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Boundary conditions of four-dimensional $\mathcal{N} = 2$ gauge theories

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Introduction

Quantum Field Theory (QFT) is the modern language of theoretical physics. It proved successful, both on conceptual and phenomenological grounds, across several different areas such as Condensed Matter Physics, Statistical Physics and Particle Physics. QFT is however 80 years in the making and multiple interesting phenomena, like the dynamical generation of a mass gap and confinement in four dimensional gauge theories, are still not understood. Numerical methods have been developed, but with our theoretical knowledge it is hard even to argue that such a transition should be present from first principles. For this reason, in this thesis, we are interested in analytical methods for strongly interacting gauge theories in four dimensions.

Another important question to be asked is how a theory depends on boundary conditions. This question is relevant because in nature all systems are bounded. Boundary conditions are crucial not only for phenomenological reasons, but they also stand at the core of the definition of relevant theoretical concepts (to name a few, monopoles, defects and more recently topological operators implementing non-invertible symmetries). Moreover, some important problems in un-bounded physical systems, even related to quark (de-)confinement, can be cast more precisely when we close the system in a "box".

For these reasons in this thesis we study analytically strongly interacting 4d gauge theories and their dependence on boundary conditions. We aim to study these from different theoretical angles, namely Quantum Field Theory and String Theory.

It is unrealistic to aim for exact results in general set-ups: we need to study more constrained models and relax the restrictions over time. These constraints come in the form of symmetries and our focus is on supersymmetry and conformal symmetry.

Supersymmetry was first introduced in string-theoretic scenarios. It was later discovered to be a consistent way of escaping Coleman-Mandula's no go theorem [1], which restricted the possible symmetries for scattering amplitudes. The theorem assumes that the symmetry algebra is a Lie algebra. An extension, due to Haag–Łopuszański–Sohnius [2], allows for Lie superalgebra structures. The most general structure that they have found, consistent with some basic scattering amplitudes assumptions, is the one we call supersymmetry algebra.

It was first introduced for phenomenological reasons, from being a solution to the hierarchy problem to a way to unify all fundamental forces known so far and eventually embed the Standard Model into Superstring Theory, it has also been incredibly successful as a benchmark to test and study non-perturbative physics. The presence of non-renormalization theorems allows to have a strong theoretical grip that often results in a complete analytical understanding.

Furthermore, in the past fifteen years, a new tool has been developed, supersymmetric localization [3, 4]. With it, for a class of interesting observables, it is possible to reduce the path integral to a finite dimensional one.

In particular in this thesis the focus is on four dimensional theories enjoying (bulk) $\mathcal{N} = 2$ supersymmetry. In placing a boundary we also necessarily break at least part of the supersymmetry (and also at least part of the conformal symmetry). We are interested in studying boundary conditions that are going to preserve the maximal amount of supersymmetry possible, namely half of it, which are called $\frac{1}{2}$-BPS boundary conditions. In this way we are going to study four dimensional systems that preserve the least amount of supersymmetry compatible with using localization. The fact that we have so few supersymmetries in 4d leads to new technical difficulties which are discussed in this thesis.

Conformal symmetry is another symmetry that avoids Coleman-Mandula's no go theorem [1], but in this case it is a symmetry that we already know to not be phenomenologically viable for Fundamental Physics. It is however enjoyed by a system that undergoes second order phase transitions and as such it is extremely important for Statistical Physics and Condensed Matter Physics. Conformal Field Theories (CFTs) are symmetric under local rescaling of the metric and they have special place in theoretical physics for the following reasons.

Lagrangians or Hamiltonians have been used extensively in modeling physical systems, especially to describe locally interacting degrees of freedom in QFTs. It is undeniable that they proved, and still prove, to be extremely powerful tools, but we still lack a deep understanding of QFTs without these. This is incredibly unsatisfactory since there are interactions that are not mutually local, as interactions among electrically and magnetically charged particles, and models that in principle do not have a Lagrangian description (or at least not a natural one). It is also possible that the theory is Lagrangian but strongly coupled from the get go or that one might want to study the physics at some point along the RG-flow. For all these cases, developing other tools outside the Lagrangian or Hamiltonian ones, would be extremely insightful.

CFTs precisely furnish infinite examples of non Lagrangian theories. The extremely constraining symmetry is such that, even without a Lagrangian description or a natural way to apply Lagrangian techniques, it is nevertheless possible to talk about the complete operator spectrum, its algebra and its relation to the Hilbert space. The conformal bootstrap, both numerical [5, 6] and analytical [7, 8], is a program that in the last ten years has been studying the solution for this class of theories by using mathematical analysis approaches.

For these reasons a large portion of the thesis is devoted in studying Boundary Conformal Field Theories (BCFTs). These theories enjoy a rich class of observables, boundary conditions and relations between dynamics in different dimensions thanks to bulk-to-boundary interactions. However it is still challenging to compute observables in these models, especially in d>2 and in non-maximally supersymmetric systems, as a result we can still learn a lot by studying these theories.

In this thesis we present a way to compute exactly the one-point function of chiral operators, which parametrize the Coulomb branch of 4d $\mathcal{N} = 2$ gauge theories with $\frac{1}{2}$-BPS boundary conditions, in the BCFT fixed point [9]. Via a Ward identity we link derivatives with respect to chiral couplings to insertions of chiral operators at the pole of a hemisphere HS^4. We then map everything back to $H\mathbb{R}^4$ using conformal symmetry to obtain the one-point function's BCFT data. We test our results with an abelian theory in the bulk, also at strong coupling, by employing the $SL(2, \mathbb{Z})$ duality that this system enjoys [10, 11].

In this thesis we also focus on running gauge theories, to study phenomena related to (de-)confinement. In order to tame infrared divergences Callan and Wilzcek introduced the concept of placing a theory on Anti de Sitter (AdS) [12]. In recent years a new idea emerged [13], to use this framework to study (de-)confinement transition. This is because in AdS asymptotic states can be identified with boundary operators, that can be studied reliably with CFT techniques, and in particular one can check the existence of colored asymptotic states. The latter can exist only if the theory is deconfined, thus by studying possible inconsistencies in the correlators of the boundary operators, once we reach strong coupling, one can argue precisely the disappearance in the IR of the free-gluon and with that the fact that the theory is no longer deconfined. One can then try to take at the end the flat-space limit to recover the physics observed in our universe. This was recently used to study non-supersymmetric scenarios in [14].

Within this context, in this thesis we present a work in progress [15] regarding 4d $\mathcal{N}=2$ $SU(2)$ pure Super Yang-Mills theory. This theory it is known to Higgs to $U(1)$ at the origin of the Coulomb branch [16]. We wish to see this analytically by computing explicitly the AdS$_4$ partition function.

All the projects explained so far regard Quantum Field Theory, but there is another important framework where theoretical advancement can be made. String Theory, which played a central role in Mathematics and in our growing understanding of Quantum Gravity, in the last thirty years also proved incredibly successful for studying Quantum Field Theories (e.g. non-Lagrangian QFTs). More interestingly for this project's aims, String Theory possesses a plethora of dualities that often translate into field theoretic ones. The poorly understood strongly coupled dynamics can be mapped by such dualities into a weakly coupled description, where our knowledge is firm. For 4d theories with boundaries the duality acts non-trivially on the boundary modes in a way which is not yet comprehended, except in $\mathcal{N}=4$ [17, 18].

A specific rich class of models, that enjoy these kind of dualities but in $\mathcal{N}=2$, are A_{N-1} circular quivers. These are very well understood without a boundary, also in M-theory [19], but so far it is still not clear what types of boundary conditions they enjoy and how these are mapped by these dualities. This is precisely what is going to be addressed in this thesis as we discuss a work in progress on brane-engineering [20].

In summary this thesis aims to sharpen our understanding of field and string theoretical tools to study analytically a class of 4d $\mathcal{N}=2$ gauge theories with $\frac{1}{2}$-BPS boundary conditions. Our intention is to improve our understanding on their dependence on the coupling, on the boundary conditions and their transformations under strong-weak dualities.

Structure of the thesis

In Chapter 1 we are going to present an introduction to supersymmetry in flat 4d and 3d. We then describe how to construct 4d supersymmetric theory with a boundary, in the specific example of bulk $\mathcal{N} = 2$ supersymmetry and $\frac{1}{2}$-BPS boundary conditions, and then we describe superconformal symmetry.

In Chapter 2 we move toward explaining how to put a theory on a rigid curved manifold while preserving supersymmetry. We then talk about the hemisphere HS^4 and AdS$_4$ as specific backgrounds and we describe supersymmetric localization on these spaces.

In Chapter 3 we describe how to compute exactly one-point functions of chiral operators in 4d $\mathcal{N} = 2$ BCFTs.

In Chapter 4 we describe, as a case study, a bulk abelian theory coupled to boundary modes. We test the results with Feynman diagrams, we prove the expected $SL(2, \mathbb{Z})$-covariance analytically and we test strong coupling expansions via Padé extrapolations.

In Chapter 5 we introduce the notion of using AdS as infrared regulator and as a mean to quantitatively study (de-)confinement. We then present a work in progress regarding analytically studying the Higgs process of pure $\mathcal{N} = 2$ $SU(2)$ SYM down to $U(1)$.

In Chapter 6 we describe how to engineer 4d $\mathcal{N} = 2$ A_{N-1} circular quiver theories in different string frames and in M-theory, we discuss the duality group and in particular its strong-weak duality. Finally we describe a work in progress regarding introducing boundary branes to the system and how we expect the duality group to act on these.

In Chapter 7 we conclude by describing the obtained results, their importance and the future directions.

We relegate to the appendices relevant computations that would break the flow of the discussion.

Chapter 1

Supersymmetry

In this chapter we give a survey on supersymmetry. We do not go into details, the reader is encouraged to refer to the common literature which we also employ [21–24]. We only focus on what is relevant for the aims of this thesis. In particular, since the supersymmetry algebra depends strongly on the signature of space-time and on the number of dimensions, we specialize to supersymmetry in 4 and 3 Euclidean dimensions.[1]

The supersymmetry algebra is a Lie superalgebras. As a vector space it decomposes in two sub-spaces, $L = L_0 \oplus L_1$, which correspond to bosonic and fermionic transformations respectively. They are endowed with a bilinear product, that in physics we realize via commutations and anti-commutation of the transformations, and whose restriction to the bosonic sub-space coincides with the product in the Lie algebra. More formally, defining $|x| = i$ if $x \in L_i$, this product satisfies

$$
\begin{aligned}
&[,]_\pm : L_i \times L_j \longrightarrow L_{i+j\big|_{\text{mod } 2}}\,, \\
&[x,y]_\pm = -(-1)^{|x||y|}[y,x]_\pm\,, \\
&(-1)^{|x|\,|z|}[x,[y,z]_\pm]_\pm + (\text{cyclic}) = 0\,.
\end{aligned}
$$

Supersymmetry has been extensively used as a theoretical laboratory, because it implies several non-renormalization theorems and often it allows quantitative non-perturbative predictions.

The particular reason we are interested in this symmetry is because in some rigid curved manifolds it was exploited, via what is known as supersymmetric

[1] To be more precise, the multiplets we describe in future sections come after a Wick rotation from the ones in Minkowski space-time.

localization [3,4], to compute exactly the partition function as well as a large class of observables [25–29].

We apply this technique to 4d theories with a boundary, we therefore start our discussion by describing supersymmetry in 4d Euclidean flat-space.

1.1 Supersymmetry in 4d flat-space

The supersymmetry algebra contains the Poincaré algebra obeyed by the generators of translations P_μ and of rotations/boosts $M_{\mu\nu}$, together with possible internal symmetries B_a

$$\begin{aligned}
&[P_\mu, P_\nu] = 0\,,\\
&[M_{\mu\nu}, M_{\rho,\sigma}] = -i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\sigma}M_{\mu\rho} + i\eta_{\nu\rho}M_{\mu\sigma}\,,\\
&[M_{\mu\nu}, P_\rho] = -i\eta_{\mu\rho}P_\nu + i\eta_{\nu\rho}P_\sigma\\
&[B_a, B_b] = if_{ab}{}^c B_c\,,\\
&[P_\mu, B_a] = 0\,,\\
&[M_{\mu\nu}, B_a] = 0\,,
\end{aligned} \tag{1.1.1}$$

but is supplemented with additional symmetry generators that respect [21]

$$\begin{aligned}
&[P_\mu, Q^A_\alpha] = [P_\mu, \bar{Q}^A_{\dot\alpha}] = 0\,,\\
&[M_{\mu\nu}, Q^A_\alpha] = \frac{i}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q^A_\beta\,, \quad [M_{\mu\nu}, \bar{Q}^{\dot\alpha}_A] = \frac{i}{2}(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}_A\,,\\
&[B_a, Q^A_\alpha] = (b_a)^A{}_B Q^B_\alpha\,, \quad [B_a, \bar{Q}_{A\dot\alpha}] = -\bar{Q}_{B\dot\alpha}(b_a)^B{}_A\,,\\
&\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta}Z^{AB}\,, \quad \{\bar{Q}_{A\dot\alpha}, \bar{Q}_{B\dot\beta}\} = \epsilon_{\dot\alpha\dot\beta}\bar{Z}_{AB}\,,\\
&\{Q^A_\alpha, \bar{Q}_{B\dot\beta}\} = -2\sigma^\mu_{\alpha\dot\beta}P_\mu\delta^A_B\,.
\end{aligned} \tag{1.1.2}$$

The Q^A_α and $\bar{Q}^{\dot\alpha}_A$ are called *supercharges* and the Z^{AB} are called *central charges*, since it is possible to prove that these commute with all the other generators. From the supersymmetry algebra it is possible to see that the action of supercharges raises or lowers the spin of a particle by $\frac{1}{2}$.

Q^A_α and $\bar{Q}^{\dot\alpha}_A$ are related by complex conjugation in Minkowski space-time but this is not true in Euclidean signature. In the supersymmetry algebras, among the generators B_a, there can be some special ones that generate what is called $\mathcal{R}$-symmetry, defined by the property that the supercharges are charged under it.

Namely, the commutator between the $\mathcal{R}$-symmetry generator and the supercharge gives a matrix rotation acting on the indices $A\,,B\,,\dots$ of the supercharge.

The irreducible representations of the supersymmetry algebra are called *supermultiplets*, or *multiplets* for short, and can be seen as collection of particles.

The $\mathcal{R}$-index $A \in \{1\,,\dots\,,\mathcal{N}\}$ "counts" the amount of supersymmetry. In this thesis we mainly focus our attention to $\mathcal{N}=2$ supersymmetry in 4d and 3d. In order to be self-contained, and in order to introduce the reader to the concept of supersymmetry, we start with 4d $\mathcal{N}=1$ supersymmetry. We show the supersymmetry transformations and some of the canonical techniques to construct both multiplets and supersymmetric invariants.

These constructions are not fundamentally different with different amount of supersymmetry. For this reason, after having showed the 4d $\mathcal{N}=1$ constructions, we just state the results for the 4d and 3d $\mathcal{N}=2$ cases.

1.2 $\mathcal{N}=1$ supersymmetry in 4d flat-space

With this amount of supersymmetry the only supercharges are Q_α and $\bar{Q}_{\dot\alpha}$, that can be rotated by $U(1)_\mathcal{R}$ $\mathcal{R}$-symmetry.

Omitting spinor indices, one can construct the so called *general off-shell multiplet*[2]

$$\begin{aligned}
\delta q &= i\epsilon\psi - i\bar\epsilon\bar\psi\,,\\
\delta\psi &= \epsilon F + \sigma^\mu\bar\epsilon(iA_\mu + \partial_\mu q)\,,\\
\delta\bar\psi &= \bar\epsilon\bar F + \bar\sigma^\mu\epsilon(iA_\mu - \partial_\mu q)\,,\\
\delta F &= 2\bar\epsilon\bar\lambda + 2i\bar\epsilon\bar\sigma^\mu\partial_\mu\psi\,,\\
\delta\bar F &= 2\epsilon\lambda + 2i\epsilon\sigma^\mu\partial_\mu\bar\psi\,,\\
\delta A_\mu &= i(\epsilon\sigma_\mu\bar\lambda + \bar\epsilon\bar\sigma_\mu\lambda) + \partial_\mu(\epsilon\psi + \bar\epsilon\bar\psi)\,,\\
\delta\lambda &= i\epsilon\mathrm{D} + 2\sigma^{\mu\nu}\epsilon\partial_\mu A_\nu\,,\\
\delta\bar\lambda &= -i\bar\epsilon\mathrm{D} + 2\bar\sigma^{\mu\nu}\bar\epsilon\partial_\mu A_\nu\,,\\
\delta\mathrm{D} &= -\partial_\mu(\epsilon\sigma^\mu\bar\lambda - \bar\epsilon\bar\sigma^\mu\lambda)\,.
\end{aligned} \tag{1.2.1}$$

This multiplet can be constructed by acting with a supersymmetry transformation on the first field, which is called *bottom component*. This generates new fields on which we can apply another supersymmetry transformation while imposing

[2]To be specific, this is the most general one with a scalar field as first component.

consistency conditions (e.g. $\{\epsilon Q, \bar{\epsilon}\bar{Q}\}q = -2\epsilon\sigma^\mu\bar{\epsilon}P_\mu Q = 2i\partial_\mu q$). One proceeds until no new fields need to be introduced in order to preserve the algebra. The last field that is found in this process is called *top component*.

This multiplet in general contains too much content to construct toy models that encompass interesting examples (e.g. gauge theories interacting with matter). We therefore impose on this some supersymmetry-consistent constrains.

The first interesting multiplet we consider is the *(anti-)chiral multiplet*, which is obtained by requiring that the bottom component is annihilated by $\bar{Q}$ (or by Q for the anti-chiral). Fixing the attention to the chiral multiplet, this means requiring $\bar{\psi} = 0$ on the general off-shell supermultiplet. By self-consistence we also need to require that $\delta\bar{\psi} = 0$, which impose new constraints, and so on. The end result is displayed below.

Chiral multiplet

$$\begin{aligned} \delta q &= i\epsilon\psi\,, \\ \delta\psi &= \epsilon F + 2\sigma^\mu\bar{\epsilon}\partial_\mu q\,, \\ \delta F &= 2i\partial_\mu(\bar{\epsilon}\bar{\sigma}^\mu\psi)\,. \end{aligned} \tag{1.2.2}$$

Anti-chiral multiplet

$$\begin{aligned} \delta\bar{q} &= -i\bar{\epsilon}\bar{\psi}\,, \\ \delta\bar{\psi} &= \bar{\epsilon}\bar{F} - 2\bar{\sigma}^\mu\epsilon\partial_\mu\bar{q}\,, \\ \delta\bar{F} &= 2i\partial_\mu(\epsilon\sigma^\mu\bar{\psi})\,. \end{aligned} \tag{1.2.3}$$

where we have, in terms of the general off shell multiplet,

Chiral multiplet

$$(q, \psi, \bar{\psi}, F, \bar{F}, A_\mu, \lambda, \bar{\lambda}, \mathrm{D}) = (q, \psi, 0, F, 0, -i\partial_\mu q, 0, 0, 0)\,. \tag{1.2.4}$$

Anti-chiral multiplet

$$(q, \psi, \bar{\psi}, F, \bar{F}, A_\mu, \lambda, \bar{\lambda}, \mathrm{D}) = (\bar{q}, 0, \bar{\psi}, 0, \bar{F}, i\partial_\mu\bar{q}, 0, 0, 0)\,. \tag{1.2.5}$$

A supersymmetric Lagrangian for the free-theory of a chiral field can be found either by enforcing supersymmetry on the scalar kinetic term or by using the method explained in Section 1.2

$$\mathcal{L} = \partial^\mu q\partial_\mu\bar{q} + \frac{i}{4}\big(\psi\sigma^\mu\partial_\mu\bar{\psi} - \partial_\mu\psi\sigma^\mu\bar{\psi}\big) - \frac{1}{4}F\bar{F}\,. \tag{1.2.6}$$

We can clearly see that the field F is auxiliary. This is a general feature of supersymmetry, we need typically to introduce these kind of fields in order to close off-shell the supersymmetry algebra [30].

When present these auxiliary fields can be integrated away exactly, even in the presence of interaction terms, so that physical interactions become manifest. The theory remains a supersymmetric one but the supersymmetry algebra is improved by adding on the right hand side of the (anti-)commutators (1.1.2) terms related to the equations of motion (this does not change the action of supersymmetry on the physical states nor on the scattering amplitudes).

It is not always possible to construct off-shell multiplets but, since off-shell supersymmetry is needed for supersymmetric localization [3], we study cases where this is possible and thus we stress the off-shell formalism.

Constructing supersymmetric invariants

There is a notion of "multiplication of multiplets" where one simply imposes the bottom component of the resulting multiplet to be the multiplication of the initial two [31]. The remaining components of the new multiplet can be found by enforcing supersymmetry. Below we display the result for the multiplication of a general off-shell multiplet

$$
\begin{aligned}
& q = q_1 q_2\,, \quad \psi = \psi_1 q_2 + q_1 \psi_2\,, \quad \bar\psi = \bar\psi_1 q_2 + q_1 \bar\psi_2\,, \\
& F = F_1 q_2 + q_1 F_2 - i\psi_1\psi_2\,, \quad \bar F = \bar F_1 q_2 + q_1 \bar F_2 + i\bar\psi_1\bar\psi_2\,, \\
& A_\mu = A_{1\mu} q_2 + q_1 A_{2\mu} - \frac{1}{2}(\psi_1\sigma_\mu\psi_2 + \bar\psi_1\bar\sigma_\mu\psi_2)\,, \\
& \lambda = \left(\lambda_1 q_2 + \frac{i}{2}\bar F_1\psi_2 - \frac{1}{2}\sigma^\mu\bar\psi_1(A_{2\mu} - i\partial_\mu q_2) + \frac{i}{2} z_2\bar\psi_1 q_2\right) + (1\leftrightarrow 2)\,, \\
& \bar\lambda = \left(\bar\lambda_1 q_2 - \frac{i}{2}F_1\bar\psi_2 - \frac{1}{2}\sigma^\mu\psi_1(A_{2\mu} + i\partial_\mu q_2) - \frac{i}{2} z_2\psi_1 q_2\right) + (1\leftrightarrow 2)\,, \\
& \mathrm{D} = \mathrm{D}_1 q_2 + q_1 \mathrm{D}_2 + \frac{1}{2}F_1\bar F_2 + \frac{1}{2}\bar F_1 F_2 - A_1^\mu A_{2\mu} - \partial^\mu q_1\partial_\mu q_2 \\
& \qquad - \psi_1(\lambda_2 + \frac{i}{2}\sigma^\mu\partial_\mu\bar\psi_2) + \bar\psi_1(\bar\lambda_2 - \frac{i}{2}\sigma^\mu\partial_\mu\psi_2) \\
& \qquad - (\lambda_1 - \frac{i}{2}\partial_\mu\bar\psi\bar\sigma^\mu)\psi_2 + (\bar\lambda_1 + \frac{i}{2}\partial_\mu\psi_1\bar\sigma^\mu)\bar\psi_2\,.
\end{aligned}
\tag{1.2.7}
$$

We can combine this notion with the fact that the supersymmetric variation of the last component of a multiplet is a total derivative (1.2.1).

In this way, starting from some "fundamental multiplets", we can construct other ones via multiplication and declare the resulting top component to be (part of) a supersymmetric Lagrangian.

A concrete example is the free chiral Lagrangian in (1.2.6) that actually is obtained, apart from a $-\frac{1}{2}$ factor, as the top-component of the multiplication of a chiral and anti-chiral field (1.2.4)-(1.2.5).

These kind of invariants are called *D-terms*, essentially because this is the usual name of the top-component in the general off-shell supermultiplet (1.2.1). One can actually notice that there is an additional invariant that it possible to obtain by multiplying only chiral multiplets, the F-component in (1.2.1). Terms constructed in this way are called *F-terms* or *superpotentials*.

Adding gauge interactions

The supersymmetric transformations for gauge-charged fields are clearly not gauge-invariant due to the lack of gauge-covariant derivatives. This can be solved by modifying the supersymmetry algebra (1.2.1) in order to contain in the right hand side some field-dependent gauge transformations. This does not change the action on physical states, since these are gauge-singlets.

In practice, in 4d $\mathcal{N} = 1$ supersymmetry, this only amounts on improving the algebra showed in (1.1.2) by requiring[3]

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = -2\sigma^\mu_{\alpha\dot\alpha}\big(P_\mu + \mathrm{Gauge}(A_\mu)\big)\,, \tag{1.2.8}$$

where $\mathrm{Gauge}(\theta)$ is a gauge transformation on a field defined in the following way

$$\mathrm{Gauge}\big(\theta(x)\big)\phi = \theta(x)\phi\,, \tag{1.2.9}$$

with $\theta(x) = \theta^I(x)T^R_I$ and T^R_I the generators in the representation R under which the field ϕ transforms. Thus $\mathrm{Gauge}(A_\mu)$ is a field dependent gauge transformation, with vector parameter, which we impose to be present in the anti-commutation of Q_α and $\bar{Q}_{\dot\alpha}$ in such a way that

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\}\phi = -2\sigma^\mu_{\alpha\dot\alpha}(-i\partial_\mu + A_\mu)\phi = 2i\sigma^\mu_{\alpha\dot\alpha}D_\mu\phi\,. \tag{1.2.10}$$

[3] In the literature there are other equivalent procedures to deal with the lack of gauge covariance, as the Wess-Zumino compensation [32].

In principle one can construct the full general off-shell multiplet and repeat the discussion we had in Section 1.2, but we refrain from doing that.

The *vector multiplet*, which contains the gauge field, has the following content and transformations

$$\begin{aligned}
&\textbf{Vector multiplet}\\
&\delta A_\mu = i(\epsilon\sigma_\mu\bar{\lambda} + \bar{\epsilon}\bar{\sigma}_\mu\lambda)\,,\\
&\delta\lambda = i\epsilon \mathrm{D} + \sigma^{\mu\nu}\epsilon F_{\mu\nu}\,,\\
&\delta\bar{\lambda} = -i\bar{\epsilon}\mathrm{D} - \bar{\sigma}^{\mu\nu}\bar{\epsilon}F_{\mu\nu}\,,\\
&\delta \mathrm{D} = -D_\mu(\epsilon\sigma^\mu\bar{\lambda} - \bar{\epsilon}\bar{\sigma}^\mu\lambda)\,.
\end{aligned} \tag{1.2.11}$$

The D-field is an auxiliary field precisely as the F-field in the chiral multiplet (1.2.2).

We can also write the supersymmetric transformations for a (anti-)chiral multiplet charged under a gauge group

$$\begin{aligned}
&\textbf{Chiral multiplet}\\
&\delta q = i\epsilon\psi\,,\\
&\delta\psi = \epsilon F + 2i\sigma^\mu\bar{\epsilon}D_\mu q\,,\\
&\delta F = 2iD_\mu(\bar{\zeta}\bar{\sigma}^\mu\psi) + 2i\bar{\zeta}\bar{\lambda}q\ .
\end{aligned} \tag{1.2.12}$$

$$\begin{aligned}
&\textbf{Anti-chiral multiplet}\\
&\delta\bar{q} = -i\bar{\epsilon}\bar{\psi}\,,\\
&\delta\bar{\psi} = \bar{\epsilon}\bar{F} - 2i\bar{\sigma}^\mu\epsilon D_\mu\bar{q}\,,\\
&\delta\bar{F} = 2iD_i(\zeta\gamma^i\bar{\psi}) - 2i\bar{q}\zeta\lambda\ .
\end{aligned} \tag{1.2.13}$$

The Lagrangian for the vector multiplet can be obtained with a procedure similar to the one showed above for the chiral multiplet.

1.3 $\mathcal{N}=2$ supersymmetry in 4d flat-space

With this amount of supersymmetry we have Q^A_α and $\bar{Q}^A_{\dot{\alpha}}$ as supercharges that can be rotated by $U(1)_\mathcal{R}\times SU(2)_\mathcal{R}$ $\mathcal{R}$-symmetry, where A is an index in the fundamental of $SU(2)_\mathcal{R}$, see Appendix A for the conventions.[4]

[4]Note that in the following we are not always consistent with the normalization of the elements in the algebra displayed in (1.1.2).

The transformations of a 4d $\mathcal{N}=2$ vector multiplet are [33]

Vector multiplet

$$\begin{aligned}
&\delta\phi = -i\epsilon^A\lambda_A\,,\\
&\delta\bar\phi = i\bar\epsilon^A\bar\lambda_A\,,\\
&\delta\lambda_A = \frac{1}{2}\sigma^{\mu\nu}\epsilon_A F_{\mu\nu} + 2\sigma^\mu\bar\epsilon_A D_\mu\phi + 2i\epsilon_A[\phi,\bar\phi] + D_{AB}\epsilon^B\,,\\
&\delta\bar\lambda_A = \frac{1}{2}\bar\sigma^{\mu\nu}\bar\epsilon_A F_{\mu\nu} + 2\bar\sigma^\mu\epsilon_A D_\mu\bar\phi - 2i\bar\epsilon_A[\phi,\bar\phi] + D_{AB}\bar\epsilon^B\,,\\
&\delta A_\mu = i\epsilon^A\sigma_\mu\bar\lambda_A - i\bar\epsilon^A\bar\sigma_\mu\lambda_A\,,\\
&\delta D_{AB} = -2i\bar\epsilon_{(A}\bar\sigma^\mu D_\mu\lambda_{B)} + 2i\epsilon_{(A}\sigma^\mu D_\mu\bar\lambda_{B)} - 4[\phi,\bar\epsilon_{(A}\bar\lambda_{B)}] + 4[\bar\phi,\epsilon_{(A}\lambda_{B)}]\,.
\end{aligned} \tag{1.3.1}$$

The matter content of a gauge theory is described by the *hypermultiplet*. In Euclidean signature the on-shell content is given by complex scalars q_{IA} and a spinor ψ_I, with $I\,,J\,,\ldots$ being a $USp(N_F)$ flavor index [33]. This index is raised and lowered via the symplectic form Ω^{IJ}/Ω_{IJ}.

In order to couple the hypermultiplets to gauge fields, we gauge a subgroup of $USp(N_F)$. The commutant is the new flavor symmetry group. We always take the gauge-algebra matrices as embedded in the $USp(N_F)$ ones.

The supersymmetry algebra needs infinitely many auxiliary fields in order to close off-shell for this multiplet. However, if one focuses to some particular 4d $\mathcal{N}=1$ subalgebra (or a 3d $\mathcal{N}=2$ one), then one can close the transformations and represent them as [33]

Hypermultiplet

$$\begin{aligned}
&\delta q_{IA} = -i\epsilon_A\psi_I + i\bar\epsilon_A\bar\psi_I\,,\\
&\delta\psi_I = 2\sigma^\mu\bar\epsilon_A D_\mu q_I^A - 4i\epsilon_A\bar\phi q_I^A + 2\epsilon_A F_I^{\hat A}\,,\\
&\delta\bar\psi_I = 2\bar\sigma^\mu\epsilon_A D_\mu q_I^A - 4i\bar\epsilon_A\phi q_I^A + 2\bar\epsilon_A F_I^{\hat A}\,,\\
&\delta F_{I\hat A} = i\epsilon_A\sigma^\mu D_\mu\bar\psi_I - 2\epsilon_A\phi\psi_I - 2\epsilon_A\lambda_B q_I^B\\
&\qquad\quad - i\bar\epsilon_A\bar\sigma^\mu D_\mu\psi_I + 2\bar\epsilon_A\bar\phi\bar\psi_I + 2\bar\epsilon_A\bar\lambda_B q_I^B\,.
\end{aligned} \tag{1.3.2}$$

Later we consider these theories on a space with a boundary in such a way to preserve $\mathcal{N}=2$ 3d supersymmetry. This is precisely the subalgebra that is preserved off-shell using the supersymmetric variations showed here, which is important in order to use the technique of supersymmetric localization [3].

Another multiplet is the *(anti-)chiral multiplet* [9, 34]

$$\begin{aligned}
&\textbf{Chiral multiplet}\\
&\delta A = -i\epsilon^A\Psi_A \ ,\\
&\delta\Psi_A = 2\sigma^\mu\bar\epsilon_A D_\mu A + B_{AB}\epsilon^B + \tfrac{1}{2}\sigma^{\mu\nu}\epsilon_A G^-_{\mu\nu} \, ,\\
&\delta B_{AB} = -2i\bar\epsilon_{(A}\bar{\not{D}}\Psi_{B)} + 2i\epsilon_{(A}\Lambda_{B)} \ ,\\
&\delta G^-_{\mu\nu} = -\tfrac{i}{2}\bar\epsilon^A\bar\sigma^\rho\sigma_{\mu\nu}D_\rho\Psi_A - \tfrac{i}{2}\epsilon^A\sigma_{\mu\nu}\Lambda_A \ ,\\
&\delta\Lambda_A = -\tfrac{1}{2}D_\rho G^-_{\mu\nu}\sigma^{\mu\nu}\sigma^\rho\bar\epsilon_A + D_\mu B_{AB}\sigma^\mu\bar\epsilon^B - C\epsilon_A \ ,\\
&\delta C = D_\mu\big(- 2i\bar\epsilon^A\bar\sigma^\mu\Lambda_A\big) \ .
\end{aligned} \tag{1.3.3}$$

$$\begin{aligned}
&\textbf{Anti-chiral multiplet}\\
&\delta\bar A = i\bar\epsilon^A\bar\Psi_A \ ,\\
&\delta\bar\Psi_A = 2\bar\sigma^\mu\epsilon_A D_\mu\bar A + \bar B_{AB}\bar\epsilon^B + \tfrac{1}{2}\bar\sigma^{\mu\nu}\bar\epsilon_A G^+_{\mu\nu} \ ,\\
&\delta\bar B_{AB} = 2i\epsilon_{(A}\not{D}\bar\Psi_{B)} + 2i\bar\epsilon_{(A}\bar\Lambda_{B)} \ ,\\
&\delta G^+_{\mu\nu} = -\tfrac{i}{2}\epsilon^A\sigma^\rho\bar\sigma_{\mu\nu}D_\rho\bar\Psi_A - \tfrac{i}{2}\bar\epsilon^A\bar\sigma_{\mu\nu}\bar\Lambda_A \ ,\\
&\delta\bar\Lambda_A = \tfrac{1}{2}D_\rho G^+_{\mu\nu}\bar\sigma^{\mu\nu}\bar\sigma^\rho\epsilon_A - D_\mu\bar B_{AB}\bar\sigma^\mu\epsilon^B + \bar C\bar\epsilon_A \ ,\\
&\delta\bar C = D_\mu\big(- 2i\epsilon^A\sigma^\mu\bar\Lambda_A\big) \ .
\end{aligned} \tag{1.3.4}$$

This multiplet is important for two main reasons. The first one is that by imposing [34][5]

$$\begin{aligned}
&B_{AB} - \varepsilon_{AC}\varepsilon_{BD}\bar B^{CD} = 0 \, ,\\
&\not\partial\Psi^i - \varepsilon^{AB}\Lambda_B = 0 \, ,\\
&\partial_\nu(G^{+\mu\nu} - G^{-\mu\nu}) = 0 \, ,\\
&C - 2\nabla^2\bar A = 0 \, .
\end{aligned} \tag{1.3.5}$$

and defining

$$\begin{aligned}
&\phi = A \, , \quad \bar\phi = \bar A \, , \quad \lambda_A = \Psi_A \, , \quad \bar\lambda_A = \bar\Psi_A \, ,\\
&F_{\mu\nu} = \frac{1}{2}(G^+_{\mu\nu} + G^-_{\mu\nu}) \, , \quad D_{AB} = B_{AB} \, ,
\end{aligned} \tag{1.3.6}$$

[5] These constraints are chosen starting from the Minkowski signature and assuming to take a anti-chiral multiplet that is the complex conjugate of a starting chiral one. In Euclidean signature this is still true with a suitable prescription of reflection positivity.

we obtain a *vector multiplet*. The second reason instead follows from the fact that we can, choosing some (anti-)holomorphic function $\mathcal{F}(\phi)$ ($\bar{\mathcal{F}}(\bar{\phi})$) of the scalar ϕ ($\bar{\phi}$) of the vector multiplet, called *prepotential*, and by defining

$$F_{I_1\dots I_n} = \frac{\partial}{\partial\phi^{I_1}}\cdots\frac{\partial}{\partial\phi^{I_n}}F(\phi)\,, \tag{1.3.7}$$

it is possible to construct a (anti-)chiral multiplet in the following way [9,34]

Chiral multiplet from prepotential

$$\begin{aligned}
&A = \mathcal{F}\,,\\
&\Psi_{A\alpha} = \mathcal{F}_I\lambda^I_{A\alpha}\,,\\
&B_{AB} = \mathcal{F}_I D^I_{AB} + \frac{i}{2}\mathcal{F}_{IJ}\lambda^I_A\lambda^J_B\,,\\
&G^-_{\mu\nu} = \mathcal{F}_I F^{-I}_{\mu\nu} + 2i\mathcal{F}_{IJ}\lambda^{IA}\sigma_{\mu\nu}\lambda^J_A\,,\\
&\Lambda_{A\alpha} = \mathcal{F}_I(\not{D}\bar{\lambda}^I_A)_\alpha - i\mathcal{F}_I[\bar{\phi},\lambda_{A\alpha}]^I + \frac{i}{4}\mathcal{F}_{IJ}F^{-I}_{\mu\nu}(\sigma^{\mu\nu}\lambda^J_A)_\alpha + \frac{1}{2}\mathcal{F}_{IJ}D^I_{AB}\lambda^{JB}_\alpha\\
&\qquad\quad - \frac{i}{12}\mathcal{F}_{IJK}\big(2(\lambda^I_A\lambda^{JB})\lambda^K_{B\alpha} - \lambda^I_{A\alpha}(\lambda^{JB}\lambda^K_B)\big)\,,\\
&C = -2\mathcal{F}_I D^\mu D_\mu\bar{\phi}^I - \mathcal{F}_I[\lambda^A,\lambda_A]^I + 2\mathcal{F}_I[[\phi,\bar{\phi}],\bar{\phi}]^I + \frac{1}{4}\mathcal{F}_{IJ}D^{IAB}D^J_{AB}\\
&\qquad - \frac{1}{2}\mathcal{F}_{IJ}F^{-I}_{\mu\nu}F^{-J\mu\nu} - i\mathcal{F}_{IJ}\lambda^{IA}\not{D}\bar{\lambda}^J_A - \mathcal{F}_{IJ}([\bar{\phi},\lambda^A]^I\lambda^J_A)\\
&\qquad + \frac{i}{4}\mathcal{F}_{IJK}D^{ABI}\lambda^J_A\lambda^K_B + \frac{i}{8}\mathcal{F}_{IJK}F^{-I}_{\mu\nu}\lambda^{JA}\sigma^{\mu\nu}\lambda^K_A\\
&\qquad + \frac{1}{24}\mathcal{F}_{IJKL}(\lambda^{IA}\lambda^{JB})(\lambda^K_A\lambda^L_B)\,.
\end{aligned} \tag{1.3.8}$$

Anti-chiral multiplet from prepotential

$$\begin{aligned}
&\bar{A} = \bar{\mathcal{F}}\,,\\
&\bar{\Psi}^{A\dot{\alpha}} = \bar{\mathcal{F}}_I\bar{\lambda}^{IA\dot{\alpha}}\,,\\
&\bar{B}^{AB} = \bar{\mathcal{F}}_I D^{IAB} - \frac{i}{2}\bar{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\lambda}^{JB}\,,\\
&G^+_{\mu\nu} = \bar{\mathcal{F}}_I F^{+I}_{\mu\nu} - \frac{i}{8}\bar{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\sigma}_{\mu\nu}\bar{\lambda}^J_A\,,\\
&\bar{\Lambda}^{A\dot{\alpha}} = -\bar{\mathcal{F}}_I(\bar{\not{D}}\lambda^I_A)^{\dot{\alpha}} + i\bar{\mathcal{F}}_I[\phi,\bar{\lambda}^{A\dot{\alpha}}]^I - \frac{i}{4}\bar{\mathcal{F}}_{IJ}F^{+I}_{\mu\nu}(\bar{\sigma}^{\mu\nu}\bar{\lambda}^{JA})^{\dot{\alpha}} + \frac{1}{2}\bar{\mathcal{F}}_{IJ}D^{IAB}\bar{\lambda}^{J\dot{\alpha}}_B
\end{aligned}$$

$$+\frac{i}{12}\bar{\mathcal{F}}_{IJK}\big(2(\bar{\lambda}^{IA}\bar{\lambda}^{JB})\bar{\lambda}^{K\dot{\alpha}}_{B}-\bar{\lambda}^{IA\dot{\alpha}}(\bar{\lambda}^{JB}\bar{\lambda}^{K}_{B})\big)\,, \tag{1.3.9}$$

$$\begin{aligned}\bar{C} = &-2\bar{\mathcal{F}}_I D^\mu D_\mu \phi^I + \bar{\mathcal{F}}_I[\bar{\lambda}^A,\bar{\lambda}_A]^I - 2\bar{\mathcal{F}}_I[\phi,[\phi,\bar{\phi}]]^I + \frac{1}{4}\bar{\mathcal{F}}_{IJ}D^{IAB}D^J_{AB}\\ &-\frac{1}{2}\bar{\mathcal{F}}_{IJ}F^{+I}_{\mu\nu}F^{+J\mu\nu} i\bar{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\not{D}}\lambda^J_A + \bar{\mathcal{F}}_{IJ}([\phi,\bar{\lambda}^A]^I\bar{\lambda}^J_A)\\ &-\frac{i}{4}\bar{\mathcal{F}}_{IJK}D^{ABI}\bar{\lambda}^J_A\bar{\lambda}^K_B + \frac{i}{8}\bar{\mathcal{F}}_{IJK}F^{+I}_{\mu\nu}\bar{\lambda}^{JA}\bar{\sigma}^{\mu\nu}\bar{\lambda}^K_A\\ &-\frac{1}{24}\bar{\mathcal{F}}_{IJKL}(\bar{\lambda}^{IA}\bar{\lambda}^{JB})(\bar{\lambda}^K_A\bar{\lambda}^L_B)\,.\end{aligned}$$

This is important because if one chooses $\mathcal{F}$ or $\bar{\mathcal{F}}$ to be gauge invariant, then the supersymmetry transformation of the component C or $\bar{C}$ is a total derivative (1.3.3)-(1.3.4). As a result, we can use it to construct supersymmetric Lagrangians, similarly to the D and F-terms in $\mathcal{N}=1$, but for the vector multiplet by choosing for instance

$$A=\mathrm{Tr}(\phi^k)\,,\quad \bar{A}=\mathrm{Tr}(\bar{\phi}^k). \tag{1.3.10}$$

For $k=2$ we obtain the renormalizable 4d $\mathcal{N}=2$ Super Yang-Mills Lagrangian (SYM) in flat space

$$S=\frac{\mathrm{Im}(\tau)}{4\pi}\int\mathcal{L}_{\mathrm{YM}}+i\frac{\mathrm{Re}(\tau)}{8\pi}\int\mathcal{L}_\theta\ , \tag{1.3.11}$$

where

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} &= \mathrm{Tr}\bigg(\frac{1}{2}F^{\mu\nu}F_{\mu\nu}-4D_\mu\bar{\phi}D^\mu\phi+i\bar{\lambda}^A\bar{\not{D}}\lambda_A-i\lambda^A\not{D}\bar{\lambda}_A\\ &\qquad -\frac{1}{2}D^{AB}D_{AB}+4[\phi,\bar{\phi}]^2-2\lambda^A[\bar{\phi},\lambda_A]+2\bar{\lambda}^A[\phi,\bar{\lambda}_A]\bigg)\,, \qquad (1.3.12)\\ \mathcal{L}_\theta &= \mathrm{Tr}\bigg(\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\bigg)\ ,\end{aligned}$$

and

$$\tau=\frac{4\pi}{g^2}i+\frac{\theta}{2\pi}\,. \tag{1.3.13}$$

The chiral operators in (1.3.10) are important because their expectation value parametrize the Coulomb branch of gauge-theories.

In order to obtain gauge theories interacting with matter, we can couple this theory to a hypermultiplet. The following action

$$
\begin{aligned}
S_{\text{hyp}} = \int & \frac{1}{2} D_\mu q^{IA} D^\mu q_{IA} - q^{IA} \{\phi, \bar{\phi}\}_I{}^J q_{JA} + \frac{i}{2} q^{IA} (D_{AB})_I{}^J q_J^B \\
& - \frac{i}{2} \bar{\psi}^I \bar{\sigma}^\mu D_\mu \psi_I - \frac{1}{2} \psi^I (\phi)_I{}^J \psi_J + \frac{1}{2} \bar{\psi}^I (\bar{\phi})_I{}^J \bar{\psi}_J \\
& - q^{IA} (\lambda_A)_I{}^J \psi_J + \bar{\psi}^I (\bar{\lambda}_A)_I{}^J q_J^A - \frac{1}{2} F^{IA} F_{IA} \,.
\end{aligned}
\tag{1.3.14}
$$

contains a finite number of auxiliary fields and preserves off-shell only a subalgebra of the full supersymmetry algebra, as we explained around (1.3.2).

1.4 $\mathcal{N}=2$ supersymmetry in 3d flat-space

The supersymmetry structure does not change drastically [31] compared to the one showed in (1.1.2)

$$
\begin{aligned}
& [Q_\alpha, P_i] = [\tilde{Q}_\alpha, P_i] = 0 \,, \\
& [M_{ij}, Q^A_\alpha] = i (\gamma_{ij})_\alpha{}^\beta Q^A_\beta \,, \quad [M_{ij}, \tilde{Q}^A_\alpha] = i (\gamma_{ij})^\alpha{}_\beta \bar{Q}^A_{\dot{\beta}} \,, \\
& \{Q_\alpha, Q_\beta\} = \{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = 0 \,, \quad \{Q_\alpha, \tilde{Q}_\alpha\} = 2 \gamma^i_{\alpha,\beta} P_i + 2 i \varepsilon_{\alpha\beta} Z \,, \\
& [R, Q_\alpha] = Q_\alpha \,, \quad [R, \tilde{Q}_\alpha] = -\tilde{Q}_\alpha \,,
\end{aligned}
\tag{1.4.1}
$$

where Z is the central charge and R is the $SO(2)_{\mathcal{R}}$-symmetry. Calling r the $\mathcal{R}$-charge, the multiplets we are interested in are [31, 35][6]

Chiral multiplet

$$
\begin{aligned}
& \delta q = \sqrt{2} \zeta \psi \,, \\
& \delta \psi = \sqrt{2} \zeta F - \sqrt{2} i z \tilde{\zeta} q - \sqrt{2} i \gamma^i \tilde{\zeta} D_i q + \sqrt{2} i \sigma \tilde{\zeta} q \,, \\
& \delta F = \sqrt{2} i z \tilde{\zeta} \psi - \sqrt{2} i D_i (\tilde{\zeta} \gamma^i \psi) - \sqrt{2} i \sigma \tilde{\zeta} \psi + 2 i \tilde{\zeta} \tilde{\lambda} q \,.
\end{aligned}
\tag{1.4.2}
$$

Anti-chiral multiplet

$$
\begin{aligned}
& \delta \tilde{q} = -\sqrt{2} \tilde{\zeta} \tilde{\psi} \,, \\
& \delta \tilde{\psi} = \sqrt{2} \tilde{\zeta} \tilde{F} + \sqrt{2} i z \zeta \tilde{q} + \sqrt{2} i \gamma^i \zeta D_i \tilde{q} - \sqrt{2} i \tilde{q} \sigma \zeta \,,
\end{aligned}
\tag{1.4.3}
$$

[6] For our conventions on covariant derivatives one can see Appendix A, we need to change the sign on the bilinear terms of the vector multiplet with respect to [35]. We also generalize the linear multiplet to one covariantly conserved, with respect to the one in [31].

$$\delta\tilde{F} = \sqrt{2}iz\zeta\tilde{\psi} - \sqrt{2}iD_i(\zeta\gamma^i\tilde{\psi}) - \sqrt{2}i\zeta\tilde{\psi}\sigma + 2i\tilde{q}\zeta\lambda \ .$$

Vector multiplet

$$\begin{aligned}
&\delta\sigma = -\zeta\tilde{\lambda} + \tilde{\zeta}\lambda \,, \\
&\delta\lambda = \left(iD - \frac{i}{2}\varepsilon^{ijk}\gamma_k F_{ij} - i\gamma^i D_i\sigma\right)\zeta \,, \\
&\delta\tilde{\lambda} = \left(-iD - \frac{i}{2}\varepsilon^{ijk}\gamma_k F_{ij} + i\gamma^i D_i\sigma\right)\tilde{\zeta} \,, \\
&\delta A_i = -i(\zeta\gamma_i\tilde{\lambda} + \tilde{\zeta}\gamma_i\lambda) \,, \\
&\delta D = \zeta\gamma^i D_i\tilde{\lambda} - \tilde{\zeta}\gamma^i D_i\lambda - [\sigma, \zeta\tilde{\lambda} + \tilde{\zeta}\lambda] \ .
\end{aligned} \tag{1.4.4}$$

Linear multiplet

$$\begin{aligned}
&\delta J = i\zeta j + i\tilde{\zeta}\tilde{j} \,, \\
&\delta j = \tilde{\zeta}K + i\gamma^i\tilde{\zeta}(j_i + iD_iJ) + \tilde{\zeta}[\sigma, J] \\
&\delta\tilde{j} = \zeta K - i\gamma^i\zeta(j_i - iD_iJ) - \zeta[\sigma, J] \,, \\
&\delta j_i = i\varepsilon_{ijk}D^j(\zeta\gamma^k j - \tilde{\zeta}\gamma^k\tilde{j}) + [\sigma, \zeta\gamma_i j + \tilde{\zeta}\gamma_i\tilde{j}] + i[J, \tilde{\zeta}\gamma_i\lambda - \zeta\gamma_i\tilde{\lambda}] \,, \\
&\delta K = -iD_i(\zeta\gamma^i j + \tilde{\zeta}\gamma^i\tilde{j}) + [\zeta\tilde{\lambda} + \tilde{\zeta}\lambda, J] \ .
\end{aligned} \tag{1.4.5}$$

The (anti-)chiral and vector multiplets play the role of elementary fields in supersymmetric gauge theories. The linear multiplet instead contains a covariantly conserved current, thus in general the fields inside the multiplet are composite operators.

All the multiplets showed here are off shell ones. For this reason, just like in (1.2.6), inside Lagrangians some of the fields are auxiliary (or in the case of the linear multiplet, there is a component that contains composite operators always made of one or more auxiliary fields).

The entire content of the (anti-)chiral multiplet is charged under the $SO(2)_\mathcal{R}$-symmetry and the auxiliary field is always played by F ($\tilde{F}$). In the vector multiplet the only elements charged under $SO(2)_\mathcal{R}$ are the spinors and the auxiliary field is D. Similarly only the spinors of the linear multiplet are charged under $SO(2)_\mathcal{R}$ and K is the auxiliary field. It is also possible to map a vector multiplet into a linear one via [31]

$$\begin{aligned} &J=\sigma\,, \quad j=i\tilde{\lambda}\,, \quad \tilde{j}=-i\lambda\,,\\ &j_\mu=-\frac{i}{2}\varepsilon_{\mu\nu\rho}F^{\mu\nu}\,, \quad K=D\,. \end{aligned} \tag{1.4.6}$$

The renormalizable supersymmetric Lagrangians for vectors interacting with matter are the following [35][7]

$$\mathcal{L}_{\text{SYM}}=\frac{ki}{4\pi}\text{Tr}\bigg(\frac{1}{4}F^{ij}F_{ij}+\frac{1}{2}D^i\sigma D_i\sigma-i\tilde{\lambda}\gamma^iD_i\lambda-\tilde{\lambda}[\sigma,\lambda]-\frac{1}{2}D^2\bigg)\,, \tag{1.4.7}$$

$$\mathcal{L}_{\text{CS}}=\frac{ki}{4\pi}\text{Tr}\bigg(\epsilon^{ijk}\big(A_i\partial_jA_k-\frac{2i}{3}A_iA_jA_k\big)+2iD\sigma+2\tilde{\lambda}\lambda\bigg)\,, \tag{1.4.8}$$

$$\begin{aligned}\mathcal{L}_{\text{chiral}}=&\,D_i\tilde{q}D^iq-i\tilde{\psi}\not{D}\psi+\tilde{q}\big(D+\sigma^2\big)q-i\tilde{\psi}\sigma\psi+\sqrt{2}i(\tilde{q}\lambda\psi+\tilde{\psi}\tilde{\lambda}q)\\ &-\tilde{F}F\,.\end{aligned} \tag{1.4.9}$$

where k needs to be an integer for gauge invariance (see e.g. [36] for more details). One can also add in principle a F-term, also called superpotential as in the 4d $\mathcal{N}=1$ case, via multiplying three chiral fields similarly as was done in (1.2.7) (the detail of the result of this multiplication does not matter in this context).

