



UNIVERSITÀ DEGLI STUDI DI TRIESTE
UNIVERSITÀ DEGLI STUDI DI UDINE

XXXVII CICLO DEL DOTTORATO DI RICERCA IN
INGEGNERIA CIVILE - AMBIENTALE E ARCHITETTURA

TESTING ON ELASTOMERS AND NUMERICAL
ANALYSES FOR THE BEHAVIOR PREDICTION
OF ELASTOMERIC SEISMIC ISOLATORS
AND APPLICATION OF THE ISOLATORS IN A
TMD SYSTEM

SETTORE SCIENTIFICO-DISCIPLINARE ICAR/09

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ANNO ACCADEMICO 2023/2024



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Abstract

Seismic isolators are increasingly used to decouple the horizontal movement of buildings from ground motion, thus protecting structures from potential earthquake-induced damage. The UNI EN 15129 standard for seismic isolation devices requires numerous time-consuming and costly tests on full-scale isolators for device qualification. From these tests, the primary mechanical properties, shear modulus and damping, that describe the isolator behavior, are derived. This research work focuses on elastomeric seismic isolators, with the main objective of predicting their mechanical properties through laboratory testing on elastomers and finite element method (FEM) simulations. This is to have information about the isolator behavior without performing experimental tests on full-scale isolators to obtain the desired characteristics. Specific laboratory tests are designed and performed for rubber, in order to determine the parameters of the viscoelastic model adopted for this material. In particular, the effect of small repeated oscillations is utilized to accelerate the stress relaxation process, allowing for a relatively rapid point-wise determination of the elastic equilibrium stress response curve. A single small thickness rubber pad specimen is designed to achieve a rapid loading ramp in the relaxation test and to highlight the viscous components of the stress that dissipate quickly. From the experimental tests, the parameters necessary to implement the rubber constitutive model in FEM simulations are derived. Numerous results from laboratory shear cyclic tests on full-scale isolators are collected in this work. Groups of these isolators are selected for FEM modeling and simulation of the same test to make a comparison between the mechanical characteristics obtained from FEM models and real ones. The prediction capacity of these models is good in many cases. Elastomeric seismic isolators are utilized in a Tuned Mass Damper (TMD) system, where they contribute to both stiffness and energy dissipation, analyzed in a numerical study that compares the seismic performance of a building equipped with the TMD system at its top to the same building with a base isolation system. A case study of a five-story reinforced concrete (RC) building seismically protected through a TMD installed at the top is presented. The aim of the study is to investigate the effectiveness of the TMD in changing the structural behavior of medium-rise existing buildings from dissipative to non-dissipative, in order to eliminate reparation or demolition costs resulting from damages caused by strong earthquakes. The TMD mass is made by a RC slab lying on flat surface sliders. Horizontal stiffness and damping of the TMD are provided by lead rubber isolators in a first proposed solution and by low-damping rubber isolators and viscous linear dampers, respectively, in a second one. The improvement in the building structural behavior attained with the installation of the TMD is assessed by considering the flexural demand over capacity ratios of structural elements and the energy dissipated by the TMD. These results are compared with those of the same building retrofitted with a base isolation system. It is demonstrated that a TMD is a valid solution for the retrofit of medium-rise existing buildings.

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Chapter 1

Introduction

1.1 Elastomeric seismic isolators

Seismic isolators are devices used as a passive isolation system in structures [1–4, 7]. When installed at the base of a building, they form a deformable layer which allows for the decoupling of the horizontal motion of the structure from that of the ground and protects the superstructure against earthquake damage. They are characterized by high deformability in the horizontal plane and low deformability in the vertical direction, allowing them to also function as supports. Additionally, isolators can be integrated into other isolation systems and positioned in various locations within the building, such as at the top or in intermediate floors, for example in the case of Tuned Mass Dampers [82, 84, 96].

The functioning of seismic isolators is based on increasing the structure period of oscillation [5], moving it away from the ground maximum horizontal accelerations frequencies. While some types of isolators, such as flat or curved surface sliders, rely on geometry to ensure their performance, elastomeric isolators behavior is fundamentally defined by the rubber material properties.

Rubber is a natural material composed of interwoven polymer chains that form a three-dimensional network. During the process of vulcanization, these chains are chemically linked together, creating a material with elastic properties [8]. There are also synthetic rubbers, such as Styrene-Butadiene Rubber (SBR) and Ethylene-Propylene Diene Monomer (EPDM), which exhibit similar properties. The behavior of rubber can be considered to be linear only for small deformations; for larger deformations, the stress response shows a phase of softening followed by hardening.

Additives can be incorporated to enhance the strength and durability of rubber; carbon black (CB), in particular, modifies both the modulus and significantly increases the damping properties. The resulting material can be considered viscoelastic, although it exhibits additional effects [9], such as the Mullins effect [10, 21, 22], and permanent deformations after an imposed load [31–33].

Elastomeric isolators are constituted by a number of alternate horizontal layers of rubber and steel, forming a cylinder or a rectangular based prism [6]. The first and last rubber layers are bonded to upper and lower rigid steel plates, which ensure the connections to the structure. A layer of rubber on the lateral surface serves to protect the steel laminates from external environmental factors. The rubber is usually CB-filled, thus demonstrating a moderate dissipative capacity when cyclically deformed. A lead core can be added in the central part of the isolator to enhance this effect. The bonding between steel and rubber is formed thanks to the process

off vulcanization.

For horizontal displacements of the superstructure, the isolator undergoes shear deformation, as shown in Figure 1.1, although localized tension and compression effects occur at the extremities.

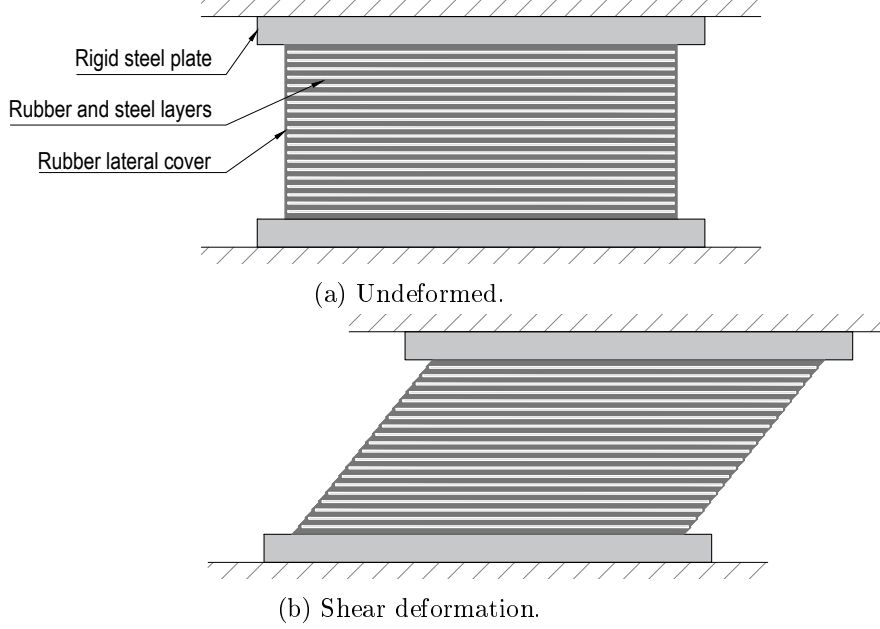


Figure 1.1: Lateral view scheme of an undeformed and shear deformed elastomeric isolator.

Standards on seismic isolation devices [11–13,86] impose numerous time-consuming and costly tests on full-scale isolators for the qualification of the devices. These tests must be performed in specialized laboratories to determine the isolator characteristics, such as vertical and horizontal stiffness, shear modulus, damping, and the constitutive relationship that adequately represents its behavior. These characteristics vary based on the geometric and mechanical configuration of the isolator and the properties of the elastomer; furthermore, the relationship between the elastomer composition and the required characteristics for the isolator is extremely complex.

Two of the characteristics that define the properties of an isolator are the dynamic shear modulus G_{dyn} and the damping ξ . According to the UNI EN 15129 standard [13], these properties are computed from data recorded during the third cycle of a shear cyclic test performed at a frequency of $f = 0.5$ Hz. The device must be loaded with a vertical force in order to have an average stress of 6 MPa normal to the cross-section. Referring to the example of hysteresis cycle shown in Figure 1.2, the horizontal stiffness K_b and equivalent viscous damping ξ of the elastomeric isolator can be calculated as

$$K_b = \frac{F^+ - F^-}{d^+ - d^-} = \frac{\Delta F}{\Delta d} \quad (1.1)$$

$$\xi = \frac{H}{2\pi K_b \Delta d^2} \quad (1.2)$$

where H is the area of the hysteresis loop, F and d are the horizontal force and displacement respectively. d^+ and d^- are the maximum and minimum values of

displacement in the cycle, and F^+ and F^- are the values of force at those displacements.

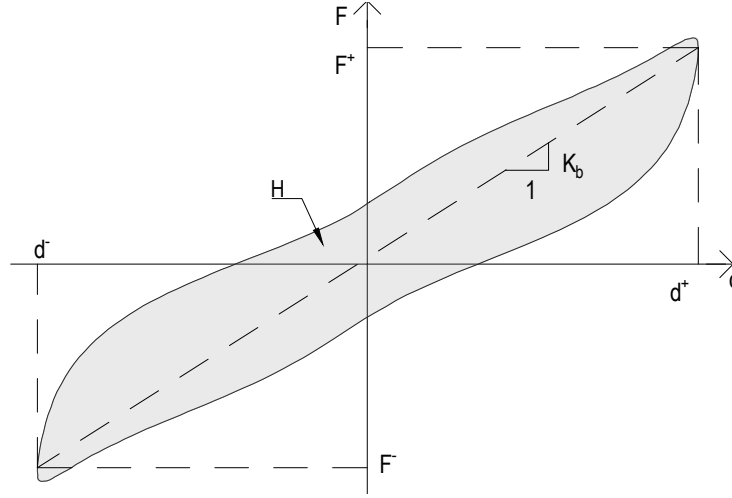


Figure 1.2: Example of hysteresis cycle of a shear cyclic test.

From the effective shear stiffness, the dynamic shear modulus can be computed as

$$G_{dyn} = K_b \frac{t}{A} = \frac{\Delta\sigma_h}{2\gamma_{max}} \quad (1.3)$$

where t is the total rubber thickness of rubber layers of thickness t_r and A is the area of the cross-section of the isolator. The expression is also written in terms of the mean shear stress $\sigma_h = F/A$ and maximum shear strain $\gamma_{max} = \Delta d/2t$. The values of shear modulus and damping vary with the maximum displacement reached by the isolator, in particular by the maximum shear strain of the rubber. In the same way, also for a direct comparison, are defined the properties of a rubber specimen.

Standard UNI EN 15129 requires that, at design displacement, value of K_b and ξ shall be within $\pm 20\%$ of design value. Knowing in advance the result of this test would ensure that these requirements are satisfied. Since the behavior of an elastomeric seismic isolator is defined by its geometry and material properties, and both the geometry and steel constitutive law are known, predicting the device response under cyclic loading would be feasible only if a model for rubber is defined.

1.2 Purpose of the Research

The objective of the doctoral research is to predict the behavior of elastomeric seismic isolators subjected to cyclic shear deformation, using finite element models. To represent the viscoelastic behavior of the elastomer in FEM simulations, the Maxwell-Wiechert constitutive model, adapted for large deformations according to the Simo-Hughes formulation [18], is adopted. This model adequately describes the dynamic response of the material. However, it does not include any parameters to describe loss of stiffness due to the Mullins effect [10, 21, 22], which refers to the stress-softening phenomenon observed in rubber-like materials under cyclic loading. The visco-elastic model parameters of the elastomer to be used in FEM simulations can be determined through laboratory tests.

During the research, three essential tests were defined to determine the complete visco-elastic model parameters of the elastomer relatively quickly and uniquely.

1. A cyclic test is necessary to account for the Mullins effect and allows for the derivation of the dynamic shear modulus and damping.
2. A relaxation test allows, at least in principle, for the determination of all model parameters, with the viscous terms highlighted by the decaying stress curve. Initial fitting attempts revealed that the optimal solution is not unique, meaning that different combinations of viscous and elastic parameters can describe the relaxation curve with similar accuracy. For this reason, a step relaxation test was added.
3. A step relaxation test allows for the independent determination of elastic parameters from viscous ones, with the latter obtained subsequently from the relaxation test.

During the relaxation tests on elastomers, it was observed that small oscillations around the target deformation drastically accelerate the relaxation process, which would otherwise take hours to complete. This observation forms the basis for a new and original multi-step relaxation testing methodology developed herein.

Previous tests were performed at the PROMAS laboratory of University of Udine. Initially, standard elastomer quadruple rubber specimens, consisting of four prisms of dimensions $40 \times 40 \times 10$ mm arranged symmetrically, commonly used to characterize the rubber material through shear tests were used. However, due to the great thickness of the specimen and the limited displacement speed of the available testing machine, performing cyclic and relaxation tests at the required deformation levels ($100 \div 200$ % shear strain) was not feasible.

A shear testing methodology for single rubber pad specimens was then developed, using $2 \div 3$ mm thick elastomer sheets adhesive bonded to steel plates. Initial tests confirmed excellent adhesion and very good repeatability of the results, with good predictive capability of the properties of the isolators, demonstrated by FEM simulations using the constitutive relationships obtained from the single specimen shear tests.

To determine these constitutive relationships, a data-fitting procedure was developed considering data from various tests, even of the same type, and applying specific constraints on the elastomer model parameters. FEM analysis was used to evaluate the behavior of the samples as the aspect ratio varied, in the presence or absence of compression, and their sensitivity to different types of geometric and mechanical defects.

The results obtained from FEM simulation of the cyclic shear tests on isolators are compared with results from tests on full-scale isolators provided by the manufacturer ILPEA S.p.A. as part of a research collaboration agreement between the company and the Polytechnic Department of Engineering and Architecture (DPIA) of the University of Udine.

Initially, the simulation of the cyclic test on isolators was modeled using the FEM software 'femux', developed by DPIA, and the parameters of the elastomer constitutive relationship were determined based on data obtained solely from tensile relaxation tests of elastomer samples. The results showed significant discrepancies between the finite element model results and those from the available full-scale tests. For this reason, tensile tests for elastomer characterization were temporarily set aside

in favor of focusing on shear tests. The FEM simulations of the shear cyclic tests on elastomeric isolators presented were performed using a commercial software.

1.2.1 Available data from cyclic tests on real scale isolators

Four types of commercial rubber are used in this study, the nominal properties of which are reported in Table 1.1 (shear modulus, G , damping ξ , hardness and black carbon content, BC). Black carbon ratio is estimated from the values of hardness.

Table 1.1: Properties of rubbers

Rubber	G [MPa]	ξ [%]	Shore A [–]	BC [%]
A	0.5	12	45	20
B	0.9	16	60	40
C	1.4	15	70	75
D	1.3	11	70	0

Reports of cyclic tests on a large number of full-scale isolators are available, all made with the four rubber compounds in Table 1.1. Isolators differ also for geometrical properties, such as the cross-section diameter or height and number of rubber and steel layers, and by the maximum shear deformation to which they are subjected during the test.

In Table 1.2 the resume of the main geometrical properties of each type of isolator is reported, together with the mean value and the standard deviation of their mechanical properties. These values are calculated after 5.8% of the isolators were considered outliers and excluded. The meaning of the symbols are:

- R: Rubber identification letter;
- I: Isolator identification number;
- $\bar{\gamma}_{max}$: mean value of the maximum shear strain on rubber layers reached during the test. The recorded value may slightly differ from the nominal one because of machine precision and unusual values can be specified by the designer;
- D_r : diameter of the rubber cross-section;
- D_s : diameter of the steel plate cross-section;
- D_h : diameter of the central hole, if present;
- t_r : thickness of one layer of rubber;
- t_s : thickness of one steel plate;
- n_r : number of rubber layers;
- $\bar{\xi}$: mean value of the damping of the isolators;
- σ_ξ : standard deviation of the damping of the isolators;
- $\bar{G}$: mean value of the shear modulus of the isolators;
- σ_G : standard deviation of the shear modulus of the isolators;

- $N_{is.}$: number of isolators of the same type.

Table 1.2: Resume of the geometric and mechanical properties of the real scale tested rubber isolators.

I	R	$\bar{\gamma}_{max}$ [%]	D_r [mm]	D_s [mm]	D_h [mm]	t_r [mm]	t_s [mm]	n_r	$\bar{\xi}$ [%]	σ_ξ [%]	$\bar{G}$ [MPa]	σ_G [%]	$N_{is.}$
A	01	159	900	880	0	11	6.8	19	14.5	3.1	0.435	4.1	4
A	02	100	900	860	51	11	3	10	11.3	10.1	0.475	2.4	10
A	03	100	400	390	0	6	2	8	18.1	0.8	0.522	0.6	2
A	04	190	440	430	41	6	3	11	15.1	6.5	0.422	4.6	6
A	05	100	500	490	51	6	3	19	17.4	13.1	0.478	4.9	12
A	06	100	550	540	41	7	3	20	16.8	11.6	0.496	4.2	8
A	07	100	550	540	41	8	3	13	19.0	1.5	0.473	0.7	2
A	07	150	550	540	41	8	3	13	17.7	0.8	0.420	0.0	2
A	07	200	550	540	41	8	3	13	16.1	0.9	0.418	0.8	2
A	08	100	550	540	0	8	3	8	19.7	0.8	0.547	2.2	2
A	09	100	650	640	41	6	3	20	14.5	2.2	0.527	4.4	10
A	10	152	700	690	41	6	3	34	17.2	0.7	0.537	4.5	2
A	11	100	750	740	0	7	3	11	16.1	0.4	0.537	1.7	2
A	12	105	900	890	41	8	4	25	15.6	2.7	0.512	2.7	2
A	12	130	900	890	41	8	4	25	11.6	2.4	0.457	3.5	2
A	12	155	900	890	41	8	4	25	12.6	0.0	0.459	0.0	2
A	12	205	900	890	41	8	4	25	11.7	0.0	0.451	0.0	2
A	13	146	900	890	41	8	4	26	14.0	3.9	0.538	0.6	2
A	14	100	900	890	0	8	3	6	13.2	-	0.537	-	1
A	15	100	900	890	0	8	3	18	11.9	0.9	0.529	1.5	2
A	15	150	900	890	0	8	3	18	9.8	3.0	0.466	2.5	2
A	15	198	900	890	0	8	3	18	8.7	5.5	0.444	2.2	2
B	01	100	450	440	41	5	3	25	14.7	4.4	0.870	4.3	21
B	02	99	700	680	0	8	4.7	25	20.1	2.1	0.980	3.9	2
B	02	166	700	680	0	8	4.7	25	17.5	3.5	0.896	6.0	59
B	02	200	700	680	0	8	4.7	25	16.3	4.3	0.782	6.2	2
B	03	100	400	390	41	5	3	25	16.7	2.3	1.020	0.9	2
C	01	115	650	630	41	12	3	8	16.6	2.6	1.210	2.6	2
C	01	179	650	630	41	12	3	8	14.2	1.3	1.063	1.0	2
C	02	120	550	540	41	6	4	18	13.0	0.2	1.445	1.4	2
C	03	149	500	490	41	6	3	15	15.9	6.6	1.307	7.6	4
C	04	138	700	690	41	8	3	15	17.4	2.2	1.314	2.4	8
C	05	134	750	740	41	9	3	10	16.4	2.0	1.424	2.7	12
C	06	115	800	790	41	9	3	9	16.8	0.8	1.428	1.5	12
C	07	122	800	790	41	9	3	12	17.1	2.4	1.339	3.1	10
C	07	126	800	790	41	9	3	12	15.7	2.3	1.356	1.3	14
C	08	101	800	790	41	9	3	23	17.2	3.1	1.378	4.0	4
C	09	60	850	840	41	10	3	12	18.9	1.3	1.907	1.0	8
C	10	136	900	890	41	10	3	10	15.6	2.3	1.313	1.1	12
C	11	56	950	940	41	10	3	16	18.3	1.5	1.983	1.4	8
C	12	100	1000	990	41	11	4	14	16.9	2.7	1.372	4.6	2
C	12	127	1000	990	41	11	4	14	15.6	1.4	1.280	3.2	2
C	13	100	1100	1090	41	12	4	16	16.4	0.4	1.479	2.2	2
C	13	117	1100	1090	41	12	4	16	15.0	0.8	1.357	2.5	2
D	01	100	650	640	0	13	4	13	12.0	-	1.130	-	1
D	02	100	600	600	0	10.5	3	10	14.1	-	1.240	-	1
D	02	150	600	600	0	10.5	3	10	12.9	-	1.166	-	1
D	02	200	600	600	0	10.5	3	10	11.4	-	1.100	-	1

In Figure 1.3 the values of dynamic shear modulus and damping of all the avail-

able elastomeric isolators are reported as a function of the maximum strain. The high dispersion in both values, particularly in damping, is evident. However, when examining the relationship between G_{dyn} and ξ in Figure 1.3c, clusters of points emerge, each representing isolators made from the same type of rubber composition. This clustering indicates that isolators with similar material properties tend to exhibit comparable values of shear modulus and damping ratio.

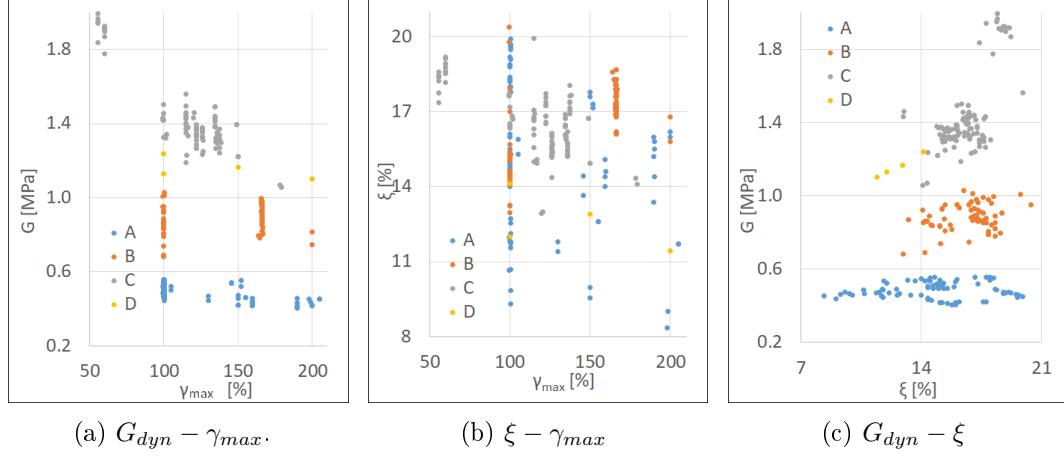


Figure 1.3: Dynamic shear modulus and damping computed from all cyclic shear tests on full-scale elastomeric isolators.

1.3 Thesis Overview

In Chapter 2 the mechanics of rubber are outlined to introduce the visco-elastic model and the parameters used to characterize its behavior. The data fitting procedures applied to the experimental tests conducted in the laboratory is presented.

The laboratory tests description is provided in Chapter 3. The experimental determination of the elastic stress response curve method and the single rubber pad designed specimen are presented. The effect of compression and defects on the specimen is investigated. Shear modulus and damping of cyclic tests on rubber specimens are compared to the ones of full-scale isolators.

In Chapter 4 the results of the fitting of the visco-elastic model to the test results are reported and used in the FEM simulations of the rubber isolators. Shear modulus and damping computed from simulations are compared to the ones of full-scale isolators.

Chapter 5 presents a numerical study investigating the seismic performance of a building equipped with a Tuned Mass Damper (TMD) system at its top. The analysis includes the use of simplified bilinear constitutive models, derived from elastomeric isolators tests, to represent their contribution to stiffness and energy dissipation. Two Tuned Mass Damper (TMD) seismic protection solutions are proposed for a reinforced concrete wall building with dissipative structural behavior. In these solutions, seismic isolators are used to provide the TMD of the design stiffness. Moreover, the results obtained with the TMD are compared with those obtained from the same building equipped with base isolation.

Chapter 2

Rubber Mechanics

To describe the mechanical behavior of rubber materials, in this work, we consider two types of mechanical deformations: simple shear and uniaxial extension. The visco-elastic constitutive model of rubber for finite strains and the stress evolution for a prescribed deformation history are derived in Section 2.1. The parameters defining such model will be calibrated on the base of the laboratory experimental tests described in Chapter 3 with the fitting process presented in Section 2.2.

2.1 Visco-elastic constitutive model

2.1.1 Mechanical deformations

Simple shear deformation

Simple shear involves parallel sliding of material layers, altering the shape without changing the volume. For an incompressible material subjected to a simple shear deformation [37] [38] of amplitude $\gamma = \tan \alpha$, as shown in Figure 2.1, the deformation gradient is

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \begin{bmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.1)$$

where $\mathbf{X}$ is the reference position of a material point and $\mathbf{x}$ is the deformed position of the same point [14]. The right Cauchy-Green deformation tensor is

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \begin{bmatrix} 1 & \gamma & 0 \\ \gamma & 1 + \gamma^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.2)$$

whose principal values (eigenvalues) are ω_j

$$\omega_{1,2} = \frac{2 + \gamma^2 \pm \gamma \sqrt{\gamma^2 + 4}}{2}, \quad \omega_3 = 1 \quad (2.3)$$

The simplest forms of elastic strain energy functions, including the one we will use, are polynomials that depend on the invariants of the deformation tensor I_k [15] [16], defined as

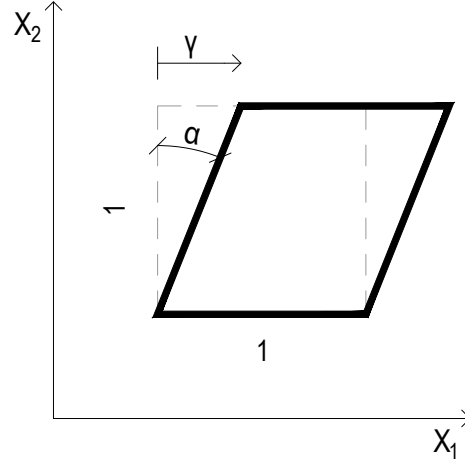


Figure 2.1: Simple shear deformation scheme.

$$I_1 = \text{tr}(\mathbf{C}) = \omega_1 + \omega_2 + \omega_3 = 3 + \gamma^2 \quad (2.4)$$

$$I_2 = \frac{1}{2} [(\text{tr}(\mathbf{C}))^2 - \text{tr}(\mathbf{C}^2)] = \omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1 = 3 + \gamma^2 \quad (2.5)$$

$$I_3 = \det(\mathbf{C}) = \omega_1\omega_2\omega_3 = 1 \quad (2.6)$$

where the third invariant is unitary because of incompressibility.

Uniaxial extension

Uniaxial extension involves stretching the material along one axis while permitting deformation in the perpendicular directions. For the uniaxial extension of stretch λ of an incompressible material, as in Figure 2.2, the deformation gradient is

$$\mathbf{F} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-\frac{1}{2}} & 0 \\ 0 & 0 & \lambda^{-\frac{1}{2}} \end{bmatrix} \quad (2.7)$$

and the right Cauchy-Green deformation tensor is

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & \lambda^{-1} \end{bmatrix} \quad (2.8)$$

whose eigenvalues ω_j are the diagonal elements, while the principal invariants I_k are

$$I_1 = \lambda^2 + 2\lambda^{-1} \quad (2.9)$$

$$I_2 = 2\lambda + \lambda^{-2} \quad (2.10)$$

$$I_3 = 1 \quad (2.11)$$

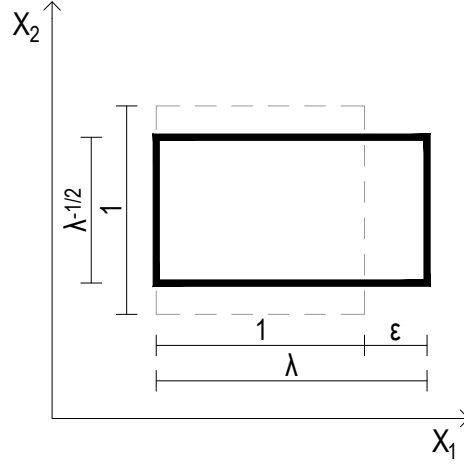


Figure 2.2: Uniaxial extension deformation scheme.

2.1.2 One-dimensional viscoelasticity

The starting point is the one-dimensional Maxwell-Wiechert visco-elastic model, which combines elastic elements (springs) and viscous elements (dashpots) as shown in Figure 2.3. The purely elastic part of the model has stiffness $E_\infty > 0$ while the N viscous elements are described by springs of stiffness $E_i > 0$ in series with dashpots of viscous coefficient $\eta_i > 0$.

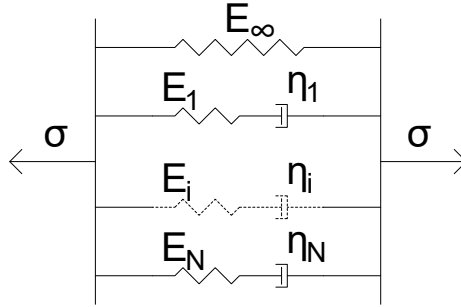


Figure 2.3: Maxwell-Wiechert visco-elastic model scheme

The total stress σ is the sum of the stress of each component of the model and it can be expressed as

$$\sigma = \sigma_\infty + \sum_{i=0}^N \sigma_i \quad (2.12)$$

$$\sigma_\infty = E_\infty \epsilon \quad (2.13)$$

$$\sigma_i = E_i (\epsilon - \beta_i) = \eta_i \dot{\beta}_i \quad (2.14)$$

where σ_∞ is the equilibrium stress and σ_i are the strain rate-dependent over-stresses. ϵ is the total strain and β_i are the internal variables describing the strain of the i -th dashpot. Starting from Equation 2.14 and defining the time constant of the i -th viscous component

$$\tau_i = \eta_i / E_i \quad (2.15)$$

we can write the differential equation governing the evolution of an internal variable

$$\dot{\beta}_i(t) + \frac{\beta_i(t)}{\tau_i} = \frac{\epsilon(t)}{\tau_i} \quad \text{for } t \in (0, \infty) \quad (2.16)$$

where the initial condition is usually the undeformed state $\beta_i(0) = 0 \quad \forall \quad i = 1 \dots N$.

The total strain energy of the system is the sum of the contributions of the purely elastic and the visco-elastic devices

$$W(\epsilon, \beta_i) = W_\infty(\epsilon) + \sum_{i=1}^N W_i(\epsilon, \beta_i) \quad (2.17)$$

$$= \frac{1}{2} E_\infty \epsilon^2 + \sum_{i=1}^N \frac{1}{2} E_i (\epsilon - \beta_i)^2 \quad (2.18)$$

The stress in 2.12 is given by

$$\sigma = \frac{\partial}{\partial \epsilon} W(\epsilon, \beta_i) \quad (2.19)$$

and the dissipation inequality still holds

$$\mathcal{D} = - \sum_{i=1}^N \frac{\partial}{\partial \beta_i} W(\epsilon, \beta_i) \dot{\beta}_i = \sum_{i=1}^N \eta_i \dot{\beta}_i^2 \geq 0 \quad (2.20)$$

Finite strain deformations

The additive decomposition is a simplification typically valid for small deformations, where the total response of the material is assumed to be the sum of elastic and viscous components, as in Equation 2.12. However, as proposed by Simo and Hughes [18] [25], we can express the model equations in terms of the free energy W and the over-stresses σ_i as follows, allowing for a more general formulation of the viscoelastic behavior.

$$W(\epsilon, \sigma_i) = W_\infty(\epsilon) + \sum_{i=1}^N W_i(\epsilon, \sigma_i) \quad (2.21)$$

$$\sigma_\infty(\epsilon) = \frac{\partial}{\partial \epsilon} W_\infty(\epsilon) \quad (2.22)$$

$$\sigma(\epsilon, \sigma_i) = \sigma_\infty(\epsilon) + \sum_{i=1}^N \sigma_i \quad (2.23)$$

$$\dot{\sigma}_i(t) + \frac{\sigma_i(t)}{\tau_i} = \mu_i \dot{\sigma}_\infty(t) \quad (2.24)$$

$$\sigma_i(0) = \mu_i \sigma_\infty(0) \quad (2.25)$$

which are compatible with a finite strain context. Equation 2.24 is derived from Equation 2.16 by setting the constant $\mu_i = E_i / E_\infty > 0$, which represents the magnitude of the i -th viscous component.

2.1.3 Three-dimensional viscoelasticity

Assuming the rubber behaves as a visco-hyperelastic material, the one-dimensional Maxwell-Wiechert model can be extended to a three-dimensional framework by replacing the total stress σ and the equilibrium stress σ_∞ with the second Piola-Kirchhoff stress tensors $\mathbf{S}$ and $\mathbf{S}_\infty$, the over-stresses σ_i with symmetric tensors $\mathbf{Q}_i$ and the strain ϵ with the right Cauchy-Green deformation tensor $\mathbf{C}$. This extension yields to the model

$$W(\mathbf{C}, \mathbf{Q}_i) = W_\infty(\mathbf{C}) + \sum_{i=1}^N W_i(\mathbf{C}, \mathbf{Q}_i) \quad (2.26)$$

$$\mathbf{S}_\infty(\mathbf{C}) = 2 \frac{\partial}{\partial \mathbf{C}} W_\infty(\mathbf{C}) \quad (2.27)$$

$$\mathbf{S}(\mathbf{C}, \mathbf{Q}_i) = \mathbf{S}_\infty(\mathbf{C}) + \sum_{i=1}^N \mathbf{Q}_i \quad (2.28)$$

$$\dot{\mathbf{Q}}_i(t) + \frac{\mathbf{Q}_i(t)}{\tau_i} = \mu_i \dot{\mathbf{S}}_\infty(t) \quad (2.29)$$

$$\mathbf{Q}_i(0) = \mu_i \mathbf{S}_\infty(0) \quad (2.30)$$

For a prescribed deformation history $\mathbf{C}(t)$, the elastic equilibrium stress $\mathbf{S}_\infty(t)$ can be derived and the response of the viscous over-stresses integrated by the convolution integral, solution of the first order differential Equation 2.29

$$\mathbf{Q}_i(t) = \mu_i \int_{-\infty}^t e^{-\frac{t-s}{\tau_i}} \dot{\mathbf{S}}_\infty(s) ds \quad (2.31)$$

which can be analytically evaluated only for simple deformation histories, such as constant deformation. For more complex histories, numerical integration is required.

The general solution Equation 2.28 of the problem (see Holzapfel [20] and Simo [19]) can be written as

$$\mathbf{S}(t) = \int_{-\infty}^t \dot{\mathbf{S}}_\infty(s) ds + \sum_{i=1}^N \mu_i \int_{-\infty}^t e^{-\frac{t-s}{\tau_i}} \dot{\mathbf{S}}_\infty(s) ds \quad (2.32)$$

$$= \int_{-\infty}^t \left[1 + \sum_{i=1}^N \mu_i e^{-\frac{t-s}{\tau_i}} \right] 2 \frac{d}{ds} \frac{d}{d\mathbf{C}} W_\infty(\mathbf{C}) ds \quad (2.33)$$

Decoupling of isochoric and volume deformations

The assumption of rubber incompressibility is acceptable for analytical models, as it is a useful simplification that closely approximates real behavior. However, in finite element (FE) analyses, it can lead to element locking. Therefore, we consider the rubber to be compressible, incorporating a linear elastic volumetric constitutive law that is uncoupled from the isochoric one. Denoting with $(\bullet)^*$ the variables referring to the total deformation and with $J = \det(\mathbf{F}^*) = \det(\mathbf{F})$ we can define

$$\mathbf{F}^* = \mathbf{F}_V \mathbf{F} = J^{\frac{1}{3}} \mathbf{F} \quad (2.34)$$

$$\mathbf{C}^* = \mathbf{C}_V \mathbf{C} = J^{\frac{2}{3}} \mathbf{C} \quad (2.35)$$

Then, the total energy W^* and the total stress $\mathbf{S}^*$ are the sums of their isochoric and volumetric components

$$W^*(J, \mathbf{C}, \mathbf{Q}_i) = W_V(J) + W_\infty(\mathbf{C}) + \sum_{i=1}^N W_i(\mathbf{C}, \mathbf{Q}_i) \quad (2.36)$$

$$\mathbf{S}^* = \mathbf{S}_V + \mathbf{S} \quad (2.37)$$

$$\mathbf{S}_V = 2 \frac{\partial}{\partial \mathbf{C}^*} W_V(J) = 2 \frac{\partial}{\partial J} W_V(J) \quad (2.38)$$

Equations 2.27 to 2.29 remain valid. In the following, we will consider volumetric deformability only in FE analyses because, for the deformations under consideration, it is not a significant factor.

2.1.4 Equilibrium energy

Although incompressible materials energy function per unit volume W_∞ is a function of I_1 and I_2 , the one here adopted follows Yeoh's formulation [16], a cubic polynomial which depends only on the first invariant of the Green deformation tensor

$$W_\infty(I_1) = \sum_{j=1}^3 C_{j0} (I_1 - 3)^j = C_{10} (I_1 - 3) + C_{20} (I_1 - 3)^2 + C_{30} (I_1 - 3)^3 \quad (2.39)$$

where C_{j0} are constants to be determined based on experimental tests. For a simple shear deformation, $I_1 = 3 + \gamma^2$ (see Equation 2.4), thus Equation 2.39 depends on the shear deformation γ as

$$W_\infty(\gamma) = C_{10}\gamma^2 + C_{20}\gamma^4 + C_{30}\gamma^6 \quad (2.40)$$

and the equilibrium shear stress is

$$S_\infty = \frac{d}{d\gamma} W_\infty(\gamma) = 2C_{10}\gamma + 4C_{20}\gamma^3 + 6C_{30}\gamma^5 \quad (2.41)$$

For an uniaxial extension W_∞ becomes a function of λ with exponents from -3 to 6 and the normal stress is

$$T_\infty = \frac{d}{d\lambda} W_\infty(\lambda) = 2 \left(\lambda - \frac{1}{\lambda^2} \right) \left[C_{10} + 2C_{20} \left(\lambda^2 + \frac{2}{\lambda} - 3 \right) + 3C_{30} \left(\lambda^2 + \frac{2}{\lambda} - 3 \right)^2 \right] \quad (2.42)$$

Constraints on the elastic parameters

The strain energy function must be zero in the undeformed state, positive and monotonously increasing for an increasing shear deformation. The following constraints apply to the material constants

$$W_\infty(\gamma = 0) = 0 \quad \rightarrow \quad \text{No constraints} \quad (2.43)$$

$$W_\infty(\gamma) \geq 0 \quad \rightarrow \quad C_{20}^2 - 4C_{10}C_{30} < 0 \quad (2.44)$$

$$\frac{d}{d\gamma}W_\infty(\gamma > 0) > 0, \quad \frac{d}{d\gamma}W_\infty(\gamma < 0) < 0 \quad \rightarrow \quad C_{20}^2 - 3C_{10}C_{30} < 0 \quad (2.45)$$

Based on experimental evidence, rubber materials subjected to simple shear and uniaxial tension consistently exhibit a hardening behavior. Hence, the shear stress must also be monotonously increasing, which imposes the most restrictive constraint

$$\frac{d}{d\gamma}S(\gamma) > 0 \quad \rightarrow \quad C_{20}^2 - \frac{5}{3}C_{10}C_{30} < 0 \quad (2.46)$$

The same constraints on the uniaxial tension stress yield to a rational expression. As, in the following, we are interested in solving a fitting problem, it is easier to check numerically that the energy and the stress are positive and monotonously increasing for an increasing stretch λ , given a set of parameters C_{j0} .

Additional constraints are imposed by experimental observations: $C_{10} > 0$ because the stress derivative is positive at the origin, $C_{30} > 0$ because at large strains the stress rises due to the finite extensibility of the polymer chains and $C_{20} < 0$ to catch the softening that usually occurs after the smaller deformations.

2.1.5 Stress evolution equations in simple shear and uniaxial extension

Once the Yeoh energy (see Equation 2.39) is chosen, the system of equations governing the stress response to a prescribed simple shear strain history $\gamma(t)$ such that $\gamma(0) = 0$, becomes:

$$S(t) = S_\infty(t) + \sum_i Q_i(t) \quad (2.47)$$

$$S_\infty(t) = 2C_{10}\gamma(t) + 4C_{20}\gamma^3(t) + 6C_{30}\gamma^5(t) \quad (2.48)$$

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = \mu_i \dot{S}_\infty(t) \quad (2.49)$$

$$\gamma(0) = S_\infty(0) = Q_i(0) = 0 \quad \forall \quad i = 1 \dots N \quad (2.50)$$

which is not analytically solvable for any given simple shear strain history $\gamma(t)$. However, the stress expressed in Equation 2.48 can be directly determined for a given shear strain and the solution of Equation 2.49 is determined through the convolution integral of Equation 2.31 for each i -th viscous term of the model, starting from the initial undeformed state. From this point onward, the variable S will represent the shear stress.

The system of equations governing the stress response to a prescribed uniaxial extension stretch history $\lambda(t)$ such that $\lambda(0) = 1$, becomes:

$$T(t) = T_\infty(t) + \sum_i P_i(t) \quad (2.51)$$

$$T_\infty(t) = 2 \left(\lambda(t) - \frac{1}{\lambda^2(t)} \right) \left[C_{10} + 2C_{20} \left(\lambda^2(t) + \frac{2}{\lambda(t)} - 3 \right) + 3C_{30} \left(\lambda^2(t) + \frac{2}{\lambda(t)} - 3 \right)^2 \right] \quad (2.52)$$

$$\dot{P}_i(t) + \frac{P_i(t)}{\tau_i} = \mu \dot{T}_\infty(t) \quad (2.53)$$

$$\lambda(0) = T_\infty(0) = P_i(0) = 0 \quad \forall \quad i = 1 \dots N \quad (2.54)$$

which can be also numerically solved once a stretch history $\lambda(t)$ is assigned. From this point onward, the variable T will represent the normal stress.

2.1.6 Theoretical simple shear cyclic test

The dynamic shear modulus G_{dyn} (Equation 1.1 and 1.3) and the damping coefficient ξ (Equation 1.2) of a rubber element subjected to a indefinite sinusoidal shear strain history with period T and amplitude γ_0 can be evaluated using closed-form expressions, once the rubber visco-elastic parameters are set. The hypotheses are that the effects of the initial conditions are negligible (steady-state solution) and that the dissipated energy does not increase the temperature of the rubber, so its mechanical properties do not change. These expressions allow to avoid the integration of a cyclic test during the data fitting of an experimental cyclic test to evaluate G_{dyn} and ξ . The complete step-by-step calculations are reported in Appendix A.

Defining

$$\phi_i = \frac{1}{\omega \tau_i} = \frac{T}{2\pi \tau_i} \quad (2.55)$$

we can evaluate the dynamic shear modulus as

$$G_{dyn} = 2 \left(C_{10} + 2C_{20}\gamma_0^2 + 3C_{30}\gamma_0^4 \right) + \sum_{i=1}^N \mu_i \left[\frac{2C_{10} + 3C_{20}\gamma_0^2 + \frac{15}{4}C_{30}\gamma_0^4}{1 + \phi_i^2} + \frac{C_{20} + \frac{15}{8}C_{30}\gamma_0^2}{1 + \left(\frac{\phi_i}{3}\right)^2} \gamma_0^2 + \frac{\frac{3}{8}C_{30}\gamma_0^4}{1 + \left(\frac{\phi_i}{5}\right)^2} \right] \quad (2.56)$$

the energy dissipated per unit volume in a single cycle

$$\mathcal{D} = \pi \gamma_0^2 \left(2C_{10} + 3C_{20}\gamma_0^2 + \frac{15}{4}C_{30}\gamma_0^4 \right) \sum_i \frac{\mu_i \phi_i}{1 + \phi_i^2} \quad (2.57)$$

and the damping

$$\xi = \frac{2C_{10} + 3C_{20}\gamma_0^2 + \frac{15}{4}C_{30}\gamma_0^4}{2G_{dyn}} \sum_i \frac{\mu_i \phi_i}{1 + \phi_i^2} \quad (2.58)$$

2.1.7 Theoretical uniaxial tension cyclic test

The same logic of previous paragraph can be applied to a uniaxial cyclic test, with the same hypothesis. The problem is set to evaluate the expressions of dynamic modulus E_{dyn} and damping ξ for a dog-bone specimen in tension, with low amplitude stretch cycles around a value of $\lambda > 1$ such that buckling is avoided, i.e. $T(\lambda) \geq 0$. While a part of the expressions obtained for the modulus and the damping are explicit, some terms can only be evaluated numerically, making the attempt to avoid integration of the whole curve futile, in fact making it computationally heavier. Nonetheless, the complete step-by-step calculations are reported in Appendix A.

2.1.8 Transition to adimensional parameters

The visco-elastic model is completely described by the 3 Yeoh elastic parameters, C_{j0} for $j = 1, 2, 3$, and $2N$ viscous parameters (τ_i, μ_i) for $i = 1 \dots N$. All values are bounded by the mechanical properties of rubber, which can be synthetized by the nominal shear modulus G , whose values range from 0.3 MPa to 1.5 MPa, and the damping ξ , ranging from 2 % to about 20 %.

For the fitting algorithm, imposing upper and lower limits can be useful to save computational time, and working with different variables can simplify the solution.

Elastic parameters

For small deformations the shear modulus is $G = 2C_{10}$ and the Young's modulus is $E = 6C_{10}$. We can impose an upper limit for the first elastic parameter, e.g. $C_{10} \leq 2 \text{ MPa}$, which is still a relatively lenient constraint. The values of the other two constants are lower in magnitude, often by one or two orders. Considering the constraint in Equation 2.46 and defining the new adimensional variables ρ and ζ we can write that

$$C_{20} = -\rho C_{10} \sqrt{\frac{5}{3}} \zeta \quad (2.59)$$

$$C_{30} = \zeta C_{10} \quad (2.60)$$

$$0 < C_{10} \leq 2 \text{ MPa} \quad (2.61)$$

$$0 \leq \rho < 1 \quad (2.62)$$

$$0 \leq \zeta \leq 1 \quad (2.63)$$

thus, narrowing the field of search of the parameters to a fixed range. An advantage is also that the physical inequality constraint of Equation 2.46 is automatically true.

The constitutive law for shear and uniaxial stresses can be re-written in terms of C_{10}, ρ, ζ

$$S_{\infty} = 2C_{10}\gamma \left(1 - 2\rho \sqrt{\frac{5}{3}} \zeta \gamma^2 + 3\zeta \gamma^4 \right) \quad (2.64)$$

$$T_{\infty} = 2 \left(\lambda - \frac{1}{\lambda^2} \right) C_{10} \left(1 - 2\rho \sqrt{\frac{5}{3}} \zeta \left(\lambda^2 + \frac{2}{\lambda} - 3 \right) + 3\zeta \left(\lambda^2 + \frac{2}{\lambda} - 3 \right)^2 \right) \quad (2.65)$$

Viscous parameters

The values of τ_i can be defined a priori, as described in [17]. The only limit to the viscous parameters μ_i is that they must be non-negative, but an upper limit is not defined. We can use the parameters $\alpha_i = E_i/E_0$, defined on the base of the instantaneous modulus E_0 , representing the total stiffness of the material immediately after a deformation is applied before any viscoelastic relaxation occurs, instead of E_∞ , where

$$E_0 = E_\infty + \sum_{i=1}^N E_i = E_\infty \left(1 + \sum_{i=1}^N \mu_i \right) \quad (2.66)$$

By the definition of α_i we obtain its relationship with μ_i , with the advantage that $0 \leq \alpha_i < 1$

$$\alpha_i = \frac{\mu_i}{1 + \sum_{i=1}^N \mu_i} \quad (2.67)$$

$$\mu_i = \frac{\alpha_i}{1 - \sum_{i=1}^N \alpha_i} \quad (2.68)$$

Equation 2.33 can be written in terms of α_i , with the addition of the term α_∞ and a strain energy Φ_∞ instead of W_∞

$$\mathbf{S}(t) = \int_{-\infty}^t \left(\alpha_\infty + \sum_{i=1}^N \alpha_i e^{-\frac{t-s}{\tau_i}} \right) 2 \frac{d}{ds} \frac{d}{d\mathbf{C}} \Phi_\infty(\mathbf{C}) ds \quad (2.69)$$

As $\mathbf{S}_\infty$ and all $\mathbf{Q}_i$ must be the same, using μ or α , for all deformations $\mathbf{C}$ and for all times t

$$\mathbf{S}_\infty(t) : \int_{-\infty}^t 2 \frac{d}{ds} \frac{d}{d\mathbf{C}} W_\infty(\mathbf{C}) ds = \int_{-\infty}^t \alpha_\infty 2 \frac{d}{ds} \frac{d}{d\mathbf{C}} \Phi_\infty(\mathbf{C}) ds \quad (2.70)$$

$$\mathbf{Q}_i(t) : \int_{-\infty}^t \mu_i e^{-\frac{t-s}{\tau_i}} 2 \frac{d}{ds} \frac{d}{d\mathbf{C}} W_\infty(\mathbf{C}) ds = \int_{-\infty}^t \alpha_i e^{-\frac{t-s}{\tau_i}} 2 \frac{d}{ds} \frac{d}{d\mathbf{C}} \Phi_\infty(\mathbf{C}) ds \quad (2.71)$$

we obtain from Equation 2.70 that

$$\frac{d}{d\mathbf{C}} W_\infty(\mathbf{C}) = \alpha_\infty \frac{d}{d\mathbf{C}} \Phi_\infty(\mathbf{C}) \quad \Rightarrow \quad W_\infty(\mathbf{C}) = \alpha_\infty \Phi_\infty(\mathbf{C}) + const \quad (2.72)$$

where the constant is zero because both expressions of energy must be zero in the undeformed state, when $\mathbf{C} = \mathbf{1}$, and α_∞ is constant. Note that Yeoh's equilibrium energy constants would be scaled as $C_{i0,\Phi} = C_{i0,W}/\alpha_\infty$. By replacing Equation 2.72 in the expression of the over-stress in Equation 2.71 we obtain that $\alpha_i = \alpha_\infty \mu_i$ and, replacing it in Equations 2.67 and 2.68, that

$$\alpha_\infty = \frac{1}{1 + \sum_{i=1}^N \mu_i} = 1 - \sum \alpha_i \quad (2.73)$$

It is worth mentioning that the sum of all parameters α is unity, each α is scaled from μ by the same quantity and a constant μ_∞ is not defined because it is just equal to unity.

2.1.9 Effect of viscous parameters

Once the values of the viscous parameters μ and τ are determined, they can be represented with a discrete spectrum, where, for each value of time constant τ , the magnitude of the associated viscous parameter μ or α is displayed. This spectrum is useful for providing a direct indication of the significance of each viscous component. As the magnitude of μ_i and α_i depends on their total number N , the spectrum of a fitted rubber is not unique and different choices of N could lead to spectrum curves that do not overlap. Typically, short-term effect components ($\tau \rightarrow 0$) are more influential but dissipate quickly, while long-term effect components ($\tau \rightarrow \infty$) continue to exert an effect over time, resulting in a hyperbola-shaped spectrum curve.

Cyclic tests on a rubber with a fixed set of elastic parameters and a single viscous component ($N = 1$) of constants (μ_1, τ_1) were simulated at different frequencies (or periods, T) to show the effect of the single viscous component on the properties of the rubber. The dynamic modulus G_{dyn} and the dissipated energy $\mathcal{D}$ were computed for each test at the third cycle, in accordance with the requirements for isolators set by the UNI EN 15129 [13] standard and when the cycle shape is stable. The simulation of cyclic tests were repeated for values of τ_1 varying in the range $10^{-2} \div 10^4$ s and $\mu_1 = 1$. The trends of G_{dyn} are shown in Figure 2.4 and those of $\mathcal{D}$ in Figure 2.5.

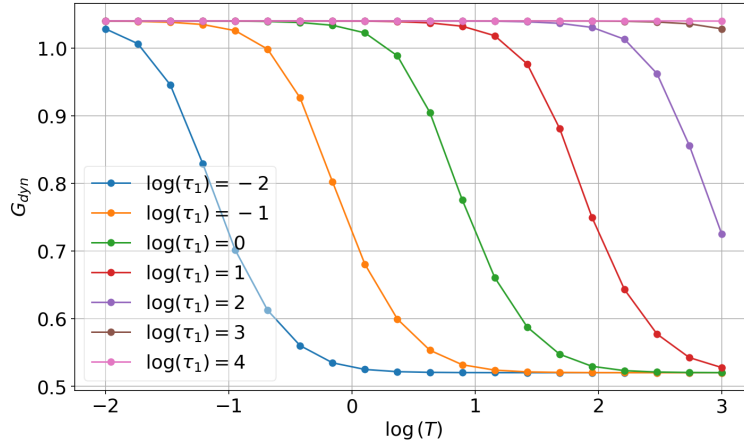


Figure 2.4: Trends of the dynamic modulus for rubbers with $N = 1$ and different values of the time constant τ_1 .

We can see that short-term viscous terms contribute to the increase of dynamic modulus with respect to the elastic terms only for high-frequency tests. For increasing values of τ_1 , the only time constant in this case, the curves shift rightwards and the values of G_{dyn} for a given value of the period T tend to increase until reaching a maximum value. In fact, in this range of periods, from the observer point of view, the viscous term $\log(\tau_1) = 4$ can be confused with an increase in the elastic contribution, as the dissipated energy related to it is negligible. This is due to the exponential term in Equation 2.31 which is almost equal to one for $\tau_i \rightarrow \infty$ and the result of the integration is that the over-stress are $\mathbf{Q}_i = \mu_i \mathbf{S}_\infty$. These trends are identical to the curves derived for the functions of ϕ that appear in Equations 2.56 and 2.58 and shown in Figure A.1.

Figure 2.5 shows that the amplitude of the hysteresis cycle reaches a maximum for a period value proportional to the time constant (see Section 2.2.9) and tends to zero as the period moves further from this value, similarly to the behavior of an

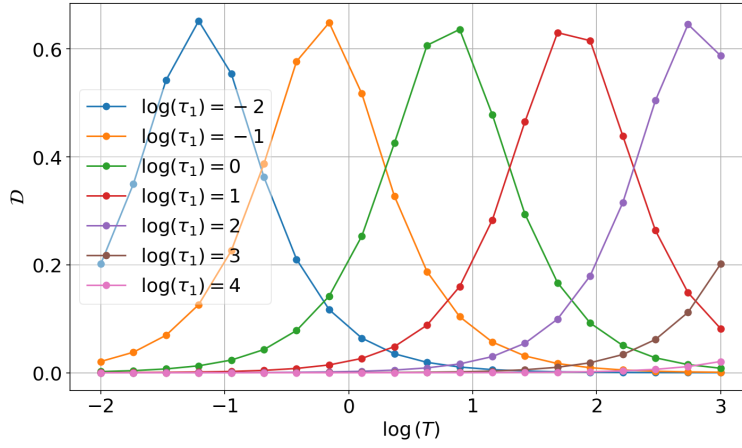


Figure 2.5: Trends of the dissipated energy for rubbers with $N = 1$ and different values of τ_1 .

oscillator excited at its resonant frequency. Again, in Section 2.2.9 this property will be exploited to determine the viscous parameters of a rubber.

The same cyclic test were numerically simulated for rubbers with $N = 2$ viscous parameters, fixed τ_2 and different values of τ_1 . Figures 2.6 and 2.7 show the trends of the dynamic modulus G_{dyn} and the dissipated energy $\mathcal{D}$ for this case. We can see that the peaks in dissipated energy are now two and there is an intermediate section in the sloped portion of the modulus curves. These curves essentially represent the sum of contributions from individual viscous terms, as shown in Figures 2.8 and 2.9, which display the trends of modulus and energy dissipation for three rubber materials: two with a single viscous term and one with two viscous terms, where the time constants of the latter match those of the former.

The behavior of real rubbers is expected to be described by many viscous terms; hence, the curves describing the dynamic modulus and damping as a function of the period of the cyclic test should show a regular trend rather than peaks for a discrete number of periods.

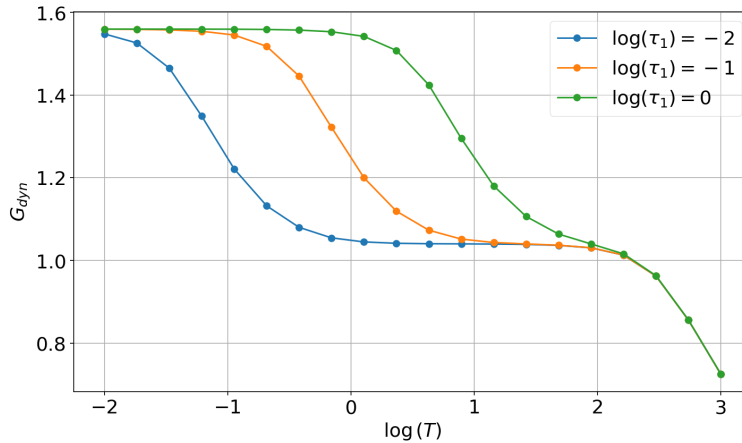


Figure 2.6: Trends of the dynamic modulus for rubbers with $N = 2$ and different values of τ_1 and $\log(\tau_2) = 2$.

In Figure 2.10 the third cycle of cyclic tests on a rubber with two viscous pa-

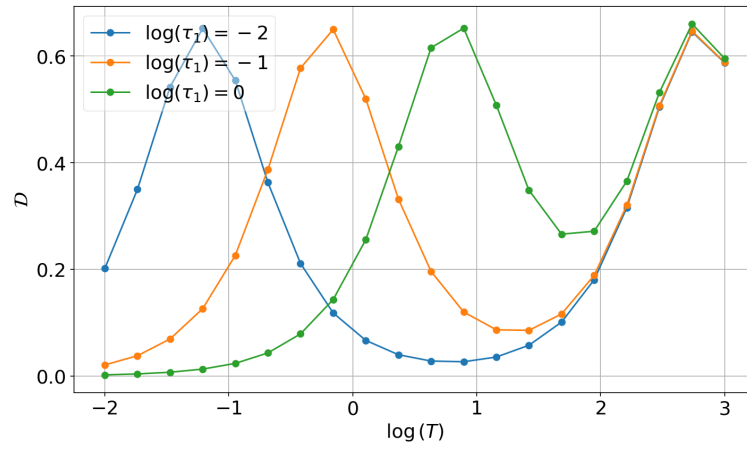


Figure 2.7: Trends of the dissipated energy for rubbers with $N = 2$ and different values of τ_1 and $\log(\tau_2) = 2$.

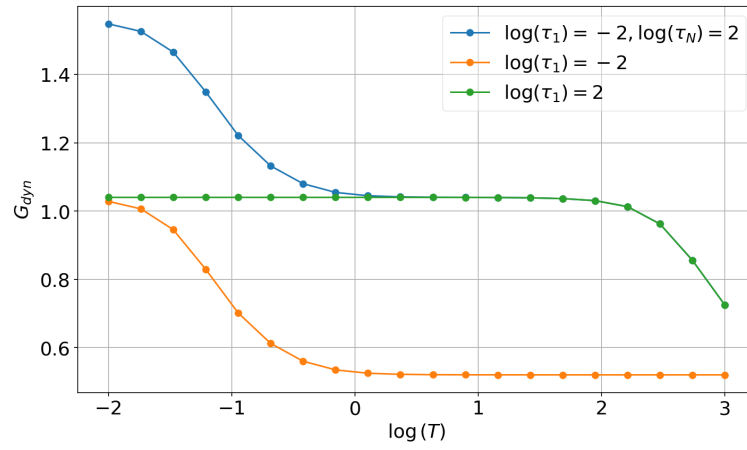


Figure 2.8: Trends of the dynamic modulus for two rubbers with $N = 1$ and one rubber with $N = 2$ with the same values of τ .

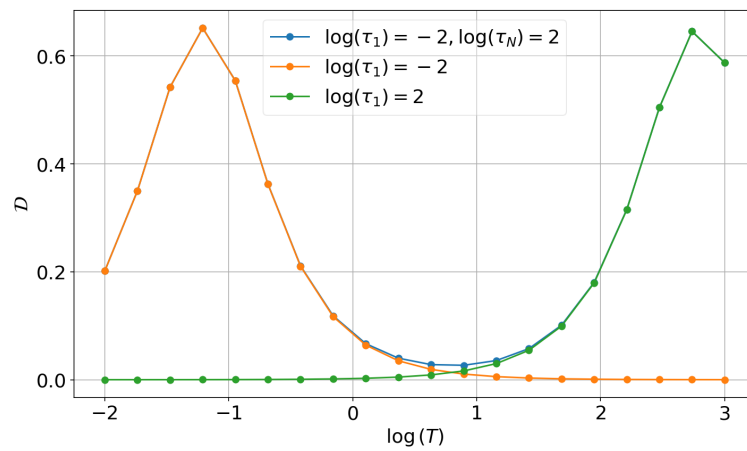


Figure 2.9: Trends of the dissipated energy for two rubbers with $N = 1$ and one rubber with $N = 2$ with the same values of τ .

rameters simulated at two different loading frequencies are shown, together with the equilibrium curve. The values of μ are the same, but the areas of the hysteresis cycles are not, because different T/τ ratios, or ϕ as defined in Equation 2.55, were set. The dynamic modulus increases for a higher loading frequency of the test, thus the damping decreases. Each component of the over-stress, for both tests, is shown in Figure 2.11. At high-frequency the short-term viscous term contributes to energy dissipation and dynamic modulus while the long-term viscous term only contributes to the value of G_{dyn} . At low frequency the long-term viscous term dissipates energy and the short-term one is practically absent. It is worth noting that for a given T/τ ratio the shape of the hysteresis cycle is constant and its amplitude, and so $\mathcal{D}$, is only proportional to μ .

