ON ASYMPTOTIC STABILITY ON A CENTER HYPERSURFACE AT THE SOLITON FOR EVEN SOLUTIONS OF THE NONLINEAR KLEIN–GORDON EQUATION WHEN $2 \geq p > \frac{5}{3}$ *

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Abstract. We extend the result of Kowalczyk, Martel, and Muñoz [*J. Eur. Math. Soc. (JEMS)*, 24 (2022), pp. 2133–2167] on the existence, in the context of spatially even solutions, of asymptotic stability on a center hypersurface at the soliton of the defocusing power nonlinear Klein–Gordon equation with $p > 3$, to the case $2 \geq p > \frac{5}{3}$. The result is attained performing new and refined estimates that allow us to close the argument for power law in the range $2 \geq p > \frac{5}{3}$.

Key words. conditional stability, nonlinear Klein–Gordon, soliton, virial estimates, Fermi golden rule

1. Introduction.

We consider the nonlinear Klein–Gordon equation (NLKG) on the line

$$\partial_t^2 u_1 - \partial_x^2 u_1 + u_1 - f(u_1) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1.1}$$

where $f(u_1) := |u_1|^{p-1} u_1$ with $p_3 := 2 \geq p > p_4 := \frac{5}{3}$. As in Krieger, Nakanishi, and Schlag [21], Kowalczyk, Martel, and Muñoz [19], and Li and Lührmann [23], we consider only even solutions, eliminating translations and simplifying the problem. Translations pose difficulties; see the complications in the virial inequalities in Kowalczyk et al. [20].

Setting $\boldsymbol{u} := (u_1, u_2)^\intercal := (u_1, \dot{u}_1)^\intercal$, where in what follows $\dot{u} = \partial_t u$ and $u' = \partial_x u$, we obtain the system

$$\dot{\boldsymbol{u}} = \mathbf{J} \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{u} + f(u_1)\mathbf{j}, \text{ with } \mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.2}$$

and with $\mathbf{j} = (0,1)^\intercal$, later also $\mathbf{i} = (0,1)^\intercal$ (not to be confused with the imaginary unit i). We denote by Q the standing wave, given (see, for example, [3, formula (3.1)]) by

$$Q(x) := \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}}\left(\frac{p-1}{2}x\right). \tag{1.3}$$

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The system (1.2) is a Hamiltonian system for the symplectic form

$$\Omega(\boldsymbol{u},\boldsymbol{v}) := \left\langle \mathbf{J}^{-1}\boldsymbol{u},\boldsymbol{v}\right\rangle, \tag{1.4}$$

$$\langle \boldsymbol{u},\boldsymbol{v}\rangle := (\boldsymbol{u},\overline{\boldsymbol{v}}), \qquad (\boldsymbol{u},\boldsymbol{v}) := \int_{\mathbb{R}} \boldsymbol{u}(x)^{\intercal}\boldsymbol{v}(x)dx, \tag{1.5}$$

and Hamiltonian, or energy function, given by

$$E(\boldsymbol{u}) = \frac{1}{2}\left(\|u_1'\|^2_{L^2(\mathbb{R})} + \|u_2\|^2_{L^2(\mathbb{R})}\right) - \int_{\mathbb{R}} \frac{|u_1|^{p+1}}{p+1}\,dx. \tag{1.6}$$

We consider the space of even functions

$$\boldsymbol{\mathcal{H}}^1_{\text{even}} = \left\{\mathbf{u} = (u_1,u_2)^{\intercal}\ \middle|\mathbf{u}\in\boldsymbol{\mathcal{H}}^1 \text{ and } \mathbf{u}(x) = \mathbf{u}(-x)\right\}$$

with norm

$$\|\boldsymbol{u}\|^2_{\boldsymbol{\mathcal{H}}^1} = \|u_1\|^2_{H^1} + \|u_2\|^2_{L^2}. \tag{1.7}$$

We recall that Kowalczyk, Martel, and Muñoz [19] proved, for $p>3$, that there exists an orbitally stable central hypersurface in $\boldsymbol{\mathcal{H}}^1_{\text{even}}$. With minor modifications, their proof extends also to our case $2\ge p>\frac{5}{3}$, so we assume it. Furthermore, Kowalczyk, Martel, and Muñoz [19] proved that Q is asymptotically stable for solutions on this hypersurface. Li and Lührmann [23] considered $p=p_3=2$, with $f(u_1)=u_1^2$. In Li and Lührmann [23] the threshold resonance $\mu_3=1$ is potentially an obstruction to the proof, but since the corresponding generalized eigenfunctions are odd in x, they are naturally orthogonal to the elements of $\boldsymbol{\mathcal{H}}^1_{\text{even}}$. So the threshold resonance $\mu_3=1$ does not affect the argument, in line with the discussion in Kowalczyk and Martel [17] on the asymptotic stability of odd perturbations of the standing kink of the ϕ^4 model, where the threshold resonance is *even,* and so does not affect odd solutions. In the case $p=2$, for the nonsmooth nonlinearity $f(u_1)=|u_1|u_1$ treated here, the same will happen in our proof.

Notice that the case $2<p<3$ is simpler. Case $p=3$ is different and still open, because of an *even* threshold resonance, certainly affecting even solutions of (1.1). For a partial finite time result, framed with very different techniques, see Lührmann and Schlag [24]. Very recently Palacios and Pusateri [26] obtained a finite time result mixing arguments similar to Kowalczyk, Martel, and Muñoz [19] and phase space analysis. Finally, as $p\to 1^+$, besides the case of the $p>1$ for which there is a resonance, which form a discrete subset in $(1,\infty)$, there is also an increase of the number of eigenvalues $p\to 1^+$, with some converging to 0, as can be seen in the pictures in Chang et al. [3]. While methods of proving dispersion by means of dispersive estimates or Strichartz estimates, with the Duhamel formula, are not particularly robust and are not applicable for p sufficiently close to 1, in fact for $p<3$, low values of p arbitrarily close to 1 could in principle be treated by the dispersion theory in Kowalczyk, Martel, and Muñoz [19]. Furthermore, the appearance of a complicated set of eigenvalues is not, by itself, an impossible problem. The corresponding discrete modes slow the rate of stabilization, as discovered computationally in [1] and observed in a rigorous context in [2, 13, 28, 29] as well as in many other papers, most of them quoted in [7], to which we refer for an extensive bibliography. But this slowing of the stabilization does not affect proofs of stabilizations similar to [6], which is older, or to [10]. In the stable case, there is a well-known stabilization mechanism that all the references utilize, called the nonlinear Fermi golden rule, which was introduced by Sigal [27], involving the nonlinear interaction of the continuous modes with the continuous ones. Using the Fermi golden rule, older references like [6], which used normal forms arguments, and more recent ones like [10], which used the so-called refined profile, were able to deal with generic configurations of eigenvalues. Finding the refined profile is a relatively simple operation involving Taylor expansions of sufficiently high order, depending on how close to the ground state the excited states get, of the function $f(\boldsymbol{\Phi}(\boldsymbol{z})_1(x))$, which appears later in (1.43). In the context of the present paper, the low regularity of the power function, however, becomes a problem. Notice though that $f(\boldsymbol{\Phi}(\boldsymbol{z})_1(x))$ is smooth at values of $(\boldsymbol{z}, x)$ for which $\boldsymbol{\Phi}(\boldsymbol{z})_1(x) \neq 0$. The gist of the argument and the intuition behind this paper is that by an appropriate partitioning of spacetime, for example with the distinction in Case 1 and Case 2 in the proof of Lemma 1.6, we can take up to two derivatives even when $p < 2$. Furthermore, to get estimates like (1.55), and specifically the fact that $Q^{p-2}\varphi_2^2 \sim e^{-|x|}$ (see below for the notation), we neutralize the growth of Q^{p-2} as $x \to \infty$ with the decay of φ_2^2; see (1.16). It is possible, in principle, that a more systematic analysis of this type will allow us to treat cases with p closer to 1. The specific idea utilized here breaks down for $p_5 = \frac{3}{2} < p < p_4 = \frac{5}{3}$, which would have an additional term $Q^{p-2}\varphi_4^2 \sim e^{(2p-3)|x|}$, unbounded as $x \to \infty$.

There is a large literature, beyond the above references, like the influential Kowalczyk, Martel, and Muñoz [18], which gave great impulse to the literature on one-dimensional stabilization problems. There are many papers with a different framework and origin, involving dispersive estimates, like [4, 5, 12, 14, 15, 22] and references therein. A quite extensive list of references known to the authors which appeared before 2019 can be found in [7].

1.1. On the linearization of (1.2) around Q. Here we give some information about the linearization of (1.2) around Q; cf. [3]. Key to the analysis is the operator (often denoted, in the literature, by L_+),

$$L_0 := -\partial_x^2 + 1 - pQ^{p-1}. \tag{1.8}$$

It is well known (see Grillakis, Shatah, and Strauss [16, section 6]) that Q is orbitally unstable. Indeed, if we set

$$\mathbf{L}_0 := \begin{pmatrix} L_0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.9}$$

the linearization of (1.2) at $Q\mathbf{i}$ is

$$\mathbf{J}\mathbf{L}_0 := \begin{pmatrix} 0 & 1 \\ -L_0 & 0 \end{pmatrix}$$

and admits the point spectrum

$$\sigma_{\mathrm{pp}}(\mathbf{J}\mathbf{L}_0) = \left\{ \nu := \pm\sqrt{-\mu} \;\middle|\; \mu \in \sigma_{\mathrm{pp}}(L_0) \right\}.$$

In particular the elements of $\sigma_{\mathrm{pp}}(L_0)$ as functions of p are

$$p_N := \begin{cases} \frac{N+1}{N-1} \text{ for } N > 1, \\ \infty \text{ if } N = 1, \end{cases} \qquad k_j := \frac{p+1}{2} - \frac{j(p-1)}{2}, \quad j \in \mathbb{N}, \tag{1.10}$$

and if $p_{N+1} < p < p_N$, then $\sigma_{\mathrm{pp}}(L_0)$ has $N+1$ elements and they are explicitly given by

$$\mu_m := 1 - k_m^2, \qquad m = 0, \ldots, N. \tag{1.11}$$

When $p = p_N$ for some $N \geq 1$, then from (1.11) we have $\mu_N = 1$ and the linear operator $\mathbf{J}\mathbf{L}_0$ has a threshold resonance at the boundary of the continuous spectrum. Here we will consider

$$N = \begin{cases} 3 \text{ if } p_3 = 2 > p > p_4 = \frac{5}{3}, \\ 2 \text{ if } p = 2 \end{cases} \tag{1.12}$$

and we will use

$$k_0 = \frac{p+1}{2}, \qquad k_1 = 1, \qquad k_2 = \frac{3-p}{2}, \qquad k_3 = 2-p, \qquad k_4 = \frac{5-3p}{2}. \tag{1.13}$$

Like in Krieger, Nakanishi, and Schlag [21], Kowalczyk, Martel, and Muñoz [19], and Li and Lührmann [23], we will prove the asymptotic stability of Q in the central hypersurface of Kowalczyk, Martel, and Muñoz [19]. The constraint $2 \geq p > \frac{5}{3}$ renders the nonlinearity more problematic compared to [19, 21, 23], in terms of lower differentiability and bigger strength. For

$$S_j := \partial_x - k_j \tanh\left(\frac{p-1}{2}x\right) \tag{1.14}$$

and for the eigenvalues μ_j in (1.11), we have $\ker(L_0 - \mu_j) = \mathrm{Span}\{\varphi_j\}$ with

$$\varphi_0 := Q^{k_0}, \quad \varphi_1 := S_0^* Q^{k_1} = \frac{p-1}{2} Q', \quad \varphi_2 := S_0^* S_1^* Q^{k_2}, \quad \varphi_3 := S_0^* S_1^* S_2^* Q^{k_3}. \tag{1.15}$$

We emphasize that the S_j^* change the symmetry of a function, so that we have the very general fact that φ_0 and φ_2 are even while φ_1 and φ_3 are odd. It is elementary and well known [11] that

$$|\varphi_j(x)| \sim e^{-k_j|x|} \text{ as } |x| \to \infty. \tag{1.16}$$

Using (1.15), for

$$\mathbf{Y}_\pm := \begin{pmatrix} \varphi_0 \\ \pm\nu_0\varphi_0 \end{pmatrix}, \qquad \mathbf{Z}_\pm := \begin{pmatrix} \varphi_0 \\ \pm\nu_0^{-1}\varphi_0 \end{pmatrix}, \tag{1.17}$$

we have

$$\mathbf{J}\mathbf{L}_0\mathbf{Y}_\pm = \pm\nu_0\mathbf{Y}_\pm.$$

We set

$$\mathbf{\Phi}_0 := \frac{\mathbf{Y}_+ - \mathrm{i}\mathbf{Y}_-}{2} = \frac{1}{2}\begin{pmatrix} 1-\mathrm{i} \\ (1+\mathrm{i})\,\nu_0 \end{pmatrix}\varphi_0 \,. \tag{1.18}$$

Then,

$$\mathbf{J}\mathbf{L}_0\mathbf{\Phi}_0 = \nu_0\bar{\mathbf{\Phi}}_0. \tag{1.19}$$

Similarly, setting $\lambda = \sqrt{\mu_2} > 0$ we define

$$\mathbf{\Phi}_2 := \begin{pmatrix} 1 \\ \mathrm{i}\lambda \end{pmatrix} \varphi_2 \tag{1.20}$$

so that

$$\mathbf{J}\mathbf{L}_0\mathbf{\Phi}_2 = \mathrm{i}\lambda\mathbf{\Phi}_2, \qquad \mathbf{J}\mathbf{L}_0\bar{\mathbf{\Phi}}_2 = -\mathrm{i}\lambda\bar{\mathbf{\Phi}}_2. \tag{1.21}$$

Notice that for

$$\mathcal{A} = S_0 \cdots S_N, \text{ for } N \text{ as in (1.12)}, \tag{1.22}$$

we have

$$\mathcal{A}^* L_0 = L_{N+1}\mathcal{A}^*, \tag{1.23}$$

where

$$L_j := -\partial_x^2 + 1 - k_{j-1}k_j \frac{2}{p-1} Q^{p-1}. \tag{1.24}$$

The procedure outlined in (1.22)–(1.24) transforms the linearization of (1.2) around Q from $\dot{U} = \mathbf{J}\mathbf{L}_0 U$ into the new linear equation, in the auxiliary variable $V := \mathcal{A}U$,

$$\dot{V} = \mathbf{J}\mathbf{L}_{N+1}V, \qquad \mathbf{L}_{N+1} := \begin{pmatrix} L_{N+1} & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.25}$$

where for $p_3 > p > p_4$ the operator $L_{N+1} = L_4$ has a repulsive potential, in the sense that

$$-k_3k_4 \frac{2}{p-1} x \left(Q^{p-1}\right)' < 0 \text{ for all } x \neq 0, \tag{1.26}$$

while for $p = 2$ we have $L_{N+1} = L_3 = -\partial_x^2 + 1$ but fortunately the parity of the solutions obviates to the lack of this repulsivity.