1.5 Coupling bulk and boundary in a supersymmetric fashion

If we consider a supersymmetric 4d theory and we study it in flat space with a boundary we necessarily break some of the supersymmetry. The reason for this statement is that, as we can see from (1.1.2), supersymmetry generates translations and the translations perpendicular to the boundary cease to be a symmetry.

If we start from a 4d $\mathcal{N}=2$ theory the best we can do is to preserve half of the supersymmetry, in particular preserving a 3d $\mathcal{N}=2$ subalgebra [37]. This can be obtained by identifying the supersymmetry parameters in the following way

$$\bar{\epsilon}_A=i\tau_{3,A}{}^B\bar{\sigma}^\perp\epsilon_B\,, \tag{1.5.1}$$

where τ_3 is one of the Pauli matrices, acting on the $\mathcal{R}$-symmetry indices, $\sigma_\mu/\bar{\sigma}_\mu$ are the 2×2 chiral sub-matrices in the gamma matrices, and $\mu=\perp$ is the perpendicular direction (see Appendix A for more details).

[7] Some signs are swapped due to what explained in footnote 6.

This reduces the amount of independent supersymmetry transformations and therefore the number of conserved charges. One can show that in this way we do not generate translations perpendicular to the boundary via anti-commutations of the conserved charges. Equation (1.5.1) breaks $U(1)_{\mathcal{R}}$-symmetry and breaks $SU(2)_{\mathcal{R}}$ down to $SO(2)_{\mathcal{R}}$ (different $SO(2)_{\mathcal{R}}$ subgroups can be obtained by substituting τ_3 with $\vec{n}\cdot\vec{\tau}$ for a generic $\vec{n}$ with $|\vec{n}|^2 = 1$).

We then identify, by our choices of normalizations, the 3d $\mathcal{N}=2$ Killing spinors and the 3d γ-matrices to be

$$\epsilon_{A=1} = \frac{1}{\sqrt{2}}\zeta\,, \qquad \epsilon_{A=2} = \frac{1}{\sqrt{2}}\tilde{\zeta}\,, \qquad \gamma^i = i\sigma^i\bar{\sigma}^\perp = -i\sigma^\perp\bar{\sigma}^i\,. \tag{1.5.2}$$

There are two separate things we need to do in order to enforce the preservation of this symmetry and we discuss them in the following subsection. Firstly, one needs to choose suitable boundary conditions on the fields. We do not discuss the most general $\frac{1}{2}$-BPS boundary conditions, but we discuss a class general enough to encompass many interesting cases. We can then notice that the bulk supersymmetric Lagrangian are constructed to be invariant modulo a total derivative which, in a space with a boundary, do not imply preservation of supersymmetry even after one imposes boundary conditions. We then need to shift the action by a boundary term.

There are additional technical comments to be made if the boundary one considers is asymptotic. We discuss this from Section 2.4 as we introduce the notion on supersymmetry on AdS_4.

1.5.1 Boundary conditions and boundary terms

Vector multiplet

On the boundary the vector multiplet can be organized in two 3d $\mathcal{N}=2$ linear multiplets as follows (1.4.5)[8]

$$\begin{aligned}
\mathcal{J}_{\mathrm{D}} &= \Big\{2\phi_1, -i\lambda_2^+, i\lambda_1^-, -\frac{i}{2}\varepsilon_{ijk}F^{jk}, -2\big(D_\perp\phi_2 - i\phi_1 + \frac{i}{2}D_{12}\big)\Big\}\,,\\
\mathcal{J}_{\mathrm{N}} &= \Big\{2\phi_2, \lambda_2^-, -\lambda_1^+, F_{\perp i}, 2(D_\perp\phi_1 + i\phi_2)\Big\}\,,
\end{aligned} \tag{1.5.3}$$

[8] Note that in order to identify the variations of $\mathcal{J}_{\mathrm{N}}$ with those of a 3d linear multiplet we need to use that the components D_{11} and D_{22} of D_{AB} vanish at the boundary. This is true in general, even in in the presence of interactions, due to the fact that the boundary condition preserves the Cartan of $SU(2)_R$ symmetry under which these components are charged.

where we have defined, in order to be consistent with the literature [33], the following quantities

$$\phi_1 = \frac{\phi + \bar{\phi}}{2i}\,, \quad \phi_2 = \frac{\phi - \bar{\phi}}{2}\,, \quad \lambda_A^{\pm} = \frac{1}{\sqrt{2}}\left(\lambda_A \pm i\sigma_4\bar{\lambda}_A\right). \tag{1.5.4}$$

For later purposes it is also possible to notice that we can construct a vector multiplet that dualizes, through (1.4.6), to $\mathcal{J}_\mathrm{D}$

$$\mathcal{V} = \{2\phi_1, -\lambda_1^-, -\lambda_2^+, A_i, -iD_{12} - 2D_\perp\phi_2\}\,. \tag{1.5.5}$$

The simplest supersymmetric boundary conditions we can impose are the Dirichlet (D) and Neumann (N) boundary conditions

$$\mathrm{D}: \quad \mathcal{J}_\mathrm{D} = 0\,, \qquad \text{and} \qquad \mathrm{N}: \quad \mathcal{J}_\mathrm{N} + \gamma\mathcal{J}_\mathrm{D} = 0\,, \tag{1.5.6}$$

where we defined

$$\gamma = \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)}\,. \tag{1.5.7}$$

These boundary conditions are the supersymmetrized versions of the Dirichlet and Neumann boundary conditions of the Maxwell theory

$$\mathrm{D}: \quad F_{ij} = 0\,, \qquad \text{and} \qquad \mathrm{N}: \quad F_{\perp i} - \frac{i}{2}\gamma\varepsilon_{ijk}F^{kl} = 0\,. \tag{1.5.8}$$

The additional terms needed to preserve 3d $\mathcal{N} = 2$ supersymmetry are

$$S_\partial = \frac{\mathrm{Im}(\tau)}{4\pi}\int_\partial \mathcal{L}^\partial_\mathrm{YM} + i\frac{\mathrm{Re}(\tau)}{8\pi}\int_\partial \mathcal{L}^\partial_\theta\,, \tag{1.5.9}$$

where

$$\begin{aligned} \mathcal{L}^\partial_\mathrm{YM} &= \mathrm{Tr}\Big[-8\phi_2\left(D_\perp\phi_2 + \tfrac{i}{2}D_{12}\right) + \lambda_1\lambda_2 + \bar{\lambda}_1\bar{\lambda}_2\Big]\,, \\ \mathcal{L}^\partial_\theta &= 2\mathrm{Tr}\Big[8i\phi_1\left(D_\perp\phi_2 + \tfrac{i}{2}D_{12}\right) - \lambda_1\lambda_2 + \bar{\lambda}_1\bar{\lambda}_2 + i\bar{\lambda}_1\bar{\sigma}^\perp\lambda_2 + i\lambda_1\sigma^\perp\bar{\lambda}_2\Big]\,. \end{aligned} \tag{1.5.10}$$

With this boundary action both N and D are supersymmetric boundary conditions according to the definition given above.

Imposing the N boundary condition we have the freedom to perform gauge transformations on the boundary, while the D boundary condition fixes it and the gauge group G becomes a global symmetry on the boundary, with associated current $\mathcal{J}_N$.[9] It is also possible to impose D boundary conditions on some of the gauge components of the gauge field and N on the others in order to preserve only a subgroup of the gauge symmetry at the boundary.

A rich class of boundary conditions for 4d $\mathcal{N}=2$ gauge theories preserving 3d $\mathcal{N}=2$ can be obtained from deformations of the N and D boundary conditions. These deformations are defined by adding local degrees of freedom on the boundary and coupling them with the boundary modes of the bulk fields. In particular, a natural deformation of the N boundary condition consists in using the boundary mode of the bulk gauge field to gauge a global symmetry of additional local boundary degrees of freedom. In the abelian case some deformation is necessary in order to introduce interactions, because the bulk theory is free and the N and D boundary conditions are linear constraints that keep the theory gaussian.

Hypermultiplet

Let us focus our attention to a theory with only 4d a single hypermultiplet. This decomposes in two 3d $\mathcal{N}=2$ chiral multiplets (1.2.12)

$$\begin{aligned}\Phi_{11} &= \left\{q_{11},\ -\frac{i}{2}(\psi_1+i\sigma_\perp\bar\psi_1),\ D_\perp q_{11}+2\phi_2 q_{11}-iF_{11}\right\}\equiv\Phi^\dagger_{22}\ ,\\ \Phi_{12} &= \left\{q_{12},\ \frac{i}{2}(\psi_1-i\sigma_\perp\bar\psi_1),\ -D_\perp q_{12}+2\phi_2 q_{12}-iF_{12}\right\}\equiv-\Phi^\dagger_{21}\ ,\end{aligned}\tag{1.5.11}$$

where ϕ_2 is defined in (1.5.4).

The simplest boundary conditions for a single 4d hypermultiplet sets to zero either Φ_{11} or Φ_{12}. We refer to them as boundary conditions 1 and 2, respectively. These conditions are easily generalized to the case of N_F hypermultiplets. Namely, a similar binary choice is obtained by picking N_F components labeled by i among the $2N_F$ components labeled by I, and setting to zero one between $\Phi_{i1}=(\Omega^{iJ}\Phi_{J2})^\dagger$ or $\Phi_{i2}=-(\Omega^{iJ}\Phi_{J1})^\dagger$, where Ω^{IJ} is the symplectic matrix acting on the flavour indices.

The boundary term that needs to be added is

$$S_{\text{hyp},\partial}=\int_\partial -\frac{1}{8}\psi^I\psi_I+\frac{1}{8}\bar\psi^I\bar\psi_I+\frac{i}{4}\psi^I\sigma^4\bar\psi_I-2q^I_{A=1}(\phi_2)_I{}^J q_{J,A=2}\,.\tag{1.5.12}$$

Like for the vector multiplet, also for the hypermultiplets the simple boundary conditions that we just mentioned can be deformed by interactions. For instance,

[9] The operator $\mathcal{J}_D$ that survives in the case of the Neumann boundary condition is not gauge invariant and so it does not give rise to a global symmetry, except in the abelian case or more generally if the group has $U(1)$ factors so that $\mathrm{Tr}[\mathcal{J}_D]\neq 0$.

if we perturb the boundary condition 1 by a boundary superpotential $W_{3\mathrm{d}}(q_{12},\ldots)$, where the dots refer to possible local 3d degrees of freedom, then the modified boundary condition is $q_{11} \propto \frac{\partial W_{3\mathrm{d}}}{\partial q_{12}}$.

Chiral multiplet

As was discussed below equation (1.3.10), the supersymmetric variation of the bottom component C of a chiral multiplet, integrated on $\mathbb{R}^4$, is a supersymmetric invariant since its variation is a total derivative. We know however that in the presence of a boundary this does not suffice, we need to add a boundary term. This boundary term can be found in [9] after sending the radius of the HS^4, considered in that paper, to infinity

$$\begin{aligned} \mathcal{C}_\partial &= 2D_\perp A - i(\tau_3)^{AB} B_{AB} \ , \\ \bar{\mathcal{C}}_\partial &= 2D_\perp \bar{A} + i(\tau_3)^{AB} \bar{B}_{AB} \ . \end{aligned} \tag{1.5.13}$$

The addition of this term preserves supersymmetry irrespective of the boundary conditions and already off-shell.

In particular, as already commented below (1.3.10), if we define a chiral multiplet starting from $A = \mathrm{Tr}(\phi^2)$ we get something proportional to the SYM action and, by adjusting some normalizations, one can find that (1.5.13) is precisely (1.5.10).

1.6 (Super)conformal symmetry

Conformal transformations are all transformations that modify a flat space metric up to local dilatations

$$\eta'_{\mu\nu} = \frac{\partial x^\rho}{\partial x^\mu}\frac{\partial x^\sigma}{\partial x^\nu}\eta_{\rho\sigma} = \Lambda(x)\eta_{\mu\nu} \, . \tag{1.6.1}$$

These transformations are incredibly constraining for a system but not totally trivializing. They allow to define a one-to-one correspondence between operators and states in a theory and constrain the correlation functions in such a way that many theoretical challenges can be overcome even in strong coupling regimes. These theories thus furnish a natural benchmark for theoretical physicists, but also they are realized in nature in points where second order phase transition occur and in the deep IR.[10]

[10]There are some examples where in the deep IR one does not recover the full conformal invariance, to be more precise. Also it is possible that the conformal invariance in the IR is realized trivially by a free theory or by an empty theory.

Here we discuss just some general properties, for more details we refer to the literature [38, 39].

The Poincaré symmetry is clearly a sub-group of the group of conformal transformations, but there are new transformations for which the algebra of the generators is the following[11]

$$
\begin{aligned}
&[M_{\mu\nu}, K_\rho] = -i\eta_{\mu\rho}K_\nu + i\eta_{\nu\rho}K_\sigma\,,\\
&[K_\mu, P_\nu] = 2i(\eta_{\mu\nu}D - M_{\mu\nu})\,,\\
&[D, P_\mu] = iP_\mu\,,\\
&[D, K_\mu] = -iK_\mu\,,\\
&[K_\mu, K_\nu] = [D, M_{\mu\nu}] = 0\,.
\end{aligned}
\tag{1.6.2}
$$

The new operators are D, that generates dilatations, and K_μ, that generate the so-called *special conformal transformations*. For a space with signature (p, q) it is possible to prove that the conformal group is $SO(p + 1, q + 1)$.

From the algebra above it is possible to infer that P_μ raises the conformal dimension of an operator by one and K_μ lowers it by one. Unitarity bounds [39], for operators of each spin, pose lower bounds on the conformal dimension of operators. For this reason, if we assume a unitarity theory, we should assume that there is a state/operator annihilated by K_μ, this is called a *conformal primary*. Starting from this state and acting with P_μ we get an infinite tower of states called *descendants* which form the so called *conformal family*.

It is possible to see, for instance, that for scalar operators the following n-point functions are completely fixed by conformal symmetry up to overall normalizations

$$
\begin{aligned}
&\langle \mathcal{O}_\Delta(x)\rangle = C_{\mathcal{O}_\Delta}\delta_{\Delta,0}\,,\\
&\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{\Delta_1\Delta_2}}{x_{12}^{2\Delta_1}}\,,\\
&\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_3+\Delta_1-\Delta_2}}\,,
\end{aligned}
\tag{1.6.3}
$$

where Δ is the *scaling dimension* defined via

$$
\mathcal{O}(\lambda x) = \lambda^{-\Delta}\mathcal{O}(x)\,. \tag{1.6.4}
$$

$C_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ in (1.6.3), together with the scaling dimension, define the *CFT data*.

[11] Technically, in $d = 2$, there is an enlargement of the symmetry to the so-called Virasoro algebra.

For $n > 3$-point functions conformal symmetry alone is no longer able to capture the full space-time dependence. For instance for 4-point functions of scalars we have

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = f(u,v)\prod_{i<j}^{4} x_{ij}^{\frac{\Delta}{3}-\Delta_i-\Delta_j} \tag{1.6.5}$$

with

$$\Delta = \sum_{i}^{4}\Delta_i \tag{1.6.6}$$

and

$$u = \frac{x_{12}x_{34}}{x_{13}x_{24}}\,, \quad v = \frac{x_{12}x_{34}}{x_{23}x_{14}}\,. \tag{1.6.7}$$

Even though conformal symmetry constrains the functional dependence in (1.6.5), the function $f(u,v)$ can be in principle generic.

However in a CFT we have the notion of operator product expansion (OPE)[12]

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\mathcal{O}} C_{\mathcal{O},12}(x_{12},\partial_{x_2})\mathcal{O}(x_2)\,, \tag{1.6.8}$$

where the sum in principle runs over conformal primaries and descendants. Usually when we write it in this form the sum runs only over primaries, the derivatives in C give the descendants. This allows us to reduce the n-point functions down to $(n-1)$-point functions recursively, until we reduce to 3-point functions. The different ways in which one makes this operation gives equations that in principle one can try to solve (or use them to constrain numerically the CFT data), and this is called *conformal bootstrap*.

We are however interested in theories with supersymmetry, thus when we fall in a conformal point the theory becomes *superconformal*. Superconformal symmetry is not just the direct sum of supersymmetry and conformal symmetry and its algebra is very constraining. Differently than supersymmetry and conformal symmetry alone, the algebra can be realized only in $2 \leqslant d \leqslant 6$ [40]. We do not talk about all the possible superconformal algebras, we only focus on the ones relevant for this thesis in what follows.

[12] What is written below needs to be be intended as an operatorial equation, thus valid inside correlation functions

1.6.1 $\mathcal{N}=2$ 4d superconformal algebra

The superconformal algebra contains as sub-algebras the supersymmetric one with zero central charge Z^{AB} and the conformal one, but we need to introduce new (anti-)commutation relations. Defining the R-symmetry operator R^A_B, which generates $U(1)_{\mathcal{R}} \times SU(2)_{\mathcal{R}}$, such that it satisfies

$$[R^A_B, R^C_D] = i\delta^C_B R^A_D - i\delta^A_D R^C_B\,, \tag{1.6.9}$$

and such that it acts on the supercharges as

$$[R^A_B, Q^C_\alpha] = i\delta^C_B Q^A_\alpha - \frac{i}{4}\delta^A_B Q^C_\alpha\,, \quad [R^A_B, \bar{Q}^{\dot{\alpha}}_C] = -i\delta^A_C \bar{Q}^{\dot{\alpha}}_B + \frac{i}{4}\delta^A_B \bar{Q}^{\dot{\alpha}}_C\,, \tag{1.6.10}$$

we have the following 4d $\mathcal{N}=2$ superconformal algebra $\mathfrak{su}^*(4|4)$ [41]

$$\begin{aligned}
&\{S_{A\alpha}, \bar{S}^B_{\dot{\beta}}\} = -2\delta^A_B \sigma^\mu_{\alpha\dot{\beta}} K_\mu \\
&[K_\mu, S_{A\alpha}] = [K_\mu, \bar{S}^{A\dot{\alpha}}] = 0\,, \\
&\{S_{A\alpha}, S_{B\beta}\} = \{\bar{S}^{A\dot{\alpha}}, \bar{S}^{B\dot{\beta}}\} = \{Q^A_\alpha, \bar{S}^{B\dot{\beta}}\} = \{\bar{Q}^{A\dot{\alpha}}, S_{B\beta}\} = 0\,, \\
&[M_{\mu\nu}, S_{A\alpha}] = \frac{i}{2}(\sigma_{\mu\nu})_\alpha^{\ \beta} S_{A\beta}\,, \quad [M_{\mu\nu}, \bar{S}^{A\dot{\alpha}}] = \frac{i}{2}(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \bar{S}^{A\dot{\alpha}} \\
&[R^A_B, S_{C\alpha}] = -i\delta^A_C S_{B\alpha} + \frac{i}{4}\delta^A_B S_{C\alpha}\,, \quad [R^A_B, \bar{S}^{C\dot{\alpha}}] = i\delta^C_B \bar{S}^{A\dot{\alpha}} - \frac{i}{4}\delta^A_B \bar{S}^{C\dot{\alpha}}\,, \\
&[D, Q^A_\alpha] = \frac{i}{2} Q^A_\alpha\,, \quad [D, \bar{Q}^{\dot{\alpha}A}] = \frac{i}{2}\bar{Q}^{\dot{\alpha}A}\,, \\
&[D, S_{A\alpha}] = -\frac{i}{2} S_{A\alpha}\,, \quad [D, \bar{S}^{A\dot{\alpha}}] = -\frac{i}{2}\bar{S}^{A\dot{\alpha}} \\
&[K_\mu, Q^A_\alpha] = -\sigma_{\mu,\alpha\dot{\beta}} \bar{S}^{A\dot{\beta}}\,, \quad [K_\mu, \bar{Q}^{\dot{\alpha}}_A] = \bar{\sigma}^{\dot{\alpha}\beta}_\mu S_{A\beta}\,, \\
&[P_\mu, S_{A\alpha}] = -\sigma_{\mu\alpha\dot{\beta}} \bar{Q}^{\dot{\beta}}_A\,, \quad [P_\mu, \bar{S}^{A\dot{\alpha}}] = \bar{\sigma}^{\dot{\alpha}\beta}_\mu Q^A_\beta \\
&\{Q^A_\alpha, S_{B\beta}\} = i\delta^A_B \sigma^{\mu\nu}_{\alpha\beta} M_{\mu\nu} + 2i\varepsilon_{\alpha\beta}\delta^i_j D - 4i\varepsilon_{\alpha\beta} R^A_B\,, \\
&\{\bar{Q}^{\dot{\alpha}}_A, \bar{S}^{B\dot{\beta}}\} = i\delta^B_A \bar{\sigma}^{\mu\nu\dot{\alpha}\dot{\beta}} M_{\mu\nu} - 2i\varepsilon^{\dot{\alpha}\dot{\beta}}\delta^B_A D - 4i\varepsilon^{\dot{\alpha}\dot{\beta}} R^B_A\,.
\end{aligned} \tag{1.6.11}$$

There are new symmetry operators, $S_{A\alpha}$ and $\bar{S}^{A\dot{\alpha}}$, that generate the so called *special superconformal transformations*. Note the symmetry between K_μ and P_μ as well as between Q_α and S_α.

As in a conformal field theory we have conformal primaries, in a superconformal theory we have the notion of a *superconformal primary*. The superconformal primary is a conformal primary such that

$$[S_{a\alpha}, \Phi] = [\bar{S}^{A\dot{\alpha}}, \Phi] = 0\,. \tag{1.6.12}$$

From this we can define the so called *superconformal family.* This can be seen as generated by a superconformal primary from which, by supersymmetric transformations, we find superpartners (that are conformal primaries but not superconformal ones) and from each of which we can generate with P_μ a conformal family. In short we have a collection of conformal families linked via supersymmetry.

There is a special class of operators, which we are interested in, called *chiral operators.* These are defined to be annihilated by all supercharges of a given chirality

$$[\bar{Q}^{\dot{\alpha}}_A, \Phi] = 0\,. \tag{1.6.13}$$

Mutatis mutandis we can define the anti-chirals ones. These class of operators satisfy nice properties. For instance the conformal dimension is fixed by the $U(1)_{\mathcal{R}}$-charge

$$\Delta = \frac{R}{2} \tag{1.6.14}$$

and the OPE of two chiral can be proven to be non-singular.

1.6.2 $\mathcal{N} = 2$ 3d superconformal algebra

The 3d $\mathcal{N} = 2$ superconformal algebra is $\mathfrak{osp}(4|2,2)$. We do not describe the algebra in full detail, only concepts we consider novel as compared to the ones in (1.6.11).

The first is that this symmetry is one of a pure 3d theory but it can also be the one for a 4d one with a boundary. Secondly there is a comment to be made regarding the $U(1)_{\mathcal{R}}$-symmetry. This we could have made also in the 4d case but it is going to be more relevant for us in this superalgebra.

In supersymmetric algebras we can re-define the $U(1)_{\mathcal{R}}$ generator with possible other $U(1)$ global symmetry generators G_i

$$R \longrightarrow R' = R + \sum_i \Delta^i G_i\,. \tag{1.6.15}$$

This happens because the $\mathcal{R}$-symmetry only appears in the commutators of the supersymmetry algebra (1.1.2) and thus

$$[R', \ldots] = [R, \ldots]\,, \tag{1.6.16}$$

because the G_i commute with everything. This shows that the $\mathcal{R}$-charge can be redefined since this does not alter the supersymmetry algebra.

This however is not true in the superconformal algebra since the $\mathcal{R}$-charge appears also outside commutators (1.6.11).

Thus, if we consider a theory in some UV point with 3d $\mathcal{N}=2$ supersymmetry and we run the RG flow to a putative superconformal fixed point, it is important to identify correctly the $\mathcal{R}$-symmetry. This can in principle be done by starting from some trial $\mathcal{R}$-charge and by identifying the collection of the Δ^i such that

$$R_{\text{SCFT}} = R_{\text{trial}} + \sum_i \Delta^i G_i \,. \tag{1.6.17}$$

The correct Δ_i can be identified by some maximization process, as we discuss in Section 2.6.1.

1.6.3 Boundary Conformal Field Theories (BCFTs)

As the name suggests, these are theories that preserve some amount of conformal symmetry despite the boundary (namely the translations $P_\perp$, $K_\perp$ and $M_{\perp i}$ need to be projected out).

Similarly to CFTs we maintain a strong non-perturbative control over the theory but the reduction of the amount of symmetry generators adds technical challenges. For instance, as can be seen from equation (1.6.3), the 1-point function is usually zero except for the identity operator.

In BCFTs one-point functions of scalar operators can depend on the distance from the boundary and thus we have

$$\langle \mathcal{O}(x) \rangle = \frac{a_{\mathcal{O}}}{x_\perp^{\Delta_{\mathcal{O}}}} \,. \tag{1.6.18}$$

The coefficients $a_{\mathcal{O}}$ become part of the so called *BCFT data.* This particular piece of BCFT data is the one we are more interested in in this thesis, but there are others like the bulk-boundary two-point function coefficients

$$\langle \mathcal{O}(x)\hat{\mathcal{O}}(\vec{y}) \rangle = \frac{a_{\mathcal{O}\hat{\mathcal{O}}}}{x_\perp^{\Delta_1-\Delta_2}|\vec{x}-\vec{y}|^{2\Delta_2}} \,, \tag{1.6.19}$$

where the hat is used in this context to state that an operator lives on the boundary.

We already see that the loss of symmetry maps into a loss of control over some n-point functions. Indeed in a BCFT the two-point function is not completely fixed as in the CFT case (1.6.3), but similarly to the case of the four-point function (1.6.5) we only have some constraints on the functional dependence [38]

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \frac{f(\xi)}{x_{\perp,1}^{\Delta_1} x_{\perp,2}^{\Delta_2}} \tag{1.6.20}$$

with

$$\xi = \frac{(x_1 - x_2)^2}{4x_{\perp,1}x_{\perp,2}} \,. \tag{1.6.21}$$

The actual two-point function can be constrained via bootstrap [42, 43], even though in this context the bootstrap equation are much more difficult.

1.6.4 Superconformal symmetry in presence of a boundary

As discussed in Section 1.6.2, if the theory is 4d $\mathcal{N} = 2$ and we study it in the presence of a boundary, we have as maximal superconformal symmetry algebra preserved the 3d $\mathcal{N} = 2$ superconformal one.

We are not able in general, differently from what was discussed in 1.5.1, to start from a 4d $\mathcal{N} = 2$ superconformal Lagrangian theory and engineer a way, via boundary conditions and boundary terms, to preserve 3d $\mathcal{N} = 2$ superconformal symmetry. This is not really surprising, since conformal theories usually do not have a Lagrangian formulation from the get go. At most one has a Lagrangian UV point and can argue, or hope, that in the IR one hits a conformal fixed point.

Chapter 2

Supersymmetry and curved manifolds

In the following Chapter we discuss how to put a supersymmetric theory on a rigid curved manifold while preserving some global supercharges. This allows us to employ supersymmetric localization, a powerful tool to compute a large class of supersymmetric partition functions and observables.

2.1 Supersymmetry in rigid curved space

We start this discussion by introducing the notion of background multiplet. We then describe how to use this to preserve global supercharges on rigid manifolds. We discuss multiplets and supersymmetric Lagrangians in curved space.

Further details can be found in the following references [30, 31, 34, 44].

2.1.1 Background multiplets and supersymmetry

It is always possible to re-interpret couplings as background values of non-dynamical fields. This turns out to be useful for supersymmetric models to construct novel supersymmetric invariants.

An example of this is the *3d $\mathcal{N} = 2$ real mass*. Consider a chiral multiplet that possesses a flavor symmetry. We can couple this flavor symmetry current to a vector multiplet, thus constructing the Lagrangian showed in (1.4.9), but we can fix the vector multiplet to the following background

$$\sigma = m\,, \quad \lambda = \tilde{\lambda} = A_i = D = 0\,. \tag{2.1.1}$$

This background is chosen to be consistent with the 3d $\mathcal{N} = 2$ vector supersymmetry transformations showed in (1.4.4). In particular, in order for a background to be compatible with supersymmetry, there cannot be a spinor acquiring expectation value. For this reasons the only condition, which fixes the bosonic background, has the following general form

$$\delta_{\text{SUSY}}(\text{ferm.} = 0) = 0 \,. \tag{2.1.2}$$

This procedure allows to write a mass term in a supersymmetric fashion, as we argue below.

Suppose to apply a fictitious supersymmetry transformation on the action that also acts on the background and not only on the dynamical fields. This fictitious supersymmetry transformation annihilates the action by construction. A genuine supersymmetry transformation in principle would lead to a different result since it only acts on dynamical fields. This fictitious action however commutes with the background, indeed it annihilates the background fermions because their background is already vanishing and it annihilates the background bosons since their transformation is a background fermion. This means that the fictitious transformation acts precisely as the genuine supersymmetric transformation, thus also the true supersymmetry transformation annihilates the action and for this reason supersymmetry is preserved.

Depending on the context these background fields are also called *spurions*.

2.1.2 Rigid geometry and supersymmetry

The natural way to put a supersymmetric theory on a rigid manifold is consider this theory in a supergravity setting and then fixing a background for the bosonic fields. The relevant bosonic gravitational content for $\mathcal{N} = 2$ theories in 4d and 3d respectively is listed in Table 2.1 and in Table 2.2 [31, 45].

In order to preserve the desired amount of supersymmetry one might try to follow what streamlined in Section 2.1.1. We focus on preserving 3d $\mathcal{N} = 2$ supersymmetry[13] and the equivalent of equation (2.1.2) is played by the variation

[13] We refrain from considering the 4d case because the common way one considers a background for these theories is by starting from superconformal gravity. The same conceptual steps we are explaining here are valid but the process is much more cumbersome; one starts with many more fields and then one needs to gauge fix conformal symmetry.

Bosons	**Description**	$\mathbf{U(1)}_{\mathcal{R}}$ charge	$\mathbf{SU(2)}_{\mathcal{R}}$ repr.
$g_{\mu\nu}$	metric	0	$\mathbf{1}$
$\mathcal{A}_\mu$	$\mathfrak{u}(1)_{\mathcal{R}}$ gauge field	0	$\mathbf{1}$
$\mathcal{V}_\mu$	$\mathfrak{su}(2)_{\mathcal{R}}$ gauge field	0	$\mathbf{3}$
$T_{\mu\nu}$	anti-self-dual tensor	$+2$	$\mathbf{1}$
$\bar{T}_{\mu\nu}$	self-dual tensor	-2	$\mathbf{1}$
$\tilde{M}$	scalar	0	$\mathbf{1}$

Table 2.1: Bosonic content of the 4d $\mathcal{N}=2$ supergravity multiplet.

Bosons	**Description**	**$\mathcal{R}$-charge**
g_{ij}	metric	0
B_{ij}	2-form gauge field	0
C_i	photino	0
$A_i^{\mathcal{R}}$	$\mathcal{R}$-gauge field	0

Table 2.2: Bosonic content of the 3d $\mathcal{N}=2$ supergravity multiplet.

of the gravitini

$$\begin{aligned} 0 &= \delta(\psi_i = 0) = 2(\nabla_i - iA_i)\zeta + H\gamma_i\zeta + 2iV_i\zeta + \varepsilon_{ijk}V^j\gamma^j\zeta\,, \\ 0 &= \delta(\tilde{\psi}_i = 0) = 2(\nabla_i + iA_i)\tilde{\zeta} + H\gamma_i\tilde{\zeta} - 2iV_i\tilde{\zeta} - \varepsilon_{ijk}V^j\gamma^j\tilde{\zeta}\,. \end{aligned} \tag{2.1.3}$$

The difference with respect to what explained in Section 2.1.1 is that, since we want to fix the background metric and possibly other background fields of the supergravity multiplet, we then need to solve a differential equation for the local spinor "$\zeta(x)$", which in supergravity would be generic (indeed it would correspond to a gauge transformation).

This is precisely what give us global supercharges, indeed for each linearly independent solutions the equations above, we obtain linearly independent transformations $c\zeta(x)$ which annihilate the action.

The equations above are called *Killing spinor equations* and the solution is thus called *Killing spinor*.

Notice how, differently from the case discussed in Section 2.1.1, the rest of the bosonic background is not obtained in a univocal way following equation (2.1.2), as it happens for the real mass in (2.1.1) after having chosen $\sigma = m$. The only requirement with the remaining bosonic background is that it still leads to a differential equation with the desired number of independent solutions. For this reason, once chosen a particular background metric, there might be inequivalent choices of backgrounds for the other bosons in the supergravity multiplet which might preserve the same supersymmetries [34]. This is a reflection of the fact that in general there are ambiguities, e.g. different possible choice, when we place a theory on a curved manifold.

The supersymmetry algebra is similar to the one already discussed but it is going to be improved, as happens in (1.2.8), with some additional transformations in order to have covariant derivatives also with respect to the geometry (and with respect to the frozen $\mathcal{R}$-symmetry gauge fields). Essentially we end up with a supergravity algebra but with the gravitational fields set to the chosen background.

In general it is not possible to obtain the same superalgebra of symmetries in different manifolds, even when we are able to preserve the same amount of supercharges. For instance, one might be able to preserve different subgroups of the $\mathcal{R}$-symmetry in different backgrounds. We do not describe in full details the mechanism of why this happens, also because the end-result depends often by the chosen manifold, we limit ourselves to cite the result for our cases of interest. We refer the reader to [34] for further details.

2.1.3 Rigid supersymmetry transformations and Lagrangians

In what follows we generalize some of the results encountered in Chapter 1 to curved space. We present general formulæ that can be specialized to the desired geometry. In this section we keep the Killing spinor unspecified, and also the corresponding background for the supergravity multiplet. In later sections we present both the Killing spinors and the supergravity background for the examples of interest.

4d $\mathcal{N} = 2$ supersymmetry

The supersymmetry transformations we have encountered before are modified in the following way.

Vector multiplet

$\delta\phi = -i\epsilon^A\lambda_A\,,$

$$
\begin{aligned}
&\delta\bar{\phi} = i\bar{\epsilon}^A\bar{\lambda}_A\,,\\
&\delta\lambda_A = \frac{1}{2}\sigma^{\mu\nu}\epsilon_A(F_{\mu\nu}+8\bar{\phi}T_{\mu\nu}) + 2\sigma^\mu\bar{\epsilon}_A D_\mu\phi + \sigma^\mu D_\mu\bar{\epsilon}_A\phi\\
&\qquad + 2i\epsilon_A[\phi,\bar{\phi}] + D_{AB}\epsilon^B\,,\\
&\delta\bar{\lambda}_A = \frac{1}{2}\bar{\sigma}^{\mu\nu}\bar{\epsilon}_A(F_{\mu\nu}+8\phi\bar{T}_{\mu\nu}) + 2\bar{\sigma}^\mu\epsilon_A D_\mu\bar{\phi} + \bar{\sigma}^\mu D_\mu\epsilon_A\bar{\phi}\\
&\qquad - 2i\bar{\epsilon}_A[\phi,\bar{\phi}] + D_{AB}\bar{\epsilon}^B\,,\\
&\delta A_\mu = i\epsilon^A\sigma_\mu\bar{\lambda}_A - i\bar{\epsilon}^A\bar{\sigma}_\mu\lambda_A\,,\\
&\delta D_{AB} = -2i\bar{\epsilon}_{(A}\bar{\sigma}^\mu D_\mu\lambda_{B)} + 2i\epsilon_{(A}\sigma^\mu D_\mu\bar{\lambda}_{B)} - 4[\phi,\bar{\epsilon}_{(A}\bar{\lambda}_{B)}] + 4[\bar{\phi},\epsilon_{(A}\lambda_{B)}]\,.
\end{aligned}
\tag{2.1.4}
$$

Hypermultiplet

$$
\begin{aligned}
&\delta q_{IA} = -i\epsilon_A\psi_I + i\bar{\epsilon}_A\bar{\psi}_I\,,\\
&\delta\psi_I = 2\sigma^\mu\bar{\epsilon}_A D_\mu q_I^A + \sigma^\mu D_\mu\bar{\epsilon}_A q_I^A - 4i\epsilon_A\bar{\phi}q_I^A + 2\hat{\epsilon}_{\hat{A}}F_I^{\hat{A}}\,,\\
&\delta\bar{\psi}_I = 2\bar{\sigma}^\mu\epsilon_A D_\mu q_I^A + \bar{\sigma}^\mu D_\mu\epsilon_A q_I^A - 4i\bar{\epsilon}_A\phi q_I^A + 2\bar{\hat{\epsilon}}_{\hat{A}}F_I^{\hat{A}}\,,\\
&\delta F_{I\hat{A}} = i\hat{\epsilon}_{\hat{A}}\sigma^\mu D_\mu\bar{\psi}_I - 2\hat{\epsilon}_{\hat{A}}\phi\psi_I - 2\hat{\epsilon}_{\hat{A}}\lambda_B q_I^B + 2i\hat{\epsilon}_{\hat{A}}(\sigma^{\mu\nu}T_{\mu\nu})\psi_I\\
&\qquad - i\bar{\hat{\epsilon}}_{\hat{A}}\bar{\sigma}^\mu D_\mu\psi_I + 2\bar{\hat{\epsilon}}_{\hat{A}}\bar{\phi}\bar{\psi}_I + 2\bar{\hat{\epsilon}}_{\hat{A}}\bar{\lambda}_B q_I^B - 2i\bar{\hat{\epsilon}}_{\hat{A}}(\bar{\sigma}^{\mu\nu}\bar{T}_{\mu\nu})\bar{\psi}_I\,,
\end{aligned}
\tag{2.1.5}
$$

Chiral multiplet

$$
\begin{aligned}
&\delta A = -i\epsilon^A\Psi_A\,,\\
&\delta\Psi_A = 2\sigma^\mu\bar{\epsilon}_A D_\mu A + B_{AB}\epsilon^B + \tfrac{1}{2}\sigma^{\mu\nu}\epsilon_A G^-_{\mu\nu} + w\,\sigma^\mu D_\mu\bar{\epsilon}_A A\,,\\
&\delta B_{AB} = -2i\bar{\epsilon}_{(A}\bar{\sigma}^\mu D_\mu\Psi_{B)} + 2i\epsilon_{(A}\Lambda_{B)} + i(1-w)D_\mu\bar{\epsilon}_{(A}\bar{\sigma}^\mu\Psi_{B)}\,,\\
&\delta G^-_{\mu\nu} = -\tfrac{i}{2}\bar{\epsilon}^A\bar{\sigma}^\rho\sigma_{\mu\nu}D_\rho\Psi_A - \tfrac{i}{2}\epsilon^A\sigma_{\mu\nu}\Lambda_A - \tfrac{i}{4}(1+w)D_\rho\bar{\epsilon}^A\bar{\sigma}^\rho\sigma_{\mu\nu}\Psi_A\,,\\
&\delta\Lambda_A = -\tfrac{1}{2}D_\rho G^-_{\mu\nu}\sigma^{\mu\nu}\sigma^\rho\bar{\epsilon}_A + D_\mu B_{AB}\sigma^\mu\bar{\epsilon}^B - C\epsilon_A\\
&\qquad + \tfrac{1}{2}(1+w)B_{AB}\sigma^\mu D_\mu\bar{\epsilon}^B + \tfrac{1}{4}(1-w)G^-_{\mu\nu}\sigma^{\mu\nu}\sigma^\rho D_\rho\bar{\epsilon}_A\\
&\qquad + 4D_\rho A\,\bar{T}^{\mu\nu}\sigma^\rho\bar{\sigma}_{\mu\nu}\bar{\epsilon}_A + 4wAD_\rho\bar{T}^{\mu\nu}\sigma^\rho\bar{\sigma}_{\mu\nu}\bar{\epsilon}_A\,,\\
&\delta C = -2i\bar{\epsilon}^A\bar{\sigma}^\mu D_\mu\Lambda_A + 4i\bar{T}^{\mu\nu}\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\sigma}^\rho D_\rho\Psi_A\\
&\qquad - iwD_\mu\bar{\epsilon}^A\bar{\sigma}^\mu\Lambda_A + 4i(w-1)D_\rho\bar{T}^{\mu\nu}\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\sigma}^\rho\Psi_A\,.
\end{aligned}
\tag{2.1.6}
$$

Anti-chiral multiplet

$$
\begin{aligned}
&\delta\bar{A} = i\bar{\epsilon}^A\bar{\Psi}_A\ ,\\
&\delta\bar{\Psi}_A = 2\bar{\sigma}^\mu\epsilon_A D_\mu\bar{A} + \bar{B}_{AB}\bar{\epsilon}^B + \tfrac{1}{2}\bar{\sigma}^{\mu\nu}\bar{\epsilon}_A G^+_{\mu\nu} + w\,\bar{\sigma}^\mu D_\mu\epsilon_A\bar{A}\ ,\\
&\delta\bar{B}_{AB} = 2i\epsilon_{(A}\sigma^\mu D_\mu\bar{\Psi}_{B)} + 2i\bar{\epsilon}_{(A}\bar{\Lambda}_{B)} - i(1-w)D_\mu\epsilon_{(A}\sigma^\mu\bar{\Psi}_{B)}\ ,\\
&\delta G^+_{\mu\nu} = -\tfrac{i}{2}\epsilon^A\sigma^\rho\bar{\sigma}_{\mu\nu}D_\rho\bar{\Psi}_A - \tfrac{i}{2}\bar{\epsilon}^A\bar{\sigma}_{\mu\nu}\bar{\Lambda}_A + \tfrac{i}{4}(1+w)D_\rho\epsilon^A\sigma^\rho\bar{\sigma}_{\mu\nu}\bar{\Psi}_A\ ,\\
&\delta\bar{\Lambda}_A = \tfrac{1}{2}D_\rho G^+_{\mu\nu}\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho\epsilon_A - D_\mu\bar{B}_{AB}\bar{\sigma}^\mu\epsilon^B + \bar{C}\bar{\epsilon}_A\\
&\qquad\quad - \tfrac{1}{2}(1+w)\bar{B}_{AB}\bar{\sigma}^\mu D_\mu\epsilon^B - \tfrac{1}{4}(1-w)G^+_{\mu\nu}\bar{\sigma}^{\mu\nu}\bar{\sigma}^\rho D_\rho\epsilon_A\\
&\qquad\quad + 4D_\rho\bar{A}\,T^{\mu\nu}\bar{\sigma}^\rho\sigma_{\mu\nu}\epsilon_A + 4w\bar{A}D_\rho T^{\mu\nu}\bar{\sigma}^\rho\sigma_{\mu\nu}\epsilon_A\ ,\\
&\delta\bar{C} = -2i\epsilon^A\sigma^\mu D_\mu\bar{\Lambda}_A + 4iT^{\mu\nu}\epsilon^A\sigma_{\mu\nu}\sigma^\rho D_\rho\bar{\Psi}_A\\
&\qquad\quad - iwD_\mu\epsilon^A\sigma^\mu\bar{\Lambda}_A + 4i(w-1)D_\rho T^{\mu\nu}\epsilon^A\sigma_{\mu\nu}\sigma^\rho\bar{\Psi}_A\ .
\end{aligned}
\tag{2.1.7}
$$

with [45]

$$
\hat{\epsilon}_{\hat{A}} = c^{\frac{1}{2}}\epsilon_A, \qquad \hat{\bar{\epsilon}}_{\hat{A}} = -c^{-\frac{1}{2}}\bar{\epsilon}_A, \qquad c = -\frac{\bar{\epsilon}^A\bar{\epsilon}_A}{\epsilon^B\epsilon_B}\,. \tag{2.1.8}
$$

The w in the chiral multiplet refers to the classical Weyl weight of the operator. Furthermore, as we stated in Section 1.3, the hypermultiplet are written with respect to a sub-algebra of the total 4d $\mathcal{N}=2$ that closes off shell.

As we saw in the flat space case in (1.3.8)-(1.3.9), it is possible to define a chiral multiplet by defining the bottom component $A = \mathcal{F}(\phi)$, with $\mathcal{F}$ a holomorphic function called prepotential (mutatis mutandis for the anti-chiral multiplet). Dubbing $\mathscr{R}$ the Ricci scalar, we present the generalization in rigid curved space of the chiral multiplet.

Chiral multiplet from prepotential

$$
\begin{aligned}
&A = \mathcal{F}(\phi)\,,\\
&\Psi_{A\alpha} = \mathcal{F}_I\lambda^I_{A\alpha}\,,\\
&B_{AB} = \mathcal{F}_I D^I_{AB} + \frac{i}{2}\mathcal{F}_{IJ}\lambda^I_A\lambda^J_B\,,\\
&G^-_{\mu\nu} = \mathcal{F}_I(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu}) + 2i\mathcal{F}_{IJ}\lambda^{IA}\sigma_{\mu\nu}\lambda^J_A\,,\\
&\Lambda_{A\alpha} = \mathcal{F}_I(\not{D}\bar{\lambda}^I_A)_\alpha - i\mathcal{F}_I[\bar{\phi},\lambda_{A\alpha}]^I + \frac{i}{4}\mathcal{F}_{IJ}(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu})(\sigma^{\mu\nu}\lambda^J_A)_\alpha\\
&\qquad\quad + \frac{1}{2}\mathcal{F}_{IJ}D^I_{AB}\lambda^{JB}_\alpha - \frac{i}{12}\mathcal{F}_{IJK}\big(2(\lambda^I_A\lambda^{JB})\lambda^K_{B\alpha} - \lambda^I_{A\alpha}(\lambda^{JB}\lambda^K_B)\big)\,,
\end{aligned}
\tag{2.1.9}
$$

$$
\begin{aligned}
C = & -2\mathcal{F}_I\big(D^\mu D_\mu - \frac{\mathscr{R}}{6} + \frac{\tilde{M}}{2}\big)\bar{\phi}^I - 8\mathcal{F}_I(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu})\bar{T}^{\mu\nu} \\
& + \mathcal{F}_I[\bar{\lambda}^A, \bar{\lambda}_A]^I - 2\mathcal{F}_I[[\bar{\phi}, \phi], \bar{\phi}]^I + \frac{1}{4}\mathcal{F}_{IJ}D^{IAB}D^J_{AB} \\
& - \frac{1}{2}\mathcal{F}_{IJ}(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu})(F^{-J\mu\nu} + 8\bar{\phi}^J T^{\mu\nu}) + i\mathcal{F}_{IJ}\lambda^{IA}\not{D}\bar{\lambda}^J_A \\
& - \mathcal{F}_{IJ}([\bar{\phi}, \lambda^A]^I\lambda^J_A) + \frac{i}{4}\mathcal{F}_{IJK}D^{ABI}\lambda^J_A\lambda^K_B \\
& + \frac{i}{8}\mathcal{F}_{IJK}(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu})\lambda^{JA}\sigma^{\mu\nu}\lambda^K_A + \frac{1}{24}\mathcal{F}_{IJKL}(\lambda^{IA}\lambda^{JB})(\lambda^K_A\lambda^L_B)\,.
\end{aligned}
$$

Anti-chiral multiplet from prepotential

$$
\begin{aligned}
\bar{A} = & \,\bar{\mathcal{F}}(\bar{\phi})\,, \\
\bar{\Psi}^{A\dot{\alpha}} = & \,\bar{\mathcal{F}}_I\bar{\lambda}^{IA\dot{\alpha}}\,, \\
\bar{B}^{AB} = & \,\bar{\mathcal{F}}_I D^{IAB} - \frac{i}{2}\bar{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\lambda}^{JB}\,, \\
G^+_{\mu\nu} = & \,\bar{\mathcal{F}}_I(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu}) - \frac{i}{8}\bar{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\sigma}_{\mu\nu}\bar{\lambda}^J_A\,, \\
\bar{\Lambda}^{A\dot{\alpha}} = & -\bar{\mathcal{F}}_I(\bar{\not{D}}\lambda^I_A)^{\dot{\alpha}} + i\bar{\mathcal{F}}_I[\phi, \bar{\lambda}^{A\dot{\alpha}}]^I - \frac{i}{4}\bar{\mathcal{F}}_{IJ}(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu})(\bar{\sigma}^{\mu\nu}\bar{\lambda}^{JA})^{\dot{\alpha}} \\
& + \frac{1}{2}\bar{\mathcal{F}}_{IJ}D^{IAB}\bar{\lambda}^{J\dot{\alpha}}_B + \frac{i}{12}\bar{\mathcal{F}}_{IJK}\big(2(\bar{\lambda}^{IA}\bar{\lambda}^{JB})\bar{\lambda}^{K\dot{\alpha}}_B - \bar{\lambda}^{IA\dot{\alpha}}(\bar{\lambda}^{JB}\bar{\lambda}^K_B)\big)\,, \qquad (2.1.10) \\
\bar{C} = & -2\bar{\mathcal{F}}_I\big(D^\mu D_\mu - \frac{\mathscr{R}}{6} + \frac{\tilde{M}}{2}\big)\phi^I - 8\bar{\mathcal{F}}_I(F^{-I}_{\mu\nu} + 8\bar{\phi}^I T_{\mu\nu})T^{\mu\nu} \\
& - \bar{\mathcal{F}}_I[\lambda^A, \lambda_A]^I + 2\bar{\mathcal{F}}_I[\phi, [\phi, \bar{\phi}]]^I + \frac{1}{4}\bar{\mathcal{F}}_{IJ}D^{IAB}D^J_{AB} \\
& - \frac{1}{2}\bar{\mathcal{F}}_{IJ}(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu})(F^{+J\mu\nu} + 8\phi^J\bar{T}^{\mu\nu}) - i\bar{\mathcal{F}}_{IJ}\bar{\lambda}^{IA}\bar{\not{D}}\lambda^J_A \\
& + \bar{\mathcal{F}}_{IJ}([\phi, \bar{\lambda}^A]^I\bar{\lambda}^J_A) - \frac{i}{4}\bar{\mathcal{F}}_{IJK}D^{ABI}\bar{\lambda}^J_A\bar{\lambda}^K_B \\
& + \frac{i}{8}\bar{\mathcal{F}}_{IJK}(F^{+I}_{\mu\nu} + 8\phi^I\bar{T}_{\mu\nu})\bar{\lambda}^{JA}\bar{\sigma}^{\mu\nu}\bar{\lambda}^K_A - \frac{1}{24}\bar{\mathcal{F}}_{IJKL}(\bar{\lambda}^{IA}\bar{\lambda}^{JB})(\bar{\lambda}^K_A\bar{\lambda}^L_B)\,.
\end{aligned}
$$

In the limit of flat space $\delta_{\text{SUSY}}C = D_\mu V^\mu$, and thus once embedded the vector multiplet in the (anti-)chiral multiplet, on $\mathbb{R}^4$ it was sufficient to use C and $\bar{C}$ as SYM Lagrangians. In curved space instead one needs a linear combination of the (anti-)chiral content in order to construct a supersymmetric invariant. We do not discuss the most general case, we only discuss later the case of S^4, the hemisphere

HS^4 and AdS_4, which are relevant for the thesis.