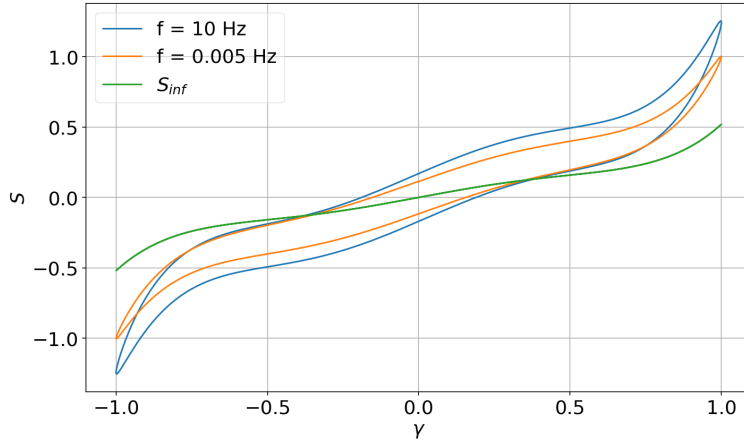


Figure 2.10: Third cycle in the total stress-strain plane of two simulated cycle tests at different frequencies and the adopted equilibrium elastic stress curve.

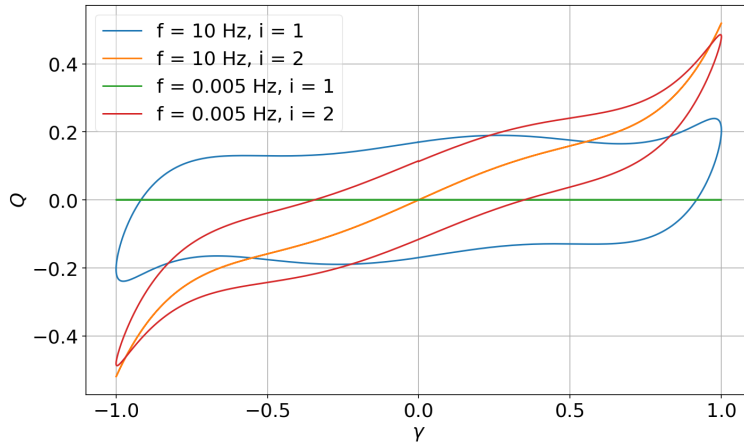


Figure 2.11: Third cycle in the viscous stress-strain plane of two simulated cycle tests at different frequencies and the adopted equilibrium elastic stress curve.

2.1.10 Relationship between γ_{max} and λ_{max}

As the maximum deformation reached by the rubber affects its behavior because of Mullins effect [22], shear tests and tension tests have to be conducted on specimens

conditioned at an equivalent level of strain, in order to consider them together in the fitting process of the visco-elastic model. An approximation of the relationship between a shear strain γ and a stretch λ can be obtained considering a material fiber of an element of unit length sides: if deformed in simple shear by a value γ , it has a stretch $\lambda = \sqrt{1 + \gamma^2}$. Another expression is obtained equating the first eigenvalue ω_1 of Equation 2.2 to the one in Equation 2.8, obtaining the expression in Equation 2.74.

$$\lambda = \sqrt{1 + \gamma^2 \frac{1 + \sqrt{1 + \frac{4}{\gamma^2}}}{2}} \quad (2.74)$$

γ is explicitly derived from the expression in Equation 2.74 to find a simple relationship

$$\gamma = \lambda - \frac{1}{\lambda} \quad (2.75)$$

The values obtained with these expressions of the relationship $\gamma_{max} - \lambda_{max}$ are compared in Figure 2.12 for values of deformation lower than failure of the rubber.

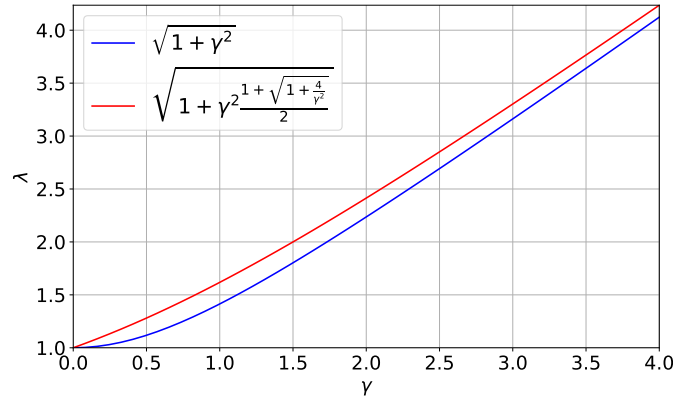


Figure 2.12: $\gamma - \lambda$ relationship.

2.2 Data fitting

Having described the viscoelastic model of rubber, the next step involves determining the model parameters. This is achieved through a data fitting process, where the parameters are calibrated based on experimental test results. By calibrating the model by means of the experimental data, we ensure that the model accurately represents the material response under the given conditions.

The laboratory experimental tests which allow to do this consist in imposing deformation histories to the specimen, during which time t , displacement d , and force F are recorded at each sampling point. Displacement and force are then converted into mean strain and stress values, to work directly with quantities defined in the material model previously introduced. This conversion is valid only if the experimental test closely replicates the theoretical deformation, i.e. simple shear or uniaxial tension, ensuring that the mean values accurately reflect the behavior of

the specimen. The fitting process essentially involves replicating the laboratory test, specifically in terms of mean strain and stress, by imposing the same recorded strain history and calculating the model stress response. This modeled stress response is then compared to the experimental data, and an error is calculated to assess the accuracy of the fit.

2.2.1 Error function

To calibrate the model parameters, we define an error function χ that measures the discrepancy between the model predictions and the experimental data. The minimization of this error function allows to determine the parameters that most accurately represent the material behavior. The most straightforward form of this error function is the Root Mean Square Error (RMSE), which is the square root of the mean of the squared differences between the observed stress and predicted one. Naming $\mathbf{x}$ the vector of A elements containing the experimental values and $\tilde{\mathbf{x}}$ the vector of the same length containing the corresponding values predicted by the model, χ is defined as

$$\chi(\mathbf{x}, \tilde{\mathbf{x}}) = \sqrt{\frac{\sum_{k=1}^A (x_k - \tilde{x}_k)^2}{A}} \quad (2.76)$$

where x_k and $\tilde{x}_k$ are the k -th components of $\mathbf{x}$ and $\tilde{\mathbf{x}}$, respectively.

In general, $\mathbf{x}$ and $\tilde{\mathbf{x}}$ can include not only stress values but also any other quantities that can be recorded from tests and then predicted by the model, e.g. the shear modulus or damping from a cyclic test. Also, multiple tests can be concatenated and constraints, e.g. on the elastic constants in Equation 2.46, can be added to the error function in the form of penalty functions with high weights.

The following definitions are considered.

- $\mathbf{x}_l, \tilde{\mathbf{x}}_l$ for $l = 1 \dots B$ are the vectors of the l -th experimental test, each of length A_l . Each vector component is denoted $x_{l,k}$ and $\tilde{x}_{l,k}$ for $l = 1 \dots B, k = 1 \dots A_l$. Each vector is normalized to the mean value $\bar{x}_l$ of $\mathbf{x}_l$ to account for the relative error of the test. The vectors $\mathbf{x}$ and $\tilde{\mathbf{x}}$ are obtained by concatenating all B vectors.
- $\mathbf{p}_l$ for $l = 1 \dots B$ are the normalized weight vectors for the l -th test, with components $p_{l,k}$. These weights control the importance of each point within a test, e.g. allowing one part of the test to be excluded from the fitting or to have more or less emphasis compared to others.
- $\mathbf{w}$ is a vector of length B containing the weights of each test. These weights can be normalized or not, as normalization simply scales the error function.

The general error function is then defined as:

$$\chi^2(\mathbf{x}, \tilde{\mathbf{x}}, \mathbf{p}, \mathbf{w}) = \sum_{l=1}^B \sum_{k=1}^{A_l} \left(\frac{x_{l,k} - \tilde{x}_{l,k}}{\bar{x}_l} \right)^2 w_l p_{l,k} \quad (2.77)$$

This error function incorporates the relative importance of different data points, weights for different tests, and ensures that the contribution of each point to the overall error is appropriately scaled.

2.2.2 Time constants

In the modeling of viscoelastic material behavior, we fix the N time constants τ_i to simplify the fitting process and reduce the number of variables, thereby enhancing computational efficiency. Although it is possible to leave these constants free and determine them through the fitting algorithm, manually specifying the time constants allows for better control over model complexity and ensures accurate representation across both short and long time scales. The time constants are chosen to be equidistant on a logarithmic scale, i.e. such that the difference $\log(\tau_{i+1}) - \log(\tau_i)$ is constant, with the minimum and maximum values determined based on the sampling frequency and the duration of the test, respectively. Values of time constants smaller than the sampling interval may not be accurately captured and can be influenced by noise, while constants greater than the duration of the test may yield misleading results if extrapolated beyond the test period.

To validate this assumption, an experimental shear relaxation test was fitted with fixed values of τ_i for $i = 1 \dots 6$, where $N = 6$ was set. Then, only the values of τ_i for $i = 2 \dots 5$ were scaled by the same factor, and the fitting was performed again for different scaling factors. For example, if in a logarithmic equidistant distribution of time constants is $\tau_i = [1.0 \cdot 10^{-2}, 1.1 \cdot 10^{-1}, 1.3 \cdot 10^0, 1.4 \cdot 10^1, 1.6 \cdot 10^2, 1.8 \cdot 10^3]$, the 50 % scaled distribution is $\tau_i = [1.0 \cdot 10^{-2}, 5.6 \cdot 10^{-2}, 6.3 \cdot 10^{-1}, 7.1 \cdot 10^0, 8.0 \cdot 10^1, 1.8 \cdot 10^3]$. The responses shown in Figure 2.13 confirm that each fitted curve closely matches the experimental test stress curve. In Figure 2.14 the spectra of all the fitted models are presented, with each showing similar trends. It can be noted that the value of μ_1 decrease as τ_2 decreases. This occurs because the weight of the second term increases, causing its contribution to the overall model to become more significant and compensating for the reduction in μ_1 . As a result, the sum of the contributions remains relatively constant.

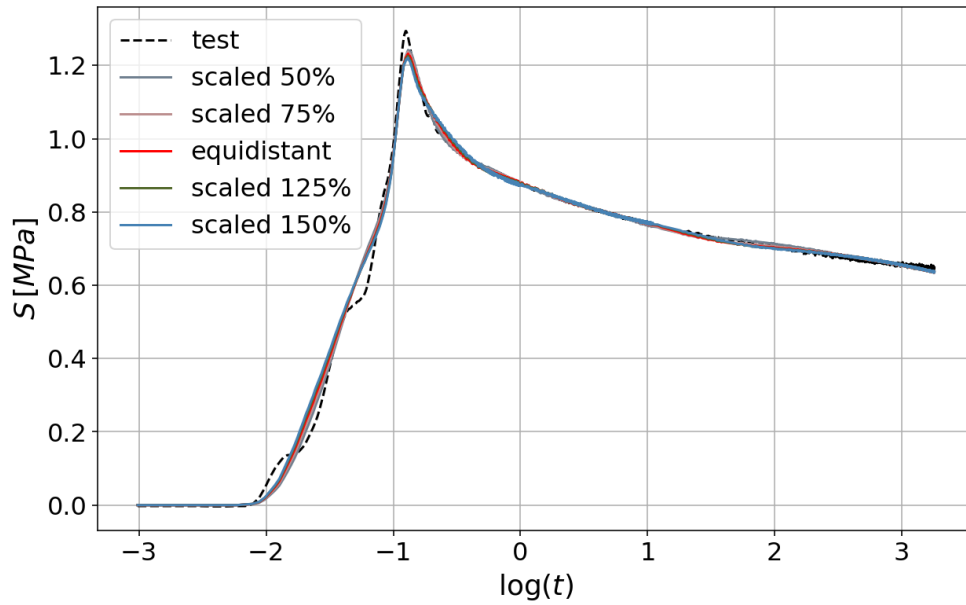


Figure 2.13: Comparison of the model stress response with different values of N to an experimental shear relaxation test.

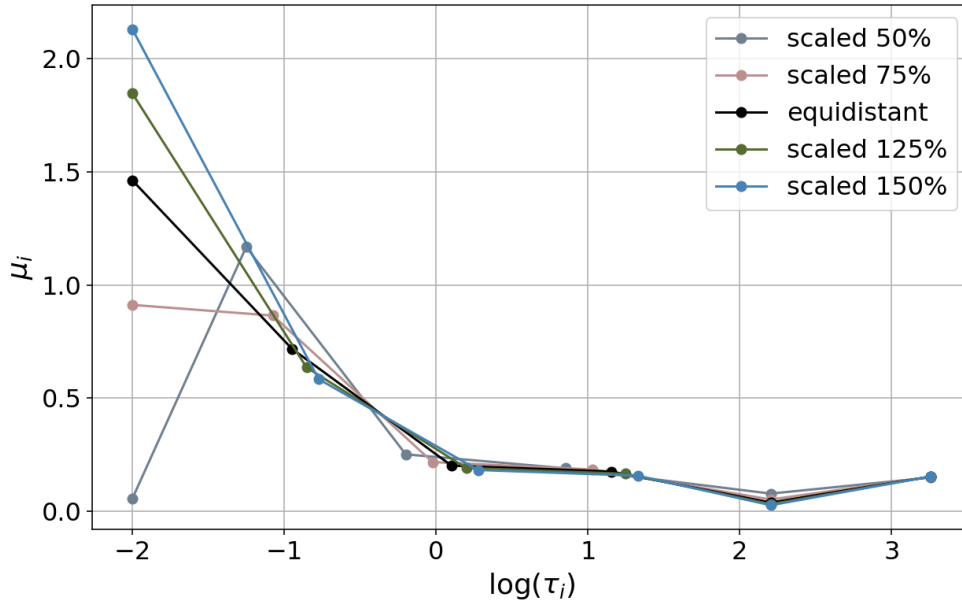


Figure 2.14: Comparison of the model spectrum $\tau_i - \mu_i$ for different values of N .

2.2.3 Number of viscous parameters

The selection of the number N of viscous parameters (μ_i, τ_i) has direct impact on the accuracy of the stress response curve prediction of an experimental test. Black carbon-filled rubbers exhibit non-linear viscous behavior, thus one viscous component is not sufficient. Increasing N improves the model ability to represent the real rubber behavior. However, this also significantly increases the computational cost associated with fitting the experimental data and performing subsequent FEM simulations, which can become prohibitive.

To determine the minimum value of N that adequately describes the rubber behavior, the model was fitted to the same shear relaxation test in Figure 2.13 with different values of N , where the elastic equilibrium curve was previously defined. In Figure 2.15, the model stress response during time is compared to the recorded experimental stress. It can be observed that for $N = 3$ the model does not follow the experimental curve relaxation tail, while for $N = 5 \div 6$ the fitting is acceptable. Further increase of N value appears to improve the approximation, but this improvement is only apparent, as the spectrum in Figure 2.16 reveals that many terms are zero and therefore do not contribute to the stress response.

On the basis of these observations, it can be said that selecting too few terms might limit the model ability to accurately approximate the experimental data. Conversely, as noted by Feng and Hallquist [17], using too many terms can lead to numerical instability and overfitting issues, where some terms may have zero coefficients during the fitting process, rendering them inactive and complicating the model unnecessarily.

2.2.4 Fitting of the elastic curve

The experimental elastic curve is determined by a few discrete points in the stress-strain plane. However, if these points are well spaced within the range from zero to maximum strain, they are sufficient to determine the elastic coefficients. Since the

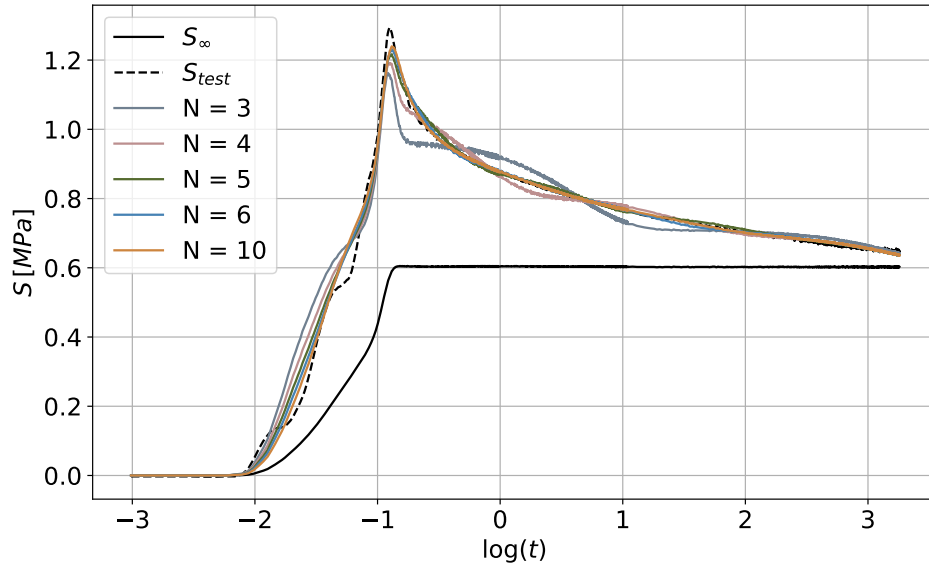


Figure 2.15: Comparison of the model stress response with different values of N to an experimental shear relaxation test.

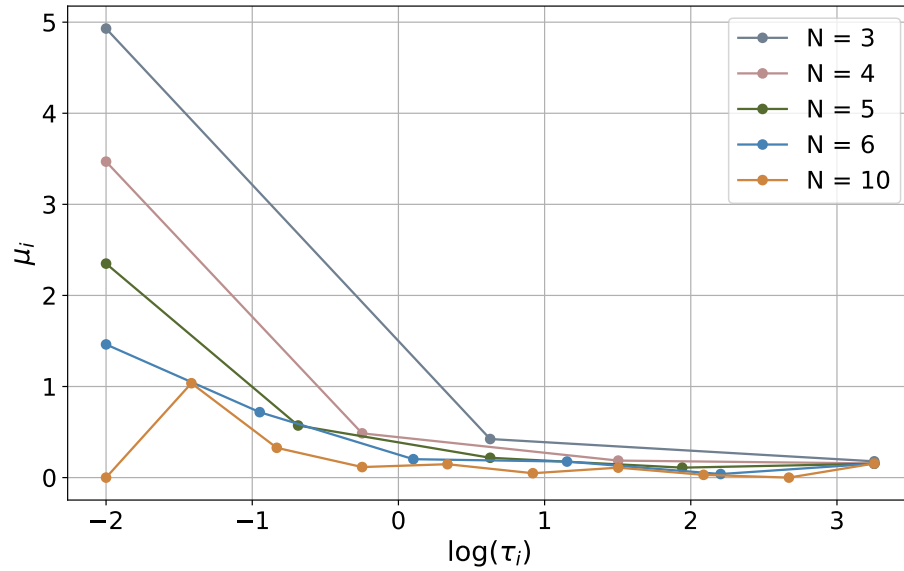


Figure 2.16: Comparison of the model spectrum $\tau_i - \mu_i$ for different values of N .

Yeoh expression for equilibrium stress depends on three coefficients C_{j0} , six points, excluding the origin, are enough to solve a least squares problem. The points near the maximum deformation have the highest uncertainty due to the Mullins effect, so they should be assigned a lower weight in the analysis.

A fit on the experimental curve can be performed independently setting a least square problem to determine a starting point for the general fitting, during which the parameters can vary within a limited range or be fixed, or the experimental data can directly contribute to the error function.

2.2.5 Fitting of a simple shear or a uniaxial tension test

Solutions for a given deformation history $\mathbf{C}(t)$ are obtained through the numerical integration of Equation 2.33, with the determination of over-stresses being the most delicate step. The duration of the experimental test is divided into A discrete steps, with each step corresponding to a specific deformation value. Given the initial conditions, or the calculated parameters at step $k-1$, the procedure to obtain the values at the following k -th step can be summarized as follows:

- the increment of time is $\Delta t^{(k)} = t^{(k)} - t^{(k-1)}$ and the deformation is $\mathbf{C}^{(k)} = \mathbf{C}(t^{(k)})$, as in Equations 2.2 and 2.8 for simple shear ($\gamma^{(k)} = \gamma(t^{(k)})$) and uniaxial extension ($\lambda^{(k)} = \lambda(t^{(k)})$), respectively;
- The equilibrium stress $\mathbf{S}_{\infty}^{(k)}(\mathbf{C}) = \mathbf{S}_{\infty}(\mathbf{C}^{(k)})$ is independent on the previous step so it is directly calculated as defined in Equation 2.27 and, specifically, with expressions in Equation 2.48 with $\gamma = \gamma^{(k)}$ for simple shear and Equation 2.52 with $\lambda = \lambda^{(k)}$ for uniaxial extension;
- each over-stress $\mathbf{Q}_i^{(k)}$ is calculated starting from the previous step $\mathbf{Q}_i^{(k-1)}$ as

$$\mathbf{Q}_i^{(k)} = \mathbf{Q}_i(t^{(k)}) = \mu_i \left[\int_{t^{(0)}}^{t^{(k-1)}} e^{-\frac{t-s}{\tau_i}} \dot{\mathbf{S}}_{\infty}(s) ds + \int_{t^{(k-1)}}^{t^{(k)}} e^{-\frac{t-s}{\tau_i}} \dot{\mathbf{S}}_{\infty}(s) ds \right] \quad (2.78)$$

$$= \mathbf{Q}_i^{(k-1)} + \mu_i e^{-\frac{\Delta t^{(k)}}{2\tau_i}} \left(\mathbf{S}_{\infty}^{(k)} - \mathbf{S}_{\infty}^{(k-1)} \right) \quad (2.79)$$

where the second term is numerically integrated with the midpoint rule.

- the total stress $\mathbf{S}^{(k)}$ is the sum of the equilibrium stress and the N over-stresses evaluated at the step n , as in Equation 2.28.

Repeating this process for $k = 1 \dots A$, starting from the initial conditions at time $t^{(0)}$, we can determine the stress response of the model. The value of the time increment Δt can vary at each step, due to a change in the sampling frequency or a downsampling of the recorded data, and it should be small so that the midpoint rule used for integration gives a good approximation of the true value. Otherwise, a division in sub-steps has to be performed while the procedure remains the same as described previously.

Fast fitting

Fixing the time constants τ_i allows for a faster fitting process for any generic deformation history. In Equation 2.31 the result of the integral, $\mathbf{R}_i(t) = \mathbf{Q}_i(t)/\mu_i$, is constant for a given deformation history, if also the time constants τ_i and the elastic equilibrium curve constants C_{j0} are set. Thus, we can write Equation 2.32 as the sum

$$\mathbf{S}(t) = \mathbf{S}_\infty(t) + \sum_{i=1}^N \mu_i \mathbf{R}_i(t) \quad (2.80)$$

where the only variables are μ_i , to be determined solving a linear least square problem. This method allows to avoid integration of the deformation history at each iteration of the fitting.

A fitting that requires 3 integrations of the time history for each viscous term only in the first step is possible if the elastic constants C_{j0} are not fixed, as well as the viscous μ_i , but only the time constants τ_i are so. In this case, the problem is not a linear system because of the product of viscous and elastic variable terms. The stress response estimated by the model would be expressed as

$$\mathbf{S}(t) = \mathbf{S}_\infty(t) + \sum_{i=1}^N \sum_{j=1}^3 \mu_i C_{j0} \mathbf{R}_{j,i} \quad (2.81)$$

$$\mathbf{R}_{j,i} = \int_{-\infty}^t e^{-\frac{t-s}{\tau_i}} 2 \frac{d}{ds} \frac{d}{d\mathbf{C}} \frac{d}{dC_{j0}} W_\infty(\mathbf{C}) ds \quad (2.82)$$

where the elastic terms do not appear in the integration because W_∞ is assumed to depend linearly on them.

For example, for the fitting of a simple shear test, the expression would become

$$S(t) = S_\infty(t) + 2 \sum_{i=1}^N \mu_i [C_{10} R_{1,i}(t) + C_{20} R_{2,i}(t) + C_{30} R_{3,i}(t)] \quad (2.83)$$

where

$$R_{j,i}(t) = j \int_{-\infty}^t e^{-\frac{t-s}{\tau_i}} \dot{\gamma}^{j-1}(s) ds \quad (2.84)$$

2.2.6 Fitting of a relaxation test

A particular case for the problem of fitting is the relaxation test, which consists of a loading ramp followed by a constant deformation state (see Section 3.1.2). When the deformation is constant, in the “tail” of the test, the variation of equilibrium stress $\dot{\mathbf{S}}_\infty$ is zero. Hence, the solution of the differential Equation 2.29 has the form of a decaying exponential, as shown in Equation 2.85

$$\mathbf{Q}_i(t) = \mathbf{Q}_{i0} e^{-\frac{t-t_0}{\tau_i}} \quad (2.85)$$

where $\mathbf{Q}_{i0}$ is the amplitude of the i -th over-stress at the beginning of the relaxation, at time $t = t_0$. Assuming to determine the equilibrium stress curve with a different test, thus knowing the value of $\mathbf{S}_\infty$ at each instant of time, we can translate the origin to the point $(t_0, \mathbf{S}_\infty(t_0))$ in the stress-strain plane. Now the curve to be

fitted is just a sum of decaying exponentials expressed as in Equation 2.85. With fixed time constants τ_i , the initial over-stresses can be estimated with a linear least square problem like $\mathbf{Y} = \mathbf{A}\mathbf{X}$ to be solved as $\mathbf{X} = (\mathbf{A}^T \mathbf{A})^{-1} \mathbf{A}^T \mathbf{Y}$.

To simplify the notation, we report the case of a simple shear test, where the system of equation components are

$$[\mathbf{Y}]_k = S(t_k) - S_\infty(t_0) = \sum_{i=1}^N Q_i(t_k) \quad (2.86)$$

$$[\mathbf{A}]_{ki} = e^{-\frac{t_k - t_0}{\tau_i}} \quad (2.87)$$

$$[\mathbf{X}]_i = Q_{i0} \quad (2.88)$$

for $i = 1 \dots N$ and $k = 1 \dots A$. Here A is the number of the points sampled after t_0 and not the total length of the test. The solution of the problem is computationally fast, due to the fact that $\mathbf{A}^T \mathbf{A}$ to be inverted is a square matrix of size $N \times N$.

The values of these over-stresses provide an estimation of the coefficients $\mu_i \simeq Q_{i0}/S_\infty(t_0)$ which improves after performing a correction based on the stress recorded during the loading ramp segment. Starting from Equation 2.31, we numerically integrate from the beginning of the test to $t = t_0$, to have

$$Q_{i0} = \mu_i \psi_i = \mu_i \int_0^{t_0} e^{-\frac{t_0 - s}{\tau_i}} \dot{S}_\infty(s) ds \quad \text{for } i = 1 \dots N \quad (2.89)$$

and determine each $\mu_i = Q_{i0}/\psi_i$, as these are the only unknowns in the equation.

In Figure 2.17 an experimental test stress response is compared with the fitting of a visco-elastic model with $N = 6$ viscous parameters using the following procedures: the general procedure of Section 2.2.5 (“General” labeled curve); the estimation of the μ_i , with (“Corrected”) and without (“Raw”) the correction ψ_i ; the curve fitted only in the tail of the relaxation test (“Tail”), shown for $t > t_0$. This is a particular case of the procedure described in the previous section for a generic deformation history with the advantage that the integration of the time history is performed only on the loading ramp.

2.2.7 Fitting of multiple tests

If multiple tests are performed, as defined later in Section 3.1, on the same type of rubber and at the same maximum deformation, all of them can contribute to determine the model optimum parameters. The general fitting procedure involves the following main steps:

- Determine the shear modulus G and damping ξ from cyclic shear tests. These values can serve as additional constraints on the model parameters, using Equations 2.56 and 2.58;
- Derive equilibrium stresses at different deformation levels from a step relaxation test (see Section 3.2). The elastic parameters C_{j0} of the model can be determined independently from the other parameters and can either be fixed or used as initial values in the general fitting process.
- A relaxation test can be used to estimate the viscous parameters through the “tail” fitting method described previously;

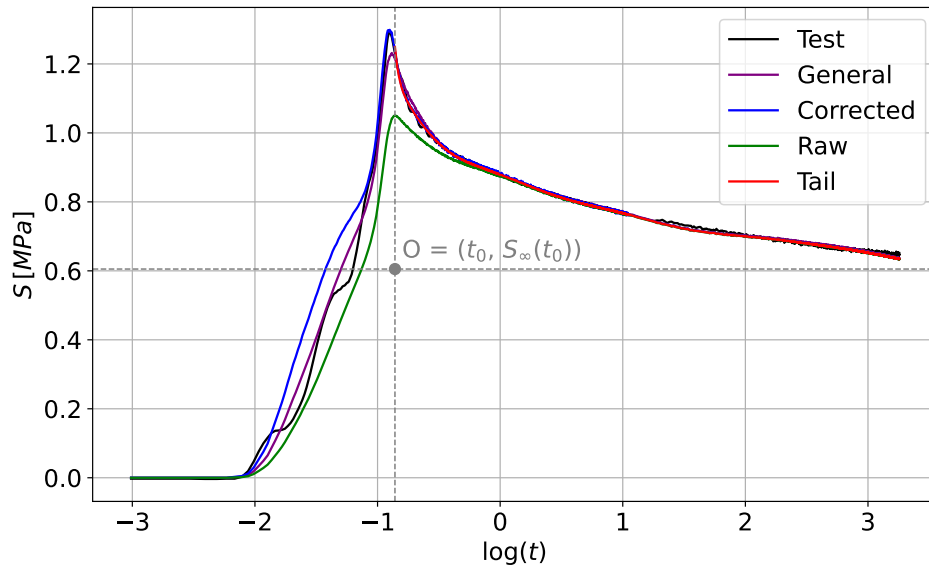


Figure 2.17: Example of an experimental shear relaxation test fitted with different methods.

- Constraints on the model parameters, i.e. C_{j0} and α_i , are applied as penalty values added to the error function of Equation 2.77 with higher weight relative to those of the other tests.
- If multiple tests of the same kind are performed on different specimens, an “average” test can be computed as described in the next Section 2.2.8, so that only one test of each kind is used in the next step.
- All tests and constraints contribute to the error function defined in Equation 2.77 and a standard fitting algorithm can be used to find the optimum values that minimize it.

2.2.8 Fitting of multiple tests of the same kind

The same test can be performed on multiple specimens of the same rubber at the same maximum deformation. While all these tests can be included in the general fitting procedure, each one must be simulated and adds several terms to the error function, thereby increasing the computational time required for fitting.

Even with identical test settings, such as cycle frequency, loading duration or sampling frequency, directly averaging the stress curves is impossible due to noise in the recorded data.

To represent multiple tests of the same kind, a preliminary fitting can be performed using only these tests to determine the model parameters, particularly the viscous ones, that best describe them. With these parameters, the stress response to the same deformation history can be numerically evaluated, producing a single “averaged” test that effectively represents the multiple tests. This can be done using a recorded deformation history or an artificial one, which has the advantage of being noise-free. This single “averaged” test is then used as input to the general fitting algorithm.

2.2.9 Determination of μ_i from multiple cyclic tests

The expressions obtained for the dynamic shear modulus, Equation 2.56, and damping, Equation 2.58, for a theoretical cyclic test can be useful to determine the rubber $3+2N$ visco-elastic parameters, valid for a maximum shear strain γ_0 . Performing M cyclic tests at different angular frequencies ω_k for $k = 1 \dots M$ we generate a system of $2M$ non-linear equations of variables C_{j0} for $j = 1, 2, 3$ and (τ_i, μ_i) for $i = 1 \dots N$. Depending on how many viscous parameters we use to describe the rubber behavior, we have to perform at least $M = 2+N$ cyclic tests at frequencies ω_k and solve the system as a non-linear least squares problem.

If the elastic parameters C_{j0} are determined independently with a different test, as described later in Section 3.2, and the time constants τ_i are priorly fixed (see [17]), the system becomes linear in the N variables μ_i and the minimum number of tests needed reduces to $M \geq N/2$. The system of equations can be summarized as follows

$$\sum_{i=1}^N \mu_i f_G(C_{j0}, \phi_{i,k}) = G(\omega_k) - c_G(C_{j0}) \quad (2.90)$$

$$\sum_{i=1}^N \mu_i f_\xi(\phi_{i,k}) = \frac{G(\omega_k) \xi(\omega_k)}{c_\xi(C_{j0})} \quad (2.91)$$

where

$$c_G(C_{j0}) = 2(C_{10} + 2C_{20}\gamma_0^2 + 3C_{30}\gamma_0^4) \quad (2.92)$$

$$f_G(C_{j0}, \phi_{i,k}) = \frac{2C_{10} + 3C_{20}\gamma_0^2 + \frac{15}{4}C_{30}\gamma_0^4}{1 + \phi_{i,k}^2} + \frac{\gamma_0^2(C_{20} + \frac{15}{8}C_{30}\gamma_0^2)}{1 + \left(\frac{\phi_{i,k}}{3}\right)^2} + \frac{\frac{3}{8}C_{30}\gamma_0^4}{1 + \left(\frac{\phi_{i,k}}{5}\right)^2} \quad (2.93)$$

$$c_\xi(C_{j0}) = C_{10} + \frac{3}{2}C_{20}\gamma_0^2 + \frac{15}{8}C_{30}\gamma_0^4 \quad (2.94)$$

$$f_\xi(\phi_{i,k}) = \frac{\phi_{i,k}}{1 + \phi_{i,k}^2} \quad (2.95)$$

The system of Equations 2.90 and 2.91 can also be summarized in matrix form as

$$\begin{aligned} \mathbf{f}\boldsymbol{\mu} &= \mathbf{c} \\ \text{or} \\ \begin{bmatrix} \mathbf{f}_G \\ \mathbf{f}_\xi \end{bmatrix} \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_i \\ \vdots \\ \mu_N \end{bmatrix} &= \begin{bmatrix} \mathbf{c}_G \\ \mathbf{c}_\xi \end{bmatrix} \end{aligned} \quad (2.96)$$

and solved as

$$\boldsymbol{\mu} = (\mathbf{f}^T \mathbf{f})^{-1} \mathbf{f}^T \mathbf{c} \quad (2.97)$$

where

$$[\mathbf{f}_G]_{ki} = f_G(C_{j0}, \phi_{i,k}) \quad (2.98)$$

$$[\mathbf{f}_\xi]_{ki} = f_\xi(\phi_{i,k}) \quad (2.99)$$

$$[\mathbf{c}_G]_k = G(\omega_k) - c_G(C_{j0}) \quad (2.100)$$

$$[\mathbf{c}_\xi]_k = \frac{G(\omega_k) \xi(\omega_k)}{c_\xi(C_{j0})} \quad (2.101)$$

The frequencies ω_k of the tests emphasize different variables for each equation of the system through the coefficients of the matrix $\mathbf{f}$. In Equation 2.90 the coefficients f_G assume the maximum values for $\phi_{i,k} \rightarrow 0$, thus the terms with the highest time constants τ_i will prevail. In Equation 2.91 each coefficient f_ξ has a maximum when $\phi_{i,k} \rightarrow 1$, thus when ω_k is approximately the inverse of the time constants τ_i . This is a confirmation of what Figures 2.4 to 2.9 show.

To evaluate this method, hereafter referred to as the Multi-Frequency Cyclic Test (MFCT), a hypothetical rubber material was created by defining realistic parameters for the viscoelastic model. Cyclic tests were then simulated at various frequencies to replicate experimental conditions and the method was subsequently applied to determine the material parameters.

A simple case was initially considered, where $C_{j0} = [0.2, -0.12, 0.1]$ MPa and a small set of time constants τ was chosen to be equidistant on a logarithmic scale, ranging from $\tau_1 = 10^{-3}$ s to $\tau_N = 10^3$ s. The values of μ decrease as τ increases (Figure 2.18a). For this test, $N = M = 5$, the values of ω_k were chosen as the inverse of τ_i and the tests were simulated at $\gamma_0 = 1$. Shear modulus and damping were then computed for each test (Figures 2.18b and 2.18c). The values of τ_i chosen to solve the system were the same of the used hypothetical rubber, so that a direct comparison of the results can be made. The computed values of μ are very close to the chosen values, with an error ranging from less than 1% for lower τ values to 25% for higher τ values. Nonetheless, the errors in the calculated values of the modulus and damping remain below 5% and 20%, respectively, compared to those obtained from cyclic tests.

These results are reported graphically in Figure 2.18, where the trends of the the viscous constants μ over the time constants τ (Figure 2.18a), the dynamic moduli G_{dyn} (Figure 2.18b) and of the damping coefficients ξ (Figure 2.18c) calculated for simulated cyclic tests at different frequencies ω are shown, compared to the ones computed from cyclic tests simulated on the hypothetical rubber.

In Figure 2.19 the third cycle of the simulation of the cyclic test at maximum frequency with the parameters of the hypothetical and fitted rubber is shown. Even if the errors on the viscous parameters, shear modulus and damping reach relatively high values, the curves stay close to each other. The other tests, which are not shown, have the same trend.

In order to provide a visual representation of the distribution of coefficients within the matrix, a colormap of the values of the elements in the matrix $\mathbf{f}$ composed for the solution is shown in the plot of Figure 2.20a. The upper half is the matrix $\mathbf{f}_G$, which is right triangular, and the lower half is $\mathbf{f}_\xi$, predominantly diagonal. The corresponding vector of constants is shown in Figure 2.20b. The same representation is given for the solving system matrix $\mathbf{f}^T \mathbf{f}$ in Figure 2.20c and for the vector of constants $\mathbf{f}^T \mathbf{c}$ in Figure 2.20d.

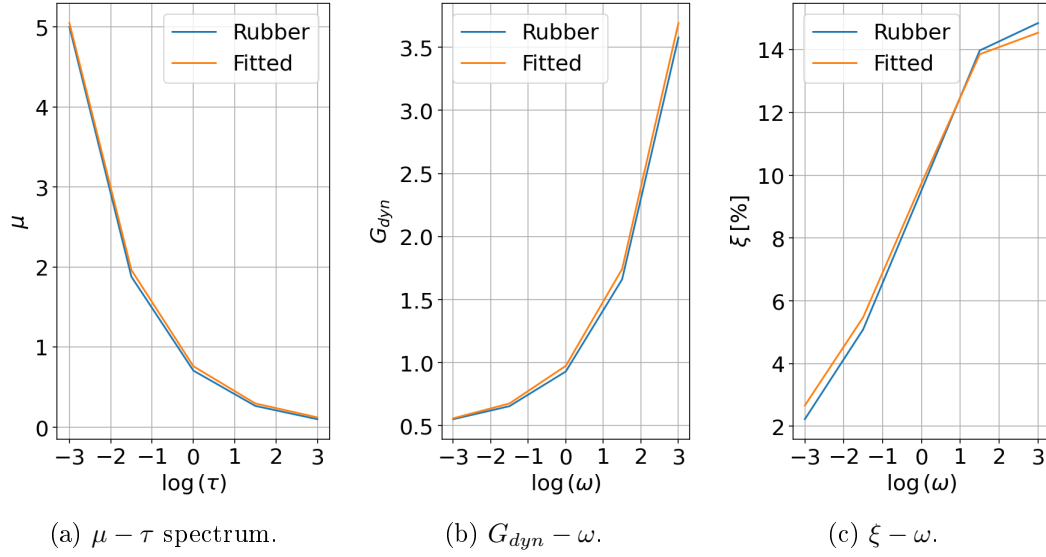


Figure 2.18: Comparison of the properties of the hypothetical rubber and the properties computed with the MFCT method.

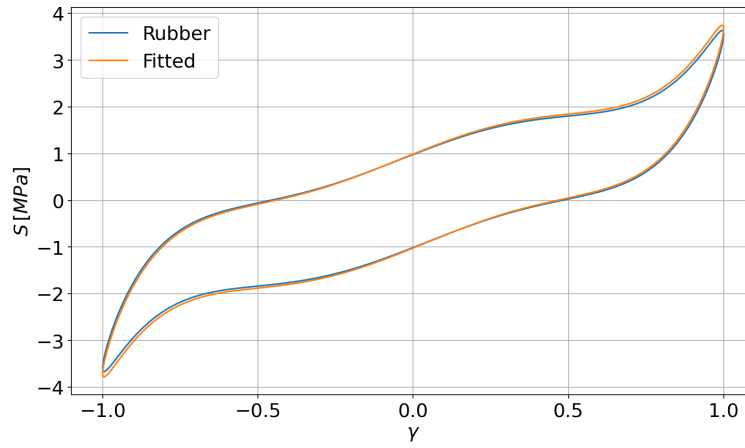


Figure 2.19: Third stress - strain cycle of the simulated test at maximum frequency of the hypothetical and fitted rubber.

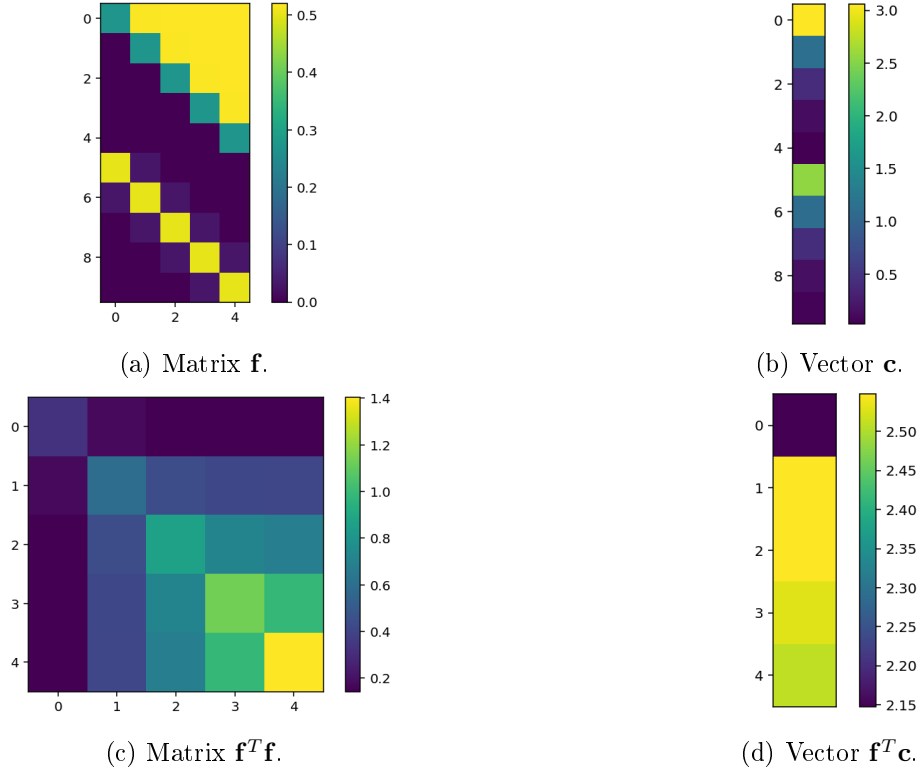


Figure 2.20: Colormap of examples of matrices $\mathbf{f}$ and $\mathbf{f}^T \mathbf{f}$ and of vectors $\mathbf{c}$ and $\mathbf{f}^T \mathbf{c}$.

In this example the frequencies are included between $1.6 \cdot 10^{-4}$ Hz and $1.6 \cdot 10^2$ Hz. For the real application of the MFCT method, there is a physical upper limit to the value of ω_k due to the machine maximum velocity and inertia, which prevents the deformation to follow a sinusoidal shape and to reach the value of maximum deformation imposed.

A possible solution is to conduct the high-frequency cyclic tests at a strain $\gamma_0 < \gamma_{max}$ so that the machine has to travel a shorter stroke, which is permissible if we assume that the material parameters are constant for a given maximum deformation.

Low-frequency tests can be performed at any level of deformation, but the lower limit of ω_k is constrained by the test duration. For the MFCT method to be a viable alternative to others, such as the relaxation test, the total test duration should be comparable to theirs. This imposes a limit on the period of the slowest test, which is on the order of 10^3 s, and consequently on the value of the highest time constant τ_N that can be determined using this method.

Cyclic tests on the same rubber of the first example were then simulated in a frequency range $f = [10, \dots, 10^{-3}]$ Hz and at the same maximum strain, except for the first two high-frequency tests which had a maximum strain of $\gamma_0 = 0.2$ and 0.5 . The system was solved, imposing the same values of τ as before, and the maximum errors of the obtained parameters were on the same range as the previous case.

Lastly, $M = 20$ simulations of cyclic tests were performed on a rubber with a wide range and high number of time constants τ , included between 10^{-3} and 10^5 with $N_{rubber} = 20$. The maximum strain was again $\gamma_0 = 1$ and the frequencies of the cycles were between 10 Hz and 10^{-3} Hz. The aim of this test is to show that instability of the solution can arise when the number of viscous terms of the model $N \leq M$ is too high, independently on the number of tests performed, or when the choice of the time

constants τ is unfortunate. In this context, the instability manifests as one or more viscous terms μ becoming non-positive or alternating in sign. The first case bears a resemblance to the case described in Section 2.2.3, the latter specifically involves the coefficients of the matrix $\mathbf{f}$ and its conditioning number.

The conditioning number of a matrix $\mathbf{A}$ is a measure of how sensitive the solution of a linear system is to small changes in the input data or the matrix itself. It is commonly defined as the product of the norm of the invertible matrix A and the norm of its inverse, $\text{cond}(\mathbf{A}) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\| > 1$, where $\|\mathbf{A}\|_2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{A}\mathbf{x}\|_2$ for any vector $\mathbf{x}$ of norm 1. A high condition number indicates a nearly singular matrix, leading to potential large errors due to small input or computational errors. A low condition number implies a well-conditioned matrix with more stable and accurate solutions.

The condition number of $\mathbf{f}^T \mathbf{f}$ is significantly influenced by the relationship between the time constants τ_i and the frequencies ω_k . When the range of values of τ_i and ω_k is too narrow, or that of τ_i is too wide, the matrix becomes ill-conditioned. This results in a higher sensitivity of the solution to small perturbations in the data, leading to numerical instability. In particular, when the highest time constant τ_N dominates the system, it exacerbates the conditioning problem, making it difficult to obtain a reliable solution. Therefore, careful attention must be given to the choice of τ_i values to ensure that the matrix remains well-conditioned and the solution is stable. The choice of the frequencies of the tests and of the values of the time constants determine the coefficients of the system, especially the ones of the matrix $\mathbf{f}^T \mathbf{f}$ that solves the linear system in Equation 2.97.

Through the comparison between the spectrum of the hypothetical elastomer model and that of the obtained solution, six cases of solution instabilities, which must be avoided, are presented as observed during the tests of this method. These instabilities are primarily attributed to the choice of the number N and the extreme values of the time constants τ_i . In Figure 2.21 the spectrum of the hypothetical rubber is compared to those of some models fitted with the MFCT method:

Figure 2.21a: $N = 20$ and τ_i are the inverse of ω_k to show that, even if we choose $\phi_{i,k} \rightarrow 1$, instability can still occur.

Figure 2.21b: $N = 20$ and τ_i are equal to the rubber ones to show that, even if we choose each $\tau_i = \tau_{i,rubber}$, instability can still occur.

Figure 2.21c: $N = 6$ and a limited range $\tau_i = [10^{-2}, \dots, 10]$ s, to represent the case in which the extreme values of the time constants are too close with respect to those of the tested rubber. Note that the conditioning number is relatively low but one value of μ is negative.

Figure 2.21d: $N = 6$ and a wide range $\tau_i = [10^{-4}, \dots, 10^6]$ s, to represent the case in which the extreme values of the time constants are too far with respect to those of the tested rubber.

Figure 2.21e: $N = 6$ and tests frequencies limited to $f_k < 1$ Hz, to represent the case in which the cyclic tests are performed at low frequencies short-term viscous parameters are sought.

Figure 2.21f: $N = 6$ and tests periods limited to $T_k < 100$ s, to represent the case in which the cyclic tests are performed with short periods and long-term viscous parameters are sought.

G_{dyn} and ξ can also be influenced by the instability, resulting in a significant increase in associated errors.

To ensure that the above conditions are met, the problem resolution was redefined to include the minimization of an error function χ^2 . This function incorporates the previous approach as part of its formulation, calculating a solution and its associated error, which contributes to the total error alongside additional terms. The function was designed to address all highlighted cases of instability by constraining the solution through penalties in the scenarios of a large norm of the residual vector, a high condition number of the system matrix, cases where one or more μ values are negative, a wide range between the maximum and minimum values of τ_i .

In particular, the error function χ^2 in Equation 2.102 takes into account not only the norm $\|\mathbf{r}\|_2$ of the residual vector, $\mathbf{r} = \mathbf{f}\boldsymbol{\mu} - \mathbf{c}$, but also the value of the conditioning number of $\mathbf{f}^T\mathbf{f}$ and a penalty function if one or more μ_i are negative. The difference between the highest and lowest time constants was added to the function to encourage the solution to span a wide range of τ . The error is defined as a product of these contributions, which vary significantly with the input parameters

$$\chi^2(\mathbf{r}, \boldsymbol{\mu}, \text{cond}(\mathbf{f}^T\mathbf{f}), \tau) = (1 + \|\mathbf{r}\|_2) \log(\text{cond}(\mathbf{f}^T\mathbf{f})) (1 + \text{penalty}) \frac{1}{\tau_N - \tau_1} \quad (2.102)$$

where $\text{penalty} \gg 1$ if any $\mu_i < 0$, else it is zero. A classic minimization algorithm does not work well in this case due to the penalty function and the instabilities of the system. However, thanks to the fast solvable system, an exhaustive search approach can be employed by varying the extreme values of the time constants, solving the system, and calculating the error χ .

In Figure 2.22 the optimum spectra determined with the MFCT method through the minimization of χ , for $N = 5, \dots, 10$, are compared to the rubber spectrum. Higher values of N lead to instability of the solution for any choice of τ_1 and τ_N . All their trends are decreasing with increasing τ , except for $\tau_N \ll \tau_{N,rubber}$, as this term has to compensate the effect of longer term effects of the rubber. Its value seems to be limited by the fact that bad conditioning occurs for higher values.

In Figure 2.23 the heatmaps of the values of χ , as defined in Equation 2.102, of the residuals $\mathbf{r}$, of the conditioning number of $\mathbf{f}^T\mathbf{f}$ and of the assigned penalty value are reported for variations of the extreme values of the time constants τ_1 and τ_N and for $N = 6$ fixed. The same is shown in Figure 2.24 for $N = 10$.

Note that the areas not subject to the penalty function vary, becoming narrower as N increases. The minimum points marked are all located in the central region of the graphs. Ideally, we would like to shift towards the upper-left corner to achieve the maximum range in τ , but this is primarily limited by the conditioning value, in addition to the penalty value. In comparison, the residual value remains relatively constant and would otherwise push the solution in that direction.

A last comparison is made between the simulated relaxation test of the rubber with the given viscous parameters and the ones determined by solving the minimum error problem. The curves in Figure 2.25 are almost overlapped, except for the tail, where the fitted rubber curve tends to the equilibrium stress as, eventually, the long term over-stresses fade.

In conclusion, the MFCT method enables the immediate determination of the correct viscous parameters, offering a substantial reduction in computational time and cost compared to the integration of the entire test step-by-step as previously described. However, it is crucial to meet specific conditions, such as appropriately

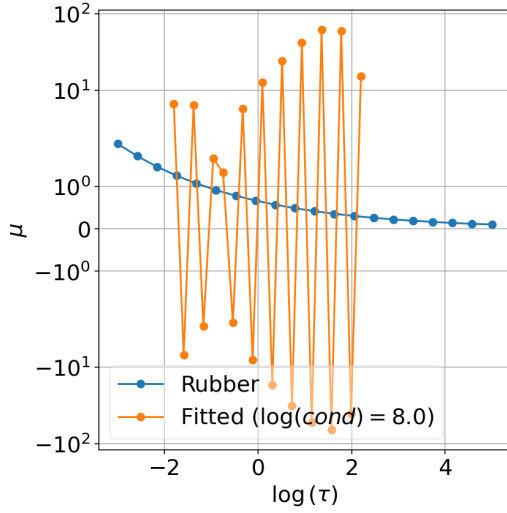
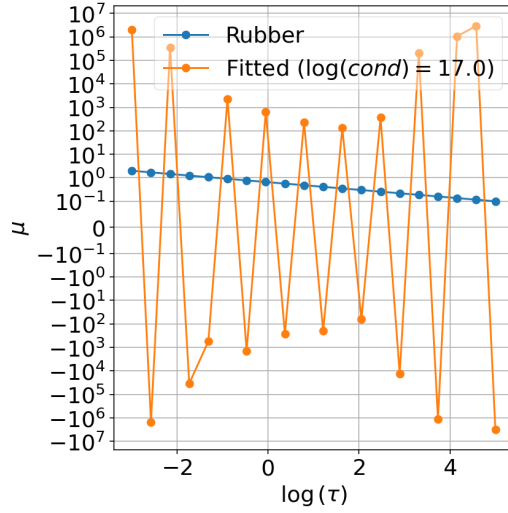
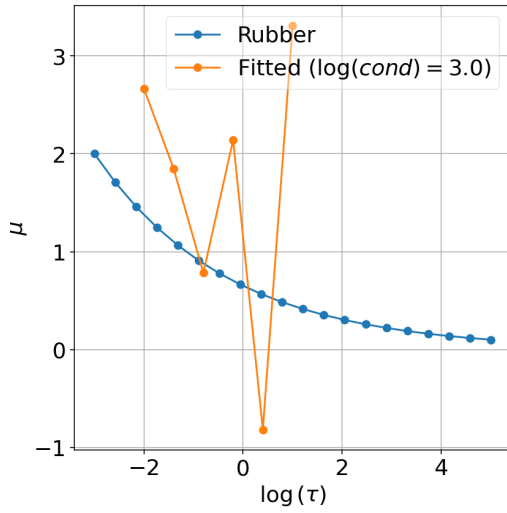
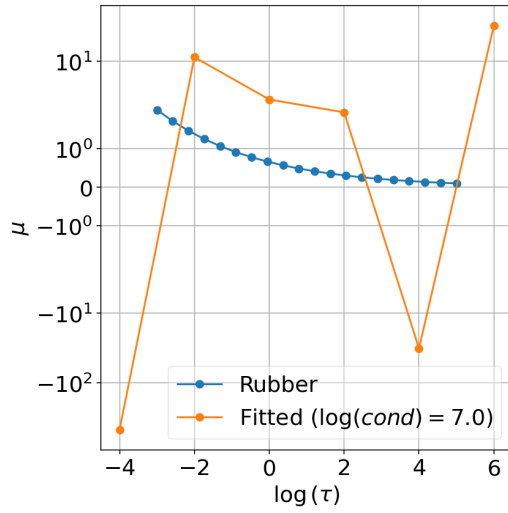
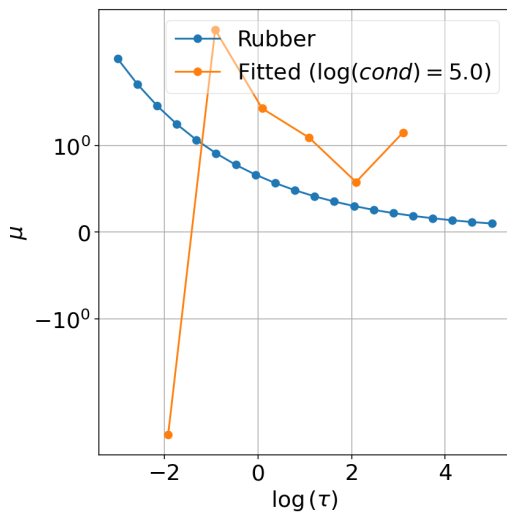
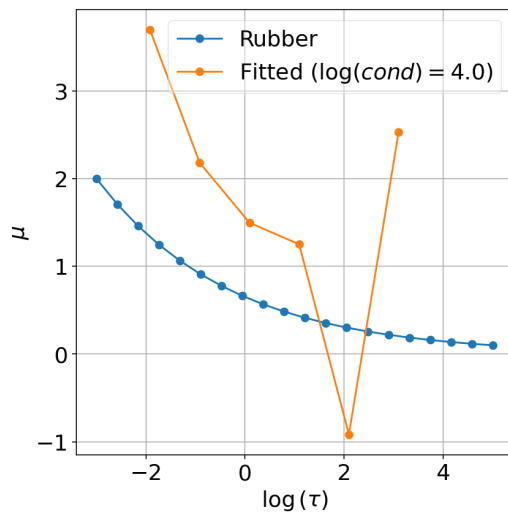
(a) $N = 20, \tau_i = 1/\omega_k$.(b) $N = 20, \tau_i = \tau_{i,rubber}$.(c) $N = 6, 10^{-2} \text{ s} \leq \tau_i \leq 10 \text{ s}$.(d) $N = 6, 10^{-4} \text{ s} \leq \tau_i \leq 10^6 \text{ s}$.(e) $N = 6, f_k < 1 \text{ Hz}$.(f) $N = 6, T_k < 100 \text{ s}$.

Figure 2.21: Comparison of the spectra of the hypothetical rubber and multiple models which induce instabilities on the solution.

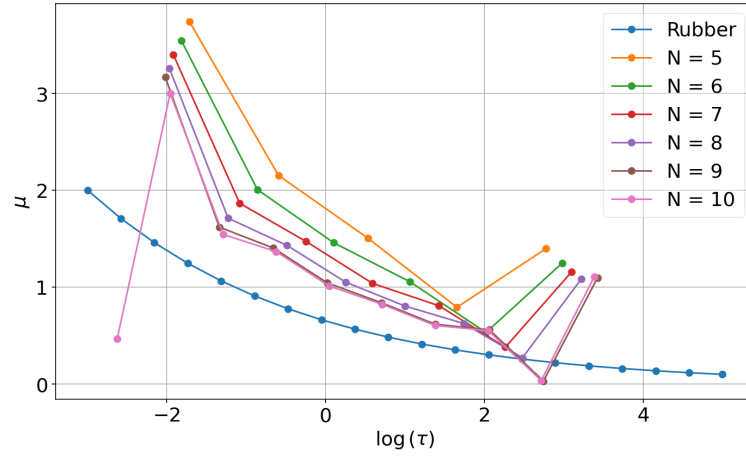


Figure 2.22: Comparison of the spectra of the hypothetical rubber and those of the models determined minimizing the error defined in Equation 2.102 for different values of N .

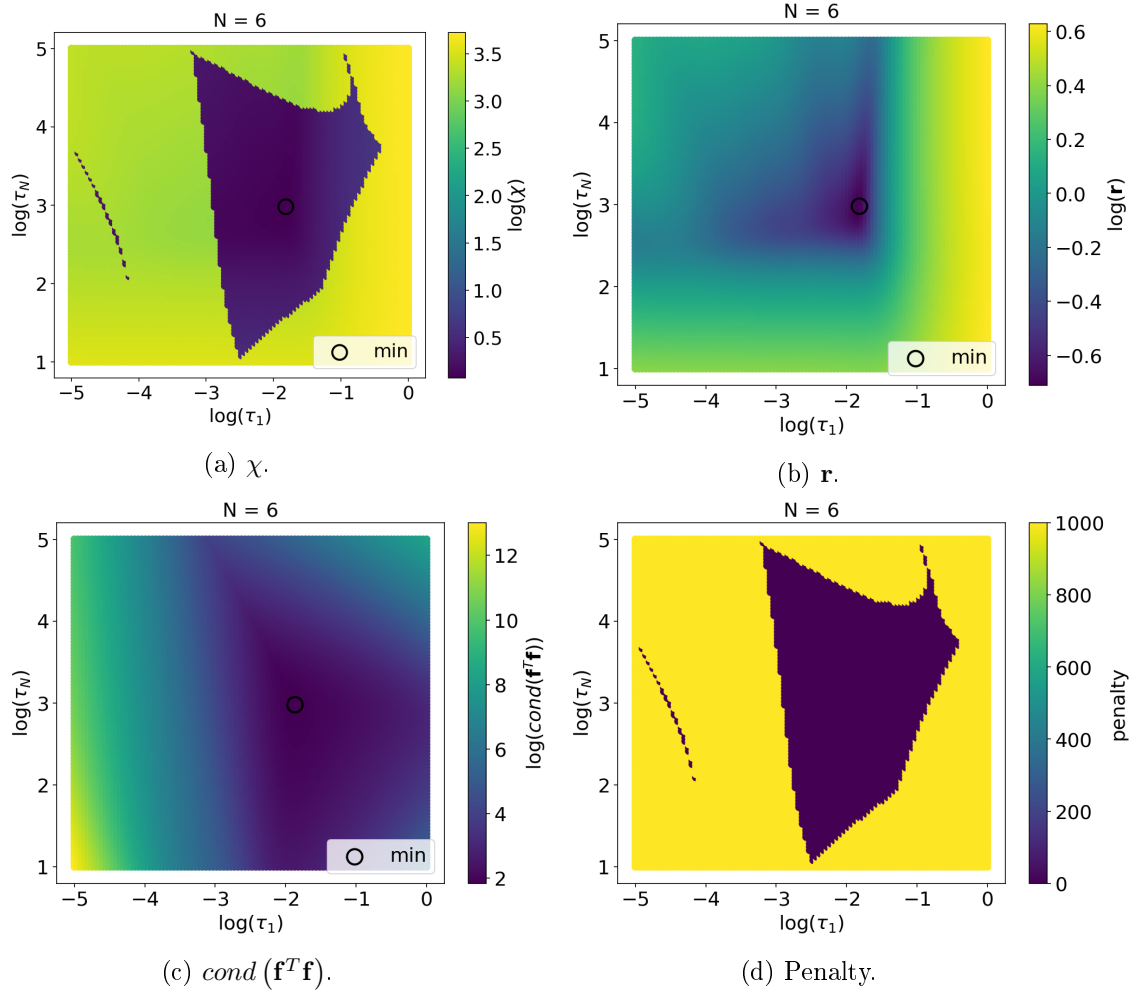


Figure 2.23: Colormap of the values of the error function, the residual, the conditioning number and the penalty value assigned for different extreme values of τ_1 and τ_N , $N = 6$.

