A Generalized Passivity Theory Over Abstract Time Domains

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***Abstract*—Passivity is a well-established concept for continuous-time systems. Yet, its application to discrete time, delay, or other classes of systems is somewhat limited, leading to inconsistencies and disparities. In this article, we study a new notion, ϱ-passivity, which reduces to standard passivity in the continuous-time case but addresses some of the aforementioned limitations when applied to other classes of systems. In particular, in an abstract input–output setting, we show that ϱ-passivity is preserved under a class of interconnections, thereby extending the existing passivity results. Moreover, we explore the relationship between ϱ-passivity and stability, and we derive sufficient conditions for high-gain, low-gain, and causal stabilizability by static output feedback. Finally, in contrast to the standard passivity notion, we prove that ϱ-passivity is preserved under sampling for a class of nonlinear systems and discretization methods. Overall, the results of this article constitute the first step toward a unifying passivity theory embracing all the different domains and systems classes relevant to systems and control theory.**

***Index Terms*—High-gain stabilization, input–output systems, low-gain stabilization, output-feedback stabilization, passivity, sampled-data systems.**

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I. Introduction

A. Problem Overview and Motivations

THE concept of passivity can be traced back to the early 20th century for the analysis of analog electrical circuits [1], [2]. Passive electrical components or circuits are those in which the energy is not generated or amplified but, instead, is transferred, stored, or dissipated. The control theoretic formalization of the notion of passivity was given by Zames in the 60s in terms of an abstract input–output property [3], [4]. Specifically, Zames described a continuous-time system as a relation between input signals u and output signals y, and defined it *passive* if there exists $\beta \in \mathbb{R}$ such that the condition

$$\int_0^t \langle y(\tau), u(\tau)\rangle \, \mathrm{d}\tau \geq -\beta \tag{1}$$

holds for all $t \geq 0$ and all input–output pairs (u, y) (see also [5], [6], [7], and [8]). A nonzero β may account for nonzero initial conditions or any transient dynamics, and $\langle y(\tau), u(\tau)\rangle$ is the instantaneous power provided to (or extracted from) the system. A state-space theory of passivity is due to Willems, who in the 70s introduced the more general concept of *dissipativity* for continuous-time state-space models [9], [10]. In particular, a continuous-time state-space system with state x is called *passive* if there exists a nonnegative function S (the *storage function*) such that the condition

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} \langle y(\tau), u(\tau)\rangle \, \mathrm{d}\tau \tag{2}$$

holds for every initial time t_0, every input u, all times $t_1 \geq t_0$, and every initial state $x(t_0)$. Hence, a system is passive if the variation of the storage function does not exceed the total power entering/exiting the system in the interval $[t_0, t_1)$. In view of the intimate connection between the inequality (2) and Lyapunov's direct method [7], [10], Willems' state-space passivity theory has had a tremendous impact in the control community [11], [12], [13], [14], [15], [16], [17], [18].

The study of passivity for discrete-time dynamical systems has a much shorter history. Byrnes and Lin first introduced passivity for discrete-time state-space models in their works [19], [20]. They defined passivity by mimicking Willems' continuous-time definition (2). Namely, a discrete-time system is defined as passive if there exists a storage function S that satisfies the condition

$$S(x(k)) \leq S(x(0)) + \sum_{\tau=0}^{k-1} \langle y(\tau), u(\tau)\rangle, \tag{3}$$

for all $k \in \mathbb{N}$, every initial state $x(0)$, and every input u. The corresponding input–output notion, paralleling (1), is represented by the inequality

$$\sum_{\tau=0}^{k-1} \langle y(\tau), u(\tau)\rangle \geq -\beta, \tag{4}$$

for some constant $\beta \in \mathbb{R}$, for every $k \in \mathbb{N}$, and every input u; see [8], [21], and [22]. Like the continuous-time definition, also the discrete-time definitions (3) and (4) have had a significant impact on the analysis and synthesis of modern control systems relying on digital technology [23], [24], [25], [26], [27], [28], [29], [30], [31], [32]. Despite the widespread use of the definitions (3) and (4), the preliminary version of this article [33] highlighted several known discrepancies between (3) and its continuous-time counterpart (2). One notable difference is that only relative-degree-zero systems meet (3), whereas (2) is also satisfied by relative-degree-one systems. Furthermore, when a passive continuous-time system (i.e., one that satisfies (2)) is discretized using an exact discretization method, the resulting system may not satisfy (3). This means that *passivity is not generally preserved under the sampling process*, even for arbitrarily small sampling times.

The phenomenon of loss of passivity under sampling has been extensively analyzed in various works, e.g., in [27], [28], [34], [35], [36], [37], [38], and [39]. In these studies, the problem has been circumvented by "redefining the output" with the objective of guaranteeing the property (3) for the newly defined output. Yet, from an input–output perspective, the limitation persists in cases in which the entire state of the system cannot be measured or when the output mapping cannot be changed. On the other hand, the "discontinuity" that passivity experiences under sampling, and the resulting necessity of redefining the output, further emphasize the limitation of the notions (3) and (4), as they render passivity representation-dependent.

In the context of linear time-invariant systems, a different solution to this problem has been proposed by the preliminary version [33] of this article, in which the discrete-time passivity notion (4) has been replaced by the relation

$$\sum_{\tau=0}^{k-1} \langle y(\tau+\varrho), u(\tau)\rangle \geq -\beta, \tag{5}$$

where ϱ denotes the relative degree of the discretized system. The following example better illustrates the motivation and the solution of [33].

Example 1: Consider a scalar continuous-time system described by the equations

$$\dot{x}(t) = -x(t) + u(t), \quad y(t) = x(t), \quad x(0) = 0, \tag{6}$$

with state $x(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$, and output $y(t) \in \mathbb{R}$. This system is passive (in the sense of (2)) with storage function $S(x) := \frac{1}{2}x^2$. For a fixed sampling time $\delta > 0$, the sampled-data system associated with (6) is obtained by sampling the state x at times $t = k\delta$ and by keeping the input u constant in the intervals $[k\delta, k\delta+\delta)$, for all $k \in \mathbb{N}$. This yields a discrete-time system described by the equations

$$x_{\mathrm{d}}(k+1) = \mathrm{e}^{-\delta} x_{\mathrm{d}}(k) + (1-\mathrm{e}^{-\delta}) u_{\mathrm{d}}(k), \quad x_{\mathrm{d}}(0) = 0, \tag{7}$$

with input $u_{\mathrm{d}}(k) = u(k\delta)$ and output $y_{\mathrm{d}}(k) = x_{\mathrm{d}}(k)$. It should be noted that system (7) does not exhibit passivity in the sense of (3) or (4). Indeed, for each value of $\beta \in \mathbb{R}$, by taking any u such that $u(0) = (1-\mathrm{e}^{-\delta})^{-1}$ and $u(\delta) = -\beta - 1$, we obtain

$$\langle y_{\mathrm{d}}(0), u_{\mathrm{d}}(0)\rangle + \langle y_{\mathrm{d}}(1), u_{\mathrm{d}}(1)\rangle = -\beta - 1 < -\beta,$$

which violates (4) for $k = 2$.

On the contrary, as shown in [33], system (7) satisfies condition (5) with $\varrho = 1$. $\triangleleft$

Another motivation behind the investigation of a different notion of passivity lies in the discrepancy between the theory of high-gain stabilization for continuous-time and discrete-time systems. Notably, a passive continuous-time system can be stabilized via high-gain feedback. However, this is not the case for discrete-time systems, for which the very notion of high-gain stabilization—at least to the best of the authors' knowledge—is still poorly understood. This is illustrated in the following example.

Example 2: As (6) is passive, any output-feedback control law of the form

$$u = -\kappa y, \tag{8}$$

with $\kappa > 0$ arbitrarily large, exponentially stabilizes its origin with a rate proportional to κ. Yet, the very same output feedback law (8) applied to the sampled-data system (7) associated with (6) has a destabilizing effect if κ is large. Indeed, the discrete-time closed-loop system induced by (8) is

$$x_{\mathrm{d}}(k+1) = (\mathrm{e}^{-\delta} + \kappa\mathrm{e}^{-\delta} - \kappa) x_{\mathrm{d}}(k), \tag{9}$$

which is unstable for all $\kappa > \frac{1+\mathrm{e}^{-\delta}}{1-\mathrm{e}^{-\delta}}$. Instead, the output-feedback law $u(\cdot) = -\kappa y(\cdot + 1)$ always yields exponential stability of the origin of the discrete-time closed-loop system for every $\kappa > 0$ and $\delta > 0$. This controller is obtained by shifting the output in time according to (5), and yields a rate increasing with κ, thereby recovering a deadbeat controller in the limit case of $\kappa \to \infty$. Note that, since (7) is strictly causal, the output-feedback law $u(\cdot) = -\kappa y(\cdot + 1)$ is still causal. High-gain stabilization will be further discussed in Section VI-C, in which the shifting is justified by the fact that (7) satisfies (5) with $\varrho = 1$. $\triangleleft$

In addition to the previous two examples, as shown in [33], reasoning in terms of (1) or (2), namely, in terms of the instantaneous product of $y(t)$ and $u(t)$, appears to be unsuitable not only for discrete-time systems but also for all systems that do not have the instantaneous reachability property of continuous-time systems by which the effect of the input is immediately reflected in the corresponding output. An important class of such systems is, for instance, continuous-time systems with delays. As the example below shows, passivity can be recovered also in this case by "shifting the output."

Example 3: Consider system (6) with delayed measurements, that is $y(t) = x(t-\varrho)$, for all $t \geq 0$ and some $\varrho \geq 0$. Although the nominal system obtained for $\varrho = 0$ is passive, this property is lost for every $\varrho > 0$. This can be shown by taking, for each $\beta \geq 0$ and $\epsilon \in (0, \varrho)$, any u such that $u(t) = 1$ for all $t \in [0, \varrho)$, and $u(t) = -(\epsilon + \mathrm{e}^{-\varrho}(\mathrm{e}^{-\epsilon} - 1))^{-1}(\beta + 1)$ for $t \in [\varrho, 2\varrho]$. Indeed, in such a case one has (assuming $x(t) = 0$ for all $t \in [-\varrho, 0]$)

$$\int_0^{\varrho+\epsilon} \langle y(\tau), u(\tau)\rangle \,\mathrm{d}\tau = \int_{\varrho}^{\varrho+\epsilon} (1 - \mathrm{e}^{-\tau}) u(\varrho)\,\mathrm{d}\tau < -\beta,$$

which violates (1) at $t = \varrho + \epsilon$.

While passivity in the sense of (1) is therefore lost for any delay $\varrho > 0$, an equality of the same form as (1) can be obtained if (1) is replaced by the condition

$$\int_0^t \langle y(\tau + \varrho), u(\tau)\rangle \,\mathrm{d}\tau \geq -\beta. \tag{10}$$

$\triangleleft$

The evident resemblance between conditions (5) and (10), along with the fact that they seem to accommodate the mentioned discrepancies between the continuous-time and the discrete-time theories of passivity, calls for the investigation of a broader concept of passivity embracing both (5) and (10) and similar cases. To this end, this article introduces the notion of *ϱ-passivity* that, once formulated in an abstract time domain as preliminarily elucidated in [33], provides a unifying framework that encompasses the aforementioned scenarios.

B. Contribution and Organization

This article develops the basic theoretical foundations of the notion of ϱ-passivity, extending the preliminary version [33] as detailed in the following. The rest of this article is organized as follows. Section II gives the mathematical preliminaries and notations. In Section III, we introduce the notion of ϱ-passivity for input–output and input-state-output systems. Our definition closely resembles the preliminary one given in [33], except that herein we omit the notion of reachability index. This omission is justified by the fact that the results presented in this article do not rely on such a notion. In Section IV, we study sufficient conditions under which an interconnection of a finite number of systems is ϱ-passive. The presented results generalize in many ways the preliminary results given in [33], and also extend the corresponding results for classical passivity. Overall, Section IV shows that, like classical passivity, ϱ-passivity is well behaved under a class of interconnections. Section V explores the relation between ϱ-passivity, boundedness, internal and external stability, and finite gain. In particular, it is shown that ϱ-passive systems are internally stabilizable and that *output-strictly ϱ-passive* systems are externally stabilizable. Here, stability is meant as a topological property in the spirit of [40]. Sections VI and VII deal with high-gain and low-gain stabilizability and stabilization. Specifically, Section VI shows that any *output-shortly ϱ-passive* system is high-gain stabilizable and, dually, Section VII shows that any *input-shortly ϱ-passive* system is low-gain stabilizable. Moreover, Section VI discusses the connection between continuous- and discrete-time high-gain stabilization, and Section VII provides sufficient conditions for *causal* stabilization in the case $\varrho \neq 0$. Section VIII provides sufficient conditions for preserving ϱ-passivity under sampling in an input–output sampled-data framework. Finally, Section IX concludes this article and gives some perspectives on ϱ-passivity.

II. Mathematical Preliminaries and Notations

This section introduces the basic mathematical tools and notations used throughout this article. The need to adopt an abstract time domain is motivated by the fact that the proposed notion of ϱ-passivity must encompass, systems defined on different time domains.

A. Spaces and Signals

Throughout we consider a *time domain* defined as a triple $(\mathbb{T}, \leq, \mu)$, in which the following hold.