The supersymmetric action for the vector multiplet can be written as in (1.3.11)

$$S = \frac{\mathrm{Im}(\tau)}{4\pi}\int \mathcal{L}_{\mathrm{YM}} + i\frac{\mathrm{Re}(\tau)}{8\pi}\int \mathcal{L}_\theta \ , \tag{2.1.11}$$

with the curved-space version of the Lagrangian

$$\begin{aligned}\mathcal{L}_{\mathrm{YM}} = \mathrm{Tr}\bigg(&\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - 4D_\mu\bar{\phi}D^\mu\phi + 2\big(\tilde{M} - \frac{\mathscr{R}}{3}\big)\bar{\phi}\phi + i\bar{\lambda}^A\bar{\not{D}}\lambda_A - i\lambda^A\not{D}\bar{\lambda}_A \\ &- \frac{1}{2}D^{AB}D_{AB} + 4[\phi,\bar{\phi}]^2 - 2\lambda^A[\bar{\phi},\lambda_A] + 2\bar{\lambda}^A[\phi,\bar{\lambda}_A] \\ &+ 16F_{\mu\nu}(\bar{\phi}T^{\mu\nu} + \phi\bar{T}^{\mu\nu}) + 64(\bar{\phi}^2T^{\mu\nu}T_{\mu\nu} + \phi^2\bar{T}^{\mu\nu}\bar{T}_{\mu\nu})\bigg) \ , \\ \mathcal{L}_\theta = \mathrm{Tr}\bigg(&\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}\bigg) \ .\end{aligned} \tag{2.1.12}$$

The hypermultiplet action, realizing explicitly off-shell only part of the 4d $\mathcal{N} = 2$ supersymmetry algebra, is

$$\begin{aligned}S_{\mathrm{hyp}} = \int &\frac{1}{2}D_\mu q^{IA}D^\mu q_{IA} - q^{IA}\{\phi,\bar{\phi}\}_I{}^J q_{JA} + \frac{i}{2}q^{IA}(D_{AB})_I{}^J q_J^B \\ &- \frac{i}{2}\bar{\psi}^I\bar{\sigma}^\mu D_\mu\psi_I - \frac{1}{2}\psi^I(\phi)_I{}^J\psi_J + \frac{1}{2}\bar{\psi}^I(\bar{\phi})_I{}^J\bar{\psi}_J \\ &- q^{IA}(\lambda_A)_I{}^J\psi_J + \bar{\psi}^I(\bar{\lambda}_A)_I{}^J q_J^A - \frac{1}{2}F^{IA}F_{IA} \\ &+ \frac{1}{8}(\tilde{M} + \frac{2}{3}\mathscr{R})q^{IA}q_{IA} + \frac{i}{2}\psi^I\sigma^{\mu\nu}T_{\mu\nu}\psi_I - \frac{i}{2}\bar{\psi}^I\bar{\sigma}^{\mu\nu}\bar{T}_{\mu\nu}\bar{\psi}_I \ .\end{aligned} \tag{2.1.13}$$

3d $\mathcal{N} = 2$ supersymmetry

The supersymmetry transformations in curved space are

Chiral multiplet

$$\begin{aligned}&\delta q = \sqrt{2}\zeta\psi \, , \\ &\delta\psi = \sqrt{2}\zeta F - \sqrt{2}i(z - rH)\tilde{\zeta}q - \sqrt{2}i\gamma^i\tilde{\zeta}D_i q + \sqrt{2}i\sigma\tilde{\zeta}q \, , \\ &\delta F = \sqrt{2}i(z - (r-2)H)\tilde{\zeta}\psi - \sqrt{2}iD_i(\tilde{\zeta}\gamma^i\psi) - \sqrt{2}i\sigma\tilde{\zeta}\psi + 2i\tilde{\zeta}\tilde{\lambda}q \ .\end{aligned} \tag{2.1.14}$$

Anti-chiral multiplet

$$\begin{aligned}
&\delta\tilde{q} = -\sqrt{2}\tilde{\zeta}\tilde{\psi}\,,\\
&\delta\tilde{\psi} = \sqrt{2}\tilde{\zeta}\tilde{F} + \sqrt{2}i(z-rH)\zeta\tilde{q} + \sqrt{2}i\gamma^i\zeta D_i\tilde{q} - \sqrt{2}i\tilde{q}\sigma\zeta\,,\\
&\delta\tilde{F} = \sqrt{2}i(z-(r-2)H)\zeta\tilde{\psi} - \sqrt{2}iD_i(\zeta\gamma^i\tilde{\psi}) - \sqrt{2}i\zeta\tilde{\psi}\sigma + 2i\tilde{q}\zeta\lambda\;.
\end{aligned} \tag{2.1.15}$$

Vector multiplet

$$\begin{aligned}
&\delta\sigma = -\zeta\tilde{\lambda} + \tilde{\zeta}\lambda\,,\\
&\delta\lambda = \Big(i(D+H\sigma) - \frac{i}{2}\varepsilon^{ijk}\gamma_k F_{ij} - i\gamma^i(D_i\sigma + iV_i\sigma)\Big)\zeta\,,\\
&\delta\tilde{\lambda} = \Big(-i(D+H\sigma) - \frac{i}{2}\varepsilon^{ijk}\gamma_k F_{ij} + i\gamma^i(D_i\sigma - iV_i\sigma)\Big)\tilde{\zeta}\,,\\
&\delta A_i = -i(\zeta\gamma_i\tilde{\lambda} + \tilde{\zeta}\gamma_i\lambda)\,,\\
&\delta D = \zeta\gamma^i D_i\tilde{\lambda} - iV_i(\zeta\gamma^i\tilde{\lambda} + \tilde{\zeta}\gamma^i\lambda) - H(\zeta\tilde{\lambda} - \tilde{\zeta}\lambda) - \tilde{\zeta}\gamma^i D_i\lambda - [\sigma, \zeta\tilde{\lambda} + \tilde{\zeta}\lambda]\,.
\end{aligned} \tag{2.1.16}$$

Linear multiplet

$$\begin{aligned}
&\delta J = i\zeta j + i\tilde{\zeta}\tilde{j}\,,\\
&\delta j = \tilde{\zeta}K + i\gamma^i\tilde{\zeta}(j_i + iD_iJ) + \tilde{\zeta}[\sigma, J]\\
&\delta\tilde{j} = \zeta K - i\gamma^i\zeta(j_i - iD_iJ) - \zeta[\sigma, J]\,,\\
&\delta j_i = i\varepsilon_{ijk}D^j(\zeta\gamma^k j - \tilde{\zeta}\gamma^k\tilde{j}) + [\sigma, \zeta\gamma_i j + \tilde{\zeta}\gamma_i\tilde{j}] + i[J, \tilde{\zeta}\gamma_i\lambda - \zeta\gamma_i\tilde{\lambda}]\,,\\
&\delta K = -iD_i(\zeta\gamma^i j + \tilde{\zeta}\gamma^i\tilde{j}) + 2iH(\zeta j + \tilde{\zeta}\tilde{j}) - V_i(\zeta\gamma^i j + \tilde{\zeta}\gamma^i\tilde{j}) + [\zeta\tilde{\lambda} + \tilde{\zeta}\lambda, J]\;.
\end{aligned} \tag{2.1.17}$$

The mapping from the linear multiplet to the vector one is modified from (2.1.18) to

$$\begin{aligned}
&J = \sigma\,, \quad j = i\tilde{\lambda}\,, \quad \tilde{j} = -i\lambda\,,\\
&j_\mu = -\frac{i}{2}\varepsilon_{\mu\nu\rho}F^{\mu\nu}\,, \quad K = D + H\sigma\,.
\end{aligned} \tag{2.1.18}$$

The Lagrangians (1.4.7)-(1.4.8)-(1.4.9) now read[14]

[14]The covariant derivatives here displayed are covariant also with respect to the background gravitational multiplet, please refer to [35] for details.

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SYM}} = \frac{ki}{4\pi}\mathrm{Tr}\bigg(&\frac{1}{4}F^{ij}F_{ij} + \frac{1}{2}D^i\sigma D_i\sigma - i\tilde{\lambda}\gamma^i D_i\lambda + \frac{i}{2}\sigma\varepsilon^{ijk}V_iF_{jk} \\
&- \frac{1}{2}V^iV_i\sigma^2 + \frac{i}{2}H\tilde{\lambda}\lambda - \tilde{\lambda}[\sigma,\lambda] - \frac{1}{2}(D + H\sigma)^2\bigg)\,,
\end{aligned} \tag{2.1.19}
$$

$$
\mathcal{L}_{\mathrm{CS}} = \frac{ki}{4\pi}\mathrm{Tr}\bigg(\epsilon^{ijk}\big(A_i\partial_jA_k - \frac{2i}{3}A_iA_jA_k\big) + 2iD\sigma + 2\tilde{\lambda}\lambda\bigg)\,, \tag{2.1.20}
$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{chiral}} = {}& D_i\tilde{q}D^iq - i\tilde{\psi}\not{D}\psi - i\tilde{\psi}\sigma\psi + \sqrt{2}i(\tilde{q}\lambda\psi + \tilde{\psi}\tilde{\lambda}q) \\
&+ \tilde{q}\bigg(D + \sigma^2 + \frac{r}{2}\mathscr{R} + \frac{1}{2}\big(r - \frac{1}{2}\big)V^iV_i + r\big(r - \frac{1}{2}\big)H^2 \\
&+ 2H\big(r - \frac{1}{2}\big)\sigma\bigg)q - \tilde{F}F\,.
\end{aligned} \tag{2.1.21}
$$

2.1.4 Conformal anomalies in curved spaces

A classically conformal theory usually is not conformal at a quantum level. An example of this is a non-abelian Yang-Mills theory in 4d, indeed such a theory undergoes strict confinement and generates a mass gap.

This breaking can be seen even in perturbation theory by the fact that the trace of the energy-stress tensor is non-vanishing. This is what usually one refers to as *conformal anomaly* or *trace anomaly.*

Using the strong grip that 4d $\mathcal{N} = 2$ supersymmetry gives us, it is possible to prove analytically that the conformal anomaly is completely captured by the 1-loop correction to the two point function of scalars in the vector multiplet, which gives the first and only contribution to the β-function of the Yang-Mills coupling.

In these contexts it is thus easy to construct conformal theories, one simply needs to choose wisely the matter content of the theory and the gauge symmetry representations in such a way that the 1-loop coefficient is zero. This is also consistent with the fact that the β-function should be zero since, in a theory that possesses no notion of scale, we cannot have running couplings.

There are however other types of anomalies we refer to as *Weyl anomaly.* These are 't Hooft anomalies, meaning that the partition function with all the sources for all the operators turned off is indeed invariant under the conformal symmetry. Another way of saying this is that the conformal Ward identities are violated by terms proportional to sources of operators. For this reason the theory is still conformal and thus still possesses CFT-data [26, 46–48].

The canonical examples of Weyl anomalies are given by local terms built out of the background metric, such as the Euler density which gives rise to the coefficient a in 4d CFTs. These kind of anomalies appear in the correlations functions of the stress tensor. Weyl anomalies can also depend on the sources for other operators, such as scalar operators. In this case correlation functions of these scalar operators fail to be covariant under Weyl rescalings. More generally, in the presence of curvature, the correlation functions of scalar operators may receive contributions that are scheme-dependent and not covariant under Weyl rescalings. What happens can be explained with the following example.

Imagine turning on a local source $\lambda_\Delta(x)$ for some operator $\mathcal{O}_\Delta$ of conformal dimension of Δ

$$Z_{\text{CFT}} = \int \mathcal{D}\phi \; e^{-S[\phi]} \longrightarrow Z_{\text{CFT}}[\lambda_\Delta(x)] = \int \mathcal{D}\phi \; e^{-S[\phi]+\int d^dx\sqrt{g}\lambda_\Delta(x)\mathcal{O}_\Delta(x)} . \tag{2.1.22}$$

In general this source is dimensionful and this spoils conformal symmetry. However we use this source only as a formal tool to insert this integrated operator in correlators by taking derivatives and then setting it to zero. In the particular case where we consider the source to be constant we can compute

$$\frac{\partial^2_{\lambda_\Delta} Z(\lambda_\Delta)}{Z(\lambda_\Delta)}\bigg|_{\lambda_\Delta=0} = \langle \int d^dx\sqrt{g(x)}\;\mathcal{O}(x) \int d^dy\sqrt{g(y)}\;\mathcal{O}(y)\rangle . \tag{2.1.23}$$

This integrated two-point function necessarily posses some UV divergences which we need to regulate. This regularization can be performed in a diffeomorphism-invariant way [47], but in general we need to redefine the source λ_Δ as being coupled also to other operators. Apart from the terms that are proportional to the regulator and are needed to cancel the divergences, we have in general residual scheme dependent finite pieces. Schematically, without taking into account the terms proportional to the regulator, we have the following mixing of the source

$$\lambda_\Delta(x)\mathcal{O}_\Delta(x) \longrightarrow \lambda_\Delta(x)\Big(\mathcal{O}_\Delta(x) + \mathscr{R}(x)\mathcal{O}_{\Delta-2}(x) + \mathscr{R}^2(x)\mathcal{O}_{\Delta-4}(x)\dots\Big), \tag{2.1.24}$$

where with $\mathscr{R}^k(x)\mathcal{O}_{\Delta-2k}(x)$ we indicate some linear combination of operators of dimension $\Delta - 2k$ coupled to some curvature scalar of dimension $2k$. Notice that since curvature scalars are of dimension 2, the mixing always happens with operators of dimension $2k$ lower than the original one until we end the operators in the spectrum (as we usually do due to constraints coming from unitarity).

This effect, that we refer to as *mixing*, indeed causes a lack of covariance under Weyl rescalings. For instance, we know from Section 1.6 that the one point function of an operator with $\Delta \neq 0$ should be zero. At the same time, ignoring the mixing, a derivative with respect to λ_Δ would only lead to the insertion of the corresponding operator $\mathcal{O}$, and therefore it would vanish. This is contrary to the expectation that the partition function does in general depend on the couplings, as can be verified explicitly in concrete examples. This apparent paradox is resolved taking into account the mixing mentioned above. Indeed, if we take Δ to be even, the source λ_Δ can mix with the identity and thus if one computes $\frac{\partial_{\lambda_\Delta} Z(\lambda_\Delta)}{Z(\lambda_\Delta)}\big|_{\lambda_\Delta=0}$ one obtains

$$
\begin{aligned}
\frac{\partial_{\lambda_\Delta} Z(\lambda_\Delta)}{Z(\lambda_\Delta)}\bigg|_{\lambda_\Delta=0} &= \langle \int d^d x \sqrt{g(x)} \, \big(\mathcal{O}_\Delta(x) + \ldots + \mathscr{R}^{\frac{\Delta}{2}}(x)\mathbf{1}\rangle \\
&= \langle \int d^d x \sqrt{g(x)} \, \mathscr{R}^{\frac{\Delta}{2}}(x)\mathbf{1}\rangle \neq 0 \,.
\end{aligned}
\tag{2.1.25}
$$

This is in contrast with the naive expectations of conformal symmetry, which is why we can call this a conformal anomaly.

As previously discussed, the theory is still a conformal one since when the sources are turned off the conformal Ward identities are still valid. We thus simply need to understand how to overcome the technical difficulties that this mixing poses and this can be done via a Gram–Schmidt-like process, as explained in [26].

Suppose for instance to have a scalar operator of dimension 2 "$\mathcal{O}_2$". Then for unitarity and for what stated in (2.1.24) we have

$$
\lambda_2(x)\mathcal{O}_2(x) \longrightarrow \lambda_2(x)\bigg(\mathcal{O}_2(x) + \mathscr{R}(x)\mathbf{1}\bigg) \,, \tag{2.1.26}
$$

thus if one is interested in computing

$$
\langle \int d^d x \sqrt{g(x)} \; \mathcal{O}(x) \int d^d y \sqrt{g(y)} \; \mathcal{O}(y)) \rangle \tag{2.1.27}
$$

one cannot simply use compute $\frac{\partial^2_{\lambda_2} Z}{Z}$. However one can see that it is possible to subtract the additional undesired terms in the following way

$$
\partial^2_{\lambda_2} \log(Z) = \frac{\partial^2_{\lambda_2} Z}{Z} - \left(\frac{\partial_{\lambda_2} Z}{Z}\right)^2
$$

$$
\begin{aligned}
&= \Big\langle \int d^dx \sqrt{g(x)}\, \big(\mathcal{O}(x) + \mathscr{R}(x)\mathbf{1}\big) \int d^dy \sqrt{g(y)}\, \big(\mathcal{O}(y) + \mathscr{R}(y)\mathbf{1}\big) \Big\rangle \\
&\quad - \Big\langle \int d^dx \sqrt{g(x)}\, \big(\mathcal{O}(x) + \mathscr{R}(x)\mathbf{1}\big) \Big\rangle^2 \qquad (2.1.28)\\
&= \Big\langle \int d^dx \sqrt{g(x)}\, \mathcal{O}(x) \int d^dy \sqrt{g(y)}\, \mathcal{O}(y)) \Big\rangle .
\end{aligned}
$$

The idea is to employ a series of derivative over sources in a way that removes what "pollutes" the source of the operator one is interested in.

2.2 $(H)S^4$, AdS$_4$ and S^3 backgrounds and Killing spinors

In this section we discuss 4d $\mathcal{N} = 2$ supersymmetry on the hemisphere $(H)S^4$,[15] on AdS$_4$ and also the 3d $\mathcal{N} = 2$ one on S^3 we want to preserve explicitly in our models. The reader should refer to [9, 33, 35, 45, 49, 50] for additional information and relevant references.

We discuss initially $(H)S^4$ and AdS$_4$ together, we later specify to each of them singularly.

The gravitational background for $(H)S^4$ and AdS$_4$ is

$$
\tilde{M} = 0\,, \quad \mathcal{A}_\mu = 0\,, \quad \mathcal{V}_\mu = 0\,, \quad T_{\mu\nu} = 0\,, \quad \bar{T}_{\mu\nu} = 0\,, \quad g_{\mu\nu} = g_{\mu\nu}^{(H)S^4/\mathrm{AdS}_4}\,, \qquad (2.2.1)
$$

while the bulk Killing spinor equations are

$$
\begin{aligned}
D_\mu \epsilon_{A\alpha} &= \frac{i}{2} s_1 (\sigma_\mu \bar{\epsilon}_B)_\alpha \tau_{3,A}{}^B\,, \\
D_\mu \bar{\epsilon}^{\dot\alpha}_A &= \frac{i}{2} s_2 (\bar{\sigma}_\mu \epsilon_B)^{\dot\alpha} \tau_{3,A}{}^B\,,
\end{aligned}
\qquad \text{with } s_1 s_2 = \begin{cases} 1 \text{ on } (H)S^4 \\ -1 \text{ on AdS}_4 \end{cases} . \qquad (2.2.2)
$$

The Killing spinors of S^4 and AdS$_4$, respectively, generate the fermionic transformations of the $OSp(2|4)$ and $OSp(2|2,2)$ superalgebras. The $\mathcal{R}$-symmetry of both these algebras is a $SO(2)_\mathcal{R}$ included in the $SU(2)_\mathcal{R}$ $\mathcal{R}$-symmetry of flat space.

We now discuss S^3 background. In the 3d gravity multiplet it is customary to redefine the graviphoton and the two-form as

$$
V^i = -i\varepsilon^{ijk}\partial_j C_k\,, \qquad H = \frac{i}{2}\varepsilon^{ijk}\partial_i B_{jk}\,. \qquad (2.2.3)
$$

[15] The discussion for S^4 is equivalent to the one of HS^4, it just lacks the considerations regarding the boundary

With these the gravitational background for S^3 can be written as

$$H = i\,, \qquad V_i = A_i^{\mathcal{R}} = 0\,, \qquad g_{ij} = g_{ij}^{S^3}\,, \tag{2.2.4}$$

and the Killing spinor equations as[16]

$$D_i\zeta = -\frac{i}{2}\gamma_i\zeta\,, \qquad D_i\tilde{\zeta} = -\frac{i}{2}\gamma_i\tilde{\zeta}\,, \tag{2.2.5}$$

which generate $OSp(2|2)\times SU(2)$ that possesses $SO(2)_{\mathcal{R}}$ $\mathcal{R}$-symmetry and preserves the same maximal supersymmetry as on HS^4.

If the theory on S^4 is superconformal, since S^4 is conformally flat, we obtain the 4d $\mathcal{N}=2$ superconformal algebra in flat space, namely $SU^*(4|4)$.

If instead the theory on HS^4 is superconformal we have the same superconformal symmetry as the one on S^3 or $\mathbb{R}^3$, namely $OSp(2|2,2)$. This superconformal algebra is the same as the massive supersymmetric one on AdS$_4$, which we expect for instance for holography. Note that for conformal theories, AdS$_4$ and HS^4 are equivalent and thus bulk conformal symmetry in AdS does not lead to a symmetry enhancement. Bulk conformal symmetry in AdS simply restricts the class of possible correlators and adds a notion of bulk OPE [51].

We proceed by discussing more carefully HS^4 and AdS$_4$ separately.

2.3 Supersymmetry on HS^4

We now discuss how to obtain 3d $\mathcal{N}=2$ supersymmetry on the boundary S^3 starting from the 4d $\mathcal{N}=2$ supersymmetry on the hemisphere HS^4. We then discuss how to preserve explicitly supersymmetry in a Lagrangian formalism.

2.3.1 Boundary Killing spinors

On HS^4 we need to impose boundary conditions [9] in such a way that the restrictions of the bulk Killing spinors become S^3 Killing spinors. This allows us to preserve a supersymmetry algebra compatible with the boundary. It is possible to see that by requiring

$$\bar{\epsilon}_A - is_1(\tau_3)_A{}^B\bar{\sigma}_4\epsilon_B\big|_{S^3=\partial HS^4} = 0\,, \tag{2.3.1}$$

[16] As can be seen in [35], there are two possible Killing spinor equations but these differ simply by the choice of the orientation of the S^3.

and by defining

$$
\begin{aligned}
&\epsilon_{A=1}\big|_{S^3=\partial HS^4} = \frac{1}{\sqrt{2}}\zeta\,, \qquad \epsilon_{A=2}\big|_{S^3=\partial HS^4} = \frac{1}{\sqrt{2}}\tilde{\zeta}\,,\\
&(\gamma^i)_\alpha^{\ \beta} = i(\sigma^i\bar{\sigma}^\perp)_\alpha^{\ \beta}\big|_{S^3=\partial HS^4} = -i(\sigma^\perp\bar{\sigma}^i)^\beta_{\ \alpha}\big|_{S^3=\partial HS^4}\,,
\end{aligned}
\tag{2.3.2}
$$

the differential equations for HS^4 (2.2.2), with covariant derivatives tangent to the boundary, become Killing spinor equations on S^3 in (2.2.5).

The four linearly independent bulk Killing spinor, that reduce to the ones generating the 3d $\mathcal{N}=2$ algebra, are such that the Killing spinor equations with derivatives perpendicular to the boundary are immediately satisfied.

It is then possible to see that this sub-algebra of bulk-spinors, from (2.5.6) and equations below, generates the one on S^3.

2.3.2 Preserving explicitly 3d $\mathcal{N}=2$ supersymmetry

In $(H)S^4$ [26] the following linear combination of the components of a (anti-)chiral multiplet (2.1.6)-(2.1.7)

$$
\begin{aligned}
\mathcal{C} &= C - s_2 i(w-2)(\tau_3)^{AB}B_{AB} + 2(w-2)(w-3)A\,,\\
\bar{\mathcal{C}} &= \bar{C} - s_1 i(w-2)(\tau_3)^{AB}\bar{B}_{AB} + 2(w-2)(w-3)\bar{A}\,,
\end{aligned}
\tag{2.3.3}
$$

is such that

$$
\begin{aligned}
\delta\mathcal{C} &= -D_\mu\big(2i\bar{\epsilon}^A\sigma^\mu\Lambda_A + 2s_2(w-2)(\tau_3)^{AB}\bar{\epsilon}_{(A}\bar{\sigma}^\mu\Psi_{B)}\big)\,,\\
\delta\bar{\mathcal{C}} &= -D_\mu\big(2i\epsilon^A\sigma^\mu\bar{\Lambda}_A - 2s_1(w-2)(\tau_3)^{AB}\epsilon_{(A}\sigma^\mu\bar{\Psi}_{B)}\big)\,.
\end{aligned}
\tag{2.3.4}
$$

Thus if we wish to preserve some of the bulk supersymmetries, we need to add a boundary term $\mathcal{C}_\partial$ and $\bar{\mathcal{C}}_\partial$ such that

$$
\int\delta\mathcal{C} = -\int_\partial\delta\mathcal{C}_\partial\,, \qquad \int\delta\bar{\mathcal{C}} = -\int_\partial\delta\bar{\mathcal{C}}_\partial\,. \tag{2.3.5}
$$

In this way

$$
\int\mathcal{C} + \int_\partial\mathcal{C}_\partial\,, \qquad \int\bar{\mathcal{C}} + \int_\partial\bar{\mathcal{C}}_\partial\,, \tag{2.3.6}
$$

become supersymmetric invariants.[17] On HS^4 this is achieved by choosing

$$
\begin{aligned}
\mathcal{C}_\partial &= is_1(\tau_3)^{AB}B_{AB} + 2D_\perp A - 4(w-2)A\ ,\\
\bar{\mathcal{C}}_\partial &= -is_1(\tau_3)^{AB}\bar{B}_{AB} + 2D_\perp\bar{A} + 4(w-2)\bar{A}\ .
\end{aligned}
\tag{2.3.7}
$$

[17] In order for this to be true we choose $\int D_\mu V^\mu = \int_\partial V^\perp$.

As discussed around equation (1.3.11), we can use this to define the SYM action on HS^4.

The boundary condition multiplets in equation (1.5.3)-(1.5.11) translate naturally to the ones modified due to the curved background.

2.4 Supersymmetry on AdS$_4$

Among all spaces with constant curvature in Euclidean signature, AdS$_d$ is characterized by the existence of an asymptotic boundary, namely S^d.

It is thus difficult to preserve via Lagrangian formalism the full bulk supersymmetries. Indeed all 8 supercharges can be reorganized, as happens in holography, in a 3d $\mathcal{N} = 2$ superconformal algebra. Preserving this algebra with a Lagrangian is as difficult as doing it for boundary superconformal theories on HS^4, furthermore it could be very restrictive in principle.

Moreover, due to the fact that the boundary is asymptotic, we have infinite volume which produces IR divergences in our theory. We regulate these by placing a spherical cut-off. We do not give a full review in this thesis, we suggest the standard literature on holographic renormalization [52, 53].

We start a discussion on asymptotics in AdS, we later relate the 4d $\mathcal{N} = 2$ AdS$_4$ algebra to the 3d $\mathcal{N} = 2$ superconformal one formulated on S^3. We then restrict to the 3d $\mathcal{N} = 2$ on S^3 and we finally discuss the asymptotic boundary conditions.

2.4.1 Asymptotics in AdS

In order to understand what are the possible boundary conditions on bulk fields one needs to solve the equations of motion. For the bosons, with the coordinate choices of Appendix A.1, one finds two independent solutions[18]

$$\Phi(\eta, \vec{x}) = e^{\Delta_+\eta}(\Phi_+^{(0)} + e^{-2\eta}\Phi_+^{(2)} + \dots) + e^{\Delta_-\eta}(\Phi_-^{(0)} + e^{-2\eta}\Phi_-^{(2)} + \dots)\,, \tag{2.4.1}$$

with $\Delta_+ > \Delta_-$, where the subleading modes are related to $\partial_\perp^k\Phi$ and thus are related to the leading ones via their second order equations of motion.[19]

Usually what happens is that the solution associated to Δ_+ would be divergent and as such we declare that we can turn it on only as a source, so we do not path

[18]There are diverse subtleties we are not going to address in the following, like the fact that if $\Delta_+ - \Delta_- \in 2\mathbb{N}$ then the expansion can have a different form.

[19]We can think of this as a necessity to satisfy all the possible Schwinger-Dyson equations when we place bulk fields to the boundary.

integrate over it. The other one instead produces finite results and thus the modes corresponding to this solution are fluctuating.

It is also possible that both modes are normalizable and as such one can take either one of the two to be the fluctuating mode and the orthogonal one to be fixed. This happens for the scalar field or the massless vector field in 4d. For massless fermions there is only one asymptotic behavior and we split that in two modes.

Below we write the asymptotic behavior for the class of fields we are interested in. We gauge-fix $A_\perp = 0$ and we take A_i to be the components of the gauge field parallel to the boundary. Using (2.4.1), and referring to its comments below, we obtain

$$\begin{aligned}
\varphi(\eta,\vec{x}) &= e^{-\eta}(\varphi^{(0)} + e^{-2\eta}\varphi^{(2)} + \ldots) + e^{-2\eta}(D_\perp\varphi^{(0)} + e^{-2\eta}D_\perp\varphi^{(2)} + \ldots)\,,\\
\psi(\eta,\vec{x}) &= e^{-\frac{3}{2}\eta}(\psi_+^{(0)} + e^{-2\eta}\psi_+^{(2)} + \ldots) + e^{-\frac{3}{2}\eta}(\psi_-^{(0)} + e^{-2\eta}\psi_-^{(2)} + \ldots)\,, \qquad (2.4.2)\\
A_i(\eta,\vec{x}) &= (A_i^{(0)} + e^{-2\eta}A_i^{(2)} + \ldots) + e^{-\eta}(j_i^{(0)} + e^{-2\eta}j_i^{(2)} + \ldots)\,.
\end{aligned}$$

The notation $D_\perp\varphi$ for the subleading mode of the scalar should not be confused with an actual derivative of the leading mode (which does not depend on η) and is simply inspired by the case of a conformally coupled scalar. The notation j_i, in the expansion of the vector field, is related to the fact that via the equations of motion we can relate $\partial_\perp A_k$ to the conserved current $j_k = \varepsilon^{ij}{}_k F_{ij}$. In all of these cases both modes can be chosen to be fluctuating or not.

In presence of supersymmetry we should also care about relating the sources and the fluctuating modes via supersymmetry.

2.4.2 Boundary 3d $\mathcal{N} = 2$ superconformal symmetry

The 3d superconformal symmetry algebra is generated by the following conformal Killing spinors [49]

$$D_i\xi_A = -\frac{i}{2}\gamma_i\xi_A\,, \quad D_i\tilde{\xi}_A = \frac{i}{2}\gamma_i\tilde{\xi}_A\,, \qquad (2.4.3)$$

where A is the $\mathbf{2}_{SO(2)_\mathcal{R}}$ symmetry index, the one preserved by the $\mathcal{N} = 2$ supersymmetry algebra in AdS$_4$.

We make the following ansatz in order to solve the equations (2.2.2)

$$\begin{aligned}
\epsilon_A &= e^{\frac{\eta}{2}}\epsilon_{+A} + e^{-\frac{\eta}{2}}\epsilon_{-A}\,,\\
\bar{\epsilon}_A &= e^{\frac{\eta}{2}}\bar{\epsilon}_{+A} + e^{-\frac{\eta}{2}}\bar{\epsilon}_{-A}\,.
\end{aligned} \qquad (2.4.4)$$

We can see that the equations for the perpendicular variable are solved by

$$\epsilon_{\pm A} = \pm i s_1 \tau_{3A}{}^{B} \sigma_4 \bar{\epsilon}_{\pm B}\,. \tag{2.4.5}$$

With this the remaining independent equations are three and can be taken to be the following

$$D_i \epsilon_A = \frac{i}{2} s_2 \tau_{3A}{}^{B} \sigma_4 \bar{\epsilon}_B\,. \tag{2.4.6}$$

We then find

$$D_i^{3\mathrm{d}} \epsilon_{\pm A} = \frac{i}{2} \gamma_i^{3\mathrm{d}} \epsilon_{\mp A}\,, \tag{2.4.7}$$

where with "3d" we here stress that we have removed all notions of the perpendicular coordinate, namely we define

$$\begin{aligned} \frac{(\gamma^i_{3\mathrm{d}})_\alpha{}^\beta}{\sinh(\eta)} &= i(\sigma^i \bar{\sigma}^\perp)_\alpha{}^\beta = -i(\sigma^\perp \bar{\sigma}^i)^\beta{}_\alpha\,, \\ \sinh(\eta)(\gamma_i^{3\mathrm{d}})_\alpha{}^\beta &= i(\sigma_i \bar{\sigma}^\perp)_\alpha{}^\beta = -i(\sigma^\perp \bar{\sigma}_i)^\beta{}_\alpha\,. \end{aligned} \tag{2.4.8}$$

and we define

$$D_i \chi_\alpha = D_i^{3\mathrm{d}} + \frac{1}{2} \omega_{i,a4} \sigma^{a4}\,. \tag{2.4.9}$$

In all future 3d formulæ for AdS, in the derivative D with the index i, j, …, the superscript "3d" is omitted.

By choosing

$$\xi_A = \epsilon_{+A} - \epsilon_{-A}\,, \quad \tilde{\xi}_A = \epsilon_{+A} + \epsilon_{-A}\,, \tag{2.4.10}$$

we see that (2.4.7) gives (2.4.3). We can then also write the 4d $\mathcal{N} = 2$ Killing spinors in function of the 3d $\mathcal{N} = 2$ superconformal ones, with our choice of coordinates in Appendix A.1

$$\begin{aligned} \epsilon_{A\alpha} &= \sinh\left(\frac{\eta}{2}\right) \xi_{A\alpha} + \cosh\left(\frac{\eta}{2}\right) \tilde{\xi}_{A\alpha}\,, \\ \bar{\epsilon}_{A\dot{\alpha}} &= i s_1 \tau_{3,A}{}^{B} \left(\cosh\left(\frac{\eta}{2}\right) \xi_A^\beta + \sinh\left(\frac{\eta}{2}\right) \tilde{\xi}_A^\beta \right) \sigma_{\perp,\beta\dot{\alpha}}\,. \end{aligned} \tag{2.4.11}$$

Each of the two independent 3d spinors ξ_A and $\tilde{\xi}_A$ preserve a different 3d $\mathcal{N} = 2$ supersymmetry algebra. As already explained, it is not in general possible to preserve the full 3d $\mathcal{N}=2$ superconformal symmetry from the Lagrangian, we thus focus on preserving just $\mathcal{N}=2$ supersymmetry as done for HS^4. The localization charge $Q_{\text{loc.}}$, defined by the Killing spinors in (A.1.8), is generated by the ξ_A-sub-algebra and as such this is the one we want to preserve. We thus consider $\tilde{\xi}_A = 0$.

We now need to relate the ξ_A with the ζ, $\tilde{\zeta}$ 3d $\mathcal{N}=2$ Killing spinors in (2.2.5). Their $SO(2)_{\mathcal{R}}$-symmetry transformation is a phase transformation [31] which we define to be the following

$$\zeta' = e^{i\theta_{\mathcal{R}}}\zeta\,, \quad \tilde{\zeta}' = e^{-i\theta_{\mathcal{R}}}\tilde{\zeta}\,. \tag{2.4.12}$$

The $SO(2)_{\mathcal{R}} \subset SU(2)_{\mathcal{R}}$ symmetry rotation acts on ξ_A via the generator τ_3 as displayed below

$$\xi'_A = \left(e^{i\theta_{\mathcal{R}}\tau_3}\right)_A^{\ B}\xi_B = \begin{pmatrix} e^{i\theta_{\mathcal{R}}}\xi_1 \\ e^{-i\theta_{\mathcal{R}}}\xi_2 \end{pmatrix}_A . \tag{2.4.13}$$

We can then argue that we have

$$\xi_1 = \zeta\,, \quad \xi_2 = \tilde{\zeta}\,. \tag{2.4.14}$$

From this relation it follows that

$$D_i\zeta = -\frac{i}{2}\gamma_i\zeta\,, \quad D_i\tilde{\zeta} = -\frac{i}{2}\gamma_i\tilde{\zeta}\,, \tag{2.4.15}$$

which are precisely the S^3 $\mathcal{N}=2$ Killing spinor equations in their canonical form [31].

2.4.3 Boundary 3d $\mathcal{N}=2$ supersymmetry

As previously stated we preserve 3d $\mathcal{N}=2$ supersymmetry by setting $\tilde{\xi}_A = 0$ in (2.4.11). If we do this we then have

$$\begin{aligned} \epsilon_{A\alpha} &= \sinh\left(\frac{\eta}{2}\right)\xi_{A\alpha} = \sinh\left(\frac{\eta}{2}\right)\begin{pmatrix} \zeta_\alpha \\ \tilde{\zeta}_\alpha \end{pmatrix}_A , \\ \bar{\epsilon}_{A\dot{\alpha}} &= is_1\tau_{3,A}^{\ \ B}\cosh\left(\frac{\eta}{2}\right)(\xi_B\sigma_\perp)_\alpha = is_1\tau_{3,A}^{\ \ B}\begin{pmatrix} (\zeta\sigma_\perp)_{\dot{\alpha}} \\ (\tilde{\zeta}\sigma_\perp)_{\dot{\alpha}} \end{pmatrix}_B . \end{aligned} \tag{2.4.16}$$

It is also possible to prove that in our coordinates the spinors $\xi_A = (\zeta, \tilde{\zeta})$ are constant.

The condition $\tilde{\xi} = 0$ imposes the following relation among the 4d Killing spinors valid for all η

$$\bar{\epsilon}_{A\dot{\alpha}} = is_1 \tau_{3,A}{}^{B} \coth\big(\frac{\eta}{2}\big)(\epsilon_B \sigma_\perp)_\alpha \,. \tag{2.4.17}$$

It is possible to see, using (2.5.6)-(2.5.8) that the restriction to this sub-algebra generates η-independent transformations and that these do not involve the index perpendicular to the boundary. We are thus able to preserve this amount of supersymmetry for each S^3 slice at any η.

This also means that the regulator we are employing does not break 3d $\mathcal{N} = 2$ supersymmetry.

2.4.4 3d $\mathcal{N} = 2$ asymptotic boundary conditions

Due to the fact that we preserve 3d $\mathcal{N} = 2$ supersymmetry for each S^3 slice, we can construct 3d $\mathcal{N} = 2$ linear multiplets, similarly to what we have done in (1.5.3), but these are η-dependent.

Using (2.4.16) we can see that

$$\begin{aligned}
&\boldsymbol{\mathcal{J}_{\mathrm{D}}(\eta)}\\
&J = 2\phi_1 \,,\\
&j = -i\frac{\sinh\left(\frac{\eta}{2}\right)}{\sqrt{2}}\big(\lambda_2 - is_1 \coth\big(\frac{\eta}{2}\big)\sigma_\perp \bar{\lambda}_2\big) \,,\\
&\tilde{j} = i\frac{\sinh\left(\frac{\eta}{2}\right)}{\sqrt{2}}\big(\lambda_1 + is_1 \coth\big(\frac{\eta}{2}\big)\sigma_\perp \bar{\lambda}_1\big) \,,\\
&j_k = -\frac{\cosh(\eta)}{2\sinh^2(\eta)} i\varepsilon^{ij}{}_k F_{ij} - \frac{i}{\sinh(\eta)} F_{k\perp} \,,\\
&K = -2\sinh(\eta) D_\perp \phi_2 - i\cosh(\eta) D_{12} + 2i\phi_1 - 2\cosh(\eta)\phi_2 \,.
\end{aligned} \tag{2.4.18}$$

$$\begin{aligned}
&\boldsymbol{\mathcal{J}_{\mathrm{N}}(\eta)}\\
&J = 2\phi_2 \,,\\
&j = \frac{\sinh\left(\frac{\eta}{2}\right)}{\sqrt{2}}\big(\lambda_2 + is_1 \coth\big(\frac{\eta}{2}\big)\sigma_\perp \bar{\lambda}_2\big) \,,\\
&\tilde{j} = -\frac{\sinh\left(\frac{\eta}{2}\right)}{\sqrt{2}}\big(\lambda_1 - is_1 \coth\big(\frac{\eta}{2}\big)\sigma_\perp \bar{\lambda}_1\big) \,,
\end{aligned} \tag{2.4.19}$$

$$
\begin{aligned}
j_k &= -\frac{1}{2\sinh^2(\eta)}\varepsilon^{ij}{}_k F_{ij} - \frac{\cosh(\eta)}{\sinh(\eta)} F_{k\perp}\,,\\
K &= 2\sinh(\eta) D_\perp\phi_1 - D_{12} + 2i\phi_2 + 2\cosh(\eta)\phi_1\,,
\end{aligned}
$$

are 3d linear multiplets. Similarly we can define a 3d $\mathcal{N}=2$ vector multiplet as follows[20]

$$
\begin{aligned}
&\boldsymbol{\mathcal{V}}(\boldsymbol{\eta})\\
\sigma &= 2\cosh(\eta)\phi_1\,,\\
j &= \frac{\sinh\left(\frac{\eta}{2}\right)}{\sqrt{2}}\left(\lambda_2 + i s_1 \coth\left(\frac{\eta}{2}\right)\sigma_\perp\bar\lambda_2\right)\,,\\
\tilde{j} &= -\frac{\sinh\left(\frac{\eta}{2}\right)}{\sqrt{2}}\left(\lambda_1 - i s_1 \coth\left(\frac{\eta}{2}\right)\sigma_\perp\bar\lambda_1\right)\,,\\
j_k &= \frac{1}{2\sinh^2(\eta)}\varepsilon^{ij}{}_k F^{ij} - \frac{\cosh(\eta)}{\sinh(\eta)} F_{k\perp}\,,\\
K &= 2\cosh(\eta) D_\perp\phi_1 - D_{12} + 2i\phi_2 + 2\cosh(\eta)\phi_1\,.
\end{aligned}
\tag{2.4.20}
$$

These vector multiplets are also appearing in the non-linear terms of the transformations of the linear multiplets (2.1.17). At large η we see that $\frac{\boldsymbol{\mathcal{V}}(\boldsymbol{\eta})}{\cosh(\eta)}$ dualizes to $\boldsymbol{\mathcal{J}}_{\mathbf{N}}(\boldsymbol{\eta})$ via (2.1.18).

These multiplets show how all the modes in (2.4.2) transform under the supersymmetry we enforce at each η, thus at each point of the bulk.[21]

For the application that we discuss in Chapter 5, we intend to compute the AdS_4 partition function with Dirichlet boundary condition for the gauge field. Since we want to preserve explicitly 3d $\mathcal{N}=2$ supersymmetry, this implies fixing the asymptotics of ϕ_1 and leaving $D_\perp\phi_1$ fluctuating (just like on HS^4). This can be seen in the following way.

We can fix $\mathcal{J}_{\mathrm{D}}(\eta)$ to a background consistent with supersymmetry, namely to the dualization of a real mass (2.1.1)-(1.4.6), but in curved space [31]. Then, after gauge-fixing $A_4 = 0$, we can see that the leading mode at large η of the field strength needs to vanish.

[20] To find this vector multiplet we defined $i\cosh(\eta)\boldsymbol{\mathcal{J}}_{\mathbf{D}}(\boldsymbol{\eta}) + \boldsymbol{\mathcal{J}}_{\mathbf{N}}(\boldsymbol{\eta})$, which is a linear multiplet such that $j_i = -\frac{i}{2}\epsilon_i{}^{jk}F_{jk}$. We then tested the transformations in (2.1.16) by "anti-dualizing it" via (2.1.18).

[21] This is also consistent with the fact that the subleading modes are constrained by the equations of motions, since these respect necessarily the same amount of supersymmetry.

With this boundary condition we can then interpret the boundary modes of the multiplets $\mathcal{J}_{\rm D}(\eta)$ and $\mathcal{J}_{\rm N}(\eta)$, respectively, as the sources and the fluctuating modes. In the scalar sector, the leading mode of ϕ_1 is a source, and the subleading is fluctuating, while the opposite is true for ϕ_2.

Similar comments would be valid for Neumann boundary conditions by fixing $\mathcal{J}_{\rm N}(\eta)$ and looking at its components at large η.

2.4.5 Lagrangian boundary term and 3d $\mathcal{N}=2$ supersymmetry

In this section discuss the boundary term needed in AdS$_4$ to preserve explicitly 3d $\mathcal{N}=2$ supersymmetry. Keeping the regulator explicit in (2.3.7) for AdS$_4$ we get

$$\begin{aligned}
\mathcal{C}_\partial &= \coth\left(\frac{\eta_0}{2}\right)\bigg(is_1(\tau_3)^{AB}B_{AB}+2D_\perp A+4(w-2)A \\
&\qquad - \left(1+\coth^2\left(\frac{\eta_0}{2}\right)-6\coth\left(\frac{\eta_0}{2}\right)\coth(\eta_0)\right)A\bigg)\,, \\
\bar{\mathcal{C}}_\partial &= \tanh\left(\frac{\eta_0}{2}\right)\bigg(-is_1(\tau_3)^{AB}\bar{B}_{AB}+2D_\perp\bar{A}+4(w-2)\bar{A} \\
&\qquad - \left(1+\tanh^2\left(\frac{\eta_0}{2}\right)-6\tanh\left(\frac{\eta_0}{2}\right)\coth(\eta_0)\right)\bar{A}\bigg)\,.
\end{aligned} \tag{2.4.21}$$

This modifies the boundary action written in (1.5.10) into

$$\begin{aligned}
\mathcal{L}_{\rm YM,\partial} &= \frac{2}{\sinh(\eta_0)}\bigg(\coth(\eta_0)\Big(\frac{3+\cosh(2\eta_0)}{\sinh(\eta_0)}(\phi_1^2-\phi_2^2)-8i\coth(\eta_0)\phi_1\phi_2\Big) \\
&\qquad -\big(\cosh(\eta_0)\phi_2+i\phi_1\big)\Big(4i\frac{D_\perp\phi_1}{\sinh(\eta_0)}+4\coth(\eta_0)D_\perp\phi_2+2iD_{12}\Big) \\
&\qquad +\cosh^2\left(\frac{\eta_0}{2}\right)\lambda_1\lambda_2+\sinh^2\left(\frac{\eta_0}{2}\right)\bar{\lambda}_1\bar{\lambda}_2\bigg)\,,
\end{aligned} \tag{2.4.22}$$

$$\begin{aligned}
\mathcal{L}_{\theta,\partial} &= \frac{2}{\sinh(\eta_0)}\bigg(2\coth(\eta_0)\Big(2\coth(\eta_0)(\phi_1^2-\phi_2^2)-\frac{3+\cosh(2\eta_0)}{\sinh(\eta_0)}i\phi_1\phi_2\Big) \\
&\qquad -\big(\phi_2+i\cosh(\eta_0)\phi_1\big)\Big(4i\frac{D_\perp\phi_1}{\sinh(\eta_0)}+4\coth(\eta_0)D_\perp\phi_2+2iD_{12}\Big) \\
&\qquad +\cosh^2\left(\frac{\eta_0}{2}\right)\lambda_1\lambda_2-\sinh^2\left(\frac{\eta_0}{2}\right)\bar{\lambda}_1\bar{\lambda}_2 \\
&\qquad -\frac{i}{2}\sinh(\eta_0)(\lambda_1\sigma_\perp\bar{\lambda}_2-\lambda_2\sigma_\perp\bar{\lambda}_1)\bigg)\,,
\end{aligned}$$

after some integration by parts in the action.

2.5 Supersymmetric localization

Supersymmetric localization is a powerful technique to compute a class of observables exactly. It was firstly employed by Witten in the context of topological theories [54] and it was also successfully implemented by Nekrasov to compute the path integral on the moduli space of instantons [55]. It is however thanks to Pestun [3], who computed the partition function and a Wilson loop exactly for 4d $\mathcal{N} = 2$ theories on the four sphere S^4, that this technique became an essential tool for studying a large class of observables in quantum field theories in various dimensions and with various amounts of supersymmetry [4].

The general argument of localization needs a conserved supercharge $\mathcal{Q}$ with certain properties. One needs to be able to construct a $\mathcal{Q}$-exact quantity such that $\int \mathcal{Q}^2 V = 0$ and such that $\int \mathcal{Q}V$ is semi-positive definite. In this way

$$Z = \int \mathcal{D}\Psi \; e^{-S[\Psi]} = \int \mathcal{D}\Psi \; e^{-S[\Psi] - t\int \mathcal{Q}V} \,, \; \forall t \in \mathbb{R} \tag{2.5.1}$$

since

$$\begin{aligned} \frac{d}{dt}\int \mathcal{D}\Psi \; e^{-S[\Psi]-t\int \mathcal{Q}V} &= \int \mathcal{D}\Psi \; \mathcal{Q}V e^{-S[\Psi]-t\int \mathcal{Q}V} \\ &= \int \mathcal{D}\Psi \; \mathcal{Q}\Big(V e^{-S[\Psi]-t\int \mathcal{Q}V}\Big) \\ &= \mathcal{Q}\Big(\int \mathcal{D}\Psi \; V e^{-S[\Psi]-t\int \mathcal{Q}V}\Big) \\ &= 0 \,. \end{aligned} \tag{2.5.2}$$

In the second row we use the fact that $\mathcal{Q}S = \int \mathcal{Q}^2 V = 0$. In the third row we use the fact that the path integral measure and the possible boundary conditions are compatible with $\mathcal{Q}$. In the fourth row we use the fact that the $\mathcal{Q}$-variation of a function is zero.

The argument can be repeated with the insertion of $\mathcal{Q}$-closed operators. One can then send $t \to +\infty$ and compute the integral via an *exact* saddle-point evaluation with the action $\int \mathcal{Q}V$.

What usually happens is that the path integral then localizes to a finite dimensional integral over some domain, called locus, and we obtain what we represent schematically below

$$Z = \int \mathcal{D}\Psi \; e^{-S[\Psi]} = \sum_{\text{loc.}} \int da_{\text{loc.}} \; e^{-S[\Psi(a_{\text{loc.}})]} Z_{\text{1-loop}}(a_{\text{loc.}}) \, . \tag{2.5.3}$$

We get a contribution coming from simply inserting the locus inside the integrand and another contribution from the quadratic fluctuations, which leads to the one-loop determinant $Z_{\text{1-loop}}(a)$. In the case of 4d partition function these saddles are labeled by instanton sectors and the final result takes the form of

$$Z = \int da \; e^{-S[\Psi(a)]} Z_{\text{1-loop}}(a) Z_{\text{Nekrasov}}(a) \, , \tag{2.5.4}$$

where $Z_{\text{Nekrasov}}(a)$ is the celebrated Nekrasov partition function in a suitable Ω-background [55].

In the following sections we display very explicitly the technique, also in known scenarios, in order to apply it to a novel case. In particular we review the cases of S^4 [3] and of the hemisphere HS^4 [56,57]. Then we discuss a new example, namely SYM on AdS$_4$ with Dirichlet boundary conditions [15].[22]

2.5.1 Setting up the localization computation

A generic supersymmetry transformation is given by

$$\mathcal{Q} = \epsilon^A Q_A + \bar{\epsilon}_A \bar{Q}^A \, . \tag{2.5.5}$$

We consider the Killing spinors as Grassmann even and the supercharges as Grassmann odd.