Given a constant $\kappa > 0$, we consider the spaces defined by the following norms:

$$\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{H}}^1_{-\kappa}} := \|\mathrm{sech}\,(\kappa x)\,\boldsymbol{u}\|_{\boldsymbol{\mathcal{H}}^1}, \qquad \|\boldsymbol{u}\|_{L^2_{-\kappa}} := \|\mathrm{sech}\,(\kappa x)\,\boldsymbol{u}\|_{L^2}. \tag{1.27}$$

In what follows, $\kappa > 0$ is fixed and smaller than a certain finite set of positive numbers coming up in the proof. For $\boldsymbol{z} = (z_1, z_2)^\intercal \in \mathbb{C}^2$, we set

(1.28)

$$\begin{aligned}\mathbf{\Phi}(\boldsymbol{z}) &:= Q\mathbf{i} + \tilde{\mathbf{\Phi}}[\boldsymbol{z}], \text{ where } \tilde{\mathbf{\Phi}}[\boldsymbol{z}] := z_1\mathbf{\Phi}_0 + z_2\mathbf{\Phi}_2 + \overline{z}_1\bar{\mathbf{\Phi}}_0 + \overline{z}_2\bar{\mathbf{\Phi}}_2 = 2\mathrm{Re}\,(z_1\mathbf{\Phi}_0 + z_2\mathbf{\Phi}_2) \\ &= z_{1R}\mathbf{Y}_+ - z_{1I}\mathbf{Y}_- + 2\varphi_2 \begin{pmatrix} z_{2R} \\ -\lambda z_{2I} \end{pmatrix}, \text{ where } z_{jR} = \mathrm{Re} z_j \text{ and } z_{jI} = \mathrm{Im} z_j.\end{aligned}$$

By the linearity of $\tilde{\mathbf{\Phi}}[\boldsymbol{z}]$ in $\boldsymbol{z}$, we obviously have

$$D_{\boldsymbol{z}}\mathbf{\Phi}(\boldsymbol{z})\boldsymbol{w} = \tilde{\mathbf{\Phi}}[\boldsymbol{w}], \tag{1.29}$$

where $D_{\boldsymbol{z}}\varphi(\boldsymbol{z})\boldsymbol{w} := .\frac{d}{ds}|_{s=0}\varphi(\boldsymbol{z} + s\boldsymbol{w})$. We set

$$\boldsymbol{\mathcal{H}}^1_c := \left\{\boldsymbol{\eta} \in \boldsymbol{\mathcal{H}}^1_{\mathrm{even}}(\mathbb{R}, \mathbb{C}^2) \;\middle|\; \langle \mathbf{J}\boldsymbol{\eta}, D_{\boldsymbol{z}}\mathbf{\Phi}(\boldsymbol{z})\mathbf{\Theta}\rangle = 0 \text{ for all } \mathbf{\Theta} \in \mathbb{C}^2\right\}. \tag{1.30}$$

There exists a nonzero $\mathbf{g}_2 \in L^\infty(\mathbb{R}; \mathbb{C}^2)$ such that $\mathbf{J}\mathbf{L}_0\mathbf{g}_2 = \mathrm{i}\ 2\lambda\mathbf{g}_2$. It is of the form $\mathbf{g}_2 = \binom{1}{2\mathrm{i}\lambda} g$ with $L_0 g = 4\lambda^2 g$, i.e., g is a *distorted plane-wave* for the Schrödinger operator L_0; see [11].

Assumption 1.1 (Fermi golden rule). We will consider the $p\in\left(5/3\,,2\right)$ for which

$$-p\,(p-1)\left\langle Q^{p-2}\varphi_2^2,\ g\right\rangle\neq 0,$$

where Q and φ_2 are defined in (1.3) and (1.15), respectively.

Remark 1.2. Assumption 1.1 is proved for $p=2$ by Li and Lührmann [23, Lemma 3.1] and for p close to 2 remains true by continuous dependence on p. Since this dependence is also analytic, Assumption 1.1 is true for all but a discrete subset of $p\in[5/3,2)$. It would be interesting to check computationally if Assumption 1.1 holds for all $p\in\left(5/3\,,2\right]$. This is feasible especially because Q, φ_2, and also g are known explicitly, but we did not try it.

Our main result is the following.

THEOREM 1.3. *Under Assumption* 1.1, *for any $a>0$ and $\epsilon>0$ there exists $\delta_0>0$ such that for $\delta\in(0,\delta_0)$ if*

$$\sup_{t\geq 0}\|\boldsymbol{u}(t)-Q\mathbf{i}\|_{\boldsymbol{\mathcal{H}}^1_{\text{even}}(\mathbb{R},\mathbb{R}^2)}<\delta, \tag{1.31}$$

then there exists $\boldsymbol{z}\in C^1\left(\mathbb{R},\mathbb{C}^2\right)$ and $\boldsymbol{\eta}\in C^0\left(\mathbb{R},\boldsymbol{\mathcal{H}}^1_c\right)$ such that we have a global representation

$$\boldsymbol{u}(t)=\boldsymbol{\Phi}\left(\boldsymbol{z}(t)\right)+\boldsymbol{\eta}(t) \tag{1.32}$$

such that

$$\int_I\left(\|\boldsymbol{\eta}(t)\|^2_{\boldsymbol{\mathcal{H}}^1_{-a}(\mathbb{R})}+|z|^4\right)\mathrm{d}t\leq\epsilon, \tag{1.33}$$

$$\lim_{t\to+\infty}\|\boldsymbol{\eta}(t)\|^2_{\boldsymbol{\mathcal{H}}^1_{-a}(\mathbb{R})}=0 \text{ and} \tag{1.34}$$

$$\lim_{t\to\infty}\boldsymbol{z}(t)=0. \tag{1.35}$$

For $\delta_0>0$ we set

$$\mathcal{B}_0:=\left\{\boldsymbol{\varepsilon}\in\boldsymbol{\mathcal{H}}^1_{\text{even}}(\mathbb{R},\mathbb{R}^2)\;\middle|\;\|\boldsymbol{\varepsilon}\|^2_{\boldsymbol{\mathcal{H}}^1}<\delta_0\text{ and }\langle\boldsymbol{\varepsilon},\mathbf{Z}_+\rangle=0\right\}. \tag{1.36}$$

The following existence and uniqueness of a stable central hypersurface can be proved like Theorem 2 in Kowalczyk, Martel, and Muñoz [19].

THEOREM 1.4. *There exist $C,\delta_0>0$ and a Lipschitz function $h:\mathcal{B}_0\to\mathbb{R}$ with*

$$h(0)=0 \text{ and } |h(\boldsymbol{\varepsilon})|\leq C\|\boldsymbol{\varepsilon}\|^{\frac{p+1}{2}}_{\boldsymbol{\mathcal{H}}^1} \tag{1.37}$$

such that, for

$$\mathcal{N}:=\left\{Q\mathbf{i}+\boldsymbol{\varepsilon}+h(\boldsymbol{\varepsilon})\mathbf{Y}_+\;\middle|\;\boldsymbol{\varepsilon}\in\mathcal{B}_0\right\}, \tag{1.38}$$

we have the following:

(a) *If $\boldsymbol{u}_0\in\mathcal{N}$, then the corresponding solution of* (1.2) *is defined for all $t\geq 0$ and*

$$\sup_{t\geq 0}\|\boldsymbol{u}(t)-Q\mathbf{i}\|_{\boldsymbol{\mathcal{H}}^1_{\text{even}}(\mathbb{R},\mathbb{R}^2)}\leq C\|\boldsymbol{u}_0-Q\mathbf{i}\|_{\boldsymbol{\mathcal{H}}^1_{\text{even}}(\mathbb{R},\mathbb{R}^2)}. \tag{1.39}$$

(b) *If a solution* $\boldsymbol{u}(t)$ *defined for all* $t \geq 0$ *satisfies*

$$\|\boldsymbol{u}(t) - Q\mathbf{i}\|_{\boldsymbol{\mathcal{H}}^1_{\mathrm{even}}(\mathbb{R},\mathbb{R}^2)} < \frac{\delta_0}{2}, \tag{1.40}$$

then $\boldsymbol{u}(t) \in \mathcal{N}$ *for all* $t \geq 0$.

Remark 1.5. Notice that by $p \leq 2$ the exponent $\frac{p+1}{2}$ in (1.37) is smaller than the exponent $3/2$ in [19]. In fact here we have inequalities, for example, like

$$|f(Q+v) - f(Q+\tilde{v}) - f'(Q)(v-\tilde{v})| \lesssim \left(|v|^{\frac{p-1}{2}} + |\tilde{v}|^{\frac{p-1}{2}}\right)|v-\tilde{v}|,$$

while in Proposition 4 [19] the exponent $\frac{p-1}{2}$ is replaced by the larger value 1, and Theorem 1.4 can be obtained mimicking almost exactly the discussion in sections 6.2 and 6.3 [19], with only some minor modifications in the exponents.

As a corollary, it follows that the conclusions of Theorem 1.3 are true for all the solutions of (1.2) in the stable center hypersurface $\mathcal{N}$. This paper is focused uniquely on the proof of Theorem 1.3 and, for dispersion, uses the framework of Kowalczyk, Martel, and Muñoz, even though for technical results it often refers to [8, 9, 10].

1.2. Refined profile and Fermi golden rule. In our earlier work [10], we referred to the notion of *refined profile.* Our problem here requires only a special case of this notion. We introduce the notation

$$\tilde{\boldsymbol{z}}_0 := \tilde{\boldsymbol{z}}_0[\boldsymbol{z}] := (\nu_0 \overline{z}_1, \mathrm{i}\lambda z_2). \tag{1.41}$$

Then $\boldsymbol{\Phi}(\boldsymbol{z}) = (\boldsymbol{\Phi}(\boldsymbol{z})_1, \boldsymbol{\Phi}(\boldsymbol{z})_2)^{\intercal}$ satisfies, using (1.19), (1.21), (1.28), and (1.41), the equation

$$D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\,\tilde{\boldsymbol{z}}_0 = \mathbf{J}\mathbf{L}_0\tilde{\boldsymbol{\Phi}}[z]. \tag{1.42}$$

Thus using the fact that $Q\mathbf{i}$ is a stationary solution of (1.2) we obtain

$$D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\,\tilde{\boldsymbol{z}}_0 = \mathbf{J}\begin{pmatrix} -\partial_x^2+1 & 0 \\ 0 & 1 \end{pmatrix}\boldsymbol{\Phi}(\boldsymbol{z}) + f(\boldsymbol{\Phi}(\boldsymbol{z})_1)\mathbf{j} + \widehat{\boldsymbol{R}}(\boldsymbol{z}) \text{ with} \tag{1.43}$$

$$\widehat{\boldsymbol{R}}(\boldsymbol{z}) := -\left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[z]_1\right)\mathbf{j}. \tag{1.44}$$

It is useful to improve the remainder $\widehat{\boldsymbol{R}}(\boldsymbol{z})$ as follows.

Lemma 1.6. *There exist constants* $C_1, \delta_1 > 0$ *and a map* $\tilde{\boldsymbol{z}}_R : D_{\mathbb{C}^2}(0,\delta_1) \to \mathbb{C}^2$ *such that the following holds:*

(i) *for*

$$\boldsymbol{R}(\boldsymbol{z}) := D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\ \tilde{\boldsymbol{z}}_R(\boldsymbol{z}) + \widehat{\boldsymbol{R}}(\boldsymbol{z}), \tag{1.45}$$

we have

$$\langle \mathbf{J}\boldsymbol{R}(\boldsymbol{z}), D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\rangle = 0 \text{ for all } \Theta \in \mathbb{C}^2 \text{ and all } \boldsymbol{z} \in D_{\mathbb{C}^2}(0,\delta_1); \tag{1.46}$$

(ii) *we have*

$$|\tilde{\boldsymbol{z}}_R(\boldsymbol{z})| \leq C_1|\boldsymbol{z}|^2 \text{ for any } \boldsymbol{z} \in D_{\mathbb{C}^2}(0,\delta_1); \tag{1.47}$$

(iii) $\tilde{\boldsymbol{z}}_R \in C^1\left(D_{\mathbb{C}^2}(0,\delta_1), \mathbb{C}^2\right)$*;*

(iv) $\tilde{\boldsymbol{z}}_R$ *is twice differentiable in* 0.

Proof. We prove item (i). First of all, (1.46) is equivalent to

$$\begin{aligned}(1.48)\qquad &\langle \mathbf{J} D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z})\, \tilde{\boldsymbol{z}}_R, D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z})\,\Theta\rangle \\ &= -\left\langle \mathbf{J}\hat{\boldsymbol{R}}(\boldsymbol{z}), D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\,\Theta\right\rangle \text{ for all } \Theta\in\mathbb{C}^2 \text{ and all } \boldsymbol{z}\in D_{\mathbb{C}^2}(0,\delta_1).\end{aligned}$$

By the definition of $\boldsymbol{\Phi}(\boldsymbol{z})$, varying Θ in an $\mathbb{R}$-basis of $\mathbb{C}^2$, (1.48) is a linear system with unknown $\tilde{\boldsymbol{z}}_R$ with constant coefficients. The linear operator acting on $\tilde{\boldsymbol{z}}_R$ is invertible, because the affine space spanned by $\boldsymbol{\Phi}(\boldsymbol{z})$ is symplectic. In particular, varying Θ in the standard $\mathbb{R}$-basis of $\mathbb{C}^2$ and utilizing the notation in (1.28), the matrix is, where $z_{a,R}=\mathrm{Re} z_a$ and $z_{a,I}=\mathrm{Im} z_a$, for $a=0,2$,

(1.49)
$$\left\{\left\langle \mathbf{J}\partial_{z_{a,A}}\boldsymbol{\Phi}(\boldsymbol{z}), \partial_{z_{b,B}}\boldsymbol{\Phi}(\boldsymbol{z})\right\rangle\right\}_{a,b=0,2 \text{ and } A,B=R,I} = \begin{pmatrix} \langle J\mathbf{Y}_-,\mathbf{Y}_+\rangle J & 0 \\ 0 & \lambda\|\varphi_2\|^2_{L^2(\mathbb{R})}J\end{pmatrix},$$

which is obviously invertible since $\langle J\mathbf{Y}_-,\mathbf{Y}_+\rangle = -2\nu_0\|\varphi_0\|^2_{L^2}$. This matrix reappears later in Lemma 3.3.

We prove item (ii). From the above we have

(1.50)
$$|\tilde{\boldsymbol{z}}_R(\boldsymbol{z})|\lesssim \int |I\varphi_0|+|I\varphi_2|\mathrm{d}x \qquad \text{with} \qquad I:=f(Q+\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1)-f(Q)-f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1.$$

Notice that

$$(1.51)\qquad \left|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right|\lesssim |z_1|\;|\varphi_0|+|z_2|\;|\varphi_2|$$

with $|\varphi_0|\sim e^{-\frac{p+1}{2}|x|}$ and $|\varphi_2|\lesssim e^{-k_2|x|}=e^{-\frac{3-p}{2}|x|}$ by (1.16). Now we estimate pointwise I.

Case 1. Let $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1|\le Q/2$. By $Q\sim e^{-|x|}$, $k_2=\frac{3-p}{2}$, and $|f''(Q+\tau\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1)|\sim Q^{p-2}$, for $\tau\in[0,1]$ we have

(1.52)
$$\begin{aligned}|I|&\lesssim \sup_{\tau\in[0,1]}\left|f''\left(Q+\tau\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)\right|\;\left(|z_1|^2|\varphi_0|^2+|z_2|^2|\varphi_2|^2\right)\lesssim |Q|^{p-2}\left(|z_1|^2|\varphi_0|^2+|z_2|^2|\varphi_2|^2\right)\\ &\lesssim Q^{p-2}|\boldsymbol{z}|^2e^{-2k_2|x|}\sim |\boldsymbol{z}|^2e^{|x|(2-p-(3-p))}=|\boldsymbol{z}|^2e^{-|x|}.\end{aligned}$$

Case 2. Let $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1|>Q/2$. Then

$$(1.53)\qquad |z_1|\;|\varphi_0|+|z_2|\;|\varphi_2|\gtrsim|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1|\gtrsim Q \qquad\Longrightarrow\qquad |z_2|\;|\varphi_2|\gtrsim Q$$

by $|\varphi_0|\sim e^{-\frac{p+1}{2}|x|}$, $Q\sim e^{-|x|}$, and $|z_1|\lesssim\delta\ll 1$, which obviously yield $|z_1|\;|\varphi_0|\ll Q$ always. Then

(1.54)
$$\begin{aligned}|I|\lesssim\left|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right|^p\lesssim|z_2\varphi_2|^p\lesssim|z_2|^pe^{-p\frac{3-p}{2}|x|}&=|z_2|^pe^{-(2-p)\frac{p-1}{2}|x|}e^{2^{-1}((2-p)(p-1)-p(3-p))|x|}\\ &=|z_2|^pe^{-(2-p)\frac{p-1}{2}|x|}e^{-|x|}.\end{aligned}$$

It is easy to conclude from (1.53) that $|z_2|e^{-\frac{3-p}{2}|x|}\gtrsim e^{-|x|}$, and so $|z_2|\gtrsim e^{-\frac{p-1}{2}|x|}$ or $|z_2|^{2-p}\gtrsim e^{-(2-p)\frac{p-1}{2}|x|}$. This proves that $|I|\lesssim|\boldsymbol{z}|^2e^{-|x|}$ when $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1|>Q/2$.

Cases 1 and 2 yield

$$|I| \lesssim |\boldsymbol{z}|^2 e^{-|x|} \text{ for all } x \in \mathbb{R}. \tag{1.55}$$

This yields (1.47) and, by dominated convergence, $\tilde{\boldsymbol{z}}_R \in C^0\left(D_{\mathbb{C}^2}(0,\delta_1), \mathbb{C}^2\right)$, proving item (ii).

