momentum operators. One major question to address is how the modified position and momentum operators affect the observables associated with modified angular momentum and boost operators. To obtain the modified boost operators we will need to also define modified time and energy operators that fit in with the tanh or arctan operators of equations (9) and (10).

Another related issue is the modification of commutation relations of position and momentum operators among themselves. The standard result is $[\hat{x}_i, \hat{x}_j] = 0$ and $[\hat{p}_i, \hat{p}_j] = 0$. While this is still true for the modified momentum operators (9) and (10), it is no longer true for the modified position operators *i.e.*, $[\hat{P}_i, \hat{P}_j] = 0$ but $[\hat{X}_i, \hat{X}_j] \neq 0$. This non-commutativity of the modified position operators provides a potential link between GUP models and non-commutative geometry models [24,25] which is another approach to minimal lengths.

Also, the modified position operators from (9) and (10) satisfy $[\hat{X}_i, \hat{X}_j] \propto \epsilon_{ijk}\hat{l}_k$, so that the commutators are proportional to the angular momentum operators. For the related uncertainty relationships, this implies that $\Delta X_i \Delta X_j \propto \epsilon_{ijk}\langle \hat{l}_k \rangle$; the uncertainty in area is related to the expectation of the angular momentum operator. This provides a potential connection between the 3D GUP models in this work and spin foam models [26,27] which have a similar connection between area and angular momentum.

We leave both of these questions – the modification of the Lorentz algebra and the connection of 3D GUP models to non-commutative geometry and spin foam models – for a subsequent, companion paper.

Declaration of competing interest

Data availability

No data was used for the research described in the article.

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