Following the supersymmetry algebra in (2.1.4) one can see that $\mathcal{Q}^2$ is given by [33]

$$\begin{aligned} \mathcal{Q}^2 =& i\mathcal{L}_v + \text{Gauge}(\Phi) + \text{Lorentz}(L_{ab}) + \text{Scale}(w) + \mathcal{R}_{U(1)}(\Theta) \\ &+ \mathcal{R}_{SU(2)}(\tilde{\Theta}_{AB}) \, , \end{aligned} \tag{2.5.6}$$

with

$$\begin{aligned} v^\mu &= 2\bar{\epsilon}^A \bar{\sigma}^\mu \epsilon_A \, , \\ \Phi &= 2i\bar{\phi}\epsilon^A \epsilon_A + 2i\phi \bar{\epsilon}_A \bar{\epsilon}^A + v^\mu A_\mu \, , \end{aligned}$$

[22] A localization computation of AdS$_4$ was done in [50], but that work focuses on a specific abelian model.

$$\begin{aligned} L_{ab} &= D_{[a}v_{b]} + v^{\mu}\Omega_{\mu,ab}\,, \\ w &= -\frac{i}{2}(\epsilon^A\sigma^\mu D_\mu\bar{\epsilon}_A + D_\mu\epsilon^A\sigma^\mu\bar{\epsilon}_A)\,, \\ \Theta &= -\frac{i}{4}(\epsilon^A\sigma^\mu D_\mu\bar{\epsilon}_A - D_\mu\epsilon^A\sigma^\mu\bar{\epsilon}_A)\,, \\ \tilde{\Theta}_{AB} &= -i\epsilon_{(A}\sigma^\mu D_\mu\bar{\epsilon}_{B)} + iD_\mu\epsilon_{(A}\sigma^\mu\bar{\epsilon}_{B)}\,. \end{aligned} \tag{2.5.7}$$

$i\mathcal{L}_v$ is a Lie derivative along v generates a rotation on S^4/AdS_4. For HS^4 this rotation can be chosen to be a symmetry, in order to employ localization, by imposing as a boundary an equator of S^4 that is preserved by this rotation.

Using the above formulæ it is also possible to prove that a generic anti-commutator "$\{\mathcal{Q},\mathcal{Q}'\}$" can be obtained by simply sending each bilinear of (2.5.7) into

$$\begin{aligned} \mathcal{Q}^2 &\longrightarrow \{\mathcal{Q},\mathcal{Q}'\}\,, \\ \epsilon_{A\alpha}\epsilon_{B\beta} &\longrightarrow \epsilon_{A\alpha}\epsilon'_{B\beta} + \epsilon'_{A\alpha}\epsilon_{B\beta}\,, \\ \bar{\epsilon}^{\dot{\alpha}}_A\bar{\epsilon}^{\dot{\beta}}_B &\longrightarrow \bar{\epsilon}^{\dot{\alpha}}_A\bar{\epsilon}'^{\dot{\beta}}_B + \bar{\epsilon}'^{\dot{\alpha}}_A\bar{\epsilon}^{\dot{\beta}}_B\,, \\ \epsilon_{A\alpha}\bar{\epsilon}^{\dot{\beta}}_B &\longrightarrow \epsilon_{A\alpha}\bar{\epsilon}'^{\dot{\beta}}_B + \epsilon'_{A\alpha}\bar{\epsilon}^{\dot{\beta}}_B\,. \end{aligned} \tag{2.5.8}$$

Since we are interested in a single supercharge for localization purposes, we can actually impose the following constraints for the localization's Killing spinor

$$(\epsilon_{A\alpha})^* = \epsilon^{A\alpha}\,,\quad (\bar{\epsilon}_{A\dot{\alpha}})^* = \bar{\epsilon}^{A\dot{\alpha}}\,,\quad \epsilon^A\sigma^\mu D_\mu\bar{\epsilon}_A = D_\mu\epsilon^A\sigma^\mu\bar{\epsilon}_A = 0\,. \tag{2.5.9}$$

A canonical choice for the functional V is

$$V = \sum_{\text{ferm.}} \text{ferm.}(\mathcal{Q}\text{ferm.})^*\,, \tag{2.5.10}$$

which is such that $\int \mathcal{Q}^2 V = 0$ thanks to the constraints in (2.5.9), which allow for $\mathcal{Q}^2$ to not generate some of the transformations showed in (2.5.7). Furthermore the bosonic part of $\int \mathcal{Q}V$ is manifestly positive semi-definite

$$\int \mathcal{Q}V\bigg|_{\text{bos.}} = \int \sum_{\text{ferm.}} \mathcal{Q}\text{ferm.}(\mathcal{Q}\text{ferm.})^*\,. \tag{2.5.11}$$

The specific choice of coordinates and Killing spinors can be seen in Appendix A.1, in particular in equations (A.1.7)-(A.1.8). Here we focus only on the vector multiplet part but a similar argument can be performed also on the hypermultiplets on $(H)S^4$.

A remark that needs to be made is that the localization procedure should be performed on the fully gauge-fixed path integral. We should thus also need the ghost fields and define for these consistent supersymmetry transformations. This leads to a series of technical comments, which we discuss later in Section 2.5.2.

In order to compute

$$\mathcal{Q}V\Big|_{\text{even}} = (\mathcal{Q}\lambda_{A\alpha})^* \mathcal{Q}\lambda_{A\alpha} + (\mathcal{Q}\bar{\lambda}^{\dot{\alpha}}_A)^* \mathcal{Q}\bar{\lambda}^{\dot{\alpha}}_A\,, \tag{2.5.12}$$

one needs the conjugation properties of the bosonic fields which are fixed by the convergence of the path integral:

$$(\phi)^* = -\bar{\phi}\,, \quad (A_\mu)^* = A_\mu\,, \quad (D_{AB})^* = -D^{AB}\,. \tag{2.5.13}$$

Putting everything together and using (1.5.4) we have

$$\begin{aligned}
\mathcal{Q}V\Big|_{\text{even}} =& \epsilon^A\epsilon_A\left(F^-_{\mu\nu} + \frac{2}{\epsilon^A\epsilon_A} v_{[\mu}D_{\nu]^-}\phi_2 + s_2 i\frac{\epsilon^A\sigma^{\mu\nu}\epsilon^B\tau_{3,AB}}{\epsilon^C\epsilon_C}\phi_2\right)^2 \\
&+ \bar{\epsilon}_A\bar{\epsilon}^A\left(F^+_{\mu\nu} - \frac{2}{\bar{\epsilon}_A\bar{\epsilon}^A} v_{[\mu}D_{\nu]^+}\phi_2 + s_1 i\frac{\bar{\epsilon}^A\bar{\sigma}^{\mu\nu}\bar{\epsilon}^B\tau_{3,AB}}{\bar{\epsilon}_C\bar{\epsilon}^C}\phi_2\right)^2 \\
&+ \frac{4}{\epsilon^A\epsilon_A + \bar{\epsilon}_A\bar{\epsilon}^A}\big(D_\mu[(\epsilon^A\epsilon_A + \bar{\epsilon}_A\bar{\epsilon}^A)\phi_1]\big)^2 + \left(\frac{1}{\epsilon^A\epsilon_a} + \frac{1}{\bar{\epsilon}_A\bar{\epsilon}^A}\right)(v\cdot D\phi_2)^2 \\
&+ \frac{1}{2(\epsilon^A\epsilon_A + \bar{\epsilon}_A\bar{\epsilon}^A)}\big|2\phi_1(s_2\epsilon^C\epsilon_C + s_1\bar{\epsilon}_C\bar{\epsilon}^C)\tau_{3,AB} + (\epsilon^C\epsilon_C + \bar{\epsilon}_C\bar{\epsilon}^C)D_{AB}\big|^2 \\
&- 16(\epsilon^A\epsilon_A + \bar{\epsilon}_A\bar{\epsilon}^A)[\phi_1,\phi_2]^2
\end{aligned} \tag{2.5.14}$$

where the $\pm$ stands for (anti-)self-dual part of a tensor which can be obtained by using the following projector

$$P^{\mu\nu,\rho\sigma}_\pm = \frac{1}{4}(g^{\mu\nu,\rho\sigma} \pm \varepsilon^{\mu\nu\rho\sigma}) = \frac{1}{4}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho} \pm \varepsilon^{\mu\nu\rho\sigma})\,. \tag{2.5.15}$$

In what follows we briefly discuss the gauge-fixing part of the path integral and how this enters in the localization computation. We then study the locus with the standard argument on S^4, extended to HS^4. Later we study the locus on AdS$_4$. Finally we discuss the classical contribution, the 1-loop determinant computation in general and in the novel case of AdS$_4$.

2.5.2 Gauge-fixing

In finding the locus with the previous argument we were still considering the non-gauged-fixed path integral. In doing so we actually have not found a single locus but all the gauge equivalent ones. The resulting localized partition function thus remains some path integration we are not able to perform explicitly.

We thus need to upgrade our argument to the gauge fixed path integral

$$\int \mathcal{D}\Psi \mathcal{D}\Psi_{\text{ghost}} e^{-S[\Psi]-t\mathcal{Q}V-\mathcal{Q}_{\text{BRST}}V_{\text{ghost}}} \,. \tag{2.5.16}$$

Defining c and $\bar{c}$ as the ghost and anti-ghost, B as the Lagrange multiplier field, κ some constant and $G[\Psi-\Psi_{\text{loc.}}]$ some gauge-fixing functional compatible with localization, namely such that $G(0)=0$, we have

$$\int V_{\text{ghost}} = \int \bar{c}\Big(-G[\Psi-\Psi^{\text{loc}}]+\frac{\kappa}{2}B\Big)\,, \tag{2.5.17}$$

with

$$\begin{aligned}
&\mathcal{Q}_{\text{BRST}}c = icc\,,\\
&\mathcal{Q}_{\text{BRST}}\bar{c} = B\,,\\
&\mathcal{Q}_{\text{BRST}}B = 0\,,\\
&\mathcal{Q}_{\text{BRST}}\Psi = \text{Gauge}(c)[\Psi] = i[c,\Psi]_{\pm} \quad (\text{except for } \mathcal{Q}_{\text{BRST}}A_\mu = D_\mu c)\,,
\end{aligned} \tag{2.5.18}$$

The path integral does not depend on κ even though it is best, in order to run correctly the localization argument, to think of it as a real and strictly positive number. One can then upgrade the localization argument by defining

$$\begin{aligned}
\hat{\mathcal{Q}} &= \mathcal{Q}+\mathcal{Q}_{\text{BRST}}\,,\\
\hat{V} &= V+V_{\text{ghost}}\,,\\
V &= \sum_{\text{ferm.}} \text{ferm.}(\hat{\mathcal{Q}}\text{ferm.})^*\,,
\end{aligned} \tag{2.5.19}$$

and, referring to (2.5.7), the following supersymmetry transformations for the ghosts

$$\begin{aligned}
\mathcal{Q}c &= \Phi^{\text{loc.}}-\Phi\,,\\
\mathcal{Q}\bar{c} &= 0\,,\\
\mathcal{Q}B &= i\mathcal{L}(\bar{c})+iv^\mu A_\mu^{\text{loc.}}\bar{c}\,.
\end{aligned} \tag{2.5.20}$$

With these we can write our original path integral in the following way

$$\int \mathcal{D}\Psi \mathcal{D}\Psi_{\text{ghost}} e^{-S[\Psi]-t\hat{\mathcal{Q}}\hat{V}} \,. \tag{2.5.21}$$

We can trade $\mathcal{Q}V$ for $\hat{\mathcal{Q}}\hat{V}$ because in this way we are just adding either $\mathcal{Q}$-exact of $\mathcal{Q}_{\text{BRST}}$-exact quantities. Furthermore with the choice of supersymmetry transformations (2.5.20) for the ghosts we end up having[23]

$$\begin{aligned}\hat{\mathcal{Q}}^2 &= i\mathcal{L}_v + \text{Gauge}(\Phi^{\text{loc.}}) + \text{Lorentz}(L_{ab}) + \mathcal{R}_{SU(2)}(\tilde{\Theta}_{AB}) \\ &= i\mathcal{L}_v + \text{Gauge}(a) + \text{Lorentz}(L_{ab}) + \mathcal{R}_{SU(2)}(-s_2\tau_{3,AB})\end{aligned} \tag{2.5.22}$$

which is such that $\int \hat{\mathcal{Q}}^2\hat{V} = 0$,[24] allowing us to run the localization argument provided that $\hat{Q}\hat{V}\big|_{\text{even}}$ is positive semi-definite. $\hat{Q}\hat{V}\big|_{\text{even}}$ is indeed positive semi-definite since

$$\int \hat{\mathcal{Q}}\hat{V}\big|_{\text{even}} = \int \mathcal{Q}V\big|_{\text{even}} + \int B\big(-G[\Psi - \Psi^{\text{loc.}}] + \frac{\kappa}{2}B\big) \tag{2.5.23}$$

and thus the argument for the locus is not affected by the gauge fixing. The only (good) difference is that now we do not have gauge redundancy in the locus.

Remark on ghosts for ghosts

In the case of the partition function on S^4 [3], as well as for Neumann boundary conditions on HS^4 and AdS$_4$, there is an additional gauge fixing for constant gauge transformations to be made. This happens because, differently from what happens in flat space or with Dirichlet boundary conditions, where these transformations are associated to asymptotic symmetries [58], these transformations are still redundancies.

When these are gauge transformations, we need to add ghosts for ghosts. This can be seen from the fact that the (partially) gauge fixed path integral is invariant under $c \to c + c_0$, for some constant c_0. Indeed c plays the role of the gauge symmetry transformation in the BRST formalism.

However the intent in this thesis is to present in some details how one performs localization computations, while specifying to the case of pure SYM on AdS$_4$ with Dirichlet boundary conditions we are interested in.

In the case of Neumann boundary conditions one should just add the terms explained in [3] and the rest of the argumentation would follow in the same way.

[23] The "Lorentz" is not present in [33] since for them $L_{ab} = 0$, in our case this is not true but that term does not play any role.

[24] It might seem that $\int \hat{\mathcal{Q}}^2\hat{V} \neq 0$ since Gauge$(\Phi^{\text{loc.}}) \subset \hat{\mathcal{Q}}$ and we have gauge fixing part in V_{ghost}. However Gauge$(\Phi^{\text{loc.}})$ is actually a constant gauge transformation that corresponds to a genuine global asymptotic symmetry of the theory. By construction this should not be fixed and thus the above remains a symmetry also for V_{ghost}.

2.5.3 Finding the locus (and the instantons) for $(H)S^4$

We propose here the same argument of Pestun [3], to argue the locus for S^4, and we adjust it in order to justify the result for HS^4. This argument would not run however for AdS$_4$, thus in the next section we give a different argument that can be applied in all of these cases.

We expand "$v_{[\mu}D_{\nu]^\pm}\phi_2$" inside the squares in the first two lines of (2.5.14) and write

$$\begin{aligned}\mathcal{Q}V'\Big|_{\text{even}} =&4(\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A)(D\phi_2)^2+8D^\mu(\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A)D_\mu\phi_2\phi_2-4D_\nu(\tilde F^{\mu\nu}v_\mu\phi_2)\\&+\epsilon^A\epsilon_A\left(F^-_{\mu\nu}+s_2 i\frac{\epsilon^A\sigma^{\mu\nu}\epsilon^B\tau_{3,AB}}{\epsilon^C\epsilon_C}\phi_2\right)^2+\bar\epsilon_A\bar\epsilon^A\left(F^+_{\mu\nu}+s_1 i\frac{\bar\epsilon^A\bar\sigma^{\mu\nu}\bar\epsilon^B\tau_{3,AB}}{\bar\epsilon_C\bar\epsilon^C}\phi_2\right)^2\\&+\frac{4}{\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A}\big(D_\mu[(\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A)\phi_1]\big)^2-16(\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A)[\phi_1,\phi_2]^2\\&+\frac{1}{2(\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A)}\left|2\phi_1(s_2\epsilon^C\epsilon_C+s_1\bar\epsilon_C\bar\epsilon^C)\tau_{3,AB}+(\epsilon^C\epsilon_C+\bar\epsilon_C\bar\epsilon^C)D_{AB}\right|^2\end{aligned}\tag{2.5.24}$$

where we have defined

$$\tilde F^{\mu\nu}=\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,.\tag{2.5.25}$$

We see that, since $\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A=1$ in this space, and thus $D_\mu(\epsilon^A\epsilon_A+\bar\epsilon_A\bar\epsilon^A)=0$, we have that (2.5.24) equals

$$\begin{aligned}\mathcal{Q}V\Big|_{\text{even}} =&4(D\phi_2)^2-4D_\nu(\tilde F^{\mu\nu}v_\mu\phi_2)+4\big(D_\mu\phi_1\big)^2-16[\phi_1,\phi_2]^2\\&+\epsilon^A\epsilon_A\left(F^-_{\mu\nu}+s_2 i\frac{\epsilon^A\sigma^{\mu\nu}\epsilon^B\tau_{3,AB}}{\epsilon^C\epsilon_C}\phi_2\right)^2+\bar\epsilon_A\bar\epsilon^A\left(F^+_{\mu\nu}+s_1 i\frac{\bar\epsilon^A\bar\sigma^{\mu\nu}\bar\epsilon^B\tau_{3,AB}}{\bar\epsilon_C\bar\epsilon^C}\phi_2\right)^2\\&+\frac{1}{2}\left|2\phi_1(s_2\epsilon^C\epsilon_C+s_1\bar\epsilon_C\bar\epsilon^C)\tau_{3,AB}+(\epsilon^C\epsilon_C+\bar\epsilon_C\bar\epsilon^C)D_{AB}\right|^2.\end{aligned}\tag{2.5.26}$$

The total derivative we can see above is not a problem for the localization argument in S^4 but it adds in principle a boundary contribution to the localization argument on HS^4. Thanks to the boundary conditions (1.5.3) however this additional term vanishes. We thus have obtained a new sum of squares whose locus is easier to find.

Thanks to the terms proportional to $(D\phi_{1,2})^2$ we must have $\phi_{1,2}$ covariantly constant. Then the terms

$$\epsilon^A\epsilon_A\left(F^-_{\mu\nu}+s_2 i\frac{\epsilon^A\sigma^{\mu\nu}\epsilon^B\tau_{3,AB}}{\epsilon^C\epsilon_C}\phi_2\right)^2\,,\quad \bar\epsilon_A\bar\epsilon^A\left(F^+_{\mu\nu}+s_1 i\frac{\bar\epsilon^A\bar\sigma^{\mu\nu}\bar\epsilon^B\tau_{3,AB}}{\bar\epsilon_C\bar\epsilon^C}\phi_2\right)^2\,,\tag{2.5.27}$$

give further constraints on the locus, except on the poles where either $\epsilon^A\epsilon_A$ or $\bar\epsilon_A\bar\epsilon^A$ is zero (A.1.7). The study of the poles leads to the instanton contributions [3], but let us here focus on all the points except these.

If we do so then we get

$$\tilde F_{\mu\nu}=i\left(-s_1\frac{\bar\epsilon_A\bar\sigma_{\mu\nu}\bar\epsilon_B\tau_3^{AB}}{\bar\epsilon_C\bar\epsilon^C}+s_2\frac{\epsilon_A\sigma_{\mu\nu}\epsilon_B\tau_3^{AB}}{\epsilon^C\epsilon_C}\right)\phi_2\,.\tag{2.5.28}$$

Using the Bianchi identity and the fact that ϕ_2 is covariantly constant one can obtain $\phi_2=0$. The commutator constraint becomes automatically solved and then, due to (2.5.27), we also have $F_{\mu\nu}=0$ which means that ϕ_1 can be taken to be a constant matrix. We then have as locus

$$\begin{aligned}\phi_1&=-\frac{a}{2}\,,\\ D_{AB}&=-\tau_{3,AB}\,a\,,\end{aligned}\tag{2.5.29}$$

where a is a matrix in the gauge-algebra. We thus see that the path integral localizes to

$$\int\mathcal{D}\Phi\ldots\longrightarrow\int_{\mathfrak{g}}da\ldots\,,\tag{2.5.30}$$

where the integration runs over the matrices of the gauge Lie algebra $\mathfrak{g}$. We can trade the integral over the space of these matrices to integrals on $\mathbb{R}^r$, with $r=\mathrm{rank}(\mathfrak{g})$, by reducing the integration to the Cartan of the Lie algebra

$$\int_{\mathfrak{g}}da\ldots=\int_{\mathfrak{h}\subset\mathfrak{g}}da\prod_{\alpha\in\Delta^+}(\alpha\cdot a)^2\ldots\,,\tag{2.5.31}$$

where $\alpha\in\Delta^+$ runs over the positive roots of the Lie algebra.

Instanton contributions

On S^4, since we have some single points where $\epsilon^A\epsilon_A$ or $\bar\epsilon_A\bar\epsilon^A$ is zero, we have some single points where we cannot follow the argument written above and argue that $F_{\mu\nu}=0$. We can only argue that in the point where $\epsilon^A\epsilon_A=0$ we have $F^+_{\mu\nu}=0$

and in the point where $\bar{\epsilon}_A\bar{\epsilon}^A = 0$ we have $F^-_{\mu\nu} = 0$, which correspond to point-like instanton and anti-instanton contributions.

For this reason, if we allow the path integral to run over these configurations, we have a path integral over each instanton sector with its path integration over the instanton moduli. The localization procedure we are following then ends up localizing as well the instanton moduli just as it happens in [55]. One obtains a localization procedure equivalent to the $\Omega_{\epsilon_1,\epsilon_2}$-background of [55], but with different ϵ_1 and ϵ_2 parameters. These parameters can be found by seeing if $i\mathcal{L}_v \subset \mathcal{Q}^2$ acts, near the fixed points, as some rotation which in some wise choice of coordinates can be rewritten as [33]

$$i\mathcal{L}_v = i\epsilon_1\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) + i\epsilon_2\left(x^3\frac{\partial}{\partial x^4} - x^4\frac{\partial}{\partial x^3}\right). \tag{2.5.32}$$

It is possible to see that for us $\epsilon_1 = \epsilon_2 = -1$ in units of the radius for both $(H)S^4$ and AdS_4. In the literature usually one has $\epsilon_1 = \epsilon_2 = 1$ but there is no substantial difference.

For HS^4 the argument is essentially the same but we simply have one pole, thus we end up with the partition function having only the instanton or anti-instanton contributions depending on the choice of the orientation of the boundary.

2.5.4 Finding the locus (and the instantons) for AdS_4

We cannot repeat the argument for the locus of $(H)S^4$ on AdS_4 since in this space $D^\mu(\epsilon^A\epsilon_A + \bar{\epsilon}_A\bar{\epsilon}^A)$ is no longer zero. We thus decide to use a novel argument, which in principle could also be used for the cases of HS^4.

This argument relies on the fact that we want to enforce 3d $\mathcal{N} = 2$ supersymmetry and all the modes, continuously for each value of η, transform as in (2.4.18)-(2.4.19), see Section 2.4.4.

From (2.5.14) we know that

$$\begin{aligned} \phi_1 &= \frac{\tilde{\phi}_1}{(\epsilon^A\epsilon_A + \bar{\epsilon}_A\bar{\epsilon}^A)} = \frac{\tilde{\phi}_1}{2\cosh(\eta)}\,, \\ D_{AB} &= -2\frac{s_2\epsilon^A\epsilon_A + s_1\bar{\epsilon}_A\bar{\epsilon}^A}{(\epsilon^A\epsilon_A + \bar{\epsilon}_A\bar{\epsilon}^A)^2}\tau_{3,AB}\,\tilde{\phi}_1 = \frac{2}{\cosh^2(\eta)}\tau_{3,AB}\,\tilde{\phi}_1\,, \end{aligned} \tag{2.5.33}$$

with $\tilde{\phi}_1$ a covariantly constant field. As argued in Section 2.4.4, we have two main case scenarios. We can fix the leading mode of ϕ_1, we then need to match the value of the locus to the one of the boundary condition. Another possibility is that it remains fluctuating and we have to path integrate over these modes, together with all the other ones linked by 3d $\mathcal{N} = 2$ supersymmetry, but on a smaller set of these thanks to localization.

In both case the original choice of our path integral forces (2.5.33) to be part of a consistent 3d $\mathcal{N} = 2$ background.

We can then insert (2.5.33) into (2.4.19) and, since we know from [31] that a 3d $\mathcal{N} = 2$ linear multiplet enforces $K = 0$, we get

$$2\cosh(\eta) D_{\perp}\phi_1 - D_{12} + 2i\phi_2 + 2\cosh(\eta)\phi_1\Big|_{\text{loc.}} = 0\,. \tag{2.5.34}$$

Using (2.5.33), we see that the locus for ϕ_2 needs to be zero. This in turn, except in the center of AdS where in (2.5.14) we have $\epsilon^A\epsilon_A$, means that $F_{\mu\nu}$ is zero from (2.5.14). In the center of AdS$_4$ where this argument cannot run and we find instantons. This means that we can consider ϕ_1 and D_{AB}, by the same arguments in $(H)S^4$, to localize to

$$\begin{aligned} \phi_1 &= -\frac{a}{2\cosh(\eta)}\,, \\ D_{AB} &= -\frac{a}{\cosh^2(\eta)}\tau_{3,AB}\,, \end{aligned} \tag{2.5.35}$$

where a is a matrix in the gauge-algebra.

One can notice that the result is similar to the one of $(H)S^4$ but with an additional $\frac{1}{\cosh(\eta)}$ dependence, which is actually expected as one can see in the literature for the locus of vector fields in AdS in other dimensions [59, 60] or for abelian gauge theories in [50].

2.5.5 Classical contribution

After we localize the path integral we need to insert in the action, as well as in the other $\mathcal{Q}$-closed operators inside the path integral, the value of the locus.

In general this is what one should do but one should remember the fact that there are some points, like the poles of S^4, where the locus is not actually the one displayed in (2.5.29). This is an issue, in particular if one wants to study non-perturbative corrections to local operators inserted in these points. This is precisely what happens when one wants to apply the canonical technique, discussed in [26], used to compute a class of extremal correlators exactly with localization.

This also becomes important if one want to study one-point functions, as we did in [9] and as we discuss in Section 3.

Focusing for the moment only on the partition function, that thus has only the action as integrand of the path integral, and counting instanton contributions, we see that on S^4 the result is the following

$$Z^{S^4} = \int_{\mathfrak{h}\subset\mathfrak{g}} da \prod_{\alpha\cdot\Delta^+} (\alpha \cdot a)^2 \; e^{-2\pi \mathrm{Im}(\tau)\mathrm{Tr}(a^2)} Z_{\text{1-loop}}(a) |Z_{\text{Nekrasov}}(a,\tau)|^2 \,, \tag{2.5.36}$$

where we have used (2.5.31) and with $Z_{\text{1-loop}}(a)$ which we discuss in Section 2.5.6. The exponential comes from the action showed in (2.1.11)-(2.1.12), with the background showed in (2.2.1) and after one inserts the locus of equation (2.5.29). The $Z_{\text{Nekrasov}}(a,\tau)$ is the Nekrasov partition function [55] with the topological twist given by $\epsilon_1 = \epsilon_2 = -1$.

For HS^4 the argument is the same but we have an additional boundary term in the action (1.5.9)-(1.5.10) and we have the contribution coming from the instantons of only one of the poles. With our conventions on the orientation of the boundary, specified by our choice of choosing $\int D_\mu V^\mu = \int_\partial V^\perp$, we see that the result on HS^4 is given by

$$Z^{HS^4} = \int_{\mathfrak{h}\subset\mathfrak{g}} da \prod_{\alpha\cdot\Delta^+} (\alpha \cdot a)^2 \; e^{i\pi\tau \mathrm{Tr}(a^2)} Z_{\text{1-loop}}(a) Z_{\text{Nekrasov}}(a,\tau) \,. \tag{2.5.37}$$

The 1-loop result is different from the case of S^4 and depends on the choice of boundary conditions, as we discussed in more details in Section 2.5.7. In particular, if we have Dirichlet boundary conditions, the integration ceases to exist due to the supersymmetrization of the condition that freezes the gauge field (1.5.6). One can also notice that the partition function is no longer periodic in θ and this is expected from the fact that, in presence of a boundary, the 4d θ term can be seen as a 3d Chern-Simons term.

A discussion analogue to the one for HS^4 follows also for AdS_4.

2.5.6 1-loop determinant

The path integral degrees of freedom are divided in terms of cohomological variables, namely divided in terms of a $\hat{\mathcal{Q}}$-singlet

$$\Phi = 2i\bar{\epsilon}_A\bar{\epsilon}^A\phi + 2i\epsilon^A\epsilon_A\bar{\phi} - iv^\mu A_\mu \,, \tag{2.5.38}$$

which parametrizes the locus, and Grassmann even and Grassmann odd quantities

$$X_{\text{even}} = (\phi_2, A_\mu) \qquad X_{\text{odd}} = (\Theta_{AB}, \bar{c}, c)\,, \tag{2.5.39}$$

with

$$\Theta_{AB} = \epsilon_{(A}\lambda_{B)} + \bar{\epsilon}_{(A}\bar{\lambda}_{B)}\,. \tag{2.5.40}$$

All the other elements can be obtained by applying $\hat{\mathcal{Q}}$.

It is possible to then write the quadratic part of $\hat{V}$ and $\hat{\mathcal{Q}}\hat{V}$ as

$$\begin{aligned}
\hat{V}\Big|_{\text{quad.}} &= \begin{pmatrix}\hat{\mathcal{Q}}X_{\text{even}} & X_{\text{odd}}\end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11}\end{pmatrix} \begin{pmatrix} X_{\text{even}} \\ \hat{\mathcal{Q}}X_{\text{odd}}\end{pmatrix}\,, \\
\hat{\mathcal{Q}}\hat{V}\Big|_{\text{quad.}} &= \begin{pmatrix}X_{\text{even}} & \hat{\mathcal{Q}}X_{\text{odd}}\end{pmatrix} \begin{pmatrix} -\hat{\mathcal{Q}}^2 & 0 \\ 0 & 1\end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11}\end{pmatrix} \begin{pmatrix} X_{\text{even}} \\ \hat{\mathcal{Q}}X_{\text{odd}}\end{pmatrix} \\
&\quad - \begin{pmatrix}\hat{\mathcal{Q}}X_{\text{even}} & X_{\text{odd}}\end{pmatrix} \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11}\end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \hat{\mathcal{Q}}^2\end{pmatrix} \begin{pmatrix} \hat{\mathcal{Q}}X_{\text{even}} \\ X_{\text{odd}}\end{pmatrix}\,.
\end{aligned} \tag{2.5.41}$$

For this reason, letting $K_{\text{even/odd}}$ being the total quadratic part of the even/odd fields, we can write the 1-loop determinant as

$$Z_{\text{1-loop}} = \sqrt{\frac{\det(K_{\text{odd}})}{\det(K_{\text{even}})}} = \sqrt{\frac{\det_{X_{\text{odd}}}(\hat{\mathcal{Q}}^2)}{\det_{X_{\text{even}}}(\hat{\mathcal{Q}}^2)}} = \sqrt{\frac{\det_{\text{Coker}.D_{10}}(\hat{\mathcal{Q}}^2)}{\det_{\text{Ker}.D_{10}}(\hat{\mathcal{Q}}^2)}}\,. \tag{2.5.42}$$

The first equality is just due to the fact that $\det(AB) = \det(A)\det(B)$. The second one is due to the fact that D_{10} maps X_{even} into X_{odd} and $[\hat{Q}, D_{10}] = 0$. For this reason in the ratio of the determinants only the unpaired eigenmodes, which by definition belong to the kernel and cokernel, survive.

Calling λ_n the n^{th} eigenvalue, k_n and c_n its multiplicity in the kernel and cokernel respectively, we can write

$$Z_{\text{1-loop}} = \sqrt{\prod_n \lambda_n^{c_n - k_n}}\,. \tag{2.5.43}$$

The eigenvalues λ_n can be computed for each of the fields, what is missing is the knowledge of $c_n - k_n$. To this aim, one can compute the index

$$\text{Ind}_{D_{10}}(t) = \text{Tr}_{\text{Ker}.D_{10}}(e^{-i\hat{\mathcal{Q}}^2 t}) - \text{Tr}_{\text{Coker}.D_{10}}(e^{-i\hat{\mathcal{Q}}^2 t}) = -\sum_n (c_n - k_n) e^{-i\lambda_n t}\,. \tag{2.5.44}$$

From the formal expansion one can obtain $c_n - k_n$. For elliptic or transversally elliptic operators, which we define below, there are considerable simplifications in computing the index.

Informally, one can construct a matrix, called symbol, by sending $\partial_\mu \to ip_\mu$ in D_{10}. If the symbol is non-invertible when all $p_\mu = 0$, the operator is said to be *elliptic* and the index is a finite Laurent series in e^{-it}. If the same is true but with a singled out $\hat{p} \in \{p_\mu\}$ that can remain diverse from zero, then the operator is said to be *transversally elliptic* along $\hat{p}_\mu$ and the index is in general a formal Laurent series in e^{-it}. This is precisely the case on $(H)S^4$ and also in AdS_4, where $\hat{p}$ is in the direction of the Killing vector generated by $\hat{\mathcal{Q}}^2$.

When we have an operator that acts on functions on a manifold with at least a free $U(1)$ action generated by some Killing vector $\hat{v}$, and when this operator is elliptic or transversally elliptic along $\hat{v}$, one can use (a generalization of) the Atiyah-Bott fixed point formula. This is precisely the case for D_{10} and the $U(1)$ action defined by the Killing vector generated by $\hat{\mathcal{Q}}^2$. Letting x_0 be the fixed points of this $U(1)$ action, we have

$$\mathrm{Ind}_{D_{10}}(t) = \sum_{x_0} \frac{\mathrm{Tr}_{X_{\text{even}}}(e^{-i\hat{\mathcal{Q}}^2 t})\big|_{x_0} - \mathrm{Tr}_{X_{\text{odd}}}(e^{-i\hat{\mathcal{Q}}^2 t})\big|_{x_0}}{\det(\delta^\mu_\nu - \partial_\nu x'^\mu)}\,, \tag{2.5.45}$$

where x'^μ is the transformation of x^μ under the $U(1)$ action induced by $\hat{\mathcal{Q}}^2$. For S^4 there are two fixed points which specify antipodal poles, thus on HS^4 we have one fixed point. We then have similarly a single fixed point for AdS_4, the center of the $U(1)$ rotation. We relegate the analysis and technical comments to the Appendix D and we write the result below

$$Z_{\text{1-loop}} = \prod_{\alpha\in\Delta^+} \frac{\alpha\cdot a}{\sinh(\pi\alpha\cdot a)} G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)\,, \tag{2.5.46}$$

with $\alpha \cdot a$ is defined to be a eigenvalue of the matrix a while Δ is the set of roots of the gauge algebra.

The Atiyah-Bott formula can be applied technically only on compact spaces. It was however pointed out in different localization computations in AdS [59, 61] that the index result matches the expectations of the partition function computation performed with different techniques. Furthermore it was shown in [61] that the partition function of AdS_2 matches the result of the one in HS^2, as our result matches the one of HS^4. This is a further check of the applicability of the formula because in the case of superconformal theories the partition functions on the two geometries should match. Indeed, while a generic boundary partition function of an SCFT receives both vector and hypermultiplet contributions, the theory of massless hypermultiplets is superconformal by itself, and therefore also the vector partition functions must match.

2.5.7 Localization formulæ

We describe in this section the known localization results for the partition function on HS^4 [56, 57]. These results are valid also for AdS$_4$ for the reasons explained at the end of Section 2.5.6.

Let us first consider the case of a Dirichlet boundary condition for the gauge field, deformed by the addition of boundary degrees of freedom in a 3d $\mathcal{N} = 2$ theory, in such a way that a subgroup G of the global symmetry of the boundary degrees of freedom is identified with the global symmetry G of the Dirichlet condition (this can be achieved through boundary superpotential couplings). Without loss of generality, we can choose a in the Cartan. This fugacity can be interpreted as a background value for the vector multiplet in (1.5.5), which couples to $\mathcal{J}_{\mathrm{N}}$ to create a real mass compatible with 3d $\mathcal{N} = 2$ supersymmetry for one of the bulk scalars. In this case the partition function can be written as following product:

$$\mathcal{Z}^{\mathrm{D}}_{HS^4}(\tau, a, z_i) = Z^{\mathrm{D}}_{\text{vector}}(\tau, a) \cdot Z^{1,2}_{\text{hypers}}(a) \cdot Z_{\text{inst}}(\tau, a) \cdot Z_{\text{3d}}(a, z_i)\,. \tag{2.5.47}$$

Let us detail the form of each factor in (2.5.47):

- The contribution of the bulk vector is given by

$$Z^{\mathrm{D}}_{\text{vector}}(\tau, a) = e^{i\pi\tau\,\mathrm{Tr}\,a^2} \prod_{\alpha\in\Delta^+} H(ia\cdot\alpha)\frac{a\cdot\alpha}{\sinh(\pi a\cdot\alpha)}\,, \tag{2.5.48}$$

  where the product is taken over the set of positive roots Δ^+ of the gauge group, and the function $H(x)$ is determined in terms of Barnes G-functions

$$H(x) = G(1 + x)G(1 - x)\,. \tag{2.5.49}$$

  Note that the classical action contributes a term holomorphic in τ, namely $e^{i\pi\tau\,\mathrm{Tr}\,a^2}$, as discussed around (2.5.37).

- The hypermultiplet contribution depends on the choice of boundary condition 1 or 2 (1.5.11). Fixing Φ_{i1} and Φ_{i2} to transform under the representation R of the gauge symmetry with weights w, the result is

$$Z^1_{\text{hypers}} = \prod_{w\in R} \frac{1}{H^{\frac{1}{2}}(ia\cdot w)} e^{-\frac{1}{2}\ell(ia\cdot w)}\,, \; Z^2_{\text{hypers}} = \prod_{w\in R} \frac{1}{H^{\frac{1}{2}}(ia\cdot w)} e^{\frac{1}{2}\ell(ia\cdot w)}\,, \tag{2.5.50}$$

with ℓ being the following function

$$\ell(z) = -z\log(1-e^{2\pi i z}) + \frac{i}{2}\Big(\pi z^2 + \frac{1}{\pi}\text{Li}_2(e^{2\pi i z})\Big) - \frac{i\pi}{12}\,. \tag{2.5.51}$$

- $Z_{\text{inst}}(\tau, a)$ is Nekrasov's instanton partition function [55] already discussed in (2.5.37). Both in the contribution of the classical action and in the instanton contribution makes the partition function holomorphic in τ. This holomorphicity is however broken after F_∂-maximization explained in Section 2.6.

- The factor $Z_{\text{3d}}(a, z_\kappa)$ denotes the S^3 partition function of the boundary degrees of freedom. For Lagrangian theories it can also be computed using localization [62]. The partition function depends on the fugacity a for the global G symmetry. It moreover depends on the fugacities u_i for the remaining continuous global symmetries of the 3d theory.[25] The $\mathcal{N} = 2$ 3d partition function depends finally on the coefficients Δ_κ that defines the R-symmetry. The 3d R-charge is a linear combination of the $U(1)$ symmetries:

$$R = \sum_\kappa \Delta_\kappa R_\kappa\,, \tag{2.5.52}$$

where R_0 is a reference R-charge and $\Delta_0 = 1$. In [62] it was found that the sphere 3d partition functions depend on u_κ and Δ_κ only through the complex combination $z_\kappa = \Delta_\kappa - iu_\kappa$ and that Z_{3d} is actually holomorphic in z_κ. In principle one should also consider the mixing coefficient Δ_κ related to the G symmetry. If however G does not contain abelian factors then it cannot mix with the 3d R-symmetry under the assumption that the boundary condition preserves the entire G.

Next, we consider the Neumann boundary condition. Also in this case we consider a possible deformation due to the presence of local 3d degrees of freedom. This 3d theory is assumed to have a G global symmetry and the bulk-boundary interaction

[25] A fugacity is an expectation value for the scalar component of a background vector multiplet associated to a global $U(1)$ symmetry.

is defined by gauging the G symmetry with the boundary mode of the bulk gauge field. Since the latter is still fluctuating on the boundary, the localization formula in this case takes the form of an integral over the Cartan-valued matrix a[26]

$$\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau, z_\kappa) = \int da \prod_{\alpha\in\Delta^+} (\alpha\cdot a)^2 \; Z^{\mathrm{N}}_{\text{vector}}(\tau,a)\cdot Z^{1,2}_{\text{hypers}}(\tau,a)\cdot Z_{\text{inst}}(\tau,a)\cdot Z_{\text{3d}}(a,z_\kappa)\,, \tag{2.5.53}$$

where the only new ingredients compared to the Dirichlet case are the Vandermonde determinant and the contribution of the vector multiplet, namely

$$Z^{\mathrm{N}}_{\text{vector}}(\tau,a) = e^{i\pi\tau\,\mathrm{Tr}\,a^2} \prod_{\alpha\in\Delta^+} H(ia\cdot\alpha)\frac{\sinh(\pi a\cdot\alpha)}{a\cdot\alpha}\,. \tag{2.5.54}$$

Abelian bulk theory

In the special case of the abelian theory, the product over roots is absent and one gets $Z^{\mathrm{D}}_{\text{vector, abelian}}(\tau,a) = Z^{\mathrm{N}}_{\text{vector, abelian}}(\tau,a) = e^{i\pi\tau a^2}$, where a is a real number. Let us focus on the case of a Neumann boundary condition, with boundary degrees of freedom that form a certain 3d $\mathcal{N}=2$ CFT with a $U(1)$ symmetry, coupled to the bulk by gauging the $U(1)$ symmetry with the boundary components of the bulk gauge field. The resulting modified Neumann boundary condition is

$$\frac{1}{g^2}(\mathcal{J}_{\mathrm{N}} + \gamma\mathcal{J}_{\mathrm{D}}) = \mathcal{J}_{\mathrm{CFT}}\ , \tag{2.5.55}$$

where $\mathcal{J}_{\mathrm{CFT}}$ denotes the linear multiplet of the $U(1)$ symmetry that is gauged by the bulk gauge field. We can think of this boundary condition as setting to zero a linear combination of the two global symmetry with currents $\mathcal{J}_{\mathrm{N}} + \gamma\mathcal{J}_{\mathrm{D}}$ and $\mathcal{J}_{\mathrm{CFT}}$, while an independent linear combination survives as a $U(1)$ global symmetry of the boundary condition. The partition function of the 3d theory is denoted as $Z_{\text{3d}}(a, z_\kappa)$, where a is the fugacity for the $U(1)$ symmetry coupled to the bulk gauge field, and the z_κ denote fugacities for additional $U(1)$ symmetries. In addition, we can turn on a fugacity a' for the global symmetry with current $\mathcal{J}_D$ (see footnote 9), and a fugacity x' for the surviving combination of $\mathcal{J}_{\mathrm{N}}$ and $\mathcal{J}_{\mathrm{CFT}}$, which can take

[26] More generally, also global symmetries acting on the hypermultiplets might contain $U(1)$ factors that are preserved by the boundary condition and can mix with the R symmetry. In that case we need to introduce appropriate z_κ's also in the hypermultiplet partition function.

for definiteness to be simply $\mathcal{J}_{\rm CFT}$. The resulting partition function is [63]

$$\mathcal{Z}^{\rm N}_{HS^4,\text{abelian}}(\tau,a',x',z_\kappa)=\int da\,e^{-2\pi i a a'}\,e^{i\pi\tau a^2}\,Z_{\rm 3d}(a-x',z_\kappa)\ . \tag{2.5.56}$$

The two symmetries with fugacities a' and x' descend from magnetic and electric one-form symmetries of the bulk theory, and therefore can be called magnetic and electric $U(1)$ symmetries. In future chapters indeed we argue how these are indeed exchanged by electric-magnetic duality. Note that the instanton contribution in the abelian case is a-independent [3], and it only contributes an overall factor holomorphic in the τ that can be reabsorbed by a counterterm, see (3.2.8). As a result it does not contribute to physical quantities and it can be omitted.

Also the magnetic and the electric $U(1)$'s, like any other $U(1)$ symmetry of the boundary condition, can in principle mix with the 3d R-symmetry. The mixing parameters, which we denote as $\tilde{\Delta}$ and $\hat{\Delta}$, respectively, appear together with a' and x' in a holomorphic combination in the partition function. The expression of $\mathcal{Z}^{\rm N}_{HS^4,\text{abelian}}$ where $\tilde{\Delta}$ and $\hat{\Delta}$ explicitly appear is then

$$\mathcal{Z}^{\rm N}_{HS^4,\text{abelian}}(\tau,a'-i\tilde{\Delta},x'-i\hat{\Delta},z_\kappa)=\int da\,e^{-2\pi i a a'}\,e^{-2\pi a\tilde{\Delta}}\,e^{i\pi\tau a^2}\,Z_{\rm 3d}(a-x'+i\hat{\Delta},z_\kappa)\ . \tag{2.5.57}$$

2.6 F_∂-maximization

In this section e describe a process, known as F_∂-maximization [63], that allows to specify the BCFT partition function on HS^4 from one that preserves 3d $\mathcal{N}=2$ supersymmetry.

In this thesis we also propose a way to re-interpret the same procedure in AdS$_4$, with the aim of studying the $SU(2)\to U(1)$ Higgsing of pure SYM (see Chapter 5).

2.6.1 F_∂-maximization on HS^4

We have given formulas for general $\mathcal{N}=2$ bulk gauge theories on HS^4, but now let us further suppose that the bulk theory is conformal, and the gauge coupling τ is an exactly marginal parameter. There are many examples in this class, to mention a few: the abelian theory, $SU(N)$ SQCD with $2N$ flavors, $\mathcal{N}=4$ SYM and its orbifold.

In this case we can require that the boundary condition on the S^3 preserves the maximal possible subgroup, namely $OSp(2|2,2)$, making it a $\frac{1}{2}$-BPS superconformal boundary condition. This happens only when the $U(1)_R$ symmetry of the 3d $\mathcal{N}=2$ algebra preserved by the boundary is the superconformal R-symmetry. As we have stated above, different values of the mixing parameters Δ_κ correspond to different choices of the 3d $U(1)_R$ symmetry. A criterion to determine the superconformal values of Δ_κ of equation (1.6.15) was given in [63] and it is analogue to the method used on S^3 partition functions [25, 62]. The BCFT values for the Δ_κ are the ones maximizing the following boundary free energy

$$F_\partial = -\frac{1}{2}\log\frac{\mathcal{Z}_{HS^4}\bar{\mathcal{Z}}_{HS^4}}{\mathcal{Z}_{S^4}} \,. \tag{2.6.1}$$

Here $\bar{\mathcal{Z}}_{HS^4}$ is simply the complex conjugate of the hemisphere partition function, which by reflection positivity can equivalently be thought of as the partition function on the orientation-reversed hemisphere, we come back to this point in Section 3.2. On the other side, $\mathcal{Z}_{S^4}$ is the partition function of the bulk theory on the full sphere. The modulus squared ensures that F_∂ is a real function, and the normalization by $\mathcal{Z}_{S^4}$ ensures that contributions from bulk Weyl anomalies cancel. F_∂ is sensitive only to the boundary modes of the bulk fields and possibly their interaction with boundary-localized degrees of freedom, so it is analogous to the quantity $F = -\log\mathcal{Z}_{S^3}$ defined for local 3d theories.

The function F_∂ evaluated at the maximum is an interesting observable of the boundary conformal field theory (BCFT), which can be defined for any BCFT as the quantity (2.6.1) in which the coupling of the BCFT to the curved space are fixed by Weyl rescaling. We use the subscript BCFT to distinguish partition functions before and after specifying Δ to the conformal value, if an ambiguity might arise. The procedure of maximization in the supersymmetric context is then a way to fix the curvature couplings compatibly with conformal symmetry. The latter are indeed related to the choice of R-symmetry because the background gauge field for the R-symmetry is in the same multiplet as the metric. An important property of F_∂ for BCFT's is that it is monotonic under boundary RG flows, as conjectured in [63] and proved in [64] using entropic arguments.

When we apply F_∂-maximization to our setup, the resulting value of Δ_κ is a real function of the marginal coupling, that we write as $\Delta_\kappa^{\max}(\tau,\bar{\tau})$. This has a very important consequence: unlike what one might have thought from the expressions (2.5.48)-(2.5.56), the hemisphere partition function of the BCFT is *not* a holomorphic function of the complexified gauge coupling τ, namely

$$\mathcal{Z}_{HS^4,\,\mathrm{BCFT}}(\tau,\bar{\tau}) = \mathcal{Z}_{HS^4}(\tau)|_{\Delta_\kappa=\Delta_\kappa^{\max}(\tau,\bar{\tau})} \ . \tag{2.6.2}$$

The additional $(\tau,\bar{\tau})$ dependence due to $\Delta_\kappa^{\max}(\tau,\bar{\tau})$ plays an important role in the following when we use the BCFT partition function to extract BCFT one-point functions in $H\mathbb{R}^4$.

2.6.2 F_∂-maximization on AdS$_4$

As already discussed in 2.4, it is hard to preserve the full AdS$_4$ $\mathcal{N}=2$ supersymmetry algebra, which is equivalent to the 3d $\mathcal{N}=2$ superconformal one, from a Lagrangian formulation. We thus preserve explicitly only a 3d $\mathcal{N}=2$ one on S^3.

We then might ask what can we do to preserve the full bulk superalgebra and we suggest [15] that this happens after a F_∂-maximization procedure, as the one explained for HS^4.

It is very tempting to think this because, as explained at the end of Section 2.5.6, we expect the partition function of a conformal theory on AdS$_4$ to be the same as the conformal one on HS^4.

The difference would be in the interpretation of this procedure, indeed we conjecture that this is just a way to preserve the supersymmetry associated to the full group of isometries of AdS$_4$ and not to actually specify a conformal theory on AdS$_4$. Thus this procedure would also apply for massive theories in AdS$_4$.

Chapter 3

One-point functions from $\mathcal{Z}_{HS^4}$

In this section we derive the relation between the derivatives of the hemisphere partition function and the one-point function of the Coulomb branch operators, namely chiral operators constructed as in (1.3.10) out of a vector multiplet.

We do this in order to compute the one-point functions BCFT-data (1.6.18)

$$\langle A_w(\vec{x}, x_\perp)\rangle = \frac{a_{A_w}}{x_\perp^w}\,, \tag{3.0.1}$$

for a chiral operator of Weyl weight w, via supersymmetric localization on HS^4.

We show two derivations, one that is an adaptation of the approach used in [26] to compute two-point functions $\langle A\,\bar{A}\,\rangle$, and a second one based purely on localization.

3.1 SUSY Identities for the derivatives of $\mathcal{Z}_{HS^4}$

As was discussed in Section 2.1.4, we can start from a BCFT fixed point and then add some deformations that we use to compute correlation functions, as in (2.1.22). In particular, just like in equation (2.1.23), we study deformations of integrated correlators but with the intention of only taking one derivative over the global source.

We do this on HS^4, which is conformally flat, because our aim is to be able to find the formulæ that can be used within the supersymmetric localization framework. For this reason we choose these deformations in such a way to preserve 3d $\mathcal{N}=2$ supersymmetry. We then use a modification of the argument presented in [26] in order to obtain (3.0.1).

Following what stated in Section 2.2 the deformation that we employ is the following

$$Z_{\rm BCFT} = \int \mathcal{D}\phi \; e^{-S[\phi]} \longrightarrow Z_{\rm BCFT}[\lambda_w, \bar{\lambda}_w] = \int \mathcal{D}\phi \; e^{-S[\phi] + \lambda(\int \mathcal{C}_w + \int_\perp \mathcal{C}_{w,\perp}) + \bar{\lambda}(\int \bar{\mathcal{C}}_w + \int_\perp \bar{\mathcal{C}}_{w,\perp})} \,. \tag{3.1.1}$$

where we find $\mathcal{C}$ and $\bar{\mathcal{C}}$ in (2.3.3), $\mathcal{C}_\partial$ and $\bar{\mathcal{C}}_\partial$ in (2.3.6) and the embedding of the vector in the chiral and anti-chiral multiplet in (2.1.9)-(2.1.10).