We prove item (iii). It suffices to prove that the right-hand side in (1.48) is in $C^1\left(D_{\mathbb{C}^2}(0,\delta_1), \mathbb{C}^2\right)$. We have using (1.29) and (1.44)

$$D_{\boldsymbol{z}} \hat{\boldsymbol{R}}(\boldsymbol{z})\ \boldsymbol{\zeta} = \left(f'\left(\boldsymbol{\Phi}(\boldsymbol{z})_1\right) - f'(Q)\right) \tilde{\boldsymbol{\Phi}}\left[\boldsymbol{\zeta}\right]_1\ \mathbf{j}. \tag{1.56}$$

For $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1| \ll Q$ we have

$$|f'(Q) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)| \lesssim \left|f''(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right| \lesssim |\boldsymbol{z}| e^{|x|\left(2-p-\frac{3-p}{2}\right)} = |\boldsymbol{z}| e^{-\frac{p-1}{2}|x|} \tag{1.57}$$

while for $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1| \gtrsim Q$, by $|z_2| \gtrsim e^{-\frac{p-1}{2}|x|}$, and so, by $|z_2|^{2-p} \gtrsim e^{-(2-p)\frac{p-1}{2}|x|}$, we have

$$\begin{aligned} |f'(Q) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)| &\lesssim |\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1|^{p-1} \lesssim |z_2|^{p-1} e^{-(3-p)\frac{p-1}{2}|x|} = |z_2|^{p-1} e^{-(2-p)\frac{p-1}{2}|x|} e^{-\frac{p-1}{2}|x|} \\ &\lesssim |\boldsymbol{z}| e^{-\frac{p-1}{2}|x|}. \end{aligned} \tag{1.58}$$

So, since the differentiation in (1.56) is well defined and is uniformly bounded by an integrable function, by dominated convergence we can commute derivative and integral and conclude

$$D_{\boldsymbol{z}} \left\langle \mathbf{J}\hat{\boldsymbol{R}}(\boldsymbol{z}), D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta \right\rangle \boldsymbol{\zeta} = \left\langle \mathbf{J} D_{\boldsymbol{z}}\hat{\boldsymbol{R}}(\boldsymbol{z})\boldsymbol{\zeta}, D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta \right\rangle$$

with, furthermore, the differential continuous in $\boldsymbol{z}$. So, from (1.48) we obtain $\tilde{\boldsymbol{z}}_R \in C^1\left(D_{\mathbb{C}^2}(0,\delta_1), \mathbb{C}^2\right)$.

We prove item (iv). From (1.56) and the linearity in $\boldsymbol{z}$ of $\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]$,

$$D^2_{\boldsymbol{z}} \hat{\boldsymbol{R}}(\boldsymbol{z})\left(\boldsymbol{\zeta}^1, \boldsymbol{\zeta}^2\right) = -f''\left(\boldsymbol{\Phi}(\boldsymbol{z})_1\right) \tilde{\boldsymbol{\Phi}}\left[\boldsymbol{\zeta}^1\right]_1\ \tilde{\boldsymbol{\Phi}}\left[\boldsymbol{\zeta}^2\right]_1 \mathbf{j}. \tag{1.59}$$

Now, for fixed $A = R, I$, $z_{aR} := \mathrm{Re} z_a$, $z_{aI} := \mathrm{Im} z_a$, and $a = 1, 2$, we have

$$\begin{aligned} &\left\langle \mathbf{J}\left(\partial_{z_{aA}}\hat{\boldsymbol{R}}(\boldsymbol{z}) - D_{\boldsymbol{z}}\partial_{z_{aA}}\hat{\boldsymbol{R}}(0)\boldsymbol{z}\right), D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta \right\rangle = -\langle II, 1\rangle \\ &\text{with } II := \left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q) - f''(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right) \partial_{z_{aA}} \tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1 \overline{D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})_1\,\Theta}. \end{aligned} \tag{1.60}$$

To show that the above expression is $o(|\boldsymbol{z}|)$ we bound II. For $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1| \ll Q$, we have

$$|II| \lesssim |f'''(Q)|\ \left|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1 e^{-k_2|x|}\right|^2 \lesssim |\boldsymbol{z}|^2 e^{(3-p-2(3-p))|x|} = |\boldsymbol{z}|^2 e^{-(3-p)|x|}.$$

For $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1| \gtrsim Q$, like in (1.58) we have

$$\begin{aligned} |II| &\lesssim \left|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right|^{p-1} e^{-2k_2|x|} \sim |z_2\psi_2|^{p-1} e^{-2k_2|x|} \lesssim |z_2|^p |z_2|^{-1} e^{-\frac{(3-p)^2}{2}|x|} \\ &\lesssim |\boldsymbol{z}|^p e^{\frac{1}{2}\left(p-1-(3-p)^2\right)|x|} \le |\boldsymbol{z}|^p e^{-|x|}. \end{aligned}$$

So we can use dominated convergence obtaining

$$\lim_{\boldsymbol{z}\to 0} \frac{1}{|\boldsymbol{z}|} \left\langle \mathbf{J}\left(\partial_{z_{aA}}\hat{\boldsymbol{R}}(\boldsymbol{z}) - D_{\boldsymbol{z}}\partial_{z_{aA}}\hat{\boldsymbol{R}}(0)\boldsymbol{z}\right), D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta \right\rangle = \lim_{\boldsymbol{z}\to 0} O\left(|\boldsymbol{z}|^{p-1}\right) = 0.$$

This shows that the function in (1.60) is differentiable in $\boldsymbol{z}=0$ for any $a=1,2$ and any $A=R,I$ or, equivalently, that the right-hand side in (1.48), and so also $\tilde{\boldsymbol{z}}_R$, is twice differentiable in $\boldsymbol{z}=0$.

We set now

$$\tilde{\boldsymbol{z}} := \tilde{\boldsymbol{z}}_0 + \tilde{\boldsymbol{z}}_R \tag{1.61}$$

and we write part of the square component of the Taylor expansion at 0 of $\boldsymbol{R}(\boldsymbol{z})$ considering

$$z_2^2\mathbf{G}_2(x)+\overline{z}_2^2\overline{\mathbf{G}}_2(x), \qquad \text{where} \qquad \mathbf{G}_2 := D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\,\partial_{z_2}^2\tilde{\boldsymbol{z}}_R\big|_{\boldsymbol{z}=0} - f''(Q)\,\varphi_2^2\mathbf{j}, \tag{1.62}$$

where, using the notation above (1.49) we applied $\partial_{z_2} := 2^{-1}\left(\partial_{z_{2R}} - \mathrm{i}\partial_{z_{2I}}\right)$ and we used (1.59) with $\boldsymbol{\zeta}^1=\boldsymbol{\zeta}^2=(0,\frac{1-\mathrm{i}}{2})^\intercal$ to get

$$\partial_{z_2}^2\hat{\boldsymbol{R}}(\boldsymbol{z})\Big|_{\boldsymbol{z}=0} = -f''(Q)\,\varphi_2^2\mathbf{j}.$$

We remark that $f''(Q)\,\varphi_2^2=-p(p-1)Q^{p-2}\varphi_2^2\sim -e^{-|x|}$ for $x\to\infty$.

Remark 1.7. Notice that $\partial_{z_2}^2\tilde{\boldsymbol{z}}_R\big|_{\boldsymbol{z}=0}$ is well defined by the double differentiability of $\tilde{\boldsymbol{z}}_R$ in $\boldsymbol{z}=0$, but its exact formula is immaterial because its contribution to the Fermi golden rule is null and its only purpose is to give the orthogonality in (1.46), which in turn is quite useful later in Lemma 3.3. For the sake of completeness we show how to compute $\tilde{\boldsymbol{z}}_R$, which in turn explicates the expression in (1.62): by the definition of $\boldsymbol{\Phi}(\boldsymbol{z})$ in (1.28) and (1.29), the left-hand side in (1.48) is constant in $\boldsymbol{z}$. in particular the left-hand side of (1.48) is equal to

$$\begin{aligned}
&\langle \mathbf{J}D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\,\tilde{\boldsymbol{z}}_R, D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\,\Theta\rangle = \left(2\mathrm{Re}\left(\tilde{z}_{R,1}\mathbf{J}\boldsymbol{\Phi}_\mathbf{0}+\tilde{z}_{R,2}\mathbf{J}\boldsymbol{\Phi}_\mathbf{2}\right),\ 2\mathrm{Re}\left(\Theta_1\boldsymbol{\Phi}_\mathbf{0}+\Theta_2\boldsymbol{\Phi}_\mathbf{2}\right)\right)\\
&\quad = 4\left(\mathrm{Re}\tilde{z}_{R,1}\frac{\mathbf{JY}_+}{2}+\mathrm{Im}\tilde{z}_{R,1}\frac{\mathbf{JY}_-}{2}-\mathrm{Re}\tilde{z}_{R,2}\mathbf{j}\varphi_2-\mathrm{Im}\tilde{z}_{R,2}\lambda\mathbf{i}\varphi_2 \ \middle|\right.\\
&\qquad\quad \left.\mathrm{Re}\Theta_1\frac{\mathbf{Y}_+}{2}+\mathrm{Im}\Theta_1\frac{\mathbf{Y}_-}{2}+\mathrm{Re}\Theta_2\mathbf{i}\varphi_2-\mathrm{Im}\Theta_2\lambda\mathbf{j}\varphi_2\right).
\end{aligned} \tag{1.63}$$

Letting $\boldsymbol{\Theta}\in\{(1,0),(\mathrm{i},0),(0,1),(0,\mathrm{i})\}$ we obtain (cf. (1.15), (1.18), and (1.44))

$$\begin{aligned}
&\mathrm{Re}\tilde{z}_{R,1} = -\frac{1}{2\nu_0\left\|\varphi_0\right\|_{L^2}^2}\left(\mathbf{J}\hat{\boldsymbol{R}}(\boldsymbol{z})\ \middle|\ \mathbf{Y}_-\right), &&\mathrm{Im}\tilde{z}_{R,1} = \frac{1}{2\nu_0\left\|\varphi_0\right\|_{L^2}^2}\left(\mathbf{J}\hat{\boldsymbol{R}}(\boldsymbol{z})\ \middle|\ \mathbf{Y}_+\right),\\
&\mathrm{Re}\tilde{z}_{R,2} = \frac{1}{4\lambda\left\|\varphi_2\right\|_{L^2}^2}\left(\mathbf{J}\hat{\boldsymbol{R}}(\boldsymbol{z})\ \middle|\ \begin{pmatrix}0\\-2\lambda\varphi_2\end{pmatrix}\right)=0, &&\mathrm{Im}\tilde{z}_{R,2} = \frac{1}{4\lambda\left\|\varphi_2\right\|_{L^2}^2}\left(\mathbf{J}\hat{\boldsymbol{R}}(\boldsymbol{z})\ \middle|\ \begin{pmatrix}2\varphi_2\\0\end{pmatrix}\right).
\end{aligned} \tag{1.64}$$

Remark 1.8 (Fermi golden rule). Notice that the existence of a function $\mathbf{g}_2\in L^\infty(\mathbb{R})$ which is a solution of $\mathbf{JL}_0\mathbf{g}_2=\mathrm{i}2\lambda\mathbf{g}_2$ such that

$$\langle \mathbf{JG}_2,\mathbf{g}_2\rangle \neq 0 \tag{1.65}$$

is equivalent to Assumption 1.1 since $\mathbf{g}_2=(1,2\mathrm{i}\lambda)^\intercal\ c\,g$ with $L_0g=4\lambda^2g$ and $c\in\mathbb{C}\backslash\{0\}$ and the definition of $\boldsymbol{\Phi}(\boldsymbol{z})$ gives a cancellation and leads to

$$\langle \mathbf{JG}_2,\mathbf{g}_2\rangle = -\overline{c}\,p\,(p-1)\left\langle Q^{p-2}\varphi_2^2, g\right\rangle,$$

where we used

$$\langle \mathbf{J} D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z}) \boldsymbol{\Theta}, \mathbf{g}_2 \rangle = 0 \text{ for any } \boldsymbol{\Theta} \in \mathbb{C}^2$$

by the well-known, elementary to check by a simple integration by parts, simplectic orthogonality between discrete and continuous modes of $\mathbf{JL}_0$.

We end this section observing that from (1.43) and (1.45) we have

$$D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z}) \tilde{\boldsymbol{z}} = \mathbf{J} \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\Phi}(\boldsymbol{z}) + f(\boldsymbol{\Phi}(\boldsymbol{z})_1) \mathbf{j} + \boldsymbol{R}(\boldsymbol{z}). \tag{1.66}$$

2. Main estimates and proof of Theorem 1.3. Consider a solution $\boldsymbol{u} \in C^0\left([0, +\infty]; \boldsymbol{\mathcal{H}}^1_{\mathrm{even}}\right)$ gravitating around $Q\mathbf{i}$ like in (1.31). From the spectral decomposition of $\boldsymbol{u} - Q\mathbf{i}$, we have $\boldsymbol{u} = \boldsymbol{\Phi}(\boldsymbol{z}) + \boldsymbol{\eta}$ with $\boldsymbol{\eta} \in C^0([0, +\infty), \boldsymbol{\mathcal{H}}^1_c)$ and the NLKG (1.2) is rewritten

$$\begin{aligned} \dot{\boldsymbol{\eta}} + D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z})(\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}) + \cancel{D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z}) \tilde{\boldsymbol{z}}} &= \mathbf{JL}_0 \boldsymbol{\eta} + \cancel{\mathbf{JL}_0 \boldsymbol{\Phi}(\boldsymbol{z})} + \cancel{f(\boldsymbol{\Phi}(\boldsymbol{z})_1) \mathbf{j}} + \cancel{\boldsymbol{R}(\boldsymbol{z})} - \boldsymbol{R}(\boldsymbol{z}) \\ &+ \underbrace{\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)\right) \eta_1 \mathbf{j} + \left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) \eta_1\right) \mathbf{j}}_{:= \boldsymbol{F}_Q(\boldsymbol{z}, \boldsymbol{\eta})} \end{aligned} \tag{2.1}$$

for $\boldsymbol{R}(\boldsymbol{z})$ defined in (1.44) and (1.45) and where the canceling follows from (1.44). Equivalently, we have also

$$\dot{\boldsymbol{\eta}} + D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z})(\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}) = \mathbf{J} \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\eta} - \boldsymbol{R}(\boldsymbol{z}) + \left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1)\right) \mathbf{j}. \tag{2.2}$$

In what follows we will consider constants $A, B, \varepsilon > 0$ satisfying

$$\log(\delta^{-1}) \gg \log\left(\epsilon^{-1}\right) \gg A \gg B^2 \gg B \gg \exp\left(\varepsilon^{-1}\right) \gg 1. \tag{2.3}$$

We will fix A, B, and ε satisfying the above relation always retaking A, B larger and ε smaller if necessary. Then, we will take ϵ sufficiently small and finally choose δ sufficiently small. Exploiting the double differentiability of $\tilde{\boldsymbol{z}}_R$ in $\boldsymbol{z} = 0$, the choices of the constants are made so that

$$A \left| \tilde{\boldsymbol{z}}_R(\boldsymbol{z}) - \frac{D^2_{\boldsymbol{z}} \tilde{\boldsymbol{z}}_R(0)(\boldsymbol{z}, \boldsymbol{z})}{2} \right| = o\left(\delta^2\right) \text{ for } |\boldsymbol{z}| \lesssim \delta. \tag{2.4}$$

We will denote by $o_\alpha(1)$ constants depending on $\alpha > 0$ such that

$$o_\alpha(1) \xrightarrow{\alpha \to 0^+} 0. \tag{2.5}$$

We will consider the norms

$$\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} := \left\| \operatorname{sech}\left(\frac{2}{A} x\right) \eta_1' \right\|_{L^2} + A^{-1} \left\| \operatorname{sech}\left(\frac{2}{A} x\right) \boldsymbol{\eta} \right\|_{L^2}, \tag{2.6}$$

$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2_{-\kappa}} := \|\operatorname{sech}(\kappa x) \boldsymbol{\eta}\|_{L^2}. \tag{2.7}$$

We will prove the following continuation argument.