1) $\mathbb{T}$ is a monoid with operation denoted by $+$ and identity by 0.
2) $\leq$ is a linear order on $\mathbb{T}$ such that $0 = \inf \mathbb{T}$ and, for every $s, \tau \in \mathbb{T}$, $s \geq \tau \Rightarrow t + s \geq t + \tau$ for all $t \geq 0$.
3) μ is a measure on $(\mathbb{T}, \Sigma)$, in which Σ is a σ-algebra on $\mathbb{T}$ with the property that every translation of the form $t \mapsto t + \varrho$, with $\varrho \in \mathbb{T}$, is measurable.

When $\leq$ and μ are clear or irrelevant, we refer to a time domain simply by $\mathbb{T}$. Given a time domain $(\mathbb{T}, \leq, \mu)$ and a separable real Hilbert space $\mathbb{H}$, we denote by $\mathcal{S}(\mathbb{T}, \mathbb{H})$ the vector space (under pointwise operations) of all (Borel) measurable functions $s : \mathbb{T} \to \mathbb{H}$ such that, for all $t \in \mathbb{T}$, the Lebesgue integral[1]

$$\langle s, s\rangle_t := \int_{\mathbb{T}_{<t}} \langle s(\tau), s(\tau)\rangle_{\mathbb{H}} \,\mathrm{d}\mu(\tau) \tag{11}$$

exists in $\mathbb{R}$, in which $\langle \cdot, \cdot\rangle_{\mathbb{H}}$ denotes the inner product on $\mathbb{H}$, and $\mathbb{T}_{<t} := \{s \in \mathbb{T} \mid s < t\}$. If $\mathbb{H} = \mathbb{H}_1 \times \cdots \times \mathbb{H}_n$, we let $\langle \cdot, \cdot\rangle_{\mathbb{H}} := \langle \cdot, \cdot\rangle_{\mathbb{H}_1} + \cdot + \langle \cdot, \cdot\rangle_{\mathbb{H}_n}$, if not otherwise specified. To ease the notation, we shall omit the subscript $\mathbb{H}$ from scalar products whenever clear from the context. The space $\mathcal{S}(\mathbb{T}, \mathbb{H})$ is an abstraction of an extended Lebesgue space. To shorten the notation, given $s, z \in \mathcal{S}(\mathbb{T}, \mathbb{H})$, we shall write

$$s^{\varrho}(t) := s(t + \varrho), \quad \langle s, z\rangle_t := \int_{\mathbb{T}_{<t}} \langle s(\tau), z(\tau)\rangle_{\mathbb{H}} \,\mathrm{d}\mu(\tau).$$

Because of the definition of $\mathcal{S}(\mathbb{T}, \mathbb{H})$ and the properties of the inner product, for every $t \in \mathbb{T}$ and $s, z \in \mathcal{S}(\mathbb{T}, \mathbb{H})$, $\langle s, z\rangle_t$ does exist in $\mathbb{R}$. Moreover, for every $\varrho \in \mathbb{T}$, $s^{\varrho} \in \mathcal{S}(\mathbb{T}, \mathbb{H})$. Indeed, $t \mapsto s(t + \varrho)$ is the composition of s, which is measurable as it is in $\mathcal{S}(\mathbb{T}, \mathbb{H})$, and $t \mapsto t + \varrho$, which is measurable by definition of time domain. Furthermore, $\langle s^{\varrho}, s^{\varrho}\rangle_t$ exists since so does $\langle s, s\rangle_t$. If not otherwise specified, $\mathcal{S}(\mathbb{T}, \mathbb{H})$ shall denote a set defined as above for a generic separable real Hilbert space $\mathbb{H}$ and a generic time domain $\mathbb{T}$.

We conclude this section by stating the following technical lemma that is often used in the forthcoming derivations.

Lemma 1: Let $\mathbb{H}$ be a real Hilbert space. For every $y, u \in \mathbb{H}$, and every $\varepsilon > 0$, the condition

$$\langle y, u\rangle \leq \varepsilon\langle y, y\rangle + \frac{1}{2\varepsilon}\langle u, u\rangle \tag{12}$$

holds.

[1]Note that $t \mapsto \langle s(t), s(t)\rangle_{\mathbb{H}}$ is measurable since $h \mapsto \langle h, h\rangle_{\mathbb{H}}$ is continuous.

Proof: The proof is a direct consequence of Young's inequality. Specifically, from the positive-definiteness property of the inner product it follows that:

$$\forall \varepsilon > 0, \quad \langle \varepsilon y - u, \varepsilon y - u\rangle \geq 0$$
$$\implies 2\varepsilon\langle y, u\rangle \leq \varepsilon^2\langle y, y\rangle + \langle u, u\rangle.$$

Thus, we get

$$\langle y, u\rangle \leq \frac{\varepsilon}{2}\langle y, y\rangle + \frac{1}{2\varepsilon}\langle u, u\rangle \leq \varepsilon\langle y, y\rangle + \frac{1}{2\varepsilon}\langle u, u\rangle,$$

which completes the proof. □

B. Systems and Feedback

An *input–output system* Ψ is a relation on $\mathcal{S}(\mathbb{T},\mathbb{H})$, namely, a subset $\Psi \subseteq \mathcal{S}(\mathbb{T},\mathbb{H}) \times \mathcal{S}(\mathbb{T},\mathbb{H})$, see [3] and [15, Ch. 1]. The domain of Ψ is defined as $\operatorname{dom}\Psi := \{u \in \mathcal{S}(\mathbb{T},\mathbb{H}) \mid \exists y \in \mathcal{S}(\mathbb{T},\mathbb{H}),\ (u, y) \in \Psi\}$, and its range as $\operatorname{ran}\Psi := \{y \in \mathcal{S}(\mathbb{T},\mathbb{H}) \mid \exists u \in \mathcal{S}(\mathbb{T},\mathbb{H}),\ (u, y) \in \Psi\}$. Given a set $\mathbb{X}$, let $\mathbb{X}^{\mathbb{T}}$ denote the set of all functions $\mathbb{T} \to \mathbb{X}$. An *input-state-output system* Σ with state-space $\mathbb{X}$ is a subset of $\mathcal{S}(\mathbb{T},\mathbb{H}) \times \mathbb{X}^{\mathbb{T}} \times \mathcal{S}(\mathbb{T},\mathbb{H})$ such that, for some $h : \mathbb{X} \times \mathbb{H} \to \mathbb{H}$, every $(u, x, y) \in \Sigma$ satisfies $y(t) = h(x(t), u(t))$ for all $t \in \mathbb{T}$. Clearly, for each $x_0 \in \mathbb{X}$, the projection $\Psi_{x_0}(\Sigma) := \{(u, y) \in \mathcal{S}(\mathbb{T},\mathbb{H}) \times \mathcal{S}(\mathbb{T},\mathbb{H}) \mid \exists x : \mathbb{T} \to \mathbb{X},\ x(0) = x_0 \wedge (u, x, y) \in \Sigma\}$ is an input–output system.

For $k \in \mathbb{R}_{>0}$, we define the *negative feedback operator* as

$$F_k : \mathbb{H} \to \mathbb{H}^2, \qquad a \mapsto (-ka, a).$$

A linear operator $M : \mathbb{H}^2 \to \mathbb{H}^2$ is called *stabilizable* if it is bounded, self-adjoint, and if

$$\forall k > 0, \quad \forall a \in \mathbb{H}\setminus\{0\}, \quad \langle F_k a, M F_k a\rangle_{\mathbb{H}^2} < 0. \tag{13}$$

For the ease of notation, given a stabilizable operator M and $(\mu, \gamma) \in \mathbb{H}^2$, we let

$$w_M(\mu, \gamma) := \langle(\mu, \gamma), M(\mu, \gamma)\rangle_{\mathbb{H}^2}.$$

The intuition underlying the definition of the negative feedback operator is the following. Let $\Psi \subset \mathcal{S}(\mathbb{T},\mathbb{H})^2$ be a system. Given a $y \in \operatorname{ran}\Psi$ and $k > 0$, the negative feedback operator F_k maps y to the pair $(-ky, y) \in \mathcal{S}(\mathbb{T},\mathbb{H})^2$. If $(-ky, y) \in \Psi$, then y has the interpretation of an output of Ψ corresponding to the static control $u = -ky$. In particular, the closed-loop system induced by the negative feedback operator F_k is defined as $\Psi \cap \operatorname{ran}F_k = \{(u, y) \in \mathcal{S}(\mathbb{T},\mathbb{H}) \mid u = -ky\}$.

Stabilizable operators, instead, are operators such that, for every nonzero closed-loop pair $(-ky, y) \in \Psi \cap \operatorname{ran}F_k$, one has $\langle(-ky, y), M(-ky, y)\rangle < 0$. As we illustrate later in Sections IV–VII, this property is associated with stability and with stabilizability via static output feedback.

Other notations used throughout this article are as follows: $\mathbb{R}$ and $\mathbb{N}$ denote the set of real and natural numbers, respectively $(0 \in \mathbb{N})$, $\subset$ denotes nonstrict inclusion, $|x|$ denotes the Euclidean norm of x, and $|M|$ denotes the operator norm of a bounded linear operator M. If $\mathcal{N} \subset \mathbb{N}$ is a finite set, and $(A_i)_{i\in\mathcal{N}}$ is a family of square matrices, we let $\operatorname{diag}(A_i \mid i \in \mathcal{N})$ denote their diagonal concatenation. If A and B are square matrices of the same dimension, $A > B$ means that $A - B$ is positive definite.

III. Concept of ϱ-Passivity

In this section, we introduce the concept of ϱ-passivity. As customary in the literature on passivity, we distinguish the input–output and the input-state-output cases.

A. Input–Output ϱ-Passivity

Throughout this section, Ψ denotes a system on $\mathcal{S}(\mathbb{T},\mathbb{H})$.

Definition 1: Let $\varrho \in \mathbb{T}$. System Ψ is called *(input–output) ϱ-passive* if there exist $\beta \geq 0$ and a stabilizable operator M such that

$$\int_{\mathbb{T}_{<t}} w_M\left(u(\tau), y(\tau + \varrho)\right) \mathrm{d}\mu(\tau) \geq -\beta, \tag{14}$$

for every $(u, y) \in \Psi$ and every $t \in \mathbb{T}$.

If a system Ψ is ϱ-passive and M is the stabilizable operator for which the definition is fulfilled, we shall say that Ψ is *ϱ-passive with operator M*.

Remark 1: A system Ψ is 0-passive (i.e., ϱ-passive with $\varrho = 0$) with operator M satisfying

$$\langle(\mu, \gamma), M(\mu, \gamma)\rangle = \frac{1}{2}\langle(\mu, \gamma), (\gamma, \mu)\rangle \tag{15}$$

if and only if it is passive in the usual sense, i.e.,

$$\int_{\mathbb{T}_{<t}} \langle u(\tau), y(\tau)\rangle \, \mathrm{d}\mu(\tau) \geq -\beta.$$

The following lemma proves that the integral in (14) is well defined and, hence, that Definition 1 is well posed.

Lemma 2: For every $(u, y) \in \Psi$, and every $t, \varrho \in \mathbb{T}$, $\tau \mapsto w_M(u(\tau), y(\tau + \varrho))$ is measurable and μ-integrable on $\mathbb{T}_{<t}$.

Proof: Measurability of $\tau \mapsto w_M(u(\tau), y(\tau + \varrho))$ follows from the fact that it is the composition of $\tau \mapsto (u(\tau), y(\tau + \varrho))$ (which is measurable since so are u and y^ϱ) and $(\mu, \gamma) \mapsto w_M(\mu, \gamma)$, which is continuous (hence, measurable) since M is bounded. Instead, μ-integrability of $w_M(u(\cdot), y(\cdot + \varrho))$ holds since, in view of the Cauchy–Schwartz inequality,

$$w_M(\mu, \gamma) \leq \sqrt{\langle(\mu, \gamma), (\mu, \gamma)\rangle\langle M(\mu, \gamma), M(\mu, \gamma)\rangle}$$
$$\leq |M|\left(\langle\mu, \mu\rangle + \langle\gamma, \gamma\rangle\right),$$

and both $\langle u, u\rangle_t$ and $\langle y, y\rangle_t$ are finite for all $t \in \mathbb{T}$ by definition of $\mathcal{S}(\mathbb{T},\mathbb{H})$. □

Except for the presence of ϱ, Definition 1 boils down to an *integral quadratic constraint* [41] like other passivity notions, see [42], [43], [44], and [45]. Moreover, for $\mathbb{T} = \mathbb{N}$, Definition 1 characterizes discrete-time passivity in terms of a bilinear form defined by a self-adjoint operator M, as in [46]. Unlike [46], however, here we have introduced the time-shift ϱ and, while [46] focuses on the case in which M is unitary, here we are instead interested in the cases in which M is stabilizable (see, e.g., Section VIII). While unitary operators are advantageous for characterizing all the facets of the classical definition of passivity, we have chosen here to adopt the requirement of stabilizability because it is better suited for the sought extension to ϱ-passivity and the related stability and stabilizability notions on an abstract time domain.

B. Input-State-Output ϱ-Passivity

While Definition 1 deals with input–output systems and parallels the input–output notion of passivity in (1) and (4), the next definitions relate to input-state-output systems and parallel the state-space notion in (2) and (3). In the following, Σ denotes an input-state-output system with state-space $\mathbb{X}$.