The case $w = 2$ is special because the deformation is marginal and preserves superconformal symmetry. The correlation functions of the marginal operators compute interesting geometric quantities on the conformal manifold [46, 47], and this also applies to the boundary one-point function, as we discuss in Section 3.5. The gauge coupling in conformal gauge theories is an example of such an exactly marginal coupling, and it is obtained from the chiral and anti-chiral multiplets with bottom components $A = \mathrm{Tr}[\phi^2]$ and $\bar{A} = \mathrm{Tr}[\bar{\phi}^2]$, respectively, and with the identification of couplings

$$\lambda = i\frac{\tau}{8\pi} \,. \tag{3.1.2}$$

In the following, to avoid possible confusion, we use the letter λ when referring to an arbitrary chiral deformation, and τ in the specific case of an exactly marginal one.

Taking derivatives of the partition function with respect to the coupling λ gives integrated one-point functions

$$\begin{aligned} \partial_\lambda(-\log \mathcal{Z}_{HS^4}) &= \int_{HS^4} d^4x\sqrt{g}\,\langle\mathcal{C}\rangle + \int_{S^3} d^3x\sqrt{h}\,\langle\mathcal{C}_\partial\rangle \,, \\ \partial_{\bar{\lambda}}(-\log \mathcal{Z}_{HS^4}) &= \int_{HS^4} d^4x\sqrt{g}\,\langle\bar{\mathcal{C}}\rangle + \int_{S^3} d^3x\sqrt{h}\,\langle\bar{\mathcal{C}}_\partial\rangle \,. \end{aligned} \tag{3.1.3}$$

Let us focus first on the bulk integral term. It is possible to prove that $\mathcal{C}$ respects the following Ward identity[27]

$$\begin{aligned}\mathcal{C} =& D_\mu\left[2B_{AB}\frac{\epsilon^A\sigma^\mu\bar{\epsilon}^B}{\epsilon^A\epsilon_A} + 2i\frac{\tau_{3,AB}\epsilon^A\sigma^\mu\bar{\epsilon}^B}{\epsilon^A\epsilon_A}\left((2w-3) - \frac{\bar{\epsilon}_A\bar{\epsilon}^A}{\epsilon^A\epsilon_A}\right)A - 2\frac{\bar{\epsilon}_A\bar{\epsilon}^A}{\epsilon^A\epsilon_A}D^\mu A\right] \\ &+ \delta\left(-\frac{\epsilon^A\Lambda_A}{\epsilon^A\epsilon_A} - i(w-1)\frac{\tau_{3,AB}\epsilon^A\Psi^B}{\epsilon^A\epsilon_A} + \frac{D_\mu\Psi^A\sigma^\mu\bar{\epsilon}_A}{\epsilon^A\epsilon_A}\right),\end{aligned} \tag{3.1.4}$$

with δ a Q-supersymmetry variation. Since the boundary condition preserves Q, the Q-exact term drops from the one-point function, and we are only left with the total derivative.

The argument of the total derivative however is singular at the north pole and following [46] we regulate it by removing a small ball of radius δ around the pole and taking $\delta \to 0$ after performing the integral. As a result the integral of the total derivative equals the sum of two boundary contributions: one at the S^3 of radius δ around the singularity which gives the same result as in [46], namely the insertion of the bottom component A at the north pole N, and a new second one at the equatorial boundary S^3. We then obtain

$$\int_{HS^4} d^4x\sqrt{g}\,\langle\mathcal{C}\rangle = 32\pi^2\langle A(N)\rangle - \int_{S^3} d^3x\sqrt{h}\,\langle -2D_\perp A + i(\tau_3)^{AB}B_{AB} + 4(w-2)A\rangle\,. \tag{3.1.5}$$

Since bulk one-point functions diverge like a power-law when the operator approaches the boundary, this equation contains divergent contributions both from the integral on the left-hand side and from the restriction of the operators to the boundary on the right-hand side. Both of these singularities can be regulated by e.g. a minimal distance from the boundary. Using the relation between the one-point function of $\mathcal{C}$ and the one-point function of A dictated by supersymmetry, these divergent contributions, as well as the finite terms, must match between the left and the right hand side. We have checked this explicitly in the case of $w = 2$.

The computation of the integrated one-point function of $\bar{\mathcal{C}}$ is analogous, and

[27] Our expression for $\mathcal{C}$ as a total derivative plus a Q-exact term, while analogous to the one presented in [46], does not agree in details with it. The results in [46] are not affected because the total derivative term that is singular at the pole is identical to ours. However the contribution on the other boundary S^3 is affected by this difference.

starts from the expression similar to (3.1.4)

$$\bar{\mathcal{C}} = D_\mu\Big[-2\bar{B}_{AB}\frac{\epsilon^A\sigma^\mu\bar{\epsilon}^B}{\bar{\epsilon}_A\bar{\epsilon}^A} - 2i\frac{\tau_{3,AB}\epsilon^A\sigma^\mu\bar{\epsilon}^B}{\bar{\epsilon}_A\bar{\epsilon}^A}\Big((2w-3) - \frac{\epsilon^A\epsilon_A}{\bar{\epsilon}_A\bar{\epsilon}^A}\Big)\bar{A} - 2\frac{\epsilon^A\epsilon_A}{\bar{\epsilon}_A\bar{\epsilon}^A}D^\mu\bar{A}\Big] \\ + \delta\Big(\frac{\bar{\epsilon}_A\bar{\Lambda}^A}{\bar{\epsilon}_A\bar{\epsilon}^A} + i(w-1)\frac{\tau_{3,AB}\bar{\epsilon}^A\bar{\Psi}^B}{\bar{\epsilon}_A\bar{\epsilon}^A} + \frac{D_\mu\bar{\Psi}_A\bar{\sigma}^\mu\epsilon^A}{\bar{\epsilon}_A\bar{\epsilon}^A}\Big)\,. \tag{3.1.6}$$

The important difference with $\bar{\mathcal{C}}$ instead of $\mathcal{C}$ is that now the argument of the total derivative is not singular anywhere in northern hemisphere. The singularity on S^4 would be at the south pole. As a result in this case, after dropping the Q-exact term, we only receive a contribution from the equatorial boundary S^3. We then obtain

$$\int_{HS^4} d^4x\sqrt{g}\,\langle\bar{\mathcal{C}}\rangle = \int_{S^3} d^3x\sqrt{h}\,\langle -2D_\perp\bar{A} - i(\tau_3)^{AB}\bar{B}_{AB} - 4(w-2)\bar{A}\rangle\,. \tag{3.1.7}$$

Plugging this relation back into (3.1.3) we finally obtain

$$\begin{aligned} \partial_\lambda(-\log\mathcal{Z}_{HS^4}) &= 32\pi^2\langle A(N)\rangle\,, \\ \partial_{\bar{\lambda}}(-\log\mathcal{Z}_{HS^4}) &= 0\,. \end{aligned} \tag{3.1.8}$$

Just like in the case of S^4 two-point functions of [47], the above results can be generalized to include the additional insertion of generic Q-invariant operators.

3.2 Flat space one-point functions from SUSY identities

In this section we show how to combine the two Ward Identities (3.1.8) in such a way to obtain the flat space one-point function from the BCFT partition function (2.6.2).

Using (3.1.8) we obtain

$$\partial_\lambda(-\log\mathcal{Z}_{HS^4}) = 32\pi^2\langle A(N)\rangle - \partial_\lambda(-\log\bar{\mathcal{Z}}_{HS^4})\,. \tag{3.2.1}$$

Defining

$$f_\partial = -\frac{1}{2}\log(\mathcal{Z}_{HS^4,\mathrm{BCFT}}\bar{\mathcal{Z}}_{HS^4,\mathrm{BCFT}})\,, \tag{3.2.2}$$

we obtain

$$\partial_\lambda f_\partial|_{\tau,\bar{\tau}} = 16\pi^2\langle A(N)\rangle_{HS^4}\,, \tag{3.2.3}$$

where by the evaluation in $\tau, \bar{\tau}$ we mean that all the dimensionful couplings with $w \neq 2$ are turned off while the marginal deformations $\tau, \bar{\tau}$ corresponding to $w = 2$ can take an arbitrary value, parametrizing a point on the conformal manifold. One might be worried about the special case in which we are computing the one-point function for a chiral with $w = 2$, and $\lambda = i\frac{\tau}{8\pi}$ is one of the marginal couplings, because $\mathcal{Z}_{HS^4,\mathrm{BCFT}}$ has additional dependence on τ through $\Delta_\kappa^{\max}(\tau, \bar{\tau})$. On the other hand we can write this additional dependence as

$$\frac{1}{2\pi i}\frac{\partial \Delta_\kappa}{\partial \tau}\partial_{\Delta_\kappa} f_\partial(\tau,\bar{\tau})|_{\Delta_\kappa=\Delta_\kappa^{\max}} = \frac{1}{2\pi i}\frac{\partial \Delta_\kappa}{\partial \tau}\partial_{\Delta_\kappa} F_\partial(\tau,\bar{\tau})|_{\Delta_\kappa=\Delta_\kappa^{\max}} \ , \tag{3.2.4}$$

and this vanishes thanks to the maximization condition. Eq. (3.2.3) allows us to compute the BCFT one-point functions on HS^4, and we added a subscript to stress that this is the result on the curved background.

This is very close to our goal of obtaining the BCFT one-point functions in flat space, the only remaining task is to take care of the possible mixing induced by the curvature as we reviewed in Section 2.1.4. As we mentioned there, this can be done with the standard Gram-Schmidt procedure that is also used for correlation functions without a boundary. After subtracting the mixing, a Weyl transformation shows that the quantity we are left with is the coefficient

$$a_A = y^w \langle A(y, x^i)\rangle_{H\mathbb{R}^4} \ , \tag{3.2.5}$$

of the one-point function in the flat-space BCFT. Here $y \geqslant 0$ is the distance from the boundary at $y = 0$, and $x^{i=1,2,3}$ are the coordinates parallel to the boundary.

We can be more explicit in the special case of $w = 2$. In this case the only possible mixing is between A and the identity operator, and this can be undone simply by subtracting the one-point function of A on S^4. Using (3.1.2) we obtain

$$w = 2: \quad a_A(\tau,\bar{\tau}) = \frac{1}{2\pi i}\partial_\tau\left(-\frac{1}{2}\log\left(\frac{\mathcal{Z}_{HS^4,\mathrm{BCFT}}\bar{\mathcal{Z}}_{HS^4,\mathrm{BCFT}}}{\mathcal{Z}_{S^4}}\right)\right) = \frac{1}{2\pi i}\partial_\tau F_\partial(\tau,\bar{\tau}) \ . \tag{3.2.6}$$

Therefore we see that, for a Coulomb branch operator residing in the same multiplet of an exactly marginal operator, the coefficient of the one-point function is directly related to the quantity F_∂ that we can assign to the BCFT.

Another advantage of expressing the one-point function in terms of F_∂ is that it makes the scheme-independence manifest. The scheme dependence of $\mathcal{Z}_{S^4}$ was analyzed in [46], where the addition of certain chiral density built out of curvature invariants was found to affect the coupling dependence as

$$\mathcal{Z}_{S^4}(\tau,\bar{\tau}) \to e^{f(\tau)+\bar{f}(\bar{\tau})}\mathcal{Z}_{S^4}(\tau,\bar{\tau}) \tag{3.2.7}$$

where $f(\tau)$ is an arbitrary holomorphic function. This matches with the interpretation of $\mathcal{Z}_{S^4}$ in terms of the Kähler potential on the conformal manifold, the ambiguity represents a Kähler transformation. It can be easily checked that evaluating the same chiral density on HS^4, with the appropriate boundary term as discussed above, gives the following transformation

$$\mathcal{Z}_{HS^4}(\tau,\bar{\tau}) \to e^{f(\tau)}\mathcal{Z}_{HS^4}(\tau,\bar{\tau}) \ . \tag{3.2.8}$$

As a result this ambiguity cancels in the combination F_∂.

3.3 Multiple derivatives

At least heuristically, we would expect that by taking multiple derivatives of the partition function we compute one-point functions of products of Coulomb branch operators (a concept that makes sense thanks to the non-singular OPE, as discussed below equation (1.6.13)). Indeed by applying additional holomorphic derivatives to (3.1.8) and using the non-singular OPE we obtain

$$\partial_\lambda^n(-\log \mathcal{Z}_{HS^4}) = (32\pi^2)^n\langle A^n(N)\rangle \ , \tag{3.3.1}$$

for any $n \geqslant 1$. However there is an important difference between $n=1$ and $n>1$. In the first case, we showed in the previous section how to express the result directly in terms of the BCFT partition function, evaluated at $\Delta_\kappa = \Delta_\kappa^{\max}(\tau,\bar{\tau})$. For $n>1$ derivatives, instead, there is no analogous simple manipulation that allows to cancel the additional dependence on the marginal parameter due to the maximization, and re-express this result just in terms of derivatives of the BCFT partition function. One possible way to proceed is to evaluate the result at $\Delta_\kappa = \Delta_\kappa^{\max}(\tau,\bar{\tau})$ after taking the derivative. However the result obtained in this way is not manifestly scheme-independent, unlike the expression for $n=1$ found in the previous section.

In the next section we use the explicit localization results, which apply in a specific scheme and have a completely determined τ-dependence, to compute the one-point functions of products of chiral operators. This confirms the procedure of taking multiple derivatives of the localized partition function and afterwards plugging in the extremized BCFT values of Δ_κ.

3.4 Flat space one-point functions from localization

One-point functions on HS^4

While the approach in the last section was just based on symmetry and did not assume any Lagrangian formulation or the existence of a path integral that localizes to a matrix model, it is also natural to derive the expression for the one-point functions using the localization formulas for the theory on HS^4. In this approach, one uses that in a $\mathcal{N}=2$ gauge theory the general Coulomb branch operator $A_{n_1,\ldots,n_k;m_1,\ldots,m_k} = \prod_{i=1}^k \mathrm{Tr}[\phi^{n_i}]^{m_i}$, labeled by the k-tuples of positive integers n_i (all distinct) and m_i (not necessarily distinct), at the localization locus reduces to the monomial $\prod_{i=1}^k \mathrm{Tr}[(-\frac{i}{2}a)^{n_i}]^{m_i}$ of the integration variable a. More precisely, this is true if one ignores the contribution from instantons localized at the north-pole. Therefore, at least up to non-perturbative corrections, the one point function is directly computed by inserting the monomial in the matrix model. Let us now discuss how to connect this localization result to the previous approach based on taking derivatives of the partition function. This also allow us to suggest how to include the non-perturbative corrections. Deforming the theory by couplings

$$S_{\mathrm{CFT}} \to S_{\mathrm{CFT}} + \sum_{i=1}^{k} \left[\int_{HS^4} d^4x \sqrt{g}\, (\lambda_{n_i} \mathcal{C}_{n_i} + \bar{\lambda}_{n_i} \bar{\mathcal{C}}_{n_i}) + \int_{S^3} d^3x \sqrt{h}\, (\lambda_{n_i} \mathcal{C}_{n_i\,\partial} + \bar{\lambda}_{n_i} \bar{\mathcal{C}}_{n_i\,\partial}) \right], \tag{3.4.1}$$

where $\mathcal{C}_{n_i}$ is the top component of the chiral multiplet with $\mathrm{Tr}[\phi^{n_i}]$ as bottom component, and $\mathcal{C}_{n_i\,\partial}$ is the associated boundary term, we have that the contribution of the classical action on HS^4 is given by

$$Z_{\mathrm{class}}(\lambda, a) = e^{-32\pi^2 \sum_{i=1}^k \lambda_{n_i} \mathrm{Tr}[(-\frac{i}{2}a)^{n_i}]} . \tag{3.4.2}$$

We are using the entry λ to indicate the dependence on the full set of couplings λ_{n_i}. Notice that here, unlike the case of the full S^4, the dependence on these couplings is holomorphic.

As a result we can trade the insertion of the monomial in the localization integral with derivatives of the classical action (ignoring for the moment instanton contributions, which we denote with subscript "pert")

$$
\begin{aligned}
&\langle A_{n_1,\dots,n_k;m_1,\dots,m_k}(N)\rangle_{HS^4,\text{pert}} \\
&= \frac{1}{\mathcal{Z}^{\text{N}}_{HS^4,\text{BCFT}}}\int da \prod_{i=1}^{k} \text{Tr}[(-\tfrac{i}{2}a)^{n_i}]^{m_i} Z_{\text{class}}(\tau,a) Z_{\text{1loop}}(a) Z_{\text{3d}}(a,\Delta^{\max}(\tau,\bar\tau)) \\
&= \frac{1}{\mathcal{Z}^{\text{N}}_{HS^4,\text{BCFT}}}\int da \left(\prod_{i=1}^{k}\left(-\tfrac{1}{32\pi^2}\tfrac{\partial}{\partial\lambda_{n_i}}\right)^{m_i} Z_{\text{class}}(\lambda,a)\right)\Big|_{\tau} Z_{\text{1loop}}(a) Z_{\text{3d}}(a,\Delta^{\max}(\tau,\bar\tau))\ .
\end{aligned}
\tag{3.4.3}
$$

Here again the evaluation at τ denotes that all the dimensionful couplings are turned off while the marginal one denoted by τ is allowed to vary on the conformal manifold. This expression also suggests a way to include the non-perturbative corrections, by including the instanton partition function in the derivative[28], obtaining

$$
\begin{aligned}
&\langle A_{n_1,\dots,n_k;m_1,\dots,m_k}(N)\rangle_{HS^4} \\
&= \frac{1}{\mathcal{Z}^{\text{N}}_{HS^4,\text{BCFT}}}\int da \left(\prod_{i=1}^{k}\left(-\tfrac{1}{32\pi^2}\tfrac{\partial}{\partial\lambda_{n_i}}\right)^{m_i} Z_{\text{class}}(\lambda,a) Z_{\text{inst}}(\lambda,a)\right)\Big|_{\tau} \\
&\qquad \times Z_{\text{1loop}}(a) Z_{\text{3d}}(a,\Delta^{\max}(\tau,\bar\tau))\ .
\end{aligned}
\tag{3.4.4}
$$

We almost obtained a derivative of the full integrand, which would allow us to rewrite the final result as a derivative of the HS^4 partition function. The obstruction to do so is the additional dependence on $\tau,\bar\tau$ coming from the conformal value of Δ, i.e. the function $\Delta^{\max}$ obtained from the extremization. We can circumvent this problem by writing the derivative as acting on the partition function with generic Δ, and plug in the conformal value only after the evaluation of the derivative. In this way we arrive at

$$
\langle A_{n_1,\dots,n_k;m_1,\dots,m_k}(N)\rangle_{HS^4} = \frac{1}{\mathcal{Z}^{\text{N}}_{HS^4,\text{BCFT}}}\left(\prod_{i=1}^{k}\left(-\tfrac{1}{32\pi^2}\tfrac{\partial}{\partial\lambda_{n_i}}\right)^{m_i} \mathcal{Z}^{\text{N}}_{HS^4}(\lambda,\Delta)\right)\Bigg|_{\tau,\Delta=\Delta^{\max}}\ .
\tag{3.4.5}
$$

Note that for single-trace operators, when we have a single derivative acting on $\mathcal{Z}$, this formula coincides with (3.2.3). This is because the additional dependence due to $\Delta^{\max}_\kappa$, which is potentially dangerous for $w=2$, cancels thanks to the extremization condition, see the discussion around eq. (3.2.4). Moreover this formula also gives a general result for multi-trace one-point functions.

[28] Furthermore this is also how these operators are inserted in extremal correlators, as explained in [26].

From HS^4 to $H\mathbb{R}^4$

Having derived the one-point functions on HS^4, the remaining step is to perform correctly the Weyl rescaling to obtain the one-point functions for the BCFT in flat space $H\mathbb{R}^4$. As we have already reviewed, this amounts to applying a certain matrix to the vector of one-point functions. This matrix is the change of basis determined by the Gram-Schmidt procedure to the matrix of two-point functions $\langle A_I(N)\bar{A}_J(S)\rangle$ on S^4, without a boundary. In formulas, using $I, J, K, \dots$ as indices that run over the full set of Coulomb branch operators, we have

$$\begin{aligned} &{U_I}^L {U^*_J}^M \langle A_L(N)\bar{A}_M(S)\rangle_{S^4} = C_I\delta_{IJ} = \langle A_L(0)\bar{A}_M(\infty)\rangle_{\mathbb{R}^4} \ , \\ &\Rightarrow {U_I}^L \langle A_L(N)\rangle_{HS^4} = y^{\Delta_I}\langle A_I(y,x^i)\rangle_{H\mathbb{R}^4} = a_{A_I} \ . \end{aligned} \tag{3.4.6}$$

We stress that the change of basis ${U_I}^J$, unlike the vector $\langle A_I(N)\rangle_{HS^4}$, does not depend on the boundary condition, it is uniquely fixed by the data of the theory in S^4 and the choice of normalization C_I in flat space.

Abelian bulk theory

We now specialize to the case of the boundary conditions for the free bulk CFT of a $\mathcal{N}=2$ vector multiplet. The S^4 partition function is simply a Gaussian integral

$$\mathcal{Z}_{S^4}(\tau,\bar{\tau}) = \int da e^{-2\pi\,\mathrm{Im}(\tau)\,a^2} = \frac{1}{\sqrt{2\mathrm{Im}(\tau)}} \ , \tag{3.4.7}$$

where τ is the exactly marginal gauge coupling. The Coulomb branch operators are

$$A_n = \phi^n \ , \quad \bar{A}_n = (-\bar{\phi})^n = A_n^\dagger \ , \tag{3.4.8}$$

i.e. powers of the generator ϕ, $\bar{\phi}$ which is the scalar field in the vector multiplet (the choice of the "$-\bar{\phi}$" is made by referring to (2.5.13)). For even n, their correlation functions are obtained taking derivatives w.r.t. to τ and $\bar{\tau}$. The matrix of two-point functions on the sphere is

$$\langle A_n(N)\bar{A}_m(S)\rangle_{S^4} = \frac{1}{\mathcal{Z}_{S^4}(\tau,\bar{\tau})}\int da\ \left(-\tfrac{i}{2}a\right)^n\ \left(\tfrac{i}{2}a\right)^m\ e^{-2\pi\,\mathrm{Im}(\tau)\,a^2} \ . \tag{3.4.9}$$

The Gram-Schmidt orthogonalization problem in this case is thus solved by finding a basis of polynomials which are orthogonal w.r.t. to the Gaussian measure. Hermite polynomials $H_n(x)$ satisfy this condition, so the change of basis is fixed by requiring

$$\sum_{m=0}^{n} U_n{}^m \left(-\tfrac{i}{2}a\right)^m = \left(\frac{-i}{4\sqrt{2\pi\,\mathrm{Im}(\tau)}}\right)^n H_n(\sqrt{2\pi\,\mathrm{Im}(\tau)}a)\ . \tag{3.4.10}$$

The normalization is such that $U_n{}^n = 1$. This choice gives the following coefficient for the two-point function in flat space: $C_n = n!\big(2\pi\mathrm{Im}(\tau)\big)^n$.

In a generic boundary condition given by Neumann deformed by gauging a 3d $\mathcal{N}=2$ SCFT, we therefore arrive at the following formula for the flat-space one-point function of the Coulomb branch operators

$$\begin{aligned} a_{A_n}(\tau,\bar\tau) &= y^n \langle A_n(y,x^i)\rangle_{H\mathbb{R}^4}(\tau,\bar\tau) \\ &= \frac{1}{\mathcal{Z}^{\mathrm{N}}_{HS^4,\mathrm{BCFT}}} \int da \left(\frac{-i}{4\sqrt{2\pi\,\mathrm{Im}(\tau)}}\right)^n H_n(\sqrt{2\pi\,\mathrm{Im}(\tau)}a) e^{i\pi\tau a^2} e^{2\pi a \tilde\Delta^{\max}(\tau,\bar\tau)} Z_{3d}(a,\Delta^{\max}(\tau,\bar\tau))\ . \end{aligned} \tag{3.4.11}$$

In the following we use interchangeably the notation a_{A_n} and $\langle A_n\rangle_{H\mathbb{R}^4}$ to denote the coefficient of the one-point function in flat space with a boundary.

3.5 $\mathcal{Z}_{HS^4}$ and the geometry of the conformal manifold

In this section we make some remarks about the geometric interpretation of the hemisphere partition function as a function defined on the space of exactly marginal couplings, i.e. the conformal manifold of the CFT. The marginal couplings, denoted as λ^I with an index I running over the multiple independent couplings, define a system of coordinates on this manifold. Conformal manifolds are equipped with the Zamolodchikov metric, computed by the two-point correlation function of the marginal operators $\langle O_I(x)O_J(y)\rangle = g_{IJ}(\lambda)|x-y|^{-8}$ [65–67]. For $\mathcal{N}=2$ SCFTs in 4d these conformal manifolds are further constrained to be Kähler-Hodge [48, 68], a result that has recently been extended to the case with only 4 supercharges [69]. A Kähler-Hodge manifold is a Kähler manifold with the property that the flux of the Kähler form on any 2 cycle is quantized, or equivalently that the Kähler form is the first Chern class of a holomorphic line bundle on the manifold.

The Kähler potential $K(\lambda^I,\bar\lambda^{\bar I})$ on the conformal manifold is directly related to the supersymmetric partition function of the theory on S^4 [47]

$$\log Z_{S^4} = \frac{K(\lambda^I,\bar\lambda^{\bar I})}{12}\,. \tag{3.5.1}$$

Given this direct relation between the geometry of the conformal manifold and the sphere partition function, it is natural to ask whether a similar connection exists for the hemisphere partition function.

In two dimensions a similar question was studied in [70] for superconformal interfaces $\mathcal{I}_{12}$ separating two $\mathcal{N} = (2, 2)$ SCFTs, denoted as CFT_1 and CFT_2. This setup can be placed on S^2, with the interface on the equator and the two CFTs on the two hemispheres, and an interface partition function analogous to the boundary one (2.6.1) can be defined as

$$F_{\mathcal{I}_{12}} = -\frac{1}{2} \log \frac{Z_{S^2,\mathcal{I}_{12}} \overline{Z_{S^2,\mathcal{I}_{12}}}}{Z_{S^2,\text{CFT}_1} Z_{S^2,\text{CFT}_2}} \,. \tag{3.5.2}$$

A relation similar to (3.5.1) is also valid for the Kähler potential on the moduli space of 2d $\mathcal{N} = (2, 2)$ SCFTs, namely

$$\log Z_{S^2} = -K(\lambda^I, \bar{\lambda}^{\bar{I}}) \,. \tag{3.5.3}$$

Ref. [70] generalized this relation to the case of an interface that separates two CFTs belonging to the same conformal manifold and only differing by the values of the moduli λ_1^I and λ_2^I, either Kähler or complex structure moduli. In this case the interface sphere partition function computes an analytic continuation of the Kähler potential to independent values of the two entries λ^I and $\bar{\lambda}^I$, in such a way that the first entry is computed at λ_1^I and the second at $\bar{\lambda}_2^I$. This leads to the following identity

$$-2F_{\mathcal{I}_{12}} = K(\lambda_1^I, \bar{\lambda}_1^{\bar{I}}) + K(\lambda_2^I, \bar{\lambda}_2^{\bar{I}}) - K(\lambda_1^I, \bar{\lambda}_2^{\bar{I}}) - K(\lambda_2^I, \bar{\lambda}_1^{\bar{I}}) \equiv \mathcal{D}_{12} \,, \tag{3.5.4}$$

where λ_a^I are the points in moduli space giving CFT_a, and K is the Kähler potential. The function appearing on the right-hand side is the so called Calabi's diastasis function $\mathcal{D}_{12}$ on the moduli space. This function exists on any Kähler manifold, and it is independent of any Kähler transformation [71]. This function can be thought of as measuring the distance between the two CFTs separated by the interface. When the moduli become infinitesimally close it indeed agrees with the geodesic distance on the Kähler manifold determined by the Zamolodchikov metric, more precisely denoting with d_{12} the geodesic distance the relation is $\mathcal{D}_{12} = d_{12}^2 + \mathcal{O}(d_{12}^4)$ [71]. As proposed in [70], the fact that the diastasis can be expressed as the interface partition function invites a generalization to general pairs of CFTs: minimizing the partition function (3.5.2) over the set of interfaces connecting the two CFTs, one gets the following notion of distance between the two CFTs

$$d(\mathrm{CFT}_1, \mathrm{CFT}_2) = \min_{\mathcal{I}_{12}} \sqrt{-2F_{\mathcal{I}_{12}}}\,. \tag{3.5.5}$$

As observed already in [70] this quantity however fails to define a distance in the rigorous mathematical sense because it does not satisfy positivity and the triangular inequality.

What we stated about interface in 2d $\mathcal{N} = (2,2)$ CFTs can be readily imported to the setup with $\frac{1}{2}$-BPS interfaces between $\mathcal{N} = 2$ CFTs in 4d. In fact a similar interface partition function can be defined [63]

$$F_{\mathcal{I}_{12}} = -\frac{1}{2} \log \frac{Z_{S^4,\mathcal{I}_{12}} \overline{Z_{S^4,\mathcal{I}_{12}}}}{Z_{S^4,\mathrm{CFT}_1} Z_{S^4,\mathrm{CFT}_2}}\,. \tag{3.5.6}$$

Using the 4d relation (3.5.1), one gets that for interfaces defined by changing the coupling within the same conformal manifolds this function computes the Calabi diastasis [72, 73], like in 2d. More generally, we get a similar notion of "distance" between 4d $\mathcal{N} = 2$ CFTs

$$d(\mathrm{CFT}_1, \mathrm{CFT}_2) = \min_{\mathcal{I}_{12}} \sqrt{24\, F_{\mathcal{I}_{12}}}\,. \tag{3.5.7}$$

The case of a boundary CFT can be seen as a special case of the interface in which one of the two sides is the empty theory. Restricting the definition (3.5.7) to this case, one defines a function that assigns a number to any CFT

$$||\mathrm{CFT}|| = d(\mathrm{CFT}, \mathrm{empty}) = \min_{\mathcal{B}} \sqrt{24 F_{\partial,\mathcal{B}}}\,. \tag{3.5.8}$$

where now the minimization is over the set of boundary conditions $\mathcal{B}$.

An notable property of the geometry of the $\mathcal{N} = 2$ conformal manifold is the existence of the tt^*-equations [68, 74], which strongly constrain the dependence of observables on the marginal couplings, see e.g. [75]. It would be interesting to explore the interplay of the hemisphere partition function with the tt^*-equations.

Chapter 4

Case study: Abelian theories

In this section we discuss 4d $\mathcal{N} = 2$ abelian theories interacting with boundary modes via gauging, with the bulk vector, a $U(1)$ boundary current. We discuss the known properties of $SL(2,\mathbb{Z})$ dualities of these theories and how these match with the expectations we have for what stated in Section 3. We also perform explicit extrapolations and Feynman diagrams tests of our results.

4.1 Dualities in the Abelian theories with a boundary

We now specify the bulk theory to be the CFT of an $\mathcal{N} = 2$ vector multiplet. This theory enjoys the electric-magnetic duality group $SL(2,\mathbb{Z})$ that acts on the exactly marginal coupling τ as

$$\tau \to g[\tau] \equiv \frac{a\tau + b}{c\tau + d} \ , \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \ . \tag{4.1.1}$$

In this section we study the interplay between the $SL(2,\mathbb{Z})$ duality group and the boundary conditions, in particular at the level of the HS^4 partition function and bulk one-point functions.

4.1.1 Action of $SL(2,\mathbb{Z})$ on the boundary conditions

Let us briefly review the action of $SL(2,\mathbb{Z})$ on the boundary conditions for the $\mathcal{N} = 2$ vector multiplets. Very similar results apply to the 4d free vector CFT for any number of suspersymmetries and have been extensively discussed in the literature also for $\mathcal{N} = 0$ and $\mathcal{N} = 4$ [10, 11, 17, 76, 77].

Consider a modified Neumann boundary condition, in which the boundary component of the bulk gauge field gauges the $U(1)$ symmetry of a certain 3d $\mathcal{N} = 2$

SCFT $\mathcal{T}_{\text{3d}}$. Let us denote this boundary condition as $B\left(\tau, \mathcal{T}_{\text{3d}}\right)$. The following duality holds

$$B\left(\tau, \mathcal{T}_{\text{3d}}\right) \simeq B\left(g[\tau], g\left[\mathcal{T}_{\text{3d}}\right]\right) \ . \tag{4.1.2}$$

Here $g\left[\mathcal{T}_{\text{3d}}\right]$ is 3d SCFT with a $U(1)$ symmetry obtained acting on $\mathcal{T}_{\text{3d}}$ with Witten's $SL(2,\mathbb{Z})$ action [10]. To define this action, it is sufficient to say what is the action of the S and T transformations. The action of S consists in gauging the $U(1)$ symmetry with 3d gauge fields, and defining the new $U(1)$ global symmetry to be the topological $U(1)$. The action of T only changes the theory by shifting the contact term in the two-point function of the conserved $U(1)$ current. This shift amounts on adding to the partition function of the theory with a background gauge field A_μ the Chern-Simons counterterm $\frac{1}{4\pi}\int(AdA+\dots)$, where the dots stand for the supersymmetrization according to the 3d $\mathcal{N}=2$ symmetry preserved by the boundary condition.

An important consequence of the duality (4.1.2) is that the boundary condition admits several decoupling limits, whenever $g[\tau] \to +i\infty$ for any g, in which one is left with a pure Neumann boundary condition for the bulk field in the appropriate duality frame, and a decoupled 3d SCFT $g\left[\mathcal{T}_{\text{3d}}\right]$ on the boundary.

4.1.2 $SL(2,\mathbb{Z})$ and the hemisphere partition function

The $SL(2,\mathbb{Z})$ action we have just described translates into specific operations done on the partition functions computed through localization. In particular we use the modified Neumann boundary condition with partition function given in (2.5.56). To describe the action on this function, we now consider the action of the generators T and S.

For sake of notation, in this section we leave the variables z_κ implicit, as they do not play any role in the duality transformations.

T-transformation

As we have explained above, the T element acts on the 3d CFT$_3$ by adding a Chern-Simons term for the background vector multiplet coupled to $U(1)$ current. At the level of 3d partition function this simply means

$$Z_{T[\mathcal{T}_{3d}]}(a) = e^{-i\pi a^2} Z_{\mathcal{T}_{3d}}(a) \, . \tag{4.1.3}$$

Inverting the relation (4.1.3) and plugging the result into (2.5.56), one obtains

$$\begin{aligned}\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau,a',x') &= \int da\, e^{-2\pi i a a'}\, e^{i\pi\tau a^2}\, Z_{T[\mathcal{T}_{3d}]}(a-x')\, e^{i\pi(a-x')^2} \\ &= e^{i\pi(x')^2}\int da\, e^{-2\pi i a(a'+x')}\, e^{i\pi(\tau+1)a^2}\, Z_{T[\mathcal{T}_{3d}]}(a-x')\,.\end{aligned} \tag{4.1.4}$$

In (4.1.4) we have obtained the expression for a 4d supersymmetric Maxwell theory coupled to a boundary CFT$_3$ $T\left[\mathcal{T}_{3d}\right]$ with gauge coupling $T[\tau]=\tau+1$. In particular we obtain the relation

$$T\left[\mathcal{Z}^{\mathrm{N}}_{HS^4}\right](\tau+1,a'+x',x') = e^{-i\pi(x')^2}\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau,a',x') \tag{4.1.5}$$

where by $T\left[\mathcal{Z}^{\mathrm{N}}_{HS^4}\right]$ we mean the 4d partition function of the 4d Maxwell theory coupled to $T\left[\mathcal{T}_{3d}\right]$.

S-transformation

We now turn to the action of the S element. As we stated above, the S-dual CFT$_3$ is obtained by taking $\mathcal{T}_{3d}$ and coupling the $U(1)$ symmetry to a dynamical 3d gauge boson. Localization then tells us that the partition functions of the dual 3d theories are connected by a Fourier transform

$$Z_{S[\mathcal{T}_{3d}]}(x) = \int da\, e^{-2\pi i a x}\, Z_{\mathcal{T}_{3d}}(a)\ . \tag{4.1.6}$$

We would expect that the 4d partition function (2.5.56) can be rewritten in an analogous form, but in terms of $Z_{S[\mathcal{T}_{3d}]}(x)$. We now show that this is indeed the case. By inverting (4.1.6), and inserting the result for $Z_{\mathcal{T}_{3d}}(a-x')$ we obtain

$$\begin{aligned}\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau,a',x') &= \int da\, e^{-2\pi i a a'}\, e^{i\pi\tau a^2}\int dx\, e^{2\pi i(a-x')x} Z_{S[\mathcal{T}_{3d}]}(x) \\ &= \frac{1}{\sqrt{-i\tau}}\int dx\, e^{-2\pi i x x'} e^{-\frac{i\pi(x-a')^2}{\tau}} Z_{S[\mathcal{T}_{3d}]}(x) \\ &= \frac{e^{-2\pi i a' x'}}{\sqrt{-i\tau}}\int dx\, e^{-2\pi i x x'} e^{-\frac{i\pi x^2}{\tau}} Z_{S[\mathcal{T}_{3d}]}(x+a'),\end{aligned} \tag{4.1.7}$$

where we did the following steps: in the second line we performed the Gaussian integral over the variable a; in the third line we shifted the integration variable $x \mapsto x+a'$.

In (4.1.7) we have obtained the expression for a 4d supersymmetric Maxwell theory coupled to a boundary CFT$_3$ $S\left[\mathcal{T}_{3d}\right]$ with gauge coupling $S[\tau]=-1/\tau$. In particular we obtain the relation

$$S\left[\mathcal{Z}^{\mathrm{N}}_{HS^4}\right](-1/\tau, x', -a') = \sqrt{-i\tau}e^{2\pi i a' x'}\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau, a', x')\,, \tag{4.1.8}$$

with the two fugacities a' and x' exchanged. Notice that the factor $\sqrt{-i\tau}\,e^{2\pi i a' x'}$ cancels out of F_∂, that then is left invariant under S-duality.

Simple Example. We now consider a simple case, i.e. $\mathcal{N}=2$ supersymmetric Maxwell theory with Neumann boundary conditions and no 3d degrees of freedom. In this case $Z^{N}_{3\mathrm{d}}(a) = 1$. Moreover there is no $U(1)$ symmetry in the 3d theory (that is empty), hence we need to set $x'=0$ in the 4d partition function, that is:

$$\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau, a') = \int da\, e^{-2\pi i a a'}\, e^{i\pi\tau a^2} \;=\; \frac{1}{\sqrt{-i\tau}}e^{-i\pi(a')^2/\tau}\;. \tag{4.1.9}$$

Using (4.1.6), we can write the S-dual 3d partition function:

$$Z^{\mathrm{D}}_{3\mathrm{d}}(x) = \int da\, e^{-2\pi i a x} = \delta(x)\,. \tag{4.1.10}$$

This induces Dirichlet boundary conditions. Inserting this expression in (2.5.56) (and setting $a'=0$ because now there is no topological symmetry generated by the boundary value of A^{4d}_μ), one obtains

$$\mathcal{Z}^{\mathrm{D}}_{HS^4}(\tau, x') = \int da\, e^{i\pi\tau a^2}\,\delta(a-x') \;=\; e^{i\pi\tau(x')^2}\;, \tag{4.1.11}$$

that is in fact the 4d partition function for supersymmetric Maxwell theory with Dirichlet boundary conditions.

Notice that (4.1.9) and (4.1.11) satisfy the relation (4.1.8), showing that Neumann and Dirichlet boundary conditions are exchanged by S-duality. For this simple case, we have reproduced the S-dual transformation laws for the partition function that were first written in [63].

4.1.3 $SL(2,\mathbb{Z})$ and one-point functions

We now turn to how the one-point functions of Coulomb branch operators transform under $SL(2,\mathbb{Z})$. We remind their expression here, writing explicitly the dependence on the electric and magnetic fugacities

$$\langle A_n\rangle_{H\mathbb{R}^4}(\tau,a',x') = \frac{1}{\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau,a',x')}\cdot \\ \cdot\int da\left(\frac{-i}{4\sqrt{2\pi\,\mathrm{Im}(\tau)}}\right)^n H_n(\sqrt{2\pi\,\mathrm{Im}(\tau)}a)e^{i\pi\tau a^2}e^{-2\pi i a a'}Z_{\mathcal{T}_{3d}}(a-x')\,. \tag{4.1.12}$$

where again we are omitting the dependence of the 3d partition function on the complexified mixing parameters z_κ.

4.1.4 T-transformation and one-point functions

We follow the same steps done in (4.1.4) in order to write the correlator (4.1.12) in terms of the correlator $T\left[\langle A_n\rangle_{H\mathbb{R}^4}\right]$ in the 4d theory coupled to the 3d boundary theory $T\left[\mathcal{T}_{3d}\right]$. We invert (4.1.3), we plug it into (4.1.3) and proceed as done above for the partition function:

$$\begin{aligned}\langle A_n\rangle_{H\mathbb{R}^4}(\tau,a',x') =& \frac{e^{i\pi(x')^2}}{\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau,a',x')}\cdot \\ &\cdot\int da\left(\frac{-i}{4\sqrt{2\pi\,\mathrm{Im}(\tau)}}\right)^n H_n(\sqrt{2\pi \mathrm{Im}(\tau)\mathrm{Im}(\tau)}a)\,e^{i\pi(\tau+1)a^2}e^{-2\pi i a(a'+x')}Z_{T[\mathcal{T}_{3d}]}(a-x') \\ =& \frac{1}{T\left[\mathcal{Z}^{\mathrm{N}}_{HS^4}\right](\tau+1,a'+x',x')}\cdot \\ &\cdot\int da\left(\frac{-i}{4\sqrt{2\pi\,\mathrm{Im}(\tau)}}\right)^n H_n(\sqrt{2\pi\,\mathrm{Im}(\tau)}a)\,e^{i\pi(\tau+1)a^2}e^{-2\pi i a(a'+x')}Z_{T[\mathcal{T}_{3d}]}(a-x')\,,\end{aligned} \tag{4.1.13}$$

where in the last step we have used (4.1.5).

The equation (4.1.13) implies the following relation between the one-point functions of the T-dual theories:

$$\langle A_n\rangle_{H\mathbb{R}^4}(\tau,a',x') = T\left[\langle A_n\rangle_{H\mathbb{R}^4}\right](\tau+1,a'+x',x')\,. \tag{4.1.14}$$

4.1.5 S-transformation and one-point functions

We again proceed as in (4.1.7), in order to write this correlator in terms of the correlator $S\left[\langle A_n\rangle_{H\mathbb{R}^4}\right]$ in the 4d theory coupled to the 3d boundary theory $S\left[\mathcal{T}_{3d}\right]$. Inverting (4.1.6), plugging the result for $Z_{\mathcal{T}_{3d}}(a-x')$ into (4.1.12) and rearranging the integrals, we obtain

$$\langle A_n \rangle_{H\mathbb{R}^4}(\tau, a', x') = \frac{1}{\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau, a', x')} \int dx \left(\frac{-i}{4\sqrt{2\pi \operatorname{Im}(\tau)}} \right)^n e^{-2\pi i x x'} Z_{S[\mathcal{T}_{3d}]}(x) \cdot \\ \cdot \int da \, e^{i\pi\tau a^2} e^{2\pi i a(x-a')} H_n(\sqrt{2\pi \operatorname{Im}(\tau)} a) \, . \tag{4.1.15}$$

To perform the integration over a on the second line, we use the following result for the Hermite polynomials:

$$\int dy \, e^{iky} \, e^{-\frac{y^2(1-i\gamma)}{2}} \, H_n(y) = \frac{i^n \sqrt{2\pi}}{\sqrt{1-i\gamma}} \frac{(1+i\gamma)^{n/2}}{(1-i\gamma)^{n/2}} e^{-\frac{k^2}{2(1-i\gamma)}} H_n \left(\frac{k}{\sqrt{1+\gamma^2}} \right) \, . \tag{4.1.16}$$

By redefining the integration variable as $a = \frac{y}{\sqrt{2\pi \operatorname{Im}(\tau)}}$ we can then write

$$\int da \, e^{i\pi\tau a^2} e^{2\pi i a(x-a')} H_n(\sqrt{2\pi \operatorname{Im}(\tau)} a) = \int \frac{dy}{\sqrt{2\pi \operatorname{Im}(\tau)}} \, e^{\frac{2\pi(x-a')}{\sqrt{2\pi \operatorname{Im}(\tau)}} iy} \, e^{\frac{i\pi\tau}{2\pi \operatorname{Im}(\tau)} y^2} \, H_n(y) \\ = \frac{i^n}{\sqrt{-i\tau}} \frac{(1+i\gamma)^{n/2}}{(1-i\gamma)^{n/2}} \, e^{-\frac{i\pi(x-a')^2}{\tau}} H_n \left(\sqrt{2\pi \operatorname{Im}\left(-\frac{1}{\tau}\right)} (x-a') \right) \, , \tag{4.1.17}$$

where in the last line we have applied (4.1.16) with $k = \frac{\sqrt{2\pi}(x-a')}{\sqrt{\operatorname{Im}(\tau)}}$ and, as done before, $\gamma = \frac{\operatorname{Re}(\tau)}{\operatorname{Im}(\tau)}$. Moreover we have used the fact that $(1-i\gamma)\operatorname{Im}(\tau) = -i\tau$ and $\frac{1}{\operatorname{Im}(\tau)(1+\gamma^2)} = \frac{|\tau|^2}{\operatorname{Im}(\tau)} = \operatorname{Im}\left(-\frac{1}{\tau}\right)$.

Plugging (4.1.17) into (4.1.15) and shifting $x \mapsto x + a'$, we obtain

$$
\begin{aligned}
\langle A_n \rangle_{H\mathbb{R}^4}(\tau, a', x') = & \frac{e^{-2\pi i a' x'}}{\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau, a', x')} \int dx \left(\frac{-i}{4\sqrt{2\pi \operatorname{Im}(\tau)}} \right)^n e^{-2\pi i x x'} Z_{S[\mathcal{T}_{3d}]}(x + a') \cdot \\
& \cdot \frac{i^n}{\sqrt{-i\tau}} \frac{(1 + i\gamma)^{n/2}}{(1 - i\gamma)^{n/2}} e^{-\frac{i\pi x^2}{\tau}} H_n \left(\sqrt{2\pi \operatorname{Im}\left(-\frac{1}{\tau}\right)}\, x \right) \\
= & \frac{e^{-2\pi i a' x'}}{\sqrt{-i\tau}\, \mathcal{Z}^{\mathrm{N}}_{HS^4}(\tau, a', x')} \left(-\frac{1}{\tau}\right)^n \int dx \left(\frac{-i}{4\sqrt{2\pi \operatorname{Im}(-1/\tau)}} \right)^n \cdot \\
& \cdot H_n \left(\sqrt{2\pi \operatorname{Im}\left(-\frac{1}{\tau}\right)}\, x \right) e^{-\frac{i\pi x^2}{\tau}} e^{-2\pi i x x'} Z_{S[\mathcal{T}_{3d}]}(x + a') \\
= & \frac{1}{S\left[\mathcal{Z}^{\mathrm{N}}_{HS^4}\right](-1/\tau, x', -a')} \left(-\frac{1}{\tau}\right)^n \int dx \left(\frac{-i}{4\sqrt{2\pi \operatorname{Im}(-1/\tau)}} \right)^n \cdot \\
& \cdot H_n \left(\sqrt{2\pi \operatorname{Im}\left(-\frac{1}{\tau}\right)}\, x \right) e^{-\frac{i\pi x^2}{\tau}} e^{-2\pi i x x'} Z_{S[\mathcal{T}_{3d}]}(x + a')\,,
\end{aligned}
\tag{4.1.18}
$$

where in the first step we have used $\operatorname{Im}(\tau)\left(\frac{1-i\gamma}{1+i\gamma}\right) = -\operatorname{Im}(\tau)\left(\frac{\tau}{\bar{\tau}}\right) = -\left(\frac{\operatorname{Im}(\tau)}{|\tau|^2}\right)\tau^2 = -\tau^2 \operatorname{Im}\left(-\frac{1}{\tau}\right)$ and in the second step the relation (4.1.8).

We have then obtained the following expression that relates the one-point functions of the S-dual theories

$$
\langle A_n \rangle_{H\mathbb{R}^4}(\tau, a', x') = \left(-\frac{1}{\tau}\right)^n S\left[\langle A_n \rangle_{H\mathbb{R}^4}\right](-1/\tau, x', -a')\,. \tag{4.1.19}
$$

The overall τ-dependent factor is consistent with the action of the S transformation on the vector multiplet, namely the self-dual components of the field strength under electric-magnetic duality transform as $S[F^+_{\mu\nu}] = -\tau F^+_{\mu\nu}$, and by supersymmetry the scalar ϕ must transform in the same way $S[\phi] = -\tau\phi$.

4.2 Boundary free energy and one-point functions in abelian theories

In this section we study different BCFTs to explicitly demonstrate our findings in the previous sections. In particular, we study $\mathcal{N} = 2$ supersymmetric $U(1)$ gauge theory in the bulk of HS^4, coupled to different CFTs on the boundary.[29]

Before proceeding with the examples, let us point out a general result about the mixing parameters $\tilde{\Delta}$ and $\hat{\Delta}$ for the magnetic and electric $U(1)$'s in (2.5.57). The values fixed by the maximization are both zero when $Z_{\mathcal{T}_{3d}}(a-i\hat{\Delta})$ is even under $a-i\hat{\Delta}\mapsto-(a-i\hat{\Delta})$. To see this, we use that when this condition is satisfied the boundary free energy (2.6.1) is even under the simultaneous change of sign of both parameters i.e. $F_\partial(-\tilde{\Delta},-\hat{\Delta})=F_\partial(\tilde{\Delta},\hat{\Delta})$. Since the maximum is unique, it must then be at $(\tilde{\Delta},\hat{\Delta})=(0,0)$. To prove this parity property of F_∂ one can simply change the integration variable in (2.5.57) by setting $a\mapsto-a$ to obtain that $\mathcal{Z}^{\mathrm{N}}_{HS^4}(-\tilde{\Delta},-\hat{\Delta})=\mathcal{Z}^{\mathrm{N}}_{HS^4}(\tilde{\Delta},\hat{\Delta})$.

4.2.1 Chirals on the boundary

Let us start by considering the 3d $\mathcal{N}=2$ SCFT of $N_F+N_{\bar{F}}$ free chiral fields. We need to specify the coupling to the bulk gauge field by picking a $U(1)$ global symmetry, and we take it to be the symmetry that assigns charge $+1$ to N_F chirals and -1 to the remaining $N_{\bar{F}}$ of them. We denote this theory as $\mathcal{T}_{3d}=(N_F,N_{\bar{F}})$. The hemisphere partition function and the values of the mixing parameters at the maximum were computed previously using a saddle-point analysis in [78], and also in [79] which in addition studied the two-point functions of boundary currents, and performed an extensive numerical analysis. Our overlapping results are in agreement with those presented in these references.

The 3d partition function with fugacity a for the selected $U(1)$ global symmetry is [62, 80]

$$\begin{gathered}Z_{(N_F,N_{\bar{F}})}(a-i\hat{\Delta},\Delta)=\Big(Z^{+}_{\mathrm{chiral}}(-a-i\hat{\Delta}-i\Delta)\Big)^{N_F}\Big(Z^{-}_{\mathrm{chiral}}(a+i\hat{\Delta}-i\Delta)\Big)^{N_{\bar{F}}}\;,\\ Z^{\pm}_{\mathrm{chiral}}(\xi)=e^{\pm\left[-\frac{i\pi}{12}+\frac{i\pi}{2}(i\xi-1)^2\right]}e^{\ell(1-i\xi)}\;,\end{gathered}\tag{4.2.1}$$

with

$$\ell(z)=-z\log(1-e^{2\pi iz})+\frac{i}{2}\left(\pi z^2+\frac{1}{\pi}\mathrm{Li}_2(e^{2\pi iz})\right)-\frac{\pi}{12}i\,.\tag{4.2.2}$$

Δ and $\hat{\Delta}$ parametrize the most general choice of the R-charge for the chiral fields that is compatible with the global symmetry when we couple to the bulk gauge field

[29] In this section, the partition functions and the one-point functions are evaluated at zero fugacities for the global symmetries.

with the above assignment of charges. The choice of $+$ or $-$ in Z_{chiral} corresponds to the choice of background Chern-Simons contact terms that are needed to make the partition function coupled to the background gauge field invariant under background gauge transformations, see e.g. [36]. The choice made here ensures that the theory is parity symmetric for $N_F = N_{\bar{F}}$.