Proposition 2.1. *Under Assumption* 1.8, *for any small* $\epsilon > 0$ *there exists a* $\delta_0 := \delta_0(\epsilon) \ll \epsilon$ *such that if a solution of* (1.2) *satisfying* (1.31), *if in* $I = [0,T]$ *we have*

$$\|\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}\|^2_{L^2(I)} + \|z_1\|^2_{L^2(I)} + \|z_2\|^4_{L^4(I)} + \|\boldsymbol{\eta}\|^2_{L^2(I,\boldsymbol{\Sigma}_A \cap \boldsymbol{L}^2_{-\kappa})} \le \epsilon^2, \tag{2.8}$$

then for $\delta \in (0,\delta_0)$ *inequality* (2.8) *holds for* ϵ^2 *replaced by* $o_\varepsilon(1)\epsilon^2$ *where* $o_\varepsilon(1) \xrightarrow{\varepsilon \to 0^+} 0$.

It is elementary that Proposition 2.1 follows from the following Propositions 2.2 to 2.5, where we always assume that our solution satisfies the stability condition (1.31).

Proposition 2.2. *Under the assumptions of Proposition* 2.1 *there exists a* $\kappa > 0$ *such that*

$$\|\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}\|^2_{L^2(I)} = \delta^{2(p-1)} \|\boldsymbol{\eta}\|^2_{L^2\left(I;\boldsymbol{L}^2_{-\kappa}\right)}. \tag{2.9}$$

Proposition 2.3 (radiation damping). *We have*

$$\|z_1\|^2_{L^2(I)} + \|z_2\|^4_{L^4(I)} = o(\epsilon^2). \tag{2.10}$$

Proposition 2.4 (first virial estimate). *We have*

$$\|\boldsymbol{\eta}\|^2_{L^2(I,\boldsymbol{\Sigma}_A)} \lesssim \delta^2 + \|\boldsymbol{\eta}\|^2_{L^2(I,\boldsymbol{L}^2_{-\kappa})} + \|\mathbf{z}\|^4_{L^4(I)}. \tag{2.11}$$

Proposition 2.5 (second virial estimate). *We have*

$$\|\boldsymbol{\eta}\|^2_{L^2(I,\boldsymbol{L}^2_{-\kappa})} \lesssim \delta^2 + o_\varepsilon(1)\|\boldsymbol{\eta}\|^2_{L^2(I,\boldsymbol{\Sigma}_A)} + \|\boldsymbol{z}\|^4_{L^4(I)}. \tag{2.12}$$

Proof of Theorem 1.3. By continuity, Proposition 2.1 implies that inequality (2.8) is valid with $I = \mathbb{R}_+$. This implies, adjusting ϵ, inequality (1.33) for $I = \mathbb{R}_+$.

$$\int_{\mathbb{R}} \|\boldsymbol{\eta}(t)\|^2_{\boldsymbol{\mathcal{H}}^1_{-a}(\mathbb{R})} \mathrm{d}t + |\boldsymbol{z}|^4_{L^4(\mathbb{R})} \le \epsilon^2. \tag{2.13}$$

From (1.31), (1.61) and (3.8) below, we have $\dot{\boldsymbol{z}} \in L^\infty(\mathbb{R},\mathbb{C}^2)$ that, with (1.33) implies (1.35). To prove (1.34), let

$$\begin{aligned} \mathsf{a} := \|\boldsymbol{\eta}\|^2_{\mathcal{H}^1_{-a}} &= \left\langle \operatorname{sech}^2(ax)\,\eta_1',\ \eta_1' \right\rangle + \left\langle \operatorname{sech}^2(ax)\,\eta_1,\ \eta_1 \right\rangle + \left\langle \operatorname{sech}^2(ax)\,\eta_2,\ \eta_2 \right\rangle \\ &=: \ \mathsf{a}_1(t) + \mathsf{a}_2(t) + \mathsf{a}_3(t). \end{aligned}$$

Using (1.29) and (2.2) we have

$$\dot{\mathsf{a}}_2 = \left\langle \operatorname{sech}^2(ax)\,\eta_1,\ -\tilde{\boldsymbol{\Phi}}\left[\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}\right]_1 + \eta_2 - \boldsymbol{R}(\boldsymbol{z})_1 \right\rangle.$$

By (1.44), (1.45), (1.47), and (3.8) we obtain

$$|\dot{\mathsf{a}}_2| \lesssim \delta^2.$$

Proceeding similarly,

$$\begin{aligned} \dot{\mathsf{a}}_1 &= \ -2\left\langle \operatorname{sech}^2(ax)\,\eta_1'',\ \eta_2 \right\rangle + \mathcal{O}\left(\delta^2\right), \\ \dot{\mathsf{a}}_3 &= 2\left\langle \operatorname{sech}^2(ax)\,\eta_1'',\ \eta_2 \right\rangle + \mathcal{O}\left(\delta^2\right), \end{aligned} \tag{2.14}$$

where, in the sum of the two terms, the explicit terms in the right-hand side cancel out, so that we obtain $|\dot{\mathsf{a}}| = \mathcal{O}\left(\delta^2\right)$. This and (1.33) yield (1.34). Notice that the computations in (2.14) are formal, but they can be made rigorous, defining $\mathsf{a}_\varepsilon := \|\left\langle \varepsilon \mathrm{i}\partial_x \right\rangle^{-1} \boldsymbol{\eta}\|^2_{\mathcal{H}^1_{-a}}$ proving $|\dot{\mathsf{a}}_\varepsilon| \le C\delta^2$ with a constant $C > 0$ independent from ε and then letting $\varepsilon \to 0^+$ to get the bound $|\dot{\mathsf{a}}| \le C\delta^2$. Similar regularizations can be used to justify rigorously some of the computations later in the manuscript. ∎

3. Proof of Proposition 2.2: Bounds for the approximate time-derivative of the discrete modes.

NOTATION 3.1. *We fix an even function* $\chi \in C_0^\infty(\mathbb{R},[0,1])$ *satisfying* $1_{[-1,1]} \le \chi \le 1_{[-2,2]}$ *and* $x\chi'(x) \le 0$ *and set* $\chi_A := \chi(\cdot/A)$.

LEMMA 3.2. *For* $\delta > 0$, $\boldsymbol{z} \in D_{\mathbb{C}^2}(0,\delta)$, *and* $\boldsymbol{\eta} \in D_{\mathcal{H}^1_c}(0,\delta)$ *then, for any* $A > 0$ *and* $\kappa > 0$,

$$\|\operatorname{sech}(\kappa x)\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)\right)\eta_1\|_{L^2} \lesssim \delta^{p-1}\|\operatorname{sech}((\kappa + (p-1)k_2)x)\eta_1\|_{L^2}, \tag{3.1}$$

$$\|\chi_A\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)\right)\eta_1\|_{L^1} \lesssim \delta^{p-1}\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}, \tag{3.2}$$

$$\|\operatorname{sech}(\kappa x)\left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\|_{L^2} \lesssim \delta^{p-1}\|\operatorname{sech}(\kappa x)\eta_1\|_{L^2}, \tag{3.3}$$

$$\|\chi_A\left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\|_{L^1} \lesssim A^{1-\frac{p}{2}}\delta^{p-1}\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}. \tag{3.4}$$

Proof.

Step 1 (proof of (3.1) and (3.2)). First of all in the points where $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1| \ll Q$,

$$|f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)| \lesssim \left|f''(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right| \sim Q^{p-2}\left|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right|^{1-(2-p)+(2-p)} \le \left|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right|^{p-1}.$$

If instead $|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1| \gtrsim Q$, then

$$|f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)| \lesssim \left|f'\left(\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)\right| \sim \left|\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right|^{p-1}.$$

Thus, by (1.16) we have

$$|\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)\right)\eta_1| \lesssim |\boldsymbol{z}|^{p-1}\operatorname{sech}((p-1)k_2 x)|\eta_1|, \tag{3.5}$$

which yields (3.1). We prove now (3.2). Using (3.5) and $\operatorname{sech}\left(\frac{2}{A}x\right) \sim 1$ on $[-A,A]$,

$$\begin{aligned}\|\chi_A\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)\right)\eta_1\|_{L^1} &\lesssim |\boldsymbol{z}|^{p-1}\int_{-A}^{A}\operatorname{sech}((p-1)k_2 x)|\eta_1(x)|\mathrm{d}x \\ &\lesssim \delta^{p-1}\left\|\operatorname{sech}\left(\frac{2}{A}\,x\right)\eta_1\right\|_{L^2}.\end{aligned}$$

Step 2 (proof of (3.3) and (3.4)). If $|\eta_1| \gtrsim |\boldsymbol{\Phi}(\boldsymbol{z})_1|$ we have

$$|f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1| \lesssim |\eta_1|^p. \tag{3.6}$$

In the points where $|\eta_1| \ll |\boldsymbol{\Phi}(\boldsymbol{z})_1|$, for some $t \in (0,1)$ we have

$$\begin{aligned}|f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1| &= |\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1 + t\eta_1) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\right)\eta_1| \\ &\lesssim \left|f''(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1^2\right| \sim |\boldsymbol{\Phi}(\boldsymbol{z})_1|^{p-2}\eta_1^2 \le |\eta_1|^p.\end{aligned} \tag{3.7}$$

Inequality (3.3) follows immediately. By $\operatorname{sech}\left(\frac{2}{A}x\right)\sim 1$ in $\operatorname{supp}\chi_A$, (1.31) and Hölder inequality, we get

$$\|\chi_A|\eta_1|^p\|_{L^1} \lesssim A^{1-\frac{p}{2}}\left\|\operatorname{sech}\left(\frac{2}{A}\ x\right)\eta_1\right\|_{L^{\frac{2}{p}}} \lesssim A^{1-\frac{p}{2}}\delta^{p-1}\left\|\operatorname{sech}\left(\frac{2}{A}\ x\right)\eta_1\right\|_{L^2},$$

which proves (3.4).

LEMMA 3.3. *With the same hypothesis of Lemma* 3.2 *and for any* $\kappa\in(0,k_2)$ *the following bound holds true:*

$$|\dot{\boldsymbol{z}}(t)-\tilde{\boldsymbol{z}}(t)|\lesssim\delta^{p-1}\left\|\operatorname{sech}(\kappa x)\boldsymbol{\eta}(t)\right\|_{L^2(\mathbb{R})}. \tag{3.8}$$

Proof. We apply $\langle\mathbf{J}\bullet,D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\rangle$ to (2.1). By $\langle\mathbf{J}\dot{\boldsymbol{\eta}},D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\rangle=0$, due to $\boldsymbol{\eta}\in\boldsymbol{\mathcal{H}}_c^1$, and $D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})$ independent of $\boldsymbol{z}$, we get

$$\begin{aligned}\langle\mathbf{J}D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})(\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}),D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\rangle&=-\cancel{\langle\mathbf{J}\boldsymbol{\eta},\mathbf{J}\mathbf{L}_0D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\rangle}-\cancel{\langle\mathbf{J}\boldsymbol{R}(\boldsymbol{z}),D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\rangle}\\&+\left\langle\mathbf{J}\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(Q)\right)\eta_1\mathbf{j},D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\right\rangle+\left\langle\mathbf{J}\left(f\left(\boldsymbol{\Phi}(\boldsymbol{z})_1+\eta_1\right)-f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\mathbf{j},D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta\right\rangle,\end{aligned}$$

where the second cancellation follows from (1.46) and the first, by (1.28), from (cf. (1.41))

$$\mathbf{J}\mathbf{L}_0D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\Theta=D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\ \ \tilde{\boldsymbol{z}}_0[\Theta],$$

and the fact that $\boldsymbol{\eta}\in\mathcal{H}_c^1$. Using Lemma 3.2, we obtain

$$\begin{aligned}|\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}|&\lesssim\left\|\operatorname{sech}(\kappa x)\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(Q)\right)\eta_1\right\|_{L^2}\\&\quad+\left\|\operatorname{sech}(\kappa x)\left(f\left(\boldsymbol{\Phi}(\boldsymbol{z})_1+\eta_1\right)-f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\right\|_{L^2}\\&\lesssim\delta^{p-1}\left\|\operatorname{sech}(\kappa x)\eta_1\right\|_{L^2}.\end{aligned}$$

∎

The proof of Proposition 2.2 follows from Lemma 3.3 setting $\kappa:=k_2/2$ and integrating-in-time.

4. Proof of Proposition 2.3: The Fermi golden rule. Here the Fermi golden rule does not refer any more to the condition in (1.65), but rather to the proof of dissipation of z_2 due, ultimately, to the nonlinear interaction of z_2 with the continuous modes of $\boldsymbol{u}$. There is an old history, starting from [27, 2], with many significant contributions, like [28, 25]. Here we use the argument of Kowalczyk and Martel [17].

To prove Proposition 2.3, for the $\mathbf{g}_2$ in Assumption 1.8, we consider

$$\mathcal{J}_{\mathrm{FGR}}:=\left\langle\mathbf{J}\boldsymbol{\eta},\chi_A\left(z_2^2\mathbf{g}_2+\overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle. \tag{4.1}$$

LEMMA 4.1. *We have*

(4.2)

$$|z_2|^4\lesssim-\dot{\mathcal{J}}_{\mathrm{FGR}}+\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Re}\left\langle 2^{-1}\lambda^{-2}z_2^4\mathbf{J}\mathbf{G}_2,\overline{\mathbf{g}}_2\right\rangle+\delta^2|z_1|^2+A^{-1}\left(\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2+\|\operatorname{sech}(k_2x)\boldsymbol{\eta}\|_{L^2}^2\right).$$

Proof. Differentiating $\mathcal{J}_{\mathrm{FGR}}$ and using (2.1), we have

$$\begin{aligned}\dot{\mathcal{J}}_{\mathrm{FGR}}&=\left\langle\mathbf{J}\dot{\boldsymbol{\eta}},\chi_A\left(z_2^2\mathbf{g}_2+\overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle+\left\langle\mathbf{J}\boldsymbol{\eta},\chi_AD_{\boldsymbol{z}}\left(z_2^2\mathbf{g}_2+\overline{z}_2^2\overline{\mathbf{g}}_2\right)\tilde{\boldsymbol{z}}_0\right\rangle\\&\quad+\left\langle\mathbf{J}\boldsymbol{\eta},\chi_AD_{\boldsymbol{z}}\left(z_2^2\mathbf{g}_2+\overline{z}_2^2\overline{\mathbf{g}}_2\right)\widetilde{\boldsymbol{z}}_R\right\rangle\\&\quad+\left\langle\mathbf{J}\boldsymbol{\eta},\chi_AD_{\boldsymbol{z}}\left(z_2^2\mathbf{g}_2+\overline{z}_2^2\overline{\mathbf{g}}_2\right)(\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}})\right\rangle=:A_1+A_2+A_3+A_4.\end{aligned}$$

By Lemma 3.3, (2.3), and the elementary inequality

$$\|\mathrm{sech}\,(\kappa x)\,\eta_1\|_{L^2} \le \left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2} \le A\,\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}\,, \tag{4.3}$$

we have

$$\begin{aligned}|A_4| &\lesssim \|\boldsymbol{\eta}\chi_A\|_{L^1}\delta^2\,|\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}|_{\mathbb{C}^2} \lesssim \delta^{p+1}A^{\frac12}\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\boldsymbol{\eta}\right\|_{L^2}\|\mathrm{sech}\,(k_2x)\,\eta_1\|_{L^2}\\ &\lesssim \delta^{p+1}A^{\frac52}\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A} \lesssim A^{-1}\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}.\end{aligned}$$

We remark that

$$A_2 = \left\langle \mathbf{J}\boldsymbol{\eta}, \chi_A\left(2\mathrm{i}\lambda z_2^2\mathbf{g}_2 - 2\mathrm{i}\lambda\overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle. \tag{4.4}$$

By (2.1) and with the cancellation in the first line due to (4.4) and $\mathbf{J}\mathbf{L}_1\mathbf{g}_2 = 2\mathrm{i}\lambda\mathbf{g}_2$,

$$\begin{aligned}A_1 + A_2 = &-\left\langle \mathbf{J}D_{\boldsymbol{z}}\boldsymbol{\Phi}\,(\boldsymbol{z})\,(\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}), \chi_A\left(z_2^2\mathbf{g}_2 + \overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle - \xcancel{\left\langle \mathbf{J}\boldsymbol{\eta}, \chi_A\mathbf{J}\mathbf{L}_0\left(z_2^2\mathbf{g}_2 + \overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle + A_2}\\ &-\left\langle \boldsymbol{\eta}, [\mathbf{L}_0,\ \chi_A]\left(z_2^2\mathbf{g}_2 + \overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle + \left\langle \mathbf{J}\boldsymbol{F}_Q\,(\boldsymbol{z},\boldsymbol{\eta}), \chi_A\left(z_2^2\mathbf{g}_2 + \overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle\\ &-\left\langle \mathbf{J}\boldsymbol{R}\,(\boldsymbol{z}), \chi_A\left(z_2^2\mathbf{g}_2 + \overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle =: A_{11} + A_{12} + A_{13} + A_{14}.\end{aligned}$$

By Lemma 3.3, (2.3), and (4.3), we have

$$|A_{11}| \lesssim |\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}||z_2|^2 \lesssim \delta^{p-1}\left(|\boldsymbol{z}|^4 + \left\|\mathrm{sech}\left(k_2x/2\right)\boldsymbol{\eta}\right\|^2_{L^2}\right) \le A^{-1}\left(|\boldsymbol{z}|^4 + \|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}\right).$$

By $[\mathbf{L}_0, \chi_A] = \left(\begin{smallmatrix} -\chi_A'' - 2\chi_A'\partial_x & 0\\ 0 & 0\end{smallmatrix}\right)$, for a fixed small $\delta_2 > 0$ we have

$$\begin{aligned}|A_{12}| &\lesssim |z_2|^2\left(\|\chi_A''\eta_1\|_{L^1} + \|\chi_A'\eta_1'\|_{L^1}\right)\\ &\lesssim |z_2|^2\left(A^{-3/2}\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2} + A^{-1/2}\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}\right)\\ &\le \delta_2|z_2|^4 + C(\delta_2)A^{-1}\left(\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\|^2_{L^2} + A^{-2}\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\|^2_{L^2}\right),\end{aligned}$$

so that we can absorb the $\delta_2|z_2|^4$ term in the left-hand side of (4.2).