Definition 2: Given $\varrho \in \mathbb{T}$ and $x_0 \in \mathbb{X}$, Σ is called *(input-state-output) ϱ-passive at x_0* if there exist a function $S : \mathbb{X} \to \mathbb{R}_{\geq 0}$ (called the *storage function*) and a stabilizable operator M such that

$$\int_{\mathbb{T}_{<t}} w_M\left(u(\tau), y(\tau+\varrho)\right) \mathrm{d}\mu(\tau) \geq S(x(t)) - S(x(0)), \tag{16}$$

for all $(u, x, y) \in \Sigma$ satisfying $x(0) = x_0$ and all $t \in \mathbb{T}$. Moreover, Σ is called *(input-state-output) ϱ-passive* if it is ϱ-passive at every $x_0 \in \mathbb{X}$ with the same storage function S.

As for the usual notion of passivity, if Σ is ϱ-passive at some $x_0 \in \mathbb{X}$, then the associated input-output system $\Psi_{x_0}(\Sigma)$ is input–output ϱ-passive with $\beta := S(x_0)$. Hence, input-state-output ϱ-passivity is generally stronger than input–output ϱ-passivity.

Remark 2: The concept of storage function was introduced in [9, Def. 2] as a nonnegative function. Later, in [47, Th. 1], it was further required that the storage function should satisfy $S(0) = 0$ and $S(x) \geq 0$ for all x. These conditions were necessary to establish a connection between storage functions and Lyapunov functions for stability analysis and control purposes, see [6, Lemma 7]. More recently, the requirement of nonnegativity has been relaxed, allowing for storage functions that are simply bounded from below, or even unbounded in the case of cyclo-passivity, as discussed in [48].

IV. ϱ-Passivity of Interconnected Systems

One of the most essential properties of passivity is that it is preserved under a relevant class of interconnections, such as parallel and negative feedback. The study of the interconnection of passive systems is rooted in the early works [42], [49], [50] and in the pioneering book [11], in which large-scale interconnections of passive systems have been studied owing to the so-called *diagonal stability property*. Later, in [51] and [52], the diagonal stability property has been combined with the *secant gain property* to significantly broaden the class of interconnections for which passivity can be established. In this section, we parallel and extend these results to the case of input–output ϱ-passivity.

A. Main Result

Let $n \in \mathbb{N}_{>0}$ and $\mathcal{N} := \{1, \ldots, n\}$. Consider a family $(\Psi_i)_{i\in\mathcal{N}}$ of n systems Ψ_i defined on the same space $\mathcal{S}(\mathbb{T}, \mathbb{H})$. We denote by (u_i, y_i) the input–output pairs of system Ψ_i, and we let $u := (u_i)_{i\in\mathcal{N}}$ and $y := (y_i)_{i\in\mathcal{N}}$. We assume that the systems in the family are interconnected according to the relation

$$u(t) = r(t) + \Phi y^{\varrho}(t), \tag{17}$$

in which $\varrho \in \mathbb{T}$, $r := (r_i)_{i\in\mathcal{N}}$ is an *open* input, with $r_i \in \mathcal{S}(\mathbb{T}, \mathbb{H})$, and $\Phi : \mathbb{H}^n \to \mathbb{H}^n$ is a linear operator specifying how the systems of the family are coupled one to another. As shown in Section IV-C, the interconnection relation (17) enables the description of a class of interconnections, such as the parallel and the feedback interconnection. The system induced by the interconnection relation (17) is a system on $\mathcal{S}(\mathbb{T}, \mathbb{H}^n)$ with input r and output y^{ϱ}.

We make the following assumption on the systems Ψ_i.

Assumption 1: For every $i \in \mathcal{N}$, there exist $a_i, b_i, c_i, \beta_i \in \mathbb{R}$, such that every $(u_i, y_i) \in \Psi_i$ satisfies

$$\forall t \in \mathbb{T}, \quad \int_{\mathbb{T}_{<t}} w_{M_i}\left(u_i(\tau), y_i(\tau+\varrho)\right) \mathrm{d}\mu(\tau) \geq -\beta_i, \tag{18}$$

in which $M_i : \mathbb{H}^2 \to \mathbb{H}^2$ is defined as

$$M_i(\mu, \gamma) = a_i(\gamma, \mu) + b_i(0, \gamma) + c_i(\mu, 0). \tag{19}$$

Assumption 1 requires that each system Ψ_i satisfies a dissipation inequality of the same kind as (14). Note that, Assumption 1 is weaker than ϱ-passivity since M_i in (19) is not assumed stabilizable. Indeed, in lieu of requiring each system Ψ_i to be ϱ-passive, we make the following more general assumption, in which we let $A := \operatorname{diag}(a_i : i \in \mathcal{N})$, $B := \operatorname{diag}(b_i : i \in \mathcal{N})$, and $C := \operatorname{diag}(c_i : i \in \mathcal{N})$.

Assumption 2: The operator M on $\mathbb{H}^n \times \mathbb{H}^n$ defined by

$$M := \begin{bmatrix} C & A + C\Phi \\ A^\top + \Phi^\top C & A\Phi + \Phi^\top A + \Phi^\top C\Phi + B \end{bmatrix} \tag{20}$$

is stabilizable.

Assumption 2 applies to the interconnection as a whole, without requiring passivity of each individual subsystem. This requirement allows handling interconnections in which some excess of passivity of one or more subsystems can compensate for a shortage of passivity of other subsystems.

The following theorem establishes the main result of this section.

Theorem 1: Let Ψ be the system defined by the interconnection relation (17). Suppose that Assumptions 1 and 2 hold. Then, Ψ is ϱ-passive.

Proof: For every $i \in \mathcal{N}$, pick $(u_i, y_i) \in \Psi_i$ arbitrarily, and pick any $r \in \mathcal{S}(\mathbb{T}, \mathbb{H}^n)$. Under Assumption 1, summing all the terms $w_{M_i}(u_i(\tau), y_i^{\varrho}(\tau))$ over $i \in \mathcal{N}$ and using (17) and (19) yields, for every $\tau \in \mathbb{T}$

$$\begin{aligned}
&\sum_{i\in\mathcal{N}} w_{M_i}(u_i(\tau), y_i^{\varrho}(\tau)) \\
&= \sum_{i\in\mathcal{N}} \big(2a_i\langle u_i(\tau), y_i^{\varrho}(\tau)\rangle + b_i\langle y_i^{\varrho}(\tau), y_i^{\varrho}(\tau)\rangle \\
&\qquad + c_i\langle u_i(\tau), u_i(\tau)\rangle\big) \\
&= 2\langle u(\tau), Ay^{\varrho}(\tau)\rangle + \langle y^{\varrho}(\tau), By^{\varrho}(\tau)\rangle + \langle u(\tau), Cu(\tau)\rangle \\
&= 2\langle r(\tau) + \Phi y^{\varrho}(\tau), Ay^{\varrho}(\tau)\rangle + \langle y^{\varrho}(\tau), By^{\varrho}(\tau)\rangle \\
&\quad + \langle r(\tau) + \Phi y^{\varrho}(\tau), C(r(\tau) + \Phi y^{\varrho}(\tau))\rangle \\
&= \langle r(\tau), Cr(\tau)\rangle + 2\langle r(\tau), (A + C\Phi)y^{\varrho}(\tau)\rangle \\
&\quad + \langle y^{\varrho}(\tau), (A\Phi + \Phi^\top A + \Phi^\top C\Phi + B)y^{\varrho}(\tau)\rangle \\
&= \langle (r(\tau), y^{\varrho}(\tau)), M(r(\tau), y^{\varrho}(\tau))\rangle \\
&= w_M(r(\tau), y^{\varrho}(\tau)).
\end{aligned}$$

Hence, by setting $\beta := \sum_{i\in\mathcal{N}} \beta_i$, from (18) we obtain

$$\forall t \in \mathbb{T}, \quad \int_{\mathbb{T}_{<t}} w_M(r(\tau), y^{\varrho}(\tau))\,\mathrm{d}\mu(\tau) \geq -\beta.$$

The claim of the theorem then follows from Assumption 2. $\square$

B. Connections With the Secant Gain Property

Assumptions 1 and 2, together, generalize the diagonal stability and the secant gain properties of [51], [52], and [53], both of which are obtained by considering an interconnection defined by an operator of the form

$$\Phi = \begin{bmatrix} 0 & 0 & \cdots & 0 & -d_n \\ d_1 & 0 & \ddots & & 0 \\ 0 & d_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_{n-1} & 0 \end{bmatrix}, \tag{21}$$

with $d_i \in \mathbb{R}_{>0}$ for each $i \in \mathcal{N}$. In particular, Arcak and Sontag [52] proved that there exist $a_1, \ldots, a_n > 0$ such that $A := \operatorname{diag}(a_i : i \in \mathcal{N})$ satisfies the diagonal stability condition

$$A\Phi + \Phi^\top A - 2A < 0 \tag{22}$$

if and only if the secant gain property

$$\prod_{i\in\mathcal{N}} d_i < \sec(\pi/n)^n \tag{23}$$

holds. In the setting of Theorem 1, if A is such that (22) holds, $B = -2A$, and $C = 0$, then the diagonal stability condition (22) is equivalent to stabilizability of M (i.e., to Assumption 2). Indeed, with such a choice of A, B, and C, the result of [51] implies that Assumption 2 holds if and only if the secant gain condition (23) holds.

C. Elementary Interconnections

As a corollary to Theorem 1, in this section, we derive ϱ-passivity conditions for parallel, series, and feedback interconnections. As a preliminary step, we prove a lemma that is useful in the forthcoming results.

Lemma 3: Suppose that the matrices $C\Phi$ and $A\Phi$ are skew-symmetric and that M_i is stabilizable for each $i \in \mathcal{N}$. Then, M, defined in (20), is stabilizable.

Proof: For every $k > 0$ and $a \in \mathbb{H}^n \setminus \{0\}$, we have that

$$\begin{aligned}\langle F_k a, M F_k a\rangle &= \langle a, (-k(C\Phi + \Phi^\top C) + A\Phi + \Phi^\top A)a\rangle \\ &\quad + \langle a, (k^2 C - 2kA + B + \Phi^\top C\Phi)a\rangle \\ &= \langle a, (k^2 C - 2kA + B + \Phi^\top C\Phi)a\rangle,\end{aligned}$$

in which we have used the fact that $C\Phi + \Phi^\top C = 0$ and $A\Phi + \Phi^\top A = 0$, since $C\Phi$ and $A\Phi$ are skew-symmetric.

If M_i is stabilizable for every $i \in \mathcal{N}$, then $\langle F_k a, M_i F_k a\rangle = (k^2 c_i - 2k a_i + b_i)\langle a, a\rangle < 0$ for all $i \in \mathcal{N}$, $k > 0$, and $a \in \mathbb{H} \setminus \{0\}$. Since the latter inequality must hold for all $k > 0$, necessarily $c_i \leq 0$. Therefore, we conclude that both $k^2 C - 2kA - B < 0$ and $\Phi^\top C\Phi \leq 0$ hold, which imply that $\langle F_k a, M F_k a\rangle < 0$ for all $a \in \mathbb{H}^n$, hence the claim. $\square$

1) Parallel Interconnections: We define the parallel interconnection Ψ_{p} of a family of systems $(\Psi_i)_{i\in\mathcal{N}}$ by taking the sum of the outputs of all systems Ψ_i corresponding to the same input. Namely

$$\Psi_{\mathrm{p}} := \left\{ \left(v, \sum_{i\in\mathcal{N}} y_i\right) \in \mathcal{S}(\mathbb{T}, \mathbb{H})^2 \;\middle|\; \forall i \in \mathcal{N},\ (v, y_i) \in \Psi_i \right\}.$$

From Theorem 1, we obtain the following result.

Corollary 1: If there exists $\varrho \in \mathbb{T}$ such that, for all $i \in \mathcal{N}$, Ψ_i is ϱ-passive with operator M_i satisfying $M_i(\mu, \gamma) = (\gamma, \mu) + b_i(0, \gamma) + c_i(\mu, 0)$ for all $\mu, \gamma \in \mathbb{H}$ and for some $b_i, c_i \leq 0$, then Ψ_{p} is ϱ-passive with operator

$$M_{\mathrm{p}} := \begin{bmatrix} \sum_{i\in\mathcal{N}} c_i & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof: By assumption, the family $(\Psi_i)_{i\in\mathcal{N}}$ satisfies Assumption 1. Set $\Phi = 0$ in (17), and let Ψ denote the resulting interconnection. Since M_i is stabilizable for each $i \in \mathcal{N}$, Lemma 3 implies that also Assumption 2 holds. Thus, Theorem 1 implies that Ψ is ϱ-passive.