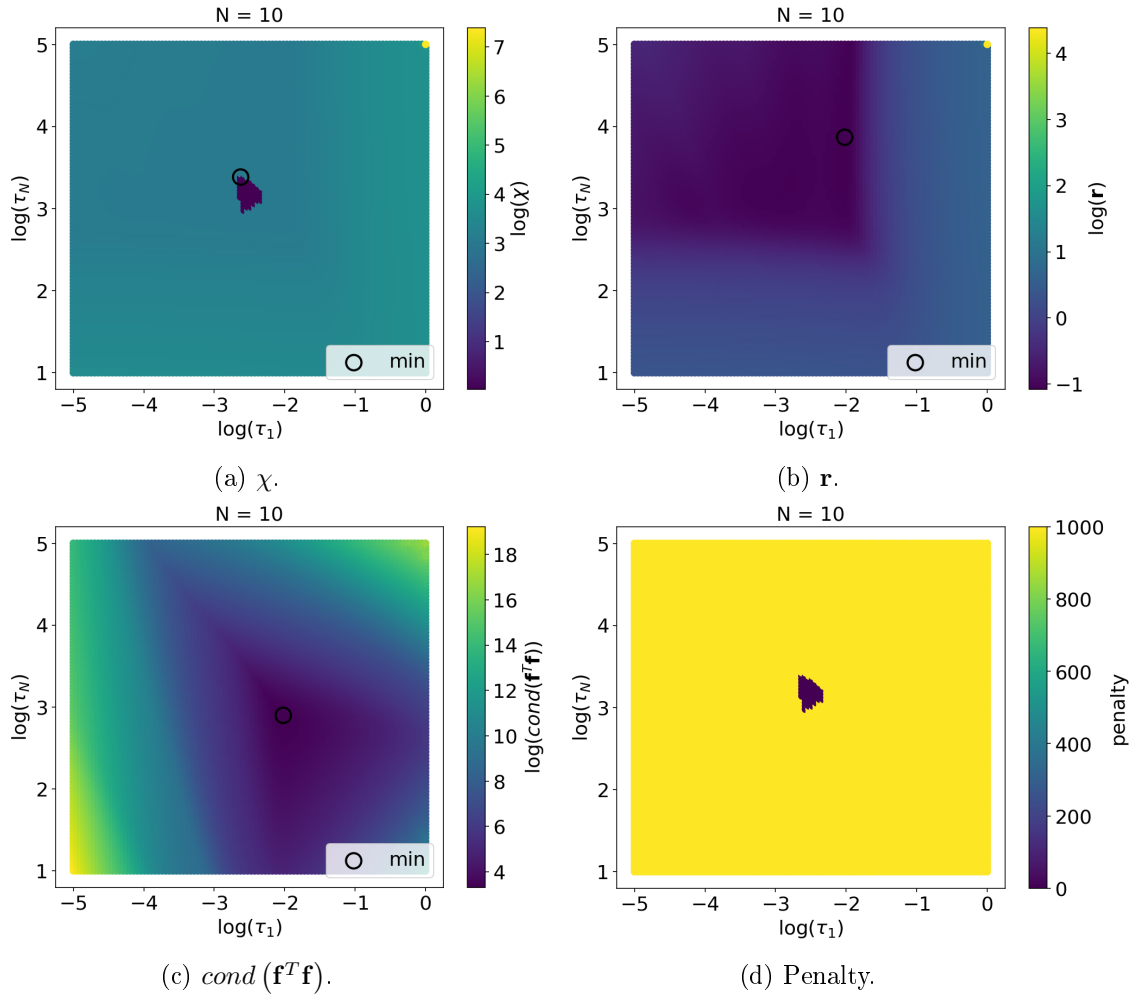


Figure 2.24: Colormap of the values of the error function, the residual, the conditioning number and the penalty value assigned for different extreme values of τ_1 and τ_N , $N = 10$.

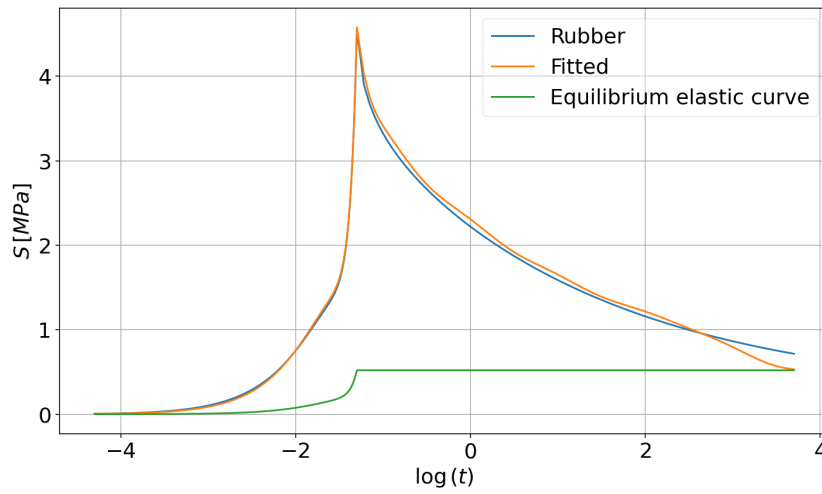


Figure 2.25: Simulated relaxation test of the hypothetical rubber and the fitted rubber.

selecting τ and ω and ensuring that the matrix used to solve the system remains well-conditioned. The application of this method to experimental cyclic tests is described in Section 3.4.1.

Chapter 3

Laboratory tests

In this chapter, the laboratory experimental tests conducted on the rubber typology of Table 1.1 used to produce the rubber isolators listed in Table 1.2 are presented. These tests vary in type, duration, and purpose, providing valuable insights into different parameters of the model. Each experiment, described in Section 3.1, was designed to obtain specific information, which contributes to the comprehensive characterization of the material properties and the overall performance of the devices. The tests have been refined over time, and the specimens have also been modified, eventually leading to the development of a testing procedure (Section 3.1.6) and of a specimen specifically designed (Section 3.3) to meet the needs of the experiments. All tests were performed on this type of specimen and all rubber visco-elastic model parameters for the FEM simulations of Chapter 4 are obtained from the fitting to the tests described herein. Particular attention is given to the determination of the elastic equilibrium stress of the rubber with a method that reduces drastically the testing time duration (Section 3.2). Section 3.4 focuses on cyclic tests conducted to determine the mechanical properties of rubber isolators, either directly or through the MFCT method (Section 2.2.9).

3.1 Tests on rubber

All tests performed to define the parameters of the rubber viscoelastic constitutive model were non-destructive and included cyclic, relaxation and step relaxation tests, as described in the following sections. Initially, only the relaxation test was used for fitting, as it allows the determination of both elastic and viscous parameters. During the fitting process, it was found that multiple, distinct sets of solutions could adequately describe the experimental relaxation stress response, yielding similar error values (χ) between the experimental and simulated data, with no clear way to select the most suitable set.

Therefore, the adopted strategy was to add constraints to the minimization problem with the dynamic shear modulus G_{dyn} and damping ξ of the cyclic tests and to include the step-relaxation test to independently determine the Yeoh model constants C_{j0} . The time constants τ_i were fixed, as described in Section 2.2.2, to reduce the number of the problem variables and the relaxation test was eventually used only to fit the viscous parameters μ_i .

All shear tests were performed in the PROMAS laboratory of the University of Udine on a computer controlled electro-hydraulic universal testing machine 'MTS 793 model', equipped with a 50 kN full-scale load cell and cross head displacement trans-

ducer. Tension tests were performed on a 'ZwickRoell ProLine Z10 TN' mechanical testing machine designed for tests on small to medium-sized samples, available in Ilpea laboratories. This machine has a maximum load capacity of 10 kN and a maximum velocity of 1000 mm/min.

Each sampled point during the tests records time, displacement, and force, the last two subsequently converted into mean strain and mean stress to ensure uniform data processing across different samples.

3.1.1 Cyclic test

The primary purpose of the cyclic test is to condition the rubber specimen, allowing the Mullins effect to develop (Section 3.1.5). At least three complete sinusoidal deformation cycles are imposed at a relatively low frequency and amplitude, defined by the maximum deformation that the sample will undergo in subsequent tests. 10 cycles are performed in about 20 seconds in order to stabilize the stress amplitude. The sampling frequency is set to $f_s = 100$ Hz to ensure adequate data resolution. The deformation frequency $f = 0.5$ Hz was chosen because it matches the frequency used in the shear cyclic tests performed on seismic isolators. Different frequencies were used only for the application of the MFCT method described in Section 2.2.9 and presented in Section 3.4.1. This test is used also to check for failures on the specimens, comparing the stress peak amplitudes recorded before and after a test.

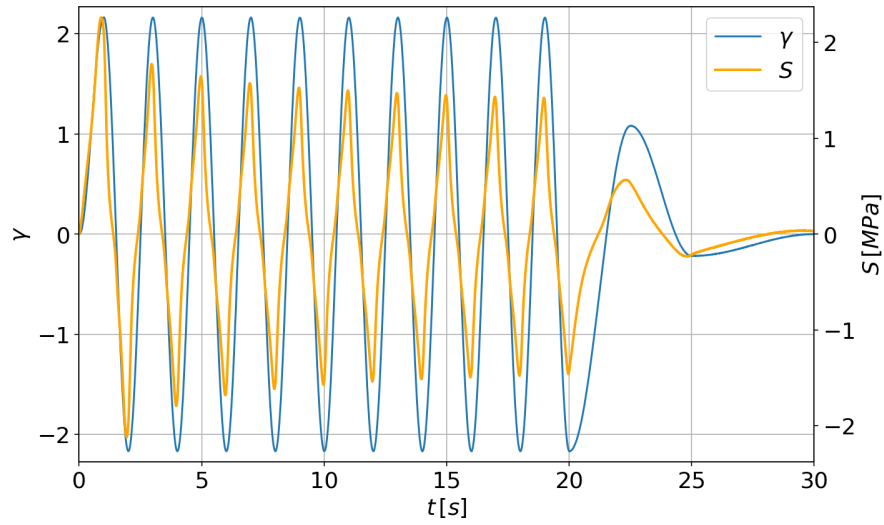
In simple shear cyclic tests, the strain oscillates between the maximum shear deformation $+\gamma_{max}$ and $-\gamma_{max}$. In uniaxial tension tests it should reach a maximum stretch λ_{max} , defined as in Equation 2.74, and a minimum stretch that prevents the instability of the slim specimen due to a compression force. Compression instability is not a concern if the cycles are performed solely for conditioning the specimen, and cyclic uniaxial tension tests were not performed for any other purpose in this work. From the cyclic test, the dynamic shear modulus G_{dyn} and the damping coefficient ξ of the rubber can also be determined, which can be used as additional constraints in the fitting process through Equations 2.56 and 2.58, or to determine the viscous parameters as described in Section 2.2.9.

Since energy is dissipated in the form of heat during each cycle, and the properties of rubber can change with temperature, the number of consecutive cycles should be limited. At maximum strain $\gamma_{max} = 2.5$, rubber C dissipates about $\mathcal{D} = 6$ J/cm³/cycle. Considering a medium specific heat capacity of rubber of $c = 1.9$ J/g/°C and a medium density of $\rho = 1.2$ g/cm³, the temperature increment at each cycle can be estimated, for the worst case and not considering the heat transfer to the metallic supports and in general to the environment, as $\Delta T = \mathcal{D}/c/\rho \simeq 2.6^\circ$ C/cycle. The temperature increase was not directly measured during these laboratory tests due to practical challenges arising from the small dimensions of the specimen.

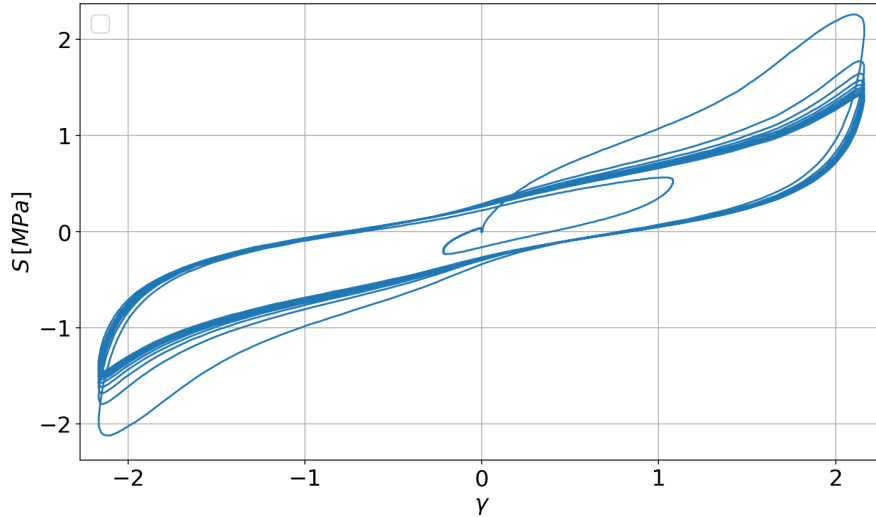
In Section 3.2 the effect of repeated small cycles performed around a mean value γ_r to accelerate the stress relaxation is presented. This technique is leveraged not only in the step relaxation test but also at the end of each cyclic test to bring the stress back nearly to zero (with a stress on the same order of magnitude as the noise being acceptable) at strain $\gamma = 0$ in a short time. This allows the subsequent test to start from the initial conditions of an undeformed specimen, in which the Mullins effect has already developed. During experimental tests, it was found that the same effect is usually obtained with a tapered wave at the end of the test.

An example of strain history and stress response of a shear cyclic test on rubber

is shown in Figure 3.1, where γ represents the assigned shear strain and S the corresponding stress. From the stress curve in Figure 3.1a it can be observed that the peaks at each cycle tend to decrease; however, even after 10 cycles, their values have not completely stabilized yet. In Figure 3.1 it is clear that the effect of the 2 tapered oscillations performed at the end of the test is to save time in waiting for the over-stresses to fade. In fact, the over-stresses caused by a deformation in one direction are almost compensated by a lower deformation in the opposite direction. This behavior is also illustrated in Figure 3.1b, where the smaller half-cycles represent the last part of the test. The path of the last branch converges towards the origin, corresponding to the point $(\gamma = 0, S = 0)$, which indicates the undeformed state of the rubber specimen.



(a) Strain history and stress response over time of a cyclic test.



(b) Stress-strain cycles in a cyclic test.

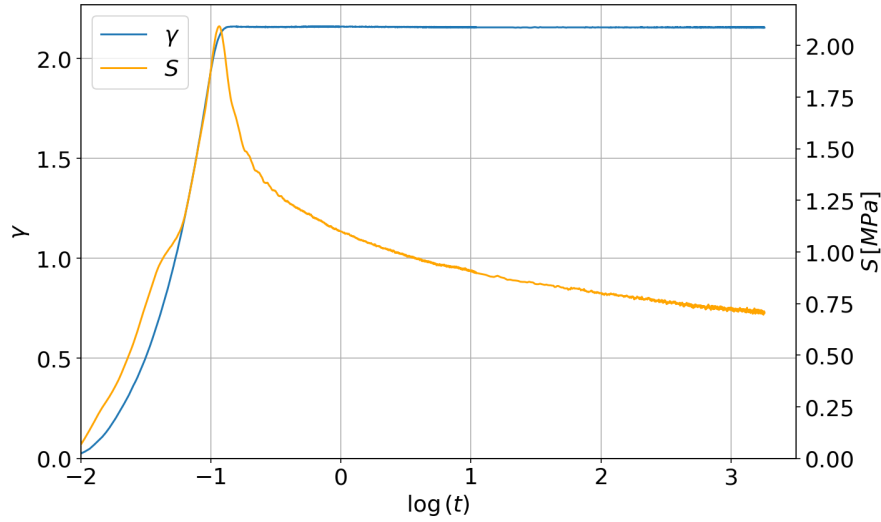
Figure 3.1: Example of recorded strain and stress of a shear cyclic test.

3.1.2 Relaxation test

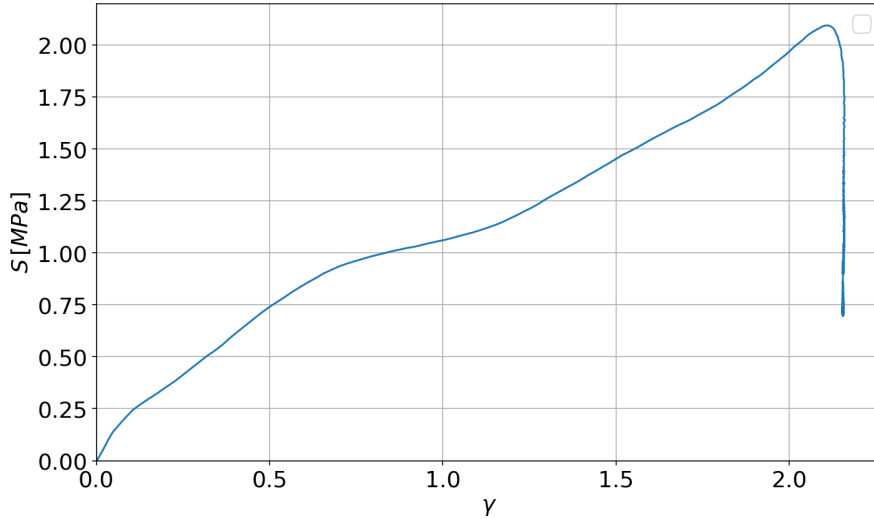
The relaxation test consists of a loading ramp that reaches the maximum deformation in a short time, followed by a constant deformation phase maintained for a set duration, after which the specimen is unloaded and returns to its undeformed state. The loading ramp is theoretically linear but, in practice, it experiences two transition phases: an acceleration phase at the beginning and a deceleration phase at the end.

Simple shear relaxation tests

An example of strain history and stress response of a shear relaxation test on rubber is shown in Figure 3.2.



(a) Strain history and stress response over time of a relaxation test.



(b) Stress-strain curve in a relaxation test.

Figure 3.2: Example of recorded strain and stress of a shear relaxation test.

To get an indication of the relaxation time length, relatively long relaxation tests were conducted, at shear strain $\gamma = 1.5$ and initial ramp strain rate $\dot{\gamma} = 0.1s^{-1}$, on compounds A, B and C. The relaxation curves are shown in Figure 3.3. It is clear

that while the low filled rubber A has substantially completed its relaxation after 1000 seconds, the heavily loaded rubber C still has a significant negative slope after 2 hours. The slope of the linear regression line calculated over the last 30 minutes relaxation data is 3 times larger for compound C compared to compound A.

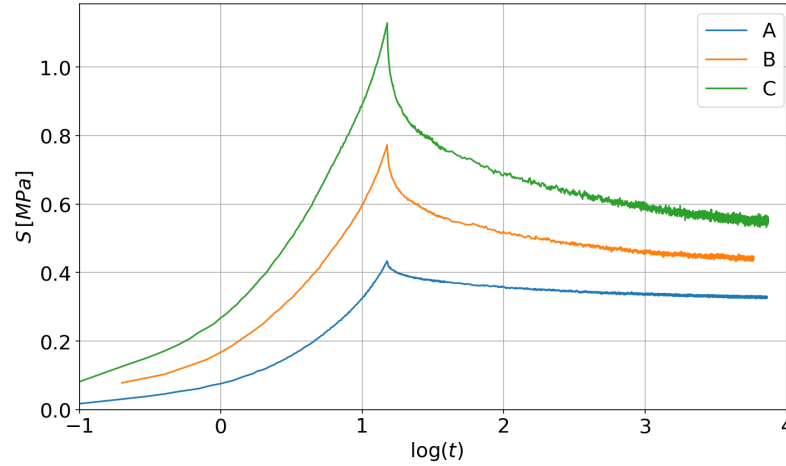


Figure 3.3: Relaxation curves for $\gamma = 1.5$, temperature 25 °C and conditioning at $\gamma = 3$.

After 15 minutes of relaxation, the shear stress in compound C is 591 kPa, after 2 hours it decreases to 545 kPa, which is 7.8 % less, and the non-negligible slope at the end of the relaxation curve indicates that it would decrease further in the following hours. This makes it evident that, in order to obtain the elastic equilibrium response, strategies based on standard relaxation are not convenient, requiring, at least in the case of rubber C, several hours for each equilibrium point.

Viscous effects occur from the very beginning of the deformation, i.e. as soon as the specimen deformation begins, and the short-term effects dissipate quickly. Consequently, the duration of the loading ramp must be as short as possible. The sampling frequency is initially high to capture the rapid stress variations and it is set to 1000 Hz from 0 to 1 second. As the curve smoothens, the frequency is reduced to 100 Hz for the next 10 seconds and then further reduced to 1 Hz for the remaining time. The constant deformation phase lasts 30 min, a time length chosen to balance the overall test duration with the residual, non-negligible over-stress due to long-term viscous effects.

The relaxation test is useful for determining all the parameters of the model, particularly the viscous ones, because it emphasizes the viscous effects. In the fitting process, the stress at the end of the relaxation was used as a constraint from above for the equilibrium stress at maximum deformation.

Tension relaxation tests

Uniaxial tension relaxation tests were performed on dog-bone specimens, whose central portion of constant section measured approximately 20 mm in length, 4 mm in width and with a thickness ranging between 2 to 3 mm, depending on the rubber. These specimens, which were die-cut from vulcanized rubber sheets of approximately 200 mm per side, are commonly used for routine tensile tests to failure as they are simple and cost-effective.

In Figure 3.4 an example of a recorded test is shown. The major issue is that, despite the test being conducted at the maximum speed of the machine, the loading ramp is very slow, with a duration on the order of several seconds during which the short-term viscous effects have already exhausted. The first FEM simulations of the isolators were based on the fitting of these tests, and since the results were poor compared to those of full-scale tests, the tensile tests were temporarily set aside from this research to focus on shear tests.

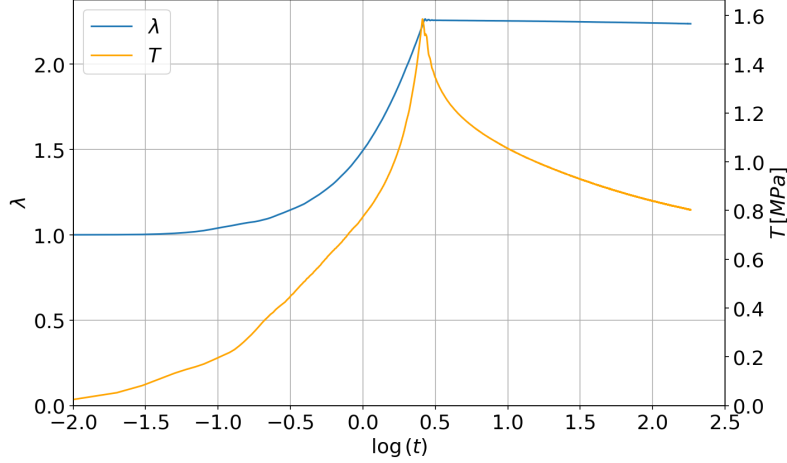


Figure 3.4: Example of recorded stretch and normal stress of a uniaxial tension relaxation test.

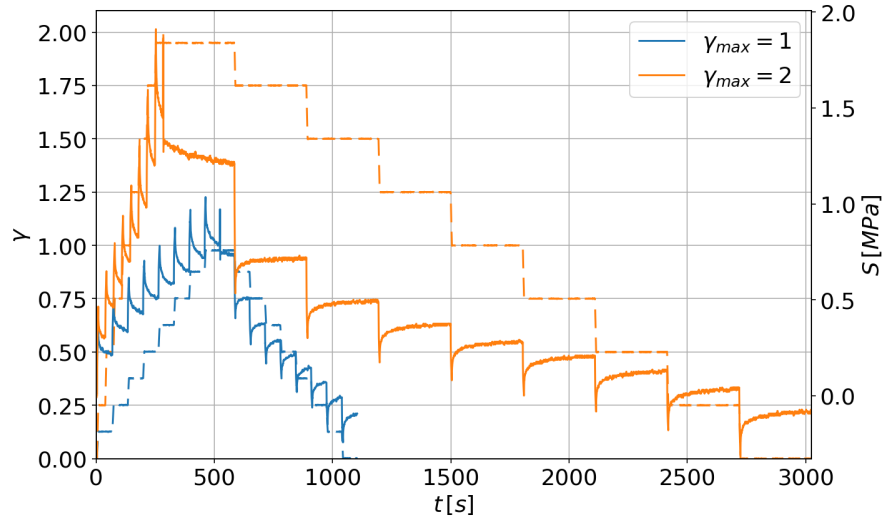
3.1.3 Step relaxation test

The step relaxation test involves repeating the relaxation test at increasing levels of deformation. The objective is to construct the elastic equilibrium curve using a finite set of points, enabling the fitting of the model parameters of expressions in Equation 2.41 or Equation 2.42 to these data points. An alternative would be to perform a quasi-static deformation test, where the total stress curve would closely follow the elastic equilibrium stress curve. However, due to the significant viscous behavior of the material, even in the long-term, such a test would take an impractically long time, making it inefficient in terms of time and equipment usage. Repeating a relaxation of 30 minutes, or even more, at each step would also require a high amount of time, and the problem of the residual over-stresses would be still present.

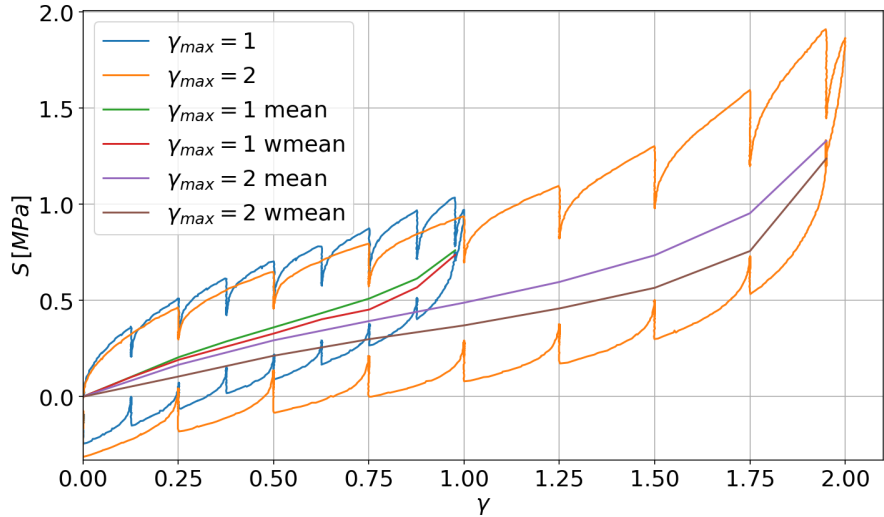
Bergström and Boyce [27] conducted a test in which uploading and unloading ramps at constant strain rate were interrupted at a discrete number of points by 120 seconds segments at constant strain to allow the specimen to relax. They observed that stress decreased during relaxation in uploading segments, and increased in unloading segments, presumably converging from above and below, respectively, to a common value representing the equilibrium elastic response at that level of strain. They also noted that the stress variation rate decreases significantly faster in the unloading relaxation curves compared to the one in the uploading curves, and this suggests that the stress could be closer to the equilibrium elastic curve.

In this work, the first attempts to determine the elastic equilibrium curve were done replicating this idea, also trying with different relaxation time durations. Two examples of strain histories and stress responses of the described shear step relaxation test on the same specimen are shown in Figure 3.5a. For the test at $\gamma_{max} = 1$

relaxation times were 60s both in loading and unloading segments while for the test at $\gamma_{max} = 2$ the uploading relaxation was halved and the downloading was increased to 300s. In Figure 3.5b the responses of both tests in the stress-strain plane are shown. Since the relaxation from a downloading branch cannot be performed exactly at γ_{max} , a point as close as possible was considered instead. However, the relaxation in the unloading branch does not always tend upwards, and therefore, this method cannot be considered reliable in this case. Also, the Bergström and Boyce limiting points along with their mean and weighted mean are shown. The mean stress value for a given deformation is computed considering the stress recorded at the end of the relaxation curves, from the uploading and downloading branches. The weighted mean (“wmean”) stress is computed considering also the stress variation rate, as defined in Equation 3.1. Note that the points obtained with the weighted average are close to the lower limit.



(a) Strain history and stress response over time.



(b) Stress-strain curve and the equilibrium stress curves determined with a mean and weighted mean of the stresses at the end of each relaxation step.

Figure 3.5: Example of recorded deformation and stress of two step relaxation tests as defined by Bergström and Boyce.

A possible estimate of the equilibrium stress arises from the observation that the slope magnitude of the relaxation segments at the same strain level are different in uploading and unloading. Then, assuming that the slope magnitude is proportional to the distance from the equilibrium point, it becomes natural to estimate the equilibrium points with the average of the limit points weighted by the inverse of their slopes. In a logarithmic time scale, the stress relaxation curves can be approximated as straight lines of slope $m_u < 0$ for the uploading branch relaxation and $m_d > 0$ for the downloading branch relaxation. This is possible because of the short duration of the relaxation segment considered, otherwise a non-linear trend should be considered.

In Figure 3.6, the recorded downwards and upwards stress relaxations for $\gamma = 0.5$ are reported for both tests shown in Figure 3.5, together with the linear regression lines that describe each of these curves. The label “UP60s, $\gamma_{max} = 1$ ” indicates the uploading branch relaxation of 60 seconds duration for the test with maximum shear deformation equal to 100 % and “DP300s, $\gamma_{max} = 2$ ” the downloading branch relaxation of 300 seconds duration for the test with maximum deformation equal to 200 %. The beginning of the relaxations has been translated to the same time origin. Calling $S_u(t_0)$ and $S_d(t_0)$ stresses on the regression lines at the same time t_0 from the beginning of the relaxation segment, it is possible to find their intersection point as a mean value weighted on the stress relaxation rates as

$$S_\infty = S_u(t_0) \frac{m_d}{m_d - m_u} - S_d(t_0) \frac{m_u}{m_d - m_u} \quad (3.1)$$

Notice that the lines in Figure 3.6 intersect for a value of time $t(S_\infty)$ in the order of 10^4 s. This suggests that, even if longer relaxation times were employed in the testing procedure, the viscous effects that mostly contribute to the stress curves depicted in the figure would dissipate. Consequently, the slope of the regression line would decrease and the intersection time would increase. This consideration illustrates that achieving equilibrium under such conditions would be impractical.

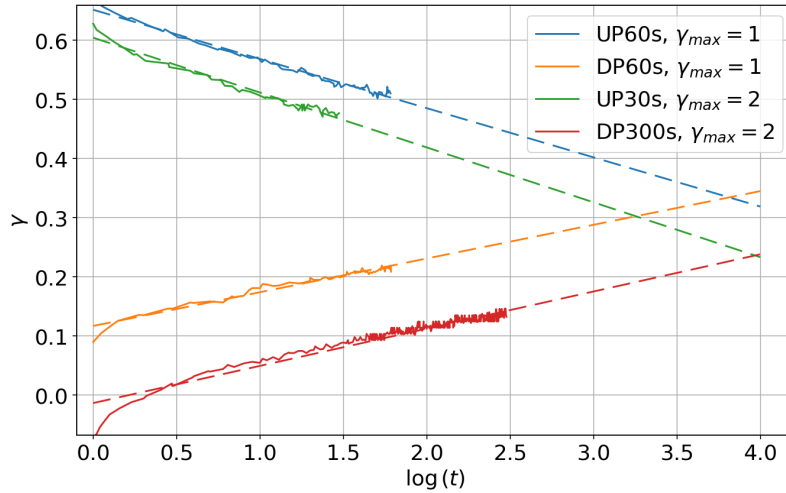


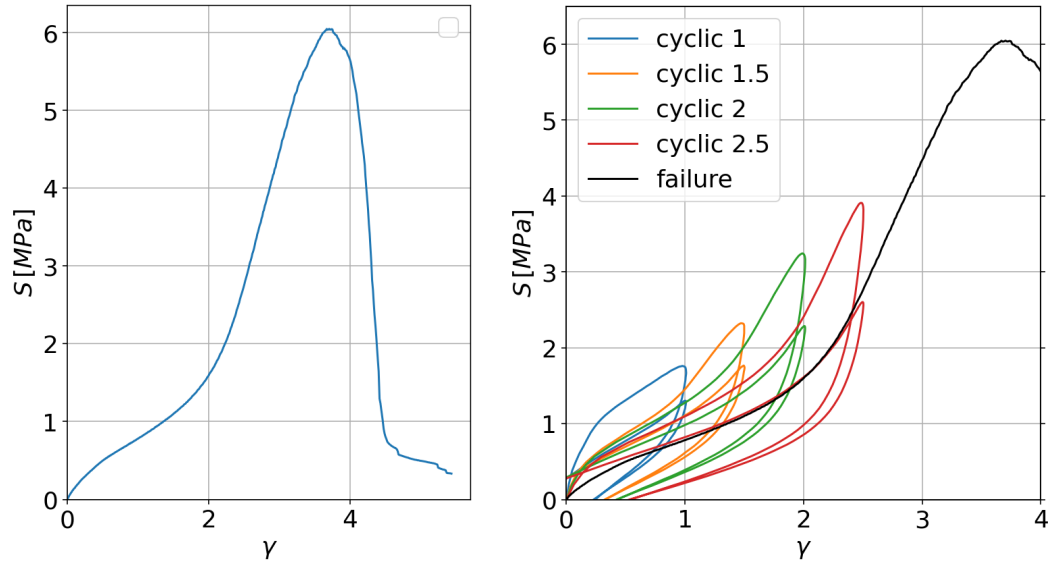
Figure 3.6: Uploading and downloading branch relaxation curves at $\gamma = 0.5$ in a step relaxation test and their respective fitted lines.

The uncertainty of the determined points is high due to the stress difference between the uploading and downloading relaxation and in both cases the relaxations are not over, forcing the extrapolation of points from the recorded data. Furthermore,

the results of the FEM simulations with the equilibrium curves so determined were not optimal. Consequently, the test was improved, as described in Section 3.2, to determine with a great approximation the equilibrium stress by using repeated oscillations to accelerate relaxation.

3.1.4 Strength test

During the strength test, the specimen is loaded at a constant rate of strain $\dot{\gamma} \simeq 0.02 \text{ s}^{-1}$ until failure occurs. In this work, this test was only conducted to check the performance of the designed specimen and to break the specimens at the end of the tests, before disassembling them. All the previously described tests are performed at a level of deformation lower and far from the failure of the rubber. An example of a strength test on rubber is shown in Figure 3.7a, where the rubber starts to fail at $\gamma \simeq 3.5$. It can be instantly noted that the loading stress curve undergo an increment in slope at about $\gamma = 2$, due to the previous deformation of the specimen in cyclic, relaxation and step relaxation tests up to $\gamma_{max} = 2.5$. In Figure 3.7b the strength test curve is compared to the first and last cycle of cyclic tests performed at increasing levels of maximum deformation. This clearly represent the Mullins effect, which is responsible for the progressive softening in repeated deformation cycles of the loading curves in the cyclic tests, which eventually affect the strength test. Notice that, once the specimen is conditioned to γ_{max} in a cyclic test, the testing procedure described in Section 3.1.6 is complete and the softening due to the Mullins effect has stabilized, the first loading branch of the subsequent test at greater deformation firstly follows the path of the last cycle of the previous cyclic test. Then, already before reaching the maximum deformation, the stress curve slope rises. At this point, the loading curve tends to the path that a virgin specimen would follow at the first load.



(a) Example of recorded strength test. (b) The same recorded strength test compared to previously performed cyclic tests (first and last cycle shown).

Figure 3.7: Example of recorded strength test of a specimen previously deformed at $\gamma_{max} = 2.5$.

3.1.5 Mullins effect

Filled vulcanized rubbers are far from ideal elastic materials since their mechanical behavior is markedly strain history dependent. Inelasticity manifests itself through a strong deformation rate dependence of the stress response, and through the Mullins effect, even at very low deformations, which causes progressive softening in repeated strain cycles and through permanent deformations after the removal of loads. The picture is further complicated by a spontaneous tendency to a partial recovery from the damage related to Mullins effect as time goes way if the material is left to rest, also at ambient temperatures.

A conditioning procedure has consolidated over time to run out the Mullins effect, consisting in stretching the sample a series of time at a certain level of strain γ_{max} before performing the real tests. After conditioning, and possibly a certain resting time to allow partial recovery, it is expected that the stress-strain response of a rubber sample becomes stable and repeatable provided that the maximum strain achieved in conditioning, γ_{max} or λ_{max} , is not exceeded.

Despite the Mullins effect manifests forcefully in filled vulcanized, to a lesser extent it manifests also in crystallizing vulcanized such as natural (NR) and nitrile (NBR) unfilled rubbers [21]. The Mullins effect can be almost entirely recovered by heat treatments at $80 \sim 100^\circ\text{C}$ for 3-24 hours [22], although, as noted by several authors, the apparent recovery at high temperatures could be an aging effect rather than a true recovery.

Holt [23] in one of the earliest observations of spontaneous recovery, stretched some specimens a series of time and then allowed to rest for some period after which he tested them again. He reported that a NR filled at 30 phr of carbon black (CB), tested at 500 % elongation recovered about 17 % of the initial stress after 2 hours of rest at room temperature and a similar recovery was observed also in pure vulcanized stretched at 700 % after 24 hours of rest.

Mullins [22] reported that a NR filled at 50 phr CB, previously stretched at 420 %, spontaneously recovered about $15 \sim 18$ % of its secant stiffness at 200 % elongation after 100 hours of recovery at 20°C , but the greatest part of the recovery, about 90 %, occurred in the first 10 hours. Furthermore he noted that the slope of time-recovery curve, although very small, was still positive at the end of 100 hours of rest, suggesting that the recovering was still continuing.

The existence of a spontaneous recovery from the internal damage caused by conditioning, makes the concept of material response somewhat indeterminate. In fact, two identical and not destructive experiments performed on the same specimen in quick succession immediately after the conditioning, yield the same results, but if the second test is conducted one hour after the first during which the specimen was let to rest, the response is different due to partial recovery during the rest pause. On the other hand, if the two experiments are performed also in rapid succession but after 24 hours of rest following the conditioning, the results are even more different because at the start of the first experiment the material has recovered as much as possible, but by the end of the experiment it has accumulated new damage that affects the second test. In phenomenological models that treat the Mullins effect, the partial recovery is often neglected, e.g. [21] and [24].

The Mullins effect cannot be described by the viscoelastic model adopted. Since the main purpose of this work is to evaluate the properties of the isolators at the third cycle, as requested by the UNI EN 15129 standard, incorporating it is unnecessary. Furthermore, adding a damage parameter, e.g. as in the one proposed

by Dorfmann and Odgen [31], would complicate the model. Hence, it is ensured that the all tests are conducted after the Mullins effect has dissipated. To achieve this, a cyclic test, as defined in Section 3.1.1, is performed before each test, also to avoid any recover of the effect.

A test battery consisting of a constant shear rate loading ramp at $\dot{\gamma} = 0.01 \text{ s}^{-1}$ and increasing deformation from $\gamma = 0$ to $\gamma = 3$ on a virgin specimen of compound B were conducted to verify the extent of the Mullins effect and the timing of partial recovery. Then, a conditioning procedure was applied to the specimen within the strain range $\gamma_{max} = 3$ and tested again the specimen immediately after. The sequence consisting of the conditioning and the constant rate test was repeated, but with a specific amount of time between the two: 90 minutes, 24 hours, 48 hours, one week, and one month. All tests were performed at room temperature $20 \pm 1 \text{ }^\circ\text{C}$ and the specimens kept at rest at the same temperature. To verify if ambient temperature had any effect on recovery, an additional test was conducted. After conditioning, the specimen was placed in an oven at $50 \text{ }^\circ\text{C}$ for 24 hours and then allowed to cool naturally for several hours before finally testing at $20 \text{ }^\circ\text{C}$. The stress responses obtained are shown in Figure 3.8.

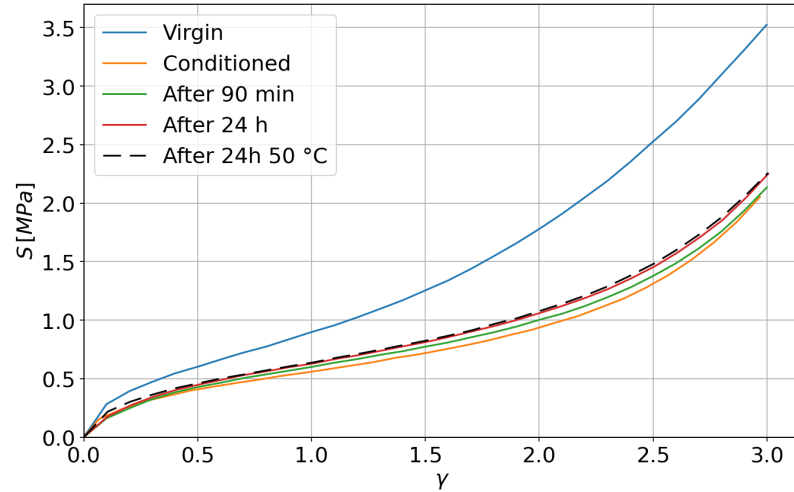


Figure 3.8: Simple shear stress-strain response to constant shear strain rate $\dot{\gamma} = 0.01 \text{ s}^{-1}$ of rubber compound B.

Curves for the 48 hours , one week and one month experiments are not reported because they were perfectly coincident with the 24-hour response. Comparing the curves, it is noticeable that the Mullins effect is substantial: after conditioning, the secant instantaneous modulus at a strain $\gamma = 3$ decreased from 1.15 MPa to 0.7 MPa which is about a 40 % reduction. The recovery is not as pronounced but is not negligible either: at $\gamma = 2.5$, the secant modulus after 24 hours increases by approximately 15 % compared to the situation immediately after conditioning. Regarding recovery timing, it can be deduced that after 24 hours, the recovery is definitely complete, and not even particularly high ambient temperatures have an effect. It is worth noting that the nearly complete recovery experiences reported in the literature [21] [22], were achieved at temperatures not less than $80 \text{ }^\circ\text{C}$.

3.1.6 Testing procedure

The final version of the testing procedure allows for the determination of the visco-elastic parameters valid for a specific rubber and for deformations below the maximum one reached by the rubber in its deformation history.

The procedure consists of:

- a cyclic test (Section 3.1.1) at maximum strain γ_{max} for the conditioning of the specimen and to compute the dynamic modulus G_{dyn} and the damping ξ at the third deformation cycle. Small cycles at the end of each test are performed if the residual stress is not negligible, in order to push the residual over-stresses to end their effect;
- a multi-step relaxation test to determine the equilibrium stress curve (see Section 3.2);
- another cyclic test to run out any possible recover in stiffness caused by the Mullins effect and to check for the integrity of the specimen;
- a relaxation test to determine the viscous constants;
- a final cyclic test to check again for the integrity of the specimen.

Considering a duration of 20s of the cyclic tests, 26 min for the UCDP2m test and 30 min for the relaxation test, this procedure lasts slightly less than one hour.

3.2 Experimental determination of the elastic equilibrium response curve

Once conditioned to a specified maximum strain magnitude, it is assumed that filled rubber behaves as a viscoelastic material described in Chapter 2; the issue is if and how it is possible to experimentally separate the elastic and viscous components of the response. Although these components are considered to be independent from each other, it is important to note that the elastic equilibrium stress curve has a direct impact on the over-stresses response of the adopted model, as the stress rate variation $\dot{\mathbf{S}}_\infty$ appears in the integration of Equation 2.31, and thus on the determination of the viscous constants μ .

In principle, by making a strain driven experiment in some simple homogeneous deformation mode (e.g. uniaxial traction/compression, simple/pure shear, equibiaxial extension), applying an increasing strain history at constant, but very slow, strain rate, one can measure directly the elastic response $\mathbf{S}_\infty(\mathbf{C}(t))$. It is expected that by conducting a strain driven experiment consisting of an uploading ramp at a “sufficiently slow” rate followed by an identical but reversed unloading ramp, the response in the two phases will tend to overlap with the elastic response.

In Figure 3.9 the stress responses in simple shear of a conditioned ($\gamma_{max} = 3$) rubber C specimen are reported for different constant shear strain rates in the range $\dot{\gamma} = 1 \div 10^{-3} \text{ s}^{-1}$. It is observed that the distance between the uploading and the unloading branches significantly decreases as the strain rate decreases, but remains substantial even at the lowest rate. Note that the slowest experiment takes $2.5 \cdot 10^3$ seconds for the uploading branch and just as much for the unloading branch to be completed, that is almost an hour and a half in total. Nevertheless, it is apparently not “sufficiently slow” to obtain an acceptable estimate of the elastic curve

because there is still a significant difference between the uploading and downloading paths, hence the viscous terms are not negligible. As noted by some authors (e.g. Bergström & Boyce, [27]), the unloading curves are much less dependent on strain rate compared to the uploading curves, and this suggests that they could be closer to the equilibrium elastic curve. However, near the origin they have the wrong sign due to the relaxation not having taken effect yet, which requires several hours to complete. In fact, although much of the relaxation occurs within the first hour, an additional relaxation of about 5% of the 60 minutes relaxation stress has been observed in the following 24 hours in preliminary experiments performed on the C compound.

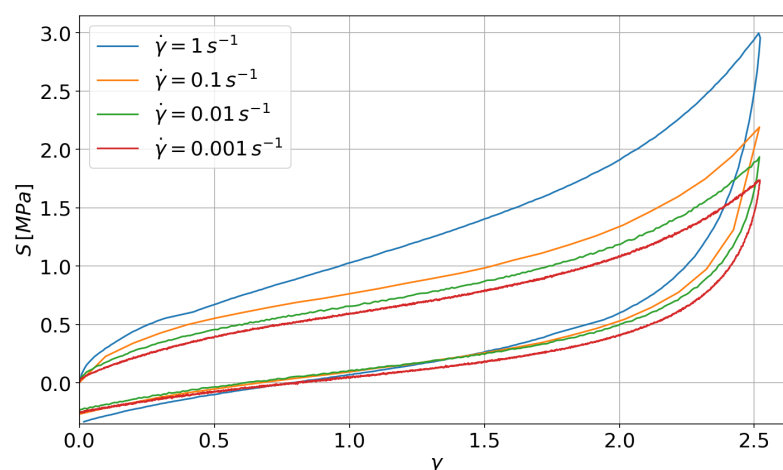


Figure 3.9: Upload and unload response curves in simple shear at different constant shear strain rates. Conditioning at $\gamma_{max} = 3$.

This indicates that, at least for heavily filled rubbers, attempting to assess the elastic curve with experiments at very low speeds is not convenient as it requires very long times. Furthermore, the curve obtained from an experiment of such duration could not be confidently attributed to the material at a single time from conditioning because other undesired phenomena could occur along the duration of the experiment, e.g. the partially recover of the Mullins effect.

One of the first authors to address the issue of elastic response determination was Gerke [28] who in an article from 1930, while explaining a multi-step creep experiment he adopted to find the elastic response of natural rubber, stated:

“To obtain points on stress-strain curve at constant temperature and without appreciable hysteresis a customary test piece of rubber is extended vertically by means of a weight attached to the lower end and the weight is oscillated up and down by hand ten to twenty times in the neighborhood of the equilibrium position ...”

He observed that small induced oscillations around an equilibrium point quickly dissipate the viscous over-stress. In the same article, he mentions that Dr. M. Mooney, a colleague from his laboratory, verified that transverse vibrations also had the same effect, and reported a table of experimental data points obtained by Mooney, showing that the distances (in elongation) between points at the same load but obtained during loading and unloading phases were indeed very close, on the order of a few percentage points of the initial length of the specimen.

It is worth noting that Gerke and Mooney only experimented with pure natural rubbers cured with moderately high amounts of sulfur (3.75 phr) for which the vis-

cous effects were presumably minimal. Gerke and Mooney’s expedient to accelerate relaxation through vibrations, seems to have fallen into oblivion and we have been unable to find its use in the literature from then until now.

Treloar, in 1944 [29], aimed at experimentally confirming the elastic behavior of natural rubber predicted by the molecular kinetic theory and completely circumvented the obstacle. In fact, he performed experiments on two natural rubbers compounds, the first one with atypically high amount of sulfur (8 phr) and very high curing times (3 hours at about 150 °C), admitting that it was a poor compound with several technical disadvantages, but chosen solely because it did not exhibit significant hysteresis effects. The second one was a standard rubber (2 phr of sulfur cured 30 minutes at 100 °C), but, as he noted, since it was affected by marked hysteresis which, however, tends to diminish as the temperature rises, tests were performed at 50 °C.

James et al., [30] performed strain driven bi-axial testing on NR (2.5 phr of sulfur, cured 50 minutes at 135 °C), measuring the forces after allowing the specimen to relax for five minutes to a “*quasi-equilibrium state*” at each measurement.

Yeoh [34], for the elastic characterization of two NR compounds filled with 40 and 70 phr CB, opted for the low strain rate approach saying “*for reasons of expediency, we decided to accept stress/strain data obtained at relatively low strain rates*”. Then, after noting that in preliminary tests at the strain rates of 10 % min^{−1} and 50 % min^{−1} the responses were not significantly different, he choose the highest rate. Since the two rates correspond to 0.00166 s^{−1} and 0.00833 s^{−1} respectively, if Yeoh had tested rubber compound C at the two speeds, he would have roughly obtained the uploading branches of the last two curves reported in Figure 3.9, which not only differ from each other by about 15%, but are also very far from any potential elastic equilibrium curve since they are very far from their respective unloading curves.

As described in Section 3.1, Bergström and Boyce [27] in 1998 present a testing procedure in which uploading and unloading ramps at constant strain rate are interrupted at a discrete number of points by 120 seconds segments at constant strain to allow the specimen to relax. They observed that stress decreased during relaxation segments in uploading, and increased in unloading segments, presumably converging from above and below, respectively, to a common value representing the equilibrium elastic response at that level of strain. Figure 3.5a illustrates both the strain history and the stress response of a rubber compound C in a simple shear experiment that roughly reproduces that of Bergström and Boyce with 60 – 300 s of relaxation at each step.

The data of an analogous experiment on a specimen of rubber compound B, are plotted as a stress-strain response curve (BB) in Figure 3.10 along with the slowest constant strain rate response of Figure 3.9. The points at the end of each relaxation segment in the uploading branch, and those in the unloading branch, form an upper limiting curve (BB+) and respectively a lower limiting curve (BB-) among which the equilibrium elastic curve should be included. Note that the two limiting curves delineate a fairly narrow region in which the elastic curve must fall, certainly much narrower than that defined by the two branches of the constant-rate response, even if the constant rate unloading branch cannot be considered a lower limit.

In [27], Bergström and Boyce make no attempt to estimate the elastic curve from the two limit curves. However, in a subsequent paper [35], after presenting the multi step relaxation curves for three chloroprene filled compounds, they state that from these, an estimate of the elastic curves can be derived and their graphs are shown, but without specifying how they were obtained. In Bergstrom’s doctoral thesis [36],

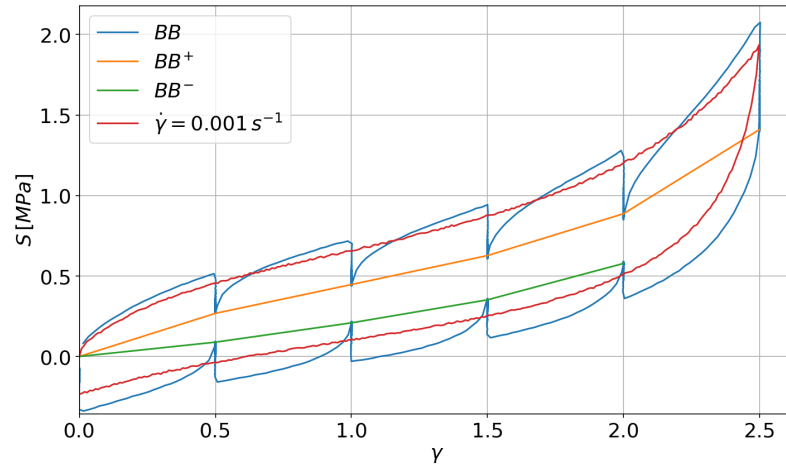


Figure 3.10: Data from a multi step simple shear relaxation experiment with ramps strain rate 0.01 s^{-1} and 15 minutes relaxation segments. Conditioning at $\gamma_{max} = 3$.

where the estimates of the equilibrium curves are reported in the same graph along with the response of the multi-step experiment, it is clearly evident that the estimated equilibrium curves are not simple averages of the limits. A possible estimate arises from the observation that the slope magnitude of the relaxation segments at the same strain level are different in uploading and unloading. Then, assuming that the slope magnitude is proportional to the distance from the equilibrium point, it becomes natural to estimate the equilibrium points with the average of the limit points weighted by the inverse of their slopes. In Figure 3.5b the Bergström and Boyce limiting points along with their mean and weighted mean are shown. Note that the points obtained with the weighted average are close to the lower limit.

It is worth to mention that, in his doctoral thesis, Bergström reports also the results of strain driven experiments where periodic oscillations of constant amplitude around a certain strain value produced stress oscillations of constant amplitude, but with a decreasing mean value that tended to stabilize. But this result, which resembles Gerke's experience, remains at the level of a simple observation and is not further developed in his work

In the following Section 3.2.1, it will be shown how to improve the Bergström and Boyce limit curves through the application of the Gerke and Mooney expedient, i.e. by introducing oscillations in the relaxation segments of the multi step experiment that accelerate relaxation. Then, an experimental procedure for a direct estimation of the elastic equilibrium points will be presented.

3.2.1 Relaxation acceleration

All of the experiments in this section were performed under simple shear on specifically designed single pad specimens. A detailed description of these specimens, as well as the preliminary analytical and numerical studies, can be found in Section 3.3, which is dedicated to this topic.

Acceleration of relaxation through oscillations

To verify Gerke and Mooney's observation [28] that oscillations around the instantaneous equilibrium point speed up the attainment of elastic equilibrium, tests are

focused on the most heavily filled compound C, since it reaches equilibrium the slowest. Gerke simply mentioned that he performed 10 to 20 oscillations around the (instantaneous) equilibrium position, without specifying their magnitude. However, the fact that he cites Mooney achieving the same effect with transverse oscillations “vibrating the rubber like a piano wire”, suggests that they were small. Therefore, after a conditioning at $\gamma_{max} = 3$, three relaxation tests were conducted, in which the strain was increased at a constant rate for 10 seconds up to a value of $\gamma = 2$, then, after a 20 second pause, the strain was varied by imposing three different oscillation types, as shown in Figure 3.11:

- ($\sin^t$) 30 sinusoidal tapered cycles at 0.5 Hz between $\gamma = 1.85$ and $\gamma = 2.15$ ($\Delta\gamma = \pm 0.15$). The first 20 cycles are pure sinusoidal, while the last 10 are tapered linearly with amplitude decreasing to zero.
- ($\sin^+$) 30 asymmetric sinusoidal cycles between $\gamma = 2.0$ and $\gamma = 2.2$ at 0.5 Hz
- ($\sin^-$) 30 asymmetric sinusoidal cycles between $\gamma = 1.8$ and $\gamma = 2.0$ at 0.5 Hz

For the sake of comparison, also a standard relaxation experiment (std) on the same specimen was made. All stress responses relative to the peak tension of each curve are reported in Figure 3.12.

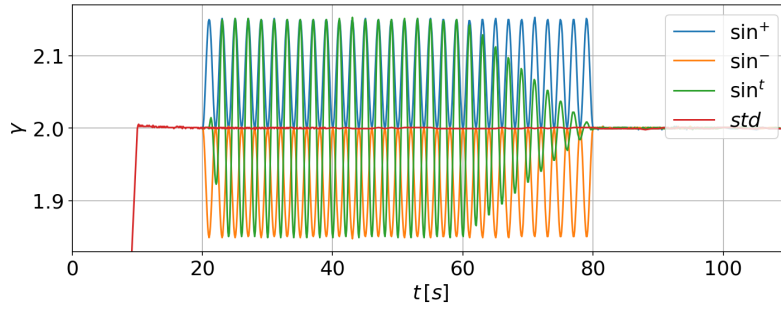


Figure 3.11: Different types of oscillations imposed around strain $\gamma = 2.0$.

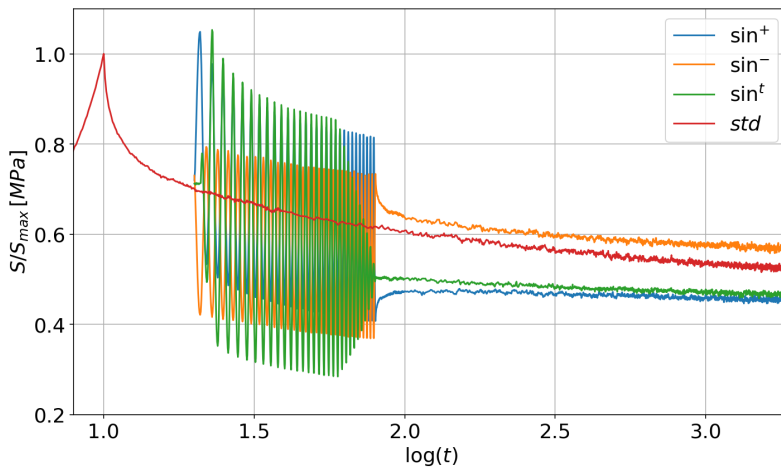


Figure 3.12: Effect on relaxation at shear strain $\gamma = 2.0$ of different types of oscillation. Conditioning at $\gamma = 3$.

The effect of the sinusoidal tapered ($\sin^t$) oscillations is remarkable: 10 seconds after the end of the cycles, the stress is significantly lower than the one reached after 10 minutes of standard relaxation, and at the end of the 15 minutes relaxation segment it has decreased further by 8%. Furthermore, the negative slope of the post-cycles relaxation curve indicates that it is also an upper estimate of the equilibrium stress. Downward cycles ($\sin^-$) produce a significant slowdown in relaxation compared to the standard procedure. On the contrary, upward cycles improve the results even more than symmetric sine tapered cycles, even if it is apparent from Figure 3.12 that the two responses tend to coincide with time. While the effect of repeated oscillations ($\sin^t$) is evident, that of ($\sin^+$) and ($\sin^-$) could be due to how the last point of the cycle is approached. In a hysteresis cycle, to the mean deformation value corresponds to values of stress, a greater one if the deformation is increasing, and a lower one vice versa. In ($\sin^+$), the last cycle ends with an unloading branch, which results in a lower value of stress. In ($\sin^-$) the last cycle ends with a loading branch, thus a greater value of stress.

Unloading relaxation

Considering the lower sensitivity to strain rate in unloading compared to that in uploading segment, we focused on the former. Immediately after a conditioning of the specimen with a cyclic test at $\gamma_{max} = 3.0$, a multi-step strain driven relaxation test was performed. The applied strain history is composed by a constant rate $\dot{\gamma} = 0.01 \text{ s}^{-1}$ uploading segment until strain $\gamma = 2.5$, followed by a partial unloading at the same rate until $\gamma = 2$ is reached. From this point, the deformation was kept constant for a 30 minutes relaxation. The partial unloading and relaxation segments are then repeated for the strains levels 1.5, 1.0, 0.5, and 0. In Figure 3.13 both the imposed strain history and the shear stress response are reported. This figure immediately strikes that the stress in the first relaxation segment ($\gamma = 2$) has a much smaller slope than the other segments suggesting that the stress reached at the end of that step is very close to the equilibrium stress for that level of deformation.