Plugging the expressions for the 3d partition functions into (2.5.57), the integral takes the form

$$\begin{aligned}
&\mathcal{Z}^{\text{N}}_{HS^4}(\tau, -i\tilde{\Delta}, -i\hat{\Delta}, \Delta) \\
&= \int da\, e^{i\pi\tau a^2} e^{-2\pi a\tilde{\Delta}} \left(Z^{+}_{\text{chiral}}(-a - i\hat{\Delta} - i\Delta)\right)^{N_F} \left(Z^{-}_{\text{chiral}}(a + i\hat{\Delta} - i\Delta)\right)^{N_{\bar{F}}} \\
&= e^{i\pi\Phi} \int da\, e^{i\pi\tau_s a^2} e^{-2\pi a\tilde{\Delta}_s} e^{N_F \ell(1-\Delta-\hat{\Delta}+ia) + N_{\bar{F}}\ell(1-\Delta+\hat{\Delta}-ia)} ,
\end{aligned} \tag{4.2.3}$$

where

$$\begin{aligned}
\Phi &= \left(\Delta^2 + \hat{\Delta}^2 - 2\Delta + \frac{5}{6}\right) \frac{N_F - N_{\bar{F}}}{2} - (1-\Delta)\hat{\Delta}(N_F + N_{\bar{F}}) , \\
\tau_s &= \tau - \frac{1}{2}(N_F - N_{\bar{F}}) , \\
\tilde{\Delta}_s &= \tilde{\Delta} - \frac{N_F + N_{\bar{F}}}{2}(\Delta - 1) - \frac{N_F - N_{\bar{F}}}{2}\hat{\Delta} .
\end{aligned} \tag{4.2.4}$$

The phase Φ does not affect the maximization, we can thus perform the maximization on the shifted variable $\tilde{\Delta}_s$, instead of $\tilde{\Delta}$. We do not consider the most general possibility but rather we restrict to two subclasses of examples: (1) $N_F = N_{\bar{F}}$, and (2) $N_{\bar{F}} = 0$, generic N_F. For the class (1) we can apply the parity condition mentioned above to the integral in terms of the shifted variables (4.2.3), because the "effective" $Z_{\mathcal{T}_{3d}}$ given by $e^{N_F \ell(1-\Delta-\hat{\Delta}+ia) + N_{\bar{F}}\ell(1-\Delta+\hat{\Delta}-ia)}$ has the desired symmetry property. As a result in this case we obtain that the maximization gives $\tilde{\Delta}_{s,\text{max}} = \hat{\Delta}_{\text{max}} = 0$ and we only need to maximize with respect to Δ. For the class (2) $\mathcal{Z}^{\text{N}}_{HS^4}$ depends on Δ and $\hat{\Delta}$ only through the combination $\Delta + \hat{\Delta}$, so we can reabsorb $\hat{\Delta}$ in Δ. As a result, to cover both classes of examples it is enough to consider

$$\mathcal{Z}^{\text{N}}_{HS^4} = e^{i\pi\Phi} \int da\, e^{i\pi\tau_s a^2} e^{-2\pi a\tilde{\Delta}_s} e^{N_F \ell(1-\Delta+ia) + N_{\bar{F}}\ell(1-\Delta-ia)} , \tag{4.2.5}$$

and perform extremization with respect to $\tilde{\Delta}_s$ and Δ. We stress that in the most general case of arbitrary N_F and $N_{\bar{F}}$ one would have instead to extremize with respect to three mixing parameters.

For simplicity of presentation we define the variables

$$\lambda^2 = \frac{\mathrm{Im}(\tau_s)}{\tau_s \bar{\tau}_s}\,, \quad \gamma = \frac{\mathrm{Re}(\tau_s)}{\mathrm{Im}(\tau_s)}\,, \quad \mu = \frac{N_F + N_{\bar{F}}}{2}\,, \quad \nu = \frac{N_F - N_{\bar{F}}}{2}\,. \tag{4.2.6}$$

The mixing parameters are then expanded in λ, with coefficients that can be an arbitrary function of the remaining parameters γ, μ, ν. Imposing the extremality constraints

$$\partial_\Delta F_\partial = 0\,, \quad \text{and} \quad \partial_{\tilde{\Delta}_s} F_\partial = 0\,, \tag{4.2.7}$$

we can determine the coefficients of the expansion, order by order in the small λ expansion. We have computed the expansions to order $\mathcal{O}(\lambda^{12})$. The full expression is, however, unwieldy to write down and instead we give up to $\mathcal{O}(\lambda^6)$

$$\begin{aligned}
\Delta_{\max} =& \frac{1}{2} - \frac{\lambda^2}{\pi} + \left(\mu + \left(\frac{2}{\pi}\right)^2 - \gamma^2(\mu+1)\right)\frac{\lambda^4}{2} - \left(\left(\frac{\pi\mu(\mu+1)}{4} + \frac{\pi\nu^2}{6\mu}\right)(1-3\gamma^2)\right. \\
&\left. - \left(\frac{2}{\pi^2} + \frac{3}{4} + \mu\right)\pi\gamma^2 + \left(\frac{2}{\pi}\right)^3 + \frac{2}{3\pi}\right)\lambda^6 + \mathcal{O}(\lambda^8)\,, \\
\tilde{\Delta}_{s,\max} =& \nu\gamma\left(\frac{\lambda^2}{2} - \frac{\pi(2\mu+1)}{4}\lambda^4 + \left(\frac{\pi^2}{24}\left(\mu(\mu+14/3) - 2\nu^2/\mu + 1\right)(1-3\gamma^2)\right.\right. \\
&\left.\left. - \frac{\pi^2}{24}\left(8\mu(\mu+8/3)+5\right) + 2\mu - 1/2\right)\lambda^6\right) + \mathcal{O}(\lambda^8)\,.
\end{aligned} \tag{4.2.8}$$

We reiterate that $\Delta_{\max}$ is the mixing parameter for an ordinary flavor symmetry when $N_F = N_{\bar{F}}$, while it should be interpreted as the mixing parameter $\hat{\Delta}_{\max}$ of the electric $U(1)$ for $N_{\bar{F}} = 0$. The leading order terms in (4.2.8) are consistent with what one expects: when $\lambda \to 0$ the bulk and boundary decouple leaving a set of free chirals with charges $\Delta_{\max} = 1/2$, and $\tilde{\Delta}_{\max} = 0$. Furthermore, for $\nu = 0$ we get $\tilde{\Delta}_{s,\max} = 0$ as expected.[30]

[30] Also the vanishing of $\tilde{\Delta}_{s,\max}$ for $\gamma = 0$ can be explained using simple symmetry arguments. Indeed, for $\gamma = 0$ we have $\bar{\tau}_s = -\tau_s$. Taking the complex conjugate of (4.2.3) and performing the change of variable $a \to -a$ we get $\bar{\mathcal{Z}}^{\mathrm{N}}_{HS^4}(\tilde{\Delta}_s) = \mathcal{Z}^{\mathrm{N}}_{HS^4}(-\tilde{\Delta}_s)$, where we used that $\overline{e^{\ell(z)}} = e^{\ell(\bar{z})}$. As a result F_∂ is even in $\tilde{\Delta}_s$ for $\gamma = 0$ and the unique maximum can only be at $\tilde{\Delta}_s = 0$.

Using the expansion of $\Delta_{\max}$ and $\tilde{\Delta}_{s,\max}$ we can directly expand the partition function itself to the same order. Here we present it up to $\mathcal{O}(\lambda^4)$:

$$\begin{aligned}\mathcal{Z}^{\mathrm{N}}_{HS^4} = \frac{2^{-\mu}e^{\frac{i\pi\nu}{12}}}{\sqrt{-i\tau}}\Bigg(1 - \left(\frac{\pi\mu}{4}(1+i\gamma) - \frac{\nu}{\pi}i\right)\lambda^2 \\ + \Bigg(-\frac{3\pi^2}{32}(-i+\gamma)^2\mu^2 - \frac{\nu^2}{2\pi^2} + \frac{-i+\gamma+2i\pi(-1+\gamma^2)}{4}\mu\nu \\ - \frac{8-\pi^2+16i\gamma-2i\pi^2\gamma+\pi^2\gamma^2}{16}\mu + i\frac{2-4\pi^2+\pi^4\gamma^2}{2\pi^3}\nu\Bigg)\lambda^4 + \mathcal{O}(\lambda^6)\Bigg).\end{aligned} \tag{4.2.9}$$

We can then compute the coefficients of the one-point functions of the Coulomb branch operators $A_{2n} = \phi^{2n}$ using (3.4.11). We have obtained the result up to order $O(\lambda^4)$ for generic n:

$$\begin{aligned}\langle A_{2n}\rangle_{H\mathbb{R}^4} = \frac{(2n)!}{(n-1)!}\left(-\frac{(1+i\gamma)^2\lambda^2}{32\pi^2}\right)^n \Bigg(\frac{1}{n} - \pi\mu\lambda^2 \\ + \frac{\mu}{6}\left(\pi^2(n-1)(2+3\mu) - 24 + 3\pi^2(1+\mu)(1+i\gamma)\right)\lambda^4 + \mathcal{O}(\lambda^6)\Bigg).\end{aligned} \tag{4.2.10}$$

As a consistency check we have performed a Feynman diagram analysis confirming the first two orders in this expression of the one point functions, which can be found in Section 4.2.6.

4.2.2 XYZ on the boundary

As a second example, we couple the bulk field to the 3d $\mathcal{N} = 2$ SCFT known as the XYZ model. Also this example was studied previously in [79] and our overlapping results, namely the perturbative expansion for the mixing parameters, are in agreement. The XYZ model is defined as the IR fixed point of a theory of three chiral fields X, Y and Z deformed by the superpotential XYZ. We need to select a $U(1)$ global symmetry to gauge with the bulk gauge field, and we take it to be the one that gives charge $+1$ to X, -1 to Y and 0 to Z (as above, its mixing parameter is called $\hat{\Delta}$). There is an additional $U(1)$ global symmetry that can mix with the R symmetry, and assigns charge 1 to X and Y, and -2 to Z. We call Δ

the corresponding mixing parameter. The 3d partition function is[31]

$$Z_{\mathrm{XYZ}}(a - i\hat{\Delta}, \Delta) = e^{\ell(\Delta-\hat{\Delta}+ia)+\ell(\Delta+\hat{\Delta}-ia)+\ell(1-2\Delta)} . \tag{4.2.13}$$

This partition function satisfies the parity relation mentioned at the beginning of this section and therefore we can set $\tilde{\Delta} = \hat{\Delta} = 0$ in (2.5.57) and maximize only with respect to Δ. Similarly to the chiral case we use the following parameters for notational convenience

$$\lambda^2 = \frac{\mathrm{Im}(\tau)}{\tau\bar{\tau}} , \quad \gamma = \frac{\mathrm{Re}(\tau)}{\mathrm{Im}(\tau)} . \tag{4.2.14}$$

We have computed the maximizing value of Δ up to order $O(\lambda^{32})$, the first terms are

$$\Delta_{\mathrm{max,XYZ}} = \frac{1}{3} - \frac{2(\sqrt{3}\pi - 9)}{9(4\pi - 3\sqrt{3})}\lambda^2 + \mathcal{O}(\lambda^4) . \tag{4.2.15}$$

It turns out that the expansion of the partition function in this case is computationally more intense. Nevertheless, we have computed it to the order $\mathcal{O}(\lambda^{32})$ as well, albeit for the coefficients of the highest orders we have only the numerical result. The first orders are

$$\mathcal{Z}^{\mathrm{N}}_{HS^4} = \frac{e^{3\ell(1/3)}}{\sqrt{-i\tau}}\left(1 + i\frac{4\sqrt{3}\pi - 9}{27(1+i\sqrt{3})}(1+i\gamma)\lambda^2 + \mathcal{O}(\lambda^4)\right) . \tag{4.2.16}$$

Finally, for the one-point functions we have computed up to order $O(\lambda^{32})$, the first few terms are

$$\langle A_{2n}\rangle_{H\mathbb{R}^4} = \frac{2n!}{(n-1)!}\left(-\frac{(1+i\gamma)^2\lambda^2}{32\pi^2}\right)^n\left(\frac{1}{n} - \frac{2}{9}(3\sqrt{3} - 4\pi)\lambda^2 + \mathcal{O}(\lambda^4)\right) . \tag{4.2.17}$$

Note that the leading order in the one-point function is just fixed by the Neumann boundary condition and does not depend on the boundary degrees of freedom, and in fact it agrees with the result (4.2.10) obtained for the boundary condition with chiral fields on the boundary.

[31] Note that this formula for $Z_{\mathrm{XYZ}}(\xi, \Delta)$ differs from

$$Z^+_{\mathrm{chiral}}(\xi - i(1-\Delta))Z^+_{\mathrm{chiral}}(-\xi - i(1-\Delta))Z^+_{\mathrm{chiral}}(-i2\Delta) , \tag{4.2.11}$$

by the prefactor

$$e^{\frac{i\pi}{4}}e^{i\pi(3\Delta^2 - 2\Delta - \xi^2)} , \tag{4.2.12}$$

which is the contribution of a properly quantized background Chern-Simons term. Therefore we make a definite choice of background contact terms in writing down this formula.

4.2.3 Dualities

As described in Section 4.1 the $SL(2,\mathbb{Z})$ duality group of the supersymmetric Maxwell theory relates different boundary conditions. In this section we consider examples of this action and use them to test the results for the partition function and for the one-point functions. We computed these observables in the previous sections perturbatively in the coupling, by expanding the localization integral. Since the $SL(2,\mathbb{Z})$ action maps weak coupling to strong coupling, the test requires extrapolating the perturbative answers. The tests we perform here therefore serve two complementary purposes: on one hand we use them to verify the localization formulas and their transformations under $SL(2,\mathbb{Z})$, and on the other hand we use them to check the validity of the extrapolations of the perturbative expansions. We consider two examples in which we can obtain predictions for the transformations of the partition functions using certain integral identities, that have already appeared in the literature in the context of 3d dualities, (see for example [11, 81]).

4.2.4 $(N_F, N_{\bar{F}}) = (1,1) \leftrightarrow$ XYZ

The first explicit example we study is the the BCFT of chiral fields with $(N_F, N_{\bar{F}}) = (1,1)$ in the notation of Section 4.2.1. The consequences of the $SL(2,\mathbb{Z})$ duality in this setup were considered before in [78, 79]. Applying an S transformation we find that the BCFT with this boundary condition and coupling τ is dual to the BCFT with coupling $-1/\tau$ and the boundary condition given by (the IR fixed point of) 3d SQED with the matter given by one chiral with charge +1 and one chiral with charge -1, coupled to the bulk via the gauging of the topological $U(1)$ symmetry. Next, we use 3d mirror symmetry that tells us that this SQED theory is dual to the XYZ model, that we considered in Section 4.2.2. Denoting the theory of chirals as $(1,1)$, this 3d duality is verified at the level of the sphere partition function using the following integral identity, called the *pentagon identity* [81],[32]

$$e^{\ell(y+z)}e^{\ell(y-z)} = e^{\ell(2y-1)}\int da\, e^{2\pi z a}\, e^{\ell(1-ia-y)}e^{\ell(1+ia-y)}\ , \tag{4.2.18}$$

which, upon setting $y = \Delta$ and $z = -ix$, implies

$$Z_{S[(1,1)]}(x,\Delta) = \int da\, e^{-2\pi i a x} Z_{(1,1)}(a,\Delta) = Z_{\text{XYZ}}(x,\Delta)\ , \tag{4.2.19}$$

[32]This is an identity of the quantum dilogarithm [82]. It was applied in this context in [11] with the following change of notation: $e^{\ell(-ix)} = s_{b=1}(x)$.

where $Z_{(1,1)}$ and $Z_{\rm XYZ}$ were given in (4.2.1) and (4.2.13), respectively.

As a result we obtain that the BCFT with $(N_F, N_{\bar{F}}) = (1,1)$ chiral fields on the boundary and coupling τ is dual to the BCFT with XYZ model on the boundary and coupling $-1/\tau$. Using (4.1.8) we then obtain

$$\mathcal{Z}^{\rm N,XYZ}_{HS^4}(-1/\tau, -i\hat{\Delta}, +i\tilde{\Delta}, \Delta) = \sqrt{-i\tau}e^{-2\pi i\hat{\Delta}\tilde{\Delta}}\mathcal{Z}^{\rm N,(1,1)}_{HS^4}(\tau, -i\tilde{\Delta}, -i\hat{\Delta}, \Delta)\ . \quad (4.2.20)$$

This equation implies the following relations between the values of the mixing parameter that extremize the partition function (we write the argument simply as τ for convenience, but remember that these functions are real and not holomorphic)

$$\begin{aligned} \tilde{\Delta}^{\rm XYZ}_{\rm max}(-1/\tau) &= -\hat{\Delta}^{(1,1)}_{\rm max}(\tau)\ , \\ \hat{\Delta}^{\rm XYZ}_{\rm max}(-1/\tau) &= \tilde{\Delta}^{(1,1)}_{\rm max}(\tau)\ , \\ \Delta^{\rm XYZ}_{\rm max}(-1/\tau) &= \Delta^{(1,1)}_{\rm max}(\tau)\ . \end{aligned} \quad (4.2.21)$$

The first two equations are trivially satisfied because in both theories $\tilde{\Delta}_{\rm max} = \hat{\Delta}_{\rm max} = 0$ by the symmetry argument explained above. The third line is a non-trivial constraint that relates the weak coupling behavior in one theory to the strong coupling one in the other theory. We test it by performing two independent Padé extrapolations of the perturbative results obtained for $\Delta^{\rm XYZ}_{\rm max}$ and $\Delta^{(1,1)}_{\rm max}$ in the previous section, and checking if the extrapolated functions satisfy the relation (4.2.21). We plot the result as a function of purely imaginary $\tau = \frac{4\pi i}{g^2}$ in Figure 4.1. We see a good agreement between the two sides of (4.2.21) for all values of g^2, interpolating from the value of $1/2$ in the free $(1,1)$ theory to the value $1/3$ of the XYZ theory.

We then plug these values for $\Delta_{\rm max}$ into the partition functions and consider the two sides of Eq. (4.2.19). In order to get rid of prefactors, and to obtain a function which is analytic in the gauge coupling both in the weak and the strong coupling expansion, it turns out to be convenient to compare the combination

$$e^{F_\partial} = \left(\frac{\mathcal{Z}^{\rm N}_{HS^4}\bar{\mathcal{Z}}^{N}_{HS^4}}{\mathcal{Z}_{S^4}}\right)^{-\frac{1}{2}}\ , \quad (4.2.22)$$

which, by (4.2.19), satisfies

$$(e^{F_\partial})^{\rm XYZ}(-1/\tau) = (e^{F_\partial})^{(1,1)}(\tau)\ . \quad (4.2.23)$$

Similar to Δ, we perform independent Padé extrapolations for this quantity, on

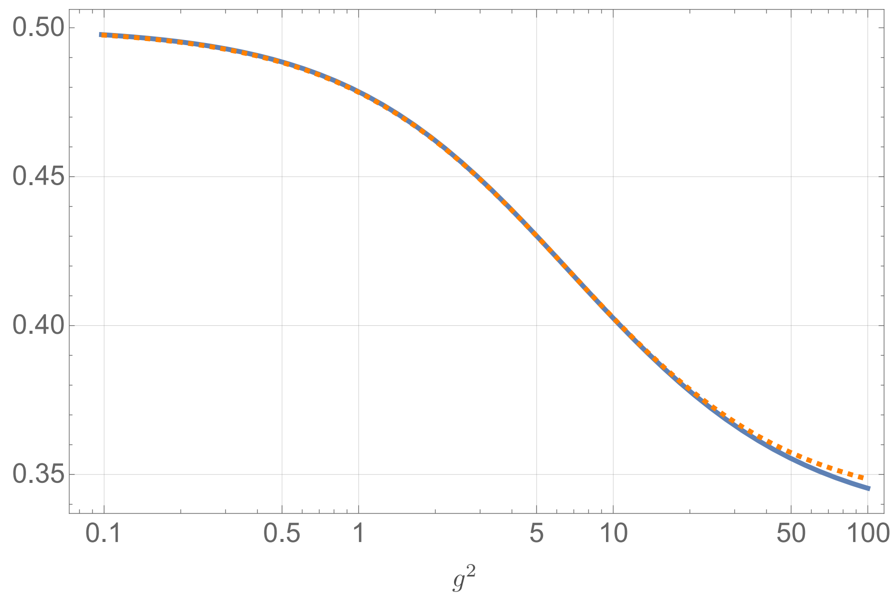


Figure 4.1: The $\{8, 8\}$ Padé approximant of $\Delta^{(1,1)}_{\text{max}}(\tau)$, in blue, and of $\Delta^{\text{XYZ}}_{\text{max}}(-1/\tau)$, in dashed orange, at $\gamma = 0$ as a function of g^2.

the two sides of the duality, using the perturbative results of the previous section and compare them. The results are showed in Figure 4.2.

Additionally, we test the S-duality action on the one-point functions. In the considered example, the S-duality relation (4.1.19) becomes

$$\langle A_n \rangle^{(1,1)}_{H\mathbb{R}^4}(\tau) = \left(-\frac{1}{\tau}\right)^n \langle A_n \rangle^{\text{XYZ}}_{H\mathbb{R}^4}(-1/\tau) \ . \tag{4.2.24}$$

We plot the results of the two independent Padé extrapolations of the two sides in Figure 4.3, for $n = 2, 4, 6, 8$. We see a good agreement for all values of g^2 for $n = 2$. The agreement deteriorates as n increase, the region where the two extrapolations agree progressively shrinks to values of g^2 of order 1. When they do not agree, this means that the number of available perturbative coefficients does not suffice to get a reliable Padé extrapolation for these observables. The lesson we draw is that, perhaps expectedly, the question of how many orders are needed in order to get a reliable strong coupling extrapolation depends on the observable. The fact that the perturbative expansion becomes less and less reliable as we increase the charge n of the Coulomb branch operators is also expected, because the perturbative coefficients grow with n, as exemplified by the first terms in (4.2.10). It should be possible to study a large charge limit of $n \to \infty$ with $g^2 n$ fixed, similarly to what

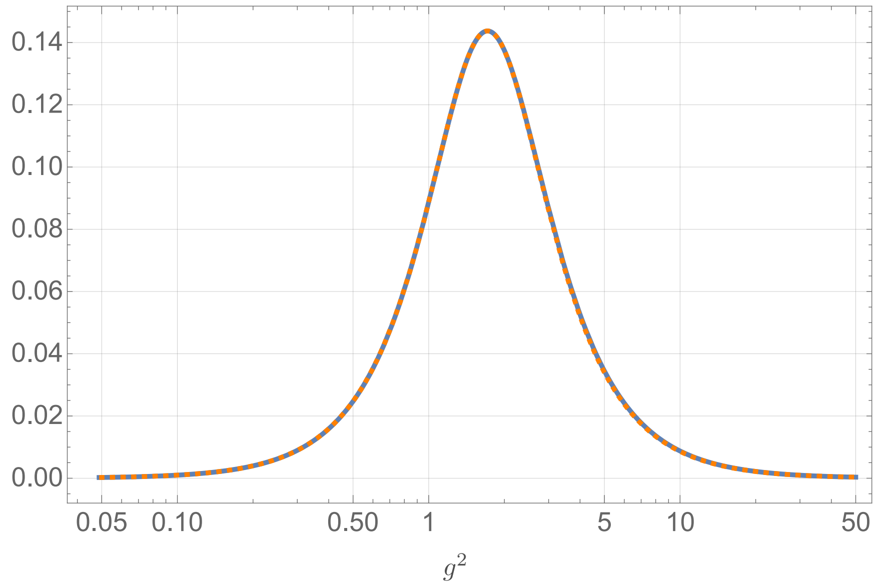


Figure 4.2: The $\{8, 8\}$ Padé approximant of $(e^{F_\partial})^{(1,1)}(\tau)$, in blue, and $(e^{F_\partial})^{\mathrm{XYZ}}(-1/\tau)$, in dashed orange, at $\gamma = 0$ as a function of g^2.

was done in [28] for the bulk two-point functions, in order to obtain exact results in $g^2 n$ in a $1/n$ expansion. We leave this as an interesting direction for the future.

4.2.5 $(N_F, N_{\bar{F}}) = (1, 0)$ triality

Next, we consider the example of the $U(1)$ $\mathcal{N} = 2$ Maxwell theory in the bulk coupled to one chiral field of charge $+1$ on the boundary, i.e. the theory $(1, 0)$ in the notation of Section 4.2.1. Applying the transformation ST^{-1} we obtain that this boundary condition at bulk coupling τ is dual to the boundary condition at bulk coupling $-\frac{1}{\tau-1}$ with the following boundary degrees of freedom: (the IR fixed point of) 3d SQED with matter given by one chiral field of charge $+1$, and one unit of Chern-Simons term, coupled to the bulk via the gauging of the topological $U(1)$ symmetry. The latter theory is typically referred to as $U(1)_{1/2}$ SQED with one chiral, where the half-integer level comes from taking the convention that a chiral field couples to the background gauge field with a $-1/2$ Chern-Simons contact term. This theory is also known as the "tetrahedron theory" because of the role it plays in the construction of 3d theories from compactifications of 6d the $(2, 0)$ theory on 3 manifolds [11].

We then use a 3d duality found in [11] that says that $U(1)_{1/2}$ SQED with one chiral is dual to the free SCFT of one chiral field. This is actually part of a triality,

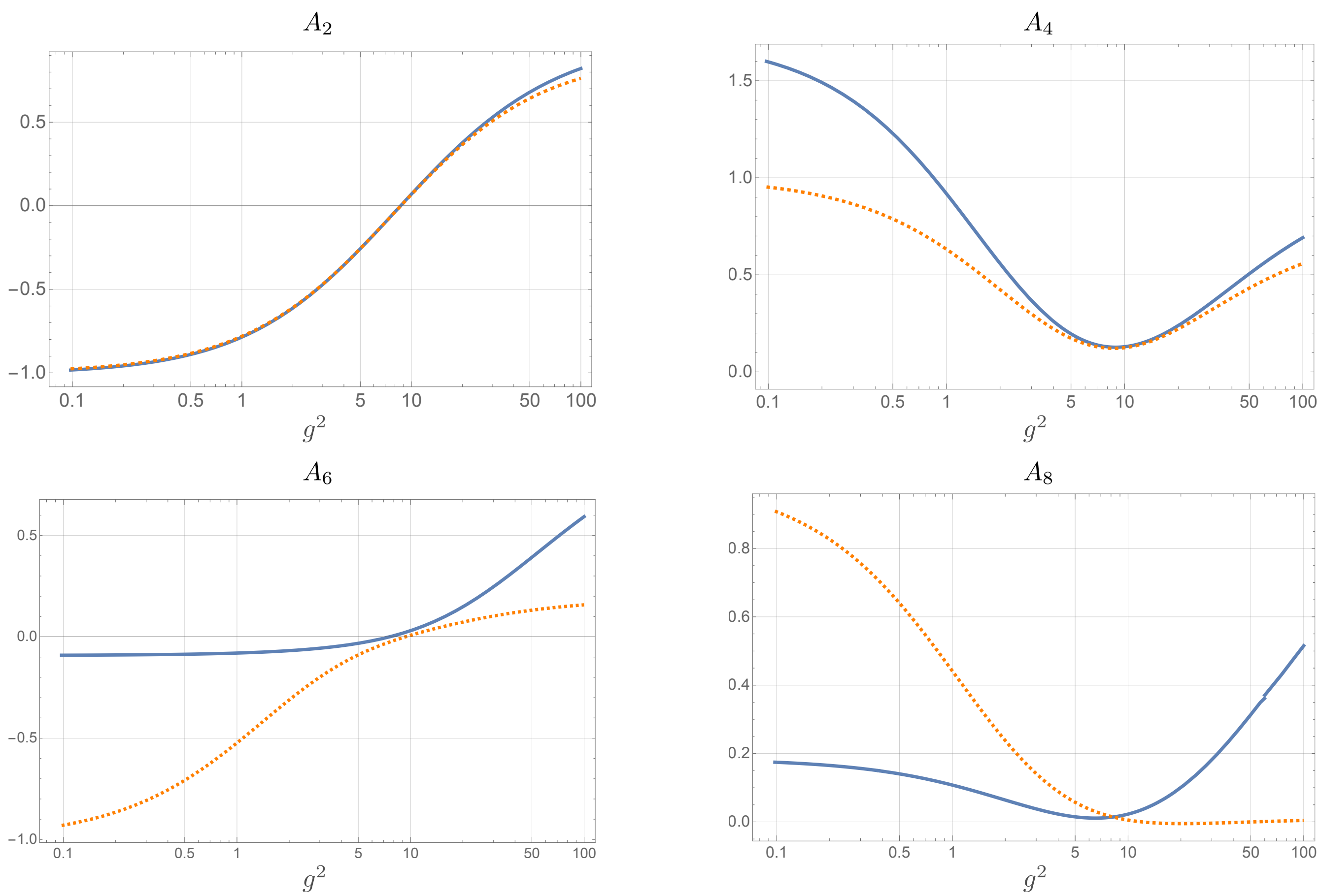


Figure 4.3: The plot labeled by A_{2n} show the $\{8,8\}$ Padé approximants for $(\frac{64\pi^2}{g^2})^n A_{2n}^{(1,1)}(\tau)$, in blue, and $(\frac{64\pi^2}{g^2\tau^2})^n A_{2n}^{XYZ}(-\frac{1}{\tau})$, in dashed orange, at $\gamma = 0$ as a function of g^2. The normalization $(\frac{64\pi^2}{g^2})^n$ is chosen for graphical convenience.

because both theories are also equivalent to $U(1)_{-1/2}$ SQED with one chiral. At the level of 3d partition functions, this duality is proven making use of the following integral identity (see e.g. [36, 82])

$$\int da\, e^{i\pi a^2} e^{2\pi az} e^{\frac{i\pi}{2}(1-r-ia)^2} e^{\ell(1-r-ia)} = e^{-\frac{7\pi i}{12}} e^{-i\pi(r^2-2rz-1)}\, e^{\frac{i\pi}{2}(r-z)^2} e^{\ell(r-z)} \ , \quad (4.2.25)$$

where, in order to make the integral convergent, the contour of the da integration needs to be slightly tilted in the complex plane, i.e. ranging from $-\infty(1+i\epsilon)$ to $+\infty(1+i\epsilon)$.[33] Setting $r = 1/2$ and $z = -ix$ this identity gives

$$Z_{ST^{-1}[(1,0)]}(x) = \int da\, e^{-2\pi iax} e^{i\pi a^2}\, Z_{(1,0)}(a - \tfrac{i}{2}) = e^{\frac{i\pi}{6}} e^{\pi x}\, Z_{(1,0)}(-x - \tfrac{i}{2}) \quad (4.2.26)$$

where $Z_{(1,0)}$ was given in (4.2.1), and the second argument of the function can be dropped in this case because Δ can be reabsorbed in $\hat{\Delta}$. The factors of $e^{\frac{i\pi}{6}}$ and $e^{\pi x}$ on the right-hand side correspond to background Chern-Simons terms that need to be added in order for the duality to work. Up to these contact terms, the transformation ST^{-1} therefore leaves the theory $(1,0)$ invariant. The fact that ST^{-1} is a transformation of order 3 implies that in fact this must be a triality, consistent with the claims that we reviewed above.

This triality implies that the BCFT with boundary condition given by the single free chiral and coupling τ is left invariant under $\tau \to ST^{-1}[\tau] = -\frac{1}{\tau-1}$ and $\tau \to (ST^{-1})^2[\tau] = \frac{\tau-1}{\tau}$. More precisely, we need to take into account also the addition of background Chern-Simons terms on the left-hand side of (4.2.26). Starting with (4.2.26) and performing manipulations analogous to those that lead to (4.1.5)-(4.1.8), we obtain the following identity for the hemisphere partition function with $(1,0)$ boundary condition

$$\begin{aligned} &\mathcal{Z}^{\mathrm{N},(1,0)}_{HS^4}(-\tfrac{1}{\tau-1}, -i\hat{\Delta}, i(1+\tilde{\Delta}+\hat{\Delta})) \\ &\qquad = \sqrt{\tau-1}\, e^{\frac{4\pi i}{3}} e^{i\pi\hat{\Delta}\left(\hat{\Delta}-2\tilde{\Delta}\right)} \mathcal{Z}^{\mathrm{N},(1,0)}_{HS^4}(\tau, -i\tilde{\Delta}, -i\hat{\Delta}) \ . \end{aligned} \quad (4.2.27)$$

Some of the consequences of this triality on the BCFT setup were studied previously in [79]. Note that the phase factor on the right hand side does not enter in the

[33] To ensure convergence, also the variable z needs to be restricted to a certain region in the complex plane, and outside of that region the identity should be interpreted in the sense of analytic continuation. We have not tried to fully determine the region of convergence, but we have checked numerically convergence and the validity of the identity in the region of small $|z|$ and $\mathrm{Arg}(z)$ in a right neighborhood of $-\frac{\pi}{2}$.

extremization. Performing extremization to determine $\tilde{\Delta}$ and $\hat{\Delta}$, this relation between the partition functions implies that

$$\begin{aligned} \tilde{\Delta}^{(1,0)}_{\max}(\tau) &= -\hat{\Delta}^{(1,0)}_{\max}(-\tfrac{1}{\tau-1})\ , \\ \hat{\Delta}^{(1,0)}_{\max}(\tau) &= 1 + \tilde{\Delta}^{(1,0)}_{\max}(-\tfrac{1}{\tau-1}) - \hat{\Delta}^{(1,0)}_{\max}(-\tfrac{1}{\tau-1})\ . \end{aligned} \tag{4.2.28}$$

This transformation on the functions $\tilde{\Delta}$ and $\hat{\Delta}$ is easily verified to have order three, as it should. Moreover, it allows to solve for the values at the self-dual point $\tau = e^{i\frac{\pi}{3}}$, giving

$$\tilde{\Delta}^{(1,0)}_{\max}(e^{\frac{i\pi}{3}}) = -\frac{1}{3}\ , \quad \hat{\Delta}^{(1,0)}_{\max}(e^{\frac{i\pi}{3}}) = \frac{1}{3}\ . \tag{4.2.29}$$

Consistently, plugging these values into (4.2.27) one finds that the whole prefactor, including the $\sqrt{\tau-1}$, simplifies to 1. We can now proceed with the numerical tests.

Firstly, we check (4.2.29) via a $\{7,7\}$ Padé extrapolation from weak coupling along the complex curve $\tau = \frac{1}{2} + i\frac{4\pi}{g^2}$

$$\begin{aligned} \tilde{\Delta}^{(1,0)}_{\max,\{7,7\}}(e^{i\frac{\pi}{3}}) + \frac{1}{3} &\approx -0.001\,, \\ \hat{\Delta}^{(1,0)}_{\max,\{7,7\}}(e^{i\frac{\pi}{3}}) - \frac{1}{3} &\approx -0.007\,, \end{aligned} \tag{4.2.30}$$

and we find reasonable accordance.

The other checks we perform utilize a $\{7,7\}$ Padé extrapolation along the complex curve $\tau = 1 + i\frac{4\pi}{g^2}$ from both duality frames. In Figure 4.4, Figure 4.5 and Figure 4.6 we show checks for (4.2.28), for e^{F_∂} and for the one-point function of ϕ^2, which satisfies the relation

$$\langle A_2^{(1,0)}\rangle(\tau) = \left(-\frac{1}{1-\tau}\right)^2 \langle A_2^{(1,0)}\rangle\left(-\frac{1}{1-\tau}\right) \tag{4.2.31}$$

that can be obtained via (4.1.14)-(4.1.19). We find a good agreement with the predictions of the triality for the real part of the one-point function and for e^{F_∂}, but generally the extrapolations appear less accurate when the θ-angle is non-vanishing.

4.2.6 Feynman diagrams calculations

In this section, we confirm the first two orders in (4.2.17) via a Feynman diagram calculation. In order to compute the one-point function of $A_{2n} = \phi^{2n}$, we repeatedly

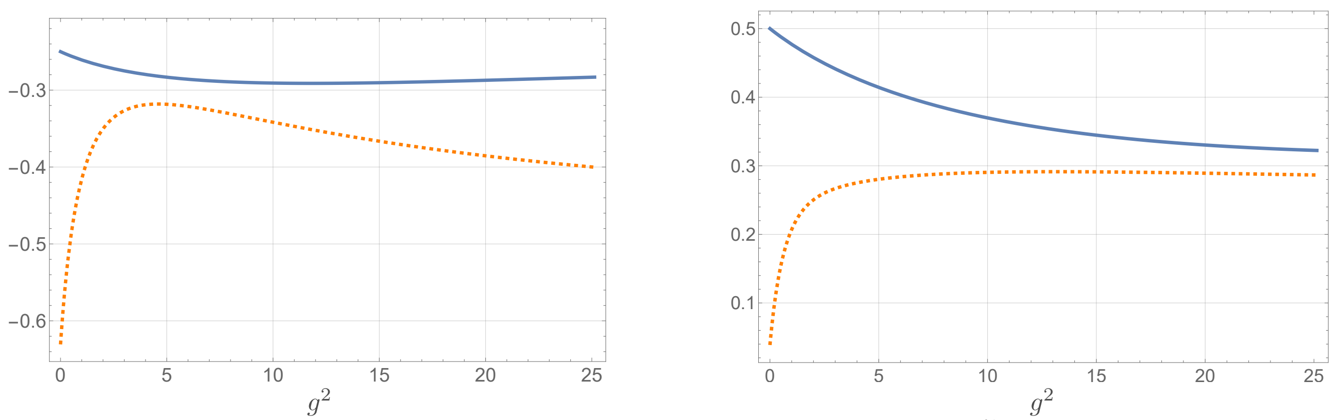


Figure 4.4: On the left, the $\{7,7\}$ Padé approximant for $\tilde{\Delta}^{(1,0)}_{\text{max}}(\tau)$, in blue, and $-\hat{\Delta}^{(1,0)}_{\text{max}}(-\frac{1}{1-\tau})$, in dashed orange. On the right, the $\{7,7\}$ Padé approximant for $\hat{\Delta}^{(1,0)}_{\text{max}}(\tau)$, in blue, and $1+\tilde{\Delta}^{(1,0)}_{\text{max}}(-\frac{1}{1-\tau})-\hat{\Delta}^{(1,0)}_{\text{max}}(-\frac{1}{1-\tau})$, in orange. The functions are evaluated at $\tau = 1 + i\frac{4\pi}{g^2}$, and plotted as a function of g^2.

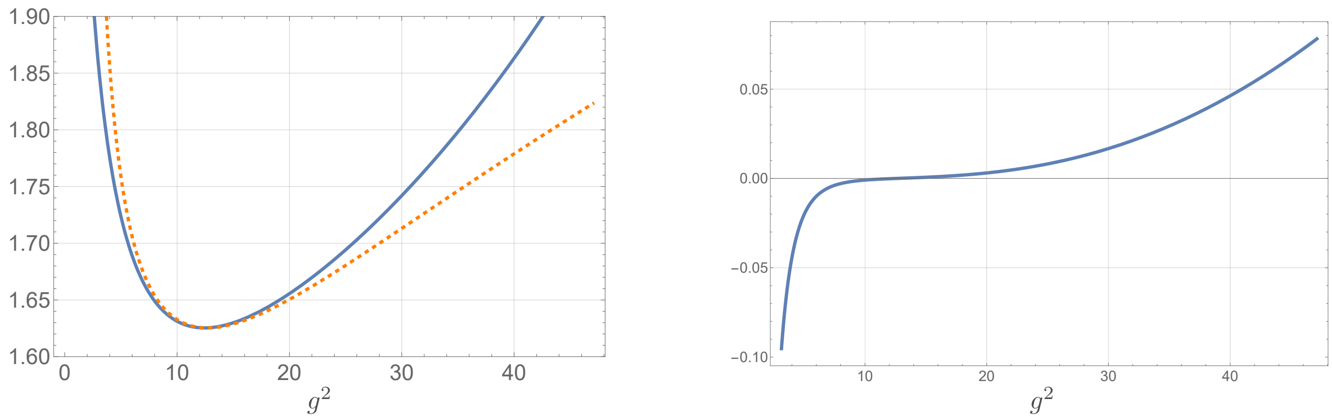


Figure 4.5: On the left, the $\{7,7\}$ Padé approximant for $(e^{F_\partial})^{(1,0)}(\tau)$, in blue, and $(e^{F_\partial})^{(1,0)}(-\frac{1}{1-\tau})$, in dashed orange. On the right, the relative error (the difference between the two functions divided by the arithmetic mean). The functions are evaluated at $\tau = 1 + i\frac{4\pi}{g^2}$, and plotted as a function of g^2.

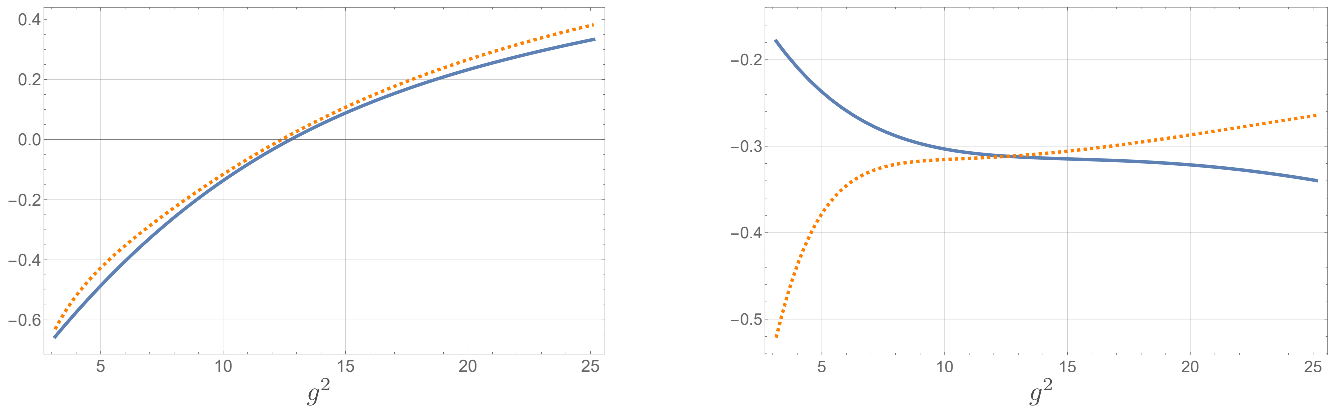


Figure 4.6: On the left, the real part of the $\{7,7\}$ Padé approximant of $\frac{64\pi^2}{g^2}A_2(\tau)$, in blue, and of $\frac{64\pi^2}{g^2}(-\frac{1}{1-\tau})^2 A_2(-\frac{1}{1-\tau})$, in dashed orange, evaluated at $\tau = 1 + i\frac{4\pi}{g^2}$, as a function of g^2. On the right the imaginary part. The normalization $\frac{64\pi^2}{g^2}$ is chosen for graphical convenience.

use the OPE in the bulk to bring $2n$ insertions of the field ϕ at the same point.

Tree-level

At tree-level we have

$$\langle\phi(x_1)\dots\phi(x_{2n})\rangle = \langle\phi(x_1)\phi(x_2)\rangle\dots\langle\phi(x_{2n-1})\phi(x_{2n})\rangle + \text{other Wick's contractions}\ , \tag{4.2.32}$$

with $(2n-1)!! = (2n-1)(2n-3)...1$ addends. Sending $x_i \to x$ we remain with

$$\langle\phi^{2n}(x)\rangle = (2n-1)!!\langle\phi^2(x)\rangle^n \tag{4.2.33}$$

which leads to

$$a_{\phi^{2n}} = (2n-1)!!a^n_{\phi^2}\ . \tag{4.2.34}$$

This matches the leading order in (4.2.17) upon using the result in (B.2.11). Note that the result of the perturbative calculation is naturally a function of τ_s rather than τ, if we include a shift of the boundary Chern-Simons term compatible with the charge assignment of the boundary chiral fields.

1-loop computations

We now need to add boundary interaction vertices that shift the above result by $\delta a^{N_F,N_{\bar{F}}}_{\phi^n}$.

A generic diagram looks like what is depicted in Figure 4.7. Each possible

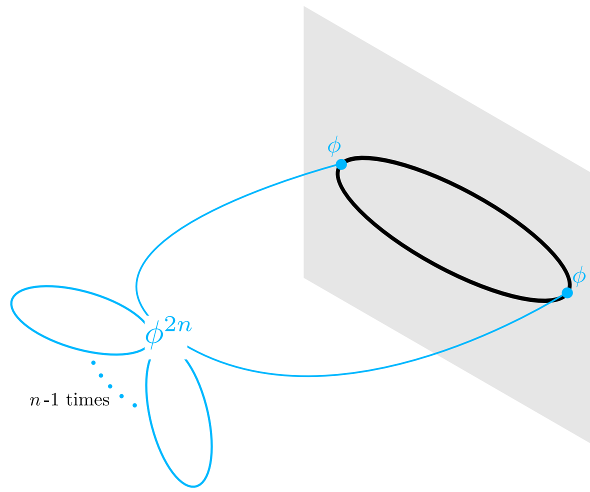


Figure 4.7: Schematic representation of the NLO diagrams needed for the computation of $\langle \phi^{2n} \rangle$.

contraction in the bulk can be seen as a correction to $a^{\text{tree}}_{\phi^2}$ multiplied by $n-1$ closed loops. Each of these contributions has multiplicity $n(2n-1)!!$[34]. The contractions in the boundary simply produce the 1-loop diagrams computed in Appendix C.3. The NLO contribution can then be written as follows

$$\delta a^{(N_F,N_{\bar{F}})}_{\phi^{2n}} = n(2n-1)!!(a^{\text{tree}}_{\phi^2})^{n-1}\delta a^{(N_F,N_{\bar{F}})}_{\phi^2} \,. \tag{4.2.35}$$

It is easy to note that at the 1-loop order holds

$$\delta a^{(N_F,N_{\bar{F}})}_{\phi^2} = (N_F + N_{\bar{F}})\delta a^{(1,0)}_{\phi^2} \,, \tag{4.2.36}$$

thus we obtain

$$\delta a^{(N_F,N_{\bar{F}})}_{\phi^{2n}} = (N_F + N_{\bar{F}})n(2n-1)!!(a^{\text{tree}}_{\phi^2})^{n-1}\delta a^{(1,0)}_{\phi^2} \,. \tag{4.2.37}$$

Using (B.2.11) and (C.3.6), again with $\tau \to \tau_s$, we find

$$\delta a^{(N_F,N_{\bar{F}})}_{\phi^{2n}} = -(N_F + N_{\bar{F}})\frac{n(2n-1)!!}{32\pi}\left(\frac{i}{8\pi(\tau_s - \bar{\tau}_s)}\frac{\bar{\tau}_s}{\tau_s}\right)^{n-1} , \tag{4.2.38}$$

in agreement with the NLO result shown in (4.2.10).

[34]There are $\frac{2n(2n-1)}{2} = n(2n-1)$ ways of choosing two ϕ operators between the $2n$ that contract on the boundary. Then there are $(2n-3)!!$ ways of pairing the remaining $2n-2$ fields in the bulk.

Chapter 5

Higgsing Transition in AdS_4

In this Chapter we first introduce the reader to the concept of using AdS as a means to study phases of strongly coupled gauge theories. We later discuss our results obtained so far using these AdS techniques in conjunction with supersymmetric localization.

The concept of AdS as an infrared regulator to study phase transitions was introduced in [12]. The reason for that is that in AdS we have exponential damping in correlators function but still infinite volume, thus, in contrast to the sphere or dS, one can still use it to study spontaneous symmetry breaking and phase transitions. Furthermore this regulator preserves a spacetime symmetry group as large as the Poincaré group (as we remove the regulator the AdS isometries generators become the Poincaré ones). Many tools that are useful to study QFT in AdS can be borrowed from holography, which indeed reduces to it in the limit when gravity decouples, which is the case we focus on. As an example of a useful concept that is familiar from holography, but still applies to QFT in AdS, correlators of boundary operators are covariant under the conformal symmetry group, simply as a consequence of the isometry of the background.[35]

In our case we study a pure (super) Yang-Mills theory and as such the theory is completely controlled by the adimensional quantity $\Lambda_{\mathrm{QCD}} L_{\mathrm{AdS}}$. Thus there is *no distinction between the flat-space limit and going to strong-coupling* or, conversely, having high curvature and having small coupling.

Interestingly, as pointed out in [13], this setup allows us to sharply define and

[35]The operators of the boundary are not the ones of a local CFT. Indeed, for examples, without dynamical gravity in the bulk we do not have a stress-tensor on the boundary. It is however true that the operators that we define/restrict on the boundary naturally transform under the conformal group.

study a confinement-deconfinement transition. When the AdS radius is small, and we are in a perturbative regime, it is possible to assign Dirichlet boundary conditions which restrict gauge transformations to be global on the boundary. This implies the existence of a boundary current associated to the gauge group, which can be thought, via state/operator correspondence, as the AdS analogue of an asymptotic gluon state in flat space. For this reason, the Dirichlet boundary condition can be thought of as a deconfined phase of the theory. As the AdS radius is increased and the coupling gets stronger, at some critical value this boundary condition should stop being available. This is required by consistency with the flat-space limit, in which no colored asymptotic states exist. Understanding the disappearance of the Dirichlet condition from the point of view of the boundary CFT is therefore very interesting because it offers a new window through which one can explore the mechanism for confinement in flat space. Various possible mechanisms for this transitions have been proposed conjecturally in [13] and have been recently explored using perturbation theory in [14]. For instance, one possible mechanism, dubbed *decoupling* in [13], concerns the vanishing of the coefficient denoted c_j, the two-point function coefficient of the conserved current. This mechanism is the one we focus on in this thesis.

The two-point function of the current is forced to be strictly positive since it is linked to the norm of the asymptotic gluon state. Then if it happens to decrease to zero at some point, as we increase the radius, this implies that the gluon state becomes null and should be excluded from the spectrum, signaling the transition to a phase without boundary global symmetry. Studying this boundary current two-point function then gives one possible quantitative way to argue the end of the deconfined phase, as was studied for instance in [14].

5.1 Higgsing transition in $SU(2)$ SYM

The transition away from the deconfined phase happens at strong coupling, which makes it hard to study it quantitatively. That is why so far there are only perturbative indications about the nature of the transition for pure YM theory. Our goal is to exploit supersymmetry and localization to make more controlled statements about this transition. The physical set up we want to consider is pure $SU(2)$ $\mathcal{N} = 2$ SYM theory in AdS$_4$ with Dirichlet boundary condition. We expect from [83] that in the flat-space limit, and thus of strong coupling for the discussion above, the theory should Higgs down to a $U(1)$ gauge theory if we stay at the origin of the Coulomb branch. Note that differently from the non-supersymmetric case of pure YM, in this case the transition is not of deconfined-confined type, but rather between a fully deconfined non-abelian Coulomb phase with $SU(2)$ symmetry, to an ordinary Coulomb phase with only $U(1)$ global symmetry on the boundary. Still, the conclusion that the (full) Dirichlet boundary condition must disappear at strong coupling remains valid.