Let $v \in \mathcal{S}(\mathbb{T}, \mathbb{H})$ and pick $r := (v, \ldots, v)$. For brevity, let $y_{\mathrm{p}} := \sum_{i\in\mathcal{N}} y_i$. Then, for all $\tau \in \mathbb{T}$, we obtain

$$\begin{aligned}&w_M(r(\tau), y^{\varrho}(\tau)) \\ &= \sum_{i\in\mathcal{N}} \left(c_i\langle r_i(\tau), r_i(\tau)\rangle + 2\langle r_i(\tau), y_i^{\varrho}(\tau)\rangle + b_i\langle y_i^{\varrho}(\tau), y_i^{\varrho}(\tau)\rangle\right) \\ &\leq \left(\sum_{i\in\mathcal{N}} c_i\right)\langle v(\tau), v(\tau)\rangle + 2\langle v(\tau), y_{\mathrm{p}}^{\varrho}(\tau)\rangle \\ &= w_{M_{\mathrm{p}}}(v(\tau), y_{\mathrm{p}}^{\varrho}(\tau)),\end{aligned}$$

from which the claim follows. $\square$

2) Feedback Interconnections: We define the negative feedback interconnection Ψ_{f} of two systems Ψ_1 and Ψ_2 by setting $u_2 = y_1^{\varrho}$ and $u_1 = v - y_2^{\varrho}$, in which v is an open input, and taking y_1 as output. Namely

$$\begin{aligned}\Psi_{\mathrm{f}} := \{&(v, y_1) \in \mathcal{S}(\mathbb{T}, \mathbb{H})^2 \mid \exists y_2 \in \mathcal{S}(\mathbb{T}, \mathbb{H}),\ (y_1^{\varrho}, y_2) \in \Psi_2, \\ &(v - y_2^{\varrho}, y_1) \in \Psi_1\}.\end{aligned}$$

A classical result is that the negative feedback interconnection of two passive systems is passive. In the ϱ-passivity setting of this article, an analogous result follows from Corollary 2 below, which provides a more general result applying to a generic finite family of systems $(\Psi_i)_{i\in\mathcal{N}}$.

Consider the system $\overline{\Psi}_{\mathrm{f}}$ obtained by interconnecting $(\Psi_i)_{i\in\mathcal{N}}$ according to (17) with a generic skew-symmetric operator Φ (note that $\overline{\Psi}_{\mathrm{f}}$ is different from Ψ_{f} defined above, and the latter is indeed a special case of the former, as detailed in Example 4). Suppose that the systems Ψ_i satisfy Assumption 1 with $a_i = 1$, $b_i \leq 0$, and $c_i = c_j \leq 0$ for all $i, j \in \mathcal{N}$ (we remark that assuming $a_i = 1$ and $c_i = c_j$ for all $i, j \in \mathcal{N}$ leads to no loss of generality). Then, the following result holds.

Corollary 2: $\overline{\Psi}_{\mathrm{f}}$ is ϱ-passive.

Proof: Consider the operator M defined in (20). Since $A = I$, $C = cI$ for some $c \leq 0$, and Φ is skew-symmetric, then $C\Phi$ and $A\Phi$ are skew-symmetric. Since $a_i = 1$, $b_i \leq 0$, and $c_i \leq 0$ imply that M_i is stabilizable, the claim follows from Lemma 3. $\square$

Example 4: Let $\mathcal{N} = \{1, 2\}$ and let the interconnection operator be defined as

$$\Phi = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

In this case, we obtain the inequality

$$\begin{aligned} w_M((v,0), y^\varrho) &\leq c\langle v, v\rangle + 2\langle v, y_1^\varrho\rangle + (b+c)\langle y^\varrho, y^\varrho\rangle \\ &= w_{M_\mathrm{f}}(v, y_1^\varrho) \end{aligned}$$

in which $c := c_1 = c_2$, $b := \max\{b_1, b_2\}$, and

$$M_\mathrm{f} := \begin{bmatrix} c & 1 \\ 1 & c+b \end{bmatrix}$$

that is stabilizable. Thus, from Corollary 2, it follows that the feedback interconnection Ψ_f of two systems Ψ_1 and Ψ_2 satisfying Assumption 1 with $a_1 = a_2 = 1$, $c_1 = c_2 \leq 0$, and $b_1, b_2 \leq 0$ is ϱ-passive. $\triangleleft$

3) Series Interconnections: We define the series interconnection Ψ_s of the family of systems $(\Psi_i)_{i\in\mathcal{N}}$ as

$$\begin{aligned} \Psi_\mathrm{s} := \big\{&(v, y_n) \in \mathcal{S}(\mathbb{T},\mathbb{H})^2 \mid \exists y_1, \ldots, y_{n-1} \in \mathcal{S}(\mathbb{T},\mathbb{H}), \\ &(v, y_1) \in \Psi_1 \wedge (\forall i \in \mathcal{N}\backslash\{1\},\ (y_{i-1}, y_i) \in \Psi_i)\big\}. \end{aligned}$$

Unlike parallel and negative feedback interconnections, series interconnections of passive systems are not always passive. In our setting, passivity of Ψ_s is implied by the forthcoming Corollary 3.

As a preliminary step, consider the system $\overline{\Psi}_\mathrm{s}$ defined by interconnecting $(\Psi_i)_{i\in\mathcal{N}}$ according to (17) with

$$\Phi = \begin{bmatrix} 0_{1\times(n-1)} & 0 \\ D & 0_{(n-1)\times 1} \end{bmatrix},$$

in which $D := \mathrm{diag}(d_1, \ldots, d_{n-1})$ with $d_i \neq 0$ for all $i = 1, \ldots, n-1$. Suppose that the systems Ψ_i satisfy Assumption 1 with a_i, b_i, c_i such that

$$C \leq 0, \quad A + C\Phi > 0, \quad A\Phi + \Phi^\top A + \Phi^\top C\Phi + B \leq 0. \tag{24}$$

Then, the following result holds.

Corollary 3: $\overline{\Psi}_\mathrm{s}$ is ϱ-passive.

Proof: The proof directly follows from Theorem 1 since (24) implies Assumption 2. $\square$

Under the previous assumptions, as a consequence of Corollary 3, we can conclude that the series interconnection Ψ_s is ϱ-passive if there exists a stabilizable operator $M_\mathrm{s} : \mathbb{H}^2 \to \mathbb{H}^2$ such that, for all $((v, 0), y) \in \Psi$ and all $t \in \mathbb{T}$, the condition

$$\int_{\mathbb{T}_{<t}} \left(w_M((v(\tau), 0), y^\varrho(\tau)) - w_{M_\mathrm{s}}(v(\tau), y_n^\varrho(\tau))\right) \mathrm{d}\mu(\tau) \leq 0$$

holds.

V. ϱ-Passivity and Stability

In this section, we connect ϱ-passivity to stability, finite-gain stability, and small-gain properties, thereby enhancing the results of classical passivity, see, e.g., [7, Ch. 2].

A. Boundedness, Stability, and Finite Gain

We begin by introducing the boundedness, stability, and finite-gain notions that will be the subjects of the forthcoming sections. To this end, consider a system Ψ defined on $\mathcal{S}(\mathbb{T},\mathbb{H})$, and fix $\varrho \in \mathbb{T}$. Let $\mathcal{S}_\mathrm{b}^\varrho(\mathbb{T},\mathbb{H})$ denote the set of all $s \in \mathcal{S}(\mathbb{T},\mathbb{H})$ for which there exists $c \geq 0$ such that

$$\forall t \in \mathbb{T},\ \langle s^\varrho, s^\varrho\rangle_t \leq c^2 \tag{25}$$

with $\langle\cdot,\cdot\rangle_t$ defined in (11). In addition, if $s \in \mathcal{S}_\mathrm{b}^\varrho(\mathbb{T},\mathbb{H})$, we define

$$|s^\varrho| := \inf\{c \geq 0 \mid (25) \text{ holds}\}.$$

Finally, we let $\mathcal{S}_\mathrm{b}(\mathbb{T},\mathbb{H}) := \mathcal{S}_\mathrm{b}^0(\mathbb{T},\mathbb{H})$.

Remark 3: The set $\mathcal{S}(\mathbb{T},\mathbb{H})$ (respectively, $\mathcal{S}_\mathrm{b}^\varrho(\mathbb{T},\mathbb{H})$) is a generalization, in the setting of ϱ-passivity, of an extended Lebesgue space (respectively a Lebesgue space), which is the usual setting of input–output control theory (see, e.g., [3] and [11, Sec. 2.1]).

Lemma 4: For every $s \in \mathcal{S}_\mathrm{b}^\varrho(\mathbb{T},\mathbb{H})$ and every $t \in \mathbb{T}$, $\langle s^\varrho, s^\varrho\rangle_t \leq |s^\varrho|^2$.

Proof: Assume, *ex absurdo*, that the claim does not hold. Then, there exist $s \in \mathcal{S}_\mathrm{b}^\varrho(\mathbb{T},\mathbb{H})$, $t \in \mathbb{T}$, and $\epsilon > 0$ such that $\langle s^\varrho, s^\varrho\rangle_t \geq |s^\varrho|^2 + \epsilon$. Let $C := \{c \geq 0 \mid (25) \text{ holds}\}$ and $k := \inf C$. By (25), for all $c \in C$, $c^2 \geq \langle s^\varrho, s^\varrho\rangle_t$. This implies $k^2 \geq \langle s^\varrho, s^\varrho\rangle_t$, as, otherwise, $\langle s^\varrho, s^\varrho\rangle_t$ would be a lower bound of C greater than k. Hence, noting that $k = |s^\varrho|$ by definition, we obtain $|s^\varrho|^2 = k^2 \geq \langle s^\varrho, s^\varrho\rangle_t \geq |s^\varrho|^2 + \epsilon > |s^\varrho|^2$, which contradicts the claim. $\square$

We now define the notion of *ϱ-bounded* system and of *internally ϱ-stable* system. Here, and in the rest of this section, for a given subset $U \subset \mathcal{S}(\mathbb{T},\mathbb{H})$ of inputs, we let $\Psi(U) := \{y \in \mathcal{S}(\mathbb{T},\mathbb{H}) \mid \exists u \in U,\ (u, y) \in \Psi\}$ denote its image under Ψ.

Definition 3: Ψ is called ϱ-bounded if

$$\forall U \subset \mathcal{S}_\mathrm{b}(\mathbb{T},\mathbb{H}),\ \Psi(U) \subset \mathcal{S}_\mathrm{b}^\varrho(\mathbb{T},\mathbb{H}).$$

If Ψ is ϱ-bounded, we call Ψ_b^ϱ the restriction of Ψ to $\mathcal{S}_\mathrm{b}(\mathbb{T},\mathbb{H})$, namely, $\Psi_\mathrm{b}^\varrho := \Psi \cap (\mathcal{S}_\mathrm{b}(\mathbb{T},\mathbb{H}) \times \mathcal{S}_\mathrm{b}^\varrho(\mathbb{T},\mathbb{H}))$.

For many systems of interest, ϱ-boundedness implies the *vanishing-input-vanishing-output* property with respect to inputs vanishing sufficiently fast. This is illustrated in the following example.

Example 5: Consider the case in which $\mathbb{T} = \mathbb{N}$, $\mathbb{H} = \mathbb{R}$, and μ is the counting measure. Then, $\mathcal{S}(\mathbb{N},\mathbb{R})$ is the space of locally square-summable sequences, and $\mathcal{S}_\mathrm{b}^\varrho(\mathbb{N},\mathbb{R}) \subset \ell^2$. If Ψ is ϱ-bounded, then for every input u vanishing sufficiently fast so that $u \in \mathcal{S}_\mathrm{b}(\mathbb{N},\mathbb{R})$, all corresponding outputs y are in ℓ^2 and, hence, they satisfy $\lim_{t\to\infty} y(t) = 0$. With additional technical conditions, a similar conclusion also holds for continuous-time systems. Let $\mathbb{T} = \mathbb{R}_{\geq 0}$, $\mathbb{H} = \mathbb{R}$, and μ the Lebesgue measure. Then, $\mathcal{S}(\mathbb{R}_{\geq 0},\mathbb{R})$ is the space of locally square-integrable functions, and every continuous $y \in \mathcal{S}_\mathrm{b}^\varrho(\mathbb{R}_{\geq 0},\mathbb{R})$ is square-integrable. If Ψ is ϱ-bounded, then for every input $u \in \mathcal{S}_\mathrm{b}(\mathbb{R}_{\geq 0},\mathbb{R})$, all corresponding outputs are in $\mathcal{S}_\mathrm{b}^\varrho(\mathbb{R}_{\geq 0},\mathbb{R})$. Then, in view of Barbălat's Lemma [54, Sec. 4.5.2], every such output y that is absolutely continuous satisfies $\lim_{t\to\infty} y(t) = 0$. $\triangleleft$

Definition 4: A system Ψ is called *internally ϱ-stable* if, for every $(u,y)\in\Psi$ with $u=0$, $y\in\mathcal{S}_{\mathrm{b}}^{\varrho}(\mathbb{T},\mathbb{H})$.

Clearly, every ϱ-bounded system is internally ϱ-stable. In addition, as established by the following theorem, an important class of ϱ-passive systems can be made internally ϱ-stable by static output feedback with any positive gain. Specifically, we call a stabilizable operator M *strictly stabilizable* if

$$\exists c>0\quad \forall a\in\mathbb{H},\ \langle F_k a, MF_k a\rangle \le -c\langle a,a\rangle. \tag{26}$$

We note that, if $\mathbb{H}$ is finite-dimensional, then every stabilizable operator is strictly stabilizable. Hence, the following result holds.

Theorem 2: Given any $k>0$, define

$$\Psi_{|k} := \{(r,y)\in\mathcal{S}(\mathbb{T},\mathbb{H})^2 \mid (-ky^{\varrho}+r,y)\in\Psi\}.$$

If Ψ is ϱ-passive with a strictly stabilizable operator M, then $\Psi_{|k}$ is internally ϱ-stable.