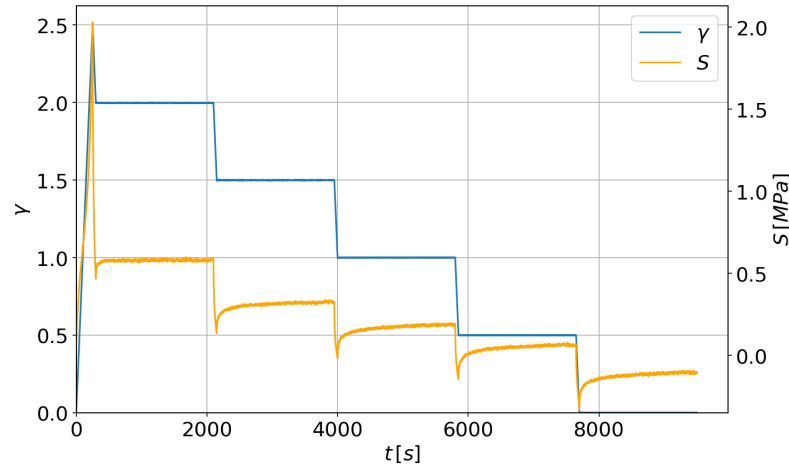


Figure 3.13: Unloading multi-step relaxation for compound C. Upload and unload ramps at constant strain rate of $\dot{\gamma} = 0.01 \text{ s}^{-1}$ and 30 minutes relaxation segments.

In Figure 3.14, only the 5 stress relaxation segments at constant deformation are shown, with the time measured from the beginning of their respective steps. The m_d values reported in the legend are the slopes of the linear regression lines evaluated

over the last 10 minutes data of each step, expressed in kPa/min . Note that the slopes at the lower strain steps are about ten times the slope at the first unloading step at $\gamma = 2$.

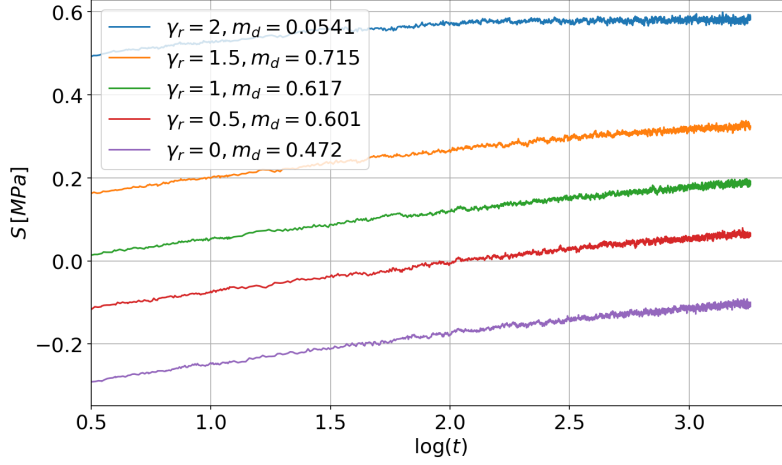


Figure 3.14: Stress relaxation of segments in the experiment of Figure 3.13 with time taken from the beginning of each segment.

The uniqueness of the first unloading step could be questioned, raising the possibility that the proximity to equilibrium is either a result of the material's ability to relax quickly at strain $\gamma = 2$, or it could be attributed to a fortunate choice in the deformation history during that step. Seeking an answer to this question, a series of strain driven experiments (UDP30m) was performed with the same structure as the first unloading step of the previous experiment but at different strain levels, consisting in the three segments, as shown also in Figure 3.15:

U uploading segment at constant strain rate $\dot{\gamma} = 0.01s^{-1}$ up to the peak strain $\gamma_p = \frac{5}{4}\gamma_r$,

D partial unloading segment at $\dot{\gamma} = 0.01s^{-1}$ to the target strain γ_r ,

P30m 30 minutes relaxation segment at constant strain γ_r ,

choosing the target strain γ_r within the set 0.5, 1.0, 1.5, 2.0.

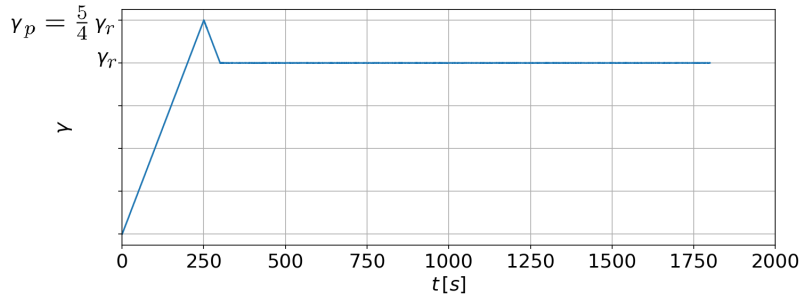


Figure 3.15: Single step UDP30m experiment deformation history.

The stress response curves are reported in Figure 3.16. Note that for all relaxation segments, there is an initial recovery of stress until reaching a maximum peak after a few tens of seconds, and then the slope becomes slightly negative, likely due to the relaxation triggered by the uploading segment not yet being exhausted.

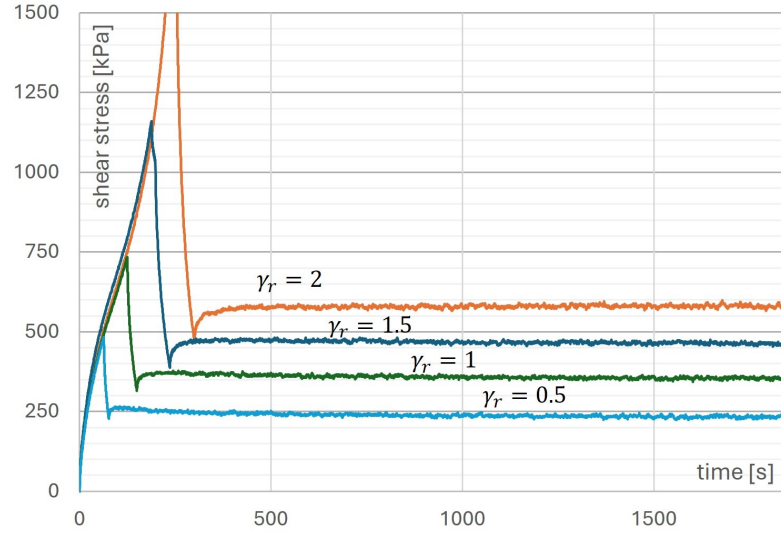


Figure 3.16: Single step UDP30m experiment at four different strain levels. Conditioning at $\gamma = 3$.

Since, as described in the previous paragraph, oscillations around the target strain in uploading drastically accelerates relaxation, we repeated the experiment for $\gamma_r = 0.5$, by introducing a new segment (C) made by 30 sinusoidal tapered oscillations of magnitude $\Delta\gamma = 0.2\gamma_r$ immediately after reaching the peak strain and before the partial unloading ramp (UCDP30m experiment). The effect of this expedient is clearly visible in the stress response shown in Figure 3.17.

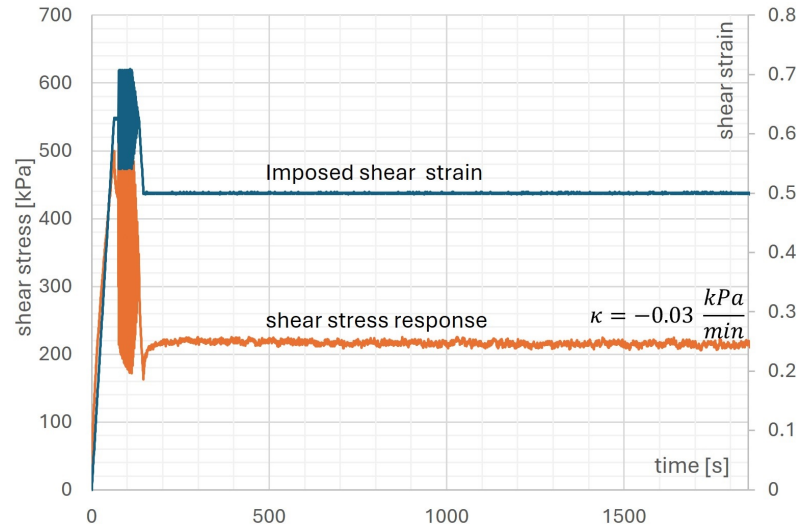


Figure 3.17: Single step UCDP30m experiment at the target shear strain $\gamma_r = 0.5$.

Note that although there is some stress recovery in the initial part of the relaxation segment after the end of the cycles, reaching a peak of 219 kPa after about two minutes, the trend then becomes and remains fairly constant until the end of the 30 minutes, where the stress assumes a value of 214 kPa , about 10% lower than the value of the stress for the relaxation at the same deformation shown in Figure 3.16. Also, the last 10 minutes data linear regression line has an extremely small slope

of $-0.03 \frac{\text{kPa}}{\text{min}}$, lower than the error and the noise of our measurement apparatus, suggesting that equilibrium is very close. Through experimentation, the best results found are obtained with cycle amplitudes equal to $10 \div 20\%$ of the target strain γ_r . Dropping below 10% , the final slope of the relaxation segment progressively increases in absolute value. For example, with an oscillation amplitude of $5\%\gamma_r$, the absolute value of the slope increases by a factor of 10 compared to the case described above.

Estimation of equilibrium elastic response

Given the excellent performance of the UCDP experimental procedure for estimating the elastic equilibrium point at a given strain level, we decided to apply this approach to all steps performed one after the other in increasing order of strain (MS-UCDP procedure). The relaxation time was halved and 6 steps at $\gamma_r \in \{0.5, 1.0, 1.5, 2.0, 2.25, 2.5\}$ were performed to a specimen conditioned at $\gamma_{max} = 2.7$. The stress response of the test is shown in Figure 3.18.

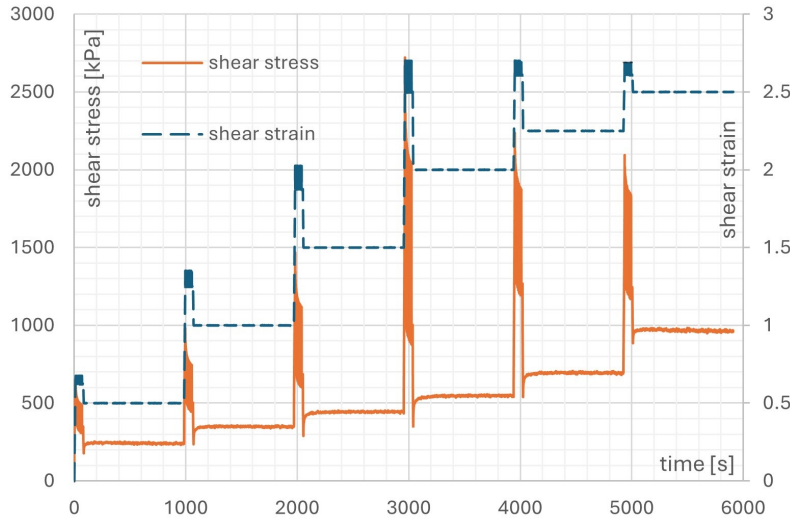


Figure 3.18: Deformation history and stress response in a MS-UCDP15m experiment.

The slopes of the relaxation segments are all slightly negative after an initial period of about 60 seconds from the start of relaxation, during which stress increased. Therefore, the estimates of the equilibrium points are, in fact, upper estimates. By closely observing the figure, it is worth noting that in the first four steps the amplitude of the oscillations is increasing and is indeed taken as $10\%\gamma_r$, while in the last two steps the oscillations are limited from above to not exceed the conditioning strain value $\gamma = 2.7$. For the target strains 2.25 and 2.5 we reduced peak strains to $\gamma_{5p} = 1.16\gamma_{5r}$ and $\gamma_{6p} = 1.05\gamma_{6r}$. The choice of sub-optimal values for the peak strain and oscillation amplitudes is clearly evident in the significant slope of the relaxation segment in the final step. Therefore, in the proposed testing procedure, the points of elastic equilibrium for strain values close to that of conditioning are inevitably estimated with greater uncertainty.

The estimated equilibrium points and their polynomial interpolation curve are shown in Figure 3.19 with the label MS-UCDP15m.

For the sake of comparison, in the same figure there are also shown the Bergström and Boyce upper limiting (BB+), lower limiting (BB-) and mean (BBm) curves, and

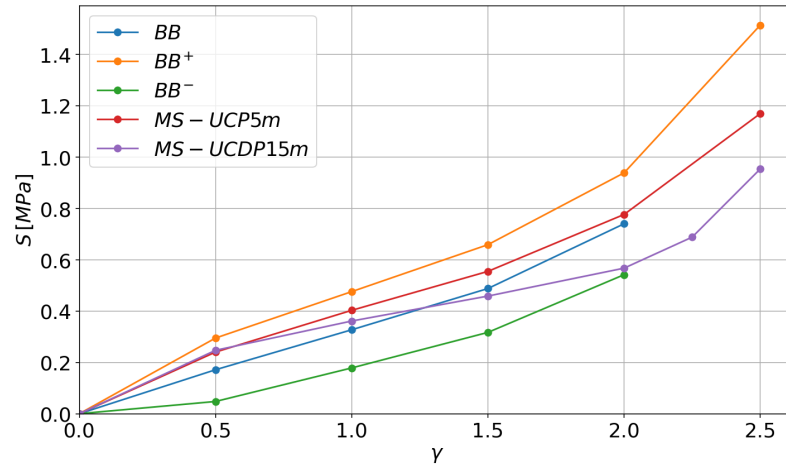


Figure 3.19: Equilibrium points obtained by the MS-UCDP15m experiment compared with other procedures.

the multi step UCP procedure results that greatly improves the BB upper limit. Note that, as expected, the MS-UCDP estimates are within the BB limits but are not centered with respect to them and differ significantly from the mean BB values.

In Figure 3.20 the results of the 5 step UCDP procedure performed to another specimen are reported along with the constant strain rate test. In the same figure there are also shown the points of elastic equilibrium obtained from a Gerke's multi step creep test performed on the same specimen with 30 stress-controlled oscillations of an amplitude equal to 10% of the target stress. It can be seen the extent to which Gerke-Mooney's oscillations, performed in each step of the UCDP test, improve the upper limit curve of Bergström and Boyce, with the same relaxation time of 5 minutes. Also, the two upper and lower limit BB curves are shown, along with the limit curve obtained by applying symmetric $\pm 10\% \gamma$ oscillations in every step of a multi-step uploading relaxation experiment (MS-UCP5m).

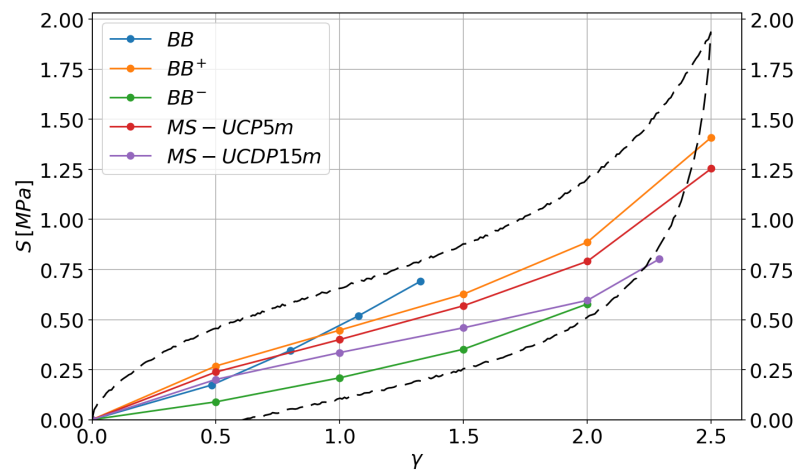


Figure 3.20: Comparison of results from different experiments.

Since a 6 step UCDP procedure with 30 minutes relaxation segments takes more than three hours to complete, the test was repeated by shortening the relaxation pauses to 5 and 2 minutes, for a total experiment duration of about 38 and 20 min-

utes, respectively. The obtained equilibrium point estimates are reported in Table 3.1 along with the percentage differences with respect to the 30 minutes procedure.

Table 3.1: Equilibrium points obtained with the multi-step UCDP procedure on rubber compound C for different relaxation times. Conditioning at $\gamma_{max} = 2.7$. The percentage variation Δ with respect to P30m is reported in parentheses.

γ	S [kPa]			
	P30m	P5m (Δ [%])	P2m (Δ [%])	P15m-24h (Δ [%])
0.5	247	252 (+2.0)	251 (+1.6)	248 (+0.4)
1	361	364 (+0.8)	365 (+1.1)	356 (-1.4)
1.5	459	459 (+0.0)	456 (-0.7)	452 (-1.5)
2	567	565 (-0.4)	560 (-1.2)	556 (-1.9)
2.25	688	692 (+0.6)	692 (+0.6)	701 (+1.9)
2.5	954	966 (+1.3)	966 (+1.3)	971 (+1.8)

It is worth noting that the two minutes estimates are significantly worse than neither the 5 nor the 30 minutes estimates. Furthermore, although the two experiments are conducted on different specimens from the same compound, note that in the curve shown in Figure 3.3 for the compound C, after two hours of relaxation at $\gamma = 1.5$, the stress has decreased to 545 kPa. In contrast, with the multi step UCDP technique, after two minutes of relaxation at the same strain, a value of 456 kPa is obtained. This despite that in the 2-hour experiment, the specimen was conditioned to a slightly higher strain $\gamma_{max} = 3$.

In the last column Table 3.1 there are reported the equilibrium points obtained with a 15 minutes relaxation multi step UCDP procedure applied to a specimen that had been conditioned 24 hours earlier and left at rest until the time of the test. Unexpectedly, the results do not reveal a partial recovery of the Mullins effect, which warrants further investigation. A possible explanation is that the small excess deformation reached at the peaks of the oscillations before reaching the target value for that step has the same effect as a complete conditioning. An alternative interpretation is that the spontaneous partial recovery is a purely viscous phenomenon rather than an elastic one. It is not possible to draw conclusions on the basis of the few tests performed on this matter, but it will certainly be worth investigating further.

For completeness and to understand its worth, the multi step UCDP procedure with parameters $\frac{\gamma_p}{\gamma_r} = \frac{5}{4}$, thirty sin tapered cycles between $\gamma_p \pm 0.1\gamma_r$ and relaxation segments of 10 minutes have also been applied to the less filled compound A, immediately after a standard conditioning at $\gamma_{max} = 2.7$. The results are shown in Figure 3.21, where, along with the BB limiting curves, the results of multi step experiments with upward oscillations ($\sin^+$) but without the partial unloading segment are also shown, for the uploading (MS-UCP2m) and unloading (MS-DCP2m) branches with 2 minutes relaxation segments. Despite the short duration of the relaxation segment, admitting these last two curves to represent upper and lower bounds, respectively, for the elastic curve, they significantly narrow down the range within which the elastic equilibrium curve must fall with respect to the five minutes relaxation BB bounds. Furthermore, since the slopes of the MS-UCDP10m relaxation segments were all slightly negative, the identified points are presumably upper bounds estimates that, together with the MS-DCP2m lower bound points, outline the even narrower region, highlighted in light gray in Figure 3.21, within which the elastic equilibrium points must fall. Unfortunately, the MS-DCP estimates only im-

proved the lower BB- bounds for compound A while for compound C the results were inconsistent and very close to the BB- points.

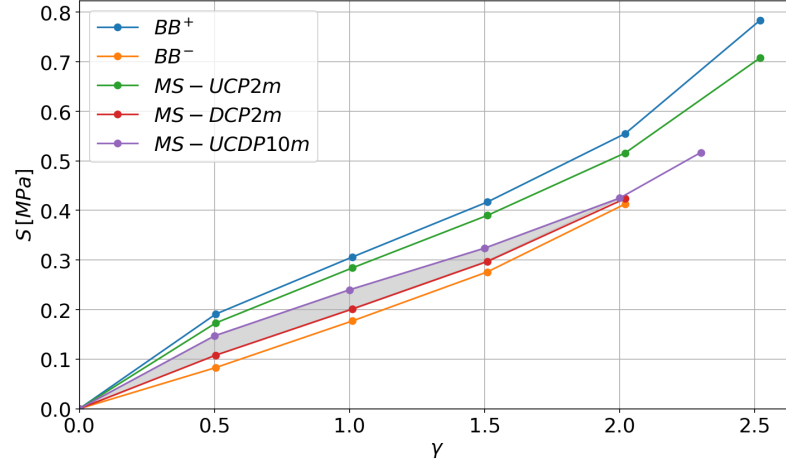


Figure 3.21: Results of different experiments on a compound A specimen.

In conclusion, the multi-step relaxation experimental procedure with oscillations and partial unloading (MS-UCDP) described in this Section has shown excellent results for the tested filled rubber samples. This procedure allows, within reasonable times, to obtain a good estimation of the elastic equilibrium points. For example, with a 2-minute relaxation segment, each equilibrium point requires just over 3 minutes, and the results differ by only a few percentage points from those of similar tests that takes hours.

Although it is not possible to determine rigorously how accurate the provided estimates are, the fact that the relaxation segments quickly assume and maintain a negative slope, albeit weak, suggests that these are upper-bound estimates of the equilibrium points. Therefore, it can be reasonably concluded that the actual elastic equilibrium points fall within the relatively narrow region between the upper MS-UCDP estimates and the lower BB- estimates, the latter possibly improved by the introduction of oscillations (MS-DCP).

A necessary improvement to this test is the determination of the elastic equilibrium stress response at γ_{max} , which is impossible to obtain with this method. Because of energy dissipation in hysteresis cycles, this point can only be reached from a greater value of stress, i.e. from a relaxation test.

The method presented in this section is used to point-wise determine the elastic equilibrium stress curves of all compounds, for different values of γ_{max} , and to be used in the FEM simulations of rubber isolators.

3.2.2 Equilibrium stress experimental curves

Using the relaxation acceleration method proposed in Section 3.2.1, the UCDP2m test was performed on several single pad shear specimens. For each rubber and each level of maximum strain γ_{max} , the elastic equilibrium stress curve coefficients C_{j0} were determined solving a least square problem based on the tests data. All values are reported in Table 3.2 and the fitted curves are compared to the recorded results in Figure 3.22, where only the last point of each relaxation step was plotted. These values are determined without considering the upper stress limit that relaxation

tests would impose. Hence, the values of stress for deformations higher than the ones recorded could carry greater uncertainty.

Table 3.2: Values of the elastic equilibrium stress curve coefficients determined from UC DP2m tests for all rubbers, valid for different levels of conditioning at strain γ_{max} .

Rubber	γ_{max} [%]	C_{10} [kPa]	C_{20} [kPa]	C_{30} [kPa]
A	100	191	-103	63
A	150	163	-41	12
A	200	140	-20	4
A	250	125	-12	1
B	100	300	-158	89
B	165	230	-60	15
B	200	196	-36	7
B	250	173	-23	3
C	100	365	-200	126
C	125	308	-140	68
C	135	283	-109	47
C	180	263	-67	16
C	250	241	-38	5
D	100	606	-411	276
D	150	466	-145	47
D	200	388	-74	14

Equilibrium stress curves as functions of γ_{max}

The trends of the C_{j0} coefficient values, shown in Figure 3.23, appear to follow regular patterns that resemble hyperbolic shapes. Curves fitted to the values of the determined coefficients with the function in Equation 3.2 are shown the same figure.

$$C_{j0}(\gamma_{max}) = \frac{a_j}{\gamma_{max}^{b_j}} \quad (3.2)$$

The shear equilibrium stress $S_{\infty,R}$ of each rubber compound R can be determined as function of the strain γ , the maximum strain reached γ_{max} and the best parameters a_j and b_j of the model function in Equation 3.2 that describe the trends of C_{j0} for $j = 1, 2, 3$.

$$S_{\infty,R}(\gamma, \gamma_{max}, a_j, b_j) = 2 \frac{a_1}{\gamma_{max}^{b_1}} \gamma + 4 \frac{a_2}{\gamma_{max}^{b_2}} \gamma^3 + 6 \frac{a_3}{\gamma_{max}^{b_3}} \gamma^5 \quad (3.3)$$

The expressions determined for rubber compounds A, B, C and D are

$$S_{\infty,A}(\gamma, \gamma_{max}) = 2 \left(\frac{0.192}{\gamma_{max}^{0.454}} \right) \gamma + 4 \left(\frac{-0.103}{\gamma_{max}^{2.308}} \right) \gamma^3 + 6 \left(\frac{0.063}{\gamma_{max}^{4.126}} \right) \gamma^5 \quad (3.4)$$

$$S_{\infty,B}(\gamma, \gamma_{max}) = 2 \left(\frac{0.301}{\gamma_{max}^{0.598}} \right) \gamma + 4 \left(\frac{-0.158}{\gamma_{max}^{2.057}} \right) \gamma^3 + 6 \left(\frac{0.089}{\gamma_{max}^{3.614}} \right) \gamma^5 \quad (3.5)$$

$$S_{\infty,C}(\gamma, \gamma_{max}) = 2 \left(\frac{0.348}{\gamma_{max}^{0.460}} \right) \gamma + 4 \left(\frac{-0.201}{\gamma_{max}^{1.849}} \right) \gamma^3 + 6 \left(\frac{0.127}{\gamma_{max}^{3.195}} \right) \gamma^5 \quad (3.6)$$

$$S_{\infty,D}(\gamma, \gamma_{max}) = 2 \left(\frac{0.606}{\gamma_{max}^{0.646}} \right) \gamma + 4 \left(\frac{-0.411}{\gamma_{max}^{2.532}} \right) \gamma^3 + 6 \left(\frac{0.276}{\gamma_{max}^{4.369}} \right) \gamma^5 \quad (3.7)$$

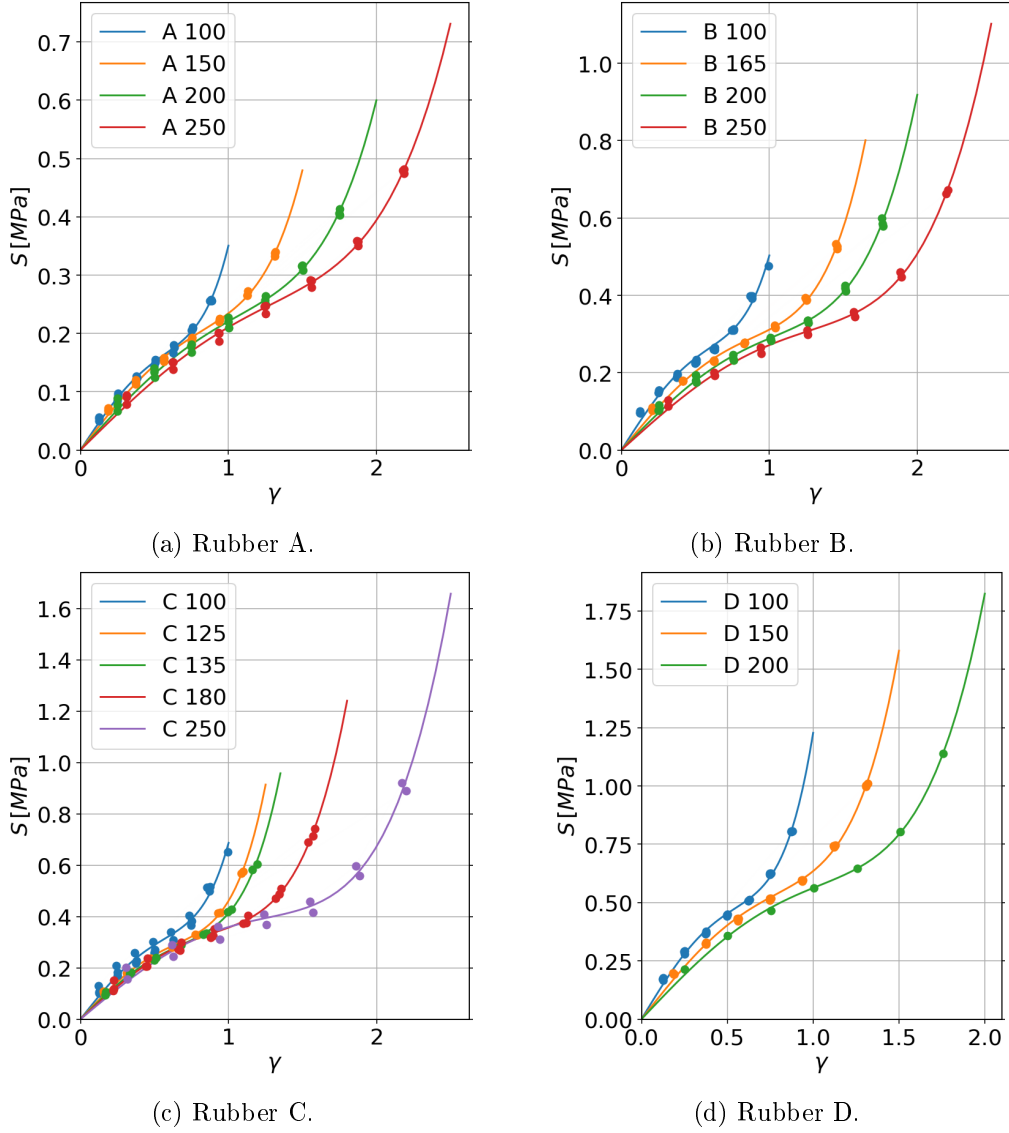


Figure 3.22: Equilibrium stress curves determined for all rubber compounds at different levels of maximum strain $\bar{\gamma}_{max}$ from results recorded through a UCDP2m test.

and could be used to determine the elastic coefficients of a rubber compound valid for a maximum strain that is not included in the performed tests.

An interesting result is obtained imposing $\gamma = \gamma_{max}$: in this case, the expression can be interpreted as the one corresponding to the first load equilibrium stress curve of a virgin specimen, i.e. the stress response during a quasi-static increasing deformation test of a previously undeformed rubber element. The generic expression for this curve, in Equation 3.8, assumes the same form of the one for simple shear in Equation 2.41, with different values as exponents.

$$S_{\infty,R}(\gamma = \gamma_{max}, a_j, b_j) = 2a_1\gamma^{1-b_1} + 4a_2\gamma^{3-b_2} + 6a_3\gamma^{5-b_3} \quad (3.8)$$

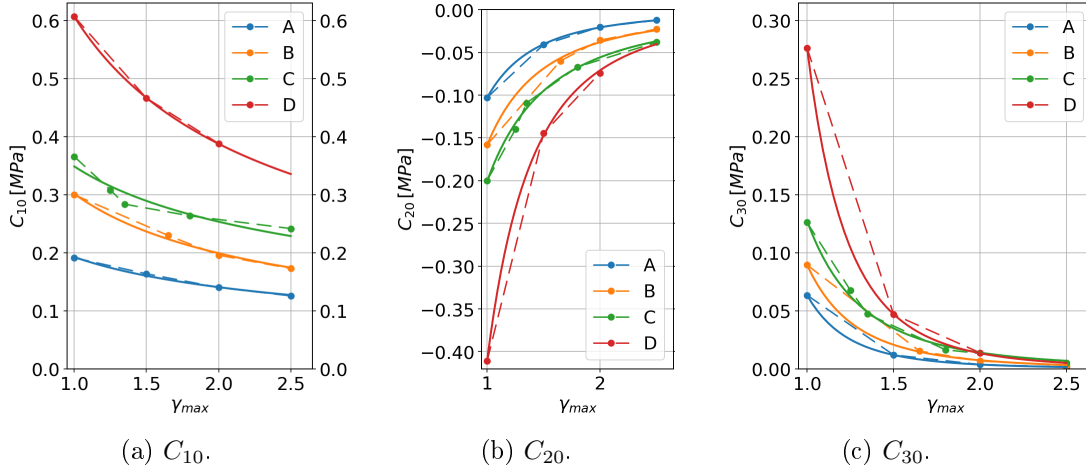


Figure 3.23: Trends of the elastic constants from data recorded with a UCDP2m test and fitted curves.

The expressions obtained for each rubber compound are

$$S_{\infty,A}(\gamma = \gamma_{max}) = 2 \cdot 0.192\gamma^{0.546} + 4(-0.103)\gamma^{0.692} + 6 \cdot 0.063\gamma^{0.874} \quad (3.9)$$

$$S_{\infty,B}(\gamma = \gamma_{max}) = 2 \cdot 0.301\gamma^{0.402} + 4(-0.158)\gamma^{0.943} + 6 \cdot 0.089\gamma^{1.386} \quad (3.10)$$

$$S_{\infty,C}(\gamma = \gamma_{max}) = 2 \cdot 0.348\gamma^{0.540} + 4(-0.201)\gamma^{1.151} + 6 \cdot 0.127\gamma^{1.805} \quad (3.11)$$

$$S_{\infty,D}(\gamma = \gamma_{max}) = 2 \cdot 0.606\gamma^{0.354} + 4(-0.411)\gamma^{0.468} + 6 \cdot 0.276\gamma^{0.631} \quad (3.12)$$

and their curves are compared to the elastic equilibrium stress curves of Figure 3.22 in Figure 3.24. For compounds A and D, the first load equilibrium stress curves seem to follow the trend of the maximum stress on the equilibrium stress curves. In compounds B and C, the curves diverge at the highest deformations because of the exponents greater than one in the last term, possibly due to their significant viscous component. It is important to remark that these curves are obtained from tests performed at strains close but not equal to γ_{max} . Thus, the first-load elastic equilibrium curve could be determined with more accuracy if some information was available at the maximum deformation, e.g. the stress value at the end of a relaxation test, which imposes an upper boundary to the elastic equilibrium stress.

3.3 Design of a single pad specimen for simple shear tests

The first relaxation tests on rubber were performed on dog-bone specimens, in Figure 3.25, deformed in uniaxial tension, as described in Section 3.1.2, but the results of fitting and FEM simulated cyclic tests on isolators were poor. The main cause was attributed to the slow loading rate, as shown in Figure 3.4.

Simple shear tests were then performed on standard quadruple specimens (Section 3.3.1), having dimensions 40 mm × 40 mm × 10 mm according to UNI EN 15129 standard, represented in Figure 3.26. Due to the specimen thickness, the machine was unable to follow the imposed load histories during the cyclic and relaxation tests because of the high stroke required to reach the maximum imposed deformation. For example, the displacement needed to reach the maximum shear strain of $\gamma_{max} = 1$, equal to 20 mm, was reached by the machine in 0.70 s.

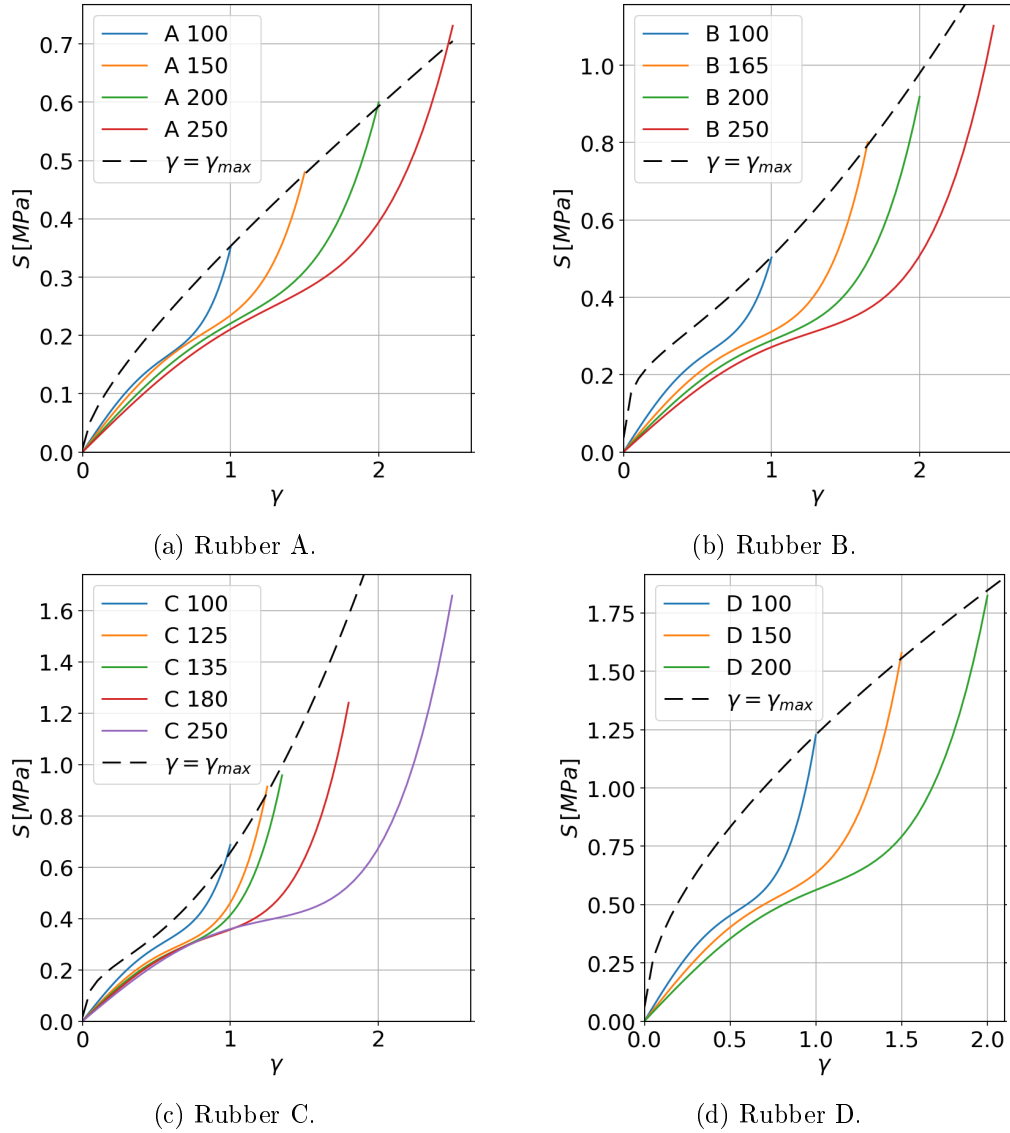


Figure 3.24: Equilibrium stress curves determined for all rubbers at different levels of maximum strain $\bar{\gamma}_{max}$ from data recorded with a UCDP2m test, compared to the first-load elastic curve determined through fitting of the elastic coefficient.

To solve this issue, a specimen with thinner rubber pads would be necessary. Consequently, after preliminary analytical and numerical studies (Sections 3.3.2 to 3.3.5), firstly a quadruple and then a single specimen were designed and manufactured using rubber sheets glued to steel plates, as described in Section 3.3.6. With this specimen a short duration of the load ramp is attained.

3.3.1 Quadruple rubber element shear test standard specimen

The standard specimen for simple shear tests is composed by 4 pieces of rubber of dimensions 40 mm $\times$ 40 mm $\times$ 10 mm disposed symmetrically and connected by steel plates, as in Figure 3.26. The scheme can be resumed by a series connection of two couples of rubber pieces disposed in parallel.

The adhesion between steel and rubber along the contact surface develops during

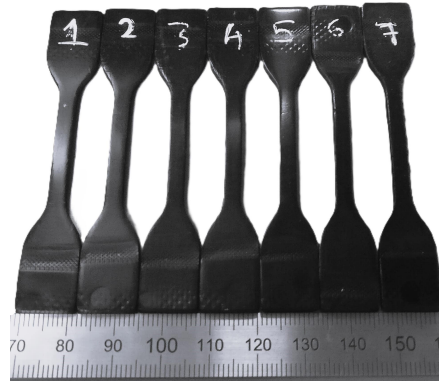


Figure 3.25: Front view of tested dog-bone rubber specimens.



Figure 3.26: Lateral view of the quadruple pad standard specimen.

the vulcanization process of the whole piece. The resulting specimen can have geometry defects, such as uneven thickness or bulges of the rubber, misalignment and rotation of the plates due to the production process.

The advantage of using quadruple specimens is their symmetry in the deformation direction, which translates in auto-equilibrated bending moments due to shear deformation, so that the clamps only need to apply a tension or compression force. Also, the pads faces bonded to the steel plates remain parallel to each other and, if all rubber pads have the same thickness, the central steel plates are aligned for clamping.

Experimental tests and numerical simulation show that the deformation field of these specimens is a mix of shear and flexural deformation, as shown in Section 3.3.2, due to the shape factor length/thickness equal to 4. In Figure 3.28 the effect of shrinkage of the front face (see Figure 3.27 for reference) of a shear deformed rubber pad is shown, highlighted by the curved edge between the front and lateral faces. This effect is due to the tension stresses developing on and near the surface perpendicular to the direction of deformation.

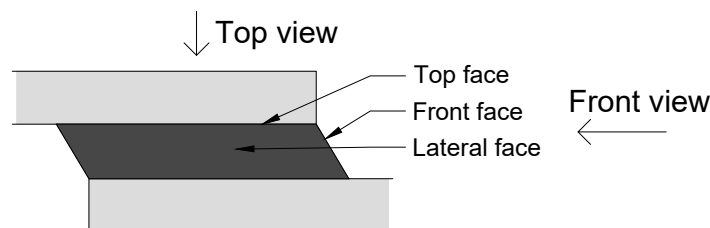


Figure 3.27: Lateral view of the rubber pad, illustrating the orientation and labeling of different perspectives.

In Figure 3.29 a single piece of rubber with a painted white grid is presented in the undeformed state and in two deformed states of different amplitudes. It is clear

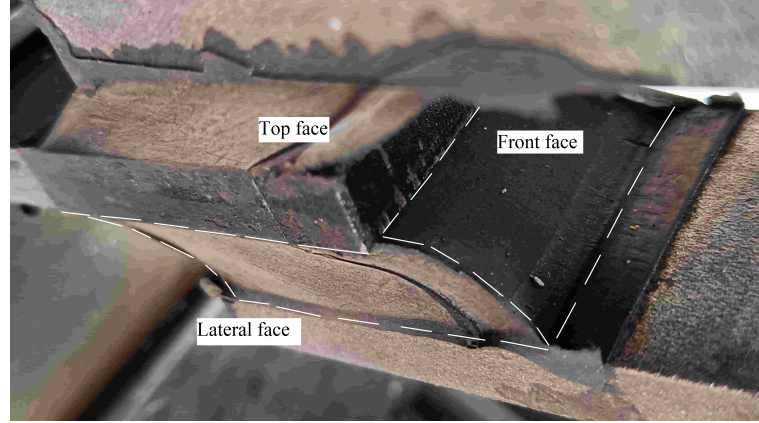


Figure 3.28: Shrinkage of a face of one rubber pad of the quadruple specimen.

that in the central part of the rubber pad the deformation is of simple shear, while bending prevails near both inclined faces and extends towards the center for a length of about the same magnitude of the thickness.

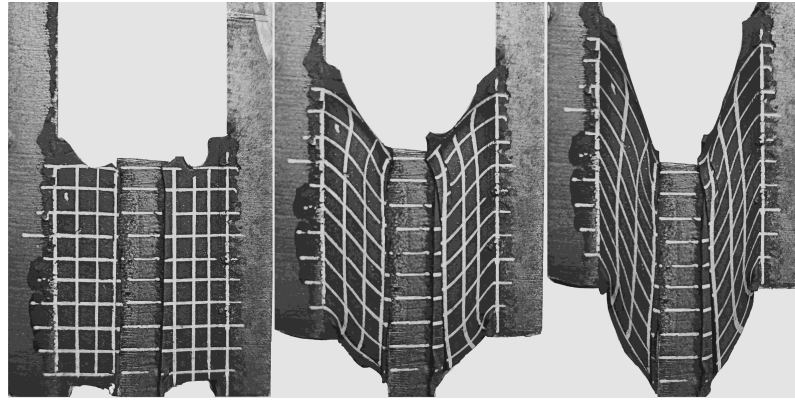


Figure 3.29: Lateral view of two rubber pieces of quadruple specimen at $\gamma = 0$, $\gamma \simeq 1$ and $\gamma \simeq 2$, respectively.

3.3.2 Stress distribution on a simple shear deformed rubber pad

In order to apply the theory presented in Section 2.1.1 for an infinitesimal element to the fitting of experimental data, the laboratory tests should replicate as best as possible the simple shear deformation [37,38].

The surface in contact with the lateral plate in a rubber pad on the quadruple specimen is free to move in the direction orthogonal to its plane. During tests on quadruple specimens it was visible by eye the transversal movement of the lateral plates towards the central axis of the specimen, so doubts that deformation did not follow the definition of simple shear arose.

A FEM simulation of one rubber pad of the quadruple specimen has been conducted to assess its stress distribution. Another simulation was performed on the same pad with a thickness reduced to 2.5 mm. In Figure 3.30 views of both models are shown. Only half of the prism was considered, due to symmetry in the XZ plane, with a normal displacement constraint applied to the symmetry face. One base was fixed, while a linear increasing deformation was applied to the opposite

face, along with a rigid body (surface) constraint. 4 nodes tetrahedrons elements were used and mesh refinement was applied to the free surfaces to better capture their deformation. Adaptive meshing was employed to handle large deformations, as the elements could become highly distorted during the loading. The rubber elastic constitutive law was described using the Yeoh model with the material constants $C_{10} = 0.2 \text{ MPa}$, $C_{20} = -0.02 \text{ MPa}$ and $C_{30} = 0.002 \text{ MPa}$, which are representative of a rubber conditioned at $\gamma_{max} = 2.5$.

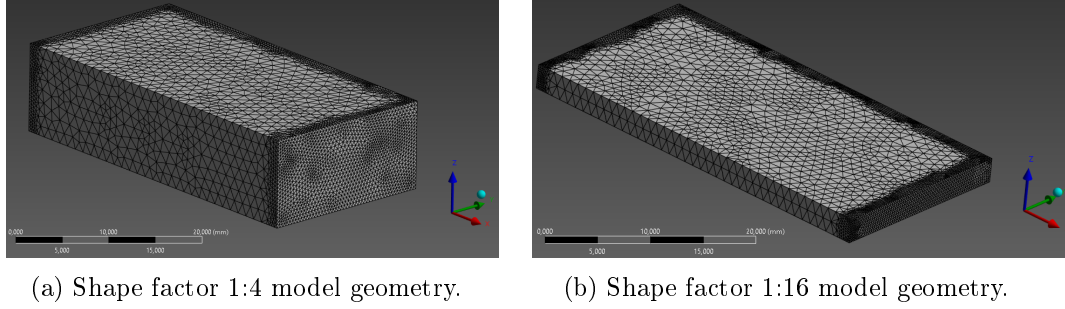


Figure 3.30: FEM geometry models of rubber elements with different shape factors.

In Figure 3.31 the deformation of the corner edge of the rubber pad is shown. Its shape is very similar to that of the experimental test in Figure 3.28, which resembles the Poisson effect that occurs when a specimen is tensioned. The same shape is obtained with the same model where the top surface is free to move in its orthogonal direction Z, which is what happens in the case of the quadruple specimen.

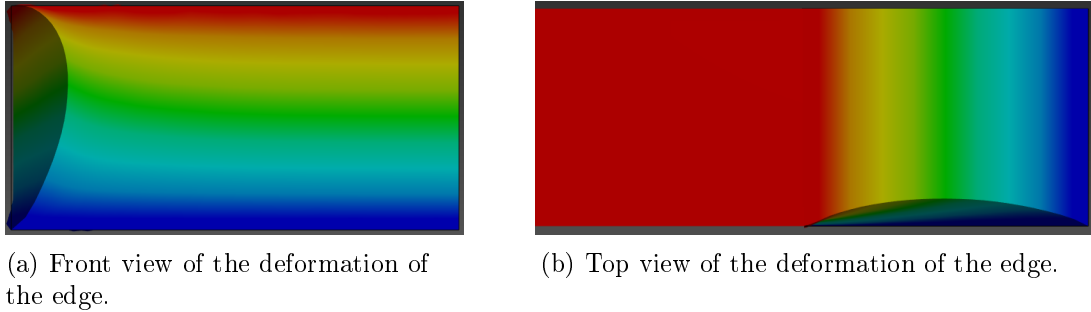


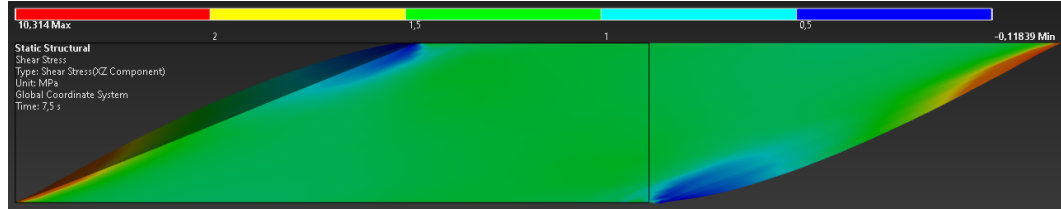
Figure 3.31: Edge deformation on a simple shear deformed rubber pad with shape factor 1:4.

In Figure 3.32 the experimental observations are confirmed by the stress field in the symmetry plane of the specimen. Constant simple shear is only present in the central part of the prism, with peaks of shear stress at the corners (Figure 3.32a). In Figure 3.32b the normal stress in Z direction is shown, highlighting that a combination of normal and shear stresses is present from the inclined faces towards the center or the rubber pad, for a depth of about that of its thickness.

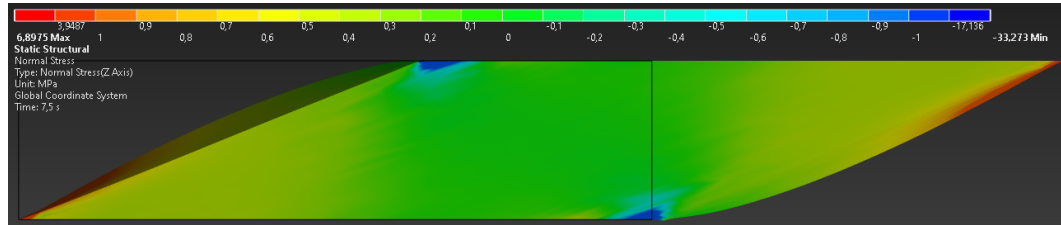
The peaks of stress at the corners, especially the tension stress, are not realistic for large deformations like in this case and only exist in the elastic model because damage is not accounted for. In experimental tests, failure on rubber or at the bonding surface between rubber and steel happens. In the second case, this damage is usually localized and the reduction in the shear area is negligible, i.e. it does not propagate towards the center of the contact surface and does not affect the shear

stiffness.

As stated by Yeoh [34] and Rivlin [39], “... *in addition to the tangential components of stress, normal components of stress are associated with simple shear. These normal components of stress do not have much effect on the overall load/deformation behavior and are ignored*”. In his article, Yeoh was discussing the stress distribution in a quadruple specimen composed by rubber blocks of dimensions $25 \times 25 \times 6$ mm with a shape factor of about 1:4.



(a) Shear stress distribution.

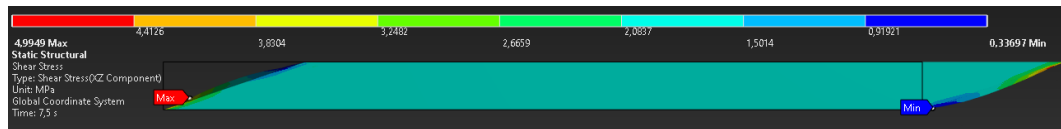


(b) Normal stress perpendicular to the mid-plane distribution.

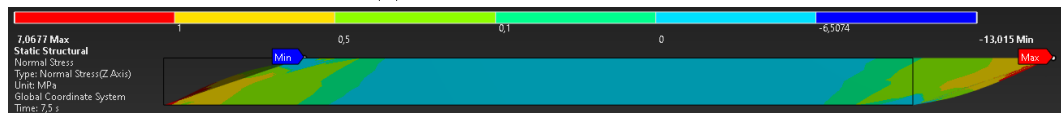
Figure 3.32: Shear and normal stress distribution on a simple shear deformed rubber pad with shape factor 1:4.

The same rubber pad with a thickness of 2.5 mm was modeled in the same way. Figures 3.33a and 3.33b show the distribution of stresses similar to those of Figure 3.32: a constant shear in almost all the pad, except for the corners, and normal stress near the inclined faces for a depth comparable to the thickness of the prism. Since the simultaneous presence of normal stress and shear stress only occurs at the ends of the specimen, in the specimen with a smaller thickness, the deformation should be closer to simple shear.

The free edge of the specimen, in Figures 3.34, seems to deform in a similar way to that of the higher specimen, in Figures 3.31. However, this distortion happens in a small portion of the former, compared to the latter.



(a) Shear stress distribution.



(b) Normal stress perpendicular to the mid-plane distribution.

Figure 3.33: Shear and normal stress distribution on a simple shear deformed rubber pad with shape factor 1:16.

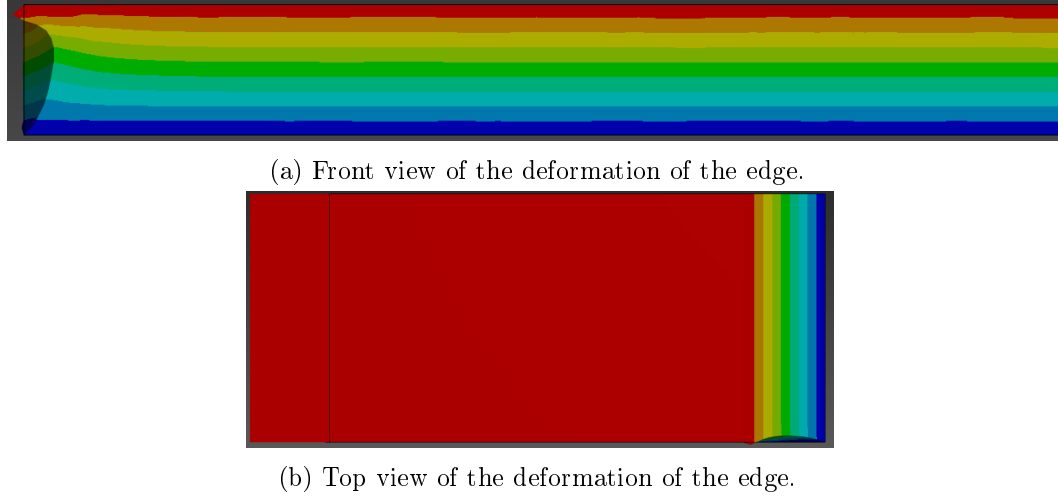


Figure 3.34: Edge deformation on a simple shear deformed rubber pad with shape factor 1:16.

3.3.3 Effect of the shape factor on the elastic equilibrium stress

The shape factor (SR) represents the ratio between the thickness of the specimen and its base length parallel to the direction of deformation. The influence of the shape factor on the elastic response of the rubber deformed until a maximum shear of $\gamma_{max} = 250\%$ was investigated numerically with FEM simulations of two specimens with aspect ratios 1:4 and 1:16. The same settings of the previous Section 3.3.2 were used for the analyses.

The stress-strain curve of each sample is shown in Figure 3.35, along with the theoretical elastic equilibrium curve (Equation 2.41). The trends of all curves are very similar to each other and they almost overlap. The curves of the relative errors between the stress-strain curves from the FEM models and the theoretical stress, also reported in Figure 3.35, show that the 1:4 SR specimen deviates from the theoretical stress by not more than about 5 % and reducing the SR to 1:16 reduces the error in a wider range of shear strains. This result confirms the observations on the stress distribution of the previous section and the deviation from the elastic equilibrium curve is presumably due only to the shape factor of the specimen.

If a 5 % deviation from the theoretical stress is acceptable, then adopting a lower SF and reducing such deviation is an advantage, because it reduces uncertainties and ensures that the deformation imposed during laboratory tests is close to the theoretical simple shear.

3.3.4 Effect of defects on the elastic equilibrium stress

Deviation from a simple shear deformation could happen if the faces of the pads are not parallel to the opposite ones, if the pads are compressed or in tension or if their thickness is not even. FEM models of the specimens with SF 1:4 and 1:16 were created with such imposed defects and a shear deformation was imposed. The settings of the simulations were the same of the ones described in Section 3.3.3.

In Figure 3.36 the resulting stress response are compared to the theoretical elastic equilibrium stress. Leaving the top surface free to move ($F_z = 0$) or having an uneven thickness of the rubber (Δt) has an influence on the stress response but it is negligible,

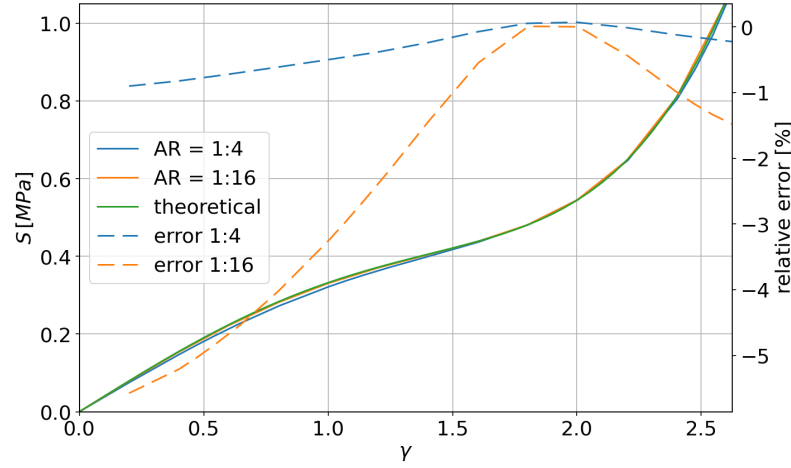
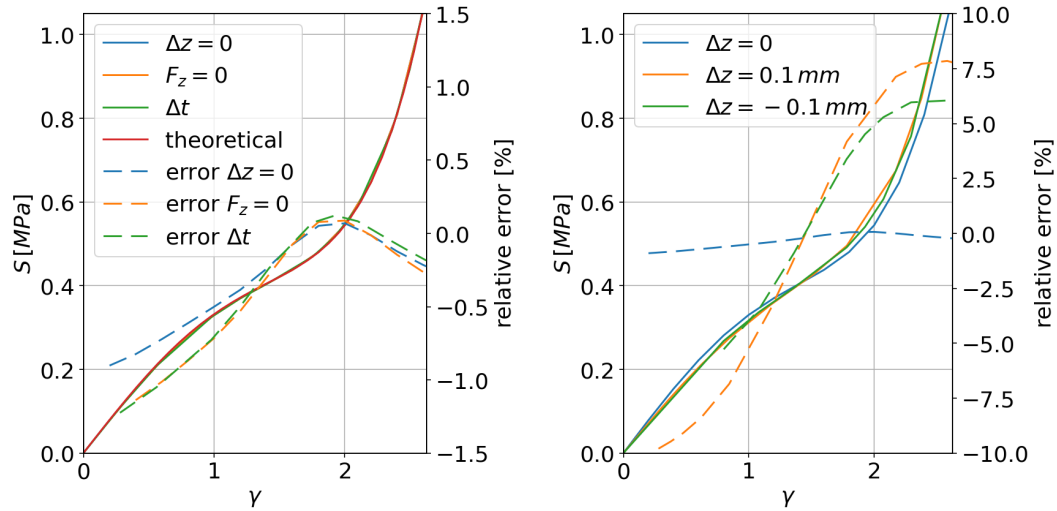


Figure 3.35: Theoretical and FEM simulated shear stress - shear strain curves and errors relative to the theoretical curve of rubber pads with different shape factors.

as the error compared to the theoretical equilibrium stress is contained around 1%. The worst effect on the elastic curve is given by an imposed uniform displacement $\Delta z \neq 0$, which causes compression or tension in the direction orthogonal to the mid-plane of the pad, and which leads to a maximum error that reaches 10%, compared to the theoretical stress.

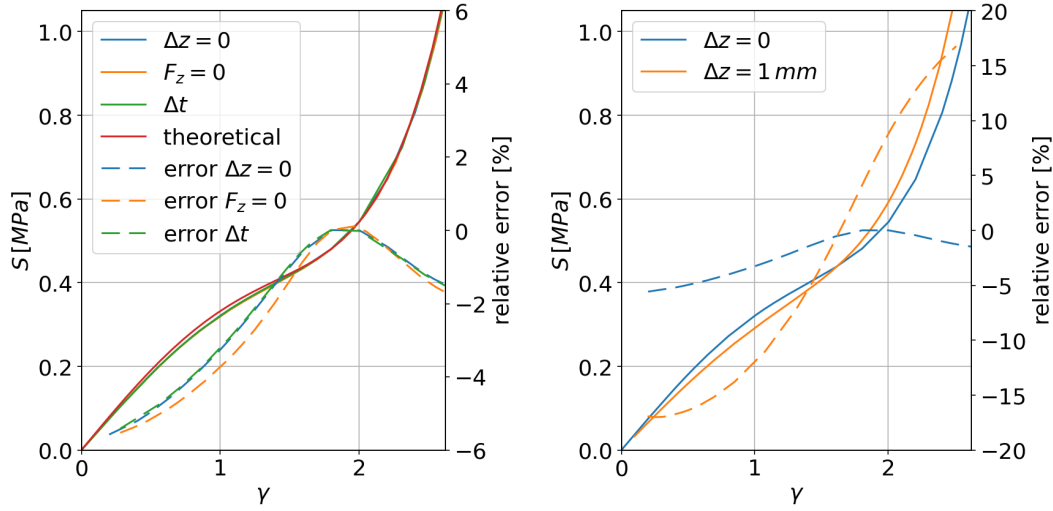
In Figure 3.37, curves obtained for the rubber pad with SF 1:4 are shown. The trend are similar and also in this case the main deviation happens for an imposed value of Δz . The error in this case is greater because the imposed displacement is greater in proportion, compared to the case of SF 1:16.