By Lemma 3.2, (3.2), and (3.4), we have

$$|A_{13}| \lesssim |z_2|^2\,\|\chi_A\boldsymbol{F}_Q\,(\boldsymbol{\eta},\boldsymbol{z})\|_{L^1} \lesssim A\delta^{p-1}\left(|\boldsymbol{z}|^4 + \|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}\right).$$

The key term is the following:

$$\begin{aligned}A_{14} = &-2^{-1}\left\langle \mathbf{J}D^2_{\boldsymbol{z}}\boldsymbol{R}[0]\boldsymbol{z}^2, \chi_A\left(z_2^2\mathbf{g}_2 + \overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle\\ &+\left\langle \mathbf{J}\left(\frac12 D^2_{\boldsymbol{z}}\boldsymbol{R}[0]\,(\boldsymbol{z},\boldsymbol{z}) - \boldsymbol{R}\,(\boldsymbol{z})\right), \chi_A\left(z_2^2\mathbf{g}_2 + \overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle =: A_{141} + A_{142}.\end{aligned}$$

We bound the term A_{142}. We have

$$\begin{aligned}D^2_{\boldsymbol{z}}\boldsymbol{R}[0]\,(\boldsymbol{z},\boldsymbol{z}) &= D^2_{\boldsymbol{z}}\left[D_{\boldsymbol{z}}\boldsymbol{\Phi}\,(\boldsymbol{z})\ \tilde{\boldsymbol{z}}_R\,(\boldsymbol{z}) - \left(f(\boldsymbol{\Phi}\,(\boldsymbol{z})_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)\mathbf{j}\right]\Big|_{\boldsymbol{z}=0}(\boldsymbol{z},\boldsymbol{z})\\ &= D_{\boldsymbol{z}}\boldsymbol{\Phi}\,(\boldsymbol{z})\,D^2_{\boldsymbol{z}}\tilde{\boldsymbol{z}}_R\,[0]\,(\boldsymbol{z},\boldsymbol{z}) - f''\,(Q)\,\mathbf{j}\ \tilde{\boldsymbol{\Phi}}\,[\boldsymbol{z}]^2,\end{aligned}$$

so that

$$
\begin{aligned}
&\frac{1}{2}D_{\boldsymbol{z}}^2\boldsymbol{R}[0]\left(\boldsymbol{z},\boldsymbol{z}\right)-\boldsymbol{R}\left(\boldsymbol{z}\right)=D_{\boldsymbol{z}}\boldsymbol{\Phi}\left(\boldsymbol{z}\right)\left(\frac{1}{2}D_{\boldsymbol{z}}^2\tilde{\boldsymbol{z}}_R\left[0\right]\left(\boldsymbol{z},\boldsymbol{z}\right)-\tilde{\boldsymbol{z}}_R\right)\\
&\qquad\qquad -\left(f(Q)+f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1+\frac{f''\left(Q\right)}{2}\tilde{\boldsymbol{\Phi}}\left[z\right]^2-f\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1\right)\right)\mathbf{j}.
\end{aligned}
\tag{4.5}
$$

In supp $\chi(\cdot/A)\subseteq[-2A,2A]$, we have $|\boldsymbol{\Phi}(\boldsymbol{z})_1|\ll Q$, so that by (4.5), $|D_{\boldsymbol{z}}\boldsymbol{\Phi}\left(\boldsymbol{z}\right)(x)\cdot w|\lesssim e^{-k_2|x|}|w|$, and (2.4), we have

$$
\begin{aligned}
|A_{142}|\lesssim|\boldsymbol{z}|^2\int_{-2A}^{2A}&\left\{\left|\frac{1}{2}D_{\boldsymbol{z}}^2\tilde{\boldsymbol{z}}_R\left[0\right]\left(\boldsymbol{z},\boldsymbol{z}\right)-\tilde{\boldsymbol{z}}_R\right|e^{-k_2|x|}\right.\\
&\left.+\left|f(Q)+f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1+\frac{f''\left(Q\right)}{2}\tilde{\boldsymbol{\Phi}}\left[z\right]^2-f\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1\right)\right|\right\}\mathrm{d}x\\
\lesssim o\left(|\boldsymbol{z}|^4\right)&+|\boldsymbol{z}|^5\left\|f'''(Q)e^{-3k_2|x|}\right\|_{L^1(\mathbb{R})},
\end{aligned}
$$

so that, by $|f'''(Q\left(x\right))e^{-3k_2|x|}|\sim e^{-k_2|x|}$, we obtain

$$
|A_{142}|\lesssim o\left(|\boldsymbol{z}|^4\right). \tag{4.6}
$$

Next we write

$$
\begin{aligned}
A_{141}=&-2^{-1}\left\langle\mathbf{J}\left(z_2^2\mathbf{G}_2+\overline{z}_2^2\overline{\mathbf{G}}_2\right),z_2^2\mathbf{g}_2+\overline{z}_2^2\overline{\mathbf{g}}_2\right\rangle\\
&-2^{-1}\left\langle\mathbf{J}\left(z_2^2\mathbf{G}_2+\overline{z}_2^2\overline{\mathbf{G}}_2\right),(1-\chi_A)\left(z_2^2\mathbf{g}_2+\overline{z}_2^2\overline{\mathbf{g}}_2\right)\right\rangle+O\left(z_1^4\right)+\delta_2O\left(z_2^4\right)
\end{aligned}
$$

for a $\delta_2>0$ arbitrarily small but fixed. The first term in the second line can be absorbed in the third. So,

$$
A_{142}=-|z_2|^4\left\langle\mathbf{J}\mathbf{G}_2,\mathbf{g}_2\right\rangle-2\mathrm{Re}\left\langle z_2^4\mathbf{J}\mathbf{G}_2,\overline{\mathbf{g}}_2\right\rangle+O\left(z_1^4\right)+\delta_2O\left(z_2^4\right).
$$

By elementary computation, for $\tilde{\boldsymbol{z}}=:((\tilde{\boldsymbol{z}})_1,(\tilde{\boldsymbol{z}})_2)^{\intercal}$ and using (1.41) and (1.61), we have

$$
\begin{aligned}
\frac{d}{\mathrm{d}t}z_2^4&=4\mathrm{i}\lambda(\tilde{\boldsymbol{z}})_2z_2^3+4\mathrm{i}\lambda\left(\dot{z}_2-(\tilde{\boldsymbol{z}})_2\right)z_2^3\\
&=-4\lambda^2z_2^4+4\mathrm{i}\lambda(\tilde{\boldsymbol{z}}_R)_2z_2^3+4\mathrm{i}\lambda\left(\dot{z}_2-(\tilde{\boldsymbol{z}})_2\right)z_2^3.
\end{aligned}
$$

Hence we conclude the following, which with (3.8) yields the statement

$$
\begin{aligned}
A_{142}=&-|z_2|^4\left\langle\mathbf{J}\mathbf{G}_2,\mathbf{g}_2\right\rangle+\frac{d}{\mathrm{d}t}\mathrm{Re}\left\langle2^{-1}\lambda^{-2}z_2^4\mathbf{J}\mathbf{G}_2,\overline{\mathbf{g}}_2\right\rangle\\
&+O\left(z_1^4\right)+\delta_2O\left(z_2^4\right)+O\left(|\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}|^2\right).
\end{aligned}
$$

∎

Lemma 4.2. *We have*

$$
|z_1|^2\lesssim\frac{d}{\mathrm{d}t}\mathrm{Re}z_1^2+|z_2|^4+\delta^{2(p-1)}\|\mathrm{sech}(k_2x)\boldsymbol{\eta}\|_{L^2}^2. \tag{4.7}
$$

Proof. Using (1.41) and (1.61), we have

$$\frac{d}{\mathrm{d}t} z_1^2 = 2\nu_0 |z_1|^2 + 2z_1(\dot{z}_1 - \nu_0 \overline{z}_1) = 2\nu_0 |z_1|^2 + 2z_1(\dot{z}_1 - (\tilde{\boldsymbol{z}})_1) + 2z_1(\tilde{\boldsymbol{z}}_R)_1.$$

Taking the real part of the above equation and using Young inequality, for any $\theta > 0$ we get

$$\begin{aligned}
2\nu_0 |z_1|^2 &\le \frac{d}{\mathrm{d}t}\mathrm{Re} z_1^2 + 2\theta |z_1|^2 + \frac{1}{\theta}|\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}|^2 + \frac{1}{\theta}|\tilde{\boldsymbol{z}}_R|^2 \\
&\overset{(1.47)}{\le} \frac{d}{\mathrm{d}t}\mathrm{Re} z_1^2 + 2\theta |z_1|^2 + \frac{1}{\theta}|\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}|^2 + \frac{2C_1^2}{\theta}|\boldsymbol{z}_1|^4 + \frac{2C_1^2}{\theta}|\boldsymbol{z}_2|^4 \\
&\le \frac{d}{\mathrm{d}t}\mathrm{Re} z_1^2 + (2\theta + \frac{2C_1^2}{\theta}\delta^2)|z_1|^2 + \frac{1}{\theta}|\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}|^2 + \frac{2C_1^2}{\theta}|\boldsymbol{z}_2|^4
\end{aligned}$$

Choosing $\theta := \frac{\nu_0}{4}$ and using $\delta \ll 1$, we get the following, which proves (4.7):

$$\begin{aligned}
|z_1|^2 &\lesssim \frac{1}{\nu_0^2}|\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}|^2 + \frac{1}{\nu_0}\frac{d}{\mathrm{d}t}\mathrm{Re} z_1^2 + \frac{1}{\nu_0^2}|\boldsymbol{z}_2|^4 \\
&\lesssim \frac{\delta^{2(p-1)}}{\nu_0^2}\,\|\mathrm{sech}\,(\kappa x)\,\boldsymbol{\eta}\,(t)\|^2_{L^2(\mathbb{R})} + \frac{1}{\nu_0}\frac{d}{\mathrm{d}t}\mathrm{Re} z_1^2 + \frac{1}{\nu_0^2}|\boldsymbol{z}_2|^4. \qquad \square
\end{aligned}$$

Proof of Proposition 2.3. Integrating (4.2) for $t \in I$ and by (2.8),

$$\begin{aligned}
\|z_2\|^4_{L^4(0,t)} &\lesssim \left| \left(\mathcal{J}_{\mathrm{FGR}} - \mathrm{Re}\left\langle 2^{-1}\lambda^{-2} z_2^4 \mathbf{J}\mathbf{G}_2, \overline{\mathbf{g}}_2\right\rangle\right)\right]_0^t\Big| + \delta^2 \|z_1\|^2_{L^2(0,t)} \\
&\quad + A^{-1/2}\|\boldsymbol{\eta}\|^2_{L^2((0,t),\boldsymbol{\Sigma}_A \cap \boldsymbol{L}^2_{-\kappa})} = o_\epsilon(1)\epsilon^2.
\end{aligned} \tag{4.8}$$

Integrating (4.7) for $t \in I$ and by (2.8) we obtain the following, which completes the proof:

$$\begin{aligned}
\|z_1\|^2_{L^2(0,t)} &\lesssim \left| \mathrm{Re} z_1^2\right]_0^t \Big| + \delta_2^{-1}\|z_2\|^4_{L^4(0,t)} + A^{-1/2}\|\mathrm{sech}(k_2 x)\boldsymbol{\eta}\|^2_{L^2((0,t),L^2)} \\
&= o_\epsilon(1)\epsilon^2. \qquad \square
\end{aligned}$$

5. Proof of Proposition 2.4: The first virial estimate.

NOTATION 5.1. *For the* χ *in Notation* 3.1 *and for* $C > 0$, *we set*

$$\zeta_C(x) := \exp\left(-\frac{|x|}{C}(1 - \chi(x))\right), \quad \varphi_C(x) := \int_0^x \zeta_C^2(y)\,\mathrm{d}y, \quad S_C := \frac{1}{2}\varphi_C' + \varphi_C \partial_x. \tag{5.1}$$

We define functionals (notice that $S_A \mathbf{J}$ and $\sigma_3 \mathbf{J}$ are self-adjoint)

$$\begin{aligned}
\mathcal{I}_{1\mathrm{st},1} &:= \frac{1}{2}\left\langle \mathbf{J}\boldsymbol{\eta}, S_A \boldsymbol{\eta}\right\rangle, \\
\mathcal{I}_{1\mathrm{st},2} &:= \frac{1}{2}\left\langle \mathbf{J}\boldsymbol{\eta}, \sigma_3 \zeta_A^4 \boldsymbol{\eta}\right\rangle, \qquad \text{where} \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}$$

LEMMA 5.2. *There exist* $\delta_0 > 0$, $A_0 > 0$, *and* $0 < \kappa_0 \ll k_2$ *such that if* $\delta \in (0, \delta_0)$, *for any* $\kappa \in (0, \kappa_0)$ *and* $A > A_0$, *we have*

$$\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|^2_{L^2} + A^{-2}\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\right\|^2_{L^2} \lesssim \dot{\mathcal{I}}_{1\mathrm{st},1} + \|\mathrm{sech}\,(\kappa x)\,\boldsymbol{\eta}\|^2_{L^2} + |\boldsymbol{z}|^4. \tag{5.2}$$

Proof. Using (2.2) and the fact that S_A is skew-adjoint, we have

(5.3)
$$\dot{\mathcal{I}}_{1\mathrm{st},1} = \langle \mathbf{J}\dot{\boldsymbol{\eta}}, S_A\boldsymbol{\eta}\rangle = \langle \mathbf{J}S_A D\boldsymbol{\Phi}(\boldsymbol{z})(\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}), \boldsymbol{\eta}\rangle + \langle \partial_x^2\eta_1, S_A\eta_1\rangle$$
$$+ \langle (f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1)), S_A\eta_1\rangle - \langle \mathbf{J}\boldsymbol{R}(\boldsymbol{z}), S_A\boldsymbol{\eta}\rangle =: B_1 + B_2 + B_3 + B_4.$$

By Lemma 1 in Kowalczyk, Martel, and Muñoz [19],

$$B_2 = -\|\zeta_A\eta_1'\|_{L^2}^2 - \frac{1}{2}\int_{\mathbb{R}} \left(\zeta_A\zeta_A'' - (\zeta_A')^2\right)\eta_1^2 dx \text{ with} \tag{5.4}$$

$$\left|\zeta_A\zeta_A'' - (\zeta_A')^2\right| \lesssim \frac{1_{1\le|x|\le 2}}{A}. \tag{5.5}$$

Using Lemma 3.3 to bound $|\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}|$, we have, for $\kappa < k_2$,

$$|B_1| \le |\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}| \|\mathrm{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2} \lesssim \delta^{p-1}\|\mathrm{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2}^2. \tag{5.6}$$