Proof: If $0\notin\operatorname{dom}\Psi_{|k}$, the claim is trivially true. Otherwise, $(-ky^{\varrho},y)\in\Psi_{|k}$. Let M be such that (14) and (26) hold. Then, the following holds

$$\begin{aligned} w_M(-ky^{\varrho}(\tau),y^{\varrho}(\tau)) &= \langle F_k y^{\varrho}(\tau), MF_k y^{\varrho}(\tau)\rangle \\ &\le -c\langle y^{\varrho}(\tau),y^{\varrho}(\tau)\rangle, \end{aligned}$$

for all $\tau\in\mathbb{T}$, and (14) implies

$$-\beta \le \int_{\mathbb{T}_{<t}} w_M(-ky^{\varrho}(\tau),y^{\varrho}(\tau))\,\mathrm{d}\mu(\tau) \le -c\langle y^{\varrho},y^{\varrho}\rangle_t.$$

Hence, $\langle y^{\varrho},y^{\varrho}\rangle_t \le \beta c^{-1}$, for all $t\in\mathbb{T}$, which proves the claim. □

Internal stability in terms of Definition 4 characterizes the behavior of the system only for $u=0$. We now introduce a notion of stability, called *ϱ-stability*, which concerns the full input–output behavior. To this end, we first define a topology on $\mathcal{S}_{\mathrm{b}}^{\varrho}(\mathbb{T},\mathbb{H})$ as follows. Let $q\ge 0$ and, for every $\varepsilon\in(q,\infty)$, define the set

$$B_{\varepsilon}^{\varrho} := \{s\in\mathcal{S}_{\mathrm{b}}^{\varrho}(\mathbb{T},\mathbb{H}) \mid |s^{\varrho}|<\varepsilon\}.$$

The family $\mathcal{B}_q^{\varrho} := \{B_{\varepsilon}^{\varrho}\}_{\varepsilon\in(q,\infty)}$ satisfies the following properties.

1) For every $s\in\mathcal{S}_{\mathrm{b}}^{\varrho}(\mathbb{T},\mathbb{H})$, there exists $\varepsilon\in(q,\infty)$ such that $s\in B_{\varepsilon}^{\varrho}$ (e.g., $\varepsilon=q+|s^{\varrho}|+1$).
2) $\mathcal{B}_q^{\varrho}$ is closed under finite intersections.

Hence, $\mathcal{B}_q^{\varrho}$ generates a topology[2] $\mathcal{T}_q^{\varrho}$ on $\mathcal{S}_{\mathrm{b}}^{\varrho}(\mathbb{T},\mathbb{H})$, and $\mathcal{B}_q^{\varrho}$ is a base for $\mathcal{T}_q^{\varrho}$ [55, Prop. 1.2.1]. Following [40], we define the following notion of input–output stability as a property similar to upper semicontinuity with respect to the previously defined topology.

Definition 5: Ψ is called *$\mathcal{T}_q^{\varrho}$-stable* if, for every $\mathcal{T}_q^{\varrho}$-open set Y, there exists a $\mathcal{T}_0^{0}$-open set U, such that $\Psi(U)\subset Y$.

Then, ϱ-stability is defined as follows.

Definition 6: Ψ is called *ϱ-stable* if there exists $q\ge 0$ such that Ψ is *$\mathcal{T}_q^{\varrho}$-stable*.

Finally, we introduce the concept of ϱ-gain, which generalizes to our setting the classical notion of finite gain [11, Sec. 3.1].

[2]In particular, $\mathcal{T}_q^{\varrho}$ is defined as the family of sets that are unions of subfamilies of $\mathcal{B}_q^{\varrho}$.

Definition 7: Ψ has the *weak finite-ϱ-gain property* if Ψ is ϱ-bounded and there exist $g\ge 0$ and $b\ge 0$ such that

$$\forall (u,y)\in\Psi_{\mathrm{b}}^{\varrho},\quad |y^{\varrho}|\le b+g|u|. \tag{27}$$

The constant g is called the *ϱ-gain* of the system, and b the *bias*. If $b=0$, we say that Ψ has the *finite-ϱ-gain property*.

Remark 4: The notions of ϱ-boundedness, ϱ-stability, and finite-ϱ-gain generalize the usual bounded-input-bounded-output, input–output stability, and finite-gain properties of a system [3], [11, Secs. 2.1 and 3.1]. For instance, Zames [3] called a system *input–output stable*, if it is bounded and continuous [3, Sec. 2.4]. ϱ-boundedness and ϱ-stability are the corresponding notions for "shifted" signals of the abstract space $\mathcal{S}(\mathbb{T},\mathbb{H})$.

We stress that every system with the weak finite-ϱ-gain property is ϱ-bounded and, hence, is internally ϱ-stable. In addition, the following result links finite-ϱ-gain, $\mathcal{T}_q^{\varrho}$-stability, and ϱ-stability.

Theorem 3: If Ψ has the weak finite-ϱ-gain property with bias b, it is $\mathcal{T}_q^{\varrho}$-stable with $q=b$. Hence, it is ϱ-stable.

Proof: Pick $q=b$. Every $\mathcal{T}_q^{\varrho}$-open set Y is a union of basic sets, namely, there exists $E\subset(b,\infty)$ such that $Y=\cup_{\varepsilon\in E}B_{\varepsilon}^{\varrho}$ [55, Prop. 1.2.1]. If $\sup E=\infty$, then $Y=\mathcal{S}_{\mathrm{b}}^{\varrho}(\mathbb{T},\mathbb{H})$. Hence, the result follows by ϱ-boundedness of Ψ since $\mathcal{S}_{\mathrm{b}}(\mathbb{T},\mathbb{H})$ is $\mathcal{T}_0^{0}$-open. Otherwise, pick any $\bar\varepsilon\in(b,\sup E)$. Then, there exists $\hat\varepsilon\in E$ such that $\hat\varepsilon\ge\bar\varepsilon$. As $\varepsilon\mapsto B_{\varepsilon}^{\varrho}$ is increasing, then $B_{\bar\varepsilon}^{\varrho}\subset B_{\hat\varepsilon}^{\varrho}\subset Y$. Hence, the set $U:=B^{0}_{(\bar\varepsilon-b)g^{-1}}$ is $\mathcal{T}_0^{0}$-open and, in view of (27), satisfies $\Psi(U)\subset B_{\bar\varepsilon}^{\varrho}\subset Y$. This proves that Ψ is $\mathcal{T}_b^{0}$-stable and, hence, ϱ-stable. □

B. Output-Strict ϱ-Passivity Implies Finite-ϱ-Gain and ϱ-Stability

As Theorem 2 shows, a ϱ-passive system (with a strictly stabilizable operator) can be made internally stable by any static output feedback. This section proves a stronger and "input–output" result showing that a system that is *strictly* ϱ-passive (in a sense made precise by Definition 8 below) has the (weak) finite-ϱ-gain property.

Definition 8: Ψ is called *strictly output ϱ-passive* if it is ϱ-passive with some operator M and there exist $m>0$ and a bounded linear operator $R:\mathbb{H}\to\mathbb{H}$ such that

$$\forall(\mu,\gamma)\in\mathbb{H}^2,\ \langle(\mu,\gamma),M(\mu,\gamma)\rangle \le \langle\mu,R\gamma\rangle - m\langle\gamma,\gamma\rangle. \tag{28}$$

Theorem 4: If Ψ is strictly output ϱ-passive, then it has the weak finite-ϱ-gain property. Hence, it is ϱ-stable.

Proof: Let m be the same as in (28). Using Lemma 1 with $\varepsilon=\frac{|R|^2}{m}$ yields

$$\langle\mu,R\gamma\rangle \le \frac{|R|^2}{m}\langle\mu,\mu\rangle + \frac{m}{2}\langle\gamma,\gamma\rangle.$$

Thus, (28) implies

$$\forall(\mu,\gamma)\in\mathbb{H}^2,\quad w_M(\mu,\gamma)\le k_1\langle\mu,\mu\rangle - k_2\langle\gamma,\gamma\rangle,$$

with $k_1=\frac{|R|^2}{m}$ and $k_2=\frac{m}{2}$. Hence, ϱ-passivity of Ψ implies that, for every $(u,y)\in\Psi$ and $t\in\mathbb{T}$, the condition

$$-\beta \le \int_{\mathbb{T}_{<t}} w_M(u(\tau),y^{\varrho}(\tau))\,\mathrm{d}\mu(\tau) \le k_1\langle u,u\rangle_t - k_2\langle y^{\varrho},y^{\varrho}\rangle_t$$

holds, which leads to

$$\forall t \in \mathbb{T}, \quad \langle y^\varrho, y^\varrho \rangle_t \le b^2 + g^2 \langle u, u \rangle_t, \tag{29}$$

with $b^2 := \max\{0, \frac{\beta}{k_2}\}$ and $g^2 := \frac{k_1}{k_2}$. Let $u \in \mathcal{S}_\mathrm{b}(\mathbb{T}, \mathbb{H})$. Lemma 4 implies that $\langle u, u \rangle_t \le |u|^2$ for all $t \in \mathbb{T}$. Thus, (29) implies that every $y \in \mathcal{S}(\mathbb{T}, \mathbb{H})$ such that $(u, y) \in \Psi$ satisfies

$$\forall t \in \mathbb{T}, \quad \langle y^\varrho, y^\varrho \rangle_t \le b^2 + g^2 |u|^2.$$

This, in turn, implies $|y^\varrho| \le b + g|u|$, which is condition (27). Finally, from Theorem 3, we conclude that Ψ is ϱ-stable. $\square$

In the same spirit of Section IV, the following corollary provides sufficient conditions for ϱ-stability of interconnected systems. Within the setting of Section IV-A, consider a finite family $(\Psi_i)_{i \in \mathcal{N}}$ of n systems Ψ_i, defined on $\mathcal{S}(\mathbb{T}, \mathbb{H})$, interconnected according to (17), and satisfying Assumptions 1 and 2. Let Ψ denote such an interconnection. As a consequence of Theorems 1 and 4, we obtain the following result.

Corollary 4: In addition to Assumptions 1 and 2, assume that there exists $m > 0$ such that

$$\forall \gamma \in \mathbb{H}, \ \langle \gamma, (A\Phi + \Phi^\top A + \Phi^\top C \Phi + B)\gamma \rangle \le -m \langle \gamma, \gamma \rangle. \tag{30}$$

Then, Ψ is ϱ-stable.

Proof: By Theorem 1, Ψ is ϱ-passive with operator M defined in (20) and, in view of (30), satisfying

$$\langle (\mu, \gamma), M(\mu, \gamma) \rangle \le \langle \mu, C\mu \rangle + 2 \langle \mu, (A + C\Phi)\gamma \rangle - m \langle \gamma, \gamma \rangle.$$

Since M is stabilizable, $\langle \mu, C\mu \rangle \le 0$ for all $\mu \in \mathbb{H}$. Then, the latter inequality implies (28) with $R = 2(A + C\Phi)$ bounded. Hence, the result follows from Theorem 4. $\square$

VI. ϱ-Passivity and High-Gain Stabilization

Building on the notion of ϱ-passivity, this section provides sufficient conditions under which a system is "high-gain stabilizable," namely, it can be stabilized by static output feedback with a sufficiently large gain to ensure that the resulting closed-loop system has the (weak) finite-ϱ-gain property with arbitrarily small gain and bias.

A. High-Gain Stabilizability

We formalize high-gain stabilizability as follows.

Definition 9: Ψ is called *weakly ϱ-high-gain stabilizable* if, for every $\epsilon > 0$, there exists $k^*(\epsilon) > 0$, such that, for every $k \ge k^*(\epsilon)$ and every $y, r \in \mathcal{S}(\mathbb{T}, \mathbb{H})$,

$$(-ky^\varrho + r, y) \in \Psi \implies \forall t \in \mathbb{T}, \ \langle y^\varrho, y^\varrho \rangle_t \le \epsilon(1 + \langle r, r \rangle_t). \tag{31}$$

Remark 5: The qualifier "weak" in Definition 9 refers to the fact that it is not required a priori that $(-ky^\varrho + r, y)$ belongs to the system. We also remark that, if Ψ has the ϱ-high-gain property, then, for every $k > 0$, the closed-loop system

$$\Psi_{|k} := \{(r, y) \in \mathcal{S}(\mathbb{T}, \mathbb{H})^2 \mid (-ky^\varrho + r, y) \in \Psi\}$$

is finite-ϱ-gain stable. Indeed, (31) implies that $\Psi_{|k}$ is ϱ-bounded and that (27) holds. Moreover, we stress that both the bias and the ϱ-gain can be taken arbitrarily small by choosing k large enough.

Remark 6: As $\Psi_{|k}$ is ϱ-bounded, it has the vanishing-input-vanishing-output property as discussed in Example 5. In addition, if Ψ has the weak ϱ-high-gain property, then the norm of y^ϱ can be arbitrarily decreased by increasing k. In some cases this may relate to the convergence rate of the autonomous behavior when $r = 0$.

B. The High-Gain Theorem

We begin with the following definition.