(a) Simple shear ($\Delta z = 0$), free surface ($F_z = 0$) and uneven thickness of the pad ($\Delta t = \pm 0.05$ mm).

(b) Different imposed values of Δz .

Figure 3.36: Stresses and errors relative to the theoretical elastic equilibrium stress curve of rubber pad of SF 1:16 with defects.



(a) Simple shear ($\Delta z = 0$), free surface ($F_z = 0$) and uneven thickness of the pad ($\Delta t = \pm 0.05$ mm).

(b) Different imposed values of Δz .

Figure 3.37: Stresses and errors relative to the theoretical elastic equilibrium stress curve of rubber pad of SF 1:4 with defects.

3.3.5 Lateral bulge of a compressed sheet

A sheet of compressed rubber exhibits a bulge on its lateral free faces and the height of the bulge is amplified relative to the uniform displacement applied perpendicular to the plane of the sheet. The surfaces where compression is applied are constrained to in-plane deformation. Some analytical estimation of the bulge height of a compressed disc are made in the following, with the simplifying hypothesis of incompressibility of the rubber, assuming a surface shape described by a parabola, a monomial of exponent n or by a circular arc. Analytical estimations are then compared with the bulge height in two FEM simulations of a compressible rubber disc, with an incompressibility modulus $\kappa = 10^{-3} \text{ MPa}^{-1}$ and of a rectangular sheet of the same thickness and of about the same area.

A simple estimation of the bulge height

The volume of an incompressible rubber disc of diameter D and thickness t has to remain constant when a uniform displacement Δt is imposed orthogonally to its circular surfaces. The equality of the volume in the undeformed initial state, V_u , and the deformed state V_d is

$$V_u = \frac{\pi}{4} D^2 t = \frac{\pi}{4} D^2 (t - \Delta t) + B\pi (D + 2s) = V_d \quad (3.13)$$

where B is the cross-section area of the bulge and s is the distance of its centroid from the undeformed lateral surface. Assuming that the bulge shape is a parabola and neglecting s , we can obtain an approximated expression of the bulge maximum height h depending on the strain $\epsilon_t = \Delta t/t$

$$\frac{h}{D} = \frac{3}{8} \frac{\epsilon_t}{1 - \epsilon_t} \quad (3.14)$$

A similar argument can be made for the rectangular based prism of sides length a and b , with the significant simplification that the shape of the bulge is constant on its perimeter, and obtain an expression similar to Equation 3.14.

$$\frac{h}{\frac{2ab}{a+b}} = \frac{3}{8} \frac{\epsilon_t}{1 - \epsilon_t} \simeq \frac{3}{8} \frac{\Delta t}{t} \quad (3.15)$$

which, for small deformations of a prism of sides $a = 49 \text{ mm}$, $b = 59 \text{ mm}$ and $t = 2 \div 3 \text{ mm}$, gives an estimation of the lateral bulge height h included between 6 and 10 times the uniform displacement Δt of the rubber. Note that, on the left side, h is divided by the harmonic mean of the length sides of the prism in place of the diameter of the disc. Also, if $a = b = D$ the estimations given for a disc would be the same as those for its circumscribed squared based prism.

Monomial surface shape

Setting a local coordinates system with origin in the middle of the lateral surface of the rubber sheet and the z axis oriented parallel to the surface, as in Figure 3.38, by the symmetry of the problem we can consider only the positive values of $z \in [0; k]$. If the shape of the bulge is described by a function $b(z)$

$$b(z) = h \left[1 - \left(\frac{z}{k} \right)^n \right] \quad (3.16)$$

where $k = (t - \Delta t)/2$, the volume of the deformed disc is given by

$$V = 2 \int_0^k \frac{\pi}{4} [D + 2b(z)]^2 dz \quad (3.17)$$

$$= 2 \frac{\pi}{4} \int_0^k \left\{ D + 2h \left[1 - \left(\frac{z}{k} \right)^n \right] \right\}^2 dz \quad (3.18)$$

$$= \frac{\pi}{4} \left[D^2 + 4D \frac{n}{n+1} h + 4 \left(1 - \frac{2}{n+1} + \frac{1}{2n+1} \right) \right] 2k \quad (3.19)$$

Equating it to the expression of the undeformed disc, the expression of the height of the bulge can be derived

$$\frac{h}{D} = \frac{2n+1}{4n} \left(\sqrt{1 + 2 \frac{n+1}{2n+1} \frac{\epsilon_t}{1 - \epsilon_t}} - 1 \right) \quad (3.20)$$

Circular arc surface shape

With the same assumption as the previous case, from the equation of a circle of center with coordinates $(z = 0, x = -(R - h))$ and a radius that satisfies the equation

$$R^2 = (R - h)^2 + k^2 \quad (3.21)$$

we can obtain the circular arc surface shape described by the function

$$b(z) = h - R + \sqrt{R^2 - z^2} \quad \text{for} \quad z \in [0; k] \quad (3.22)$$

where, from Equation 3.21,

$$R = \frac{k^2 + h^2}{2h} \quad (3.23)$$

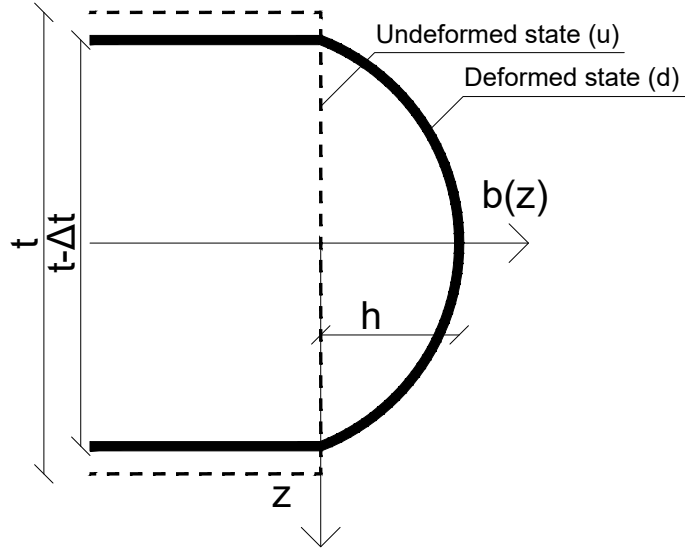


Figure 3.38: Reference system for the bulge problem.

and the volume of the deformed disc

$$V_d = 2\frac{\pi}{4} \int_0^k \left(D + 2 \left(h - R + \sqrt{R^2 - z^2} \right) \right)^2 dz \quad (3.24)$$

$$= \frac{\pi}{2} k \left\{ \left[(D + 2h - 2R)^2 + 4R^2 + \frac{4k^2}{3} \right] + \right. \\ \left. + (D + 2h - 2R) (2k)^2 \left[\left(1 + \left(\frac{R-h}{k} \right)^2 \right) \arcsin \frac{k}{R} + \frac{R-h}{k} \right] \right\} \quad (3.25)$$

which are only valid until $h \leq R$. Beyond this range, the function $b(z)$ is no longer bijective. Equating this expression to the initial volume leads to a transcendental equation in the variable h .

Hence, a numerical different solution is approached, which is valid also when $h \geq R$. The non-linear equation is obtained from Equation 3.13 imposing B as the area of the circular segment of height h and s as its centroid

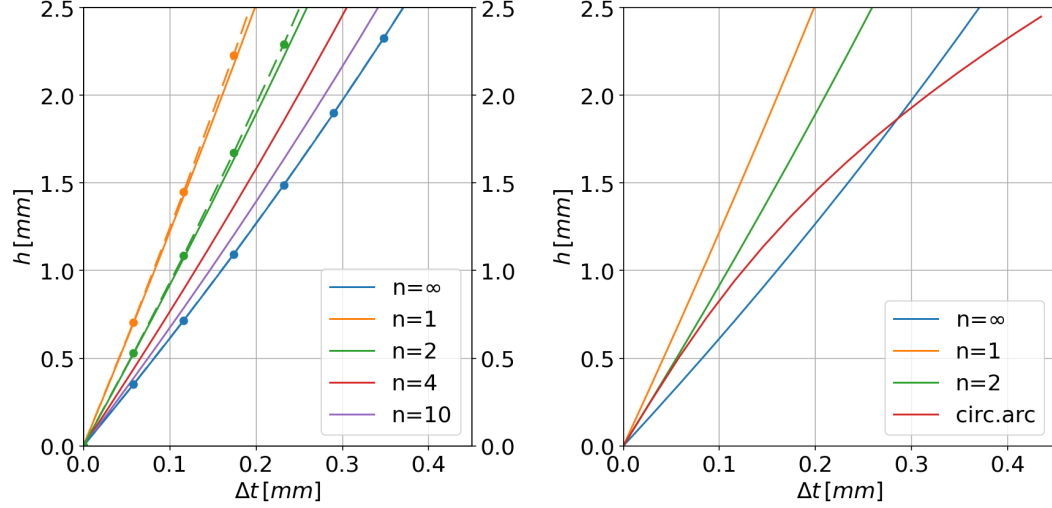
$$B = \frac{h^2 + k^2}{2h} \arccos \left(1 - \frac{2h^2}{h^2 + k^2} \right) + \frac{h^2 - k^2}{2h} k \quad (3.26)$$

$$s = \frac{2k^3}{3B} + \frac{h^2 - k^2}{2h} \quad (3.27)$$

and it is solved numerically to determine the trend of h for increasing values of Δt .

The trend of the estimated values of h at different levels of uniform displacement of a disc of diameter $D = 55$ mm and thickness $t = 2.32$ mm is shown in Figure 3.39. It can be noted that all curves derived from a monomial expression of grade n have an increasing slope while the curve from circular arc shaped bulge has a softening branch, due to the expansion of the bulge in the direction perpendicular to the plane

of the disc for large deformations. All these are upper estimations, as volumetric compressibility is not accounted for. Attempts were made to measure the bulge height during these laboratory tests. However, the results were deemed unreliable due to the specimen small dimensions and irregular surface.



(a) Bulge estimations with grade n monomial bulge section shapes; exact and approximated estimations. (b) Bulge estimations with monomial or circular arc bulge section shapes.

Figure 3.39: Bulge height theoretical estimation with respect to the imposed uniform displacement for a compressed disc with different bulge shapes.

Numerical estimation of the bulge height

Two FEM models were made to make a better estimation of the lateral bulge for the case of the rectangular based compressed sheet. One is a model of a disc of diameter $D = 55$ mm, to confirm the estimations made previously, and the other model is of a rectangular prism of sides 49 mm and 59 mm, both with a thickness $t = 2.32$ mm. The diameter of the disc was chosen close to the harmonic mean of a and b so that the estimation of the bulge of Equations 3.14 and 3.15 were similar. In the FEM simulation of the disc, axial symmetry was exploited by modeling only a circular sector of the disc. Additionally, the symmetry with respect to the mid-plane was exploited, further simplifying the geometry to just one-half of the circular sector. On the lateral faces, where the disc was sectioned to obtain the sector, boundary conditions were applied to restrict displacement normal to these surfaces. For the rectangular prism model, a quarter of the specimen was modeled due to its two axes of symmetry, while symmetry with respect to the mid-plane was also utilized. This approach significantly reduced model complexity and computational time while maintaining accuracy. To accurately capture large deformations, an adaptive meshing approach was employed. This method allowed for mesh refinement in areas of significant deformation and solid element distortions. The uniform displacement was increased linearly up to about 17% ($\Delta t = 0.4$ mm) and the reaction force was calculated through integration of the nodal reactions at the base.

The 3D model view of the deformed disc is shown in Figure 3.40 with a color map of the total radial displacement when the uniform displacement is maximum,

$\Delta t = 0.4$ mm. The border of the bulge is not where the maximum displacement happens, probably due to a tangential stretching of the rubber. The final volume is about -7.5% of the initial volume.

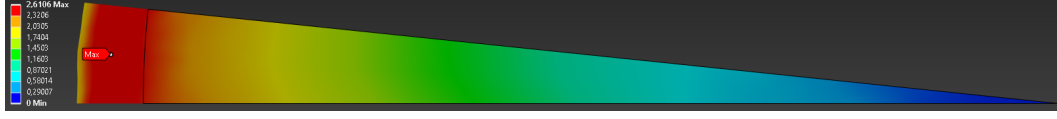
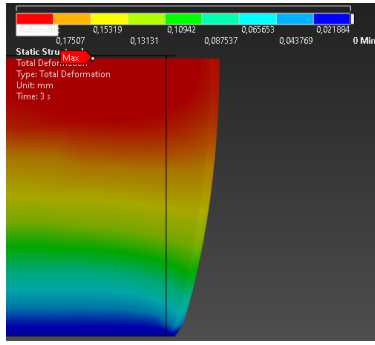
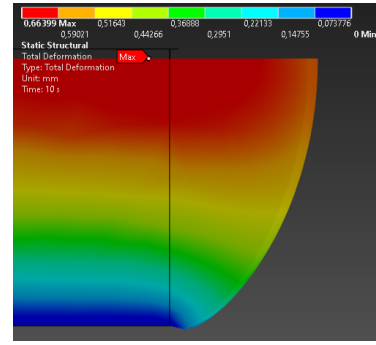


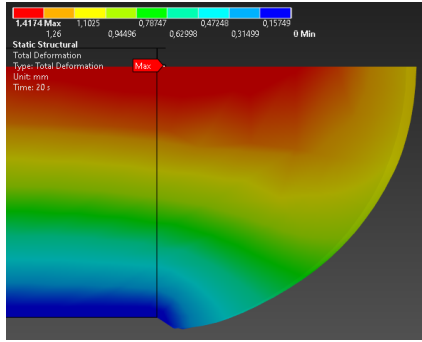
Figure 3.40: 3D model view of a deformed disc. Color map of the total displacement at $\Delta t = 0.4$ mm.



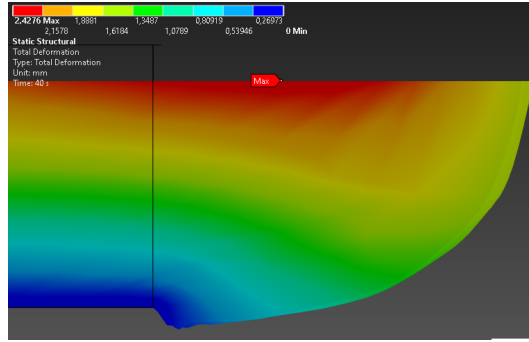
(a) $\Delta t = 0.03$ mm.



(b) $\Delta t = 0.09$ mm.



(c) $\Delta t = 0.17$ mm.



(d) $\Delta t = 0.35$ mm.

Figure 3.41: Half bulge shape for increasing values of Δt in a compressed disc. Black line shows the undeformed shape.

The shape bulge at the mid point of the sides of the rectangular based compressed prism is similar to that of the disc. Hence, the simplification of a constant bulge shape and height previously made to obtain the expression in Equation 3.15 is not correct. The top view of the mid surface of the modeled rubber pad is shown in Figure 3.42 for $\Delta t = 0.35$ mm because of unconverged solution in the following step and failed attempts to continue the analysis. From this figure, it can be seen that the bulge height is not constant along the sides of the prism, and the maximum is recorded at the mid point of the side of major length, although both directions show similar amplitudes. The final volume is about -4.9% of the initial volume, in proportion lower than the case of the disc.

In Figure 3.43 analytical and FEM analysis estimations are compared. For small deformations (Figure 3.43b), the trends of all curves are close, indicating that the simplest formula (Equation 3.15) provides a good estimation of the compression, comparable to the more refined model. As the deformation increases, non-linear

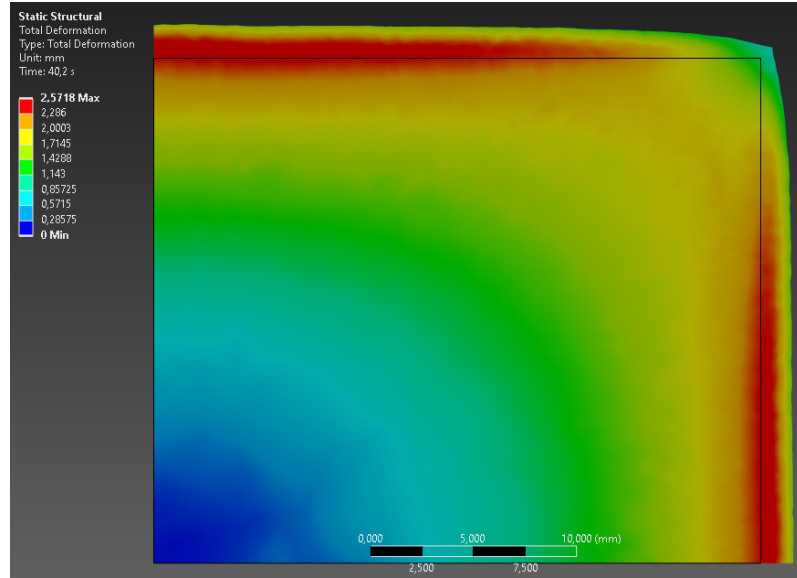


Figure 3.42: Top view of the mid surface of a quarter of compressed rectangular sheet. Color map of the total displacement at $\Delta t = 0.35$ mm.

effects become evident, and the estimation of the arc shaped bulge closely matches the FEM model of the disc. At the highest deformations, the compressibility of rubber limits the bulge height of the FEM model, a factor not captured by the analytical expression and the possible cause of divergence between the circular arc model and the FEM model.

3.3.6 Single rubber element shear test specimen

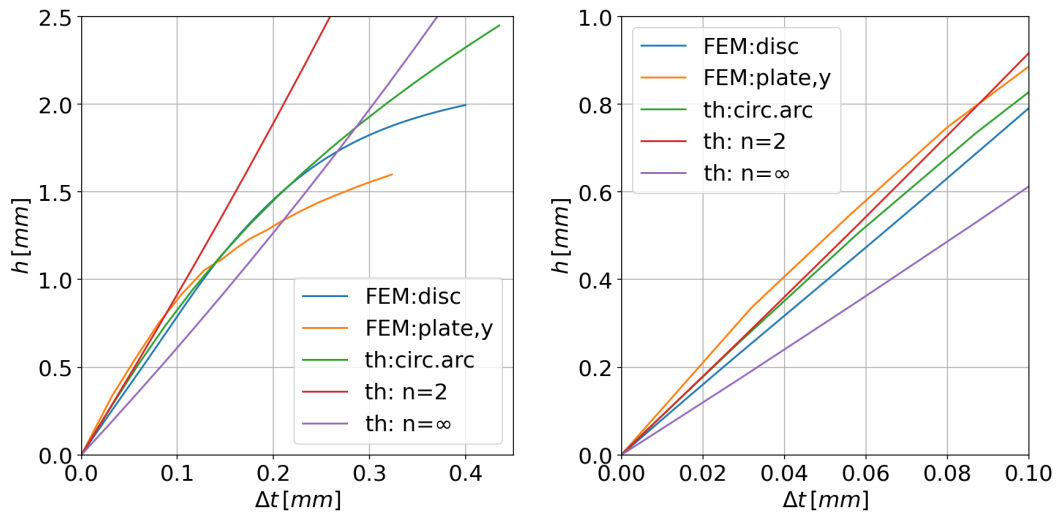
A new specimen with low thickness and shape factor was designed. The specimen is made by vulcanized rubber sheets, which are usually produced to cut out dog-bone specimens for the axial tension tests, glued to steel plates using slow-drying cyanoacrylate.

Initially, several quadruple specimens with rubber pads measuring 50×50 mm, with a medium thickness of about 2.4 mm, were manufactured and tested. An example with white painted lines on the lateral surface is shown in Figure 3.44. The high shape factor induces a simple shear deformation field in most part of the rubber pads of the specimen, as it is shown in Figure 3.44b, where the white lines of the grid remain substantially parallel to each other during deformation, except for a low extended disturbance at the ends of the rubber. In the two rubber pads on the right, the adhesion between the rubber and the steel plate in the central part has failed, as indicated by the breakage of some of the white lines in this area. This because, unfortunately, the production process involved multiple steps that required precise gluing and alignment, which proved challenging to achieve consistently.

To minimize the likelihood of defects in the specimens, the number of gluing steps was reduced to the minimum necessary. The result is a single specimen constituted by a sandwich of two steel plates and one central rubber sheet, as the one shown in Figure 3.45.

In Figure 3.46 the scheme of deformation of the specimen is represented. To guarantee that it is followed as close as possible, adequate supports must be designed.

The first prototype, shown in Figure 3.47, was a 40×50 mm rectangular base



(a) Theoretical and FEM simulation estimations of a disc and of a rectangular sheet lateral bulge. (b) Theoretical and FEM simulation estimations of the bulge at small deformations.

Figure 3.43: Bulge height theoretical estimations with respect to the imposed uniform displacement compared to FEM simulation estimations in a disc and in a rectangular sheet.



(a) Undeformed.



(b) Deformed.

Figure 3.44: Prototype of the quadruple specimen in the undeformed and deformed state.

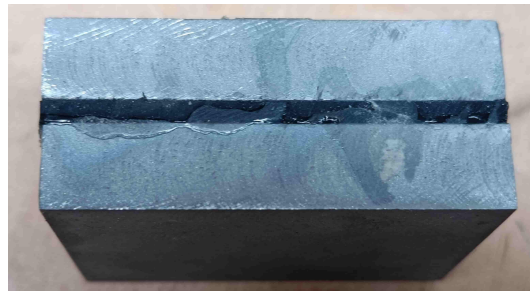


Figure 3.45: Single specimen sandwich composed by two steel plates and a central rubber layer.

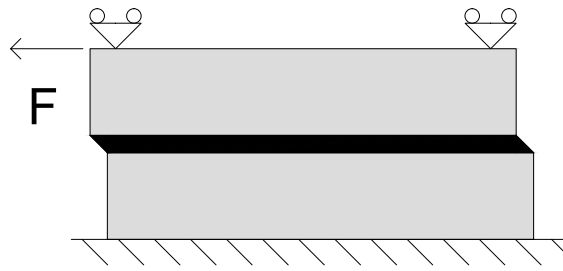


Figure 3.46: Scheme of the single specimen in the simple shear deformed state.

specimen held in place by two steel supports. The initial connection was made using two screws on the top of the supports, which pressed the head of the plate against the support to prevent slippage in the direction of the imposed displacement. The friction force developed between the support and the plate was insufficient to counteract the couple that produces a rotation of the sandwich, as shown in Figure 3.48. The connection was then improved with the use of 4 screws positioned on the face of each steel plate directly in contact with the support.

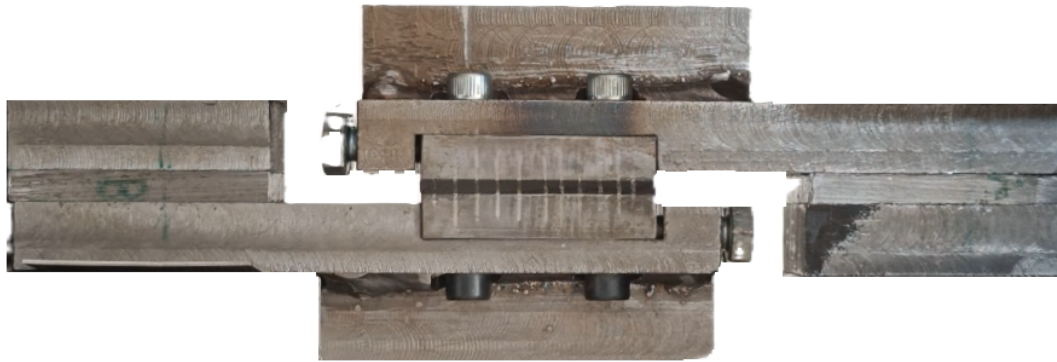


Figure 3.47: Single specimen sandwich connected to the supports.

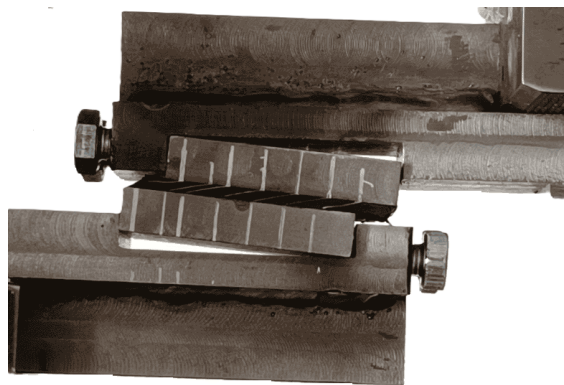


Figure 3.48: Single specimen sandwich rotation due to insufficient constraint.

The single specimen, undeformed and deformed, is shown in Figure 3.49. Also in this case, the white painted lines show that the deformation is of simple shear.

In the final version of the single specimen, presented in the next section, stiffer supports and a friction connection to the steel plates of the sandwich are adopted.

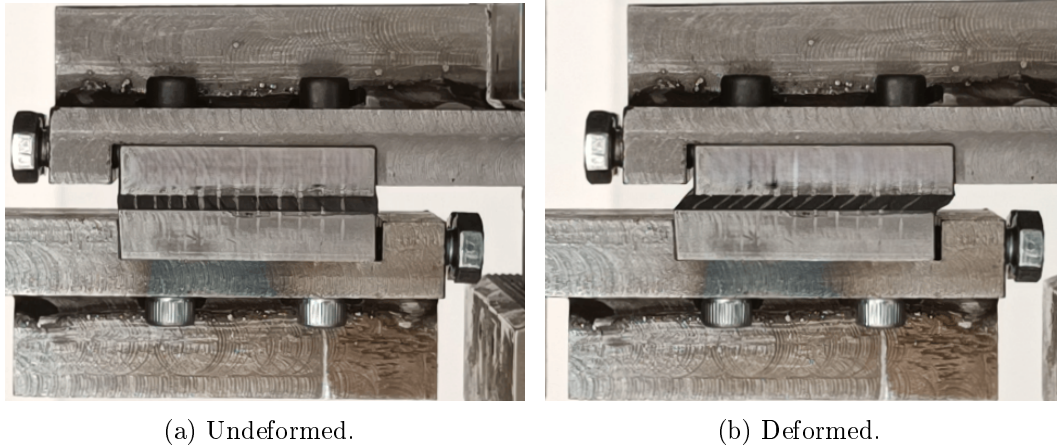


Figure 3.49: Prototype of the single specimen in the undeformed and deformed state.

The area of the rubber pads is increased to 49×59 mm. Also, the alignment of the supports is necessary in order to avoid compression or tension perpendicular to the plane of the rubber sheet. This is guaranteed by a junction that allows the support position to be adapted to the thickness of the rubber. It also allows to correct torsion around the vertical axis due to the rotation of the machine clamp and to reset the small force induced in the specimen when the clamps are closed.

The advantages of adopting the single specimen are: the rubber piece has a high aspect ratio, thus the almost uniform simple shear deformation state in the whole rubber and an elastic curve close to the theoretical curve; the low thickness allows the execution of relaxation tests with a brief loading ramp; the rubber material that is already produced for other tests and a lower volume of rubber is needed to make one specimen; the steel plates are re-usable; the effective dimensions of the rubber are easily controlled; the production time is similar to the vulcanized quadruple specimen.

The drawback is that the symmetry of the specimen is lost and stiff supports are needed to counteract its rotation caused by the couple generated during the deformation. Also, it is important to prevent compression or tension in the direction perpendicular to the mid-plane.

Geometry of the single specimen

The rubber pad is cut from sheets of about $200 \text{ mm} \times 200 \text{ mm}$ and a thickness that ranges between 2.1 mm and 2.7 mm. The variability of the thickness in a single sheet is usually contained below 5×10^{-2} mm. The steel plates are 12 mm thick, thus the medium total thickness is 26.4 mm. The plates have a base of $49 \text{ mm} \times 59 \text{ mm}$, so the shape factor of this specimen ranges between 20 and 30. The side where the rubber is glued is flat and the other has 4 threaded holes to create a bolted friction connection with nominal diameter 8 mm screws. The screws have a yielding stress $f_{yk} = 640 \text{ MPa}$ and a strength stress $f_{tk} = 800 \text{ MPa}$ and are tightened using a calibrated torque wrench to ensure controlled tensioning to a torque of 35 Nm. In Figure 3.50 the steel plates drawings are shown.

The sandwich is held in place by the steel supports, shown in Figure 3.51, which are stiffened by a transversal plate. The shape is designed to minimize the eccentricity of the axial load transmitted by the specimen and to have the supports aligned when

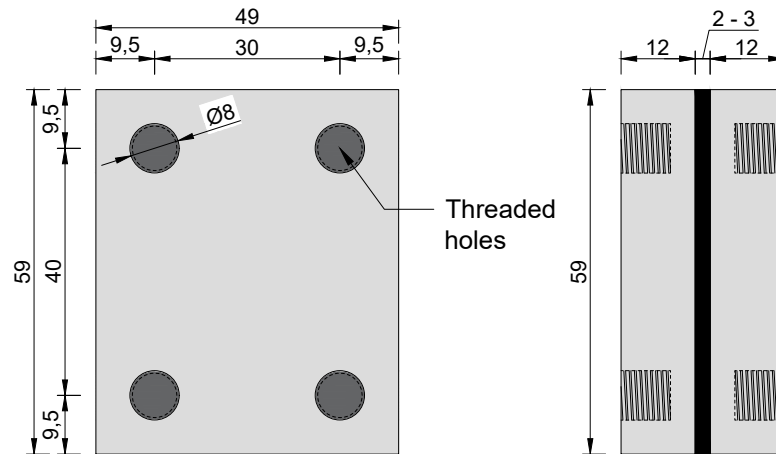


Figure 3.50: Steel plates of the single specimen.

the thickness of the rubber is 2,4 mm. The diameter 17 mm hole is present only in the one support that will be connected to the T-shaped junction.

A T-shaped steel junction, shown in Figure 3.52, with two diameter 14 mm vertical bolts connecting two L-shaped steel plates, holds the upper support with a friction connection made with a single diameter 16 mm bolt. It allows the axial alignment of the supports, adapting their position depending on the thickness of the rubber; the rotational alignment around the vertical axis, because the test machine is not constrained in this direction; to have a null force on the specimen after clamping, a phenomenon that occurs because of small displacements of the clamps and increases the force by hundreds of newtons; if the junction is already clamped when installing a specimen, it allows the horizontal alignment in the direction parallel to the plane of the rubber.

The self-weight of a single plate is of about 2.7 N, the supports about 23.6 N and the junction 40.0 N. During the installation of the specimen on the testing machine, the procedure described later, the weight of one support and one plate acts on the rubber but it has the same magnitude of the recorded data noise and causes a shear deformation on the rubber that is negligible. Anyway, once the specimen is clamped, the small recorded force is off-set to zero.

In Figure 3.53 the whole specimen installed in the testing machine is shown.

Production of the single specimen sandwich

The production of one sandwich follows these steps:

- Cleaning with isopropyl alcohol and sandblasting of the steel plates flat surface to remove all possible presence of dust and grease and to create a rough surface (Figure 3.54a), in order to enhance adhesion with rubber. The opposite face, which will be in contact with the support, has also to be sanded because it is part of the friction connection.
- Cutting of a rectangle of sides about $5 \div 10$ mm greater than the plates from rubber sheets.
- Sanding of the rubber surfaces with fine grit sandpaper (Figure 3.54b) and cleaning with isopropyl alcohol (Figure 3.54c). The sanding can also happen

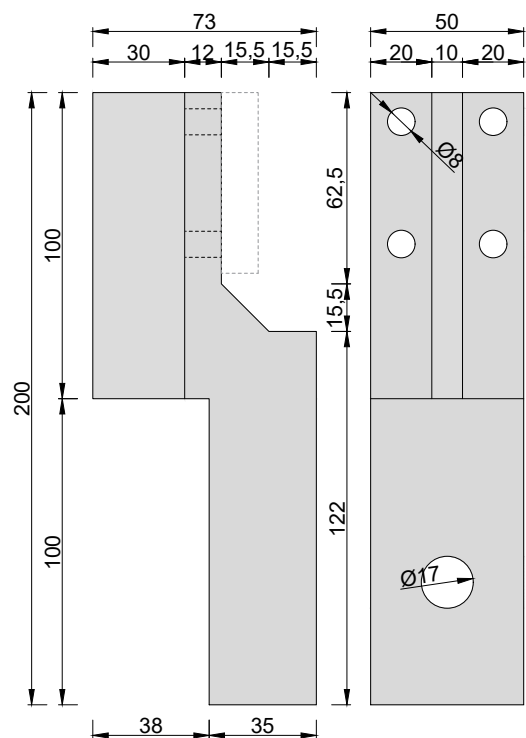


Figure 3.51: Front and lateral view of the steel supports of the sandwich.

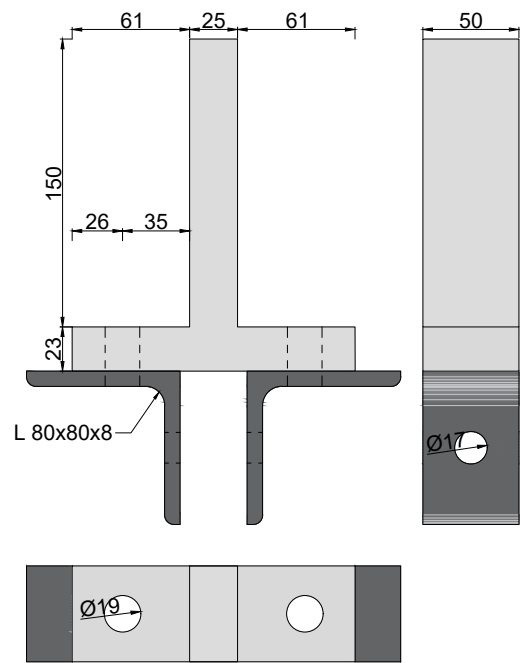


Figure 3.52: Steel junction.

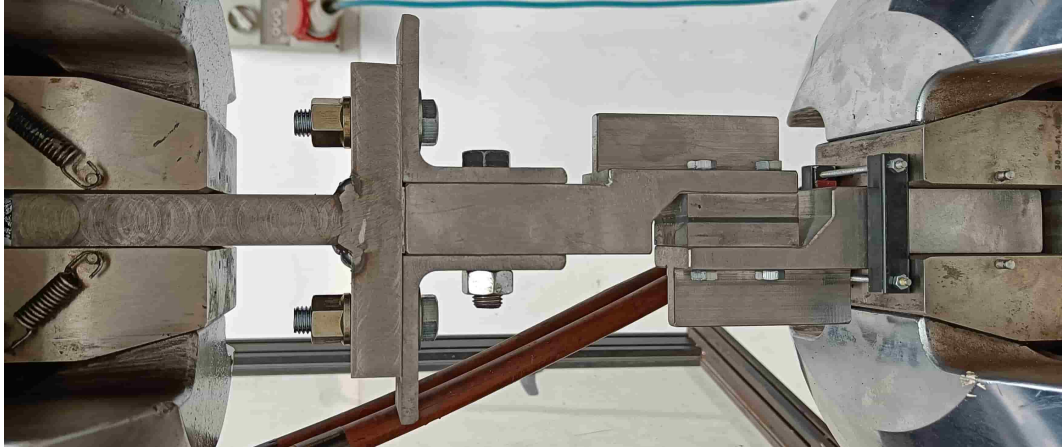


Figure 3.53: The whole specimen mounted in the test machine (view is rotated by 90°).

in the whole sheet before the cutting. Sanding of the rubber is necessary to remove the oily substance present on its surface, which compromises the bonding. The visual difference is shown in Figure 3.54d, where the right side is the one sanded.

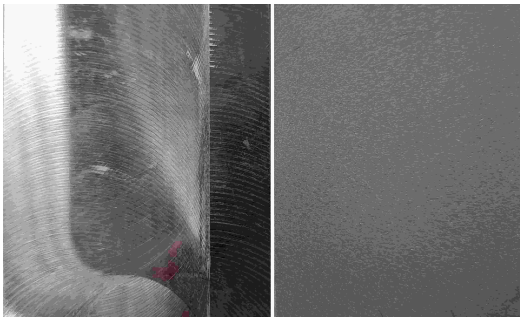
- Measuring of the rubber thickness with a dial snap meter (Figure 3.54e). A mean of 5 measurements was considered, 1 on the center and 4 towards the corners of the rubber pad.
- Gluing the rubber to the steel plates. One steel plate is placed firmly in a vise with its flat face upwards, while the other plate is anchored in an upper support and aligned to the lower plate (Figure 3.54f). This support must be able to slide vertically and press the plates on the rubber. Cleaning is repeated before uniformly spreading cyanoacrylate on the lower plate surface with a plastic sheet (Figure 3.54g). The rubber pad is positioned over the plate and it is uniformly pressed with the upper plate for a few tens of seconds. The pressure is progressively increased to avoid deforming the rubber in-plane before it is bonded. The second gluing is made spreading the cyanoacrylate on the rubber surface (Figure 3.54h) and pressing down the other plate.
- Bonding and hardening is completed in 12 hours. Then, the rubber in excess is trimmed with a cutter and alignment of the plates is checked.

In this process, the most delicate steps are the cleaning and sanding of plates and rubber and the glue spreading, all of which has influence in the final result of gluing.

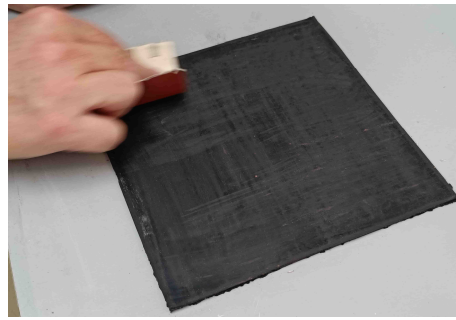
Geometrical inaccuracies and imperfections can be avoided easily by aligning the plates before gluing. A possible non-parallelism of plates could happen due to a variable thickness of the rubber layer, which should be measured beforehand on the rubber sheet. If the differences are significant, then the sheet should be discarded.

Installation of the specimen on the testing machine

The surfaces of supports in contact with the sandwich and those with the L-shaped plates of the junction must be sandblasted and periodically checked, in order to



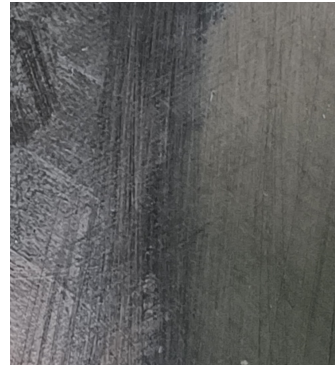
(a) New and sanded surface of the plate.



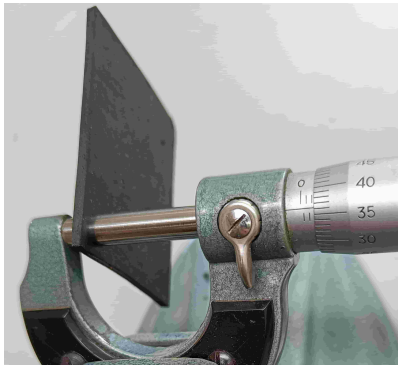
(b) Sanding of the rubber surface.



(c) Cleaning of the rubber surface.



(d) Rubber surface before and after sanding.



(e) Measuring of the rubber pad thickness.



(f) Aligning of the steel plates before gluing.



(g) Glue spreading on the steel surface.



(h) Glue spreading on the rubber surface.

Figure 3.54: Photos of the single specimen production process.

guarantee a friction coefficient of 0.5 assumed in the calculations. The sandwich is connected to one steel support at a time with 4 screws controlled tightened with 35 Nm of torque. Supports can be aligned by laying them in a horizontal surface to avoid a relative rotation of the plates during testing. The junction element is clamped onto the testing machine connected to the support; however, the bolts should not be tightened to allow for relative movement during installation. The other support is clamped in the other end of the machine. The compression or tension perpendicular to the plane of the rubber layer can be limited with the use of the junction, moving it until there are no bulges on the side of the sandwich. After checking the verticality and alignment of the supports and the absence of lateral bulges on the rubber, all 3 bolts of the junction are controlled tightened with 200 Nm of torque.

Disassembly of the specimen

A broken or defected specimen can be disassembled by heating it at 150°C in an oven for 30 minutes. When the glue melts, the rubber can be easily removed with pliers and the surface scraped to remove glue. Once cooled, the flat surface of the plates should be sanded to remove any remaining glue or rubber, cleaned with isopropyl alcohol, and sandblasted to prepare a new sandwich.

Lateral displacement of the support

The lateral displacement of the supports due to the eccentricity of the load when the specimen is loaded is unavoidable. The lateral displacement, measured during a failure test of a specimen, is plotted in Figure 3.55 as a function of the applied load. The displacement is directly proportional to the applied load and at $F = 20$ kN the lateral displacement is less than 0.1 mm. Non destructive tests are performed at lower forces. For example, in a relaxation test on the hardest rubber, compound C, at $\gamma_{max} = 2.5$ the peak force recorded was of about 12 kN. Thus, the lateral displacement of the support is negligible.

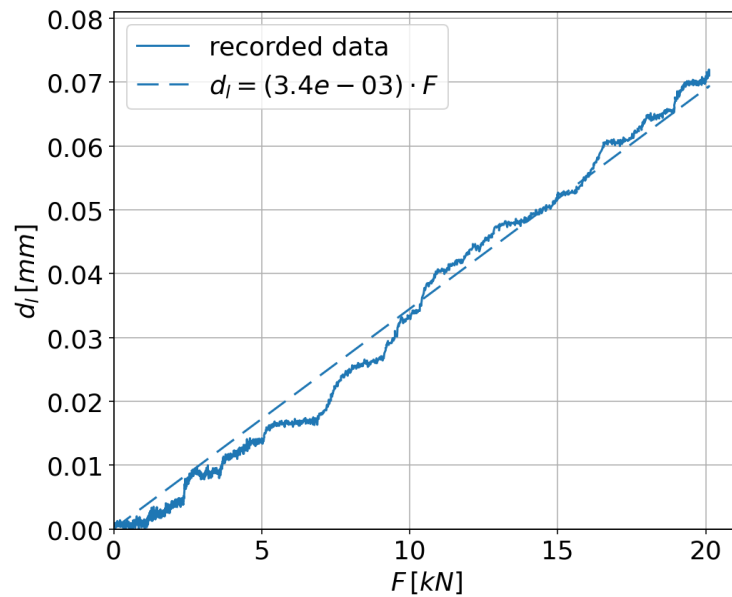


Figure 3.55: Lateral displacement of the single specimen support compared to the applied load in a simple shear test.

3.4 Shear cyclic tests on single specimens

From cyclic shear tests performed on single pad specimens the dynamic shear modulus G_{dyn} and the damping ξ can be computed. In Figures 3.56 and 3.57 the trends of G_{dyn} and ξ of each specimen (rubber compounds grouped by color), computed at the first, third and ninth cycle, are reported for increasing values of maximum strain. These results are consistent because different specimens of the same rubber compound behave similarly, exhibiting a low standard deviation in the values of G_{dyn} and ξ , which demonstrates that the designed specimen is reliable.

In Figures 3.58 and 3.59 the same values are reported as a function of the number of consecutive cycles. Both quantities exhibit a decreasing trend that stabilizes quickly, particularly the damping, which remains stable from the third cycle onward. The modulus also stabilizes, but even after the ninth cycle, the curves still show a noticeable slope.

The values of shear modulus and damping obtained at the third cycle of the cyclic tests on specimens are shown in Figures 3.60a and 3.60b, respectively, with respect to γ_{max} . Figure 3.60c the same quantities are directly compared to search for a correlation.

Second-degree polynomial regression curves were derived for modulus and damping trends (Figures 3.60a and 3.60b, respectively) with respect to γ_{max} with each expression corresponding to a specific rubber compound. These curves can be used to estimate the values of G_{dyn} and ξ even for γ_{max} values for which no tests were conducted. Modulus and damping tend to form groups in Figure 3.60c, thus a regression curve has no meaning.

The trends of these curves are directly compared with shear modulus and damping computed from the cyclic tests on isolators in Figures 3.61a, 3.61b and 3.61c, respectively. On the basis of this figure, it can be said that the shear modulus of the isolators is discretely estimated through cyclic tests on individual specimens, while the high dispersion in the damping of the isolators makes it impossible to achieve a good correlation with the results obtained from the same tests.

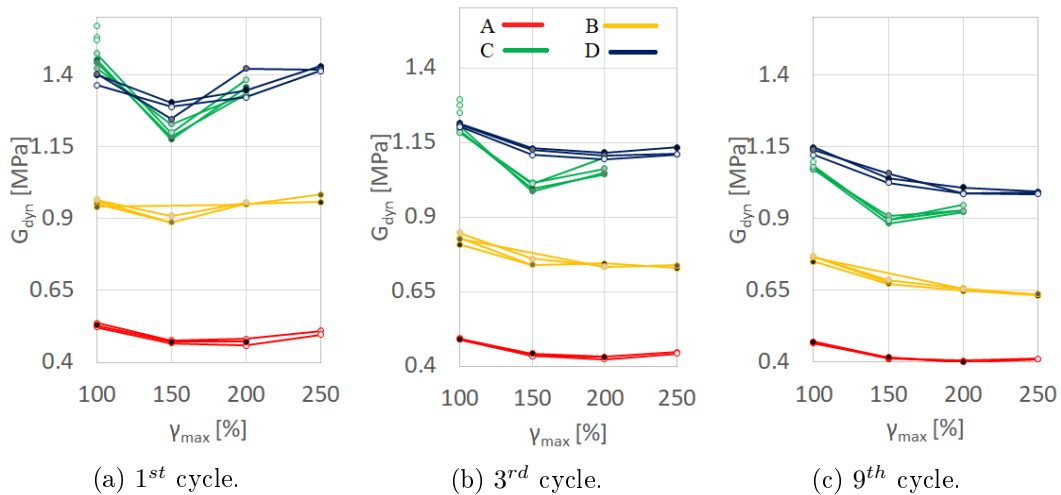


Figure 3.56: Dynamic shear modulus computed at the first, third and ninth cycle of cyclic tests on single specimens.

Table 3.3 presents the same results in numerical form, including the calculation of the error for both modulus and damping. In some cases, the estimation is satisfactory

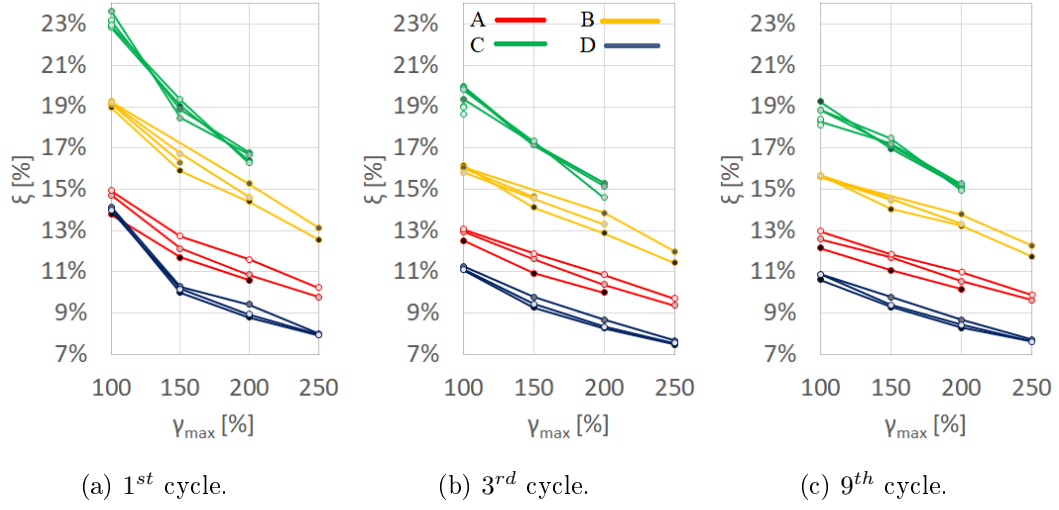


Figure 3.57: Damping computed at the first, third and ninth cycle of cyclic tests on single specimens.

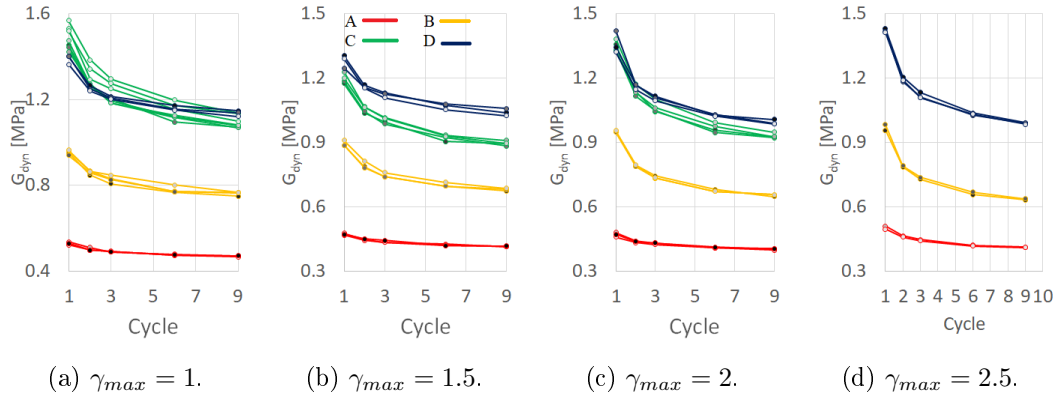


Figure 3.58: Dynamic shear modulus decrease with the number of cycles in cyclic tests on single specimens, $\gamma_{max} = 1, 1.5, 2, 2.5$.

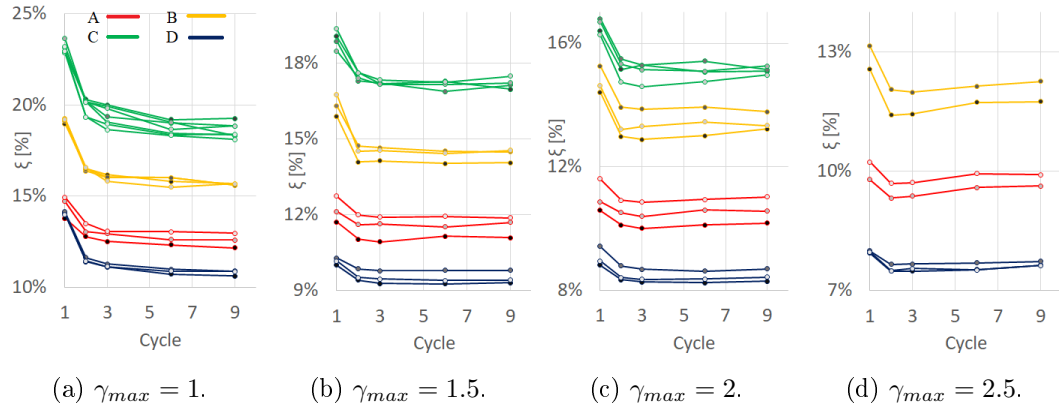


Figure 3.59: Damping decrease with the number of cycles in cyclic tests on single specimens, $\gamma_{max} = 1, 1.5, 2, 2.5$.

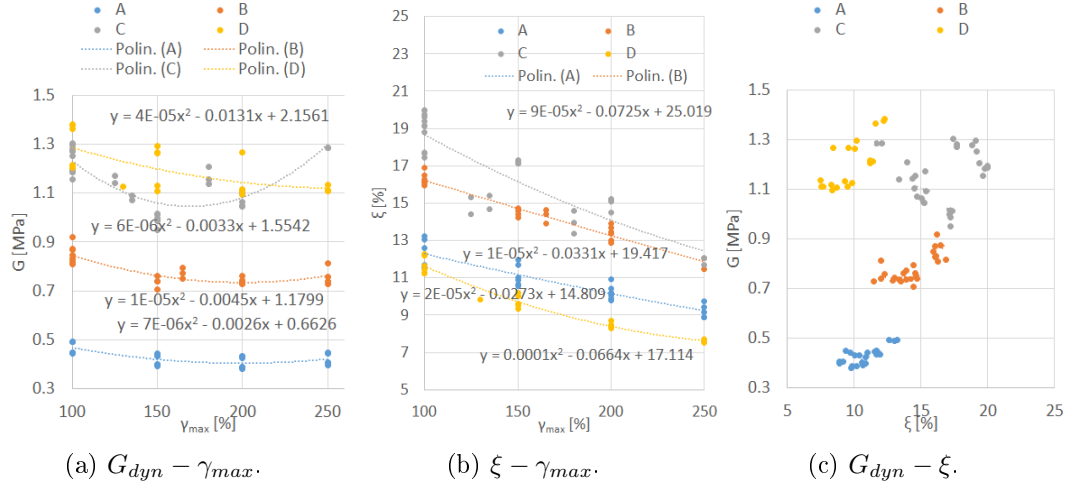


Figure 3.60: Polynomial regression curves of G_{dyn} and ξ computed at the third cycle of cyclic shear tests on single specimens, compared to the experimental data.

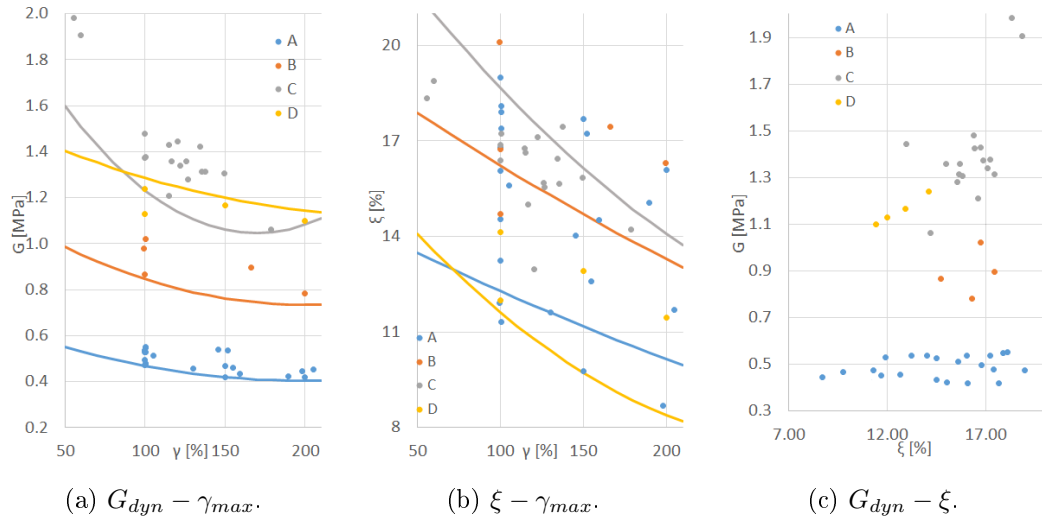


Figure 3.61: Polynomial regression curves of Figure 3.60 compared to the same quantities of the cyclic tests on full-scale isolators.

for both, but generally, the errors exceed 5 %. Therefore, the estimation based solely on cyclic tests is not sufficient, requiring more detailed analyses. These include the testing method presented in Section 3.1 and subsequent finite element simulations discussed in Chapter 4.

Table 3.3: Dynamic shear modulus and damping computed at the third cycle from all the cyclic tests performed on single specimens, compared to the cyclic tests on isolators.

γ_{max} [%]	R	I	Isolators				Specimens		Errors	
			G_{dyn} [MPa]	σ_G [%]	ξ [%]	σ_ξ [%]	G_{dyn} [MPa]	ξ [%]	ϵ_G [%]	ϵ_ξ [%]
159	A	01	0.44	4.1	14.53	3.1	0.414	10.97	-4.9	-24.5
100	A	02	0.48	2.4	11.31	10.1	0.467	12.27	-1.6	+8.6
100	A	03	0.55	0.6	18.10	0.8	0.468	12.28	-15.4	-32.2
190	A	04	0.42	4.6	15.05	6.5	0.404	10.35	-4.1	-31.2
100	A	05	0.48	4.9	17.40	13.1	0.468	12.28	-2.2	-29.5
100	A	06	0.50	4.2	16.79	11.6	0.468	12.28	-5.6	-26.8
100	A	07	0.47	0.7	19.00	1.5	0.468	12.28	-1.0	-35.4
150	A	07	0.42	0.0	17.70	0.8	0.420	11.17	-0.2	-36.9
200	A	07	0.42	0.8	16.10	0.9	0.404	10.15	-3.4	-36.9
100	A	08	0.55	2.2	17.90	0.8	0.468	12.28	-14.5	-31.4
100	A	09	0.53	4.4	14.53	2.2	0.468	12.28	-11.2	-15.5
152	A	10	0.54	4.5	17.23	0.7	0.418	11.13	-22.1	-35.4
100	A	11	0.54	1.7	16.05	0.4	0.468	12.28	-12.9	-23.5
105	A	12	0.51	2.7	15.60	2.7	0.462	12.17	-9.9	-22.0
130	A	12	0.46	3.5	11.60	2.4	0.435	11.60	-4.9	0.0
155	A	12	0.46	0.0	12.60	0.0	0.416	11.06	-9.2	-12.2
205	A	12	0.45	0.0	11.70	0.0	0.404	10.06	-10.5	-14.0
146	A	13	0.54	0.6	14.04	3.9	0.422	11.26	-21.5	-19.8
100	A	14	0.54	-	13.23	-	0.468	12.28	-12.9	-7.2
100	A	15	0.53	1.5	11.92	0.9	0.468	12.29	-11.5	+3.1
150	A	15	0.47	2.5	9.77	3.0	0.420	11.17	-10.1	+14.3
198	A	15	0.44	2.2	8.69	5.5	0.404	10.19	-9.1	+17.3
100	B	01	0.87	4.3	14.71	4.4	0.846	16.23	-2.2	+10.3
99	B	02	0.98	3.9	20.10	2.1	0.847	16.24	-13.5	-19.2
166	B	02	0.90	6.0	17.45	3.5	0.747	14.23	-16.6	-18.5
200	B	02	0.78	6.2	16.30	4.3	0.734	13.29	-6.0	-18.5
100	B	03	1.02	0.9	16.73	2.3	0.845	16.22	-17.1	-3.1
115	C	01	1.21	2.6	16.62	2.6	1.160	17.85	-4.1	+7.4
179	C	01	1.06	1.0	14.22	1.3	1.050	14.90	-1.3	+4.8
120	C	02	1.45	1.4	12.96	0.2	1.140	17.59	-21.1	+35.7
149	C	03	1.31	7.6	15.85	6.6	1.061	16.17	-18.8	+2.1
138	C	04	1.31	2.4	17.44	2.2	1.085	16.73	-17.4	-4.1
134	C	05	1.42	2.7	16.44	2.0	1.093	16.88	-23.2	+2.7
115	C	06	1.43	1.5	16.77	0.8	1.161	17.87	-18.7	+6.6
122	C	07	1.34	3.1	17.13	2.4	1.132	17.49	-15.5	+2.1
126	C	07	1.36	1.3	15.69	2.3	1.119	17.30	-17.5	+10.3
101	C	08	1.38	4.0	17.22	3.1	1.229	18.63	-10.9	+8.2
60	C	09	1.91	1.0	18.88	1.3	1.509	20.99	-20.9	+11.2
136	C	10	1.31	1.1	15.64	2.3	1.090	16.82	-17.0	+7.5
56	C	11	1.98	1.4	18.33	1.5	1.546	21.26	-22.0	+16.0
100	C	12	1.37	4.6	16.87	2.7	1.232	18.66	-10.2	+10.6
127	C	12	1.28	3.2	15.55	1.4	1.117	17.27	-12.7	+11.0
100	C	13	1.48	2.2	16.39	0.4	1.232	18.66	-16.7	+13.9
117	C	13	1.36	2.5	15.01	0.8	1.153	17.77	-15.0	+18.4
100	D	01	1.13	-	12.00	-	1.286	11.61	+13.8	-3.2

100	D	02	1.24	-	14.13	-	1.286	11.61	+3.7	-17.8
150	D	02	1.17	-	12.92	-	1.199	9.72	+2.9	-24.8
200	D	02	1.10	-	11.44	-	1.144	8.39	+4.0	-26.7

3.4.1 Application of the MFCT method to laboratory tests

Multiple cyclic tests at different frequencies were performed in one specimen of rubber compound C to apply the Multi-Frequency Cyclic Test (MFCT) method presented in Section 2.2.9 to determine the viscous parameters (μ_i, τ_i) . A total of 19 cyclic tests were performed after the testing procedure, in Section 3.1.6, at $\gamma_{max} = 1$. Hence, the specimen was already conditioned multiple times. Frequencies of the test ranged between 10^1 Hz to 10^{-3} Hz and sampling frequencies was set to $f_s = 10^3$ Hz for high frequencies and $f_s = 10^2$ Hz for the lower ones. To run out any short-term recover of the Mullins effect the tests were composed by 1.5 sinusoidal cycles at frequency $f = 0.5$ Hz, immediately after which the frequency was changed to the target of the test to perform 3 cycles. The number of cycles was limited to 1.5 in tests of frequency lower than $f = 10^{-1}$ in order to reduce the duration of the test. The deformation amplitude of the cycles were reduced tests of frequencies higher than $f = 1$ Hz to ensure that the testing machine followed a sinusoidal deformation history. Even after adopting this strategy, the recorded amplitude and frequency were slightly lower than the imposed ones, but the sinusoidal shape was still maintained.