We decompose

(5.7)
$$B_4 = \langle \mathbf{J}S_A D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\tilde{\boldsymbol{z}}_R, \boldsymbol{\eta}\rangle + \left\langle f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1, S_A\eta_1\right\rangle =: B_{41} + B_{42}.$$

By Lemma 1.6 we have

$$\begin{aligned}|B_{41}| &= |\langle \mathbf{J}S_A D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})\tilde{\boldsymbol{z}}_R, \boldsymbol{\eta}\rangle| \lesssim |\tilde{\boldsymbol{z}}_R| \; \|\mathrm{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2} \\ &\lesssim |\boldsymbol{z}|^2\|\mathrm{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2} \le |\boldsymbol{z}|^4 + \|\mathrm{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2}^2.\end{aligned} \tag{5.8}$$

By (1.55) and for $\delta_2 > 0$ small fixed,

$$\begin{aligned}|B_{42}| &\lesssim |\boldsymbol{z}|^2\|e^{-|x|}\varphi_A\cosh\left(\frac{2}{A}x\right)\|_{L^2}\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\|_{L^2} + |\boldsymbol{z}|^2\|\mathrm{sech}(\kappa x)\eta_1\|_{L^2} \\ &\le \frac{2}{\delta_2}|\boldsymbol{z}|^4 + \delta_2\|\mathrm{sech}(\kappa x)\eta_1\|_{L^2}^2 + \delta_2\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\|_{L^2}^2.\end{aligned} \tag{5.9}$$

We decompose

(5.10)
$$B_3 = \langle f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f(\eta_1), S_A\eta_1\rangle + \langle f(\eta_1), S_A\eta_1\rangle =: B_{31} + B_{32}.$$

For the pure nonlinear term B_{32}, we have

$$\begin{aligned}B_{32} =& \frac{1}{p+1}\int_{\mathbb{R}} \partial_x\left(|\eta_1|^{p+1}\right)\varphi_A(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} \zeta_A^2(x)|\eta_1|^{p+1}\,\mathrm{d}x \\ =& \frac{p-1}{2(p+1)}\int_{\mathbb{R}} \zeta_A^2(x)|\eta_1|^{p+1}\,\mathrm{d}x.\end{aligned} \tag{5.11}$$

By a clever but very simple integration by parts Kowalczyk, Martel, and Muñoz [19] (see inequality (36) therein), get the following bound, which is a striking validation of their dispersion theory:

$$|B_{32}| \lesssim \delta^{\frac{2p}{3}}\|(\zeta_A\eta_1)'\|_{L^2}^2. \tag{5.12}$$

Next, since for the concave function $t \to |t|^{p-1}$ for $1 < p \le 2$ we have

$$\left||y+h|^{p-1} - |y|^{p-1}\right| \le |h|^{p-1} \quad \text{for any} \quad y, h \in \mathbb{C},$$

we conclude

$$\begin{aligned}
|f\left(\mathbf{\Phi}\left(\boldsymbol{z}\right)_1+\eta_1\right)-f(\mathbf{\Phi}\left(\boldsymbol{z}\right)_1)-f\left(\eta_1\right)| \le & \int_0^1 \left|f'(\mathbf{\Phi}\left(\boldsymbol{z}\right)_1+\tau\eta_1)-f'(\tau\eta_1)\right|\mathrm{d}\tau\cdot|\eta_1| \\
\le & p\left|\mathbf{\Phi}\left(\boldsymbol{z}\right)_1\right|^{p-1}|\eta_1| \lesssim \mathrm{sech}(k_2(p-1)x)|\eta_1|.
\end{aligned} \tag{5.13}$$

Then, for $\frac{2}{A}<\kappa<k_2(p-1)/2$, we have

$$\begin{aligned}
|B_{31}| \lesssim & \int_{\mathbb{R}} \mathrm{sech}^2(\kappa x)|\eta_1||\eta_1'| + \int_{\mathbb{R}} \mathrm{sech}^2(\kappa x)|\eta_1|^2 \\
\le & \left(1+\tfrac{1}{\delta_2}\right)\|\mathrm{sech}(\kappa x)\eta_1\|_{L^2}^2 + \delta_2\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\|_{L^2}^2.
\end{aligned} \tag{5.14}$$

Collecting all the terms in (5.3), (5.4), (5.7), and (5.10) and their bounds in (5.6), (5.8), (5.9), (5.12), and (5.14), we get

$$\|(\zeta_A\eta_1)'\|_{L^2}^2 \lesssim -\dot{\mathcal{I}}_{1\mathrm{st},1} + \|\mathrm{sech}\left(\kappa x\right)\boldsymbol{\eta}\|_{L^2}^2 + |\boldsymbol{z}|^4 + \delta_2\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}^2.$$

Finally, we use the following, which is analogous to (19) of [17] and is proved in [10] (see formula (6.5)):

$$\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}^2 + \frac{1}{A^2}\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}^2 \lesssim \|(\zeta_A\eta_1)'\|_{L^2}^2 + \frac{1}{A}\|\mathrm{sech}\left(\kappa x\right)\eta_1\|_{L^2}^2. \tag{5.15}$$

This, choosing $\delta_2>0$ small, yields (5.2). ◻

Lemma 5.3. *With the same hypothesis of Lemma* 5.2, *we have*

$$\left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_2\right\|_{L^2}^2 \lesssim \dot{\mathcal{I}}_{1\mathrm{st},2} + \left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}^2 + \left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}^2 + \|\boldsymbol{\eta}\|_{L^2_{-\kappa}}^2 + |\boldsymbol{z}|^4. \tag{5.16}$$

Proof. We have from (2.2)

$$\begin{aligned}
\dot{\mathcal{I}}_{1\mathrm{st},2} = & -\left\langle \mathbf{J}D_{\boldsymbol{z}}\mathbf{\Phi}\left(\boldsymbol{z}\right)\left(\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}\right), \sigma_3\zeta_A^4\boldsymbol{\eta}\right\rangle - \left\langle \begin{pmatrix} -\partial_x^2+1 & 0 \\ 0 & 1\end{pmatrix}\boldsymbol{\eta}, \sigma_3\zeta_A^4\boldsymbol{\eta}\right\rangle \\
& + \left\langle \left(f\left(\mathbf{\Phi}\left(\boldsymbol{z}\right)_1+\eta_1\right)-f(\mathbf{\Phi}\left(\boldsymbol{z}\right)_1)\right), \zeta_A^4\eta_1\right\rangle - \left\langle \mathbf{J}\boldsymbol{R}\left(\boldsymbol{z}\right), \sigma_3\zeta_A^4\boldsymbol{\eta}\right\rangle \\
=: & \ C_1+C_2+C_3+C_4.
\end{aligned}$$

For the main term C_2, we have

$$C_2 = -\left\langle(-\partial_x^2+1)\eta_1, \zeta_A^4\eta_1\right\rangle + \left\|\zeta_A^2\eta_2\right\|_{L^2}^2 \tag{5.17}$$

with

$$\left|\left\langle(-\partial_x^2+1)\eta_1, \zeta_A^4\eta_1\right\rangle\right| \lesssim \left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}^2 + \left\|\mathrm{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}^2. \tag{5.18}$$

We consider the remainder terms, starting with C_1. By Lemma 3.3 and

$$\left|D_{\boldsymbol{z}}\mathbf{\Phi}\left(\boldsymbol{z}\right)(w)\zeta_A^4\right| \lesssim \mathrm{sech}\left(\frac{4}{A}x\right)\mathrm{sech}\left(\kappa x\right)|w| \quad \text{for all } w\in\mathbb{C}^2, \tag{5.19}$$

we obtain

$$|C_1| \lesssim |\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}| \, \|\mathrm{sech}\,(\kappa x)\, \boldsymbol{\eta}\|_{L^2} \lesssim \delta^{p-1} \, \|\boldsymbol{\eta}\|^2_{L^2_{-\kappa}} . \tag{5.20}$$

We next focus on C_4 with

$$\begin{aligned} C_4 \ &= C_{41} + C_{42}, \\ C_{41} \ &:= - \left\langle \mathbf{J} D_{\boldsymbol{z}} \boldsymbol{\Phi}\,(\boldsymbol{z})\, \tilde{\boldsymbol{z}}_R, \sigma_3 \zeta_A^4 \boldsymbol{\eta} \right\rangle \\ C_{42} \ &:= \left\langle f(\boldsymbol{\Phi}\,(\boldsymbol{z})_1) - f(Q) - f'(Q) \tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1, \sigma_3 \zeta_A^4 \boldsymbol{\eta} \right\rangle . \end{aligned} \tag{5.21}$$

Combining (5.19) and (1.47), we get

$$|C_{41}| \le |\boldsymbol{z}|^2 \, \|\boldsymbol{\eta}\|_{L^2_{-\kappa}} \le |\boldsymbol{z}|^4 + \|\boldsymbol{\eta}\|^2_{L^2_{-\kappa}} . \tag{5.22}$$

By (1.55), we have

$$|C_{42}| \le |\boldsymbol{z}|^2 \, \|\boldsymbol{\eta}\|_{L^2_{-\kappa}} \le |\boldsymbol{z}|^4 + \|\boldsymbol{\eta}\|^2_{L^2_{-\kappa}} , \tag{5.23}$$

so that (5.21)–(5.23) give that

$$|C_4| \lesssim |\boldsymbol{z}|^4 + \|\boldsymbol{\eta}\|^2_{L^2_{-\kappa}} . \tag{5.24}$$

To bound C_3, we notice that by

$$|f\,(\boldsymbol{\Phi}\,(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}\,(\boldsymbol{z})_1)| \le p \, |\boldsymbol{\Phi}\,(\boldsymbol{z})_1|^{p-1} \, |\eta_1| + |\eta_1|^p$$

we obtain

$$|C_3| \lesssim \|\mathrm{sech}\,(\kappa x)\, \eta_1\|^2_{L^2} + \delta^{p-1} \left\| \mathrm{sech} \left(\frac{2}{A} x \right) \eta_1 \right\|^2_{L^2} . \tag{5.25}$$

Collecting the estimates in (5.17), (5.18), (5.20), and (5.25), we have the conclusion. ∎

Proof of Proposition 2.4. It is immediate from $|\mathcal{I}_{1\mathrm{st},1}| \lesssim A\delta^2$, $|\mathcal{I}_{1\mathrm{st},2}| \lesssim \delta^2$, Lemmas 5.2 and 5.3, and the definition (2.6) of $\| \cdot \|_{\boldsymbol{\Sigma}_A}$. ∎

6. Technical estimates on the Darboux transform. For the N in (1.12) and the $\mathcal{A}$ in (1.22), we consider

$$\mathcal{T} := \langle \mathrm{i}\varepsilon\partial_x \rangle^{-(1+N)} \mathcal{A}^* . \tag{6.1}$$

We will use the following formula proved in [8], with $L^2_c(L_0)$ the continuous spectrum component in $L^2(\mathbb{R}, \mathbb{C})$ of L_0 and P_c the corresponding orthogonal projection,

$$\boldsymbol{u} = \prod_{j=0}^{N} R_{L_0}(\mu_j) P_c \mathcal{A} \, \langle \mathrm{i}\varepsilon\partial_x \rangle^{N+1} \mathcal{T}\boldsymbol{u} \qquad \text{for all } \boldsymbol{u} \in L^2_c(L_0). \tag{6.2}$$

In [8] the following result is proved.

LEMMA 6.1. *There exists a $0 < \kappa_0 \ll k_2$ such that for any $\kappa \in (0, \kappa_0)$ there exists a constant C_κ such that for all $0 < \varepsilon \le 1$ and $\boldsymbol{w} \in L^2_{-\frac{\kappa}{2}}$*

$$\left\| \prod_{j=0}^{N} R_{L_0}(\mu_j) P_c \mathcal{A}^* \, \langle \mathrm{i}\varepsilon\partial_x \rangle^{N+1} \boldsymbol{w} \right\|_{L^2_{-\kappa}} \le C_\kappa \|\boldsymbol{w}\|_{L^2_{-\frac{\kappa}{2}}} . \tag{6.3}$$

In [10, Lemma 7.5], the following is proved.

Lemma 6.2. *There exists $A_0, \varepsilon_0 > 0$ such that for any $A > A_0$ and $\varepsilon \in (0, \varepsilon_0)$ and for any $u \in H^1$ we have*

$$\left\| \operatorname{sech}\left(\frac{4}{A}x\right) \mathcal{T} u \right\|_{L^2} \lesssim \varepsilon^{-(1+N)} \left\| \operatorname{sech}\left(\frac{2}{A}x\right) u \right\|_{L^2}, \tag{6.4}$$

$$\left\| \operatorname{sech}\left(\frac{4}{A}x\right) \partial_x \mathcal{T} u \right\|_{L^2} \lesssim \varepsilon^{-(1+N)} \left\| \operatorname{sech}\left(\frac{2}{A}x\right) u' \right\|_{L^2} + \varepsilon^{-N} \|u\|_{L^2_{-\kappa}}. \tag{6.5}$$

In [10, Lemma 7.6] the following, obvious for $V_{N+1} = 0$, is proved.

Lemma 6.3. *For any $u \in H^1$,*

$$\left\| \left[\langle \mathrm{i}\varepsilon \partial_x \rangle^{-(1+N)}, \; V_{N+1} \right] \mathcal{A} u \right\|_{L^2} \lesssim \varepsilon \|\mathcal{T} u\|_{L^2_{-\kappa}}, \tag{6.6}$$

$$\left\| \cosh\left(\frac{\kappa}{2}x\right) \left[\langle \mathrm{i}\varepsilon \partial_x \rangle^{-(1+N)}, \; V_{N+1} \right] \mathcal{A} u \right\|_{L^2} \lesssim \varepsilon \|\mathcal{T} u\|_{L^2_{-\frac{\kappa}{2}}}. \tag{6.7}$$

7. Proof of Proposition 2.5: Virial estimates for the transformed equation. Using the $\mathcal{T}$ in (6.1) with $N = 3$ for $p \in \left(\frac{5}{3}, 2\right)$ and $N = 2$ for $p = 2$, we consider the new variable

$$\boldsymbol{v} := \mathcal{T} \boldsymbol{\eta}. \tag{7.1}$$

Here $\boldsymbol{v}$ is even for $5/3 < p < 2$ but, crucially, is odd for $p = 2$. For $\mathbf{L}_{N+1}$ (cf. (1.25)), $\boldsymbol{v}$ satisfies

(7.2)

$$\dot{\boldsymbol{v}} = -\mathcal{T} D\boldsymbol{\Phi}(\boldsymbol{z})(\dot{\boldsymbol{z}} - \tilde{\boldsymbol{z}}) + \mathbf{J}\left(\mathbf{L}_{N+1}\boldsymbol{v} + \begin{pmatrix} \left[\langle \mathrm{i}\varepsilon\partial_x \rangle^{-(N+1)}, \; V_{N+1} \right] & 0 \\ 0 & 0 \end{pmatrix} \mathcal{A}^* \boldsymbol{\eta} \right)$$
$$+ \mathcal{T}\left[\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(Q)\right)\eta_1 \mathbf{i} + \left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1 + \eta_1) - f(\boldsymbol{\Phi}(\boldsymbol{z})_1) - f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\mathbf{i} - \boldsymbol{R}(\boldsymbol{z}) \right].$$

From Lemma 6.1, we have

$$\|\operatorname{sech}(\kappa x)\boldsymbol{\eta}\|_{L^2} \lesssim \left\| \operatorname{sech}\left(\frac{\kappa}{2}\, x\right) \boldsymbol{v} \right\|_{L^2}. \tag{7.3}$$

Set, for the φ_B defined in (5.1),

$$\psi_{A,B} = \chi_A^2 \varphi_B, \qquad \tilde{S}_{A,B} = \frac{1}{2}\psi'_{A,B} + \psi_{A,B}\partial_x,$$

and consider the functionals

$$\mathcal{I}_{\mathrm{2nd},1} := \frac{1}{2}\left\langle \mathbf{J}\boldsymbol{v}, \tilde{S}_{A,B}\boldsymbol{v} \right\rangle, \qquad \mathcal{I}_{\mathrm{2nd},2} := \frac{1}{2}\left\langle \mathbf{J}\boldsymbol{v}, \sigma_3 e^{-\kappa\langle x \rangle}\boldsymbol{v} \right\rangle.$$