In order to apply this logic we need to choose boundary conditions in AdS in such a way to preserve the maximal amount of isometries and supersymmetries, so that in particular we have the same number of symmetry generators as for the theory in flat space. For the reasons explained in 2.6.2, we claim to achieve this by imposing boundary conditions that explicitly preserve 3d $\mathcal{N} = 2$ supersymmetry and then by employing F_∂-maximization to restore the full 4d $\mathcal{N} = 2$ supersymmetry.

5.2 Higgsing from F_∂-maximization

The partition function of the 4d $\mathcal{N} = 2$ pure SYM on AdS$_4$, as a function of the dynamically generated scale, is derived in Appendix D (see (D.4.2)). We write it below, specializing to $SU(2)$, for convenience

$$Z^{\mathrm{D}}_{\mathrm{vector}}(a, \Lambda L) = (\Lambda L)^{4a^2} \frac{2a}{\sinh(2\pi a)} G(1 + 2ia) G(1 - 2ia) Z_{\mathrm{Nekrasov}}(a, \Lambda L)\,. \quad (5.2.1)$$

The partition function depends on the asymptotics of ϕ_1 (2.5.35). This boundary value generates a profile for the scalar in AdS$_4$ which leads the theory to a Higgs phase where the gauge symmetry is broken to $U(1)$. As we explain in more details below, note that $a \neq 0$ corresponds to turning on a real mass on the boundary, which also breaks conformal symmetry.

The "explicit" breaking due to $a \neq 0$ is put by hand, while we are looking for a mechanism that dynamically breaks $SU(2)$ as we increase the radius, without affecting the conformal symmetry. We therefore concentrate on $a = 0$ in (5.2.1), thus we enforce $SU(2)$-preserving boundary conditions.

The key to this dynamical mechanism is that, as explained in Section 2.6.2, starting from a Lagrangian formulation that preserves explicitly $\mathcal{N} = 2$ 3d supersymmetry, the partition function that preserves the full AdS$_4$ supersymmetry and that allows to recover the $\mathbb{R}^4$ physics at strong coupling/large AdS radius is obtained through a maximization procedure.

If the theory Higgses down to $U(1)$, then there is a $U(1)$ global symmetry generated by the multiplet $\mathcal{J}_{\mathrm{N}}$, whose source is precisely the vector multiplet in (1.5.5) that on the boundary dualizes to $\mathcal{J}_{\mathrm{D}}$ (as explained in Section 2.5.7).

a is the real mass deformation for this $U(1)$ and, thanks to the discussion under equation (2.5.52), we know that the partition function depends holomorphically on $a - i\delta$, with δ the mixing parameter of a trial $\mathcal{R}$-charge. Since we set $a = 0$, we study F_∂ maximization with $Z^{\mathrm{D}}_{\text{vector}}(\tau, -i\delta)$ (5.2.1).

A maximum at a non-zero value of δ means that a $U(1)$ subgroup of the $SU(2)$ global symmetry mixes with the superconformal $U(1)_{\mathcal{R}}$ symmetry of the boundary. This means that the corresponding boundary condition, while preserving the superconformal symmetry, breaks $SU(2)$ to $U(1)$. We parametrize this possibility by studying the F_∂ maximization at different values of ΛL as a function of δ and, if a local maximum arises at $\delta \neq 0$ with a greater F_∂-value compared to $\delta = 0$, we interpret this as the destabilization of the $SU(2)$-preserving boundary condition towards the $U(1)$-preserving one. When this happens we declare the theory to be in the Higgs phase.

In the next section we present the plots related to this study. We write Λ instead of ΛL to ease the notation. Furthermore we call

$$F_\partial = -\frac{1}{2}\log(|Z^{\mathrm{AdS}_4}|^2) \tag{5.2.2}$$

since the partition function on S^4 we would have from equation (2.6.1) does not depend on δ and thus would not influence the study of maximization. One practical difficulty that we encounter is that Z_{Nekrasov} cannot be expressed in closed form, but rather it is known as an infinite series expansion in the instanton number. In order to study the minima numerically, we need to truncate this expansion to a certain order. To asses the reliability of the results, we then test how they change by varying the order of the truncation in the instanton number.

5.3 Results

We start a preliminary analysis with the plot in Figure 5.1. We plot three different $F_\partial(-i\delta, \Lambda)$ at different fixed values of the truncation in the instanton number and of Λ, as a function of δ.

Let us explain some qualitative features of the function. We observe that for small Λ the functions obtained by truncating the sum over instantons have a

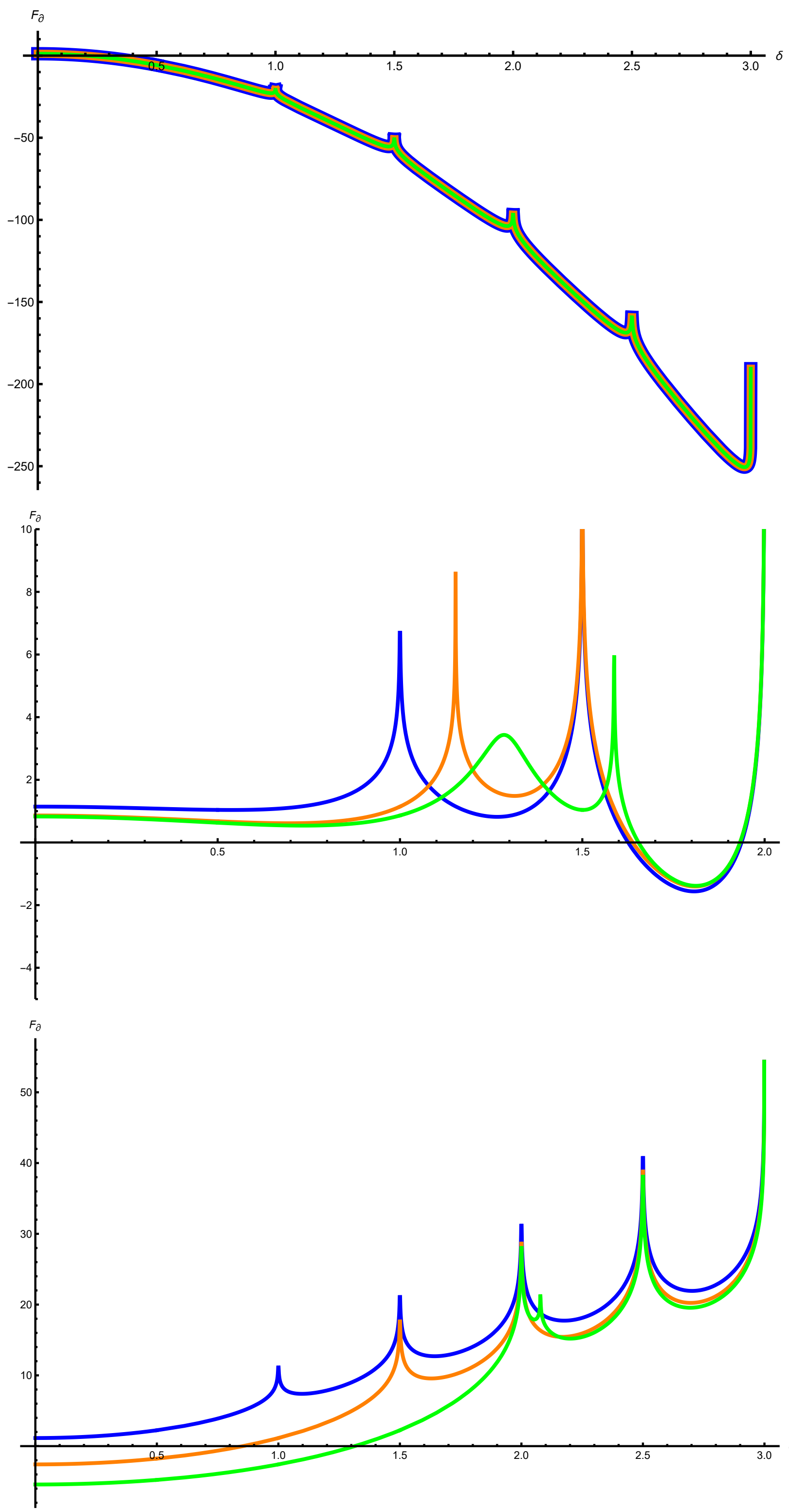


Figure 5.1: The diverse curves are $F_\partial(\Lambda, -i\delta)$ at different perturbative order in instanton number. The blue one has only the leading order, the orange one has one instanton correction and the green has two instanton corrections. The first figure is at $\Lambda = 0.001$, the second is at $\Lambda = 0.9$ and the third at $\Lambda = 3$.

local maximum at $\delta = 0$, and as Λ is increased additional local maxima emerge at non-zero value of δ. While these features depend on the truncation of the instanton series, we observe that they stabilize after a certain number of instantons is included. The number of instantons needed for the stabilization itself depends on the value of the coupling Λ: for larger Λ more terms in the instanton series are needed to stabilize the function.

We define $\Lambda_{1,\text{crit.}}$ as the value of Λ where a new maximum appears and $\Lambda_{2,\text{crit.}}$ as the value where it disappears. We also define Λ_{c_j} as the value where the local maxima at the origin disappears. We reiterate that these values can actually depend on the instanton number truncation, we will comment on their stability/instability later on.

In this preliminary analysis we notice at $\Lambda \in [\Lambda_{1,\text{crit.}}, \Lambda_{2,\text{crit.}}]$ the emergence of a bigger maxima, compatible with the fact that we know the theory Higgses in flat space, but in contradiction with this expectation $\Lambda_{2,\text{crit.}}$ is finite and thus this new maxima seems to disappear in the deep IR. Perhaps even morse surprisingly we have some finite $\Lambda_{c_j} > \Lambda_{2,\text{crit.}}$, thus no stable point seems remaining in the deep IR.

In the plot we can also see singularities, these are however pushed away to larger and larger values of δ as we increase the number of instantons. We thus interpret these just as an artifact of the truncation in the number of instantons.

To clarify the results we study $\Lambda_{1,\text{crit}}, \Lambda_{2,\text{crit}}, \Lambda_{c_j}$ as we increase the number of instantons. We also introduce $\Lambda_{\text{reliable}}$, the largest value of Λ for which we deem reliable the sum truncated up to instanton number k. We use the following criterion to asses reliability: for a given k and for $\Lambda \geqslant \Lambda_{1,\text{crit}}$, we look at the location of the new maximum $\delta_{\text{crit}} = \delta_{\text{crit}}(k, \Lambda)$. As we increase the truncation, the location of δ_{crit} initially changes significantly until at the value k it stabilizes, for all values of the coupling lower than $\Lambda_{\text{reliable}}(k)$. The results can be found in Figure 5.2.

$\Lambda_{\text{reliable}}$, as we would expect, increases as we increase the truncation. $\Lambda_{1,\text{crit.}}$ is always reliable by construction, since the definition of $\Lambda_{\text{reliable}}$ is based on the stabilization of the second maxima. It is however a non trivial confirmation of our choice to assess reliability the fact that $\Lambda_{1,\text{crit.}}$ remains stable as k is increased.

We also see that the values of $\Lambda_{2,\text{crit.}}$, Λ_{c_j}, at each fixed number of instantons, are always above $\Lambda_{\text{reliable}}$ and their value does not remain stable as we increase the number of instantons. Extrapolating from the trend displayed in Figure 5.2, this seems to be the case because all these values are pushed to infinity as we remove the truncation and we study the exact physics of this system.

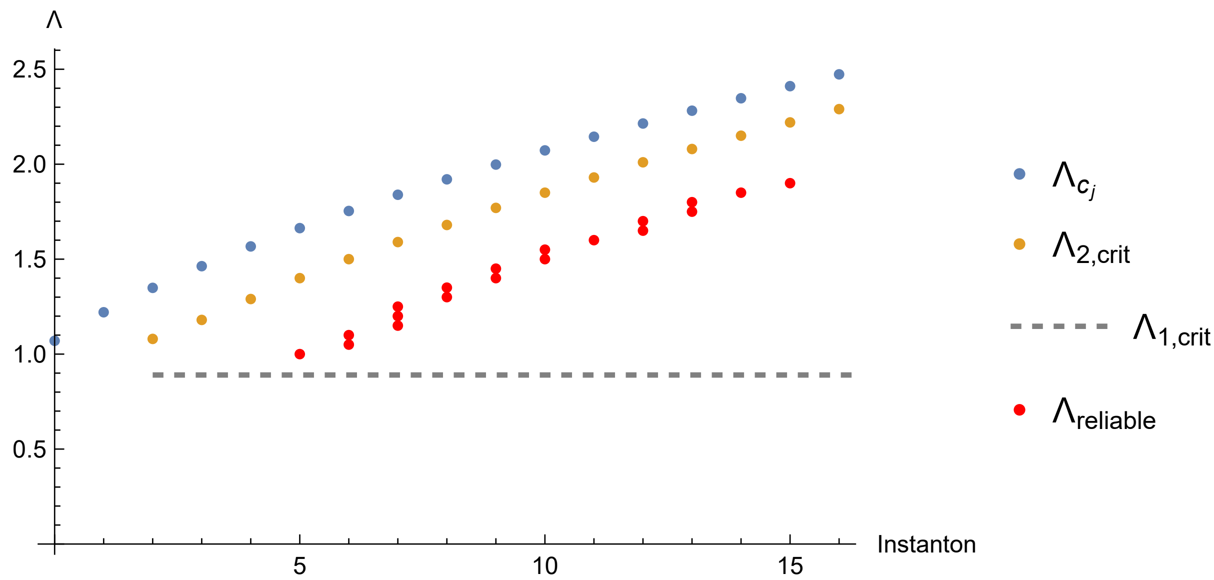


Figure 5.2: In this plot we show the values of $\Lambda_{1,\text{crit.}}$, $\Lambda_{2,\text{crit.}}$, Λ_{c_j}, $\Lambda_{\text{reliable}}$ as we increase the number of instantons. We represent $\Lambda_{1,\text{crit.}}$ as a dashed line because its value remains stationary across diverse instanton numbers.

Our analysis focuses around the position of the new maxima and on a much higher instanton truncation with respect to the preliminary analysis of Figure 5.1. The singularities are then pushed far enough from the region of interest that we thus deem reliable. From this study we can then see that this new maxima does not just come from having smoothed out a singularity, it arises from $F_\partial(\Lambda, -i\delta)$ at $\Lambda > \Lambda_{1,\text{crit.}}$ within the δ-region we can trust.

This implies that that there is a second maximum of F_∂ at $\delta \neq 0$ and that it survives in the deep IR. It is also true that the local maximum at $\delta = 0$ seems surviving as well, but it is always smaller than the second one we find. While we cannot rigorously prove that the $SU(2)$-preserving boundary condition at the origin stops being consistent, we take the emergence of this larger maximum as a strong signal that the theory Higgses after a certain value of the coupling/AdS scale and remains in the Higgs phase in the deep IR limit/flat space limit, consistently with [16].

Chapter 6

String-theoretic BCFTs and $\mathbb{S}$-duality

In this last section we aim to study string theoretic constructions of a class of 4d $\mathcal{N} = 2$ gauge theories, circular quiver theories of A_{N-1}-type, as the ones depicted in Figure 6.1.

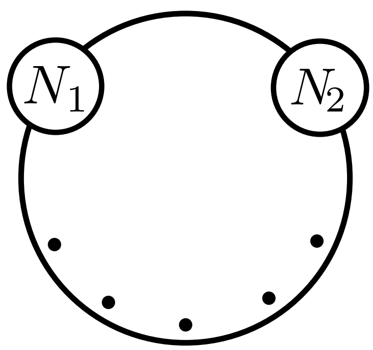


Figure 6.1: The following picture describes a gauge theory with a $SU(N)$ $\mathcal{N} = 2$ vector multiplet for each node. Every link that starts from a i^{th} note and ends into a j^{th} represents a 4d $\mathcal{N} = 2$ hypermultiplet in the $(N_i, \bar{N}_j)$ representation.

This class of theories is prototypical especially because it is possible to describe them in M-theory [19], which allows a geometrization that manifests explicitly the duality group.

We aim to add a boundary in these theories and describe a strong-weak duality action in this set up [20], *which we denote as* $\mathbb{S}$*-duality*, similarly to what was done in [17, 18]. We proceed by describing the set up from the point of view of Type IIA, its enhancement to M-theory and thus the description of the duality group for the theory without a boundary. We also describe the dual Type IIB set up and the interpretation of this $\mathbb{S}$-duality action.

We then proceed to add the boundary, discuss the preserved supersymmetry, the $\mathbb{S}$-duality and a preliminary analysis of the boundary conditions imposed by the different branes.

For convenience we call "#" the M-theory's 10^{th} direction.

6.1 Bulk construction in Type IIA and M-theory

In Figure 6.2 we depict the Type IIA brane set up, which we describe in Table 6.1.

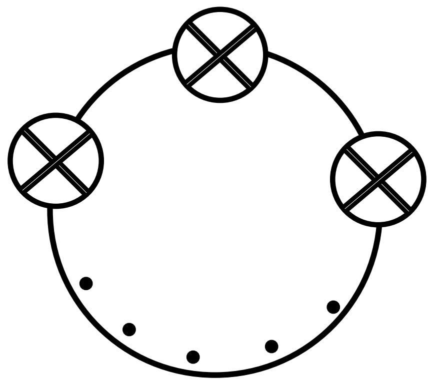


Figure 6.2: The picture shows N parallel and superposed D4, along the 4 directions, that intersect K NS5s

	012	3	$\underline{4}$	5	6	7	8	9
D4	•	•	•					
NS5	•	•		•	•			

Table 6.1: In this table we show in which directions the branes depicted in Figure 6.2 extend. The 4-direction is a circle.

We call the total length of the 4-circle as $L^{(4)}$ and the distance between the i^{th} and $(i+1)^{\text{th}}$ NS5 as $L_i^{(4)}$, so we have $\sum_{i=1}^{K} L_i^{(4)} = L^{(4)}$. In principle one could expect to have $U(N)$ gauge theories living on each slab but actually all the $U(1)$ factors are frozen except a diagonal one, that decouples (all the hypermultiplets are not charged under this [19]). For this reason it is effectively a theory of $SU(N)$ gauge nodes. On the D4 slabs there are strings that describe a gauge theory and across the NS5 there are strings which we interpret as the hypermultiplets in the bifundamental.[36]

In M-theory all of these D4s and NS5s can be seen as the complicated worldvolume of a M5 [19]. We depict the set up in Table 6.2 and in Figure 6.3

	012	3	**4**	5	**6**	7	8	9	#
M5	•	•	•						•
M5	•	•		•	•				

Table 6.2: This is the M-theory up-lift of Table 6.1. From the M-theory point of view the final theory is actually just a single M5 with a more complicated world-volume.

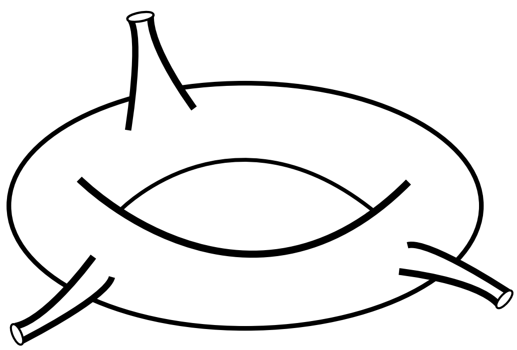


Figure 6.3: This figure displays a M5 that wraps the 4#-torus and the spikes represent the M5 leaving the 4#-directions. In the Type IIA limit we shrink the #-circle and all the points on the torus are D4s except those of the spikes which become the NS5s.

We can then define the asymptotic position p_i where the M5 extends outside the 4#-torus (and becomes a NS5 in the Type IIA limit). We decide to define and normalize this position in the following way [19, 85]

$$p_i = \frac{1}{g_s}\frac{x_i^4}{L^{(4)}}i + \frac{\theta}{2\pi}\frac{x_i^{\#}}{L^{(\#)}}\,, \tag{6.1.1}$$

[36]These states can only be argued indirectly since, in the duality frame where we quantize the F1, they probe the core of NS5s which are non-perturbative objects [84]

with[37]

$$x^a_{i+K} = x^a_i + L^{(a)}\,,\ \text{with } a = 4,\,\#\ \text{and } x^4_{i+1} > x^4_i\,. \tag{6.1.2}$$

Defining as usual

$$\tau = \frac{i}{g_S} + \frac{\theta}{2\pi}\,, \tag{6.1.3}$$

we can rewrite (6.1.2) as

$$p_{i+K} = p_i + \tau\,,\ \text{with } \mathrm{Im}(p_{i+1}) > \mathrm{Im}(p_i)\,. \tag{6.1.4}$$

As we see in Section 6.2.2, $\frac{1}{g_S}$, θ are connected to Type IIB parameters. In this context we can interpret these as suitable normalization constants such that, letting τ_i be the complexified gauge coupling of the i^{th} node of the quiver theory, we have

$$\tau_i = p_{i+1} - p_i\,. \tag{6.1.5}$$

Notice the fact that ordering the x^4_i, as shown in (6.1.2), was fundamental in order to have $\mathrm{Im}(\tau_i) > 0$.

It is possible to notice that, thanks to the periodicity condition in (6.1.4) we have

$$\sum_i^K \tau_i = \tau\,. \tag{6.1.6}$$

6.1.1 Action of $\mathbb{S}$-duality from M-theory

The action we have called $\mathbb{S}$-duality, from the M-theory point of view, is just the usual modular map of the 4#-torus. Defining a generic point on the torus to be

$$z = \sigma_1 + \tau\sigma_2\,,\ \text{with}\ \ \sigma_i \sim \sigma_i + 1\,, \tag{6.1.7}$$

the modular map simply acts as

$$z = \sigma_1 + \tau\sigma_2 \longrightarrow z' = \sigma'_1 + \frac{a\tau + b}{c\tau + d}\sigma'_2 = \frac{z}{c\tau + d}\,,\ \text{with } \sigma'_i \sim \sigma'_i + 1\,, \tag{6.1.8}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})\,. \tag{6.1.9}$$

[37]String theoretically it can be possible to have $\mathrm{Im}(p_{i+1})$=$\mathrm{Im}(p_i)$. These set up however do not have, up to now, a clear field theoretic limit and thus we do not discuss it here.

Due to the fact that all the points z are mapped by the diffeomorphism to $z' = \frac{z}{c\tau+d}$, we see that this action changes the p_i defined in (6.1.1) and thus in turn the gauge couplings.

Take the matrix of the form (6.1.9), with $a = d = 0,\ b = -c = -1$.[38] If we take $p_i\,,\ i \in \{1\,,\dots,K\}$ with $x_i^a \in [0, L^{(a)})$, then[39]

$$p_i' = \frac{p_i}{\tau}\,,\ \forall i \in \{1\,,\dots,K\} \text{ and } p_{i+K}^{'} = p_i^{'} - \frac{1}{\tau}\,. \tag{6.1.10}$$

Following (6.1.5) one might be tempted to deduce that the gauge coupling transforms as[40]

$$\tau_i' = \frac{\tau_i}{\tau}\,,\ \forall i \in \{1\,,\dots,K-1\}\,, \tag{6.1.11}$$

$$\tau_K' = -\frac{1}{\tau} - \sum_i^{K-1}\frac{\tau_i}{\tau}\,. \tag{6.1.12}$$

but this would be too hasty. Indeed one can see that, depending on the sign of θ_i, one could end up with $\mathrm{Im}(\tau_i') < 0$.

This apparent inconsistency is just due to the fact that the action of the duality group and the RG-flow action do not commute with each other. We should move string-theoretic parameters while studying the moduli of a theory, and then take the low energy limit, otherwise we can reach inconsistencies.

An example of this is moving an NS5 such that it crosses another one in Figure 6.2. In following this motion we would end up with $\frac{1}{g_i^2} \sim x_{i+1}^{(4)} - x_i^{(4)} < 0$, but clearly the end-result after crossing leads at low energy a consistent field theory analogue to the one before. The crucial point for this example is that one should re-read the position of the NS5s after the crossing and re-label the i-index correctly in order to obtain the physical couplings like in (6.1.2)-(6.1.5).

These kind of phenomena, where inconsistent field-theoretic parameters correspond to region in the physical string-theoretical moduli, was initially observed in the conifold [86, 87], later considered for Seiberg-Witten dualities in [88–90] and

[38] Choosing $a = d = 0,\ b = -c = 1$ would lead to $p_i' = -\frac{p_i}{\tau}$. For reasons we argue later, this does not change the couplings of the low energy theory. Furthermore the match of the duality transformation with the Type IIB set-up in (6.2.10) persists with both definition of S-action.

[39] Notice that it is important to specify the domain. Indeed, following the comment in footnote 38, we would have $p_i' = \pm\frac{p_i}{\tau}$ depending on the definition of the S-action. If we do not choose the fundamental domain we find $\pm\frac{p_{i+K}}{\tau} = \pm\frac{p_i}{\tau} \pm 1 \neq p_{i+K}^{'} = p_i' - \frac{1}{\tau}$, which would not be consistent with the new periodicity condition.

[40] Once one decide which is the last node, due to what stated in footnote 39, we are forced to treat the last coupling differently. Notice that we have $\sum_i^K \tau_i' = -\frac{1}{\tau}$, which mimics (6.1.6).

then generalized for a large class of 4d theories with at least $\mathcal{N} = 1$ supersymmetry in [91] (which also encompass our example). These transitions are known as *flop transitions* and, differently from what happens in our case, they do not necessarily lead to a theory with the same kind of interactions among the same types of degrees of freedom (even though, after the flop transition, one can obtain dual field theories).

6.2 Bulk construction in Type IIB

In this section we describe the set up that one obtains when we take a T-duality along the 4-circle. We start with a general discussion of T-duality in directions perpendicular to NS5s, then we focus on our set-up and we discuss the duality group from the Type IIB perspective.

6.2.1 T-duality and NS5s

Let us assume a set up like the one in Table 6.1 but with only parallel NS5s. Let us furthermore assume that these NS5s are all in different positions, not only along the circle 4, but also along the flat directions 789.

After taking a T-duality along 4 one obtains a Taub-NUT geometry of the form of [92]

$$ds^2 = V(\vec{x})d\vec{x}^2 + \frac{1}{V(\vec{x})}(Rd\theta + \vec{\omega}\cdot d\vec{x})^2\,, \tag{6.2.1}$$

with

$$\begin{aligned} V &= 1 + \sum_{i=1}^{K}\frac{R}{2|\vec{x}-\vec{x}_i|}\,, \\ V &= \vec{\nabla}\cdot\vec{\omega}\,, \end{aligned} \tag{6.2.2}$$

and $\vec{x}_i$ the positions of the NS5 along 789. In $\vec{x} = \vec{x}_i$ the S^1 fibration shrinks. From each x_i one can trace non-intersecting curves in 789 and from the fibration one obtains 2d cigars $\mathcal{C}_i$ (see Figure 6.4).

We can then define some $B_{(2)}$-fluxes and we learn from [92] that there is a connection between the distance of the NS5s in Type IIA (6.1.2) and $\int_{\mathcal{C}_i} B_{(2)}$ in Type

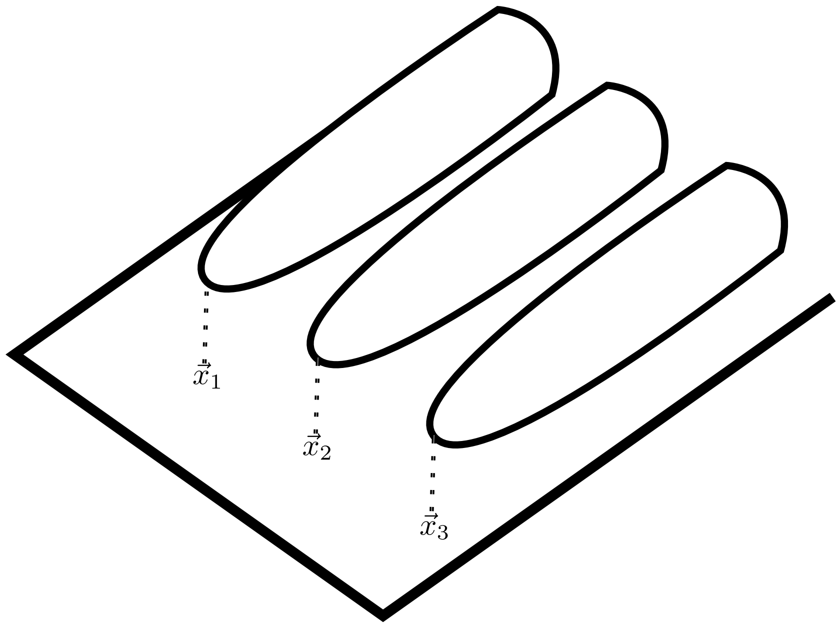


Figure 6.4: In this image we show the Taub-NUT as a S^1 fibration over a plane with different points where the fibration shrinks to a point.

IIB. In a suitable normalization, we can actually write

$$x_i^4 = \int_{\mathcal{C}_i} B_{(2)} \,. \tag{6.2.3}$$

These fluxes can be rigidly shifted by the same amount

$$B_{(2)} \longrightarrow B_{(2)} + \lambda_{(2)} \quad \text{such that } d\lambda_{(2)} = 0\,, \tag{6.2.4}$$

which parallels the rigid translations of all NS5s in the previous string frame [92].

The only thing that is gauge invariant is the subtraction among two of these fluxes, which is equivalent to a flux through $\mathcal{C}_i - \mathcal{C}_j \sim S^2$. Indeed this is related to the physical distance among the i^{th} and the j^{th} NS5.

Choosing some order of the $\mathcal{C}_i$ and defining

$$\mathcal{C}_{K+1} = -\sum_{i=1}^{K} \mathcal{C}_i \tag{6.2.5}$$

we define the following spheres

$$S_i^2 = \mathcal{C}_{i+1} - \mathcal{C}_i \quad \forall i \in 1, \dots, K\,, \tag{6.2.6}$$

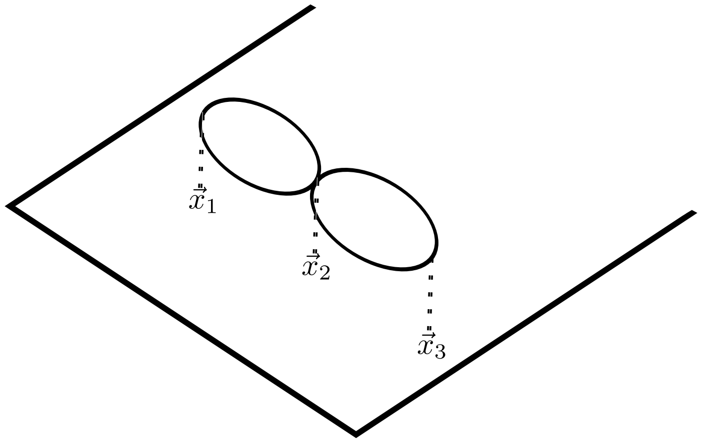


Figure 6.5: In this figure we depict the same picture of Figure 6.4 but we show the fibration along a curve that connects two different points where the fibration shrinks.

which we display in Figure 6.5.

We can also assume, for future convenience, that we could have $C_{(2)}$-fluxes as well. One can then notice that the sum of all these S_i^2 is topologically a 2-surface that can be shrunk, essentially by definition of (6.2.6). Thus we have

$$\sum_{i=1}^{K} \int_{S_i^2} B_{(2)} = \sum_{i=1}^{K} \int_{S_i^2} C_{(2)} = 0\,. \tag{6.2.7}$$

6.2.2 The orbifold and field theory limit

When in a Taub-NUT all degenerate fibers are close to a single point, with respect to the scale of the radius, this is called the *Asymptotically Locally Euclidean* limit (*ALE* limit for short). If all the points where the fiber degenerates coincide, we obtain an *orbifold* limit, which is $\frac{\mathbb{R}^4}{\Gamma} \sim \frac{\mathbb{C}^2}{\Gamma}$ with Γ some discrete subgroup of $SU(2)$. These kind of singularities have been studied and have an ADE classification, the famous McKay correspondence.

The orbifold limit is precisely what happens in our case since the Type IIA 4-circle in Table 6.1 is considered to be small and all the NS5s are considered to be in the same position along 789. With respect to what was showed in Figure 6.5, we see that all the spheres have shrunk. We however have B-fluxes on *vanishing 2-cycles* due to the fact that all the NS5 where in different positions on this circle.

The kind of orbifold we obtain is actually one of the simplest and is $\frac{\mathbb{C}^2}{\mathbb{Z}_K}$. It is interesting to notice that string theory is not actually singular in this limit. Indeed the string sigma-model has two terms contributing to the kinetic terms, the first one due to the metric $G_{MN}\partial X^M \bar{\partial} X^N$, which vanishes in the orbifold point, but also another one due to $B_{(2)}$ fluxes which are turned on.

The fields' mode expansion now contains twisted sectors. The string theory is however consistent and thus we expect all the symmetries to still hold. In particular S-duality mixes $B_{(2)}$ and $C_{(2)}$, sending the a^{th} twisted sector of $B_{(2)}$ to the same one of $C_{(2)}$ and viceversa.

The field theory limit can be obtained by studying and expanding the DBI-action of *fractional branes*, which we do not describe in detail but refer the reader to useful notes on the subject [93]. The end result is precisely the quiver gauge theory described in the previous section. In particular one can see that [90][41]

$$
\begin{aligned}
\tau_i &= \tau \int_{S_i^2} B_{(2)} + \int_{S_i^2} C_{(2)}\,, \;\; \forall i \in \{1\,, \ldots\,, K-1\}\,, \\
\tau_K &= \tau - \sum_{i=1}^{K-1} \left(\tau \int_{S_i^2} B_{(2)} + \int_{S_i^2} C_{(2)} \right)
\end{aligned}
\tag{6.2.8}
$$

with

$$
\tau = \frac{i}{g_S} + \frac{\theta}{2\pi}\,. \tag{6.2.9}
$$

The fluxes on the vanishing cycles should be interpreted more carefully as scalars in the twisted sectors coming from the mode expansion on the orbifold of $B_{(2)}$ and $C_{(2)}$. We decide however to keep explicit $B_{(2)}$ and $C_{(2)}$ because it makes the S-duality action apparent.

In this duality frame we do have a dilaton and $C_{(0)}$ profile, the latter coming from a $C_{(1)}$ flux on the 4-circle, and that comes from the twist of the M-theory 4#-torus.

[41] As argued in [94], we can think of the N fractional D3 on the orbifold singularity as the $N-1$ D5 wrapping the vanishing cycles S_i^2 $\forall i \in \{1\,, \ldots\,, K-1\}$ and an anti-D5 wrapping S_K with some F-flux from the fundamental string. The anti-D5 and the F-flux are needed to cancel the total D5 charges in the final set up with only D3s. In this picture, this is the reason why τ_K seems cast out with respect to the other τ_i in (6.2.8). We however do not use this picture because, upon S-duality, we would need to deal with a putative DBI for NS5s to obtain the low energy theory.

It could also be possible to move the $\int_{S_i^2} B_{(2)}$ module in such a way that one or more of the τ_i become negative. This would correspond to the flop transitions already discussed in Section 6.1.1.

From this it is possible to show that our $\mathbb{S}$-duality is actually just S-duality. Indeed keeping fixed the choice of the S-matrix action of $SL(2,\mathbb{Z})$ with respect to the one stated below (6.1.9), namely the one with $a = d = 0,\ b = -c = -1$, we have [94]

$$\tau \to -\frac{1}{\tau}\,, \quad B_{(2)} \to -C_{(2)}\,, \quad C_{(2)} \to B_{(2)}\,, \tag{6.2.10}$$

with which we can match (6.1.11)

$$\tau_i' = \left(-\frac{1}{\tau}\right)\left(-\int_{S_i^2} C_{(2)}\right) + \int_{S_i^2} B_{(2)} = \frac{\tau_i}{\tau}\,,\ \forall i \in \{1\,,\dots\,,K-1\}\,, \tag{6.2.11}$$

$$\tau_K' = -\frac{1}{\tau} - \sum_{i=1}^{K-1}\left(\big(-\frac{1}{\tau}\big)\big(-\int_{S_i^2} C_{(2)}\big) + \int_{S_i^2} B_{(2)}\right) = -\frac{1}{\tau} - \sum_{i=1}^{K-1}\frac{\tau_i}{\tau}\,. \tag{6.2.12}$$

We again interpret $\mathrm{Im}(\tau_i') < 0$ as flop transitions, which lead by construction to the same low energy theory's coupling described of the dual Type IIA set up, as discussed in [91]. As already stated in footnote 38, the different choice of the S action of $SL(2,\mathbb{Z})$, corresponding to $a = d = 0,\ b = -c = 1$, would still have led to a match of the physical IR coupling transformations across the two duality frames, once flop transitions are taken into account.

6.3 Adding the boundary and $\mathbb{S}$-duality

We add boundary-branes to our set up, as shown in Table 6.3.

We consider the 3-direction as the boundary where the D4s can end on a system of D6s and NS5s, much like D3s on D5s and NS5s in [17].

It is possible to check explicitly that this set up preserves 4 supercharges, half with respect to the set up in Table 6.1. This coincides with imposing $\frac{1}{2}$-BPS boundary conditions on a 4d $\mathcal{N} = 2$ theory.

Notice that we expect the $SO(2)_{\mathcal{R}}$ 3d $\mathcal{R}$-symmetry to be the Cartan of the $SU(2)_{\mathcal{R}} \subset U(2)_{\mathcal{R}}$ 4d one (see the boundary conditions on vectors an hypermultiplets in Section 1.5.1). This is precisely what happens, namely the purely bulk model

	012	3	**4**	5	**6**	7	8	9
D4	•	•	•					
NS5	•	•		•	•			
D6	•		•		•		•	•
NS5	•		•	•		•		

Table 6.3: We suppose the 4-circle to be small, with respect to the cut-off of our effective theory, while we do not need assumptions for the 6-circle. In blue we present the boundary-branes.

has a $SU(2)_{\mathcal{R}} \subset U(2)_{\mathcal{R}}$ $\mathcal{R}$-symmetry, coming from the 789 directions, and now we remain with a $SO(2)_{\mathcal{R}}$ coming from 89.

The $\mathbb{S}$-duality action we propose is shown in Table 6.4.

A crucial step is that during the modularity on the 46-torus the branes do not move. This is due to the fact that the quantization of the string-states should be fully insensitive to diffeomorphism. Indeed this can also be seen, for instance, in M-theory on a torus with a stack of M2, as in Table 6.5.

The modularity in M-theory should correspond to the S-duality in the Type IIB picture. Taking the Type IIA limit on one of the two cycles in Table 6.5 and then taking T-duality on the other cycle leaves either a F1 or a D1. A modularity action precisely swaps these two end results, just like S-duality swaps F1s and D1s. This would not work however if we assume that the modularity acts also on the branes, changing which cycles they wrap.

For our set up the effect of this action is going to be the same one already discussed for the bulk. On the boundary the action exchanges D6 and NS5s, much like the exchange of D5 and NS5 in [17].

In Appendix E we discuss which modes of the D4s survive depending on which boundary branes the D4s end (but we do not discuss boundary conditions). We argue that when the D4s end on NS5s the 3d $\mathcal{N} = 2$ vector multiplet survives while when they end on D6s the 3d $\mathcal{N} = 2$ chiral multiplet survives.

Due to the fact that the bulk bi-fundamental hypermultiplets are non-perturbative states from the canonical string-quantization frame [84], it is not possible to argue which 3d $\mathcal{N} = 2$ mode survives using the same means of Appendix E. We argue however that this does not matter.

Since the boundary breaks explicitly the $\mathcal{R}$-symmetry down to $SO(2)_{\mathcal{R}} \subset$

	012	3	$\underline{\mathbf{4}}$	5	$\underline{\mathbf{6}}$	7	8	9
D4	•	•	•					
NS5	•	•		•	•			
D6	•		•		•		•	•
NS5	•		•	•		•		

	012	3	$\underline{\mathbf{4}}$	5	$\underline{\mathbf{6}}$	7	8	9
D3	•	•	•					
TN			$\times_{\mathbf{4}}$			•	•	•
D5	•				•		•	•
NS5	•		•	•		•		

	012	3	$\underline{\mathbf{4}}$	5	$\underline{\mathbf{6}}$	7	8	9
D3	•	•	•					
TN					$\times_{\mathbf{6}}$	•	•	•
D5	•				•		•	•
NS5	•		•	•		•		

	012	3	$\underline{\mathbf{4}}$	5	$\underline{\mathbf{6}}$	7	8	9
D3	•	•	•					
TN					$\times_{\mathbf{6}}$	•	•	•
NS5	•				•		•	•
D5	•		•	•		•		

	012	3	$\underline{\mathbf{4}}$	5	$\underline{\mathbf{6}}$	7	8	9
D4	•	•	•		•			
NS5	•	•	•	•				
NS5	•				•		•	•
D6	•		•	•	•	•		

Table 6.4: The first table is the initial set-up. The second one is after T-duality along the 4-circle, with TN standing for for Taub-NUT and $\times_k$ under the j-direction means means that the k-circle shrinks along j ($j = k$ stands for the orbifold limit). The third step is modularity of the 46-torus. The fourth one is Type IIB's S-duality and in the final one we take a T-duality on 6 and we obtain back the $\mathbb{S}$-dual frame.

$SU(2)_{\mathcal{R}}$, the 4d $\mathcal{N} = 2$ hypermultiplet decomposes in two 3d $\mathcal{N} = 2$ chiral multiplets with opposite, but otherwise generic, $SO(2)_{\mathcal{R}}$-charge (see (1.5.11)).

These 3d chiral multiplets posses the same flavor and gauge symmetry representation. Even if boundary interactions require a specific $SO(2)_{\mathcal{R}}$-charge, this

	0	1...7	$\underline{8}$	9	$\underline{\#}$
M2	•		•		•

Table 6.5: We consider 8 and # to be circles.

can be assigned to either of the two boundary modes we can consider fluctuating (remembering to fix the opposite one for the degrees of freedom we freeze). For this reason the class of interactions, and triggered RG-flows, is the same.

Further analysis is required in order to study more precisely the boundary conditions, the interactions within the boundary and the bulk-to-boundary ones. It is possible that the diagonal $U(1)$ vector that survives [19], see Section 6.1, can actually interact with boundary modes and as such it would not be decoupled.

Chapter 7

Conclusions and future directions

In the present thesis we have studied, in different scenarios, four dimensional supersymetric quantum field theories in presence of a boundary.

We have presented how to obtain exact results for the one-point function BCFT observables in 4d, with the minimal supersymmetry which allows the use of supersymmetric localization.

Multiple future directions and further studies can be addressed. It could be interesting to study whether, for a class of observables, it is actually possible to obtain exact closed formulæ in some interesting regimes.

For instance it is known that theories have a universal behavior in presence of large global charges [95]. This was exploited to compute, with localization, a class of CFT observables [28, 96], but in presence of a boundary this is not well understood.

Another interesting regime is large-N. The literature is already vast in this context (see for instance [97] and references therein). It could be interesting to see if it is possible to obtain exact results in the large N limit also in the presence of a boundary. This was done in $\mathcal{N} = 4$ [29] and it would be interesting to see if there are systematic methods also in $\mathcal{N} = 2$. Moreover this can be useful for holography, to study the back-reacted string-theoretic side in set ups with less supersymmetry. Holography checks have either been made for correlation functions in 4d $\mathcal{N} = 2$ without a boundary, see for instance [98], or for the partition function in $\mathcal{N} = 4$ in presence of a $\frac{1}{2}$-BPS boundary (see [99] for a recent article).

In the context of $\mathcal{N} = 4$ it is often possible to obtain the full analytic result by employing the modular properties expected from the partition function (see [100] for a recent work where this was done). It could be interesting to understand if this is possible also in $\mathcal{N} = 2$ where there are additional technical challenges. For instance, even for conformally invariant theories, the coupling that appears in the Lagrangian is corrected by instantons [101, 102] and the modular properties, that act on the physical coupling, are less apparent (recently in [103] this was successfully taken into account for conformal bootstrap purposes in superconformal QCD without a boundary). Furthermore the partition function itself is expected to transform under modular transformations [10, 104].

In this thesis we have also began a string-theoretic analysis, for a class of 4d $\mathcal{N} = 2$ theories with a boundary, via brane engineering [20]. It would be interesting to understand better the M/F-theory point of view, in particular if it is possible to describe the shape of the branes/of the geometry, as it was done in [19] in M-theory. This can be interesting since these quantities are connected to field theoretic ones such as the β-function. It could be interesting to see if there is any connection with F_∂-maximization [63] and boundary RG-flow quantities. Further study are also needed to understand the types of interactions within the boundary and the bulk-to-boundary ones.

We have also started an analytic study of AdS RG-flow phenomena for the pure $\mathcal{N} = 2$ $SU(2)$ SYM [15]. We have conjectured a form of F_∂-maximization in AdS in order to obtain the AdS partition function, also for massive theories, that preserve the full AdS$_4$ $\mathcal{N} = 2$ supersymmetry and not only half of it. It would be interesting to make more precise this conjecture and connect it to Seiberg-Witten curves. This would allow us to study the the AdS$_4$ prepotential, its flat space limit and how the monopole\dyon points arise as we decrease the curvature.

Appendix A

Tensor conventions

1. We work in Euclidean signature, and unless otherwise stated we use the following index notations to denote the different representations:
$$\begin{aligned}
&\mu,\ \nu,\ \ldots\ \text{4d curved-space indices}\,,\\
&i,\ j,\ \ldots\ \text{3d curved-space indices}\,,\\
&a,\ b,\ \ldots\ \text{4d tangent-space indices}\,,\\
&a',\ b',\ \ldots\ \text{3d tangent-space indices}\,,\\
&\alpha,\ \beta,\ \ldots\ \text{ for } \psi_\alpha \in (\mathbf{2},\mathbf{1})_{\mathrm{Spin}(4)}\,,\\
&\dot\alpha,\ \dot\beta,\ \ldots\ \text{ for } \bar\psi^{\dot\alpha} \in (\mathbf{1},\mathbf{2})_{\mathrm{Spin}(4)}\,,\\
&A,\ B,\ \ldots\ \text{ for } \lambda_A \in \mathbf{2}_{SU(2)_\mathcal{R}}\,.
\end{aligned}$$
2. The Pauli-matrices are taken to be
$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{A.0.1}$$
3. We define the chiral sub-matrices of the 4d gamma matrices, the 3d gamma matrices, the $SU(2)_\mathcal{R}$ matrices and the anti-symmetric tensor matrices to be
$$\begin{gathered}
(\sigma^a)_{\alpha\dot\beta} = (-i\vec\tau, 1)\,, \qquad (\bar\sigma^a)^{\dot\alpha\beta} = (i\vec\tau, 1)\,, \qquad (\gamma^{a'})_\alpha{}^\beta = \tau^{a'}\,, \qquad \vec\tau_A{}^B = \vec\tau\,,\\
\sigma^{ab} = \sigma^{[a}\bar\sigma^{b]}\,, \quad \bar\sigma^{ab} = \bar\sigma^{[a}\sigma^{b]}\,, \quad \gamma^{a'b'} = \gamma^{[a'}\gamma^{b']}\,.
\end{gathered} \tag{A.0.2}$$
4. All rank n Levi-Civita tangent-space tensors are chosen such that $\varepsilon^{12\ldots n} = 1$, the space-time ones are obtained via the vielbein and become tensor densities.
5. All spinorial and $SU(2)_R$-symmetry indices are raised and lowered with the ε-tensors as in the Wess & Bagger notation. We employ the same notation for

spinor bilinears as well as for the conjugation of Grassmann odd quantities θ_i:

$$(\theta_1\theta_2)^\dagger = \theta_2^\dagger\theta_1^\dagger\,. \tag{A.0.3}$$

6. The Killing spinors are taken to be Grassmann even.
7. We define the Lie-Algebra commutators as

$$[T_a, T_b] = if_{ab}{}^c \tag{A.0.4}$$

and the gauge-covariant derivatives as

$$D_\mu = \partial_\mu - iA_\mu\,. \tag{A.0.5}$$

Since in the adjoint representation is we have $(T_a)^c{}_b = if_{ab}{}^c$ we see that on adjoint fields $D_\mu = \partial_\mu - i[A_\mu, \cdot]$.

8. We in general employ the following choice of orientation of our spaces with a boundary $\int \partial \cdot V = \int_\partial V^\perp$.

A.1 Coordinate choices and Killing spinors for localization

We employ the same coordinates used in [56] for $(H)S^4$ and similar ones for AdS$_4$. For convenience we set the radius to one. With the chosen coordinates the metrics takes the following form

$$(\mathbf{H})\mathbf{S^4} : ds^2 = dr^2 + \sin(r)^2 d\Omega^2_{S^3}\,, \tag{A.1.1}$$

$$\mathbf{AdS_4} : ds^2 = d\eta^2 + \sinh(\eta)^2 d\Omega^2_{S^3}\,, \tag{A.1.2}$$

with $r \in [0, \pi]$ on S^4, $r \in [0, \frac{\pi}{2}]$ on HS^4 and $\eta \in [0, +\infty)$ on AdS$_4$. We then choose the following Hopf fibration for parametrizing the S^3 part of the metric in both cases

$$z_1 = \sin\left(\frac{\theta}{2}\right) e^{i\frac{\psi-\phi}{2}}\,, \quad z_2 = \cos\left(\frac{\theta}{2}\right) e^{i\frac{\psi+\phi}{2}}\,, \tag{A.1.3}$$

with $0 \leqslant \theta \leqslant \pi$ $0 \leqslant \phi \leqslant 2\pi$ $0 \leqslant \psi \leqslant 4\pi$. With this the unit three-sphere has the following metric

$$d\Omega^2_{S^3} = \frac{1}{4}\big(d\theta^2 + \sin^2(\theta)d\phi^2 + (d\psi + \cos(\theta)d\phi)^2\big)\,. \tag{A.1.4}$$

A choice of vierbein can be the following

$$(e_{(H)S^4})^a{}_\mu = \begin{pmatrix} 0 & \frac{1}{2}\sin(r)\cos(\psi) & \frac{1}{2}\sin(r)\sin(\theta)\sin(\psi) & 0 \\ 0 & -\frac{1}{2}\sin(r)\sin(\psi) & \frac{1}{2}\sin(r)\sin(\theta)\cos(\psi) & 0 \\ \frac{1}{2}\sin(r) & 0 & \frac{1}{2}\sin(r)\cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{A.1.5}$$

$$(e_{\mathrm{AdS}_4})^a{}_\mu = \begin{pmatrix} 0 & \frac{1}{2}\sinh(\eta)\cos(\psi) & \frac{1}{2}\sinh(\eta)\sin(\theta)\sin(\psi) & 0 \\ 0 & -\frac{1}{2}\sinh(\eta)\sin(\psi) & \frac{1}{2}\sinh(\eta)\sin(\theta)\cos(\psi) & 0 \\ \frac{1}{2}\sinh(\eta) & 0 & \frac{1}{2}\sinh(\eta)\cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{A.1.6}$$

The explicit Killing spinors, satisfying (2.5.9), that we use for the localization computations, are written below.

$$\begin{aligned}
(\mathbf{H})\mathbf{S^4} : ds^2 &= dr^2 + \sin(r)^2 d\Omega^2_{S^3} \\
&= dr^2 + \frac{\sin^2(r)}{4}\left(d\theta^2 + \sin^2(\theta)d\phi^2 + (d\psi + \cos(\theta)d\phi)^2\right), \\
& s_1 = s_2 = -1\,, \\
& \epsilon_{A\alpha} = \frac{1}{\sqrt{2}}\sin\left(\frac{r}{2}\right)\mathbf{1}_{A\alpha}\,, \quad \bar\epsilon^{\dot\alpha}_A = \frac{i}{\sqrt{2}}\cos\left(\frac{r}{2}\right)\tau_{3,A}{}^{\alpha}\,, \\
& \epsilon^A\epsilon_A = \sin^2\left(\frac{r}{2}\right), \quad \bar\epsilon_A\bar\epsilon^A = \cos^2\left(\frac{r}{2}\right), \\
& \epsilon^A\epsilon_A + \bar\epsilon_A\bar\epsilon^A = 1\,, \quad \epsilon^A\epsilon_A - \bar\epsilon_A\bar\epsilon^A = -\cos(r)\,.
\end{aligned} \tag{A.1.7}$$

$$\begin{aligned}
\mathbf{AdS_4} : ds^2 &= d\eta^2 + \sinh(\eta)^2 d\Omega^2_{S^3} \\
&= d\eta^2 + \frac{\sinh^2(\eta)}{4}\left(d\theta^2 + \sin^2(\theta)d\phi^2 + (d\psi + \cos(\theta)d\phi)^2\right), \\
& - s_1 = s_2 = 1\,, \\
& \epsilon_{A\alpha} = \frac{1}{\sqrt{2}}\sinh\left(\frac{\eta}{2}\right)\mathbf{1}_{A\alpha}\,, \quad \bar\epsilon^{\dot\alpha}_A = \frac{i}{\sqrt{2}}\cosh\left(\frac{\eta}{2}\right)\tau_{3,A}{}^{\alpha}\,, \\
& \epsilon^A\epsilon_A = \sinh^2\left(\frac{\eta}{2}\right), \quad \bar\epsilon_A\bar\epsilon^A = \cosh^2\left(\frac{\eta}{2}\right), \\
& \epsilon^A\epsilon_A + \bar\epsilon_A\bar\epsilon^A = \cosh(\eta)\,, \quad \epsilon^A\epsilon_A - \bar\epsilon_A\bar\epsilon^A = -1\,.
\end{aligned} \tag{A.1.8}$$

Appendix B

One point functions in free BCFTs

In this appendix we determine the perturbative one-point functions of scalar operators, via their two-point functions, in the case of Dirichlet and Neumann boundary conditions. We start with a generic free scalar and we progressively move toward the free scalar of $\mathcal{N} = 2$ $U(1)$ gauge theory on $H\mathbb{R}^4$ with the Neumann boundary conditions (1.5.6).