The cycles of some experimental cyclic tests considered to apply the method are reported in Figure 3.62.

It is worth noting that, looking at the curves of the tests performed at $\gamma_{max} = 1$, two “fixed points” can be identified in the unloading branches at $\gamma \simeq 0.65$ (black circles). If additional tests were performed at lower frequencies and the stress response curves would still intersect in these points with the others, then for $f \rightarrow 0$ the elastic equilibrium stress curve would contain them. This point would also represent the value of γ for which the unloading branch curve works as a lower bound for the step relaxation test. In this work this considerations were not further investigated.

The MFCT method was applied with the values of G_{dyn} and ξ computed from all cyclic tests to determine the viscous constants ($N = 6$) of the rubber compound C, valid for $\gamma_{max} = 1$. The elastic equilibrium stress curve is determined with the UCDP test. Three different cases are compared:

- **UCDP + relax.**: the visco-elastic parameters are determined from the fitting of a UCDP test and a 30 min relaxation test. This case is only considered for comparison with the MFCT method.
- **UCDP + MFCT**: the elastic constants ($C_{j0} = [0.365, -0.200, 0.126]$ MPa) are determined from the UCDP test only. Then, the MFCT method is applied to determine (μ_i, τ_i) . This is the case in which the MFCT method would be used to determine the viscous constants as an alternative to the relaxation test.
- **UCDP + relax. + MFCT**: the elastic constants C_{j0} are the ones of case UCDP + relax ($C_{j0} = [0.308, -0.104, 0.064]$ MPa). In this case the elastic equilibrium stress evaluated for strains near γ_{max} is limited from above by the relaxation test and the overall curve is lower than the previous case. Then, MFCT method is applied to determine (μ_i, τ_i) . This case is considered to compare only the viscous components differences.

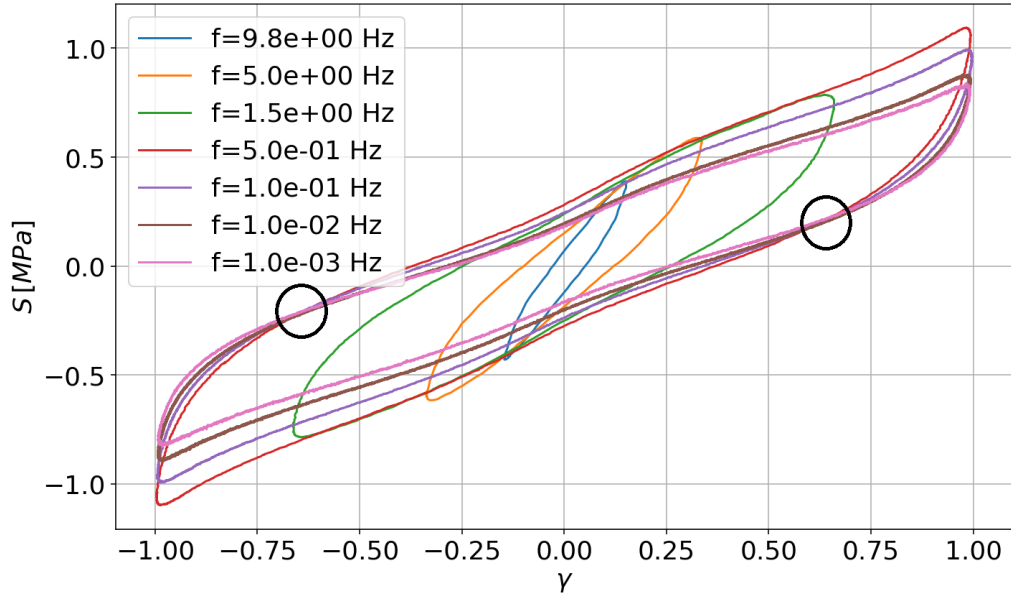


Figure 3.62: Third cycle hysteresis loop of cyclic tests at different frequencies performed on a rubber compound C specimen.

In Figure 3.63 the color maps of the error function χ , the residual $\mathbf{r}$, the conditioning number of the least square problem solving matrix $\mathbf{f}^T \mathbf{f}$ and the penalty assigned if the solution found had negative values of μ_i , as defined in Section 2.2.9, are shown.

The first comparison to evaluate the MFCT method is made between the experimental and the simulated stress responses of several cyclic tests. In Figure 3.64 two examples of low and high frequency tests are reported. While for some frequencies the curves tend to overlap, for others the opposite. Also the shape of the simulated response cycle is not similar to the experimental one. The consequence can be seen comparing the values of dynamic modulus (Figure 3.65a) and damping (Figure 3.65b). The errors in the shear modulus are acceptable, while the damping values are completely different from the experimental ones. This is likely due to the limited value of the last time constant, which results in low over-stresses for low-frequency cycles and, consequently, reduced dissipated energy.

In Figure 3.66a the spectra determined in the three cases are reported and their simulated stress response is compared to the experimental relaxation test in Figure 3.66b. The main difference in the spectra is the range of the time constants, which for the MFCT method spectra are limited between 10^{-2} and 10^2 . The consequence is that the viscous effects in the simulation of the relaxation test are substantially exhausted at time $t \simeq 100$ s. However, in the first part of the relaxation, both curves derived from the MFCT method are not far from the experimental curve, especially in the loading ramp.

In Figure 3.67 the spectra and the simulated response are shown for different choices of N in the case of UCDP + relax. + MFCT. Note that a regular trend of the relaxation curve is obtained for a number of viscous parameters $N \geq 5$, the greatest time constant τ_N is limited between $10^2 \div 10^3$, and often the viscous component $\mu_N \simeq 0$, thus the over-stresses dissipate long before the 30 minutes relaxation experimental test.

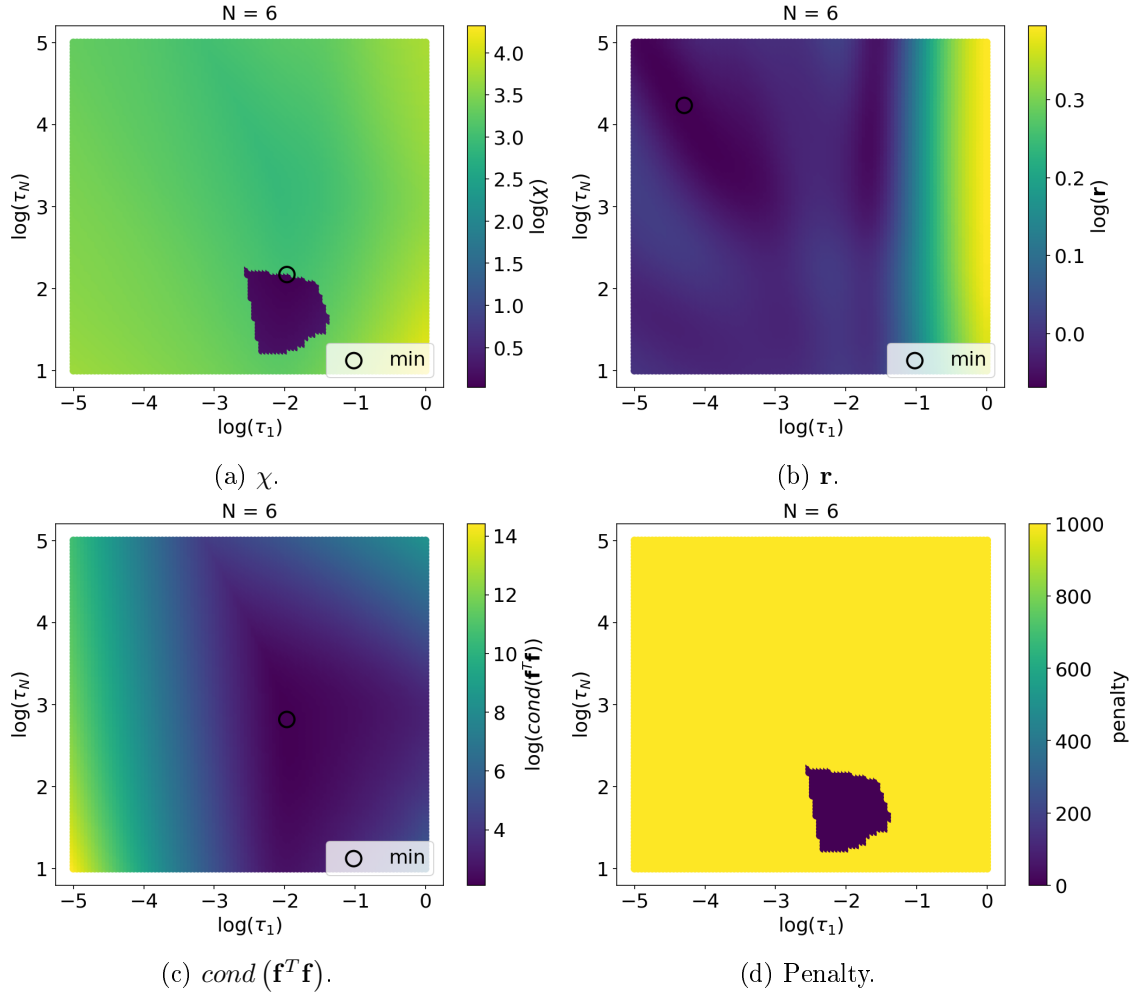


Figure 3.63: Colormap of the values of the error function, the residual, the conditioning number and the penalty value assigned for different extreme values of τ_1 and τ_N in the MFCT method applied to experimental cyclic tests.

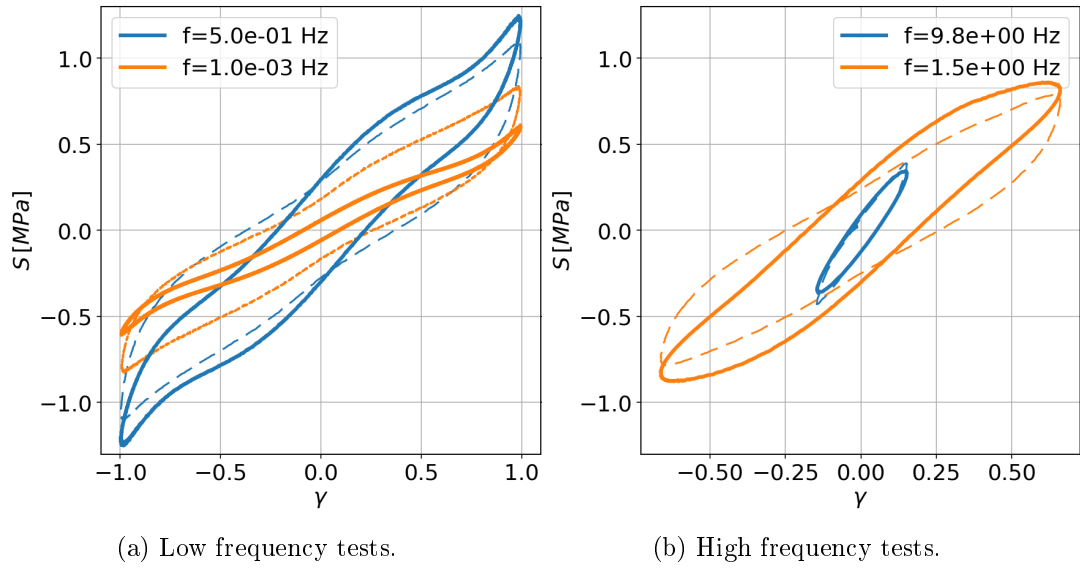


Figure 3.64: Experimental (dashed lines) and simulated stress response (UCDP + relax. + MFCT) of some of the performed cyclic tests.

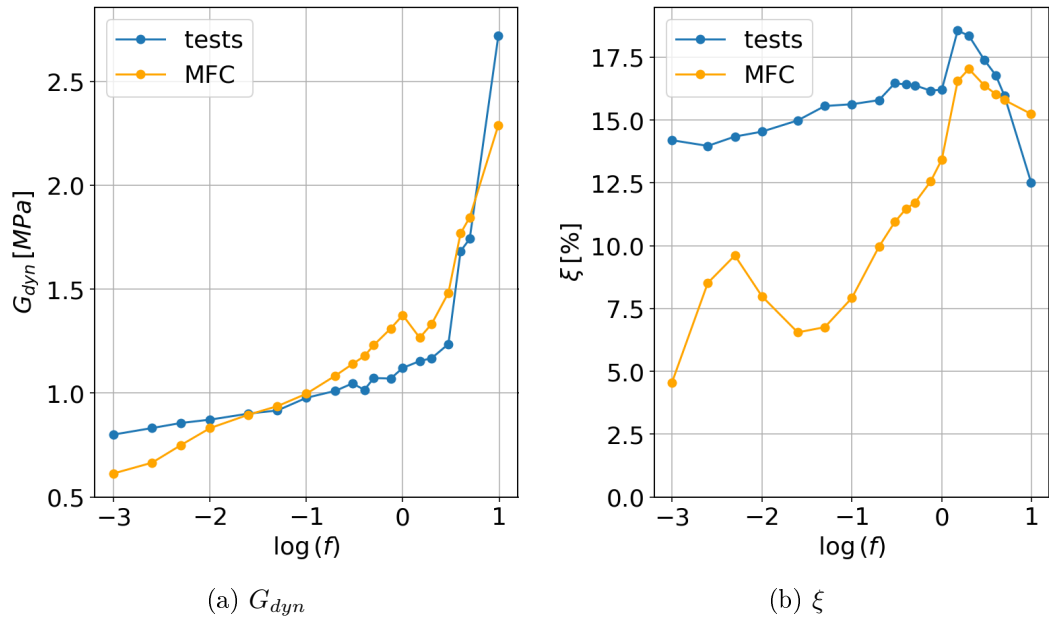


Figure 3.65: Experimental and simulated dynamic shear modulus and damping (UCDP + relax. + MFCT).

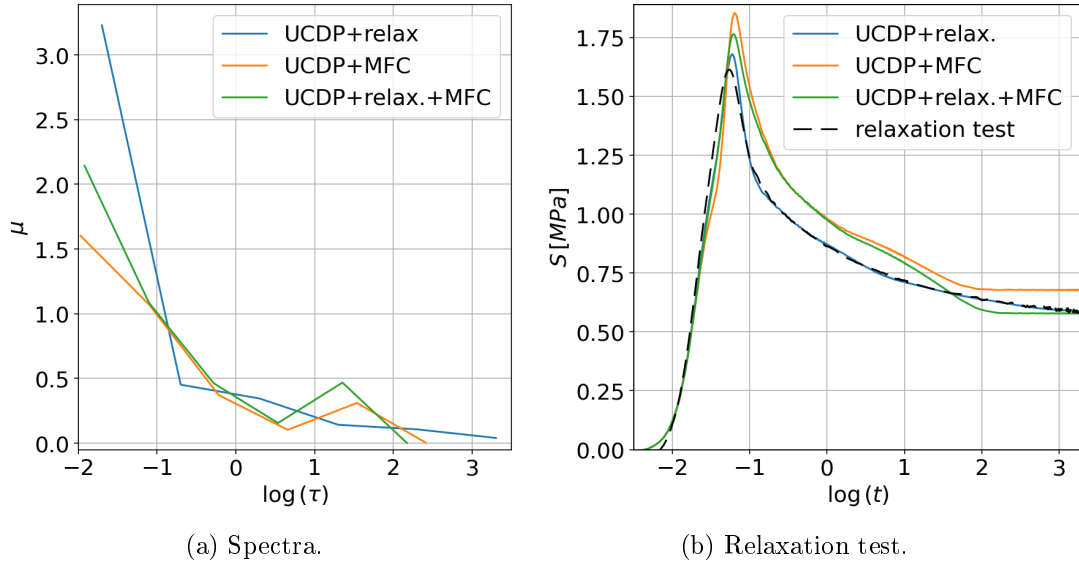


Figure 3.66: Spectra of rubber C determined with a relaxation test and MFCT. The simulated stress response is compared to the experimental relaxation test.

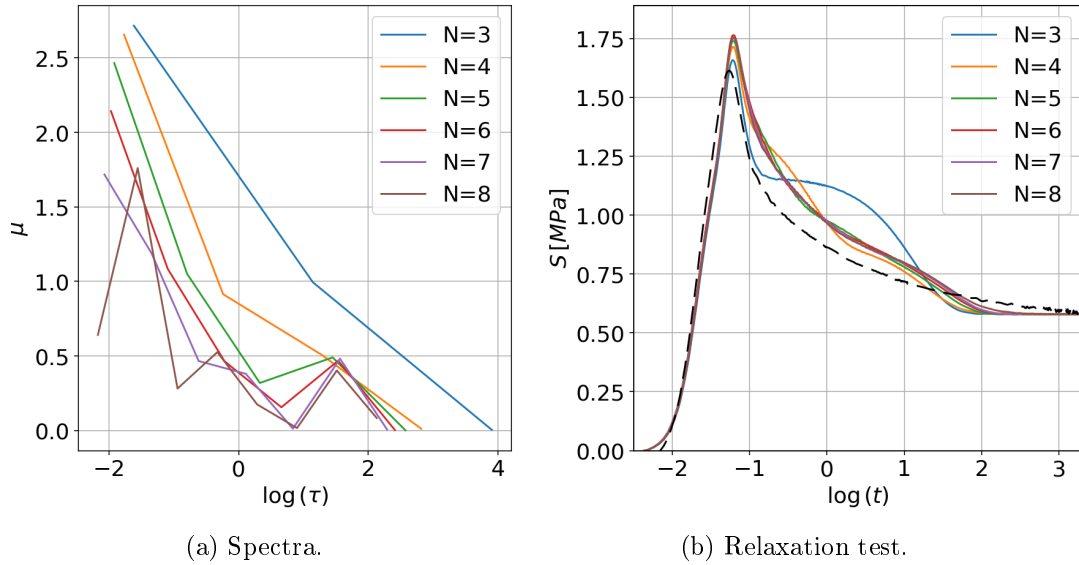


Figure 3.67: Spectra of rubber C determined with MFCTs for different values of N . The simulated stress response is compared to the experimental relaxation test.

Chapter 4

Numerical Simulations

The parameters of the visco-elastic constitutive model (Section 2.1.3) for compounds A-D reported in Table 1.1 are determined through numerical fitting of simple shear tests results at various maximum shear strain levels. Such models are then used in the FEM simulation of the shear cyclic test on a number of rubber isolators whose data are reported in Table 1.2. The simulations results are compared to the full-scale tests results in terms of dynamic shear modulus and damping.

4.1 Fitting of experimental data

The data of the simple shear tests described in Section 3.1 are used to evaluate the parameters of the visco-elastic model through the fitting process described in Section 2.2.7. The weight of each test in the error function 2.77 was adjusted to minimize the discrepancy between the model predictions and observed data for individual tests, thereby improving the accuracy of their contributions to the overall model total error.

4.1.1 Visco-elastic model parameters from multiple tests

The visco-elastic model (Section 2.1.3) was fitted to multiple tests (cyclic, UCDP, relaxation). Table 4.1 the parameters C_{j0} for $j = 1, 2, 3$ and μ_i for $i = 1, \dots, 6$ for each compound are reported. $N = 6$ time constants were fixed to $\tau_i = 2 \cdot [10^{-2}, 10^{-1}, 10^0, 10^1, 10^2, 10^3]$ seconds for all compounds.

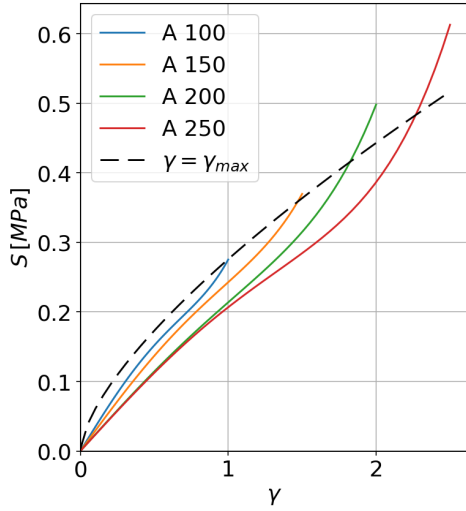
In Figures from 4.1 to 4.4 the same parameters are reported graphically in the form of elastic equilibrium stress curves and $\mu - \tau$ discrete spectra. The estimates of the first load equilibrium stress curves determined according to Section 3.2.2 are also reported with dashed lines.

The trends of C_{j0} and the hyperbolic curves, fitted according to Equation 3.2, are reported in Figure 4.5. The values of C_{10} are similar to the ones determined from the fitting of the UCDP2m tests only, although compound C trend is not very regular. Values of C_{20} and C_{30} are more than halved in modulus but the hyperbolic trends are still present.

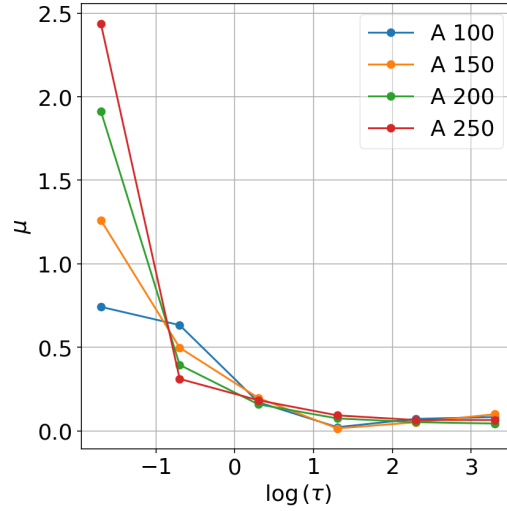
As in Section 3.2.2, the expressions for the elastic parameters C_{j0} depending on the maximum strain γ_{max} used to express the first load equilibrium stress curve estimates are determined.

Table 4.1: Compound constitutive parameters from the fitting of multiple tests (cyclic, UCDP, relaxation) used for the FEM simulation of the isolators cyclic tests.

R $\bar{\gamma}_{max}$	C_{j0} [MPa], $j = 1, 2, 3$	μ_i , $i = 1 \dots 6$
A 100	[1.727E-1,-4.970E-2,2.144E-2]	[7.419E-1,6.326E-1,1.708E-1,2.161E-2,7.131E-2,8.259E-2]
A 150	[1.448E-1,-1.728E-2,3.698E-3]	[1.261E+0,4.973E-1,1.962E-1,1.446E-2,5.095E-2,9.903E-2]
A 200	[1.179E-1,-7.630E-3,1.409E-3]	[1.913E+0,3.942E-1,1.581E-1,7.411E-2,5.145E-2,4.334E-2]
A 250	[1.161E-1,-7.689E-3,8.750E-4]	[2.437E+0,3.106E-1,1.827E-1,9.231E-2,6.424E-2,6.455E-2]
B 100	[2.735E-1,-8.910E-2,4.222E-2]	[2.072E+0,9.265E-1,3.626E-1,3.399E-2,1.508E-1,1.482E-2]
B 165	[1.949E-1,-2.579E-2,6.013E-3]	[3.704E+0,7.005E-1,2.847E-1,1.283E-1,1.047E-1,6.757E-2]
B 200	[1.638E-1,-1.494E-2,2.900E-3]	[4.363E+0,5.748E-1,2.730E-1,1.552E-1,9.454E-2,7.278E-2]
B 250	[1.488E-1,-1.228E-2,1.572E-3]	[5.091E+0,4.733E-1,3.484E-1,1.496E-1,1.129E-1,8.629E-2]
C 100	[3.240E-1,-1.124E-1,6.831E-2]	[1.881E+0,1.200E+0,3.683E-1,9.688E-3,1.452E-1,3.570E-2]
C 125	[2.668E-1,-6.584E-2,3.070E-2]	[3.031E+0,9.180E-1,2.936E-1,6.505E-2,1.211E-1,7.211E-2]
C 135	[2.443E-1,-4.915E-2,2.156E-2]	[3.268E+0,8.838E-1,2.437E-1,6.198E-2,1.180E-1,7.432E-2]
C 180	[2.192E-1,-3.337E-2,8.175E-3]	[4.552E+0,7.873E-1,3.737E-1,8.281E-2,8.587E-2,1.512E-1]
C 250	[2.214E-1,-2.702E-2,3.322E-3]	[4.330E+0,7.081E-1,4.477E-1,8.777E-2,6.206E-2,1.016E-1]
D 100	[5.243E-1,-1.715E-1,8.663E-2]	[6.859E-1,6.418E-1,1.278E-1,8.780E-10,4.289E-2,8.870E-2]
D 150	[3.988E-1,-5.977E-2,1.741E-2]	[1.470E+0,4.602E-1,1.695E-1,3.125E-2,5.754E-2,8.041E-2]
D 200	[3.329E-1,-3.581E-2,6.441E-3]	[1.919E+0,3.353E-1,2.825E-1,3.264E-2,7.803E-2,6.145E-2]



(a) Equilibrium stress curves.



(b) $\mu - \tau$ spectra.

Figure 4.1: Visco-elastic model parameters determined for compound A at different levels of maximum strain $\bar{\gamma}_{max}$, represented by equilibrium stress curves and $\mu - \tau$ spectra.

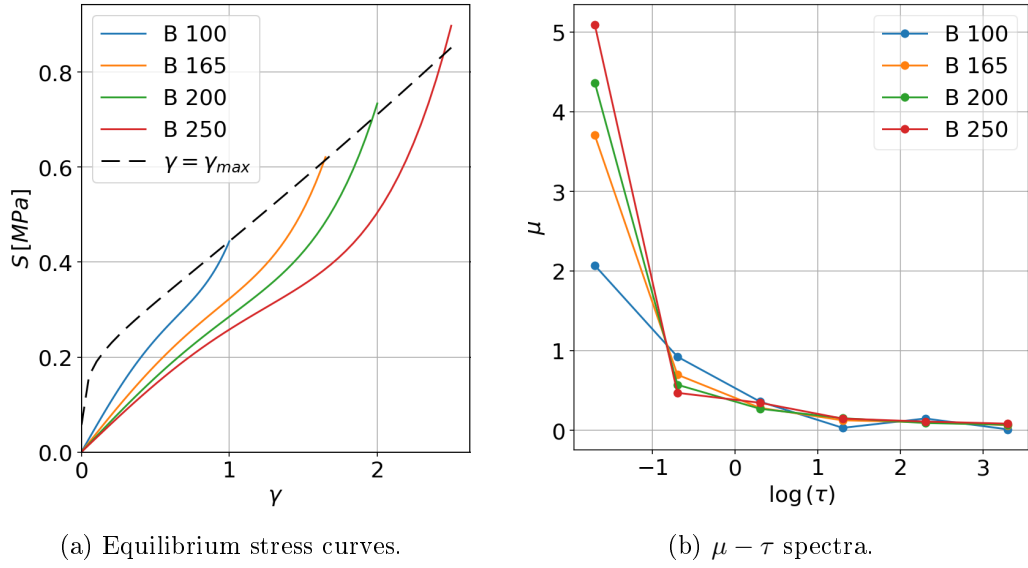


Figure 4.2: Visco-elastic model parameters determined for compound B at different levels of maximum strain $\bar{\gamma}_{max}$, represented by equilibrium stress curves and $\mu - \tau$ spectra.

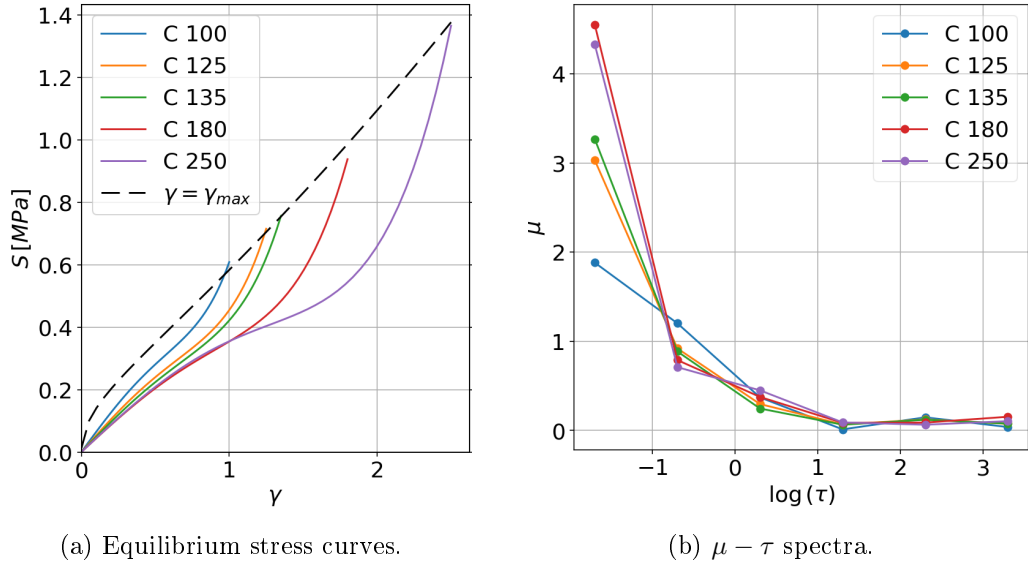


Figure 4.3: Visco-elastic model parameters determined for compound C at different levels of maximum strain $\bar{\gamma}_{max}$, represented by equilibrium stress curves and $\mu - \tau$ spectra.

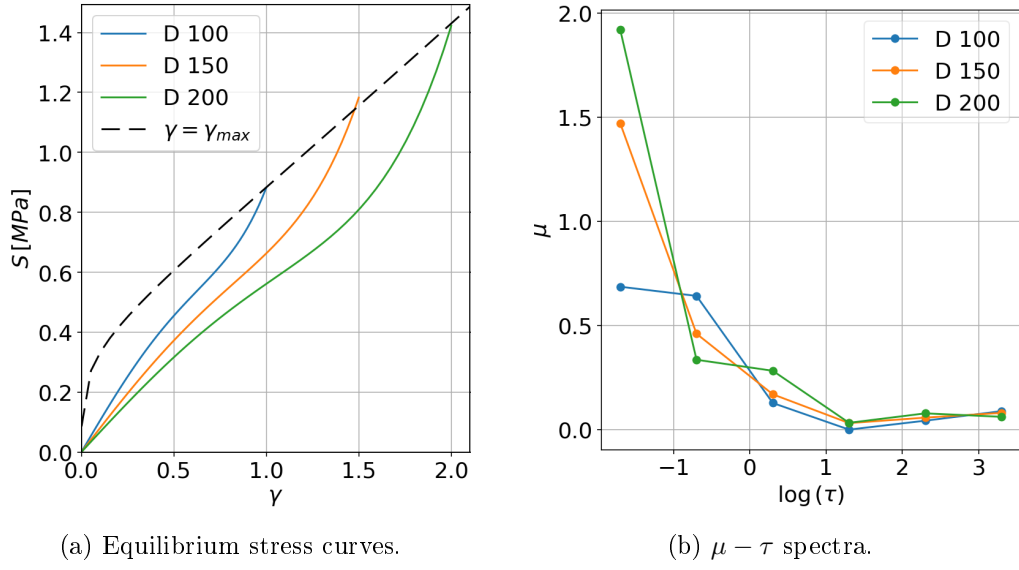


Figure 4.4: Visco-elastic model parameters determined for compound D at different levels of maximum strain $\bar{\gamma}_{max}$, represented by equilibrium stress curves and $\mu - \tau$ spectra.

$$S_{\infty,A}(\gamma = \gamma_{max}) = 2 \cdot 0.173\gamma^{0.527} + 4(-0.050)\gamma^{0.505} + 6 \cdot 0.021\gamma^{0.816} \quad (4.1)$$

$$S_{\infty,B}(\gamma = \gamma_{max}) = 2 \cdot 0.273\gamma^{0.310} + 4(-0.089)\gamma^{0.565} + 6 \cdot 0.042\gamma^{1.145} \quad (4.2)$$

$$S_{\infty,C}(\gamma = \gamma_{max}) = 2 \cdot 0.303\gamma^{0.547} + 4(-0.108)\gamma^{0.921} + 6 \cdot 0.068\gamma^{1.327} \quad (4.3)$$

$$S_{\infty,D}(\gamma = \gamma_{max}) = 2 \cdot 0.524\gamma^{0.340} + 4(-0.171)\gamma^{0.557} + 6 \cdot 0.087\gamma^{1.095} \quad (4.4)$$

The trends of the first load equilibrium stress curves in Figures from 4.1a to 4.4a are close to the stresses at the end of the elastic equilibrium stress curves. In particular, compounds B and C curves do not diverge, as happened in Figure 3.24.

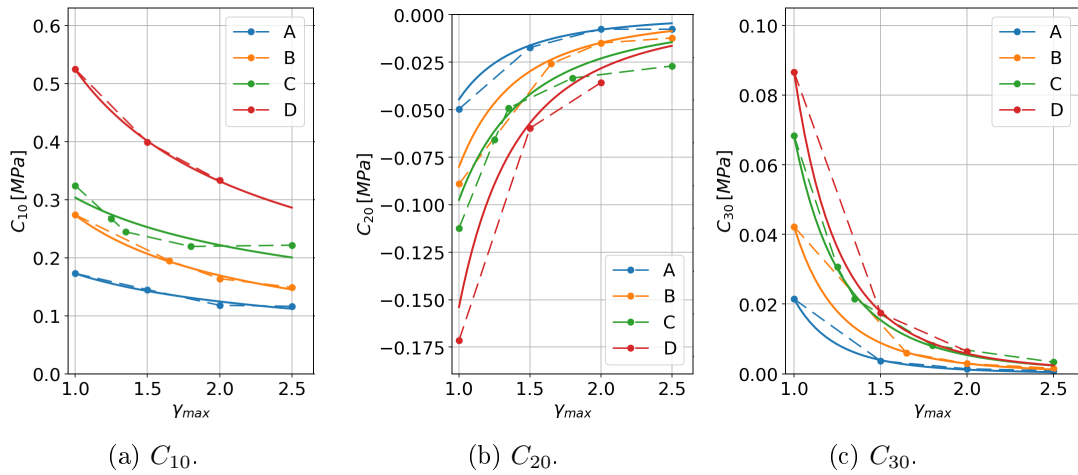


Figure 4.5: Trends of the elastic constants from data recorded with UCDP2m tests and fitted curves.

An attempt to determine the value of each viscous constant μ_i as function of γ_{max} was made, since the same time constants τ_i were assumed for all maximum strains.

However, the viscous constants did not always show a regular trend, as shown in Figure 4.6, so a fitting was not possible in this case.

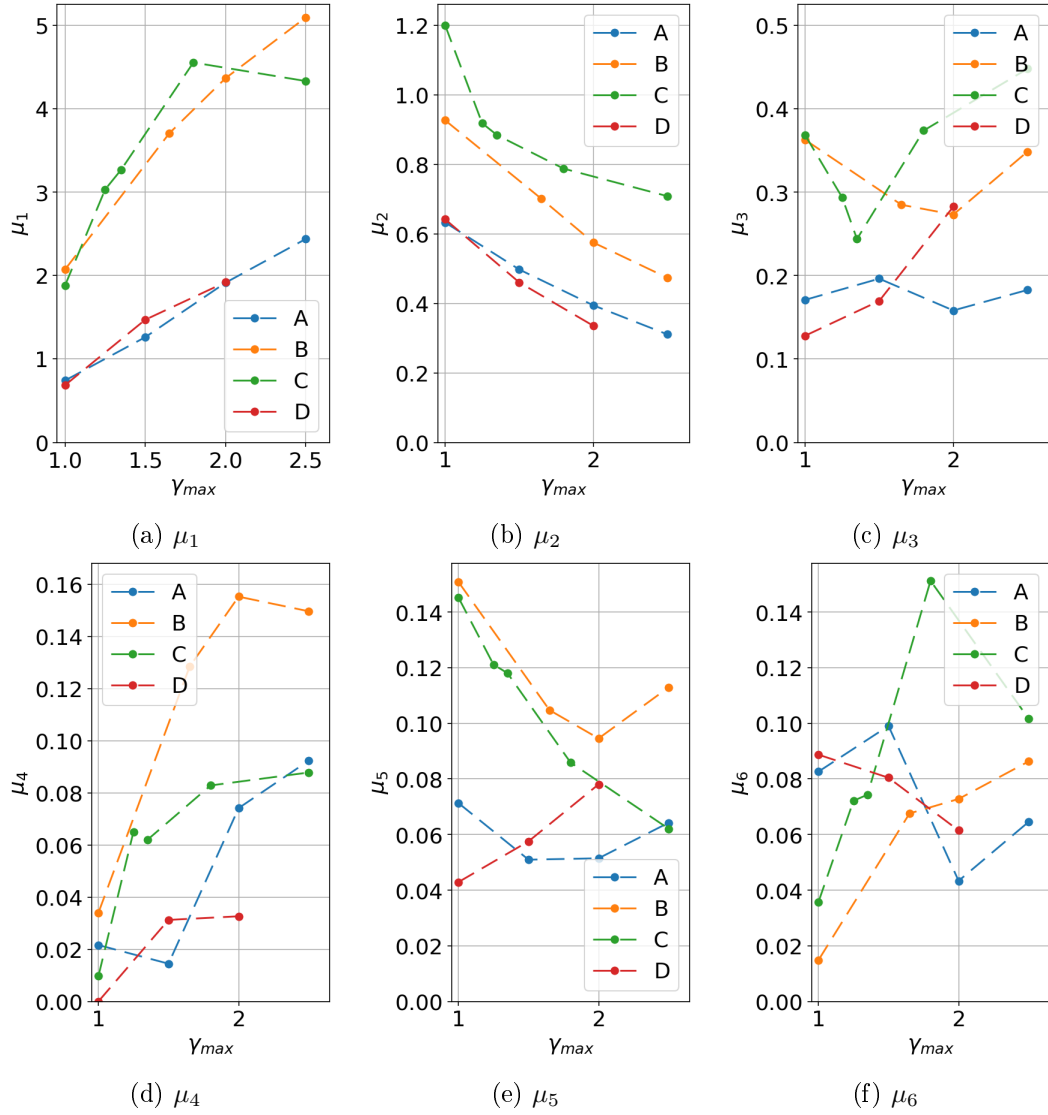


Figure 4.6: Trends of the viscous constants μ_i for $i = 1 \dots 6$ determined from fitting.

4.2 FEM analyses of rubber isolators

A subset of the isolators listed in Table 1.2 was selected for FEM simulation of the shear cyclic test, prioritizing those with a low standard deviation in the shear modulus and damping values, a high number of tests conducted on the same type of isolators, and those that were tested at different maximum strain levels. The FEM simulation results are compared to the results of the full-scale isolators shear cyclic tests.

4.2.1 FEM analysis settings

Finite element analyses were conducted on rubber isolators using the software ANSYS [40]. The 'Static Structural' module was employed and dynamic effects, i.e. in-

ertial forces, were not considered. The 'Nonlinear Mechanical' option was activated to account for material non-linearities, and the 'Large Deflections' setting was used to correctly capture the geometric non-linearities due to significant deformations.

The rubber material was modeled using the 'Yeoh third-order' constitutive law to capture the non-linear elastic behavior. Additionally, viscoelastic effects were incorporated through a 'series of Prony' terms. A linear elastic material model was associated to the steel components. These constitutive laws are defined in terms of the strain energy Φ , defined in Section 2.1.8, on the base of the instantaneous stiffness modulus. Hence, the input elastic parameters are C_{j0}/α_∞ and the viscous ones are α_i (see Section 2.1.8).

In Figure 4.7 the geometrical model of isolator A 04 is represented. The axial symmetry of the cylindrical isolator, combined with shear deformation, allowed the model to be reduced by half through the use of a single plane of symmetry (YZ), thus reducing computational cost. The model includes the portion of isolator from the first rubber layer (in red) at the base to the last at the top, alternating with steel layers (white). The rubber cover on the lateral surface was neglected in the model. The effect of the upper and lower rigid steel plates is obtained imposing a fixed support on the nodes at the base and a rigid body constraint on the nodes at the top surface, through a 'remote point' definition. To this point, the deformation history $d_y(t)$ in the Y direction and zero displacement constraints in the X direction are assigned and zero rotation constraints are applied. The deformation in the vertical (Z) direction is set free because a vertical load has to be applied on the top surface. The displacement of the nodes in the symmetry plane surface is constrained in the normal direction. The edge nodes, belonging to the top or to the base surface and to the symmetry surface, are subject to two displacement constraint (X direction and rigid body), which are compatible with the imposed deformation.

The body of the isolator is represented by solid 20-nodes hexahedral elements. In Figure 4.8 views of the mesh are reported as an example. Nodes on the separation surface between rubber and steel layers are shared by adjacent elements. Size and type of the elements are determined with a sensitivity analysis, as shown in the following.

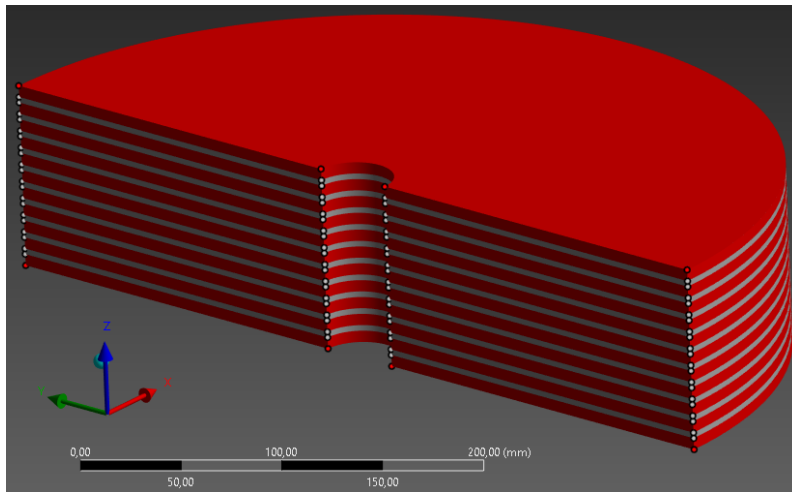


Figure 4.7: Geometrical model of half isolator A 04.

The deformation and load histories, reported in Figure 4.9, are divided in 3 steps.

1. Compression of the isolator in the vertical direction (Z), normal to the top

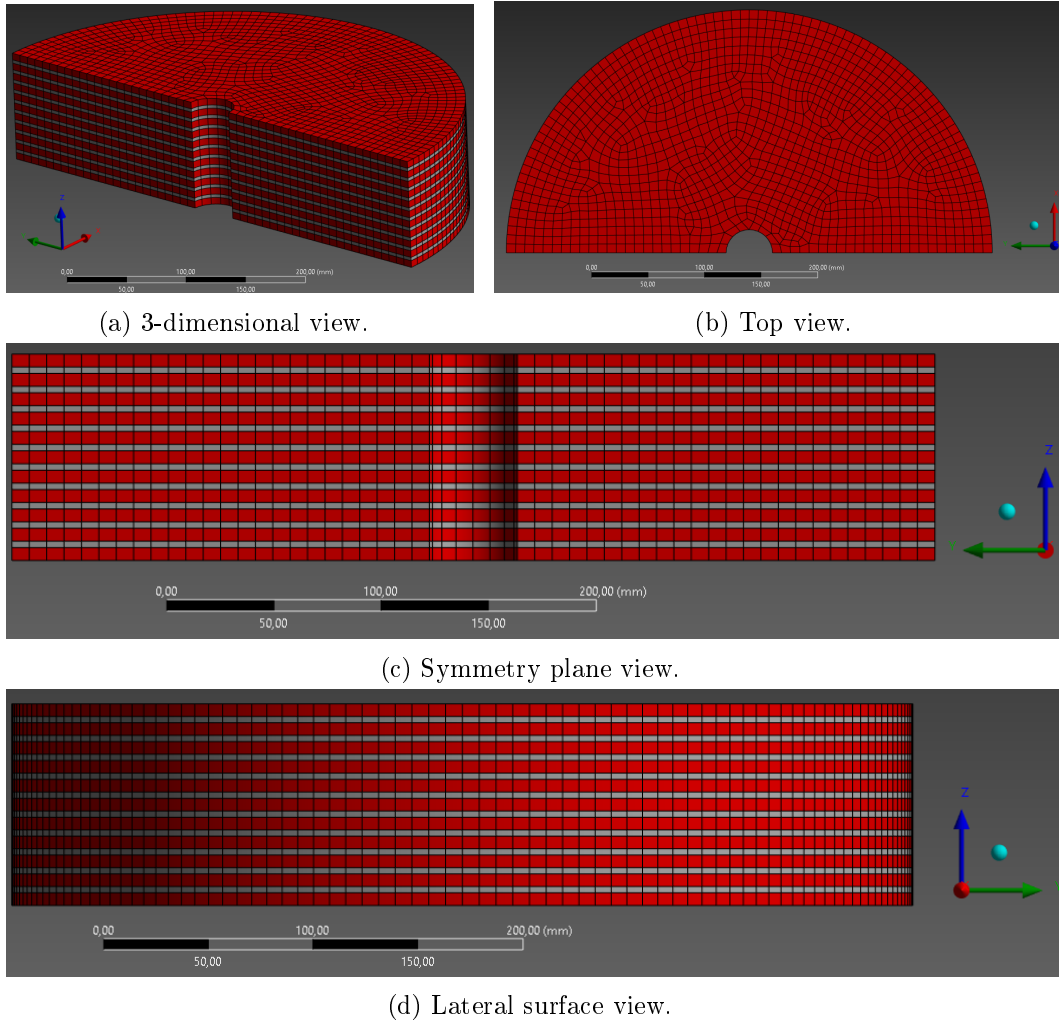


Figure 4.8: Hexahedral elements division of the geometrical model of half isolator A 04.

surface, applying a linearly increasing pressure σ_z up to 6 MPa in 10 s and maintaining it constant until the end of the analysis. Since the effect of viscous terms in this step is not relevant for the computation of shear modulus and damping of the isolator, integration time step can range between a minimum of $\Delta t_{min} = 1 \cdot 10^{-3}$ s and a maximum of $\Delta t_{max} = 2$ s.

2. A 30 s pause to allow most over-stresses to dissipate. Maximum integration time step is increased to $\Delta t_{max} = 5$ s.
3. Two shear deformation sinusoidal cycles performed imposing the displacement $d_y(t)$ on the top surface. Frequency is $f = 0.5$ Hz. Mechanical properties of the isolator are evaluated at the second cycle, instead of the third, because the Mullins effect is not included in the model, thus stability of the stress response is attained already after one cycle. The minimum integration time step is reduced to $\Delta t_{min} = 1 \cdot 10^{-3}$ s while the maximum is determined by a sensitivity analysis.

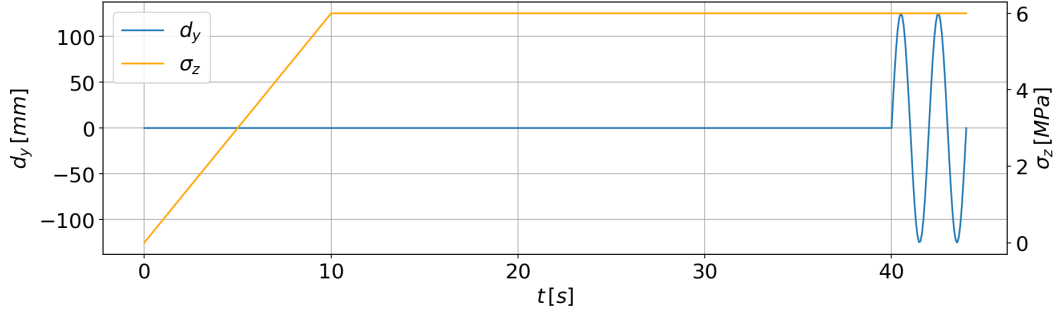


Figure 4.9: Deformation history in Y direction and vertical load (stress σ_z) imposed in the FEM simulation of the isolator A 04 cyclic test.

Sensitivity analysis to mesh size and solid element type

Multiple simulations of the cyclic test were performed on isolator A 04 modeled with 4-node tetrahedral (TETRA4), 8-node (HEXA8) or 20-node (HEXA20) hexahedral solid elements of various sizes. In Figures 4.10 and 4.11 the second cycle stress-strain and stress-time responses, respectively, for four of these analyses are reported. In all cases, each rubber and steel layer was represented with a single element through the thickness and a height/base ratio close to 1. Maximum time step integration during cyclic deformation was set to $1 \cdot 10^{-2}$ s. Since HEXA20 elements can describe quadratic deformation and, consequently, linear strain, they are preferred over linear elements such as HEXA8 or TETRA4 and used to model all the other isolators. The effect is well represented by the hysteresis cycles, which lead to a higher dynamic shear modulus and dissipated energy for the former, with respect to the latter.

With element size of 6 mm the number of nodes is significantly higher ($\sim 210,000$) than the model with 8 mm elements ($\sim 120,000$). The response difference in terms of shear modulus and damping is less than 1.5 %, as it can be seen in Table 4.2, and considered acceptable. Hence, the chosen element dimension is 8 mm. More refined models are not considered because the computational time becomes prohibitive and the advantage in better accuracy is low.

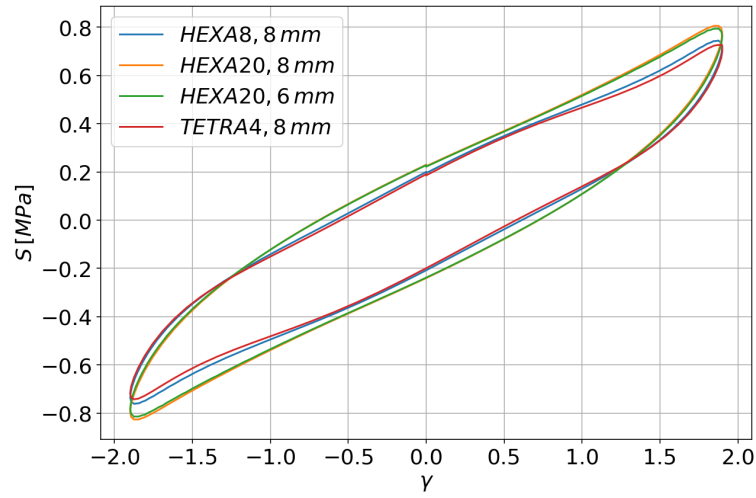


Figure 4.10: Second cycle stress-strain response curves from FEM simulations of cyclic shear tests of rubber isolator A 04 performed with different mesh sizes and element types.

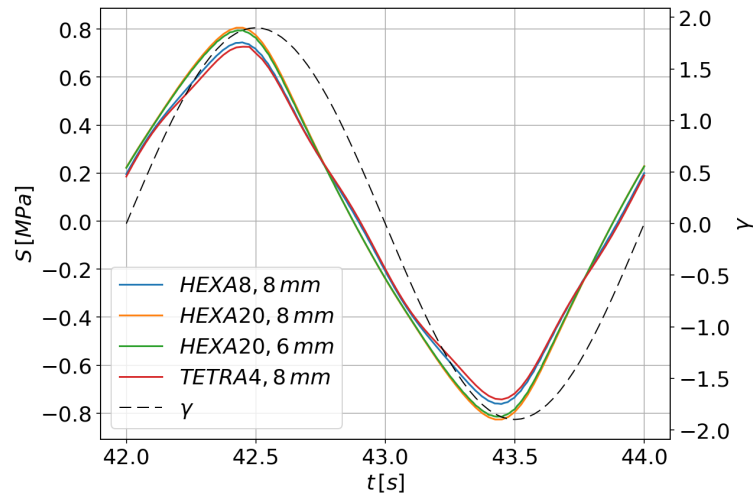


Figure 4.11: Second cycle stress-time response curves from FEM simulations of cyclic shear tests of rubber isolator A 04 performed with different mesh sizes and element types.

Table 4.2: Shear modulus and damping of the isolator A 04 in the second cycle of the FEM simulated cyclic test, modeled with different element types and sizes.

Solid element type - size	ξ [%]	G [MPa]
HEXA8, 8 mm	14.6	0.383
HEXA20, 8 mm	15.7	0.413
HEXA20, 6 mm	15.7	0.407
TETRA4, 8 mm	14.1	0.373

Sensitivity analysis to integration time step

Multiple simulations of the cyclic test with different maximum integration time steps Δt were performed on isolator A 04 modeled with HEXA20 elements of size 8 mm. In particular, the Δt considered were fixed to the values of $5.0 \cdot 10^{-2}$ s, $2.5 \cdot 10^{-2}$ s and $1.0 \cdot 10^{-2}$ s.

In Figure 4.12 the second stress-strain cycle is reported for four of these analyses and the stress versus time of the same analyses is reported in Figure 4.13. A greater time step results, other than in a lower resolution of the output data, in the loss of the shortest time viscous effects, consequently affecting the values of shear modulus and damping reported in Table 4.3. Transitioning from $\Delta t = 2.5 \cdot 10^{-2}$ s to $1.0 \cdot 10^{-2}$ s, the relative difference in shear modulus and damping is about $1.5 \div 2\%$. The maximum integration time step was then set to $1.0 \cdot 10^{-2}$ s for all analyses.

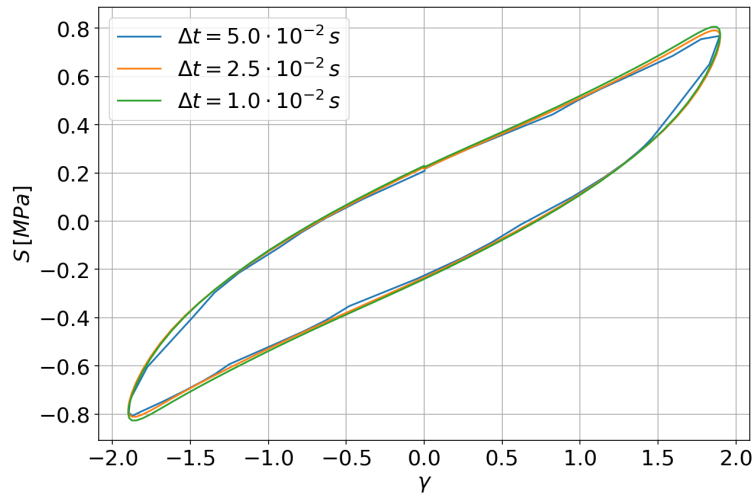


Figure 4.12: Second cycle stress-strain response curves from FEM simulations of cyclic shear tests of rubber isolator A04 performed with different integration time steps lengths.

Table 4.3: Shear modulus and damping of the isolator A 04 in the second cycle of the FEM simulated cyclic test, performed with different integration time steps.

Δt [s]	ξ [%]	G [MPa]
$5.0 \cdot 10^{-2}$ s	14.3	0.411
$2.5 \cdot 10^{-2}$ s	15.4	0.406
$1.0 \cdot 10^{-2}$ s	15.7	0.413

4.2.2 Results of cyclic test FEM simulations

Several FEM simulations of the shear cyclic test on rubber isolators were performed with the settings described previously. An example of the total deformation colormap when the isolator is compressed with a vertical load during one FEM analysis is shown in Figure 4.14a. Note that the maximum strain occurs at the center and not on the lateral surface, where it would be further constrained by the confinement of the rubber cover. The isolator deformed to γ_{max} is shown in Figure 4.14b.

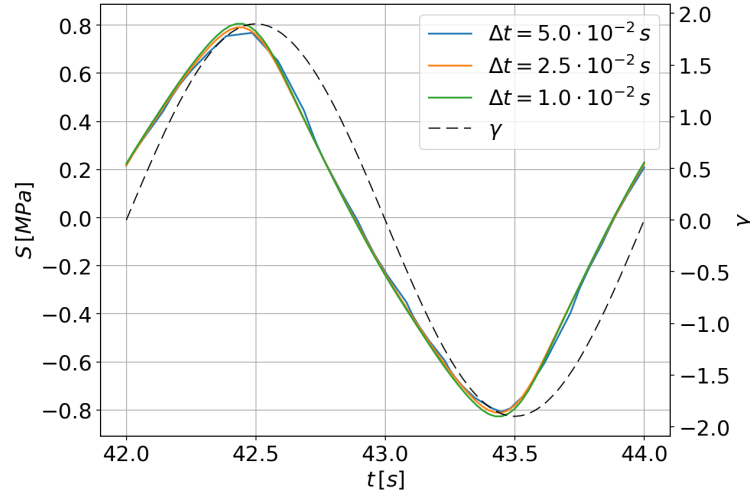
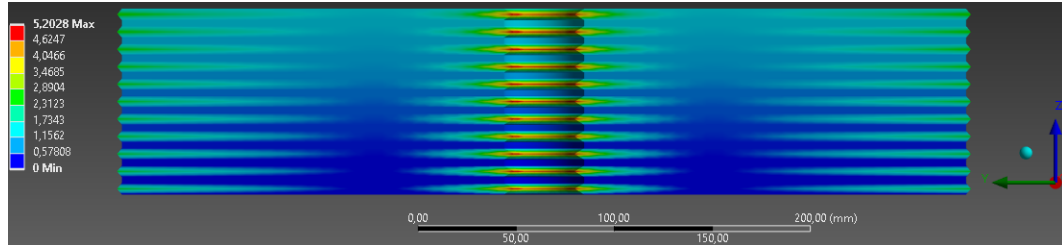
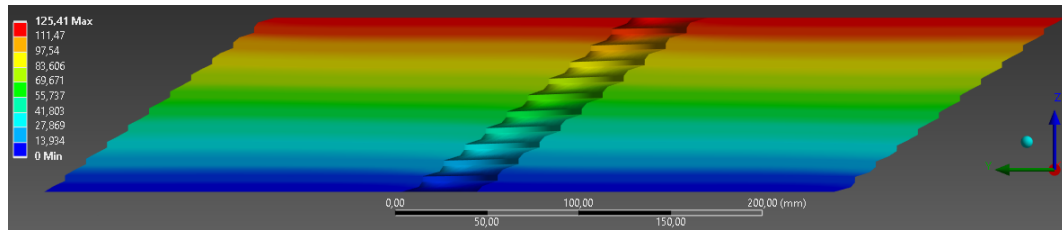


Figure 4.13: Second cycle stress-time response curves from FEM simulations of cyclic shear tests of rubber isolator A04 performed with different integration time steps lengths.



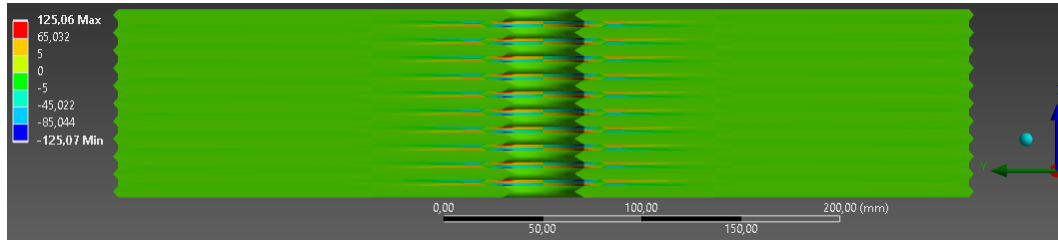
(a) Vertical compression to 6 MPa



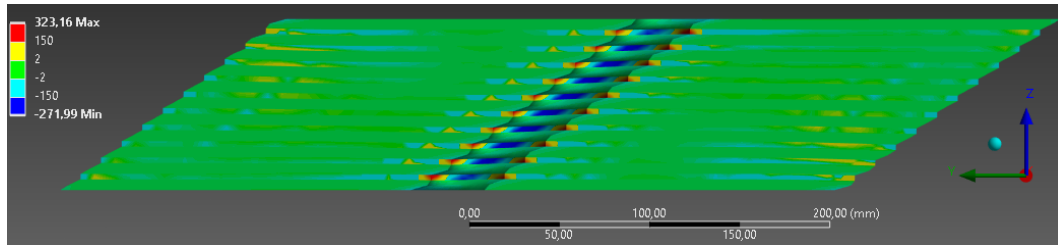
(b) Maximum shear strain γ_{max}

Figure 4.14: Total deformation (displacement magnitude) colormap of isolator A 04 during the cyclic shear test FEM simulation (symmetry surface).

Shear stress in the YZ plane and vertical normal stress distribution on the symmetry surface of the isolator are shown in Figures 4.16 and 4.15, respectively, when the isolator is deformed by compression only and at maximum deformation. Note that local, high and non realistic maximum and minimum stresses are present in both cases. Since the strain-stress relationship defined by Equation 2.48 depends on shear strain raised to the fifth power, and considering that the material model for rubber does not account for damage or softening caused by the Mullins effect, exceeding the maximum shear strain results in a significant and unbounded increase in the stress value. Expression in Equation 3.3 could represent the first step in developing a more detailed model capable of limiting such stresses.