Definition 10: A system Ψ is called *output-shortly ϱ-passive* if there exists $\beta \in \mathbb{R}$ such that every $(u, y) \in \Psi$ satisfies (14) for all $t \in \mathbb{T}$, with M satisfying the condition

$$\forall (\mu, \gamma) \in \mathbb{H}^2, \quad \langle (\mu, \gamma), M(\mu, \gamma) \rangle \le m_1 \langle \gamma, \mu \rangle + m_2 \langle \gamma, \gamma \rangle, \tag{32}$$

for some $m_1 > 0$ and $m_2 \ge 0$.

The following theorem shows that all systems that are "output-short" of ϱ-passivity are weakly ϱ-high-gain stabilizable.

Theorem 5: If Ψ is output-shortly ϱ-passive, it is weakly ϱ-high-gain stabilizable.

Proof: Pick $\epsilon > 0$ arbitrarily and, with $\beta \in \mathbb{R}$ such that (14) holds for Ψ, let $k^*(\epsilon) := \frac{m_2}{m_1} + \max\left\{\sqrt{\frac{2}{\epsilon}}, \frac{2|\beta|}{m_1 \epsilon}\right\}$.

Pick $k \ge k^*(\epsilon)$ and $r \in \mathcal{S}(\mathbb{T}, \mathbb{H})$ arbitrarily. If $y \in \mathcal{S}(\mathbb{T}, \mathbb{H})$ is such that $(-ky^\varrho + r, y) \notin \Psi$, then (31) is trivially true.

Otherwise, we obtain from the assumptions of the theorem that, for all $t \in \mathbb{T}$,

$$\begin{aligned} -\beta &\le m_1 \langle y^\varrho, -ky^\varrho + r \rangle_t + m_2 \langle y^\varrho, y^\varrho \rangle_t \\ &= (m_2 - km_1) \langle y^\varrho, y^\varrho \rangle_t + m_1 \langle y^\varrho, r \rangle_t \\ &\le -\frac{hm_1}{2} \langle y^\varrho, y^\varrho \rangle_t + \frac{m_1}{h} \langle r, r \rangle_t, \end{aligned}$$

in which $h := k - \frac{m_2}{m_1}$ is such that $h \ge \max\left\{\sqrt{\frac{2}{\epsilon}}, \frac{2|\beta|}{m_1 \epsilon}\right\}$, and where, in the last inequality, we have used Lemma 1 with $\varepsilon = \frac{h}{2}$. Then, we obtain

$$\langle y^\varrho, y^\varrho \rangle_t \le \frac{2|\beta|}{hm_1} + \frac{2}{h^2} \langle r, r \rangle_t \le \epsilon + \epsilon \langle r, r \rangle_t,$$

hence the claim. $\square$

C. High-Gain Control in Continuous- and Discrete-Time

This section links the previous notions and results with classical high-gain control ideas, see e.g. [56, Sec. 4.7]. In the basic single-input single-output setting, one starts from an equation of the form

$$\dot{y} = f(y) + bu, \tag{33}$$

in which f is Lipschitz. Clearly, the input–output system obtained for each initial condition $y(0)$ is output-shortly 0-passive. Indeed, (14) holds with $\varrho = 0$ and M that satisfies (32) with m_1 the Lipschitz constant of f and $m_2 = b$. Then, for every $\epsilon > 0$, the control

$$u = -ky + r \tag{34}$$

guarantees (31) for all sufficiently large k.

An equivalent notion of high-gain stabilization for nonlinear discrete-time systems is missing. Indeed, a control policy of the kind of (34) makes little sense in a discrete-time setting when high gain is sought, since taking k too large destabilizes the system as in Example 2. The theory of ϱ-passivity developed so far, in particular Theorem 5, gives a different perspective on discrete-time stabilization, suggesting that, in lieu of (34), one should choose (throughout this section, we let $y^+(t) := y(t+1)$)

$$u = -ky^+ + r, \tag{35}$$

since discrete-time systems with relative degree 1 are, at best, 1-passive. Specifically, consider the discrete-time system

$$y^+ = f(y) + bu. \tag{36}$$

If f is Lipschitz, the input–output system produced by (36) from every fixed initial condition is output-shortly 1-passive. Hence, Theorem 5 motivates using (35). Substituting (36) into (35) and solving for u yield

$$u = -\frac{k}{1+bk} f(y) + \frac{r}{1+bk}.$$

Substituting in (36) yields

$$y^+ = \left(1 - \frac{bk}{1+bk}\right) f(y) + \frac{b}{1+bk} r.$$

Since, as $k \to \infty$,

$$\frac{bk}{1+bk} \to 1, \qquad \frac{b}{1+bk} \to 0,$$

then the role of the gain k matches that played in continuous-time: as k grows, the eigenvalues of the linearization of the continuous-time closed-loop system (33)–(34) diverge to $-\infty$; instead, those of the linearization of the discrete-time closed-loop system (35)–(36) converge to zero. Hence, as $k \to \infty$, the control law (35) approximates a deadbeat controller.

Clearly, the control law (35) is not causal if y is the only measured output. Stabilization via causal, *low-gain* feedback is the subject of the forthcoming Section VII-B.

VII. ϱ-Passivity and Low-Gain Stabilization

This section provides a "dual" result to the high-gain Theorem 5 showing that "input-short" ϱ-passive systems are stabilizable by all sufficiently small static output feedback control policies.

A. The Low-Gain Theorem

Definition 11 below is "dual" to Definition 9 in the sense that, while the latter requires that the system can be stabilized by all sufficiently large feedback gains achieving an arbitrarily small finite gain, Definition 11 requires stabilization for all sufficiently small feedback gains achieving an arbitrarily large finite gain.

Definition 11: Ψ is called *weakly ϱ-low-gain stabilizable* if, for every $\varepsilon > 0$, there exist $k^*(\varepsilon) > 0$ and $\alpha \geq \varepsilon$, such that, for every $k \in (0, k^*(\varepsilon)]$, $y \in \mathcal{S}(\mathbb{T}, \mathbb{H})$, and $r \in \mathcal{S}(\mathbb{T}, \mathbb{H})$, the condition

$$(-ky^\varrho + r, y) \in \Psi \implies \forall t \in \mathbb{T},\ \langle y^\varrho, y^\varrho \rangle_t \leq \alpha(1 + \langle r, r \rangle_t) \tag{37}$$

holds.

The following definition parallels Definition 10 defining the shortage of ϱ-passivity in the input.

Definition 12: A system Ψ is called *input-shortly ϱ-passive* if there exists $\beta \in \mathbb{R}$ such that every $(u, y) \in \Psi$ satisfies (14) for all $t \in \mathbb{T}$, with M satisfying the condition

$$\forall (\mu, \gamma) \in \mathbb{H}^2, \quad \langle (\mu, \gamma), M(\mu, \gamma) \rangle \leq m_1 \langle \gamma, \mu \rangle + m_2 \langle \mu, \mu \rangle, \tag{38}$$

for some $m_1 > 0$ and $m_2 \geq 0$.

In the same way as output-short ϱ-passivity is associated with ϱ-high-gain stabilizability, the next result shows that input-short ϱ-passivity is associated with ϱ-low-gain stabilizability.

Theorem 6: If Ψ is input-shortly ϱ-passive, it is weakly ϱ-low-gain stabilizable.

Proof: Pick arbitrarily $\varepsilon > 0$ and, with $\beta \in \mathbb{R}$ such that (14) holds for Ψ, let

$$k^*(\varepsilon) = \min\left\{1, \frac{m_1}{4m_2}, \frac{2|\beta|}{\varepsilon m_1}, \sqrt{\frac{4m_1^2 + 2m_1 m_2 + 16m_2^2}{\varepsilon m_1^2}}\right\}.$$

Now, pick $k \in (0, k^*(\varepsilon)]$ and $r \in \mathcal{S}(\mathbb{T}, \mathbb{H})$. If $y \in \mathcal{S}(\mathbb{T}, \mathbb{H})$ is such that $(-ky^\varrho + r, r) \notin \Psi$, then (37) is trivially true. Otherwise, from the assumptions of the theorem, for all $t \in \mathbb{T}$, we obtain

$$\begin{aligned} -\beta &\leq m_1 \langle y^\varrho, -ky^\varrho + r \rangle_t + m_2 \langle -ky^\varrho + r, -ky^\varrho + r \rangle_t \\ &= k(km_2 - m_1) \langle y^\varrho, y^\varrho \rangle_t \\ &\quad + (m_1 - 2m_2 k) \langle y^\varrho, r \rangle_t + m_2 \langle r, r \rangle_t \\ &\leq -k \frac{3m_1}{4} \langle y^\varrho, y^\varrho \rangle_t + (m_1 - 2m_2 k) \langle y^\varrho, r \rangle_t + m_2 \langle r, r \rangle_t \end{aligned}$$

in which we have used the fact that $k \leq k^*(\varepsilon) \leq \frac{m_1}{4m_2}$. Note that $k \leq \frac{m_1}{4m_2}$ also implies $m_1 - 2m_2 k \geq \frac{m_1}{2} > 0$.

By using Lemma 1 with $\varepsilon = \frac{km_1}{4(m_1 - 2m_2 k)} > 0$, we get

$$\begin{aligned} &(m_1 - 2m_2 k) \langle y^\varrho, r \rangle_t \\ &\qquad \leq \frac{km_1}{4} \langle y^\varrho, y^\varrho \rangle_t + \frac{2(m_1 - 2m_2 k)^2}{km_1} \langle r, r \rangle_t. \end{aligned}$$

Since $k \leq k^*(\varepsilon) \leq 1$, the previous inequalities imply

$$\langle y^\varrho, y^\varrho \rangle_t \leq \frac{2|\beta|}{m_1 k} + \frac{4m_1^2 + 2m_1 m_2 + 16m_2^2}{(m_1 k)^2} \langle r, r \rangle_t,$$

and, hence, the claim follows from $k \leq k^*(\varepsilon)$ with $\alpha = \max\{\frac{2|\beta|}{m_1 k}, \frac{2m_1^2 + 10m_1 m_2 + 48m_2^2}{(m_1 k)^2}\} \geq \varepsilon$. □

Finally, the following corollary shows that a system that can be made input-shortly ϱ-passive by static output feedback is also robustly stabilizable by output feedback.

Corollary 5: Consider a system Ψ, and suppose that there exists $h > 0$ such that the system

$$\Psi_{|h} := \{(v, y) \in \mathcal{S}(\mathbb{T}, \mathbb{H})^2 \mid (-hy^\varrho + v, y) \in \Psi\}$$

is input-shortly ϱ-passive. Then, there exists an open set $K \subset \mathbb{R}_{>0}$ such that, for every $k \in K$, the system

$$\Psi_{|k} := \{(r, y) \in \mathcal{S}(\mathbb{T}, \mathbb{H})^2 \mid (-ky^\varrho + r, y) \in \Psi\}$$

has the finite-ϱ-gain property (hence, it is ϱ-stable).

Proof: As $\Psi_{|h}$ is input-shortly ϱ-passive, Theorem 6 guarantees that there exist $\alpha \geq 1$ and $\ell^* > 0$ such that, for every

$\ell \in (0, \ell^*]$ and $r \in \mathcal{S}(\mathbb{T}, \mathbb{H})$, we have

$$(-\ell y^\varrho + r, y) \in \Psi_{|h} \Rightarrow \forall t \in \mathbb{T},\ \langle y^\varrho, y^\varrho\rangle_t \le \alpha^2(1 + \langle r, r\rangle_t). \tag{39}$$

Let $K := (h, h + \ell^*)$ and pick $k \in K$. Moreover, pick $(r, y) \in \Psi_{|k}$ arbitrarily. Then, $(-ky^\varrho + r, y) \in \Psi$, which implies that, with $\ell := k - h \in (0, \ell^*)$, $(-\ell y^\varrho + r, y) \in \Psi_{|h}$. From (39), we then deduce that $\Psi_{|k}$ is ϱ-bounded and, by using Lemma 4 with the same arguments used in the proof of Theorem 4, we also deduce that $r \in \mathcal{S}_{\rm b}(\mathbb{T}, \mathbb{H})$ implies $|y^\varrho| \le \alpha + \alpha|r|$. Hence, $\Psi_{|k}$ has the ϱ-finite-gain property. Finally, by Theorem 3, we conclude that $\Psi_{|k}$ is ϱ-stable. □

B. Stabilization by Causal Low-Gain Feedback

The stabilization results presented in Section VII-A deal with static feedback policies of the form

$$u = -ky^\varrho + r, \tag{40}$$

which, if $\varrho \neq 0$, are *noncausal*.

While noncausality may be dealt with prediction, thereby leading to "separation principles" justifying the interest in noncausal control policies of the form of Definition 11 *per se*, it still makes sense to wonder when a system that is stabilizable by (40) is also stabilizable by a *causal* policy of the form

$$u = -ky + r.$$

This section investigates such a question and provides sufficient conditions under which causal stabilization is possible.

We consider an input-state-output system Σ that, for some $\varrho \in \mathbb{T}$, meets the following assumption.

Assumption 3: There exist $c_1 \in \mathcal{S}_{\rm b}(\mathbb{T}, \mathbb{H})$ and $c_2, c_3 \ge 0$ such that every $(u, x, y) \in \Sigma$ satisfies

$$\langle y^\varrho(t) - y(t), y^\varrho(t) - y(t)\rangle \le \langle c_1(t), c_1(t)\rangle + c_2\langle x(t), x(t)\rangle + c_3\langle u(t), u(t)\rangle,$$

for all $t \in \mathbb{T}$.