Lemma 7.1. *We have*

$$\|v_1'\|^2_{L^2_{-\frac{\kappa}{2}}} + \|v_1\|^2_{L^2_{-\frac{\kappa}{2}}} + \dot{\mathcal{I}}_{\mathrm{2nd},1} \lesssim \left(\varepsilon^{-(N+1)} A^2 \delta + A^{-1/2} \right) \|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A} + |\boldsymbol{z}|^4. \tag{7.4}$$

Proof. By (7.2), we have

$$
\begin{aligned}
\dot{\mathcal{I}}_{2\text{nd},1} = & -\left\langle \mathbf{J}\mathcal{T}D\mathbf{\Phi}(\boldsymbol{z})(\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}), \tilde{S}_{A,B}\boldsymbol{v}\right\rangle - \left\langle \mathbf{L}_{N+1}\boldsymbol{v}, \tilde{S}_{A,B}\boldsymbol{v}\right\rangle \\
& + \left\langle \begin{pmatrix} \left[\langle \mathrm{i}\varepsilon\partial_x\rangle^{-(N+1)}, \ V_4\right] & 0 \\ 0 & 0 \end{pmatrix} \mathcal{A}^*\boldsymbol{\eta}, \tilde{S}_{A,B}\boldsymbol{v}\right\rangle \\
& + \Big\langle \mathcal{T}\left((f'(\mathbf{\Phi}(\boldsymbol{z})_1) - f'(Q))\eta_1 + (f(\mathbf{\Phi}(\boldsymbol{z})_1 + \eta_1)\right. \\
& \left. - f(\mathbf{\Phi}(\boldsymbol{z})_1) - f'(\mathbf{\Phi}(\boldsymbol{z})_1)\eta_1)\right), \tilde{S}_{A,B}v_1\Big\rangle \\
& - \left\langle \mathcal{T}\boldsymbol{R}(\boldsymbol{z}), \tilde{S}_{A,B}\boldsymbol{v}\right\rangle =: \ D_1 + D_2 + D_3 + D_4 + D_5.
\end{aligned}
$$

Following [17, section 5], for the main term D_2 we have

$$
D_2 = -\left\langle L_{N+1}v_1, \widetilde{S}_{A,B}v_1\right\rangle = -\int \left(\xi_1'^2 + V_B\xi_1\right) dx + D_{21}, \qquad \xi_1 := \chi_A\zeta_B v_1, \tag{7.5}
$$

where

$$
\begin{aligned}
V_B &= \frac{1}{2}B^{-1}\left(\chi''|x| + 2\chi'\frac{x}{|x|}\right) - \frac{1}{2}\frac{\varphi_B}{\zeta_B^2}V_{N+1}' \text{ and} \\
D_{21} &= \frac{1}{4}\int (\chi_A^2)'(\zeta_B^2)'v_1^2 + \frac{1}{2}\int \left(3(\chi_A')^2 + \chi_A''\chi_A\right)\zeta_B^2 v_1^2 \\
&\quad - \int (\chi_A^2)'\varphi_B(v_1')^2 + \frac{1}{4}\int (\chi_A^2)'''\varphi_B v_1^2.
\end{aligned} \tag{7.6}
$$

We claim that

$$
\int (\xi_1'^2 + V_B\xi_1)\,dx + A^{-1}\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A} \gtrsim \left(\left\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1'\right\|^2_{L^2} + \left\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1\right\|^2\right). \tag{7.7}
$$

The proof is like in [17, Lemma 3]. We have, by $0 < \zeta_B \le 1$,

$$
\int_{|x|\le A} \operatorname{sech}(\kappa x)\, v_1^2 \le \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right)\zeta_B^2 v_1^2 \le \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right)\xi_1^2.
$$

We have

$$
\int_{|x|\le A} \operatorname{sech}(\kappa x)\, v_1'^2 \le \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right)(\xi_1' - \zeta_B' v_1)^2 \lesssim \int_{|x|\le A} \operatorname{sech}\left(\frac{\kappa}{2}x\right)(\xi_1'^2 + \xi_1^2).
$$

We have, thanks to (6.4),

$$
\begin{aligned}
\int_{|x|\ge A} \operatorname{sech}(\kappa x)\left(v_1'^2 + v_1^2\right) &\le \operatorname{sech}\left(\frac{\kappa}{2}A\right)\int_{\mathbb{R}} \operatorname{sech}\left(\frac{8}{A}x\right)\left(v_1'^2 + v_1^2\right)dx \\
&\lesssim \operatorname{sech}\left(\frac{\kappa}{2}A\right)\varepsilon^{-(N+1)}\int_{\mathbb{R}} \operatorname{sech}\left(\frac{4}{A}x\right)\left(\eta_1'^2 + \eta_1^2\right)dx \le A^{-1}\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}.
\end{aligned}
$$

Finally, we claim the following, which completes the proof of (7.7):

$$
\int_{\mathbb{R}} \operatorname{sech}\left(\frac{\kappa}{2}x\right)(\xi_1'^2 + \xi_1^2) \lesssim \int_{\mathbb{R}} (\xi_1'^2 + V_B\xi_1^2)\,dx. \tag{7.8}
$$

In the case $5/3 < p < 2$, the above inequality is true for all $\xi_1 \in H^1(\mathbb{R})$ and follows easily from the fact for B large the potential V_B is a small perturbation of the positive potential $-\frac{\varphi_B}{\zeta_B^2}V'_{N+1}>0$ for $x\neq 0$. In the case $p=2$, then $V_{N+1}=V_3=0$, and so V_B is a small potential. Then for B large, the coercivity in (7.8) is true only for $\xi_1\in H^1_{\text{odd}}(\mathbb{R})$. Fortunately, from (7.5) we see that ξ_1 has the same parity of v_1 and, as we remarked right under (7.1), v_1 is odd if $p=2$.

We next have the following, which is [17, Lemma 4] (see also the argument in [10], where $A^{-1}B\varepsilon^{-N-1}\ll A^{-\frac{1}{2}}$):

$$|D_{21}|\lesssim A^{-1/2}\left(\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}+\|\text{sech}\,(\kappa x)\,\eta_1\|^2_{L^2}\right). \tag{7.9}$$

By Lemma 3.3, $|\widetilde{S}^*_{A,B}\mathcal{T}D_{\boldsymbol{z}}\boldsymbol{\Phi}(\boldsymbol{z})|\sim e^{-\kappa|x|}$, and (6.4) we have that

$$\begin{aligned}|D_1|&\lesssim|\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}|\|\text{sech}\,(2\kappa x)\,\boldsymbol{v}\|_{L^2}\\&\lesssim\delta^{p-1}\|\text{sech}\,(\kappa x)\,\eta_1\|_{L^2}\|\text{sech}\,(2\kappa x)\,\boldsymbol{v}\|_{L^2}\lesssim\delta^{p-1}\varepsilon^{-(N+1)}\|\text{sech}\,(\kappa x)\,\boldsymbol{\eta}\|^2_{L^2}.\end{aligned} \tag{7.10}$$

Like in [10], we have (obviously here we care only of case $2>p>5/3$ where $V_{N+1}\neq 0$)

$$\begin{aligned}|D_3|&=\left|\left\langle\left[\langle\mathrm{i}\varepsilon\partial_x\rangle^{-(N+1)},\ V_4\right]\mathcal{A}^*\eta_1,\widetilde{S}_{A,B}v_1\right\rangle\right|\\&\le\left\|\cosh\left(\frac{\kappa}{2}x\right)\left[\langle\mathrm{i}\varepsilon\partial_x\rangle^{-(N+1)},\ V_4\right]\mathcal{A}^*\eta_1\right\|_{L^2}\left\|\text{sech}\left(\frac{\kappa}{2}x\right)\widetilde{S}_{A,B}v_1\right\|_{L^2}\\&\le\varepsilon\left\|\text{sech}\left(\frac{\kappa}{2}x\right)v_1\right\|_{L^2}\left(\left\|\text{sech}\left(\frac{\kappa}{2}x\right)v_1'\right\|_{L^2}+\left\|\text{sech}\left(\frac{\kappa}{2}x\right)v_1\right\|_{L^2}\right)\\&\lesssim\varepsilon\left(\left\|\text{sech}\left(\frac{\kappa}{2}x\right)v_1'\right\|^2_{L^2}+\left\|\text{sech}\left(\frac{\kappa}{2}x\right)v_1\right\|^2_{L^2}\right),\end{aligned} \tag{7.11}$$

where the upper bound can be absorbed inside the left-hand side of (7.4).

We now write $D_4=D_{41}+D_{42}$ with[1]

$$\begin{aligned}D_{41}&:=\left\langle\mathcal{T}\left(f'(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1)-f'(Q)\right)\eta_1,\tilde{S}_{A,B}v_1\right\rangle,\\D_{42}&:=\left\langle\mathcal{T}\left(\left(f\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1+\eta_1\right)-f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1)-f'(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1)\eta_1\right)\right),\tilde{S}_{A,B}v_1\right\rangle.\end{aligned}$$

By $\text{sech}\left(\frac{4}{A}x\right)\sim 1$ on $\text{supp}\,\psi_{A,B}\subseteq[-2A,2A]$, by (1.57) and (1.58), and by (6.4) and (6.5),

$$\begin{aligned}|D_{41}|\le&\left\|\text{sech}\left(\frac{4}{A}x\right)\mathcal{T}\left(f'(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1)-f'(Q)\right)\eta_1\right\|_{L^2}\\&\times\left(\left\|\text{sech}\left(\frac{4}{A}x\right)v_1\right\|_{L^2}+\left\|\text{sech}\left(\frac{4}{A}x\right)v_1'\right\|_{L^2}\right)\\&\lesssim\varepsilon^{-2(N+1)}\left\|\left(f'(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1)-f'(Q)\right)\eta_1\right\|_{L^2}\left(\left\|\text{sech}\left(\frac{2}{A}x\right)\eta_1\right\|_{L^2}\right.\\&\left.+\left\|\text{sech}\left(\frac{2}{A}x\right)\eta_1'\right\|_{L^2}\right)\\&\lesssim\varepsilon^{-2(N+1)}A|\boldsymbol{z}|\left\|e^{-\frac{p-1}{2}|x|}\eta_1\right\|_{L^2}\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}\lesssim\varepsilon^{-2(N+1)}A^2\delta\,\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}.\end{aligned} \tag{7.12}$$

[1]Notice that in the estimate of the term D_4 in [10, Lemma 8.1] the operator $\mathcal{T}$ is omitted by mistake. The proof can be corrected proceeding like here.

We have similarly, using (6.4) to get the second line and (3.6) and (3.7) the third,

(7.13)

$$\begin{aligned}
|D_{42}| &\leq \varepsilon^{-(N+1)} A \left\| \operatorname{sech}\left(\frac{4}{A}x\right) \mathcal{T}\left(f\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1 + \eta_1\right) - f\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1\right) - f'\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1\right)\eta_1\right) \right\|_{L^2} \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} \\
&\lesssim \varepsilon^{-2(N+1)} A \left\| \operatorname{sech}\left(\frac{2}{A}x\right) \left(f\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1 + \eta_1\right) - f\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1\right) - f'\left(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1\right)\eta_1\right) \right\|_{L^2} \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} \\
&\lesssim \varepsilon^{-2(N+1)} A \left\| \operatorname{sech}\left(\frac{2}{A}x\right) |\eta_1|^p \right\|_{L^2} \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A} \leq \varepsilon^{-2(N+1)} A^2 \|\eta_1\|_{L^\infty}^{p-1} \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2 \\
&\lesssim \varepsilon^{-2(N+1)} A^2 \delta^{p-1} \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2 .
\end{aligned}$$

Hence (7.12) and (7.13) yield

$$|D_4| \lesssim \left(\frac{A}{\varepsilon^4}\right)^2 \delta^{p-1} \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2. \tag{7.14}$$

Finally, we consider

$$\begin{aligned}
D_5 &= \left\langle \mathcal{T} D_{\boldsymbol{z}} \boldsymbol{\Phi}\left(\boldsymbol{z}\right) \tilde{\boldsymbol{z}}_R, \tilde{S}_{A,B} \boldsymbol{v} \right\rangle - \left\langle \mathcal{T}\left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right), \tilde{S}_{A,B} v_1 \right\rangle \\
&=: D_{51} + D_{52}.
\end{aligned}$$

We have

$$\begin{aligned}
D_{51} &= D_{511} + D_{512} \\
&= -\left\langle \partial_x \mathcal{T} D_{\boldsymbol{z}} \boldsymbol{\Phi}\left(\boldsymbol{z}\right) \tilde{\boldsymbol{z}}_R, \psi_{A,B} \boldsymbol{v} \right\rangle - \frac{1}{2} \left\langle \mathcal{T} D_{\boldsymbol{z}} \boldsymbol{\Phi}\left(\boldsymbol{z}\right) \tilde{\boldsymbol{z}}_R, \psi'_{A,B} \boldsymbol{v} \right\rangle.
\end{aligned}$$

We have

$$|D_{511}| \leq \|\cosh\left(\kappa x\right) \partial_x \mathcal{T} D_{\boldsymbol{z}} \boldsymbol{\Phi}\left(\boldsymbol{z}\right) \tilde{\boldsymbol{z}}_R\|_{L^2} \|\operatorname{sech}\left(\kappa x\right) \boldsymbol{v}\|_{L^2}.$$

In [8, Lemma 5.6] it is shown that there exist constants $C_0 > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and for $K_\varepsilon(x,y) \in \mathcal{D}'(\mathbb{R} \times \mathbb{R})$ the Schwartz kernel of $\mathcal{T}$, then we have

$$|K_\varepsilon(x,y)| \leq C_0 e^{-\frac{|x-y|}{3\varepsilon}} \text{ for all } x, y \text{ with } |x-y| \geq 1. \tag{7.15}$$

Using this fact, it is elementary to show that $|\partial_x \mathcal{T} D_{\boldsymbol{z}} \boldsymbol{\Phi}\left(\boldsymbol{z}\right)| \sim e^{-k_2|x|}$. Hence, if $0 < \kappa \ll k_2$,

$$|D_{511}| \lesssim |\tilde{\boldsymbol{z}}_R|.$$

The same argument yields the same bounds for D_{512} so we conclude that

$$|D_{51}| \lesssim |\tilde{\boldsymbol{z}}_R| \, \|\boldsymbol{v}\|_{L^2_{-\kappa}} \lesssim \delta_2^{-1} |\boldsymbol{z}|^4 + \delta_2 \|\boldsymbol{v}\|_{L^2_{-\frac{\kappa}{2}}}^2. \tag{7.16}$$

Turning to D_{52}, we have

$$\begin{aligned}
|D_{52}| &\leq \|\mathbf{C}\|_{L^2(|x| \leq 2A)} \|\operatorname{sech}\left(\kappa x\right) \left(2^{-1} \psi'_{A,B} v_1 + \psi_{A,B} v'_1\right)\|_{L^2}, \text{ where} \\
\mathbf{C} &:= \cosh\left(\kappa x\right) \mathcal{T}\left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right).
\end{aligned}$$

For a cutoff χ like in Notation 3.1, we write $\mathbf{C}=\mathbf{C}_1+\mathbf{C}_2$ with

$$\mathbf{C}_1 := \cosh(\kappa x)\,\mathcal{T}\chi(x-\cdot)\left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f(Q)-f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right),$$
$$\mathbf{C}_2 := \cosh(\kappa x)\,\mathcal{T}\chi_1(x-\cdot)\left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f(Q)-f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right) \text{ with } \chi_1 := 1-\chi.$$

For the I in (1.50), by (1.55), and using (7.15), we have

(7.17)
$$\|\mathbf{C}_2\|_{L^2} \lesssim \|\cosh(\kappa x)\int e^{-\frac{|x-y|}{3\varepsilon}}|I(y)|dy\|_{L^2} \lesssim |\boldsymbol{z}|^2\|\cosh(\kappa x)\int e^{-\frac{|x-y|}{3\varepsilon}}e^{-|y|}dy\|_{L^2} \lesssim |\boldsymbol{z}|^2.$$