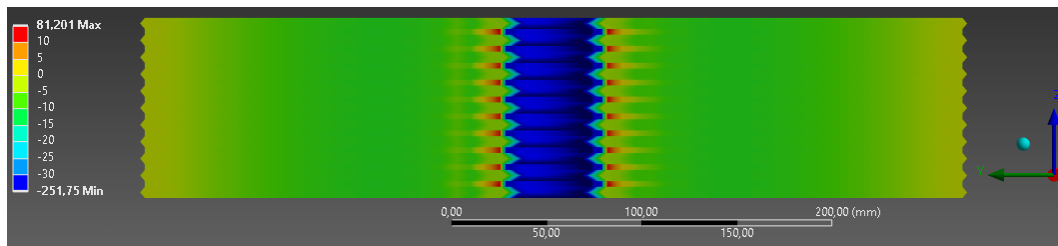


(a) Vertical compression to 6 MPa

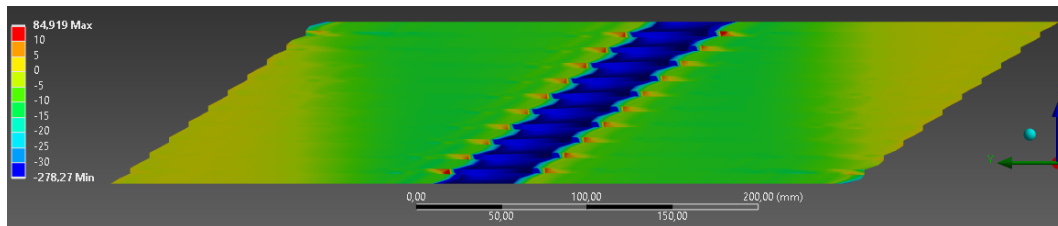


(b) Maximum shear strain γ_{max}

Figure 4.15: Shear stress σ_{yz} colormap of isolator A 04 during the cyclic shear test FEM simulation (symmetry surface).



(a) Vertical compression to 6 MPa



(b) Maximum shear strain γ_{max}

Figure 4.16: Vertical stress σ_z colormap of isolator A 04 during the cyclic shear test FEM simulation (symmetry surface).

In Table 4.4 the damping ξ and the shear modulus G computed from the FEM analyses results are compared to the ones determined from tests on full-scale isolators and relative errors are reported. Also the values from Table 3.3 are repeated for comparison.

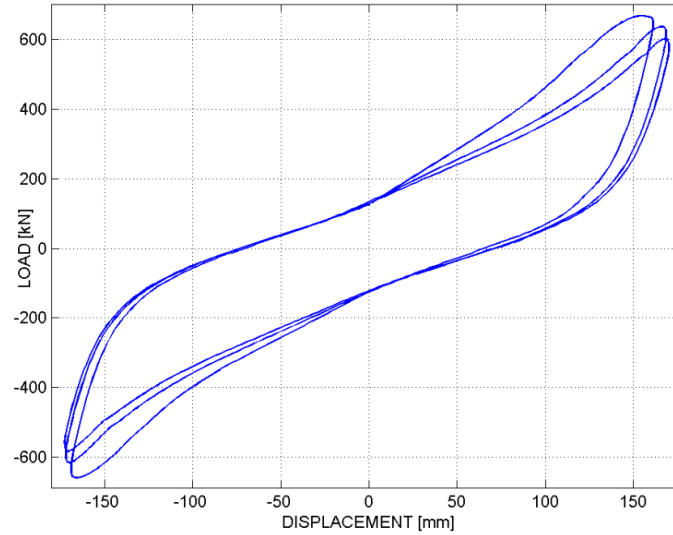
Table 4.4: Damping and shear modulus of the real scale tested rubber isolators compared to the FEM simulated and to the simple shear cyclic tests on rubber specimens ones.

R	I	$\bar{\gamma}_{max}$ [%]	$\bar{\xi}_{is}$ [%]	$\bar{G}_{is}$ [MPa]	ξ_{FEM} [%]	G_{FEM} [MPa]	ϵ_{ξ} [%]	ϵ_G [%]	ξ_{sp} [%]	G_{sp} [MPa]
A	04	190	15.1	0.422	15.7	0.413	+4.0	-2.2	10.4	0.404
C	01	115	16.6	1.210	19.0	1.062	+14.4	-12.2	17.9	1.160
C	01	179	14.2	1.063	19.1	1.207	+34.6	+13.5	14.9	1.050
C	05	134	16.4	1.424	19.1	1.12	+16.0	-21.3	16.9	1.093
C	10	136	15.6	1.313	18.7	1.145	+19.8	-12.8	16.8	1.090
D	02	100	14.1	1.240	13.4	1.349	-5.0	+8.8	11.6	1.286
D	02	150	12.9	1.166	12.7	1.266	-1.5	+8.6	9.7	1.199
D	02	200	11.4	1.100	12.3	1.272	+7.4	+15.6	8.4	1.144

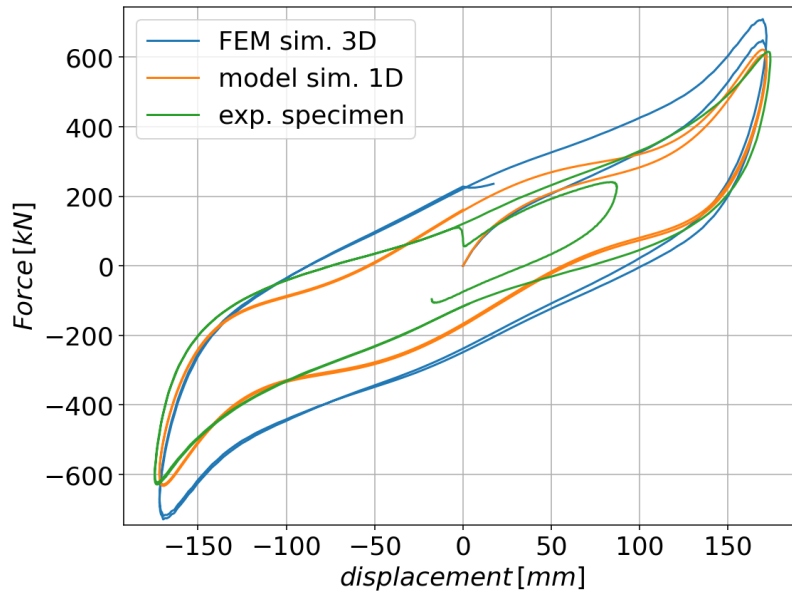
The simulation results demonstrate good accuracy in the estimation of damping and shear modulus in some cases, with errors below 5 %. Only two predicted values have an error greater than the 20 % imposed by the UNI EN 15129 standard.

Comparing the values of shear modulus and damping in Table 4.4 with those obtained from cyclic tests on the rubber specimens presented in Section 3.4 and reported in the last two columns of the table, it can be noted that while cyclic tests provide an accurate estimation of the properties of certain isolators (e.g., C01 for $\gamma_{max} = 179\%$), they may yield poor results in simulations. Conversely, in other cases (e.g., A04 for $\gamma_{max} = 190\%$), the cyclic tests do not estimate the properties accurately, but the simulations perform better.

In Figure 4.17 the force-displacement responses of real and simulated cyclic tests are shown. In Figure 4.17a the cyclic shear test response of full-scale test on isolator C 01 at $\gamma_{max} = 179\%$, which was taken from one of the available laboratory reports, is shown. Comparison with the other curves in the figure is made graphically and in terms of force and displacement, instead of stress and strain, because the recorded data is only available in this form. In Figure 4.17b the FEM simulation of the cyclic test on the same isolator is shown (FEM sim. 3D). In the same figure, the simulated response of the mono-dimensional visco-elastic model of rubber, as described in Section 2.1.5, used in the FEM simulation (model sim. 1D), and the one of a rubber specimen tested to a cyclic test (exp. specimen) are also shown. These two curves are obtained scaling the stress-strain curves by the thickness of the isolator, to obtain the displacement, and by the cross-section area to obtain the force. It can be observed that the cycle of the simple shear experimental test on the specimen aligns most closely with that of the full-scale isolator test. By contrast, the FEM simulation test cycle is broader, which accounts for the significantly deviation from the laboratory test results.



(a) Real-scale isolator stress response to a cyclic test.



(b) Three-dimensional FEM simulation of the isolator and mono-dimensional force-displacement response in a cyclic test.

Figure 4.17: Force responses during cyclic test FEM simulation and full-scale cyclic test of isolator C01 at maximum shear strain $\gamma_{max} = 179\%$, rubber visco-elastic model cyclic test simulation, rubber specimen cyclic test.

Chapter 5

Effectiveness of TMD in Reducing Damage Caused by Strong Earthquake in a Medium-Rise Building

The previous chapter focused on elastomeric isolators, exploring their mechanical behavior and finite element simulation. Simplified bilinear constitutive models of isolators can be derived from laboratory tests or finite element simulations to represent their contribution to stiffness and energy dissipation in dynamic analyses of structures equipped with base isolation or other types of seismic protection systems. In this chapter, a numerical application is presented to investigate the seismic performance of a building equipped with a Tuned Mass Damper (TMD) system at its top, where isolators are used to obtain the stiffness and damping of the system. Moreover, the performance of the building with the TMD is compared to that of the same building with a base isolation system.

A case study where a TMD was installed at the top of a five-story reinforced concrete (RC) building is presented. The aim of the study was to investigate the effectiveness of the TMD in changing the structural behavior of medium-rise existing buildings from dissipative to non-dissipative in order to eliminate reparation or demolition costs resulting from damages caused by strong earthquakes.

The TMD mass is made by a RC slab lying on flat surface sliders. Horizontal stiffness and damping of the TMD are both provided by lead rubber isolators in the first proposed solution and by low-damping rubber isolators and viscous linear dampers, respectively, in the second.

The improvement in the building structural behavior attained with the installation of the TMD was assessed by considering the flexural demand over capacity ratios of structural elements and the energy dissipated by the TMD. These results are compared with those of the same building retrofitted with a base isolation system.

In both of the proposed solutions, TMD remarkably changed the modal behavior of the considered building, improved the flexural verifications, and dissipated most of the input seismic energy. It is therefore demonstrated that a TMD is a valid solution for the retrofit of medium-rise existing buildings.

The presented study was published in Applied Sciences [96].

5.1 Introduction

Tuned mass dampers (TMDs) are mechanical systems that are widely used to reduce the amplitude of dynamic vibrations transmitted by the support to machines or structures in order to guarantee their correct functioning or avoid damages. In the field of construction, a TMD is usually placed on the top floors or on the rooftop of buildings to reduce their swaying due to wind forces. However, the seismic response of buildings can also benefit from TMDs [41, 42], as they can be used as passive supplemental energy dissipation devices. In this case, the TMD frequency is designed to be tuned with the frequency of the main vibrational mode of the building in order to resonate out of phase with the building structure. Therefore, the effects of the main vibrational mode of the building during seismic behavior are mitigated, and the inertial forces arising in the structural members due to seismic forces are reduced.

In analytical applications, the mechanical behavior of TMDs is represented by a mass connected to the main structure by a spring and a dashpot arranged in parallel. In this schematization, the TMD behavior is described by its mass, frequency and damping, which are designed on the basis of the main vibration modes of the building. In these applications, the TMD can be any device that behaves like such a system, e.g., a pendulum with dampers or a concrete slab on rubber isolators.

In the literature, several TMD design methods exist [43–79]. Among these, those developed in the last 15 years [54–79] propose advanced procedures aiming to optimize the TMD design in order to obtain the highest benefits regarding the seismic response of the building. The most recent design methods either provide formulas coming from closed-form solutions of mechanical models [55–69] or consist of genetic algorithms that output the best design parameters [70–76]. All these methods focus on the optimization of one or more control parameters characterizing the seismic response of the building equipped with the TMD, e.g., the minimization of the highest floor acceleration or displacement or the maximization of energy dissipated by the TMD. The effectiveness of the design method in reducing the internal forces in structural members is fundamental and depends both on the loads considered in the design process and on the chosen control parameter. Actually, an inadequate design due to, for example, neglecting the soil–structure interaction effects on the building in the TMD design [66], can cause the plasticization of some structural members, leading to a decrease in the vibration mode frequencies. As a consequence, the resonance between the TMD and the building main vibrational modes vanishes, and a detuning of the TMD occurs. Detuning conditions can also arise during the building lifespan due to variations in the parameters affecting the vibration frequencies. This occurs when an increase in the building mass or a change in the stiffness occurs due to degradation of the rubber layers in seismic isolators.

Since a correctly designed TMD can remarkably improve the behavior of the structure under horizontal loads, several applications to different types of structures, such as tall buildings, skyscrapers, bridges and wind turbines [80, 81], have been studied by researchers. However, only a few applications of TMD to existing buildings for their seismic retrofit have been found. These include: the installation of TMD on the top of multi-story buildings sited in Armenia (Assatourians A. and Mehrdoust M. [82]) to reduce the base shear and the top floor displacements; the roof substitution of the city tower in Rieti, Italy, including a TMD, to increase the vibrational periods and limit the displacements' amplitude of the tower (Clemente P. [83]); the application of a TMD on the top of a 13-story building to reduce structural and non-structural damage (Villaverde R. [84]); the modification of the penthouses of

ten buildings in order to make them act as masses of TMDs to reduce the overall seismic demand on structural members (Johnson J. et al. [85]). Such a limited use of TMDs in the retrofit of existing buildings is probably due to the increase in the building total mass and, thus, gravity loads.

In this chapter, a case study where a TMD is installed at the top of a medium-rise reinforced concrete (RC) building is presented. The building structure without the TMD was initially designed with dissipative behavior in accordance with the Italian Building Code [86]. The aim of the TMD application is to modify the structural behavior of the building from dissipative to non-dissipative in order to eliminate reparation or demolition costs after a strong seismic event. In the considered case study, the mass of the TMD was made by an RC slab lying on flat surface sliders (FSSs). Two different solutions are proposed to provide horizontal stiffness and damping to the added system. One involves the use of rubber isolators (RIs), which replace some of the FSSs installed along the perimeter of the roof. The other is based on the use of low-damping rubber isolators (LDRIs) and viscous linear dampers (VLDs) installed along the roof perimeter, together with the FSSs. The design of the TMD was performed in accordance with the procedure described by Christopoulos in [50]. The effects of the TMD application on the internal forces arising in the structural members will be discussed in comparison with those acting in the original building (OB). The improvement in the building structural behavior provided by the TMD is assessed on the basis of the flexural demand over capacity ratios and the energy dissipated by the TMD.

5.2 Case Study

The case study is a six-story residential building located in Tolmezzo, Italy. The building plan is rectangular with dimensions of $18.15\text{ m} \times 11.90\text{ m}$ (Figure 5.1) and is the same for all five floors. The inter-storey height is 3.00 m , except for the ground floor, which is 3.30 m high. The bearing structure against seismic actions is made by RC walls placed along the building perimeter (walls S1–S6 in Figure 5.2) and on the sides of the stairwell and the elevator shaft parallel to the Y direction (S7–S8 and A1–A2 in Figure 5.2). The resistance against vertical loads is provided by both walls and columns (P1–P9 in Figure 5.2).

Columns and walls are connected at floor level by beams oriented along the building main directions (X and Y in Figure 5.1), following the dash-dotted grid in Figure 5.2. Walls S7–S8, A1–A2 rise above the flat rooftop, as shown in Figure 5.3, to allow the access to the roof and the housing of the elevator technical compartment. According to the dimensions shown in Figures 5.1 and 5.3, the slenderness of the building is calculated as the ratio between the building height and the width of its base, which is equal to approximately 1 in the X direction and 1.5 in the Y direction. A building can be considered a skyscraper if this ratio is at least almost 10. Hence, the considered case study falls in the range of medium- or even low-rise buildings. The cross-sections of vertical elements stay the same for all the storeys; only the walls' longitudinal reinforcements are reduced from the fourth floor upward, where steel bars of diameter $\phi 20$ change to $\phi 16$. The dimensions of structural members cross-sections are shown in Table 5.1. The reinforcement layouts at the base of vertical elements are shown in Figure 5.4. Floors are made by RC joists with a depth of 20 cm , a 5 cm thick RC slab at the top, and hollow clay blocks as lightening elements. The building foundations are made of a RC slab with dimensions of

19.15 m $\times$ 12.90 m $\times$ 0.80 m. The materials used for the building design are concrete of Italian grade $C35/45$, with characteristic compressive strength $f_{ck} = 35$ MPa and Young's modulus $E_c = 34$ GPa, and reinforcing steel of Italian grade $B450C$, with characteristic yield strength $f_{yk} = 450$ MPa and Young's modulus $E_s = 210$ GPa. Both concrete and steel are assumed to be linear materials in the finite element model of the building.

Table 5.1: Cross-sections of structural members and type of finite element used for their modeling.

Element	Type	Section	Finite element
P1-9	Column	0.55 m $\times$ 0.55 m	Frame
S1-4	Wall	0.30 m $\times$ 3.80 m	Shell
S5-8	Wall	0.30 m $\times$ 3.40 m	Shell
A1-2	Wall	0.30 m $\times$ 2.10 m	Shell
(Beams)	Beam	0.25 m $\times$ 0.50 m	Frame

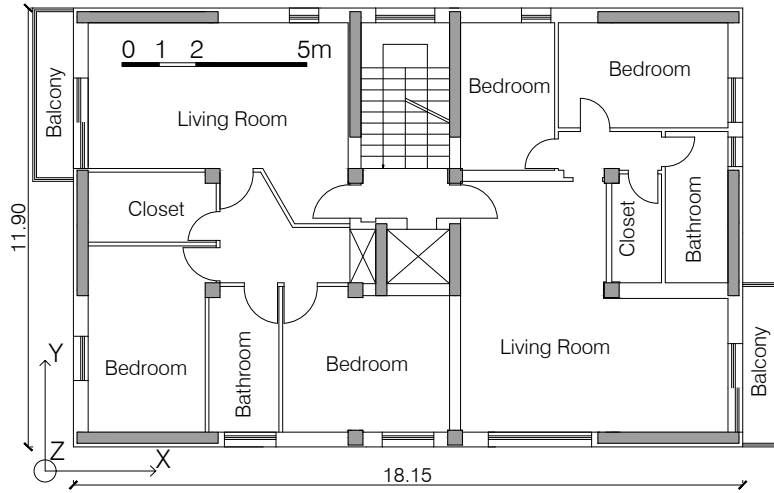


Figure 5.1: Plan of the typical building floor.

The design of the RC structure against seismic actions was performed according to the dissipative behavior required by the Italian Building Code [86] for ductile wall structures, corresponding to a behavior factor $q = 3$. The finite element model (FEM) of the building was made using the software SAP2000 [87]. Beams and columns were modeled through frame elements, while shell elements were used for wall modeling (Table 5.1). Floors were assumed to behave as rigid diaphragms. The effects of cracking on structural members were taken into account by reducing the shear and flexural stiffnesses of cross-sections by 20 % for vertical elements and by 50 % for beams. The considered cracking is mainly due to the vertical loads for beams and to the horizontal loads for columns. In no case were these cracks considered to be connected to a non-linear behavior of the structural elements, which are assumed to still be subjected to elastic deformations even after cracking.

Gravity loads per unit area applied to the FEM of the building are resumed in Table 5.2, where $g_{1,k}$ and $g_{2,k}$ are the permanent structural and non-structural characteristic loads, respectively, and q_k is the characteristic variable load. A distributed

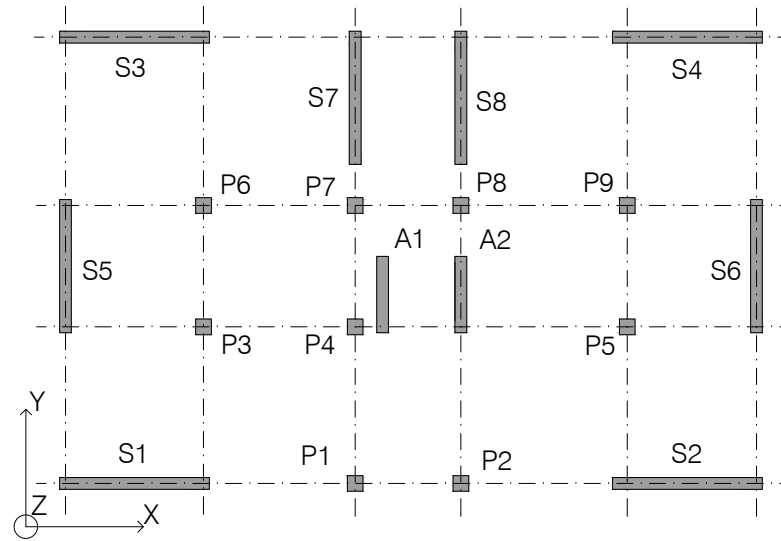


Figure 5.2: Plan view of vertical structural elements.

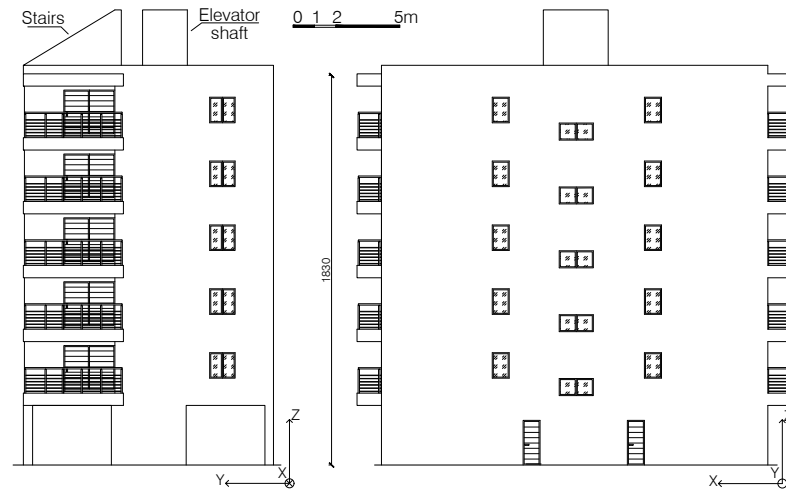


Figure 5.3: Building elevation in X direction on the left, and Y direction on the right.

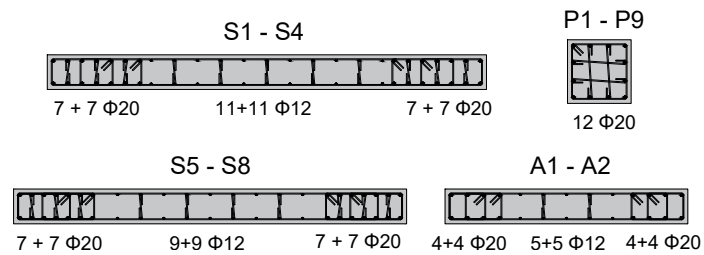


Figure 5.4: Reinforcement layouts in vertical structural elements at the ground floor.

load of 5.6 kN/m was applied to the perimeter beams to consider the self-weight of masonry external infills. The self-weight and self-mass of structural elements were calculated by the software on the basis of their geometric and material properties. The FEM analyses are performed considering the load combinations at the ultimate limit state (ULS) provided by [86] for static and seismic load conditions. For the seismic analyses, the life-safety limit state, in Italian “Stato Limite di salvaguardia della Vita” (SLV), is considered herein. The seismic action was determined considering a building design service life of 50 yr and a ground of type *B*, which involves a return period of the seismic actions of 475 yr. In the following, the ULS acronym is used to indicate the static load combination of actions.

Table 5.2: Gravity loads per unit area.

	$g_{1,k}$ [kN/m ²]	$g_{2,k}$ [kN/m ²]	q_k [kN/m ²]
Floor	3.5	2.7	2.0
Balcony	3.0	1.2	4.0
Rooftop	3.5	1.5	2.0
Stairs	4.3	1.0	4.0

Table 5.3 shows the periods and the participating mass ratios of the most significant modes of the OB, i.e., the modes having a participating mass ratio greater or equal to 5 % and that, globally, provide a mass participating ratio ≥ 85 % in each main direction. T_i indicates the period of the i -th mode, and $\alpha_{i,j}$ indicates the participating mass ratio of the i -th mode in j direction. The values of the participating mass ratio in j direction which contributes the most for each mode are underlined in Table 5.3.

Table 5.3: Periods and mass participating ratios of the most significant modes of OB.

Mode i	T_i	$\alpha_{i,X}$	$\alpha_{i,Y}$	$\alpha_{i,Rz}$	$\Sigma \alpha_{i,X}$	$\Sigma \alpha_{i,Y}$	$\Sigma \alpha_{i,Rz}$
1	0.39 s	<u>66.3</u> %	0.1 %	1.0 %	66.3 %	0.1 %	1.0 %
2	0.37 s	0.2 %	<u>68.3</u> %	0.0 %	66.5 %	68.4 %	1.1 %
3	0.34 s	1.0 %	0.2 %	<u>66.0</u> %	67.5 %	68.6 %	67.1 %
4	0.08 s	0.1 %	<u>18.7</u> %	0.0 %	67.6 %	87.3 %	67.1 %
5	0.08 s	<u>14.8</u> %	0.1 %	0. %	82.4 %	87.4 %	67.1 %
6	0.07 s	3.4 %	0.0 %	<u>6.4</u> %	85.8 %	87.4 %	73.5 %
7	0.07 s	2.0 %	0.0 %	<u>13.4</u> %	87.8 %	87.4 %	86.9 %

A fast non-linear analysis was performed. This is a modal analysis method, implemented by SAP2000, designed to be used for structural systems that are primarily linear elastic but that only have a limited number of predefined non-linear elements. FNA is well-suited for time-history analysis, which is the analysis used in the case study, because of its computationally efficient formulation. This efficiency is due to the separation of the non-linear-objects vector from the elastic stiffness matrix and the damped equations of motions. At each time increment, forces within the non-linear-objects vector are resolved through an iterative process, while the uncoupled modal equations are solved exactly. This allows the formulation to rapidly converge to the equilibrium solution. Regarding the considered case study, the only non-linear

members are the devices used to reproduce the stiffness and the dissipation properties of the TMD, which are specified in the following section. Hence, FNA is a suitable analysis method for the case study. The ground motion is modeled using three different artificial accelerograms acting simultaneously along the horizontal directions X and Y and the vertical direction Z . Three independent analyses were conducted, considering different ground motions. The accelerograms were generated by the software SIMQKE_GR [88], matching their spectrum with the site elastic response spectrum provided by the Italian Code for a 5 % damping coefficient, which is shown in Figure 5.5. The 5 % damping factor takes account of all phenomena related to the building shaking under horizontal loads contributing to energy dissipation in elastic conditions, such as concrete viscous properties and cracking. The accelerograms have a stationary part that starts at time $t = 2$ s and lasts 10 s, for a total duration of 30 s. One of the horizontal accelerograms used in the analyses is shown in Figure 5.6. The accelerometer record presented in Figure 5.6 refers to the life-safety limit state according to the Italian Building Code [86], hence with a 10 % probability of occurrence in 50 years. The construction site of the building, Tolmezzo, is located in one of the areas of Italy with the highest seismicity. Therefore, the accelerogram shown is one of the most demanding for the Italian territory. For the structural members' design, the most disadvantageous internal forces obtained from the three analyses are considered.

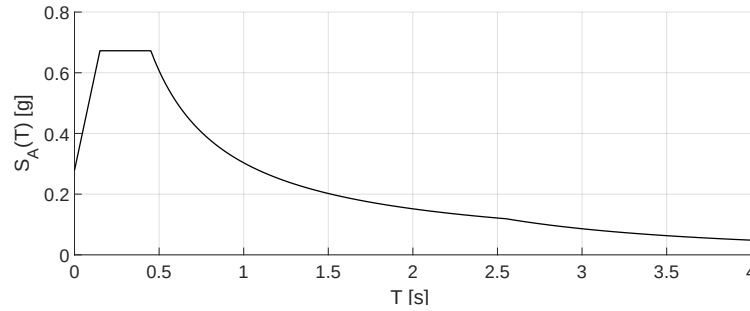


Figure 5.5: Site elastic response spectrum for the horizontal acceleration component according to [86].

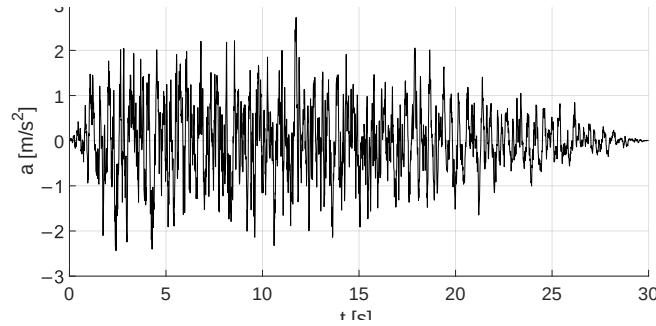


Figure 5.6: Horizontal accelerogram used in one of the seismic analyses of the building.

5.3 TMD Design Method

The mass of the TMD constituted a RC slab, 19.65 m long and 13.40 m wide, overlooking the entire rooftop and protruding by about 1 m from each façade, as shown in Figure 5.7. The slab thickness was constant, and it was assumed as a design parameter of the TMD mass. A steel structure composed of braced frames oriented in the X and Y directions was installed on the rooftop to support the TMD above the stairwell and the elevator shaft, as shown in Figure 5.8, and also to allow for inspection and replacement of the isolation and dissipation devices. The RC slab lay on flat surface sliders (FSSs) anchored to the beam-column joints of the steel structure, which allowed the RC slab to sway out of phase with the building. The correct disposition of LRIs, LDRIs and VLDs in the two proposed solutions will be described in the design process in the next section.

In the steel structure, horizontal (Figure 5.7) and vertical braces (Figure 5.8) were used to provide in-plane stiffness to the plane at the base of the FSSs and horizontal stiffness to the steel frame, respectively. The steel structure self-weight is negligible in comparison with that of the building. Therefore, the increase in the building seismic mass due to this structure is also negligible. However, the design of the TMD was performed taking into account the availability of a new external living space at the level of the original roof and, consequently, the increase in gravity loads due to the variation in the roof intended use. In the FEM of the building, the steel structure was modeled through frame elements, while the RC slab was modeled through shell elements. For the FSS modeling, non-linear link elements of ‘Friction Isolator’ type were used.

At first, a 2% damping factor was considered for the building. However, as the analyses indicated that the structure sustained damage, the damping factor was increased to 5%. A more detailed investigation should account for a lower damping value and incorporate a finite element model with the introduction of plastic hinges, either localized or distributed, to evaluate the actual performance of the TMD, including the potential for detuning.

Since the horizontal stiffness and the energy dissipation contribution of the FSSs are negligible, to provide the TMD system with these properties, two solutions are proposed. In one solution (hereinafter referred to as TMD1), horizontal stiffness and damping are both provided by RIs, which are placed on the perimeter of the steel structure (Figure 5.7). In the other solution (referred to as TMD2), horizontal stiffness is provided by LDRIs and damping by VLDs, which are installed along the perimeter of the steel structure, as can be seen in Figures 5.7 and 5.8.

The construction of the TMD may follow these steps:

- Installation of metal anchors to connect the base of the steel frame columns to the top end of the RC columns and walls of the building;
- Assemblance of the steel frame with columns, beams and vertical and horizontal braces;
- Installation of the FSSs, LRIs (in TMD1) and LDRIs (in TMD2) at the intersection nodes of beams, on the top of the steel frame, in the stairwell and in the elevator shaft, using steel plates to support them, if necessary. A way of access must be guaranteed for maintenance and to inspect and replace the isolation devices;

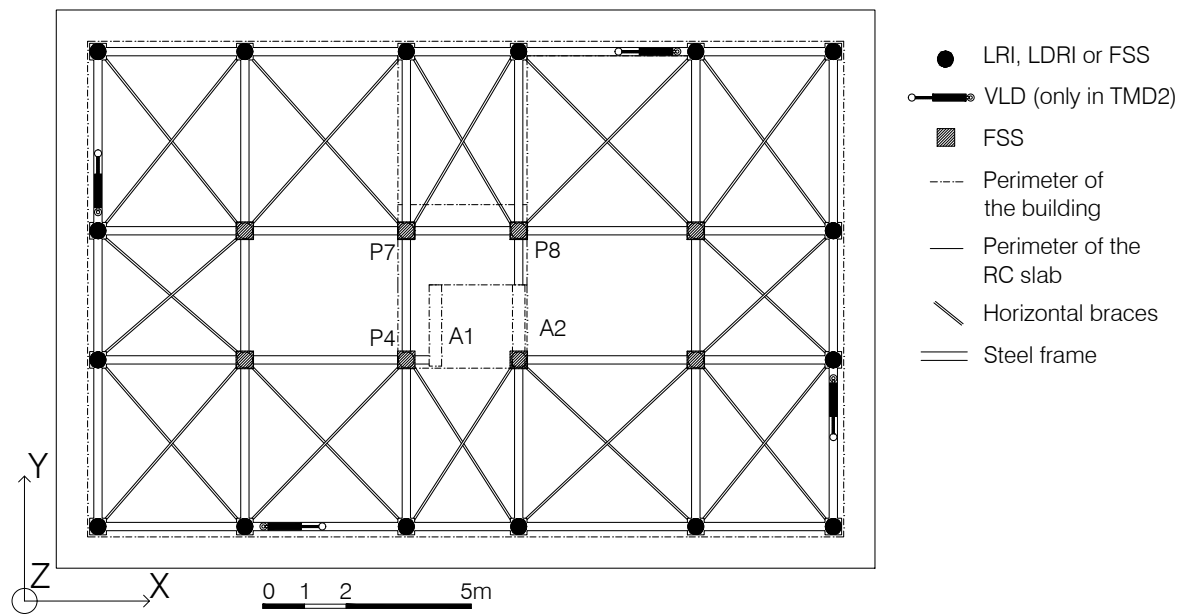


Figure 5.7: Top view of the steel structure supporting the TMD, and arrangement of dampers, isolators and sliders.

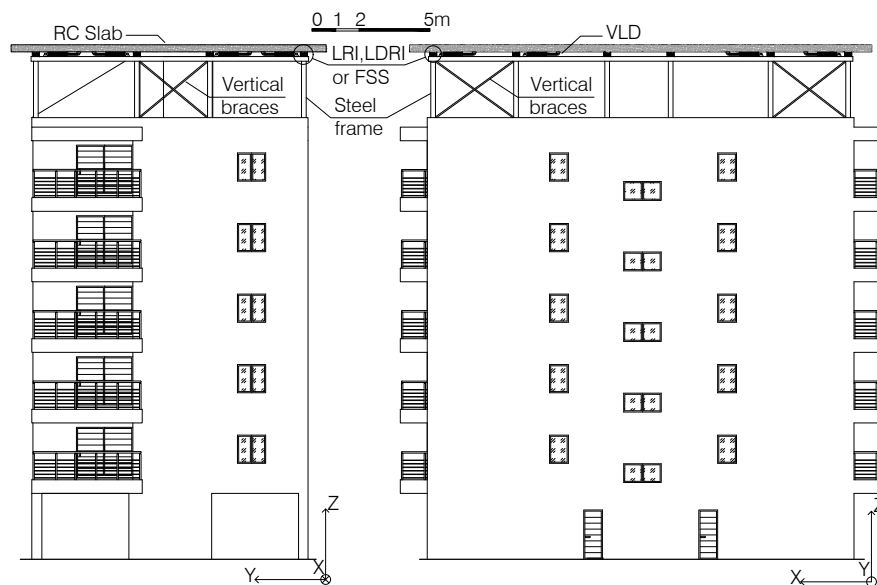


Figure 5.8: Elevation of the building equipped with the TMD in X direction on the left and Y direction on the right.

- Positioning of the form and the steel reinforcement for the concrete slab on the top of the steel frame and pouring of the concrete. A slight slope for water collection is required;
- In TMD2, VLDs can be installed after the curing of the concrete, connecting one end to a beam of the steel frame and the other to the under-surface of the RC slab, far enough away from the isolators to let them deform without making contact with the VLDs.

In both solutions, the design of the TMD is performed according to the analytical method described by Christopoulos in [50]. In this method, the building structure is modeled through a single-degree-of-freedom (DOF) system of mass M and stiffness K , connected to the TMD mass, m , by a linear elastic spring of stiffness k and a linear viscous damper with viscous coefficient c , disposed in parallel, as shown in Figure 5.9. Therefore, the TMD behavior is completely described by the three parameters m , k and c .

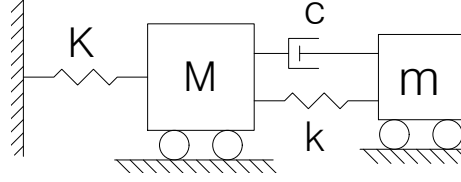


Figure 5.9: Schematization of the structure of the TMD system.

The dynamics of the two-DOF system in Figure 5.9 when it is subjected to acceleration $a(t)$ at the base are described by Equations (5.1) and (5.2).

$$Ma_{OB}(t) + c[v_{OB}(t) - v_{TMD}(t)] + Kd_{OB}(t) + k[d_{OB}(t) - d_{TMD}(t)] = -Ma(t) \quad (5.1)$$

$$ma_{TMD}(t) + c[v_{TMD}(t) - v_{OB}(t)] + k[d_{TMD}(t) - d_{OB}(t)] = -ma(t) \quad (5.2)$$

where a_{OB} , a_{OB} , v_{OB} , v_{TMD} , d_{OB} and d_{TMD} are the accelerations, velocities and displacements of OB and TMD, respectively, relative to the base.

The design method is based on the minimization of the building displacements caused by a random acceleration time history acting at its base. This criterion allows the limitation of the stress demand on structural members and the avoidance of damage to non-structural elements.

The design is carried out independently for motion along the X and Y directions. The TMD parameters obtained for each direction may be different due to the different properties (periods and participating mass ratios) of the main vibration modes along the main directions. The periods and the participating mass ratios of the main translational modes of the building considered in the case study are similar. Hence, the TMD parameters are calculated on the basis of the mode with the higher participating mass ratio. Otherwise, a TMD solution providing different stiffness and damping along the main directions of the building would have to be considered.

The determination of the TMD parameters m , k and c is divided into the following steps:

1. Definition of the mass ratio μ between the TMD mass, m , and the effective mass of the main vibration mode of the building without the TMD, M_i . In general,

the choice of m has to be made taking into account the bearing capacity of the structural members. In this case study, the structure can easily support, at ULS, an additional vertical load up to about 500% of the total mass of the building. In order to optimize the choice of m , the design procedure proposed by Luft [89] is used herein. In this procedure, the mass ratio is determined as

$$\mu = 16(\xi_{eq} - 0.8\xi_i)^2 \quad (5.3)$$

where ξ_i is the viscous damping factor of the i -th vibration mode of period T_i and the participating mass ratio α_i , and ξ_{eq} is the damping factor of the acceleration and displacement spectra, S_A and S_D , respectively. ξ_{eq} is determined so that the following inequalities are satisfied:

$$a_{max} \geq \alpha_i S_A(T_i, \xi_{eq}) \quad (5.4)$$

$$d_{max} \geq \alpha_i S_D(T_i, \xi_{eq}). \quad (5.5)$$

In (5.4) and (5.5), a_{max} and d_{max} are the target maximum acceleration and displacement at the top of the building, respectively, that can be borne without damage by the structural members. The values of a_{max} and d_{max} are determined by scaling the amplitude of the accelerograms.

2. Calculation of the frequency ratio, f , between the frequencies of the TMD and of the structure with the following equation

$$f = \frac{\sqrt{1 - \frac{\mu}{2}}}{1 + \mu} \quad (5.6)$$

and then the TMD stiffness, k ,

$$k = m\Omega_i^2 f^2 \quad (5.7)$$

where $\Omega_i = \sqrt{\frac{K_i}{M_i}}$ is the frequency of the i -th vibration mode of the building.

3. Calculation of the damping factor, ξ , and the viscous damping coefficient c of the TMD with the following equations:

$$\xi = \sqrt{\frac{\mu(1 - \frac{\mu}{4})}{4(1 + \mu)(1 - \frac{\mu}{2})}} \quad (5.8)$$

$$c = 2\xi\omega m \quad (5.9)$$

where $\omega = \sqrt{\frac{k}{m}}$ is the TMD natural frequency.

The values of k and ξ are then used in the design of isolators and dampers.

After the definition of these parameters, the design of the TMD is completed by the implementation of the FEM of the building and the performance of its analysis. The choice of the model input parameters could be improved by means of on-site experimental dynamic tests on the existing building in order to obtain information about the building vibration modes. This information could be used to calibrate the FE model. In this way, the FEM-based approach to designing the TMD would better take into account the real behavior of the building. Another possibility to improve the design of the TMD is to realize a simple to-scale experimental dynamic test of a two-DOF system, such as the one in Figure 5.9, where the two carts may be moved by a vibration table or an actuator, which can apply an accelerogram to their base. The mass M should represent the building mass activated by the main vibration mode, and K the corresponding stiffness. Different values should be chosen for the mass, the damping and the stiffness of the TMD to select the best combination that minimizes the force acting on the mass M . This force can be measured through a load cell.

In both cases, a direct-integration non-linear analysis should be conducted to assess the real behavior of the building with TMD.

5.3.1 TMD with RIs (TMD1)

In the first proposed solution, RIs are employed to provide the TMD with both stiffness and damping. The RI damping ratio, ξ_{is} , usually ranges between 10% and 15%. The use of lead-rubber isolators (LRIs) can be considered when a higher damping coefficient is required, up to 35%.

To implement RIs in the FEM of the building, the non-linear link element called ‘Rubber Isolator’ is used. This element has a bilinear force–displacement (F-d) relationship, like the one shown in Figure 5.10. The first branch of the diagram, of slope k_1 , describes the isolator behavior up to the yielding point of coordinates (d_1, F_1) , while the second branch, with slope k_2 , describes the post-yielding behavior up to the maximum force F_2 and the maximum displacement d_2 . k_{eq} is the secant stiffness corresponding to the displacement that the RI undergoes.

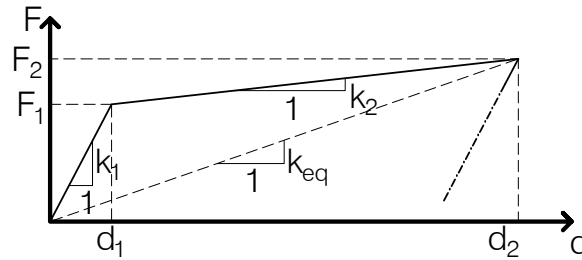


Figure 5.10: Bilinear force–displacement relationship of the ‘Rubber Isolator’ element.

The isolators’ design consists of choosing the number of devices and of determining their stiffness k_{eq} and equivalent damping factor ξ_{is} through the design method described above. In particular, k_{eq} is obtained by dividing the value of k obtained from Equation (5.7) by the number of isolators, and ξ_{is} coincides with the value of ξ calculated with Equation (5.8). Then, the values of k_1 , F_1 and k_2 are computed with the formulas provided by FEMA [90].

5.3.2 TMD with LDRI and VLDs (TMD2)

In addition to the FSSs, the second solution employs LDRI and VLDs to provide the TMD with horizontal stiffness and damping, respectively.

LDRI are composed of layers of natural rubber (without the usual additives, such as carbon black) alternating with steel shims. Since they have a damping factor of about 2%, in the FEM of the building, LDRI are modeled through linear elastic springs of stiffness k_{LDRI} . Conversely, the non-linear behavior of VLDs is modeled in SAP2000 through the non-linear link ‘Damper - Bilinear’, in which a linear elastic spring of stiffness k_{VLD} and a dashpot are connected in series. The force in the dashpot is directly proportional to the relative velocity v of its ends through a damping coefficient c_{VLD} up to the maximum force F_{max} , as shown in Figure 5.11. The values of F_{max} and k_{VLD} are provided by the manufacturers of the device.

The value of k_{LDRI} is calculated by dividing k (from Equation (5.7)) by the number of LDRI, while the value of c_{VLD} is calculated dividing c (from Equation (5.9)) by the number of VLDs.

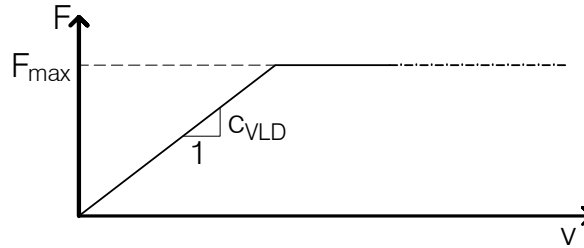


Figure 5.11: Bilinear force–velocity relationship of the ‘Damper–Bilinear’ element.

5.4 Results

The design of the TMD is performed with the aim of changing the building structural behavior under SLV seismic actions from dissipative to non-dissipative. This aim is set to avoid damages to structural and non-structural elements at the SLV in order to eliminate costs for the building reparation. In regards to structural elements, damage is considered as the exceeding of the elements’ yield strength.

It is also fundamental to avoid TMD detuning. Actually, plasticization of structural members induces a modification of the building stiffness and vibrational modes, thus changing the building properties considered in the design process. According to [86], the structure behavior is non-dissipative if its members work in the so-called ‘substantially elastic field’.

The internal forces in the isolated structure are divided by a factor of 1.5, as allowed by the Italian Code [86] and the Eurocode [91]. For all vertical structural members, strength verifications are carried out at the base (0.00 m), at the top of the building (18.30 m) and at the level where the walls’ longitudinal reinforcements are reduced (12.30 m).

The properties of TMD1 and TMD2 derived from the design process are presented. The effects of TMD1 and TMD2 on the building modal behavior, strength verifications and energy dissipation are also provided. Results obtained for the building with TMD1 and TMD2 are compared with those of the OB and with those of the building seismically protected by a base isolation system (BIS).

5.4.1 Properties of TMD1 and TMD2

As discussed in Section 5.3, the TMD design is based on the main vibration modes of the building. Since the installation of the TMD on the building rooftop requires the steel frame structure described in Section 5.3, the vibration modes to consider are those of the OB with the steel structure on the top (OBS). Table 5.4 shows the periods and the participating mass ratios of the most significant modes of the OBS.

Table 5.4: Periods and mass participating ratios of the most significant modes of OBS.

Mode i	T_i	$\alpha_{i,X}$	$\alpha_{i,Y}$	$\alpha_{i,Rz}$	$\Sigma \alpha_{i,X}$	$\Sigma \alpha_{i,Y}$	$\Sigma \alpha_{i,Rz}$
1	0.41 s	<u>66.3</u> %	0.2 %	1.0 %	66.3 %	0.2 %	1.0 %
2	0.40 s	0.2 %	<u>68.3</u> %	0.0 %	66.5 %	68.5 %	1.0 %
3	0.35 s	1.0 %	0.0 %	<u>66.2</u> %	67.5 %	68.6 %	67.3 %
4	0.09 s	0.0 %	<u>18.4</u> %	0.0 %	67.5 %	87.0 %	67.3 %
5	0.09 s	<u>19.5</u> %	0.0 %	0.1 %	87.1 %	87.0 %	67.5 %
6	0.08 s	0.1 %	0.0 %	<u>19.2</u> %	87.2 %	87.0 %	86.8 %

From Table 5.4, it can be seen that the first two modes are translational in the X and Y directions, respectively, and the third is rotational. These three modes have similar periods, ranging between 0.35 s and 0.41 s, and very similar participating mass ratios in the main direction, varying between 62.2 % and 68.3 %. The remaining modes have lower periods, ranging between 0.08 s and 0.09 s. The modes of the OBS are practically identical to those of the OB (Table 5.3).

The design method requires the values of maximum acceleration a_{max} and maximum displacement d_{max} at the top of the building to be fixed. In order to maintain the structural members in the elastic field and to avoid damage to masonry infill panels under the seismic actions at the SLV, the values $a_{max} = 3.26 \text{ m/s}^2$ and $d_{max} = 0.011 \text{ m}$ were taken. These values lead to a mass ratio equal to 0.51. As the effective mass of mode 2, M_2 equal to 1183 t, an RC slab thickness of z_{TMD} equal to 0.90 m is derived. If the TMD design was made taking different values of a_{max} and d_{max} in the X and Y directions, the values of a_{max} and d_{max} to take in the X direction would be 3.50 m/s^2 and 0.012 m, respectively, which would lead to $\mu = 0.38$. Consequently, with $M_1 = 1148 \text{ t}$, a thickness of the RC slab equal to 0.65 m would be obtained.

From the values of k and ξ , the number and characteristics of LRIs, for TMD1, and those of LDRIs and VLDs, for TMD2, can be determined. Then, the strength verifications of structural members are made, and the maximum values of the flexural demand over capacity ratio $\frac{M_{Ed}}{M_{Rd}},_{max}$ equal to 1.45 for TMD1 and equal to 1.94 for TMD2 are found.

However, the values obtained for μ and z_{TMD} are considered to be too large. Therefore, the performance of smaller TMDs is investigated. The trend of $\frac{M_{Ed}}{M_{Rd}},_{max}$ obtained for TMD1 and TMD2 as μ varies between 0 (OB) and 0.51 is shown in Figure 5.12.

In this figure, the mean values of the flexural demand over capacity ratios obtained for the two proposed solutions are also shown. From Figure 5.12, it can be seen that, in both solutions, $\frac{M_{Ed}}{M_{Rd}},_{mean}$ decreases monotonically as μ increases, passing from 1.3 for the OB to 0.53 for TMD1 and 0.52 for TMD2, but the decrease is negligible from $\mu = 0.11$. Moreover, from Figure 5.12, it can be observed that

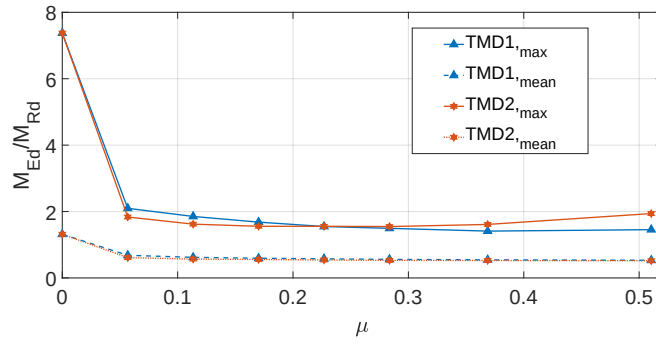


Figure 5.12: Trend of the maximum and mean values of the flexural demand over capacity ratios, $\frac{M_{Ed}}{M_{Rd}}$, as a function of the mass ratio μ .

the designed TMDs do not lead to the minimum values of $\frac{M_{Ed}}{M_{Rd}},_{max}$ as the minimum values are achieved with lower values of μ , and in particular with $\mu = 0.38$ for TMD1 and $\mu = 0.28$ for TMD2. Therefore, a greater mass of the TMD does not mean better performance. Hence, the optimum thickness is defined on the basis of the percentage of not-satisfied verifications (PNV) of the structural members. The trend of PNV as a function of μ obtained for the two solutions is shown in Figure 5.13.

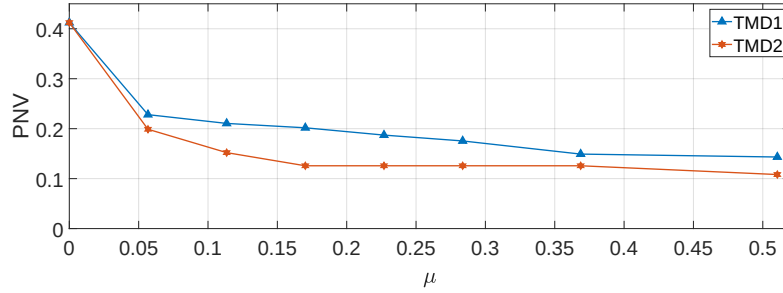


Figure 5.13: Percentage of not-satisfied verifications as a function of the mass ratio μ .

From Figures 5.12 and 5.13, it can be observed that for both TMD solutions $\mu = 0.2$ is already sufficient to largely reduce the flexural demand to structural members and the PNV. Mass ratios larger than 0.2 provide only a slight further reduction in the flexural demand and the PNV. In particular, both solutions provide the best results for $\mu > 0.35$, but, also taking into account the $\frac{M_{Ed}}{M_{Rd}},_{max}$ trend, the values $\mu = 0.38$ for TMD1 and $\mu = 0.17$ for TMD2 are considered optimal.

These values of μ are obtained from the values of z_{TMD} reported in column (1) of Tables 5.5 and 5.6 for TMD1 and TMD, respectively. The parameters of the TMD calculated with Equations (5.6)–(5.9) are reported in column (2) of Table 5.5 for TMD1 and Table 5.6 for TMD2. Then, the properties of LRIs for TMD1 (in column (3) of Table 5.5) and those of VLDs and LDRIs for TMD2 (in columns (3) and (4) of Table 5.6, respectively) are determined.

The position of LRIs and that of VLDs and LDRIs in the building plan are shown in Figures 5.14 and 5.15, respectively. The schematization of the simplified two-DOF structure-TMD systems are shown in Figure 5.16.

Table 5.5: TMD1 design properties.

TMD1			
(1) RC slab properties	(2) Design parameters from Equations (5.7) and (5.8)	(3) LRIs properties	
$z_{TMD} = 0.65$ m	$f = 0.65$	$n^o = 4$	$k_1 = 1787$ kN/m
$m = 435$ t	$k = 43,976$ kN/m	$k_{eq} = 10,994$ kN/m	$d_1 = 0.008$ m
$\mu = 0.38$	$\xi = 27.7\%$	$k_1 = 19,059$ kN/m	$d_2 = 0.083$ m
		$F_1 = 152$ kN	$F_2 = 287$ kN

Table 5.6: TMD2 design properties.

TMD2			
(1) RC slab properties	(2) Design parameters from Equations (5.7) and (5.8)	(3) VLDs properties	(4) LDRIs properties
$z_{TMD} = 0.30$ m	$f = 0.82$	$n^o/direction = 2$	$n^o = 16$
$m = 201$ t	$k = 33,389$ kN/m	$k_{VLD} = 55,000$ kN/m	$k_{LDRI} = 2,087$ kN/m
$\mu = 0.17$	$\xi = 19.5\%$	$c_{VLD} = 506$ kNs/m	
	$c = 1,011$ kNs/m	$F_{max} = 200$ kN	

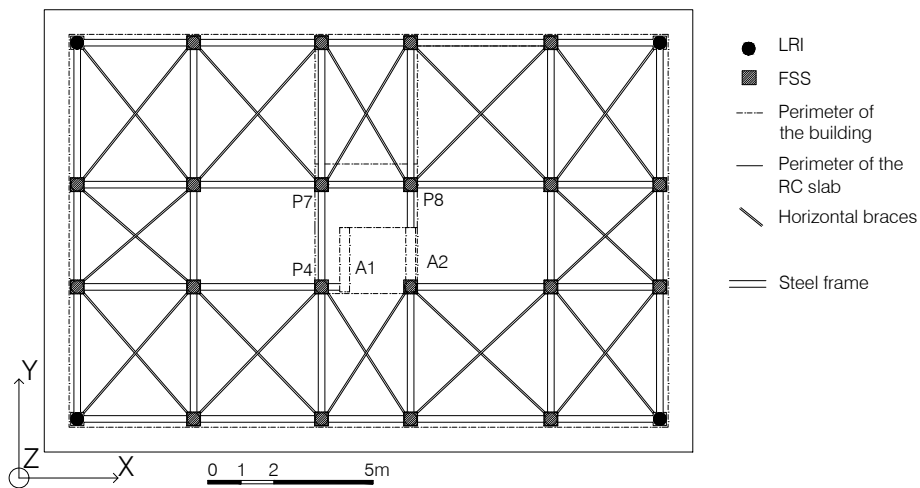


Figure 5.14: Top view of the steel structure supporting the TMD1 and arrangement of LRIs and FSSs.

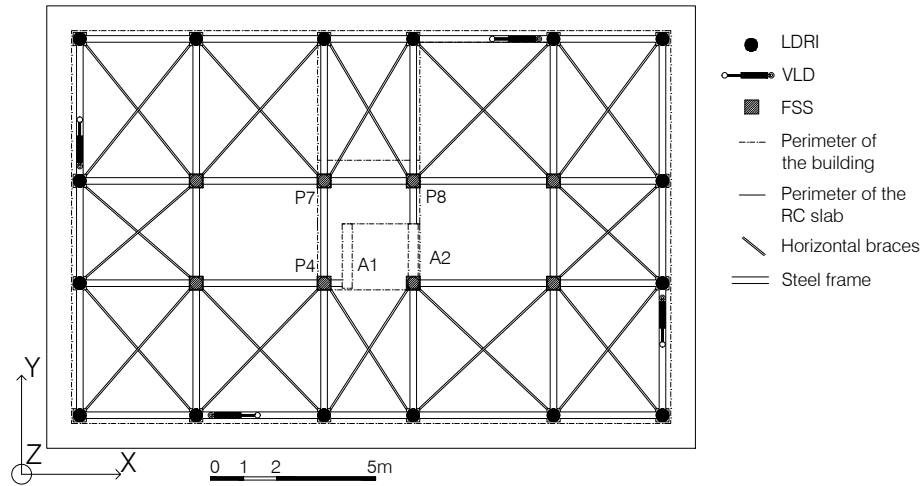


Figure 5.15: Top view of the steel structure supporting the TMD2 and arrangement of VLDs, LDRI and FSSs.

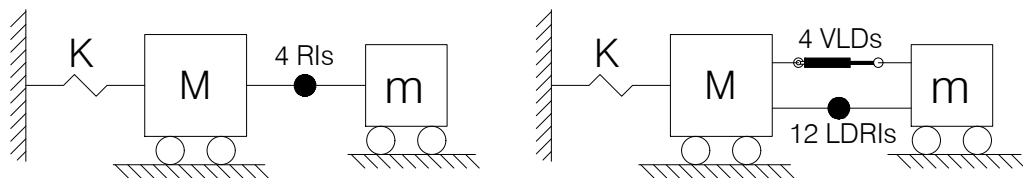


Figure 5.16: Schematization of the structure-TMD system for the building with TMD1 (left) and TMD2 (right).

TMD1 makes use of four LRIs placed at the corners of the steel structure, as shown in Figure 5.14, in order to maximize the TMD torsional stiffness.

In TMD2, due to the low shear modulus of LRIs, which is equal to 0.45 MPa, to provide the TMD with the design stiffness, 16 devices are required (Figure 5.15). Two VLDs in each main direction of the building are placed on the steel frame perimeter symmetrically with respect to the main axes of the building plan (Figure 5.15).

5.4.2 Design of the Base Isolation System (BIS)

In addition to the seismic protection strategy based on the use of TMD, the strategy based on building seismic isolation is also considered. BIS is designed to increase the fundamental period T_1 of the building up to the value T_{BIS} such that the corresponding spectral acceleration at the SLV, $S_A(T_{BIS})$, is equal to or lower than the spectral acceleration, $S_A(T_1)$, divided by the maximum value of $\frac{M_{Ed}}{M_{Rd}}$. For the OB, $\frac{M_{Ed}}{M_{Rd},max}$ is equal to 7.38. The stiffness of the BIS, k_{BIS} , is then determined from the total mass of the building, M , which is equal to 1740 t. For the considered case study, $S_A(T_{BIS}) = 0.90 \text{ m/s}^2$, which is attained for $T_{BIS} = 2.90 \text{ s}$. Consequently, $k_{BIS} = 8175 \text{ kN/m}$ is obtained. To provide the BIS with this stiffness, eight RIs with the properties shown in Table 5.7 are used. The RIs are placed at the base of the walls lying on the building perimeter. At the base of the remaining structural elements, FSSs are installed. The layout of RIs and FSSs in the building plan is shown in Figure 5.17.

Table 5.7: Properties of RIs composing the BIS.

BIS			
$\xi_{RI} = 12.5 \%$	$k_1 = 7,710 \text{ kN/m}$	$k_2 = 815 \text{ kN/m}$	$F_1 = 39 \text{ kN}$
$k_{eq,RI} = 1022 \text{ kN/m}$	$d_1 = 0.005 \text{ m}$	$d_2 = 0.167 \text{ m}$	$F_2 = 170 \text{ kN}$

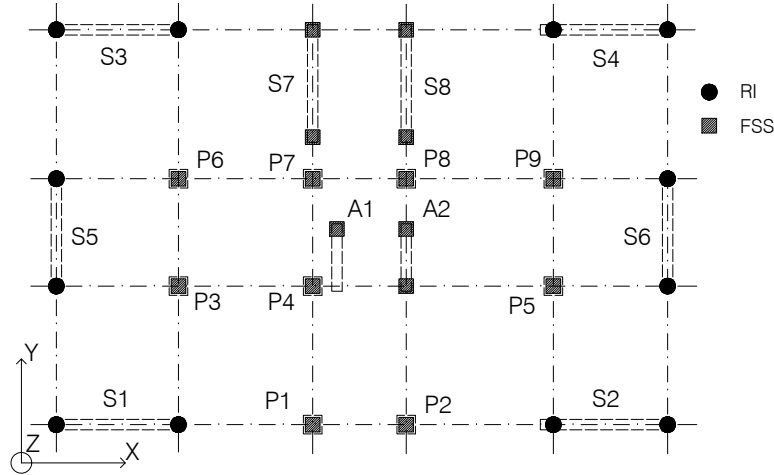


Figure 5.17: Layout of the base isolation system.

5.4.3 Vibration Modes of the Building After the Interventions

The modal information of the building equipped with TMD1 is shown in Table 5.8.

Table 5.8: Periods and mass participating ratios of the most significant modes of the building equipped with TMD1.