Assumption 3 is a growth condition on the output y relative to the state x and input u. In the case of (sufficiently regular) or (Lipschitz) continuous-time dynamics described by differential equations, this assumption is trivially satisfied, for instance, when $\varrho = 0$. Moreover, it is also satisfied in the context of discrete-time difference equations as highlighted by the following example.

Example 6: Let $\mathbb{T} = \mathbb{N}$, $\mathbb{H} = \mathbb{R}$, $\mathbb{X} = \mathbb{R}^n$ ($n \in \mathbb{N}$), and consider the system Σ defined by the difference equation

$$x^+ = f(x, u), \qquad y = h(x),$$

with f and h Lipschitz. Then, Assumptions 3 holds with $\varrho = 1$. Indeed

$$\begin{aligned}|y^+ - y|^2 &= |h(f(x, u)) - h(x) \pm h(f(x, 0))|^2\\ &\le (L_hL_f|u| + L_h(1 + L_f)|x|)^2\\ &\le L_h^2(1+L_f)(1 + 2L_f)|x|^2 + L_h^2L_f(2L_f + 1)|u|^2,\end{aligned}$$

in which L_h and L_f denote the Lipschitz constant of h and f, respectively. Hence, Assumption 3 holds with $c_1(\cdot) = 0$, $c_2 = L_h^2(1 + L_f)(1 + 2L_f)$, and $c_3 = L_h^2L_f(2L_f + 1)$. ◁

Next, we assume that Σ verifies the following *robust detectability* condition.

Assumption 4: There exist $\ell_1, \ell_2, \ell_3 > 0$ such that every $(u, x, y) \in \Sigma$ satisfies

$$\forall t \in \mathbb{T},\ \langle x, x\rangle_t \le \ell_1 + \ell_2\langle y, y\rangle_t + \ell_3\langle u, u\rangle_t.$$

We stress that Assumption 4 bounds the growth of the integral of $\langle x, x\rangle$, and not of its pointwise value as is the case in Assumption 3.

The following theorem establishes the main result of this section. Specifically, it is shown that, under Assumptions 3 and 4, and an additional small-gain-like condition, every input-shortly ϱ-passive input–output system derived from Σ is robustly stabilizable by causal output feedback.

Theorem 7: Suppose that Σ satisfies Assumptions 3 and 4 with constants c_2 and ℓ_2 fulfilling the condition

$$2c_2\ell_2 < 1. \tag{41}$$

Let $x_0 \in \mathbb{X}$ and suppose that the input–output system $\Psi_{x_0}(\Sigma)$ associated with Σ is input-shortly ϱ-passive. Then, there exists an open set $K \subset \mathbb{R}_{>0}$ such that, for every $k \in K$, the closed-loop system

$$\Psi_{x_0}(\Sigma)_{|k} := \{(r, y) \in \mathcal{S}(\mathbb{T}, \mathbb{H})^2 \ |\ (-ky + r, y) \in \Psi_{x_0}(\Sigma)\}$$

has the 0-finite-gain property and, hence, it is 0-stable.

Proof: As $\Psi_{x_0}(\Sigma)$ is input-shortly ϱ-passive, there exist $\beta \in \mathbb{R}$, $m_1 > 0$, and $m_2 \ge 0$ such that, for all $(u, y) \in \Psi_{x_0}(\Sigma)$,

$$\forall t \in \mathbb{T}, \quad -\beta \le m_1\langle u, y^\varrho\rangle_t + m_2\langle u, u\rangle_t.$$

Adding and subtracting $\langle u, y\rangle_t$ yields

$$\forall t \in \mathbb{T},\ -\beta \le m_1\langle u, y\rangle_t + m_1\langle u, y^\varrho - y\rangle_t + m_2\langle u, u\rangle_t.$$

Lemma 1 and Assumption 3 imply that, for every $\varepsilon > 0$,

$$\begin{aligned}&\langle u(t), y^\varrho(t) - y(t)\rangle\\ &\le \varepsilon\langle y^\varrho(t) - y(t), y^\varrho(t) - y(t)\rangle + \frac{1}{2\varepsilon}\langle u(t), u(t)\rangle\\ &\le \varepsilon\langle c_1(t), c_1(t)\rangle + c_2\varepsilon\langle x(t), x(t)\rangle\\ &\quad + \left(c_3\varepsilon + \frac{1}{2\varepsilon}\right)\langle u(t), u(t)\rangle.\end{aligned}$$

Hence, for all $t \in \mathbb{T}$, we obtain

$$-q_1(\varepsilon) \le \langle u, y\rangle_t + c_2\varepsilon\langle x, x\rangle_t + q_2(\varepsilon)\langle u, u\rangle_t$$

with $q_1(\varepsilon) := \frac{\beta}{m_1} + |c_1|^2\varepsilon$ and $q_2(\varepsilon) := \frac{m_2}{m_1} + c_3\varepsilon + \frac{1}{2\varepsilon}$. Assumption 4 further implies that

$$\forall t \in \mathbb{T}, -q_3(\varepsilon) \le \langle u, y\rangle_t + c_2\ell_2\varepsilon\langle y, y\rangle_t + q_4(\varepsilon)\langle u, u\rangle_t \tag{42}$$

with $q_3(\varepsilon) := q_1(\varepsilon) + c_2\ell_1\varepsilon$ and $q_4(\varepsilon) := q_2(\varepsilon) + c_2\ell_3\varepsilon$.

Now, we show that there exists $h > 0$ such that the system

$$\Psi_{x_0}(\Sigma)_{|h} := \{(v, y) \in \mathcal{S}(\mathbb{T}, \mathbb{H})^2 \ |\ (-hy + v, y) \in \Psi_{x_0}(\Sigma)\}$$

is input-shortly 0-passive. We start by noting that (41) implies that

$$\eta := \frac{1}{2} + \frac{\sqrt{1 - 2c_2\ell_2}}{4} > 0$$

satisfies the conditions

$$\eta^2 - \eta + \frac{c_2\ell_2}{2} < 0, \tag{43a}$$

$$\frac{c_2\ell_2}{\eta} < 1. \tag{43b}$$

Next, with $h > 0$ to be fixed, we set

$$\varepsilon := \frac{h\eta}{c_2\ell_2}.$$

Taking $u = -hy + v$ in (42) (with $v \in \mathcal{S}(\mathbb{T}, \mathbb{H})$) yields

$$-\tilde{q}_3(h) \leq \left((\eta-1)h + \tilde{q}_4(h)h^2\right)\langle y, y\rangle_t + (1 - 2h\tilde{q}_4(h))\langle v, y\rangle_t + \tilde{q}_4(h)\langle v, v\rangle_t$$

for all $t \in \mathbb{T}$, in which we let $\tilde{q}_i(h) := q_i(\frac{h\eta}{c_2\ell_2})$ for $i = 3, 4$.

Let $\omega > 0$ be such that $-\omega = \eta - 1 + \frac{c_2\ell_2}{2\eta}$ [which does exist in view of (43a)]. Then, let

$$h_1^* := \min\left\{\frac{\omega m_1}{2m_2}, \sqrt{\frac{\omega \ell_2 c_2}{2\eta(\ell_3 c_2 + c_3)}}\right\},$$

and note that, for all $h \in (0, h_1^*)$, we obtain

$$\begin{aligned}(\eta-1)h &+ \tilde{q}_4(h)h^2 \\ &= \left(\eta - 1 + \frac{c_2\ell_2}{2\eta} + h\frac{m_2}{m_1} + h^2\frac{\eta}{\ell_2 c_2}(\ell_3 c_2 + c_3)\right)h \\ &< 0.\end{aligned}$$

Next, let

$$h_2^* := \min\left\{\frac{m_1}{4m_2}\left(1 - \frac{c_2\ell_2}{\eta}\right), \sqrt{\frac{c_2\ell_2}{4\eta(c_2\ell_3 + c_3)}\left(1 - \frac{c_2\ell_2}{\eta}\right)}\right\}.$$

We underline that $h_2^* > 0$ in view of (43b). Then, for all $h \in (0, h_2^*)$, we get

$$\begin{aligned}1 - 2h\tilde{q}_4(h) &= 1 - \frac{c_2\ell_2}{\eta} - 2h\frac{m_2}{m_1} - 2h^2\frac{\eta}{c_2\ell_2}(c_2\ell_3 + c_3) \\ &> 0.\end{aligned}$$

Now, fix $h \in (0, \min\{h_1^*, h_2^*\})$. Then, the previous inequalities imply that every $(v, y) \in \Psi_{x_0}(\Sigma)_{|h}$ satisfies

$$\forall t \in \mathbb{T}, \quad -q_5 \leq \langle v, y\rangle_t + q_6\langle v, v\rangle_t,$$

with $q_5 := (1 - 2h\tilde{q}_4(h))^{-1}\tilde{q}_3(h)$ and $q_6 := (1 - 2h\tilde{q}_4(h))^{-1}\tilde{q}_4(h)$. Hence, $\Psi_{x_0}(\Sigma)_{|h}$ is input-shortly 0-passive as claimed. The proof of the theorem then follows from Corollary 5. □

VIII. Sampled-Data Systems

As previously mentioned in Section I-A, the classical passivity notions (3) and (4) are generally not preserved under sampling, even for small sampling times. The preliminary version of this work [33] has shown that discretizing a passive continuous-time linear time-invariant system (with unitary relative degree) leads to a 1-passive discrete-time system for all sufficiently small sampling times. In particular, consider a 0-passive input-state-output system with time domain $\mathbb{T} = \mathbb{R}_{\geq 0}$ described by the equations

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x(0) = x_0, \tag{44}$$

with state $x(t) \in \mathbb{R}^n$, output $y(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$, and in which A, B, and C are such that there exists $P = P^\top > 0$ such that $A^\top P + PA \leq -I$ and $B^\top P = C$. The sampled-data system of (44) is an input-state-output system with time domain $\mathbb{T} = \mathbb{N}$ described by the equations

$$x^+ = Fx + Gu, \quad y = Cx, \quad x(0) = x_0, \tag{45}$$

in which $F := \mathrm{e}^{Ah}$ and $G := \int_0^h \mathrm{e}^{As}B\,\mathrm{d}s$, where $h > 0$ is the sampling time. Then, the following result holds.

Theorem 8 ([33]): There exists $\bar{h} > 0$ such that, for all $h \in (0, \bar{h})$, system (45) is 1-passive.

Theorem 8 is proved in [33]. Alternatively, it can be deduced by the results presented hereafter (see Remark 8). Unlike the classical notion of passivity, Theorem 8 shows that ϱ-passivity is preserved under sampling in the case of linear time-invariant systems thereby suggesting that ϱ-passivity may be more suitable than regular notions of passivity when dealing with sampled-data systems. In this section, we strengthen such a conjecture by generalizing Theorem 8 to a class of nonlinear systems and discretization methods. To shed light on this generalization, we first provide a simple motivating example.

Example 7: Consider system (33) under the assumption that it is 0-passive with an operator M satisfying (15). The forward Euler approximation of (33) yields the discrete-time system

$$y^+ = y + h(f(y) + bu), \tag{46}$$

where $h \in \mathbb{R}_{>0}$ is a constant step-size. To guarantee accuracy of the approximate model (46) with respect to (33), we assume that $H := \{\hat{h} \in \mathbb{R}_{>0} \mid \forall y \in \mathbb{R},\ \hat{h}f(y)^2 \leq -2yf(y)\}$ is nonempty and $h \in H$. Then, by definition of H, the storage function $S(y) = y^2$ satisfies

$$\begin{aligned}S(y^+) - S(y) &= 2hb(y + hf(y) + hbu)u - h^2b^2u^2 \\ &\quad + h(hf^2(y) + 2yf(y)) \\ &\leq 2hbuy^+ - h^2b^2u^2.\end{aligned}$$

Hence, the discretized model (46) is 1-passive with operator

$$M = hb\begin{bmatrix} -hb & 1 \\ 1 & 0\end{bmatrix}. \qquad \triangleleft$$

Although elementary, Example 7 shows that 1-passivity, rather than 0-passivity, is a more natural passivity property for discretized systems. This section proposes sufficient conditions under which the reasoning behind Example 7 can be extended to more general systems. To begin with, we formalize what *sampled-data* means for a generic continuous-time input–output system Ψ defined on $\mathcal{S}(\mathbb{R}_{\geq 0}, \mathbb{H})$. For an arbitrary $h \in \mathbb{R}_{>0}$, let $Z_h : \mathcal{S}(\mathbb{N}, \mathbb{H}) \to \mathcal{S}(\mathbb{R}_{\geq 0}, \mathbb{H})$ denote the *zero-order hold* operator, mapping $\hat{u} \in \mathcal{S}(\mathbb{N}, \mathbb{H})$ to $u \in \mathcal{S}(\mathbb{R}_{\geq 0}, \mathbb{H})$ by setting $u(t) := \hat{u}(k)$, for every $t \in [kh, (k+1)h)$. Moreover, let $S_h : \mathcal{S}(\mathbb{R}_{\geq 0}, \mathbb{H}) \to \mathcal{S}(\mathbb{N}, \mathbb{H})$ denote the *sampling* operator, mapping $y \in \mathcal{S}(\mathbb{R}_{\geq 0}, \mathbb{H})$ to $\hat{y} \in \mathcal{S}(\mathbb{N}, \mathbb{H})$ by setting $\hat{y}(k) := y(kh)$, for every $k \in \mathbb{N}$. The *sampled-data system of* Ψ is defined as a system on $\mathcal{S}(\mathbb{N},\mathbb{H})$ given by

$$\Psi_h := \{(\hat{u}, S_h(y)) \in \mathcal{S}(\mathbb{N},\mathbb{H}) \times \mathcal{S}(\mathbb{N},\mathbb{H}) \mid (Z_h(\hat{u}), y) \in \Psi\}.$$

Consider an input-state-output system described by the equations

$$\dot{x} = f(x) + Bu, \qquad y = Cx, \tag{47}$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^m$, for some $n \in \mathbb{N}_{>0}$ and $m \in \mathbb{N}_{>0}$. Equation (47) is a generalization of (44). We introduce the following assumption concerning regularity and internal stability.