We bound

$$\|\mathbf{C}_2\|_{L^2(|x|\le 2A)} \lesssim \int_{\mathbb{R}} d\xi|\widehat{\chi}(\xi)|\sum_{j=0}^{\log(10A)} 2^{\kappa j}\|\varphi_j\mathcal{T}e^{\mathrm{i}\xi\sqcup}\widetilde{\varphi}_j\left(f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f(Q)-f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)\|_{L^2(|x|\le 2A)}$$

for appropriate Paley–Littlewood decompositions (in x and y space) with supp φ_j and supp $\widetilde{\varphi}_j$ in $|x|\sim 2^j$ for $j\ge 1$ and $|x|\lesssim 1$ for $j=0$. Since for $|y|\lesssim 2^j$ with $j\le \log(10A)$ we have

$$\text{(7.18)}\qquad f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f(Q)-f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1 = 2^{-1}f''(Q)\left(\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)^2+E \text{ with}$$
$$|E| \lesssim \left|f'''(Q)\left(\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)^3\right| \lesssim |\boldsymbol{z}|^3 e^{-\frac{3-p}{2}|x|},$$

we accordingly split $\mathbf{C}_2=\mathbf{C}_{21}+\mathbf{C}_{22}$. With the function $\left(e^{\mathrm{i}\xi\sqcup}\right)(x):=e^{\mathrm{i}\xi x}$, we have

$$\begin{aligned}\|\mathbf{C}_{21}\|_{L^2(|x|\le 2A)} &\lesssim \int_{\mathbb{R}} d\xi|\widehat{\chi}(\xi)|\sum_{j=0}^{\log(10A)} 2^{\kappa j}\|\langle \mathrm{i}\varepsilon\partial_x\rangle^{-(N+1)}\mathcal{A}^* e^{\mathrm{i}\xi\sqcup}\widetilde{\varphi}_j f''(Q)\left(\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)^2\|_{L^2(|x|\le 2A)}\\ &\lesssim \|\langle\xi\rangle^{N+1}\widehat{\chi}\|_{L^1(\mathbb{R}_\xi)}\sum_{j=0}^{\log(10A)} 2^{\kappa j}\|\widetilde{\varphi}_j f''(Q)\left(\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right)^2\|_{H^{N+1}(\mathbb{R})} \lesssim |\boldsymbol{z}|^2.\end{aligned}$$

We have

$$\begin{aligned}\|\mathbf{C}_{22}\|_{L^2(|x|\le 2A)} &\lesssim \int_{\mathbb{R}} d\xi|\widehat{\chi}(\xi)|\sum_{j=0}^{\log(10A)} 2^{\kappa j}\|\langle \mathrm{i}\varepsilon\partial_x\rangle^{-(N+1)}\mathcal{A}^* e^{\mathrm{i}\xi\sqcup}\widetilde{\varphi}_j E\|_{L^2(|x|\le 2A)}\\ &\lesssim \|\widehat{\chi}\|_{L^1(\mathbb{R}_\xi)}\varepsilon^{-(N+1)}\sum_{j=0}^{\log(10A)} 2^{\kappa j}\|E\|_{L^2(\mathbb{R})} \lesssim A\varepsilon^{-(N+1)}|\boldsymbol{z}|^3 \lesssim A\varepsilon^{-(N+1)}\delta|\boldsymbol{z}|^2.\end{aligned}$$

Summing up, we have

$$\text{(7.19)}\qquad |D_{52}| \lesssim \delta_2^{-1}|\boldsymbol{z}|^4+\delta_2\|\boldsymbol{v}\|^2_{L^2_{-\frac{\kappa}{2}}}+\delta_2\left\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1'\right\|^2_{L^2}.$$

Equations (7.16) and (7.19) give that

$$\text{(7.20)}\qquad |D_5| \lesssim \frac{|\boldsymbol{z}|^4}{\delta_2}+\delta_2\|\boldsymbol{v}\|^2_{L^2_{-\frac{\kappa}{2}}}+\delta_2\left\|\operatorname{sech}\left(\frac{\kappa}{2}x\right)v_1'\right\|^2_{L^2}.$$

Collecting (7.5), (7.7), (7.9)–(7.11), (7.14), and (7.20) and bootstrapping, we deduce (7.4). ∎

LEMMA 7.2. *We have*

(7.21)
$$\|e^{-\kappa\langle x\rangle/2}v_2\|_{L^2}+\dot{\mathcal{I}}_{\mathrm{2nd},2}\lesssim\|e^{-\kappa\langle x\rangle/2}v_1'\|^2_{L^2}+\|e^{-\kappa\langle x\rangle/2}v_1\|^2_{L^2}+|\boldsymbol{z}|^4+o_\varepsilon(1)\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}.$$

Proof. Differentiating $\mathcal{I}_{\mathrm{2nd},2}$, we have

$$\begin{aligned}\dot{\mathcal{I}}_{\mathrm{2nd},2}=&-\left\langle \mathcal{T}D\boldsymbol{\Phi}(\boldsymbol{z})(\dot{\boldsymbol{z}}-\tilde{\boldsymbol{z}}),\sigma_3e^{-\kappa\langle x\rangle}\boldsymbol{v}\right\rangle\\&-\left\langle \mathbf{L}_{N+1}\boldsymbol{v},\tilde{S}_{A,B}\boldsymbol{v}\right\rangle+\left\langle\begin{pmatrix}\left[\langle\mathrm{i}\varepsilon\partial_x\rangle^{-(N+1)},\ V_4\right] & 0\\ 0 & 0\end{pmatrix}\mathcal{A}^*\boldsymbol{\eta},\sigma_3e^{-\kappa\langle x\rangle}\boldsymbol{v}\right\rangle\\&+\Big\langle\mathcal{T}\left(\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(Q)\right)\eta_1+\left(f\left(\boldsymbol{\Phi}(\boldsymbol{z})_1+\eta_1\right)\right.\right.\\&\left.\left.-f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\right),e^{-\kappa\langle x\rangle}v_1\Big\rangle\\&-\left\langle\mathcal{T}\boldsymbol{R}(\boldsymbol{z}),\sigma_3e^{-\kappa\langle x\rangle}\boldsymbol{v}\right\rangle=:E_1+E_2+E_3+E_4+E_5.\end{aligned}$$

Like in [10] we have

$$E_2=-\|e^{-\kappa\langle x\rangle/2}v_2\|^2_{L^2}+\left\langle L_{N+1}v_1,e^{-\kappa\langle x\rangle}v_1\right\rangle=-\|e^{-\kappa\langle x\rangle/2}v_2\|^2_{L^2}+E_{21}\text{ with}$$
$$|E_{21}|\lesssim\|e^{-\kappa\langle x\rangle/2}v_1'\|^2_{L^2}+\|e^{-\kappa\langle x\rangle/2}v_1\|^2_{L^2}.$$

By Lemma 3.3, we have

$$|E_1|\lesssim\delta^{p-1}\|e^{-\kappa\langle x\rangle/2}\boldsymbol{v}\|_{L^2}\|e^{-\kappa\langle x\rangle}\boldsymbol{\eta}\|_{L^2}\lesssim\delta^{p-1}\varepsilon^{-N}\|e^{-\kappa\langle x\rangle/2}\boldsymbol{v}\|^2_{L^2}.$$

By (6.7), we have

$$\begin{aligned}|E_3|&=\left|\left\langle[\langle\mathrm{i}\varepsilon\partial_x\rangle^{-N},V_4]\mathcal{A}^*\eta_1,\sigma_3e^{-\kappa\langle x\rangle}v_1\right\rangle\right|\\&\lesssim\varepsilon\|e^{-\frac{\kappa}{2}\langle x\rangle}v_1\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\le\varepsilon\|e^{-\frac{\kappa}{2}\langle x\rangle}v_1\|^2_{L^2}.\end{aligned}$$

Proceeding like in Lemma 7.1, we write $E_4=E_{41}+E_{42}$ with[2]

$$\begin{aligned}E_{41}&:=\left\langle\mathcal{T}\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(Q)\right)\eta_1,e^{-\kappa\langle x\rangle}v_1\right\rangle,\\E_{42}&:=\left\langle\mathcal{T}\left(\left(f\left(\boldsymbol{\Phi}(\boldsymbol{z})_1+\eta_1\right)-f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\right),e^{-\kappa\langle x\rangle}v_1\right\rangle.\end{aligned}$$

By an analogue of (7.3) and by (6.4)–(6.5),

$$\begin{aligned}|E_{41}|&\le\varepsilon^{-N-1}\|\left(f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(Q)\right)\eta_1\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\\&\lesssim\varepsilon^{-(N+1)}A|\boldsymbol{z}|\|e^{-\frac{p-1}{2}|x|}\eta_1\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\lesssim\varepsilon^{-N}A\delta\|e^{-\frac{p-1}{4}|x|}v_1\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\\&\lesssim\varepsilon^{-N-1}A\delta\|e^{-\kappa\langle x\rangle}v_1\|^2_{L^2}.\end{aligned}$$

In analogy to estimates for D_{41} and D_{42} in Lemma 7.1, but using $\operatorname{sech}\left(\frac{4}{A}x\right)e^{-\frac{\kappa}{2}\langle x\rangle}\lesssim e^{-\kappa\langle x\rangle}$ along with (6.4)–(6.5),

$$\begin{aligned}|E_{42}|&\le\left\|\operatorname{sech}\left(\frac{4}{A}x\right)\mathcal{T}\left(f\left(\boldsymbol{\Phi}(\boldsymbol{z})_1+\eta_1\right)-f(\boldsymbol{\Phi}(\boldsymbol{z})_1)-f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\right\|_{L^2}\|e^{-\frac{\kappa}{2}\langle x\rangle}v_1\|_{L^2}\\&\lesssim\varepsilon^{-2(N+1)}A\left\|\operatorname{sech}\left(\frac{2}{A}x\right)\left(f\left(\boldsymbol{\Phi}(\boldsymbol{z})_1+\eta_1\right)-f(\boldsymbol{\Phi}(\boldsymbol{z})_1)\right.\right.\\&\qquad\left.\left.-f'(\boldsymbol{\Phi}(\boldsymbol{z})_1)\eta_1\right)\right\|_{L^2}\|e^{-\kappa\langle x\rangle}v_1\|_{L^2}\\&\lesssim\varepsilon^{-2(N+1)}A\delta^{p-1}\|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}\|e^{-\frac{\kappa}{2}\langle x\rangle}v_1\|_{L^2}\le\varepsilon^{-2(N+1)}A\delta^{p-1}\left(\|\boldsymbol{\eta}\|^2_{\boldsymbol{\Sigma}_A}+\|e^{-\frac{\kappa}{2}\langle x\rangle}v_1\|^2_{L^2}\right).\end{aligned}$$

[2]Also in the estimate of the term E_4 in [10, Lemma 8.2], like in the estimate of D_4 in [10, Lemma 8.1], the operator $\mathcal{T}$ has been omitted by mistake. The proof can be corrected proceeding like here.

Finally, we consider

$$E_5 = \left\langle \mathcal{T} D_{\boldsymbol{z}} \boldsymbol{\Phi}\left(\boldsymbol{z}\right) \tilde{\boldsymbol{z}}_R, \sigma_3 e^{-\kappa\langle x\rangle} \boldsymbol{v}\right\rangle - \left\langle \mathcal{T}\left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right), e^{-\kappa\langle x\rangle} v_1\right\rangle$$
$$=: E_{51} + E_{52}.$$

The last three terms can be bounded similarly. We have

$$|E_{51}| \le \|\mathcal{T} D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z}) \tilde{\boldsymbol{z}}_R\|_{L^2} \|e^{-\kappa\langle x\rangle} \boldsymbol{v}\|_{L^2} \lesssim \|\mathcal{A}^* D_{\boldsymbol{z}} \boldsymbol{\Phi}(\boldsymbol{z}) \tilde{\boldsymbol{z}}_R\|_{L^2} \|e^{-\kappa\langle x\rangle} \boldsymbol{v}\|_{L^2} \lesssim \delta_2^{-1} |\boldsymbol{z}|^4 + \delta_2 \|e^{-\frac{\kappa}{2}\langle x\rangle} \boldsymbol{v}\|_{L^2}^2.$$

Turning to E_{52}, we have

$$|E_{52}| \le \|\mathbf{E}\|_{L^2} \|e^{-\frac{\kappa}{2}\langle x\rangle} v_1\|_{L^2}, \text{ where}$$
$$\mathbf{E} := e^{-\frac{\kappa}{2}\langle x\rangle} \mathcal{T}\left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right).$$

Let us write $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ with

$$\mathbf{E}_1 := e^{-\frac{\kappa}{2}\langle x\rangle} \mathcal{T} \chi(x - \cdot)\left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right),$$
$$\mathbf{E}_2 := e^{-\frac{\kappa}{2}\langle x\rangle} \mathcal{T} \chi_1(x - \cdot)\left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right) \text{ with } \chi_1 = 1 - \chi.$$

Like for (7.17) and using (1.55), we have

$$\|\mathbf{E}_2\|_{L^2} \lesssim \|e^{-\frac{\kappa}{2}\langle x\rangle} \int e^{-\frac{|x-y|}{3\varepsilon}} |I(y)| dy\|_{L^2} \lesssim |\boldsymbol{z}|^2 \|e^{-\frac{\kappa}{2}\langle x\rangle} \int e^{-\frac{|x-y|}{3\varepsilon}} e^{-|y|} dy\|_{L^2} \lesssim |\boldsymbol{z}|^2.$$

In analogy to the bound of $\mathbf{C}_2$ in Lemma 7.1, we bound

$$\|\mathbf{E}_2\|_{L^2} \lesssim \int_{\mathbb{R}} d\xi |\widehat{\chi}(\xi)| \sum_{j=0}^{\infty} 2^{-\frac{\kappa}{2} j} \|\varphi_j \mathcal{T} e^{\mathrm{i}\xi\sqcup} \widetilde{\varphi}_j \left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right) \|_{L^2}$$

for appropriate Paley–Littlewood decompositions with supp φ_j and supp $\widetilde{\varphi}_j$ in $|x| \sim 2^j$ for $j \ge 1$ and $|x| \lesssim 1$ for $j = 0$. Next we split $\|\mathbf{E}_2\|_{L^2} \le A_{2l} + A_{2h}$, with A_{2l} involving the sum for $j \le \log(10A)$ and A_{2l} involving the sum for $j > \log(10A)$. Now, using the same argument in Lemma 7.1, we have

$$A_{2l} \lesssim |\boldsymbol{z}|^2.$$

Next, using also (1.52) and (1.55)

$$A_{2h} \lesssim \int_{\mathbb{R}} d\xi |\widehat{\chi}(\xi)| \sum_{j > \log(10A)} 2^{-\frac{\kappa}{2} j} \|\mathcal{T} e^{\mathrm{i}\xi\sqcup} \widetilde{\varphi}_j \left(f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\right) \|_{L^2}$$
$$\lesssim \varepsilon^{-(N+1)} A^{-1} \|f(\boldsymbol{\Phi}\left(\boldsymbol{z}\right)_1) - f(Q) - f'(Q)\tilde{\boldsymbol{\Phi}}[\boldsymbol{z}]_1\|_{L^2} \lesssim \varepsilon^{-(N+1)} A^{-1} |\boldsymbol{z}|^2.$$

Summing up

$$|E_{52}| \le \|\mathbf{E}\|_{L^2} \|e^{-\frac{\kappa}{2}\langle x\rangle} v_1\|_{L^2} \lesssim \delta_2^{-1} |\boldsymbol{z}|^4 + \delta_2 \|e^{-\frac{\kappa}{2}\langle x\rangle} v_1\|_{L^2}^2.$$

Collecting the estimates and bootstrapping, we obtain the conclusion (7.21). ∎

Combining Lemmas 7.1 and 7.2, we have the next lemma.

Lemma 7.3. *For any $\mu > 0$, we have*

$$\int_0^T \left(\left\|\operatorname{sech}\left(\frac{\kappa}{2} x\right) v_1'\right\|_{L^2}^2 + \left\|\operatorname{sech}\left(\frac{\kappa}{2} x\right) \boldsymbol{v}\right\|_{L^2}^2 \right) \lesssim \delta + o_\varepsilon(1) \int_0^T \|\boldsymbol{\eta}\|_{\boldsymbol{\Sigma}_A}^2 + \|\boldsymbol{z}\|_{L^4(0,T)}^4.$$

Proof. The claim follows from Lemmas 7.1 and 7.2 and

$$|\mathcal{I}_{\rm 2nd,1}| \lesssim B\varepsilon^{-(N+1)}\delta^2, \ |\mathcal{I}_{\rm 2nd,2}| \lesssim \varepsilon^{-(N+1)}\delta^2. \qquad \square$$

Proof of Proposition 2.5. It is a consequence of Lemma 7.3 and inequality (7.3). $\square$

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