Mode i	T_i	$\alpha_{i,X}$	$\alpha_{i,Y}$	$\alpha_{i,Rz}$	$\Sigma \alpha_{i,X}$	$\Sigma \alpha_{i,Y}$	$\Sigma \alpha_{i,Rz}$
1	0.62 s	0.0 %	<u>36.9</u> %	0.0 %	0.0 %	36.9 %	0.0 %
2	0.61 s	<u>36.7</u> %	0.0 %	0.0 %	36.8 %	36.9 %	0.0 %
3	0.47 s	0.0 %	0.0 %	<u>48.7</u> %	36.8 %	36.9 %	48.7 %
4	0.31 s	0.1 %	<u>38.3</u> %	0.0 %	36.9 %	75.2 %	48.7 %
5	0.31 s	<u>37.8</u> %	0.1 %	0.0 %	74.7 %	75.3 %	48.7 %
6	0.24 s	0.1 %	0.0 %	<u>26.2</u> %	74.7 %	75.3 %	75.0 %
7	0.08 s	0.0 %	<u>14.1</u> %	0.0 %	74.7 %	89.4 %	75.0 %
8	0.08 s	<u>14.9</u> %	0.0 %	0.1 %	89.6 %	89.4 %	75.0 %
9	0.07 s	0.1 %	0.0 %	<u>14.2</u> %	89.7 %	89.4 %	89.2 %

From this table, it can be seen that the modes can be subdivided into three groups, each constituted by three modes: the first two translational in the X and Y directions, respectively, and the third rotational. The first group, composed of modes 1 to 3, is characterized by higher periods than those of OBS (Table 5.4), ranging from 0.62 s (mode 1) to 0.47 s (mode 3). The participating mass ratios of these modes range between 36.7 % and 48.7 %. The second group of modes, composed of modes 4, 5 and 6, is characterized by periods that are slightly lower than those of modes 1, 2 and 3 of the OBS and mass participating ratios that are about half, ranging between 26.2 % (mode 6) and 38.3 % (mode 4). The periods and mass participating ratios of the last group of modes (modes 7–9) are similar to those of modes 4–6 of the OBS. These modes are fundamentally those of the OBS, since the TMD does not participate in them.

Similarly to what was performed for the building equipped with TMD1, the modal properties of the building equipped with TMD2 are shown in Table 5.9.

Table 5.9: Periods and mass participating ratios of the most significant modes of the building equipped with TMD2.

Mode i	T_i	$\alpha_{i,X}$	$\alpha_{i,Y}$	$\alpha_{i,Rz}$	$\Sigma \alpha_{i,X}$	$\Sigma \alpha_{i,Y}$	$\Sigma \alpha_{i,Rz}$
1	0.58 s	0.1 %	<u>36.2</u> %	0.0 %	0.1 %	36.2 %	0.0 %
2	0.57 s	<u>35.9</u> %	0.1 %	0.0 %	35.9 %	36.3 %	0.0 %
3	0.49 s	0.0 %	0.0 %	<u>37.6</u> %	36.0 %	36.3 %	37.7 %
4	0.32 s	0.2 %	<u>37.5</u> %	0.0 %	36.2 %	73.8 %	37.7 %
5	0.32 s	<u>37.1</u> %	0.3 %	0.0 %	73.3 %	74.1 %	37.8 %
6	0.27 s	0.2 %	0.0 %	<u>35.7</u> %	73.5 %	74.1 %	73.5 %
7	0.09 s	0.0 %	<u>14.8</u> %	0.0 %	73.5 %	88.9 %	73.5 %
8	0.08 s	<u>15.6</u> %	0.0 %	0.1 %	89.1 %	88.9 %	73.6 %
9	0.07 s	0.1 %	0.0 %	<u>15.1</u> %	89.2 %	88.9 %	88.7 %

From the comparison with Table 5.8, it can be observed that the effect of TMD2 on the vibration modes of the building is very similar to that of TMD1, since the periods and participating mass ratios of translational modes are almost identical. A slight difference emerges in rotational modes 3 and 6, whose participating mass ratios are, respectively, higher and lower than those of TMD1.

The modal information of the original building with the BIS is shown in Table 5.10.

Table 5.10: Periods and mass participating ratios of the most significant modes of the base-isolated building.

Mode i	T_i	$\alpha_{i,X}$	$\alpha_{i,Y}$	$\alpha_{i,Rz}$	$\Sigma \alpha_{i,X}$	$\Sigma \alpha_{i,Y}$	$\Sigma \alpha_{i,Rz}$
1	2.87 s	<u>99.9</u> %	0.0 %	0.1 %	99.9 %	0.0 %	0.1 %
2	2.87 s	0.0 %	<u>100.0</u> %	0.0 %	99.9 %	100.0 %	0.1 %
3	2.05 s	0.1 %	0.0 %	<u>99.9</u> %	100.0 %	100.0 %	100.0 %

It can be seen that the dynamic behavior of the building is fully described by only three modes, which altogether provide participating mass ratios of 100 % for each DOF of the building. The periods of the translational modes, equal to 2.87 s for modes 1 and 2, fall in the last branch of the acceleration site spectrum in Figure 5.5.

From the comparison of the modal analysis results of the OB (Table 5.4) with those of the building equipped with TMD1 (Table 5.8), TMD2 (Table 5.9) and BIS (Table 5.10), it can be noticed that all three interventions produce significant modifications of the building modes.

5.4.4 Strength Verifications

The flexural demand over the capacity ratios for the OB and the building equipped with TMD1, TMD2 and BIS and their percentage variation from those of the OB are shown in Tables 5.11 and 5.12, respectively, for walls and columns.

Shear demand on structural members is always lower than their current capacity for the OB, and it is reduced in the building with TMD1, TMD2 and BIS. In particular, the shear demand at the base of the vertical members is reduced by at least 50% for the building equipped with TMD and 90% for the base-isolated building. Axial forces are also reduced by at least 60% and 90%, respectively.

5.4.5 Input and Dissipated Energies

The graphs of cumulative input and dissipated energies of the OB and of the building with TMD1, TMD2 and BIS obtained from one of the three non-linear analysis cases are shown in Figure 5.18.

The force–displacement diagrams in the X and Y directions of one LRI (TMD1) and one VLD (TMD2) are shown in Figures 5.19 and 5.20, respectively. Similarly, the force–displacement diagrams in the X and Y directions of one RI of the base-isolated building are shown in Figure 5.21.

5.4.6 Maximum Accelerations

In Figure 5.22, the roof barycenter acceleration along the X direction of the OB and of the building with TMD1, TMD2 and BIS versus time graphs is shown. The peaks of maximum acceleration in each graph are highlighted with a red circle. The maximum amplitude of acceleration on the OB equals 10.9 m/s^2 , and it can be observed that the installation of TMD1 and TMD2 reduces its value to 7.4 m/s^2 and 6.3 m/s^2 , respectively, while the base isolation system binds it below 0.8 m/s^2 .

Table 5.11: Flexural demand over capacity ratios for the walls of the OB and the building equipped with TMD1, TMD2 and BIS and percentage variation in the ratios with respect to those of the OB. Ratios greater than one are underlined; maximum values are in bold.

	OB	TMD1		TMD2		BIS	
Wall Section	$\frac{M_{Ed}}{M_{Rd}}$	$\frac{M_{Ed}}{M_{Rd}}$	Δ %	$\frac{M_{Ed}}{M_{Rd}}$	Δ %	$\frac{M_{Ed}}{M_{Rd}}$	Δ %
A01(0.0)	<u>2.86</u>	0.94	-67.1 %	0.98	-65.8 %	0.21	-92.7 %
A01(12.3)	0.57	0.27	-52.2 %	0.25	-55.9 %	0.04	-92.4 %
A01(18.3)	0.72	<u>1.01</u>	+41.2 %	0.86	+20.5 %	0.06	-91.1 %
A02(0.0)	<u>7.38</u>	0.91	-87.6 %	0.99	-86.6 %	0.19	-97.4 %
A02(12.3)	<u>1.30</u>	0.25	-80.5 %	0.23	-77.2 %	0.07	-94.4 %
A02(18.3)	<u>1.34</u>	<u>1.16</u>	-17.1 %	0.99	-28.7 %	0.11	-91.9 %
S01(0.0)	<u>3.37</u>	<u>1.33</u>	-60.5 %	<u>1.42</u>	-58.0 %	0.27	-92.1 %
S01(12.3)	0.83	0.31	-62.2 %	0.27	-64.4 %	0.05	-94.6 %
S01(18.3)	0.47	<u>1.28</u>	+172.4 %	0.94	+100.8 %	0.04	-91.2 %
S02(0.0)	<u>3.42</u>	<u>1.31</u>	-61.8 %	<u>1.36</u>	-60.3 %	0.26	-92.4 %
S02(12.3)	0.85	0.30	-64.7 %	0.30	-64.7 %	0.05	-94.0 %
S02(18.3)	0.48	0.79	+63.7 %	0.64	+33.7 %	0.05	-90.2 %
S03(0.0)	<u>3.59</u>	<u>1.40</u>	-61.0 %	<u>1.44</u>	-59.9 %	0.27	-92.4 %
S03(12.3)	0.86	0.30	-64.9 %	0.29	-66.7 %	0.05	-94.4 %
S03(18.3)	0.41	<u>1.26</u>	+209.9 %	0.97	+137.8 %	0.03	-92.9 %
S04(0.0)	<u>3.63</u>	<u>1.38</u>	-62.0 %	<u>1.42</u>	-61.0 %	0.28	-92.4 %
S04(12.3)	0.88	0.30	-65.6 %	0.28	-67.9 %	0.05	-94.3 %
S04(18.3)	0.40	0.69	+72.9 %	0.63	+56.7 %	0.03	-91.5 %
S05(0.0)	<u>3.07</u>	<u>1.27</u>	-58.7 %	<u>1.34</u>	-56.3 %	0.24	-92.3 %
S05(12.3)	0.70	0.34	-51.9 %	0.30	-57.4 %	0.04	-94.6 %
S05(18.3)	0.57	0.76	+33.7 %	0.56	-1.9 %	0.04	-92.8 %
S06(0.0)	<u>3.09</u>	<u>1.27</u>	-58.9 %	<u>1.37</u>	-55.7 %	0.25	-91.9 %
S06(12.3)	0.65	0.35	-45.6 %	0.31	-52.4 %	0.04	-93.2 %
S06(18.3)	0.60	0.74	+21.9 %	0.57	-5.6 %	0.05	-92.3 %
S07(0.0)	<u>5.48</u>	<u>1.40</u>	-74.4 %	<u>1.56</u>	-71.6 %	0.26	-95.3 %
S07(12.3)	0.89	0.30	-66.6 %	0.34	-62.2 %	0.05	-94.6 %
S07(18.3)	0.74	0.70	-6.1 %	0.66	-11.1 %	0.05	-93.7 %
S08(0.0)	<u>6.70</u>	<u>1.40</u>	-79.1 %	<u>1.55</u>	-76.8 %	0.26	-96.1 %
S08(12.3)	<u>1.04</u>	0.29	-72.1 %	0.33	-68.5 %	0.05	-95.4 %
S08(18.3)	0.78	0.74	-5.3 %	0.68	-12.4 %	0.04	-94.6 %

Table 5.12: Flexural demand over capacity ratios for the columns of the OB and the building equipped with TMD1, TMD2 and BIS and percentage variation in the ratios with respect to those of the OB. Ratios greater than one are underlined, maximum values are in bold.

	OB	TMD1		TMD2		BIS	
Wall Section	$\frac{M_{Ed}}{M_{Rd}}$	$\frac{M_{Ed}}{M_{Rd}}$	$\Delta \%$	$\frac{M_{Ed}}{M_{Rd}}$	$\Delta \%$	$\frac{M_{Ed}}{M_{Rd}}$	$\Delta \%$
P01(0.0)	0.83	0.34	−59.0 %	0.35	−58.6 %	0.06	−92.9 %
P01(12.3)	0.73	0.21	−70.9 %	0.25	−65.4 %	0.05	−93.9 %
P01(18.3)	<u>1.29</u>	0.44	−65.9 %	0.48	−63.1 %	0.08	−93.8 %
P02(0.0)	0.91	0.33	−63.3 %	0.34	−62.1 %	0.06	−93.4 %
P02(12.3)	0.82	0.20	−75.1 %	0.25	−69.9 %	0.05	−94.3 %
P02(18.3)	<u>1.45</u>	0.47	−67.3 %	0.51	−65.2 %	0.09	−93.9 %
P03(0.0)	0.80	0.33	−58.5 %	0.33	−58.1 %	0.06	−92.6 %
P03(12.3)	0.90	0.25	−72.0 %	0.30	−66.3 %	0.06	−93.5 %
P03(18.3)	<u>1.68</u>	0.42	−74.9 %	0.47	−71.9 %	0.11	−93.5 %
P04(0.0)	0.91	0.33	−63.9 %	0.34	−63.1 %	0.06	−93.2 %
P04(12.3)	0.75	0.19	−74.8 %	0.23	−68.9 %	0.05	−93.3 %
P04(18.3)	<u>1.38</u>	0.54	−59.0 %	0.48	−63.6 %	0.09	−92.9 %
P05(0.0)	0.77	0.33	−57.5 %	0.33	−56.3 %	0.06	−92.3 %
P05(12.3)	0.78	0.25	−67.3 %	0.30	−61.4 %	0.06	−92.6 %
P05(18.3)	<u>1.49</u>	0.39	−73.6 %	0.44	−70.2 %	0.11	−92.6 %
P06(0.0)	0.76	0.33	−56.3 %	0.34	−55.3 %	0.06	−92.5 %
P06(12.3)	0.86	0.26	−70.3 %	0.30	−64.9 %	0.05	−94.2 %
P06(18.3)	<u>1.63</u>	0.41	−74.6 %	0.47	−71.4 %	0.09	−94.3 %
P07(0.0)	<u>7.31</u> ¹	0.49	−93.2 %	0.52	−92.9 %	0.08	−98.9 %
P07(12.3)	<u>2.17</u>	0.60	−72.4 %	0.73	−66.5 %	0.10	−95.4 %
P07(18.3)	<u>2.71</u>	<u>1.41</u>	−48.0 %	<u>1.49</u>	−45.0 %	0.16	−94.3 %
P08(0.0)	<u>1.80</u>	0.46	−74.2 %	0.49	−72.5 %	0.08	−95.4 %
P08(12.3)	<u>2.28</u>	0.69	−69.9 %	0.82	−64.3 %	0.14	−93.9 %
P08(18.3)	<u>3.24</u>	<u>1.29</u>	−60.2 %	<u>1.32</u>	−59.5 %	0.21	−93.6 %
P09(0.0)	0.84	0.33	−60.6 %	0.34	−59.9 %	0.06	−92.9 %
P09(12.3)	0.94	0.25	−73.4 %	0.31	−67.3 %	0.06	−93.9 %
P09(18.3)	<u>1.77</u>	0.37	−79.0 %	0.46	−74.3 %	0.11	−94.0 %

¹ Tensile axial force at the base of column P07 is greater than its cross-section capacity in one of the three analyses conducted.

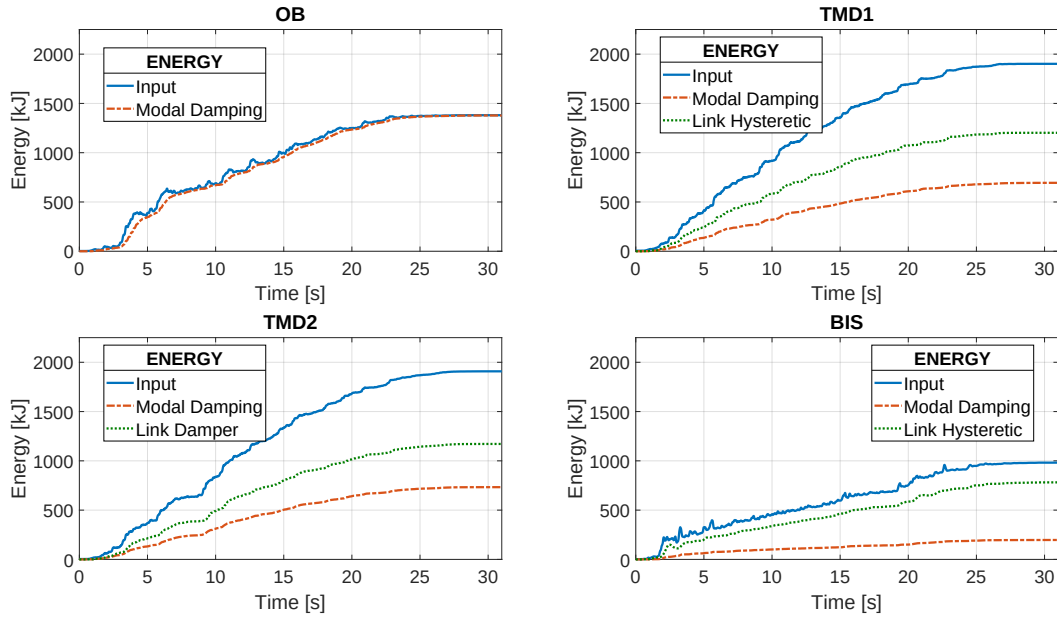


Figure 5.18: Cumulative input and dissipated energies for the OB and the building with TMD1, TMD2 and BIS.

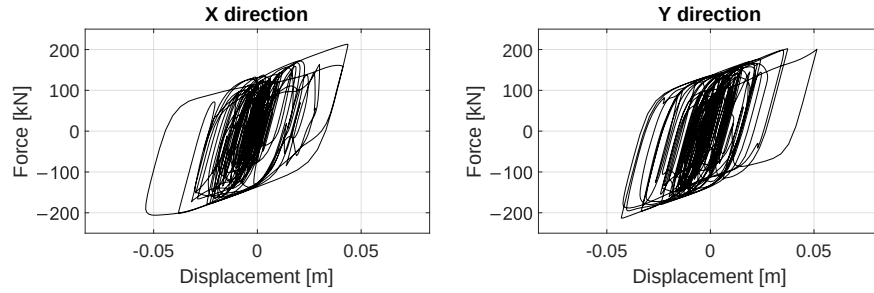


Figure 5.19: Force-displacement diagrams along X and Y directions of one LRI for the building equipped with TMD1.

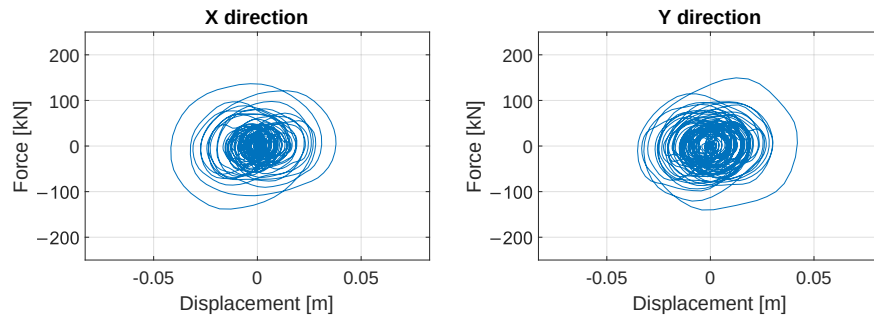


Figure 5.20: Force-displacement diagrams along X and Y directions of one VLD for the building equipped with TMD2.

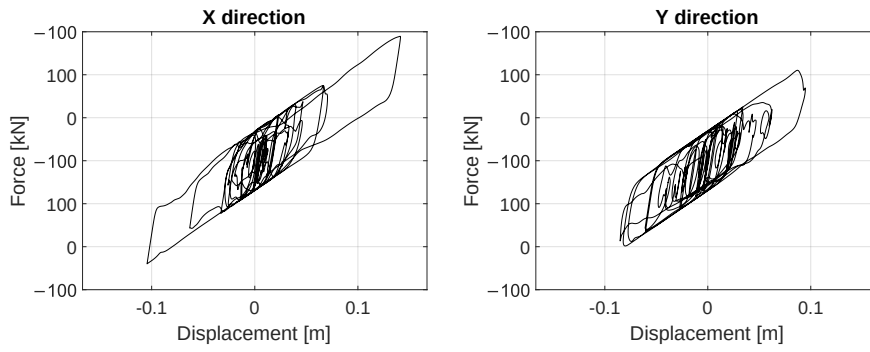


Figure 5.21: Force–displacement diagrams along X and Y directions of one RI for the building equipped with BIS.

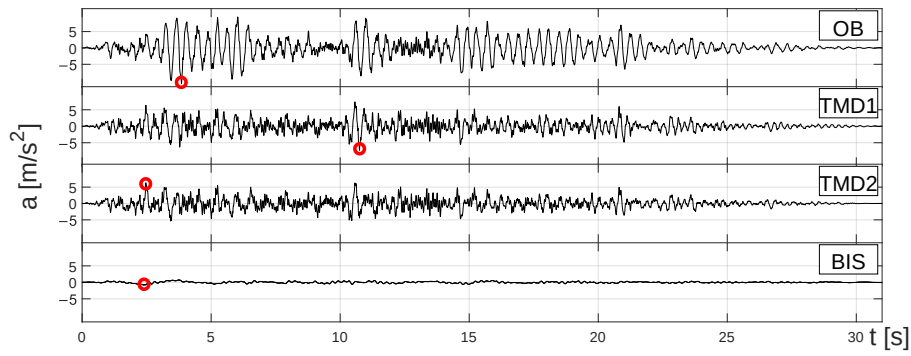


Figure 5.22: Acceleration–time graph of the barycenter of the building roof along X direction for all the considered configurations during one of the conducted analyses.

5.5 Discussion of Results

5.5.1 TMD Design Procedure

A high sensitivity of the formula proposed by Luft [89] to the maximum target displacement, d_{max} , of the building was observed. Due to this sensitivity, a high value of the mass ratio ($\mu = 0.51$) was obtained. This is probably due to the fact that Luft's formula was derived for high-rise buildings, to which TMDs with a low mass ratio (i.e., with $\mu \leq 0.10$) are usually applied. Then, the performance of lighter TMDs (i.e., with $\mu \leq 0.51$) was investigated.

5.5.2 Building Vibration Modes

In Section 5.4.3, it was observed that the rotational modes 3 and 6 of the building equipped with TMD2 have participating mass ratios that are slightly different from those of modes 3 and 6 of the building with TMD1. This is due to the different arrangement in the building plan of LDRIs in TMD2 (Figure 5.15) with respect to LRIs in TMD1 (Figure 5.14), which induces a difference in the rotational stiffness.

In Figure 5.23, the acceleration spectrum for the considered building site with 5% and 27.7% damping is shown. From this figure, it can be seen that the spectral accelerations of the first three modes of the OB belong to the constant acceleration branch. The spectral acceleration of the first mode is indicated as OBa in Figure 5.23.

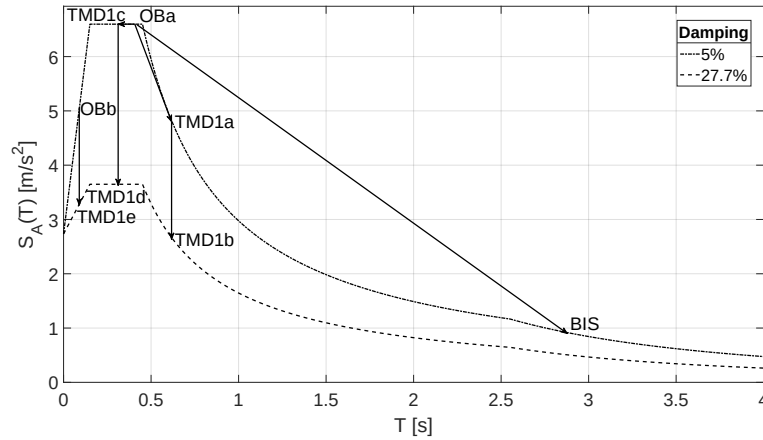


Figure 5.23: Reduction of the spectral accelerations of the modes with the most significant participating mass ratios in X direction for the building equipped with TMD1.

As observed in Section 5.4.3, the first six modes of the building equipped with TMD1 have participating mass ratios similar to those of the building with TMD2. The periods of the first three are higher than those of the OB and involve lower spectral accelerations belonging to the constant velocity branch. Conversely, the periods of the other three modes are lower than those of the OB, but involve a spectral acceleration that is still on the plateau. To show the effect of periods' change on spectral accelerations due to TMD, the spectral accelerations for $\xi = 5\%$ of modes 1 and 4 of the building with TMD1 are indicated in Figure 5.23 and labeled as TMDa and TMDc, respectively. The damping provided by TMD further reduces

spectral accelerations, thus moving points TMDa and TMDc to points TMDb and TMDd, respectively (see Figure 5.23).

The last three modes of the OB do not change remarkably with the installation of the TMD, but their spectral accelerations are reduced due to damping. Thus, it can be gathered that higher frequency modes only benefit from the TMD by damping and not by tuning. To show this, the spectral acceleration of mode 4 of the OB and that of the corresponding mode of the building with TMD1 (mode 7) are indicated by points OBb and TMDe in Figure 5.23, respectively.

From Figure 5.23, it can be seen that, thanks to the intervention with TMD1, the whole building is subjected to spectral accelerations ranging between 2.66 and 3.64 m/s², which are lower than those of the OB (5.07–6.60 m/s²). As a consequence, internal forces in structural members due to seismic action are considerably reduced.

As for the base-isolated building, from Figure 5.23, it can be seen that the spectral acceleration of the first mode (point BIS) belongs to the spectrum constant displacement branch, which is characterized by the lowest accelerations. As a consequence, very low seismic forces arise in the building. Damping brought by RIs has very little relevance to spectral acceleration reduction.

5.5.3 Strength Verifications

From Tables 5.11 and 5.12, it can be seen that both TMDs remarkably improve flexural verifications since $\frac{M_{Ed}}{M_{Rd}}$ ratios are significantly reduced for all structural members that have, in the OB, $\frac{M_{Ed}}{M_{Rd}} > 1$. However, the flexural demand on walls S01–S08, which counteract most of the seismic forces arising in the structure, is still greater than their capacity. Hence, the structure would be damaged by earthquakes with magnitudes the same as those considered. The missing gap could be reduced by using a more advanced design method for the TMD and increasing the reinforcements in the vertical members. Anyway, the flexural demand reduction is high, exceeding, in some cases, 70%.

From Tables 5.11 and 5.12, it can also be observed that $\frac{M_{Ed}}{M_{Rd}}$ ratios increase at the top of almost all the walls due to the forces incurred by the TMD. $\frac{M_{Ed}}{M_{Rd}}$ ratios and the PNV for the building equipped with TMD1 are very similar to those of the building equipped with TMD2. Thus, it can be said that the two proposed solutions, based on the use of TMD, are equally effective in reducing structural damage caused by seismic actions. The TMD2 should be preferred because the quantity of concrete required to make the RC slab is less than half of that required in TMD1. On the basis of the above, it can be concluded that the method adopted for the TMD design, though it is based on simple expressions, leads to remarkable results. Strength verifications show also that the BIS is the best solution and should be preferred if the installation is possible in an existing building.

5.5.4 Input and Dissipated Energy

From Figure 5.18, it is seen that the OB dissipates all the input energy by modal damping. In reality, part of the input energy would also be dissipated by flexural plastic hinges forming at structural members ends, which have not been modeled in the FEM of the building. The effectiveness of both TMD solutions is also proven by the energy–time graphs shown in Figure 5.18. Actually, TMD1 and TMD2, despite inducing an increase in the total input energy, halve the energy dissipated by modal damping. From Figures 5.19 and 5.20, it can be seen that the mean force and the

relative displacements are similar for the devices of TMD1 and TMD2. Thus, these devices dissipate similar energy amounts.

Among the intervention strategies considered in the presented case study, BIS is the most effective in reducing stresses arising in the structural members due to seismic actions. Indeed, $\frac{M_{Ed}}{M_{Rd}}$ ratios are lower than one tenth of those of the OB, and the highest is 0.275 (see Tables 5.11 and 5.12). This is due to the fact that BIS reduces the input energy and dissipates most of it by RI hysteretic cycles. Hence, the energy dissipated by the building through modal damping is reduced to at least one tenth of the OB. Despite the fact that the force–displacement cycles of RIs (Figure 5.21) are narrower than those of LRIs and VLDs (Figures 5.19 and 5.20), the energy dissipated by RIs is greater than that dissipated by LRIs and VLDs because RIs are more numerous than LRIs and VLDs.

TMD may also be a valid and efficient strategy for the retrofit of existing buildings as an alternative to dissipative braces [92,93] and base isolation [94,95], which often require interventions on the foundations and on the superstructure, making these solutions too expensive or technically unfeasible.

Chapter 6

Conclusions

The qualification tests on seismic isolators required by current standards present challenges due to their high demands of time and financial resources. This thesis explored the possibility of predicting the behavior of elastomeric isolators under cyclic shear tests through FEM simulations, based on constitutive relationships for the elastomers derived from experimental tests. The results obtained from these simulations are compared with full-scale tests results in terms of damping and shear modulus. The main findings are summarized as follows:

- Black carbon-filled rubbers, used in the isolators, exhibit highly nonlinear behavior, including elastic, viscous, and softening effects, with the notable presence of the Mullins effect. Extending the Maxwell-Wiechert model to finite deformations with the Simo-Hughes formulation and describing the elastic component with the Yeoh model, it is possible to effectively replicate the material response under simple deformation histories.
- A closed-form expression for the stress response in simple shear cyclic deformation was derived, allowing for direct computation of shear modulus and damping for a given set of the viscoelastic model parameters. Using these expressions, a fitting method was developed to determine viscous constants μ from cyclic tests at various frequencies (MFCT). Its application to experimental results revealed limitations due to narrow ranges for time constants and instability in the solution.
- The parameters of the viscoelastic model were calibrated through laboratory tests on rubber specimens, which include cyclic, relaxation, and step relaxation tests. Acceleration of the relaxation through repeated oscillations in a step relaxation test (UCDP) allowed for the direct determination of the elastic equilibrium stress curves. A hyperbolic trend in the elastic parameters enabled the estimation of the equilibrium stress at given shear strains and the first load equilibrium stress curve.
- A new specimen, created by gluing a rubber layer to steel plates, was designed to reduce the loading ramp duration in the relaxation test and minimize the influence of the shape factor on the stress distribution, aiming to approach simple shear deformation. Dynamic shear modulus and damping were computed from cyclic tests on these specimens, with consistent results across samples, offering fair predictions when compared to full-scale isolator cyclic tests.

- Analytical expressions for the lateral bulge height in compressed rubber discs were derived and validated through FEM simulations, which demonstrated consistency with the circular arc shape assumption for both small and large deformations.
- The viscoelastic model, calibrated on laboratory tests, was incorporated into three-dimensional FEM models of elastomeric isolators for the simulation of shear cyclic tests. Shear modulus and damping obtained from the simulations showed good agreement with laboratory tests in some cases but exceeded acceptable error limitations imposed by the standards in others. These discrepancies could be reduced with a more refined model for rubber, which incorporates a constitutive relationship accounting for the maximum strain experienced by the material, thus capturing the Mullins effect more accurately, particularly in regions with larger strains. Additionally, further investigations should focus on how geometric parameters, such as the diameter, thickness, and number of elastomer layers, influence shear modulus and damping values. It is necessary to investigate the reasons behind the exceptionally high variability, in some cases, of the mechanical properties of isolators with identical materials and geometry.
- A numerical application of elastomeric isolators investigated the seismic performance of a building equipped with a Tuned Mass Damper (TMD) system, compared to a base isolation system. Two TMD configurations were proposed: one using lead rubber isolators (TMD1) and the other using low-damping rubber isolators with viscous dampers (TMD2). Both solutions effectively reduced structural stresses and displacements during seismic events, with the solution featuring a thinner plate proving to be more effective. Additionally, the TMD can be a valid alternative to base isolation, despite its lower performance, in cases where base isolation is not feasible.

Appendix A

Theoretical cyclic tests

Theoretical cyclic simple shear test

The deformation history and the velocity of the test are

$$\gamma(t) = \gamma_0 \sin(\omega t) \quad (\text{A.1})$$

$$\dot{\gamma}(t) = \gamma_0 \omega \cos(\omega t) \quad (\text{A.2})$$

The equilibrium shear stress rate is obtained deriving Equation 2.48

$$\dot{S}_\infty(t) = 2\dot{\gamma}(t) [C_{10} + 6C_{20}\gamma^2(t) + 15C_{30}\gamma^4(t)] \quad (\text{A.3})$$

$$\begin{aligned} &= 2C_{10}\gamma_0\omega \cos(\omega t) + 12C_{20}\gamma_0^3\omega \sin^2(\omega t) \cos(\omega t) \\ &\quad + 30C_{30}\gamma_0^5\omega \sin^4(\omega t) \cos(\omega t) \end{aligned} \quad (\text{A.4})$$

We can solve the differential Equation 2.49 as the sum of the homogeneous solution $Q_{i,h}(t)$ and three particular solutions $Q_{i,p}(t)$, corresponding to the three terms of the sum in Equation A.3

- homogeneous equation:

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = 0 \quad (\text{A.5})$$

$$Q_{i,h}(t) = Q_{i0} e^{-\frac{t}{\tau_i}} \quad (\text{A.6})$$

- first particular solution:

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = 2\mu_i C_{10} \gamma_0 \omega \cos(\omega t) \quad (\text{A.7})$$

$$Q_{i,p1}(t) = 2\mu_i C_{10} \gamma_0 \frac{\sin(\omega t) + \phi_i \cos(\omega t)}{1 + \phi_i^2} \quad (\text{A.8})$$

- second particular solution:

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = 12\mu_i C_{20} \gamma_0^3 \omega \sin^2(\omega t) \cos(\omega t) \quad (\text{A.9})$$

$$Q_{i,p2}(t) = 3\mu_i C_{20} \gamma_0^3 \left[\frac{\sin(\omega t) + \phi_i \cos(\omega t)}{1 + \phi_i^2} - \frac{\sin(3\omega t) + \frac{\phi_i}{3} \cos(3\omega t)}{1 + \left(\frac{\phi_i}{3}\right)^2} \right] \quad (\text{A.10})$$

- third particular solution:

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = 30\mu_i C_{30} \gamma_0^5 \omega \sin^4(\omega t) \cos(\omega t) \quad (\text{A.11})$$

$$Q_{i,p3}(t) = \frac{15}{8} \mu_i C_{30} \gamma_0^5 \left[2 \frac{\sin(\omega t) + \phi_i \cos(\omega t)}{1 + \phi_i^2} + \right. \\ \left. - \frac{\sin(3\omega t) + \frac{\phi_i}{3} \cos(3\omega t)}{1 + \left(\frac{\phi_i}{3}\right)^2} + \frac{1}{5} \frac{\sin(5\omega t) + \frac{\phi_i}{5} \cos(5\omega t)}{1 + \left(\frac{\phi_i}{5}\right)^2} \right] \quad (\text{A.12})$$

The general solution is the sum of each solution reported above, where, as time passes, the solution of the homogeneous differential equation fades away, thus it is neglected

$$Q_{i,omog}(t) = Q_{i0} e^{-\frac{t}{\tau_i}} \longrightarrow 0 \quad \text{for} \quad t \longrightarrow \infty \quad (\text{A.13})$$

$$Q_i(t) = Q_{i,p1}(t) + Q_{i,p2}(t) + Q_{i,p3}(t) \\ = \mu_i \gamma_0 \left\{ \frac{1}{4} \frac{8C_{10} + 12C_{20}\gamma_0^2 + 15C_{30}\gamma_0^4}{1 + \phi_i^2} [\sin(\omega t) + \phi_i \cos(\omega t)] + \right. \\ \left. - \gamma_0^2 \frac{C_{20} + \frac{15}{8}C_{30}\gamma_0^2}{1 + \left(\frac{\phi_i}{3}\right)^2} \left[\sin(3\omega t) + \frac{\phi_i}{3} \cos(3\omega t) \right] + \right. \\ \left. + \frac{3}{8} \frac{C_{30}\gamma_0^4}{1 + \left(\frac{\phi_i}{5}\right)^2} \left[\sin(5\omega t) + \frac{\phi_i}{5} \cos(5\omega t) \right] \right\} \\ = \mu_i \gamma_0 \left\{ \frac{\mathcal{X}}{1 + \phi_i^2} [\sin(\omega t) + \phi_i \cos(\omega t)] + \right. \\ \left. + \frac{\mathcal{Y}}{1 + \left(\frac{\phi_i}{3}\right)^2} \left[\sin(3\omega t) + \frac{\phi_i}{3} \cos(3\omega t) \right] + \right. \\ \left. + \frac{\mathcal{Z}}{1 + \left(\frac{\phi_i}{5}\right)^2} \left[\sin(5\omega t) + \frac{\phi_i}{5} \cos(5\omega t) \right] \right\} \quad (\text{A.14})$$

where

$$\mathcal{X} = (8C_{10} + 12C_{20}\gamma_0^2 + 15C_{30}\gamma_0^4) / 4 \quad (\text{A.15})$$

$$\mathcal{Y} = -\gamma_0^2 \left(C_{20} + \frac{15}{8}C_{30}\gamma_0^2 \right) \quad (\text{A.16})$$

$$\mathcal{Z} = \frac{3}{8}C_{30}\gamma_0^4 \quad (\text{A.17})$$

The dynamic shear modulus is evaluated as the ratio

$$G_{dyn} = \frac{S_{\gamma_{max}} - S_{\gamma_{min}}}{\gamma_{max} - \gamma_{min}} = \frac{S\left(\frac{T}{4}\right)}{\gamma_0} \quad (\text{A.18})$$

where, by the symmetry of the deformation, the shear stresses $S_{\gamma_{max}}$ and $S_{\gamma_{min}}$ at the maximum and minimum deformations are equal in magnitude and have opposite sign. The maximum and minimum deformations are reached when $t = n + \frac{1}{4}$ and $t = n + \frac{3}{4}$

$$S_{\gamma_{max}} = S(\gamma = \gamma_0) = S\left(t = \left(n + \frac{1}{4}\right) T\right) \quad (\text{A.19})$$

$$S_{\gamma_{min}} = S(\gamma = -\gamma_0) = S\left(t = \left(n + \frac{3}{4}\right) T\right) = -S_{\gamma_{max}} \quad (\text{A.20})$$

So the dynamic shear modulus can be evaluated just as

$$G_{dyn} = \frac{S\left(\frac{T}{4}\right)}{\gamma_0} \quad (\text{A.21})$$

The stress at maximum deformation is the sum of the equilibrium stress and the N over-stresses

$$S\left(\frac{T}{4}\right) = S_{\infty}\left(\frac{T}{4}\right) + \sum_{i=1}^N Q_i\left(\frac{T}{4}\right) \quad (\text{A.22})$$

$$\begin{aligned} S_{\infty}\left(\frac{T}{4}\right) &= 2C_{10}\gamma_0 \sin\left(\omega\frac{T}{4}\right) + 4C_{20}\gamma_0^3 \sin^3\left(\omega\frac{T}{4}\right) + 6C_{30}\gamma_0^5 \sin^5\left(\omega\frac{T}{4}\right) \\ &= 2C_{10}\gamma_0 + 4C_{20}\gamma_0^3 + 6C_{30}\gamma_0^5 \end{aligned} \quad (\text{A.23})$$

$$Q_i\left(\frac{T}{4}\right) = \dots = \mu_i \gamma_0 \left[\frac{\mathcal{X}}{1 + \phi_i^2} - \frac{\mathcal{Y}}{1 + \left(\frac{\phi_i}{3}\right)^2} + \frac{\mathcal{Z}}{1 + \left(\frac{\phi_i}{5}\right)^2} \right] \quad (\text{A.24})$$

Thus the dynamic shear modulus is

$$\begin{aligned} G_{dyn} &= 2\left(C_{10} + 2C_{20}\gamma_0^2 + 3C_{30}\gamma_0^4\right) + \\ &+ \sum_{i=1}^N \mu_i \left[\frac{2C_{10} + 3C_{20}\gamma_0^2 + \frac{15}{4}C_{30}\gamma_0^4}{1 + \phi_i^2} + \frac{C_{20} + \frac{15}{8}C_{30}\gamma_0^2}{1 + \left(\frac{\phi_i}{3}\right)^2} \gamma_0^2 + \frac{\frac{3}{8}C_{30}\gamma_0^4}{1 + \left(\frac{\phi_i}{5}\right)^2} \right] \end{aligned} \quad (\text{A.25})$$

To evaluate the damping we start from the energy dissipated, per unit volume, in one cycle of deformation as the integration of the power $S(t)\dot{\gamma}(t)$. The elastic component S_{∞} does not dissipate energy, thus only the viscous terms Q_i participate.

$$\mathcal{D} = \oint S(t)\dot{\gamma}(t) dt = \int_{nT}^{(n+1)T} \sum_{i=1}^N Q_i(t)\dot{\gamma}(t) dt \quad \forall n \in \mathbb{N} \quad (\text{A.26})$$

From product of the expression of Q_i in Equation A.14 and the deformation velocity $\dot{\gamma}(t)$ in Equation A.2 a series of trigonometric functions are integrated, but only one yields to a non-zero value

$$\int_0^T \cos^2(\omega t) dt = \frac{T}{2} \quad (\text{A.27})$$

and only the terms multiplying it contribute to the energy dissipation

$$\mathcal{D} = \omega \gamma_0^2 \mathcal{X} \frac{T}{2} \sum_{i=1}^N \frac{\mu_i \phi_i}{1 + \phi_i^2} = \pi \gamma_0^2 \mathcal{X} \sum_{i=1}^N \frac{\mu_i \phi_i}{1 + \phi_i^2} \quad (\text{A.28})$$

$$= \pi \gamma_0^2 \left(2C_{10} + 3C_{20}\gamma_0^2 + \frac{15}{4}C_{30}\gamma_0^4 \right) \sum_{i=1}^N \frac{\mu_i \phi_i}{1 + \phi_i^2} \quad (\text{A.29})$$

and hence to the damping

$$\xi = \frac{\mathcal{D}}{2\pi S_{\gamma_{max}} \gamma_0} = \frac{\mathcal{D}}{2\pi \frac{1}{2} G_{dyn} \gamma_0^2} = \frac{2C_{10} + 3C_{20}\gamma_0^2 + \frac{15}{4}C_{30}\gamma_0^4}{2G_{dyn}} \sum_{i=1}^N \frac{\mu_i \phi_i}{1 + \phi_i^2} \quad (\text{A.30})$$

In Figure A.1 the trends of the functions of ϕ that appear in Equations A.25 and A.30.

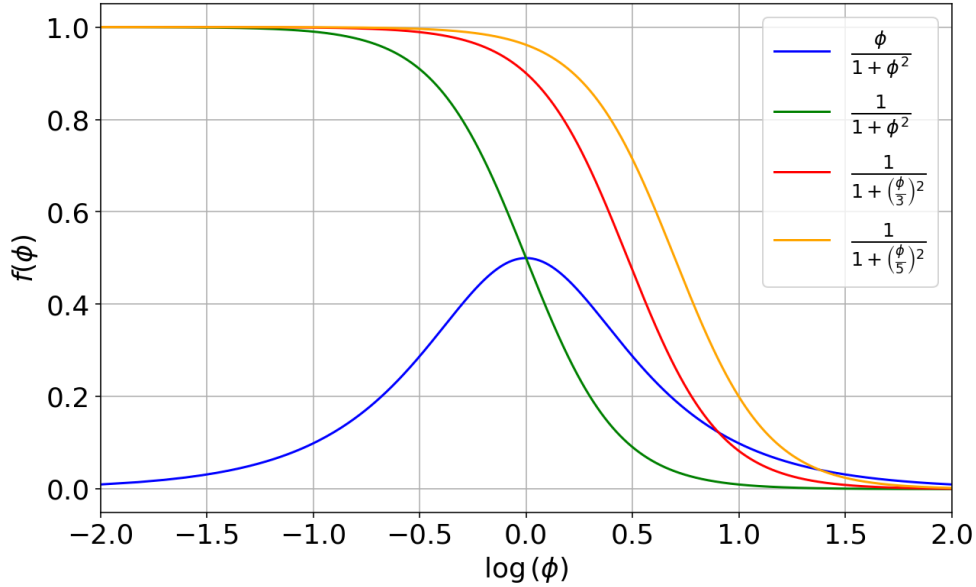


Figure A.1: Trends of the ϕ functions that determining damping and shear dynamic modulus of a cyclic test.

Theoretical cyclic uniaxial test

In this part of the appendix we will call the tension stress P and the over-stresses Q_i to maintain the variable T to indicate the period of one cycle.

The deformation history and the velocity are defined as sinusoidal functions around a mean value of stretch

$$\lambda(t) = \lambda_m + \lambda_0 \sin(\omega t) = \lambda_m (1 + l_0 \sin(\omega t)) \quad (\text{A.31})$$

$$\dot{\lambda}(t) = \lambda_0 \omega \cos(\omega t) = \lambda_m l_0 \omega \cos(\omega t) \quad (\text{A.32})$$

where

$$\lambda_m = \frac{\lambda_{max} + \lambda_{min}}{2} \quad (\text{A.33})$$

$$\lambda_0 = \frac{\lambda_{max} - \lambda_{min}}{2} \quad (\text{A.34})$$

$$l_0 = \frac{\lambda_0}{\lambda_m} \quad (\text{A.35})$$

The equilibrium axial stress rate is obtained deriving Equation 2.52

$$\begin{aligned} \dot{P}_\infty(t) = & 2\dot{\lambda}(t) \left[15C_{30}\lambda^4(t) + 6(C_{30} + 9C_{30})\lambda^2(t) + 18C_{30}\lambda(t) + C_{10} - 6C_{20} + 27C_{30} + \right. \\ & \left. + 2(C_{10} - 6C_{20} + 27C_{30})\frac{1}{\lambda^3(t)} + 12(C_{20} - 9C_{30})\frac{1}{\lambda^4(t)} + 48C_{30}\frac{1}{\lambda^5(t)} \right] \end{aligned} \quad (\text{A.36})$$

$$\begin{aligned} = & A\omega \cos(\omega t) \left[A_0 + A_1 \sin(\omega t) + A_2 \sin^2(\omega t) + A_3 \sin^3(\omega t) + A_4 \sin^4(\omega t) + \right. \\ & \left. + A_6 \frac{1}{(1 + l_0 \sin(\omega t))^3} + A_8 \frac{1}{(1 + l_0 \sin(\omega t))^4} + A_{10} \frac{1}{(1 + l_0 \sin(\omega t))^5} \right] \end{aligned} \quad (\text{A.37})$$

where the following constants are defined

$$A = 2\lambda_0 \quad (\text{A.38})$$

$$A_0 = C_{10} + 6C_{20}(\lambda_m^2 - 1) + 3C_{30}(5\lambda_m^4 - 18\lambda_m^2 + 6\lambda_m + 9) \quad (\text{A.39})$$

$$A_1 = 6\lambda_0 [2C_{20}\lambda_m + C_{30}(10\lambda_m^3 - 18\lambda_m + 3)] \quad (\text{A.40})$$

$$A_2 = 6\lambda_0^2 [C_{20} + 3C_{30}(5\lambda_m^2 - 3)] \quad (\text{A.41})$$

$$A_3 = 60C_{30}l_0^3\lambda_m \quad (\text{A.42})$$

$$A_4 = 15C_{30}l_0^4 \quad (\text{A.43})$$

$$A_6 = 2 \frac{C_{10} - 6C_{20} + 27C_{30}}{\lambda_m^3} \quad (\text{A.44})$$

$$A_8 = 12 \frac{C_{20} - 9C_{30}}{\lambda_m^4} \quad (\text{A.45})$$

$$A_{10} = 48 \frac{C_{30}}{\lambda_m^5} \quad (\text{A.46})$$

The solution of Equation 2.53 is obtained from the general solution of Equation 2.29 to evaluate the viscous components of the stress. As for the case of shear deformation, the solution is the sum of the homogeneous and the particular solutions.

- Homogeneous equation:

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = 0 \quad (\text{A.47})$$

$$Q_{i,h}(t) = Q_{i0} e^{-\frac{t}{\tau_i}} \quad (\text{A.48})$$

- Particular solution (0):

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = \mu_i A A_0 \omega \cos(\omega t) \quad (\text{A.49})$$

$$Q_{i,p1}(t) = \mu_i A A_0 \frac{\sin(\omega t) + \phi_i \cos(\omega t)}{1 + \phi_i^2} \quad (\text{A.50})$$

- Particular solution (1):

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = \mu_i A A_1 \omega \cos(\omega t) \sin(\omega t) \quad (\text{A.51})$$

$$Q_{i,p1}(t) = \mu_i A A_1 \frac{\frac{\phi}{2} \sin(2\omega t) - \cos(2\omega t)}{4 \left(1 + \left(\frac{\phi}{2}\right)^2\right)} \quad (\text{A.52})$$

- Particular solution (2):

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = \mu_i A A_2 \omega \cos(\omega t) \sin^2(\omega t) \quad (\text{A.53})$$

$$Q_{i,p2}(t) = \mu_i A A_2 \left[\frac{\sin(\omega t) + \phi_i \cos(\omega t)}{4 (1 + \phi_i^2)} - \frac{\sin(3\omega t) + \frac{\phi_i}{3} \cos(3\omega t)}{12 \left(1 + \left(\frac{\phi_i}{3}\right)^2\right)} \right] \quad (\text{A.54})$$

- Particular solution (3):

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = \mu_i A A_3 \omega \cos(\omega t) \sin^3(\omega t) \quad (\text{A.55})$$

$$Q_{i,p3}(t) = \mu_i A A_3 \left[\frac{\frac{\phi}{2} \sin(2\omega t) - \cos(2\omega t)}{4 \left(1 + \left(\frac{\phi}{2}\right)^2\right)} - \frac{\frac{\phi}{4} \sin(4\omega t) - \cos(4\omega t)}{32 \left(1 + \left(\frac{\phi}{4}\right)^2\right)} \right] \quad (\text{A.56})$$

- Particular solution (4):

$$\dot{Q}_i(t) + \frac{Q_i(t)}{\tau_i} = \mu_i A A_4 \omega \cos(\omega t) \sin^4(\omega t) \quad (\text{A.57})$$

$$Q_{i,p4}(t) = \mu_i A A_4 \left[\frac{\sin(\omega t) + \phi_i \cos(\omega t)}{8 (1 + \phi_i^2)} + \right. \\ \left. - \frac{\sin(3\omega t) + \frac{\phi_i}{3} \cos(3\omega t)}{16 \left(1 + \left(\frac{\phi_i}{3}\right)^2\right)} + \frac{\sin(5\omega t) + \frac{\phi_i}{5} \cos(5\omega t)}{80 \left(1 + \left(\frac{\phi_i}{5}\right)^2\right)} \right] \quad (\text{A.58})$$

- Particular solutions (6, 8, 10) cannot be expressed in a closed form, but for $j = 6, 8, 10$

$$\dot{Q}_{i,pj}(t) + \frac{Q_{i,pj}(t)}{\tau_i} = \mu_i A A_j \omega \frac{\cos(\omega t)}{(1 + l_0 \sin(\omega t))^{\frac{j}{2}}} \quad (\text{A.59})$$

$$Q_{i,pj}(t) = e^{-\frac{t-t_0}{\tau_i}} \left[Q_{i,pj}(t_0) + \mu_i A A_j \omega \int_{t_0}^t e^{\frac{s-t_0}{\tau_i}} \frac{\cos(\omega s)}{(1 + l_0 \sin(\omega s))^{\frac{j}{2}}} ds \right] \quad (\text{A.60})$$

If $t_0 = 0$ and $Q_{i,pj}(0) = 0$ the solution is

$$Q_{i,pj}(t) = \mu_i A A_j \omega \int_0^t e^{\frac{s-t}{\tau_i}} \frac{\cos(\omega s)}{(1 + l_0 \sin(\omega s))^{\frac{j}{2}}} ds \quad j = 6, 8, 10 \quad (\text{A.61})$$

Once the initial conditions decayed

$$Q_{i,omog}(t) = Q_{i0} e^{-\frac{t}{\tau_i}} \longrightarrow 0 \quad \text{for} \quad t \longrightarrow \infty \quad (\text{A.62})$$

the general solution for the over-stresses is the sum of each particular solution $Q_i(t) = \sum_j Q_{i,pj}(t)$

$$\begin{aligned} Q_i(t) = & \mu_i A \left\{ \frac{1}{8} \frac{8A_0 + 2A_2 + A_4}{1 + \phi_i^2} [\sin(\omega t) + \phi_i \cos(\omega t)] + \right. \\ & + \frac{1}{8} \frac{2A_1 + A_3}{1 + \left(\frac{\phi_i}{2}\right)^2} \left[\frac{\phi_i}{2} \sin(2\omega t) - \cos(2\omega t) \right] + \\ & - \frac{1}{16} \frac{\frac{3}{4}A_2 + A_4}{1 + \left(\frac{\phi_i}{3}\right)^2} \left[\sin(3\omega t) + \frac{\phi_i}{3} \cos(3\omega t) \right] + \\ & - \frac{1}{32} \frac{A_3}{1 + \left(\frac{\phi_i}{4}\right)^2} \left[\frac{\phi_i}{4} \sin(4\omega t) - \cos(4\omega t) \right] + \\ & + \frac{1}{80} \frac{A_4}{1 + \left(\frac{\phi_i}{5}\right)^2} \left[\sin(5\omega t) + \frac{\phi_i}{5} \cos(5\omega t) \right] + \\ & \left. + \omega \sum_{j=6,8,10} A_j \int_0^t e^{\frac{s-t}{\tau_i}} \frac{\cos(\omega s)}{(1 + l_0 \sin(\omega s))^{\frac{j}{2}}} ds \right\} \quad (\text{A.63}) \end{aligned}$$

The stress at maximum and minimum stretches λ_{max} and λ_{min} are evaluated at time values of one quarter and three quarters of the cycle

$$P_{\lambda_{max}} = P(\lambda = \lambda_{max}) = P(t = (n + \frac{1}{4})T) = P_{\infty}(\frac{T}{4}) + \sum_{i=1}^N Q_i(\frac{T}{4}) \quad (\text{A.64})$$

$$\begin{aligned} P_{\infty}(\frac{T}{4}) = & 2 \left(\lambda_{max} - \frac{1}{\lambda_{max}^2} \right) \left[C_{10} + 2C_{20} \left(\lambda_{max}^2 + \frac{2}{\lambda_{max}} - 3 \right) + \right. \\ & \left. + 3C_{30} \left(\lambda_{max}^2 + \frac{2}{\lambda_{max}} - 3 \right)^2 \right] \quad (\text{A.65}) \end{aligned}$$

$$\begin{aligned} Q_i(\frac{T}{4}) = & \dots = \mu_i A \left[\frac{\mathcal{V}}{1 + \phi_i^2} + \frac{\mathcal{W}}{1 + \left(\frac{\phi_i}{2}\right)^2} - \frac{\mathcal{X}}{1 + \left(\frac{\phi_i}{3}\right)^2} + \right. \\ & \left. - \frac{\mathcal{Y}}{1 + \left(\frac{\phi_i}{4}\right)^2} + \frac{\mathcal{Z}}{1 + \left(\frac{\phi_i}{5}\right)^2} + \mathcal{U}_i(\frac{T}{4}) \right] \quad (\text{A.66}) \end{aligned}$$

$$P_{\lambda_{min}} = P(\lambda = \lambda_{min}) = P(t = (n + \frac{3}{4})T) = P_{\infty}(\frac{3T}{4}) + \sum_{i=1}^N Q_i(\frac{3T}{4}) \quad (\text{A.67})$$

$$P_{\infty}(\frac{3T}{4}) = 2 \left(\lambda_{min} - \frac{1}{\lambda_{min}^2} \right) \left[C_{10} + 2C_{20} \left(\lambda_{min}^2 + \frac{2}{\lambda_{min}} - 3 \right) + 3C_{30} \left(\lambda_{min}^2 + \frac{2}{\lambda_{min}} - 3 \right)^2 \right] \quad (\text{A.68})$$

$$Q_i(\frac{3T}{4}) = \dots = \mu_i A \left[-\frac{\mathcal{V}}{1 + \phi_i^2} + \frac{\mathcal{W}}{1 + \left(\frac{\phi_i}{2}\right)^2} + \frac{\mathcal{X}}{1 + \left(\frac{\phi_i}{3}\right)^2} - \frac{\mathcal{Y}}{1 + \left(\frac{\phi_i}{4}\right)^2} - \frac{\mathcal{Z}}{1 + \left(\frac{\phi_i}{5}\right)^2} + \mathcal{U}_i(\frac{3T}{4}) \right] \quad (\text{A.69})$$

where we define

$$\mathcal{V} = A_0 + \frac{A_2}{4} + \frac{A_4}{8} \quad (\text{A.70})$$

$$\mathcal{W} = \frac{A_1}{4} + \frac{A_3}{8} \quad (\text{A.71})$$

$$\mathcal{X} = -\frac{A_2}{12} - \frac{A_4}{16} \quad (\text{A.72})$$

$$\mathcal{Y} = -\frac{A_3}{32} \quad (\text{A.73})$$

$$\mathcal{Z} = \frac{A_4}{80} \quad (\text{A.74})$$

$$\mathcal{U}_i(t) = \omega \sum_{j=6,8,10} A_j \int_0^t e^{\frac{s-t}{\tau_i}} \frac{\cos(\omega s)}{(1 + l_0 \sin(\omega s))^{\frac{j}{2}}} ds \quad (\text{A.75})$$

The dynamic modulus evaluated between the maximum and minimum stretches is

$$E_{dyn} = \frac{P_{\lambda_{max}} - P_{\lambda_{min}}}{\lambda_{max} - \lambda_{min}} = \frac{P_{\lambda_{max}} - P_{\lambda_{min}}}{2\lambda_0} \quad (\text{A.76})$$

where

$$\begin{aligned} P_{\lambda_{max}} - P_{\lambda_{min}} &= \left\{ P_{\infty}(\frac{T}{4}) - P_{\infty}(\frac{3T}{4}) \right\} + \left\{ \sum_{i=1}^N Q_i(\frac{T}{4}) - Q_i(\frac{3T}{4}) \right\} \\ &= 4\lambda_0 [3C_{30}L_4 + 2(C_{20} - 6C_{30})L_2 + 6C_{30}L_1 + \\ &\quad + C_{10} - 6C_{20} + 27C_{30} - 6C_{30}L_{-1} + \\ &\quad + (C_{10} - 6C_{20} + 27C_{30})L_{-2} + 2(2C_{20} + 9C_{30})L_{-3} + 12C_{30}L_{-4}] + \\ &\quad + 4\lambda_0 \sum_{i=1}^N \mu_i \left[\frac{\mathcal{V}}{1 + \phi_i^2} - \frac{\mathcal{X}}{1 + \left(\frac{\phi_i}{3}\right)^2} + \frac{\mathcal{Z}}{1 + \left(\frac{\phi_i}{5}\right)^2} + \mathcal{U}_i(\frac{T}{4}) - \mathcal{U}_i(\frac{3T}{4}) \right] \end{aligned} \quad (\text{A.77})$$

and

$$L_4 = (\lambda_{max}^4 + \lambda_{max}^3 \lambda_{min} + \lambda_{max}^2 \lambda_{min}^2 + \lambda_{max} \lambda_{min}^3 + \lambda_{min}^4) \quad (\text{A.78})$$

$$L_2 = (\lambda_{max}^2 + \lambda_{max} \lambda_{min} + \lambda_{min}^2) \quad (\text{A.79})$$

$$L_1 = (\lambda_{max} + \lambda_{min}) \quad (\text{A.80})$$

$$L_{-1} = \frac{1}{\lambda_{max} \lambda_{min}} \quad (\text{A.81})$$

$$L_{-2} = \frac{\lambda_{max} + \lambda_{min}}{\lambda_{max}^2 \lambda_{min}^2} \quad (\text{A.82})$$

$$L_{-3} = \frac{\lambda_{max}^2 + \lambda_{max} \lambda_{min} + \lambda_{min}^2}{\lambda_{max}^3 \lambda_{min}^3} \quad (\text{A.83})$$

$$L_{-4} = \frac{\lambda_{max}^3 + \lambda_{max}^2 \lambda_{min} + \lambda_{max} \lambda_{min}^2 + \lambda_{min}^3}{\lambda_{max}^4 \lambda_{min}^4} \quad (\text{A.84})$$

The dissipation of energy in one cycle is

$$\mathcal{D} = \oint P(t) \dot{\lambda}(t) dt = \int_{nT}^{(n+1)T} \sum_{i=1}^N Q_i(t) \dot{\lambda}(t) dt \quad \forall n \in \mathbb{N} \quad (\text{A.85})$$

and only one term is non-zero in a whole cycle, as in Equation A.27, as well as the integration of $\mathcal{U}_i(t)$

$$\mathcal{D} = 2\pi \lambda_0^2 \sum_{i=1}^N \mu_i \left[\nu \frac{\phi_i}{1 + \phi_i^2} + \frac{2}{T} \oint \mathcal{U}_i(t) \cos(\omega t) dt \right] \quad (\text{A.86})$$

The damping is eventually evaluated as

$$\xi = \frac{\mathcal{D}}{2\pi \frac{P_{\lambda_{max}} - P_{\lambda_{min}}}{2} \frac{\lambda_{max} - \lambda_{min}}{2}} \quad (\text{A.87})$$

Even if we accept to numerically integrate the $3N$ functions $\mathcal{U}_i(t)$ to evaluate the dynamic modulus E_{dyn} , the calculation of the damping imposes another integration in one cycle, so that Equation 2.31 is computationally more convenient in the case of a cyclic axial test.

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