Assumption 5: B is full-rank, f is Lipschitz, and there exist $q > 0$ and $P = P^\top > 0$ satisfying $C = B^\top P$, such that the function $S(x) := x^\top P x$ satisfies

$$\forall x \in \mathbb{R}^n, \ \ \frac{\partial S(x)}{\partial x} f(x) \le -q|x|^2. \tag{48}$$

For every initial condition $x(0) = x_0 \in \mathbb{R}^n$, system (47) defines an input–output system Ψ_{x_0} with which, for every $h > 0$, we can associate a sampled-data system $\Psi_{x_0,h}$ as described before. We say that $\Psi_{x_0,h}$ is *Euler in the limit* if it coincides with the input–output system obtained from an input-state-output system of the form

$$x^+ = x + h(f(x) + Bu) + h^2\delta(x,u), \qquad y = Cx, \tag{49}$$

from the initial condition x_0, and with δ satisfying

$$\exists \ell > 0, \ \ \forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m, \ \ |\delta(x,u)| \le \ell(|x| + |u|). \tag{50}$$

The following result proves that, under Assumption 5, every discretization of (47) that is Euler in the limit is 1-passive for all sufficiently small sampling times.

Theorem 9: Suppose that Assumption 5 holds, and consider the input–output system Ψ_{x_0} defined by (47) for some $x_0 \in \mathbb{R}^n$. Let $\Psi_{x_0,h}$ be the sampled-data system associated with Ψ_{x_0}, and suppose that $\Psi_{x_0,h}$ is Euler in the limit. Then, there exists $\bar{h} > 0$ such that, for all $h \in (0, \bar{h})$, $\Psi_{x_0,h}$ is 1-passive with operator

$$M := \frac{h}{2}\begin{bmatrix} -hB^\top PB & 2I \\ 2I & -q\bar{\lambda}^{-1} I \end{bmatrix},$$

in which $\bar{\lambda}$ denotes the largest eigenvalue of $C^\top C$.

Proof: Note that

$$\begin{aligned} S(x^+) &= (x + hf(x))^\top P (x + hf(x)) + h^2 u^\top B^\top PBu \\ &\quad + 2h(x + hf(x))^\top PBu + 2h^3 u^\top B^\top P\delta(x,u) \\ &\quad + 2h^2(x + hf(x))^\top P\delta(x,u) + h^4\delta(x,u)^\top P\delta(x,u). \end{aligned}$$

Let $\kappa > 0$ denote the Lipschitz constant of f. Then, in view of Assumption 5, the first term of the previous equality is such that

$$\begin{aligned} &(x + hf(x))^\top P(x + hf(x)) \\ &\quad = S(x) + h\frac{\partial S(x)}{\partial x} f(x) + h^2 f(x)^\top P f(x) \\ &\quad \le S(x) - h\left(q - h\kappa^2|P|\right)|x|^2, \end{aligned} \tag{51}$$

for all $h > 0$. Because of (49), $x + hf(x) = x^+ - hBu - h^2\delta(x,u)$. Hence, in view of (50), the second, third, and fourth terms satisfy

$$\begin{aligned} &h^2 u^\top B^\top PBu + 2h(x + hf(x))^\top PBu + 2h^3 u^\top B^\top P\delta(x,u) \\ &\quad = 2hu^\top B^\top P x^+ - h^2 u^\top B^\top PB \\ &\quad = 2h\langle y^+, u\rangle - h^2\langle u, (B^\top PB)u\rangle, \end{aligned}$$

for all $h > 0$, in which we have used $C = B^\top P$ thanks to Assumption 5.

As for the fifth term, by (50) and Lemma 1, we get

$$\begin{aligned} &2h^2(x + hf(x))^\top P\delta(x,u) \\ &\quad \le 2h^2\ell|P|\,(1 + h\kappa)\,(|x|^2 + |x||u|) \\ &\quad \le h^2\ell|P|\,(1 + h\kappa)\left(2(1+\varepsilon)|x|^2 + \varepsilon^{-1}|u|^2\right), \end{aligned}$$

for every $\varepsilon > 0$ and $h > 0$.

Finally, for all $h > 0$, the last term satisfies

$$h^4\delta(x,u)^\top P\delta(x,u) \le 2h^4|P|\ell^2\left(|x|^2 + |u|^2\right).$$

Thus, in view of the previous inequalities, we get

$$\begin{aligned} S(x^+) \le\ & S(x) + 2h\langle y^+, u\rangle \\ &+ h\left(h|P|\left(\kappa^2 + 2\ell(1+\varepsilon)\right)\right. \\ &\left. + h^2 2|P|\ell\kappa(1+\varepsilon) + h^3 2|P|\ell^2 - q\right)|x|^2 \\ &+ h^2\langle u, (\ell|P|\varepsilon^{-1} I + h\ell\kappa|P|\varepsilon^{-1} I \\ &+ 2h^2|P|\ell^2 I - B^\top PB)u\rangle, \end{aligned}$$

for all $\varepsilon > 0$ and $h > 0$.

Let $\underline{\sigma} > 0$ denote the minimum eigenvalue of $B^\top PB$ (which is strictly positive in view of Assumption 5), fix once for all $\varepsilon := \frac{8\ell|P|}{\underline{\sigma}}$, and let

$$\begin{aligned} \bar{h} := \min\Bigg\{ &\frac{q}{8|P|(\kappa^2 + 2\ell(1+\varepsilon))}, \sqrt{\frac{q}{8|P|\ell\kappa(1+\varepsilon)}}, \\ &\sqrt[3]{\frac{q}{16|P|\ell^2}}, \frac{\underline{\sigma}\varepsilon}{8\ell\kappa|P|}, \sqrt{\frac{\underline{\sigma}}{8|P|\ell^2}} \Bigg\}. \end{aligned}$$

Then, for all $h \in (0, \bar{h})$, we obtain

$$S(x^+) - S(x) \le -\frac{q}{2}h|x|^2 + 2h\langle y^+, u\rangle - \frac{h^2}{2}\langle u, B^\top PBu\rangle. \tag{52}$$

Let $\bar{\lambda} > 0$ denote the largest eigenvalue of $C^\top C$. Then, $\bar{\lambda}|x|^2 \ge x^\top C^\top Cx = |y|^2$. Hence, integrating (52) yields the claim with $\beta = -S(x_0)$. □

Remark 7 (Preservation of stability under sampling): Assumption 5 implies that the continuous-time system (47) is strictly output 0-passive (Section V-B). Therefore, by Theorem 4, it has the finite-gain property and it is 0-stable (hence, internally 0-stable). The result of Theorem 9 implies that its sampled-data system $\Psi_{x_0,h}$ is strictly output 1-passive for all sufficiently small h. Therefore, by Theorem 4, it has the finite-1-gain property and it is 1-stable (hence, internally 1-stable). Therefore, this shows that also ϱ-stability is preserved under sampling (for sufficiently small sampling times).

Remark 8 (Alternative proof of Theorem 8): Alternatively to the proof given in [33], Theorem 8 can be directly obtained as a consequence of Theorem 9. Indeed, it is easy to see that the linear sampled-data system (45) is Euler in the limit.

It is worth noting that the scope of Theorem 9 is limited to systems described by (47) and satisfying Assumption 5, to discretization methods that are Euler in the limit, and to 1-passivity of the obtained sampled-data system. Moreover, the proof is heavily based on the state-space representation of the involved systems. The following proposition provides another purely input–output sufficient condition for ϱ-passivity preservation under sampling that may apply to systems and discretization methods not satisfying the previous assumptions. In the following, we let Ψ denote a system on $\mathcal{S}(\mathbb{R}_{\geq 0}, \mathbb{H})$ such that, given an arbitrary, yet fixed sampling time $h > 0$, the set

$$Z_h^{-1}(\mathrm{dom}\Psi) := \{\hat{u} \in \mathcal{S}(\mathbb{N}, \mathbb{H}) \mid Z_h(\hat{u}) \in \mathrm{dom}\Psi\}$$

is nonempty.

Proposition 1: Suppose that Ψ is 0-passive with operator M satisfying $M(\mu, \gamma) = a(\gamma, \mu) + c(\mu, 0)$, for some $a > 0$ and $c \leq 0$. Moreover, assume that there exist $\varrho \in \mathbb{N}$, $\bar{b} \leq 0$, $\bar{c} \leq 0$, and $\bar{a} > 0$, such that, for every $\hat{u} \in Z_h^{-1}(\mathrm{dom}\Psi)$, we have

$$\forall K \in \mathbb{N}_{>0}, \ \ \sum_{k=0}^{K-1} \phi(k) \geq -\bar{b} \sum_{k=0}^{K-1} \langle y^{\varrho}(kh), y^{\varrho}(kh) \rangle, \tag{53}$$

where

$$\phi(k) := \langle \hat{u}(k), (\bar{c} - ch)\hat{u}(k) + 2\bar{a} y^{\varrho}(kh) - 2a \int_{kh}^{kh+h} y(\tau)\, \mathrm{d}\tau \rangle.$$

Then, the sampled-data system Ψ_h of Ψ is ϱ-passive.

Proof: Since Ψ is 0-passive, there exists $\beta \in \mathbb{R}$ such that, for every $\hat{u} \in Z_h^{-1}(\mathrm{dom}\Psi)$, and every $K \in \mathbb{N}_{>0}$, we have $\int_0^{Kh} w_M(Z_h(\hat{u})(\tau), y(\tau))\, \mathrm{d}\tau \geq -\beta$. Hence, by using condition (53), the definition of M, and by defining $\hat{y} := S_h(y)$ (which satisfies $\hat{y}(k) = y(kh)$), for all $K \in \mathbb{N}_{>0}$, we obtain

$$\begin{aligned}
-\beta &\leq \int_0^{Kh} w_M \left(Z_h(\hat{u})(\tau), y(\tau)\right) \mathrm{d}\tau \\
&= \sum_{k=0}^{K-1} \int_{kh}^{kh+h} w_M \left(\hat{u}(k), y(\tau)\right) \mathrm{d}\tau \\
&= \sum_{k=0}^{K-1} ch \langle \hat{u}(k), \hat{u}(k) \rangle + \sum_{k=0}^{K-1} 2a \int_{kh}^{kh+h} \langle \hat{u}(k), y(\tau) \rangle \, \mathrm{d}\tau \\
&= -\sum_{k=0}^{K-1} \phi(k) + \sum_{k=0}^{K} \left(\bar{c}\langle \hat{u}(k), \hat{u}(k)\rangle + 2\bar{a}\langle \hat{u}(k), \hat{y}^{\varrho}(k)\rangle\right) \\
&\leq \sum_{k=0}^{K} w_{\bar{M}}(\hat{u}(k), \hat{y}^{\varrho}(k)),
\end{aligned}$$

with $\bar{M}(\mu, \gamma) = \bar{a}(\gamma, \mu) + \bar{b}(0, \gamma) + \bar{c}(\mu, 0)$. Therefore, Ψ_h is ϱ-passive. □

IX. Conclusion

Passivity theory has pervaded control theory since its dawn. Despite its widespread use in the field, the classical notion of passivity is affected by several limitations when it is applied to systems different from continuous-time state-space models, such as discrete-time systems, and systems with delays. Motivated by the early work [33], this article developed the new notion of ϱ-passivity, which is introduced to cope with the limitations mentioned earlier. In particular, we have provided several results covering ϱ-passivity of interconnections, the relation between ϱ-passivity, stability, and finite gain, high- and low-gain stabilizability of ϱ-passive systems, and preservation of ϱ-passivity under sampling. In this way, we lay the first stone in the development of a comprehensive theory of ϱ-passive systems.

Overall, the presented results show that ϱ-passivity behaves as well as regular passivity in terms of interconnections and stabilizability. Nevertheless, ϱ-passivity has a broader scope and solves some of the issues affecting the standard notion. When $\varrho > 0$, however, static output feedback employing the ϱ-passive output may be noncausal. While this is not necessarily a problem when predictors are used, further research is needed to characterize when and how such a noncausality leads to fundamental limitations, and to investigate causal stabilization laws in the spirit of Section VII. Since ϱ-passivity is defined on an abstract time domain, it can be applied to a broader class of systems, such as sampled-data systems. In particular, unlike the standard definition, we have shown that the property of ϱ-passivity is preserved under sampling, at least for a relevant class of systems.

Ultimately, the results of this article suggest that ϱ-passivity may become a useful notion for the control community. Future research will be devoted to its connections to optimal control, discrete-time adaptive control, hybrid systems, and nonminimum-phase systems, as well as preservation of ϱ-passivity for model reduction.

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