B.1 4d free massless scalar

Suppose we have a free 4d massless scalar on $\mathbb{R}^4$ with coordinates $x^\mu = (\vec{x}, x_\perp)$. Suppose furthermore that the kinetic term in the Lagrangian is canonical, i.e. it takes the form of $\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$. Then, through the Schwinger-Dyson equations, we can write

$$\langle \Box\varphi(x)\varphi(y)\rangle = \langle \varphi(x)\Box\varphi(y)\rangle = -\delta(x-y) \tag{B.1.1}$$

and find that

$$\langle \varphi(x)\varphi(y)\rangle_{\text{no br.}} = \frac{1}{4\pi^2}\frac{1}{|\vec{x}-\vec{y}|^2 + (x_\perp - y_\perp)^2}\,. \tag{B.1.2}$$

If we add a boundary at $x_\perp = 0$ we need to solve the same equation but in the region $x_\perp, y_\perp > 0$ with some boundary conditions. For instance we can have

$$\langle \varphi(x)\varphi(y)\rangle = \frac{1}{4\pi^2}\left(\frac{1}{|\vec{x}-\vec{y}|^2 + (x_\perp - y_\perp)^2} \mp \frac{1}{|\vec{x}-\vec{y}|^2 + (x_\perp + y_\perp)^2}\right)\,, \tag{B.1.3}$$

where the sign is negative if $\varphi(\vec{x}, 0) = 0$ and positive if $\partial_y\varphi(\vec{x}, 0) = 0$. The OPE of two free scalar fields is

$$\varphi(x)\varphi(y) \sim \langle \varphi(x)\varphi(y)\rangle_{\text{no br.}}\mathbb{1} + \varphi^2(x) + \dots\,, \tag{B.1.4}$$

from which we obtain that

$$a_{\varphi^2} = \mp\frac{1}{16\pi^2} \quad \begin{cases} - \text{ if } \varphi(\vec{x},0)=0 \\ + \text{ if } \partial_y\varphi(\vec{x},0)=0 \end{cases} . \tag{B.1.5}$$

B.2 $U(1)$ $\mathcal{N}=2$ theory on $H\mathbb{R}^4$

We now move to the theory we are interested in. We start by considering Dirichlet boundary conditions, we then move to the Neumann ones (see (1.5.6)). We remind the reader that we use the following definitions [33, 56]:

$$\begin{aligned} \phi &= \phi_2 + i\phi_1 \,, \\ \bar{\phi} &= -\phi_2 + i\phi_1 \,. \end{aligned} \tag{B.2.1}$$

Dirichlet boundary conditions

In this case the boundary conditions for the scalars are

$$\begin{cases} \phi_1 = 0 \\ \partial_\perp\phi_2 + \frac{i}{2}D_{12} = 0 \end{cases} . \tag{B.2.2}$$

With these boundary conditions the θ-term is zero and so $\langle\phi_1(x)\phi_2(y)\rangle = 0$ due to a preserved $\mathbb{Z}\times\mathbb{Z}$ symmetry $\phi_i \longmapsto -\phi_i$. We can thus write:

$$\langle\phi(x)\phi(y)\rangle = \langle\phi_2(x)\phi_2(y)\rangle - \langle\phi_1(x)\phi_1(y)\rangle \,. \tag{B.2.3}$$

Notice also that we have not considered the presence of the D_{12} in the boundary conditions (B.2.2). This is because at separated points the Schwinger-Dyson equations put that field to zero. We can use the result showed in (B.1.3) with the fields normalized in the action as $\frac{4}{g^2}\partial_\mu\phi_i\partial^\mu\phi_i = -\frac{i}{2\pi}(\tau-\bar{\tau})\partial_\mu\phi_i\partial^\mu\phi_i$ to find

$$\langle\phi(x)\phi(y)\rangle = \frac{i}{2\pi(\tau-\bar{\tau})}\frac{1}{|\vec{x}-\vec{y}|^2 + (x_\perp + y_\perp)^2} \,. \tag{B.2.4}$$

In the absence of a boundary chiral operators do not have a singular OPE. Consistently then we have that (B.2.4) is non-singular in the limit $x \to y$ and we find

$$a_{\phi^2} = \frac{i}{8\pi(\tau-\bar{\tau})} \,. \tag{B.2.5}$$

Neumann boundary conditions

In this case there is no $\mathbb{Z}\times\mathbb{Z}$ symmetry. We still have the Schwinger-Dyson equations (B.1.1) with $D_{12}=0$, with the different normalization dictated by the choice of the kinetic term, which we now solve for the boundary conditions

$$\begin{cases}\gamma\phi_1+\phi_2=0\\ \partial_\perp\phi_1-\gamma\partial_\perp\phi_2=0\end{cases}. \tag{B.2.6}$$

The solution can be found with the following ansatz

$$\begin{cases}\langle\phi_1(x)\phi_1(y)\rangle=\frac{g^2}{32\pi^2}\left(\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp-y_\perp)^2}+a\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp+y_\perp)^2}\right)\\ \langle\phi_1(x)\phi_2(y)\rangle=\frac{g^2}{32\pi^2}\left(b\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp-y_\perp)^2}+c\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp+y_\perp)^2}\right)\\ \langle\phi_1(x)\phi_1(y)\rangle=\frac{g^2}{32\pi^2}\left(\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp-y_\perp)^2}+d\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp+y_\perp)^2}\right)\end{cases}. \tag{B.2.7}$$

Then we fix the constants via the boundary conditions

$$\begin{cases}\langle\phi_1(x)\phi_1(y)\rangle\Big|_{x\in\partial}=-\frac{1}{\gamma}\langle\phi_2(x)\phi_1(y)\rangle\Big|_{x\in\partial}\\ \langle\phi_2(x)\phi_2(y)\rangle\Big|_{x\in\partial}=-\gamma\langle\phi_1(x)\phi_1(y)\rangle\Big|_{x\in\partial}\\ \langle\phi_1(x)\partial_\perp\phi_1(y)\rangle\Big|_{y\in\partial}=\gamma\langle\phi_2(x)\partial_\perp\phi_1(y)\rangle\Big|_{y\in\partial}\\ \langle\partial_\perp\phi_2(x)\phi_2(y)\rangle\Big|_{x\in\partial}=\frac{1}{\gamma}\langle\partial_\perp\phi_1(x)\phi_1(y)\rangle\Big|_{x\in\partial}\end{cases}. \tag{B.2.8}$$

Solving the above we find

$$\begin{aligned}\langle\phi_1(x)\phi_1(y)\rangle&=\frac{i}{4\pi(\tau-\bar{\tau})}\left(\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp-y_\perp)^2}+\frac{1-\gamma^2}{1+\gamma^2}\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp+y_\perp)^2}\right),\\ \langle\phi_2(x)\phi_2(y)\rangle&=\frac{i}{4\pi(\tau-\bar{\tau})}\left(\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp-y_\perp)^2}-\frac{1-\gamma^2}{1+\gamma^2}\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp+y_\perp)^2}\right),\\ \langle\phi_1(x)\phi_2(y)\rangle&=-\frac{i}{2\pi(\tau-\bar{\tau})}\frac{\gamma}{1+\gamma^2}\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp+y_\perp)^2},\end{aligned} \tag{B.2.9}$$

from which, using $\phi=\phi_2+i\phi_1$, we get:

$$\langle\phi(x)\phi(y)\rangle=\frac{i}{2\pi(\tau-\bar{\tau})}\frac{\bar{\tau}}{\tau}\frac{1}{|\vec{x}-\vec{y}|^2+(x_\perp+y_\perp)^2}. \tag{B.2.10}$$

For the same reasons explained below (B.2.3), we have a non-divergent result and in the limit $x \to y$ we find

$$a_{\phi^2} = \frac{i}{8\pi(\tau - \bar{\tau})}\frac{\bar{\tau}}{\tau}\,. \tag{B.2.11}$$

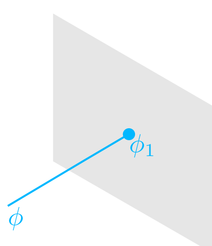
Appendix C

Diagrammatics

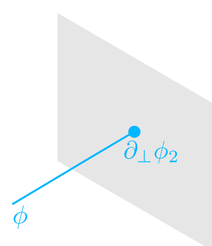
In this appendix we compute the 1-loop contributions to $\langle A_2 \rangle = \langle \phi^2(x) \rangle$ in $U(1)$ $\mathcal{N} = 2$ Maxwell theory on $H\mathbb{R}^4$ conformally coupled to a single chiral in the boundary. The complete diagrammatics needed for this calculation, using dimensional regularization

$$d = 4 - 2\epsilon \tag{C.0.1}$$

is given below

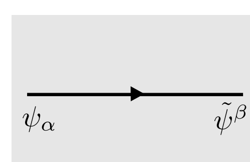


$$= \frac{\pi \mathrm{i}}{2\tau_s} \mathrm{e}^{-|\vec{p}| x_\perp},$$

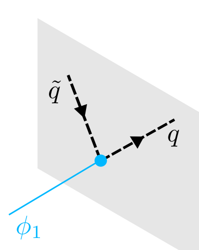


$$= -\frac{\pi}{2\tau_s} \frac{\mathrm{e}^{-|\vec{p}| x_\perp}}{|\vec{p}|}, \tag{C.0.2}$$

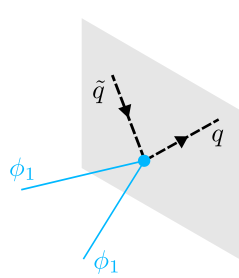
$$= \frac{1}{|\vec{p}|^2},$$



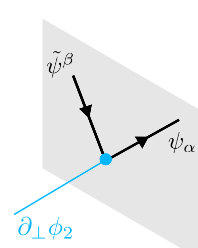
$$= \frac{p^i (\tau_i)_\alpha{}^\beta}{|\vec{p}|^2}, \tag{C.0.3}$$



$$= 2\mu^\epsilon,$$



$$= -8\mu^{2\epsilon},$$



$$= 2i\delta_\alpha^\beta \mu^\epsilon. \tag{C.0.4}$$

Here we already employed the shift $\tau \to \tau_s$ showed in (4.2.4) due to the Chern-Simons contact terms we need to add for the chirals (see Section 4.2.1).

C.1 Bulk-boundary propagators in phase space

Following what found in (B.2.9) and placing one operator on the boundary, we can perform the Fourier transformations over $\vec{x} - \vec{y}$:

$$\begin{aligned}
\langle \phi_1(x)\phi_1(\vec{y})\rangle &= \frac{i}{4\pi(\tau_s - \bar{\tau}_s)}\left(1 + \frac{1-\gamma^2}{1+\gamma^2}\right)\frac{1}{(\vec{x}-\vec{y})^2 + x_\perp^2} \\
&= \int \frac{d^{d-1}\vec{p}}{(2\pi)^{2-1}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left(-\frac{\pi i}{4}\frac{\tau_s - \bar{\tau}_s}{\tau_s\bar{\tau}_s}\frac{e^{-|\vec{p}|x_\perp}}{|\vec{p}|}\right), \\
\langle \phi_2(x)\phi_1(\vec{y})\rangle &= \langle \phi_1(x)\phi_2(\vec{y})\rangle = -\gamma\langle \phi_1(x)\phi_1(\vec{y})\rangle \\
&= \int \frac{d^{d-1}\vec{p}}{(2\pi)^{2-1}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left(-\frac{\pi}{4}\frac{\tau_s + \bar{\tau}_s}{\tau_s\bar{\tau}_s}\frac{e^{-|\vec{p}|x_\perp}}{|\vec{p}|}\right), \\
\langle \phi_1(x)\partial_\perp\phi_2(\vec{y})\rangle &= \partial_{y_\perp}\langle \phi_1(x)\phi_2(y)\rangle\Big|_{y_\perp=0} \\
&= \int \frac{d^{d-1}\vec{p}}{(2\pi)^{2-1}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left(\frac{\pi}{4}\frac{\tau_s + \bar{\tau}_s}{\tau_s\bar{\tau}_s}e^{-|\vec{p}|x_\perp}\right), \\
\langle \phi_2(x)\partial_\perp\phi_2(\vec{y})\rangle &= \partial_{y_\perp}\langle \phi_2(x)\phi_2(y)\rangle\Big|_{y_\perp=0} \\
&= \int \frac{d^{d-1}\vec{p}}{(2\pi)^{2-1}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\left(-\frac{i\pi}{4}\frac{\tau_s - \bar{\tau}_s}{\tau_s\bar{\tau}_s}e^{-|\vec{p}|x_\perp}\right).
\end{aligned} \tag{C.1.1}$$

Then, using (B.2.1), we get what is shown in (C.0.2).

C.2 Boundary-boundary propagators and vertices

We shall write the action for one $\mathcal{N} = 2$ 3d chiral multiplet interacting with a vector following the conventions in [35]. Since we have a conformal coupling with the bulk we have to choose $r = \frac{1}{2}$ and $z = 0$ in (2.1.14) and in (2.1.15).

When we impose Neumann boundary conditions $\mathcal{J}_{\mathrm{N}} = 0$ (1.5.3), we have the content in $\mathcal{J}_{\mathrm{D}}$, that is the dualized version of the 3d $\mathcal{N} = 2$ vector multiplet in (1.5.5), that is still dynamical.

Thanks to this we have, using (1.4.9)-(1.5.5)

$$
\begin{aligned}
\mathcal{L}_{\text{chiral}} &= D_i\tilde{q}D^i q - i\tilde{\psi}\not{D}\psi + \tilde{q}\Big(D+\sigma^2\Big)q - i\tilde{\psi}\sigma\psi + \sqrt{2}i(\tilde{q}\lambda\psi + \tilde{\psi}\tilde{\lambda}q) - \tilde{F}F \\
&= D_i\tilde{q}D^i q - i\tilde{\psi}\not{D}\psi + \tilde{q}\Big(-iD_{12} - 2\partial_\perp\phi_2 + 4\phi_1^2\Big)q + \\
&\quad - 2i\tilde{\psi}\phi_1\psi + \sqrt{2}i(\tilde{q}(\lambda_1 - i\sigma_4\bar{\lambda})\psi + \tilde{\psi}(\lambda_2 + i\sigma_4\bar{\lambda}_2)q) - \tilde{F}F\,,
\end{aligned}
\tag{C.2.1}
$$

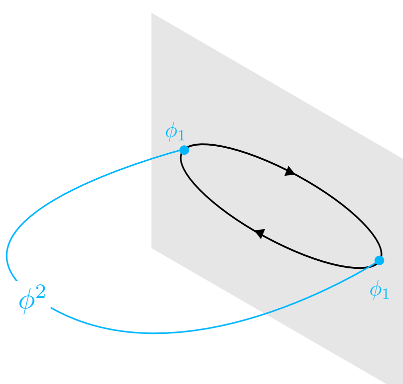
the following bulk-to-boundary interactions among the bulk vectors and the boundary matter.

With this we arrive to the boundary-boundary propagators showed in (C.0.3) and the vertices in (C.0.4).

C.3 1-loop correction to A_2, 1 chiral multiplet

We now compute the 1-loop contribution to $\langle A_2\rangle$.

There are only three diagrams contributing to the 1-loop correction. Employing dimensional regularization (C.0.1) we can write the first contribution



$$
= \frac{\pi^2}{\tau_s^2}\mu^{2\epsilon}\int\frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}}\int\frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}}\frac{\mathrm{Tr}(\vec{\not{q}}(\vec{\not{q}}-\vec{\not{p}}))}{\vec{q}^2(\vec{q}-\vec{p})^2\vec{p}^2}e^{-2|\vec{p}|x_\perp}
\tag{C.3.1}
$$

$$
= \frac{\Upsilon_1}{x_\perp^{2-4\epsilon}} + \frac{\Upsilon_2}{x_\perp^{1-2\epsilon}}
$$

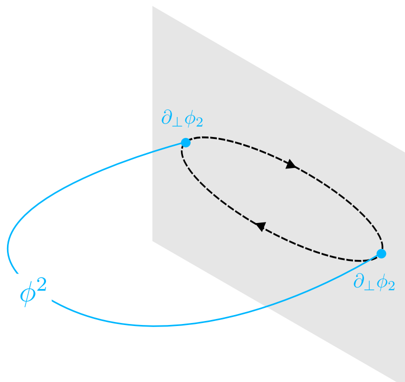
with

$$
\Upsilon_1 = -\frac{\pi^2}{2\tau_s^2}\frac{(4\mu)^{2\epsilon}\Gamma(2-2\epsilon)\Gamma\left(\frac{5-d}{2}\right)}{(4\pi)^{d-1}\Gamma\left(\frac{d-1}{2}\right)}\int_0^1 dx\,\left[x(1-x)\right]^{-\frac{1}{2}(1+2\epsilon)},
\tag{C.3.2}
$$

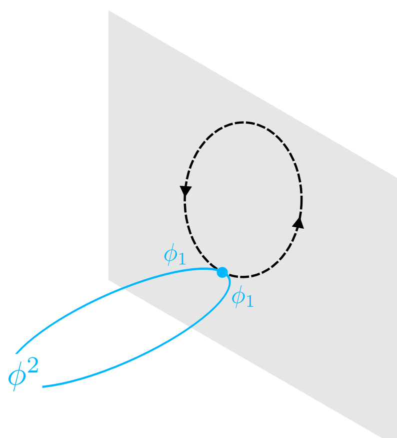
$$
\Upsilon_2 = \frac{4\pi^2}{\tau_s^2}\frac{(2\mu)^{2\epsilon}\Gamma(1-2\epsilon)}{(4\pi)^{d-1}\Gamma^2\left(\frac{d-1}{2}\right)}\int_0^{+\infty} dq\; q^{d-4}\,.
\tag{C.3.3}
$$

Note that the Υ_2 contribution must cancel in the end result. Indeed, not only it adds an IR-divergence (which is a problem for the theory at hand if we do not have other diagrams canceling it), it also adds a term scaling like $\sim \frac{1}{x_\perp}$ not allowed by conformal invariance which fixes $\langle \phi^2(x) \rangle \propto \frac{1}{x_\perp^2}$.

Indeed the following further two corrections are given by:



$$= -\frac{2\pi^2}{\tau_s^2} \mu^{2\epsilon} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{1}{\vec{q}^2} \frac{e^{-2|\vec{p}|x_\perp}}{\vec{p}^2} = -\frac{\Upsilon_2}{x_\perp^{1-2\epsilon}} \tag{C.3.4}$$



$$= -\frac{\pi^2}{\tau_s^2} \mu^{2\epsilon} \int \frac{d^{d-1}\vec{q}}{(2\pi)^{d-1}} \int \frac{d^{d-1}\vec{p}}{(2\pi)^{d-1}} \frac{e^{-2|\vec{p}|x_\perp}}{\vec{q}^2(\vec{q}-\vec{p})^2} = \frac{\Upsilon_1}{x_\perp^{2-4\epsilon}} \tag{C.3.5}$$

The total contribution goes precisely as $\sim \frac{1}{x_\perp^2}$ as it should in the limit for $\epsilon \to 0$. We thus obtain

$$\delta a_{\phi^2}^{(1,0)} = \lim_{\epsilon \to 0} 2\Upsilon_1 = -\frac{1}{32\tau_s^2} \,. \tag{C.3.6}$$

Appendix D

1-loop determinant in AdS$_4$

In this appendix we streamline the general procedure one employs, when using the index formula (2.5.45), to compute the 1-loop determinant in localization. We specify to SYM on AdS$_4$ with Dirichlet boundary conditions (meaning the gauge field is frozen on the boundary), even though this argument can be generalized to Neumann boundary conditions.

D.1 Writing the index

Using local coordinates

$$z_1 = x_1 + ix_2\,, \qquad z_2 = x_3 + ix_4 \tag{D.1.1}$$

such that the $i\mathcal{L}_v \subset \hat{\mathcal{Q}}^2$ (2.5.22) acts near the center of AdS as the following infinitesimal rotation on the complex planes z_1 and z_2

$$i\mathcal{L}_v = i\left(x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right) + i\left(x^3\frac{\partial}{\partial x^4} - x^4\frac{\partial}{\partial x^3}\right). \tag{D.1.2}$$

In this way we have that $\hat{\mathcal{Q}}^2$ acts on the coordinates as

$$z_1' = e^{it}z_1\,, \qquad z_2' = e^{it}z_2 \tag{D.1.3}$$

and thus we have

$$\det(\delta^\mu_\nu - \partial_\nu x'^\mu) = |1-e^{it}|^4\,. \tag{D.1.4}$$

We now simply need the action on X_{even} and X_{odd}, defined in (2.5.39), of $e^{-i\hat{\mathcal{Q}}^2 t}$ on the fixed points. This can be readily found using (2.5.22)

$$\begin{aligned}
e^{-i\hat{\mathcal{Q}}^2 t}A_{z_1} &= e^{-i(-1+ia)t}A_{z_1} & e^{-i\hat{\mathcal{Q}}^2 t}\Theta_{11} &= e^{-i(2+ia)t}\Theta_{11} \\
e^{-i\hat{\mathcal{Q}}^2 t}A_{z_2} &= e^{-i(-1+ia)t}A_{z_2} & e^{-i\hat{\mathcal{Q}}^2 t}\Theta_{12} &= e^{-i(ia)t}\Theta_{12} \\
e^{-i\hat{\mathcal{Q}}^2 t}A_{\bar{z}_1} &= e^{-i(1+ia)t}A_{\bar{z}_1} & e^{-i\hat{\mathcal{Q}}^2 t}\Theta_{22} &= e^{-i(-2+ia)t}\Theta_{22} \\
e^{-i\hat{\mathcal{Q}}^2 t}A_{\bar{z}_2} &= e^{-i(1+ia)t}A_{\bar{z}_2} & e^{-i\hat{\mathcal{Q}}^2 t}\bar{c} &= e^{-i(ia)t}\bar{c} \\
e^{-i\hat{\mathcal{Q}}^2 t}\phi_2 &= e^{-i(ia)t}\phi_2 & e^{-i\hat{\mathcal{Q}}^2 t}c &= e^{-i(ia)t}c
\end{aligned} \tag{D.1.5}$$

Putting everything together and using (2.5.45) we have

$$\begin{aligned}
\text{Ind}_{D_{10}}(t) &= \text{Tr}_{\text{adj.}}(e^{at})\frac{\Big(\big(2e^{it}+2e^{-it}+1\big)-\big(e^{2it}+e^{-2it}+3\big)\Big)}{|1-e^{it}|^4} \\
&= -\text{Tr}_{\text{adj.}}(e^{at})\frac{1+e^{2it}}{(1-e^{it})^2} = -\sum_{\alpha\in\Delta}e^{\alpha\cdot at}\times\frac{1+e^{2it}}{(1-e^{it})^2}\,.
\end{aligned} \tag{D.1.6}$$

$\alpha\cdot a$ is defined to be a eigenvalue of the matrix a while Δ is the set of roots of the gauge algebra.

D.2 Obtaining the products of the eigenvalues

We have different possible answers of the 1-loop determinant depending on whether we expand the denominator, for each numerator's addend of (D.1.6), in powers of e^{it} or e^{-it}. As it happens in these cases for localization's computations in AdS [59,61], one needs to find the right answer via additional arguments after having examined all possibilities.

For these different expansions one can suitably rewrite (D.1.6) in four different ways

$$\begin{aligned}
\mathrm{Ind}^{[\mathrm{a}]}_{D_{10}}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \frac{1+e^{2it}}{(1-e^{it})^2}\,,\\
\mathrm{Ind}^{[\mathrm{b}]}_{D_{10}}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \frac{1+e^{-2it}}{(1-e^{-it})^2}\,,\\
\mathrm{Ind}^{[\mathrm{c}]}_{D_{10}}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \left(\frac{1}{(1-e^{it})^2} + \frac{1}{(1-e^{-it})^2}\right),\\
\mathrm{Ind}^{[\mathrm{d}]}_{D_{10}}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at} \times \left(\frac{e^{2it}}{(1-e^{it})^2} + \frac{e^{-2it}}{(1-e^{-it})^2}\right).
\end{aligned} \tag{D.2.1}$$

For instance the resulting expansion of

$$\begin{aligned}
\mathrm{Ind}^{[\mathrm{a}]}_{D_{10}}(t) &= -\sum_{\alpha\in\Delta} e^{\alpha\cdot at}(1+e^{2it})\left(\sum_{k\geqslant 0} e^{ikt}\right)^2\\
&= -\sum_{\alpha\in\Delta} e^{-i(i\alpha\cdot a)t} \; - \; \sum_{\alpha\in\Delta}\sum_{k>0} 2k e^{-i(i\alpha\cdot a-k)t}\,,
\end{aligned} \tag{D.2.2}$$

using (2.5.43)-(2.5.44), leads to

$$Z^{[a]}_{\text{1-loop}} = \prod_{\alpha\in\Delta^+} \alpha\cdot a \prod_{k>0}\big(k^2+(\alpha\cdot a)^2\big)^k\,, \tag{D.2.3}$$

with Δ^+ being the set of positive roots. Coincidentally $\mathrm{Ind}^{[\mathrm{b}]}_{D_{10}}$ gives

$$Z^{[b]}_{\text{1-loop}} = Z^{[a]}_{\text{1-loop}}\,, \tag{D.2.4}$$

but the other two choices give different answers

$$\begin{aligned}
Z^{[c]}_{\text{1-loop}} &= \prod_{\alpha\in\Delta^+} (\alpha\cdot a)^2 \prod_{k>0} (k+i\alpha\cdot a)^{k+1}(k-i\alpha\cdot a)^{k+1}\,,\\
Z^{[d]}_{\text{1-loop}} &= \prod_{\alpha\in\Delta^+}\prod_{k>0} (k+i\alpha\cdot a)^{k-1}(k-i\alpha\cdot a)^{k-1}\,.
\end{aligned} \tag{D.2.5}$$

We now discuss the regularization of these products.

D.3 Regularizing the products

In order to regularize the 1-loop infinite products that we have found, we can use these two expressions for the Γ-function and Barnes G-function

$$\begin{aligned} \Gamma(z) &= \frac{e^{-\gamma_E z}}{z}\prod_{k>0}\left(1+\frac{z}{k}\right)^{-1} e^{\frac{z}{k}}\,, \\ G(z) &= (2\pi)^{\frac{z}{2}} e^{-\frac{1+\gamma_E z^2+z}{2}}\prod_{k>0}\left(1+\frac{z}{k}\right)^{k} e^{-z+\frac{z^2}{k}}\,, \end{aligned} \tag{D.3.1}$$

where γ_E the Euler-Mascheroni constant. With these equations we can write the following formal products

$$\begin{aligned} \prod_{k>0}(k+z) &= \prod_{k>0} k e^{\frac{z}{k}} \;\times\; \frac{e^{-\gamma_E z}}{\Gamma(z+1)}\,, \\ \prod_{k>0}(k+z)^k &= \prod_{k>0} k^k e^{z-\frac{z^2}{2k}} \;\times\; G(1+z)\frac{e^{\frac{1+\gamma_E z^2+z}{2}}}{(2\pi)^{\frac{z}{2}}}\,. \end{aligned} \tag{D.3.2}$$

Using the expression above, and keeping for now all infinite constants, one gets the following results for 1-loop determinants in (D.2.3)-(D.2.4)-(D.2.5)

$$\begin{aligned} Z^{[a]}_{\text{1-loop}} = Z^{[b]}_{\text{1-loop}} &= \Big(e\prod_{k>0}k^{2k}\Big)^{\mathrm{rk}(G)} \;\times\; \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \\ &\quad \prod_{\alpha\in\Delta^+}\alpha\cdot a\; G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-(\gamma_E\alpha\cdot a)^2}\,, \\ Z^{[c]}_{\text{1-loop}} &= \Big(e\prod_{k>0}(-1)^{1+k}k^{2k+2}\Big)^{\mathrm{rk}(G)} \;\times\; \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \\ &\quad \prod_{\alpha\in\Delta^+}\alpha\cdot a\;\; \sinh(\pi\alpha\cdot a)G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}\,, \\ Z^{[d]}_{\text{1-loop}} &= \Big(e\prod_{k>0}(-1)^{1+k}k^{2k+2}\Big)^{\mathrm{rk}(G)} \;\times\; \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \\ &\quad \prod_{\alpha\in\Delta^+}\frac{\alpha\cdot a}{\sinh(\pi\alpha\cdot a)}G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}\,, \end{aligned} \tag{D.3.3}$$

with rk(G) being the rank of the gauge group. The a-independent quantities can be safely removed

$$
\begin{aligned}
Z^{[a]}_{\text{1-loop}} = Z^{[b]}_{\text{1-loop}} &= \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \times \prod_{\alpha\in\Delta^+} \alpha\cdot a\; G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}\,,\\
Z^{[c]}_{\text{1-loop}} &= \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \times \prod_{\alpha\in\Delta^+} \alpha\cdot a\;\sinh(\pi\alpha\cdot a)G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}\,,\\
Z^{[d]}_{\text{1-loop}} &= \prod_{\alpha\in\Delta^+}\prod_{k>0} e^{\frac{(\alpha\cdot a)^2}{k}} \times \prod_{\alpha\in\Delta^+} \frac{\alpha\cdot a}{\sinh(\pi\alpha\cdot a)} G(1+i\alpha\cdot a)G(1-i\alpha\cdot a)e^{-\gamma_E(\alpha\cdot a)^2}\,.
\end{aligned}
\tag{D.3.4}
$$

We remain with an infinite quadratic part $\sim e^{\sum_k \frac{\alpha\cdot a}{k}}$ which can be absorbed, together with the finite piece coming from $\sim e^{-\gamma_E(\alpha\cdot a)^2}$, into the renormalization of g^2.

D.4 Final result

The partition function we are working with is invariant under $\phi_1 \to -\phi_1$. For this reason we rule out [a] and [b], since with these 1-loop determinants the partition function varies as we send a to $-a$.

We also do not expect the 1-loop contribution to be zero at the origin of the Coulomb branch, because this would give in that point a vanishing partition function. We thus rule out [c].

We then expect the right partition function for Dirichlet boundary conditions to be [d]. This coincides with the Dirichlet 1-loop contribution of HS^4 (2.5.48). This also is consistent with the fact that we would expect, as explained at the end of Section 2.5.6, that in superconformal fixed points the Z^{HS^4} and Z^{AdS_4} should match.

If we study Neumann boundary conditions, we need to add ghosts for ghosts, as explained in Section 2.5.2. This can be seen as adding in the index (D.1.6) a +2 [3]. Repeating thus the argument of the index we find again four possible 1-loop results which are equivalent to the ones showed in (D.3.4) apart for the multiplication by a factor of $\prod_{\alpha\in\Delta^+}\frac{1}{(\alpha\cdot a)^2}$. The only even expressions are again [c] and [d]. In this case we are not able to single out the partition function with an argument based solely on the AdS computation. The fact that the 1-loop contribution [d] becomes singular in $a = 0$, after the multiplication of $\prod_{\alpha\in\Delta^+}\frac{1}{(\alpha\cdot a)^2}$, can be interpreted as a signal for discarding this result but it can't be used as a strict argument. Indeed the partition function remains finite (this happens because of what showed in equation (2.5.31)).

We however argued in Section 2.5.6 that the 1-loop result with Neumann boundary conditions should match the one on HS^4 with the same boundary conditions. With this in mind, and referring to (2.5.54), we exclude [d] and choose [c] as the result.

We interpret already satisfactory the fact that one of the two 1-loop contributions was singled out with independent arguments while the other one still remained among the consistent possible results.

As we renormalize the coupling, due to the divergences discussed below equation (D.3.4), we obtain that the classical part should be proportional to a power of a dimensionless combination of the dynamically generated scale Λ, which in pure SYM in AdS$_4$ can only be ΛL. It is possible to see that also the instanton expansion of the Nekrasov partition is a function of ΛL. This can be seen by restoring $\frac{1}{L}$ in the values of the fields at the locus (2.5.35) and in the twisting parameters of the $\Omega_{\epsilon_1,\epsilon_2}$-background in (2.5.32)

$$Z_{\text{Nekrasov}}(a, \Lambda L) = \sum_{k \geqslant 0} (\Lambda L)^k Z_k(a) \,. \tag{D.4.1}$$

We thus obtain, for Dirichlet boundary conditions, the following partition function for pure SYM

$$Z^{\text{D}}(a, \Lambda L) = (\Lambda L)^{2\text{Tr}(a^2)} \frac{\alpha \cdot a}{\sinh(\pi \alpha \cdot a)} G(1 + i\alpha \cdot a) G(1 - i\alpha \cdot a) Z_{\text{Nekrasov}}(a, \Lambda L) \,. \tag{D.4.2}$$

Appendix E

String-theoretic boundary conditions

In this section we discuss which 3d $\mathcal{N} = 2$ boundary modes are expected to survive and which are expected to freeze when D4s intersect branes. We study this firstly when the D4 intersect the NS5 of Table 6.1 and then when D4s also end on boundary branes like the ones in Table 6.3.

We are not going to discuss in depth the boundary conditions, just which modes are fixed and which ones survive. The boundary conditions are indeed expected to at least depend on how many D4 end on a given brane, as it happens in [17, 18]. When D3s end on D5s, the 3d $\mathcal{N} = 4$ vector multiplet modes inside the 4d $\mathcal{N} = 4$ vector ones freeze. However, for instance, depending on whether a stack of D3s ends on a single D5 or if each D3 ends on a different D5, the boundary conditions are different (respectively, these give rise to Nahm or Dirichlet boundary conditions).

Following [105] and [84], we explain the amount of preserved supercharges by looking at the constraints that the branes impose on the Killing spinor. We also argue which mode freeze depending on the brane content as in [17, 84] and as summarized in [29].

For our 10d conventions, which we use only for this appendix, we let $M\,,\ N\,,\ \ldots \in \{0, ..., 9\}$ be 10d Minkowski index[42]. We also employ

$$\eta_{MN} = \mathrm{diag}(-1, 1, 1, 1, 1, 1, 1, 1, 1, 1)\,, \tag{E.0.1}$$

[42] We use the Minkowski notation to ease the relations with [105]. This is not a problem since the Euclidean theories we are considering are thought as a Wick rotation of the Minkowski ones.

with which we define the 10d gamma matrices as satisfying

$$\{\Gamma_M, \Gamma_N\} = 2\eta_{MN}\,. \tag{E.0.2}$$

We also denote the "Γ^{10}" as $\bar{\Gamma}$, which with these choices is simply

$$\bar{\Gamma} = \Gamma^0 \dots \Gamma^9\,. \tag{E.0.3}$$

E.1 Theory on a D4 and its modes

Here we discuss the theory on D4 slab. We thus start with the 5d $\mathcal{N} = 4$ SYM theory of a flat D4 in 10d space. With this we mean that, choosing a frame, we want the scalar modes to be directly related to the directions transverse to the D4. In this way, once conventions are fixed and once we place boundary-branes, it is possible to argue which modes are frozen, which not and how these are rotated by the supersymmetry preserved by the boundary-branes.

We achieve this by dimensionally reducing the 10d SYM as in [18], while keeping manifest a relation between the 10d indexes throughout the discussion (this can be thought as obtaining the theory on the D4s after five T-dualities on D9s).

The 5d $\mathcal{N} = 4$ SYM Lagrangian can then be written in 10d language as

$$\mathcal{L} = \frac{1}{2g^2}\bigg(F^{MN}F_{MN} - i\bar{\Psi}\Gamma^M D_M \Psi\bigg)\,, \tag{E.1.1}$$

where we treat the 10d Majorana-Weyl spinor Ψ as 5d symplectic-Majorana spinors rotated by $\mathfrak{usp}(2)_{\mathcal{R}} \sim \mathfrak{so}(5)_{\mathcal{R}}$. Then, defining $\mu\,,\nu\,,\ \dots \in \{0,\dots 4\}$ and $I\,,\ J\,,\ \dots \in \{5,\dots,9\}$, we write

$$A_M\big|_{M=\mu} = A_\mu\,, \qquad A_M\big|_{M=I} = \phi_I\,, \tag{E.1.2}$$

where A_μ becomes the gauge field living on the branes and the ϕ_I are going to be scalars representing the transverse directions where the brane can be displaced.

In Type IIA the supersymmetry transformations are given by two Majorana-Weyl spinors with opposite chirality

$$\epsilon = \bar{\Gamma}\epsilon\,, \quad \tilde{\epsilon} = -\bar{\Gamma}\tilde{\epsilon}\,. \tag{E.1.3}$$

while for the "$\epsilon_{\text{5d SYM}}$" one, in 10d language, we only need one Majorana-Weyl spinor which we can take to have positive chirality. The link with this parameter to the ones of Type IIA comes from the fact that the Type IIA's spinor of negative chirality $\tilde{\epsilon}$ is linked to ϵ by the following restriction imposed by the D4 [105] (refer to Table 6.1)

$$\tilde{\epsilon} = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_4\epsilon\,. \tag{E.1.4}$$

We then take the positive chirality spinor ϵ to be the one of the 5d SYM. We write below the supersymmetry transformations in 10d notation

$$\begin{aligned} \delta A_M &= \epsilon\Gamma_M\Psi\,, \\ \delta\Psi &= \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon\,. \end{aligned} \tag{E.1.5}$$

E.1.1 D4 on the slab

In Type IIA the constraints given by the NS5 in Table 6.1 are given by

$$\epsilon = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5\Gamma_6\epsilon\,, \quad \tilde{\epsilon} = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5\Gamma_6\tilde{\epsilon}\,. \tag{E.1.6}$$

Together with the constraints written in E.1.4, there are the ones of D6s that extend along 0123789 which can be proven to be redundant

$$\tilde{\epsilon} = \Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_7\Gamma_8\Gamma_9\epsilon\,. \tag{E.1.7}$$

This means that we can add D6s stretched along 0123789 without breaking further supersymmetry. This is not a surprise, this set up is T-dual to the one in [84]. Indeed we are going to obtain results analogue to the ones argue in [84], but we intend to explain everything more pedagogically both for the thesis aims and for explaining what happens when we add further branes to this system.

It is then possible to show, while keeping the 10d notation, that by defining[43]

$$\begin{aligned} &P^{\pm}_{\text{bulk}} = \frac{\mathbf{1} \pm \Gamma_4\Gamma_7\Gamma_8\Gamma_9}{2}\,, \\ &\Psi^{\pm} = P^{\pm}_{\text{bulk}}\Psi\,, \\ &a\,,\ b\,,\ \ldots\{0,\ldots,3,5,6\}\,, \\ &\tilde{a}\,,\ \tilde{b}\,,\ \ldots\{4,7,8,9\}\,, \end{aligned} \tag{E.1.8}$$

[43]The directions displayed in the product of Γ-matrices below connects to the string-theoretic parameters by being related to the directions where the NS5 does not extend, see Table 6.1.

we can construct two 4d $\mathcal{N}=2$ multiplets out of the 5d $\mathcal{N}=4$ vector one showed in (E.1.5), if we add the constraint in (E.1.6)

$$\begin{array}{ll} \textbf{4d } \boldsymbol{\mathcal{N}} \textbf{ = 2 vector multiplet} & \textbf{4d } \boldsymbol{\mathcal{N}} \textbf{= 2 hypermultiplet} \\ \delta A_a = \epsilon \Gamma_a \Psi^+ \,, & \delta A_{\tilde{a}} = \epsilon \Gamma_a \Psi^- \,, \\ \delta \Psi^+ = \frac{1}{2} F_{ab} \Gamma^{ab} \epsilon \,. & \delta \Psi^- = F_{a\tilde{b}} \Gamma^{a\tilde{b}} \epsilon \,. \end{array} \tag{E.1.9}$$

If the D4, which stretches along 01234, ends on a D6 that stretches along 0123789, then the scalars $\Phi_{5,6} \subset A_a$ should be fixed in order for the D4 to not detach from the D6. This is achieved in a supersymmetric fashion by freezing the modes of a 4d $\mathcal{N}=2$ vector multiplet while leaving the ones of a 4d $\mathcal{N}=2$ hypermultiplet fluctuating.

One can see that the opposite is true if the D4 ends on NS5s, which is precisely what we see in Section 6.1.

E.1.2 Adding boundary branes

We now want to discuss the additional boundary branes in Table 6.3. We thus are interested in which 3d $\mathcal{N}=2$ modes, out of the ones in 4d $\mathcal{N}=2$ vector multiplets ones in E.1.9, remain fluctuating on the boundary branes.

The constraints coming from the boundary D6 and NS5 are again redundant among each other, we thus write only the ones of the D6

$$\tilde{\epsilon} = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_4 \Gamma_6 \Gamma_8 \Gamma_9 \epsilon \,. \tag{E.1.10}$$

It is also possible to show explicitly that the total constraints preserve four independent supercharges,[44] namely the same amount of 3d $\mathcal{N}=2$ supersymmetry.

We then define

$$\begin{aligned} \Gamma_{\mathrm{NS5,br}} &= \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^4 \Gamma^5 \Gamma^7 \,, \\ P^{\pm}_{\mathrm{NS5,br}} &= \frac{1 + \Gamma_{\mathrm{NS5,br}} \bar{\Gamma}}{2} \,, \end{aligned} \tag{E.1.11}$$

and, by following the same steps of the previous section, we can arrive to the result

[44] We checked this claim by using an explicit spinor base and Mathematica.

schematically written below

NS5 boundary brane	**D6 boundary brane**	
Frozen modes	**Frozen modes**	
$A_{M=3,6} = 0\,,$	$A_{M=0,1,2,5} = 0\,,$	
$P^{-}_{\mathrm{NS5,br}}\Psi^{+} = 0\,.$	$P^{+}_{\mathrm{NS5,br}}\Psi^{+} = 0\,.$	(E.1.12)
Fluctuating modes	**Fluctuating modes**	
3d $\mathcal{N} = 2$ vector	**3d $\mathcal{N} = 2$ chiral**	
$A_{M=0,1,2,5}\,,$	$A_{M=3,6}\,,$	
$P^{+}_{\mathrm{NS5,br}}\Psi^{+}\,.$	$P^{-}_{\mathrm{NS5,br}}\Psi^{+}\,.$	

Bibliography

[1] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," *Phys. Rev.* **159** (1967) 1251–1256. vi

[2] R. Haag, J. T. Lopuszanski, and M. Sohnius, "All Possible Generators of Supersymmetries of the s Matrix," *Nucl. Phys. B* **88** (1975) 257. vi

[3] V. Pestun, "Localization of gauge theory on a four-sphere and supersymmetric Wilson loops," *Commun. Math. Phys.* **313** (2012) 71–129, `arXiv:0712.2824 [hep-th]`. vi, 2, 5, 8, 47, 48, 52, 53, 54, 63, 141

[4] V. Pestun *et al.*, "Localization techniques in quantum field theories," *J. Phys. A* **50** no. 44, (2017) 440301, `arXiv:1608.02952 [hep-th]`. vi, 2, 47

[5] S. Rychkov and N. Su, "New Developments in the Numerical Conformal Bootstrap," `arXiv:2311.15844 [hep-th]`. vii

[6] D. Poland, S. Rychkov, and A. Vichi, "The Conformal Bootstrap: Theory, Numerical Techniques, and Applications," *Rev. Mod. Phys.* **91** (2019) 015002, `arXiv:1805.04405 [hep-th]`. vii

[7] A. Bissi, A. Sinha, and X. Zhou, "Selected topics in analytic conformal bootstrap: A guided journey," *Phys. Rept.* **991** (2022) 1–89, `arXiv:2202.08475 [hep-th]`. vii

[8] T. Hartman, D. Mazac, D. Simmons-Duffin, and A. Zhiboedov, "Snowmass White Paper: The Analytic Conformal Bootstrap," in *Snowmass 2021*. 2, 2022. `arXiv:2202.11012 [hep-th]`. vii

[9] D. Bason, L. Di Pietro, R. Valandro, and J. van Muiden, "BCFT One-point Functions of Coulomb Branch Operators," `arXiv:2311.17888 [hep-th]`. vii, 9, 10, 18, 37, 38, 57

[10] E. Witten, "SL(2,Z) action on three-dimensional conformal field theories with Abelian symmetry," in *From Fields to Strings: Circumnavigating Theoretical Physics: A Conference in Tribute to Ian Kogan*, pp. 1173–1200. 7, 2003. arXiv:hep-th/0307041. vii, 79, 80, 124

[11] T. Dimofte, D. Gaiotto, and S. Gukov, "Gauge Theories Labelled by Three-Manifolds," *Commun. Math. Phys.* **325** (2014) 367–419, arXiv:1108.4389 [hep-th]. vii, 79, 91, 94

[12] C. G. Callan, Jr. and F. Wilczek, "INFRARED BEHAVIOR AT NEGATIVE CURVATURE," *Nucl. Phys. B* **340** (1990) 366–386. vii, 101

[13] O. Aharony, M. Berkooz, D. Tong, and S. Yankielowicz, "Confinement in Anti-de Sitter Space," *JHEP* **02** (2013) 076, arXiv:1210.5195 [hep-th]. vii, 101, 102

[14] R. Ciccone, F. De Cesare, L. Di Pietro, and M. Serone, "Exploring Confinement in Anti-de Sitter Space," arXiv:2407.06268 [hep-th]. viii, 102

[15] D. Bason, C. Copetti, L. Di Pietro, Z. Ji, and S. Komatsu, "to appear,". viii, 48, 65, 124

[16] N. Seiberg and E. Witten, "Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD," *Nucl. Phys. B* **431** (1994) 484–550, arXiv:hep-th/9408099. viii, 107

[17] D. Gaiotto and E. Witten, "S-Duality of Boundary Conditions In N=4 Super Yang-Mills Theory," *Adv. Theor. Math. Phys.* **13** no. 3, (2009) 721–896, arXiv:0807.3720 [hep-th]. viii, 79, 109, 118, 119, 143

[18] D. Gaiotto and E. Witten, "Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory," *J. Statist. Phys.* **135** (2009) 789–855, arXiv:0804.2902 [hep-th]. viii, 109, 143, 144

[19] E. Witten, "Solutions of four-dimensional field theories via M theory," *Nucl. Phys. B* **500** (1997) 3–42, arXiv:hep-th/9703166. viii, 109, 110, 111, 121, 124

[20] D. Bason and R. Valandro, "to appear,". viii, 109, 124

[21] M. Bertolini, "Lectures on supersymmetry," *Lecture notes given at SISSA* (2015) . 1, 2

[22] J. D. Lykken, "Introduction to supersymmetry," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 85–153. 6, 1996. `arXiv:hep-th/9612114`. 1

[23] S. J. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, vol. 58 of *Frontiers in Physics*. 1983. `arXiv:hep-th/0108200`. 1

[24] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg, and M. J. Strassler, "Aspects of N=2 supersymmetric gauge theories in three-dimensions," *Nucl. Phys. B* **499** (1997) 67–99, `arXiv:hep-th/9703110`. 1

[25] C. Closset, T. T. Dumitrescu, G. Festuccia, Z. Komargodski, and N. Seiberg, "Contact Terms, Unitarity, and F-Maximization in Three-Dimensional Superconformal Theories," *JHEP* **10** (2012) 053, `arXiv:1205.4142 [hep-th]`. 2, 64

[26] E. Gerchkovitz, J. Gomis, N. Ishtiaque, A. Karasik, Z. Komargodski, and S. S. Pufu, "Correlation Functions of Coulomb Branch Operators," *JHEP* **01** (2017) 103, `arXiv:1602.05971 [hep-th]`. 2, 35, 36, 39, 56, 67, 74

[27] M. Billò, F. Galvagno, and A. Lerda, "BPS wilson loops in generic conformal $\mathcal{N} = 2$ SU(N) SYM theories," *JHEP* **08** (2019) 108, `arXiv:1906.07085 [hep-th]`. 2

[28] A. Grassi, Z. Komargodski, and L. Tizzano, "Extremal correlators and random matrix theory," *JHEP* **04** (2021) 214, `arXiv:1908.10306 [hep-th]`. 2, 94, 123

[29] S. Komatsu and Y. Wang, "Non-perturbative defect one-point functions in planar $\mathcal{N} = 4$ super-Yang-Mills," *Nucl. Phys. B* **958** (2020) 115120, `arXiv:2004.09514 [hep-th]`. 2, 123, 143

[30] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 5, 2012. 5, 25

[31] C. Closset, T. T. Dumitrescu, G. Festuccia, and Z. Komargodski, "Supersymmetric Field Theories on Three-Manifolds," *JHEP* **05** (2013) 017, arXiv:1212.3388 [hep-th]. 5, 12, 14, 25, 26, 43, 45, 56

[32] J. Wess and J. Bagger, *Supersymmetry and supergravity*. Princeton University Press, Princeton, NJ, USA, 1992. 6

[33] N. Hama and K. Hosomichi, "Seiberg-Witten Theories on Ellipsoids," *JHEP* **09** (2012) 033, arXiv:1206.6359 [hep-th]. [Addendum: JHEP 10, 051 (2012)]. 8, 16, 37, 48, 52, 55, 130

[34] E. Lauria and A. Van Proeyen, *$\mathcal{N} = 2$ Supergravity in $D = 4, 5, 6$ Dimensions*, vol. 966. 3, 2020. arXiv:2004.11433 [hep-th]. 9, 10, 25, 28

[35] B. Willett, "Localization on three-dimensional manifolds," *J. Phys. A* **50** no. 44, (2017) 443006, arXiv:1608.02958 [hep-th]. 12, 14, 33, 37, 38, 134

[36] C. Closset and H. Kim, "Three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and partition functions on Seifert manifolds: A review," *Int. J. Mod. Phys. A* **34** no. 23, (2019) 1930011, arXiv:1908.08875 [hep-th]. 14, 87, 96

[37] J. Erdmenger, Z. Guralnik, and I. Kirsch, "Four-dimensional superconformal theories with interacting boundaries or defects," *Phys. Rev. D* **66** (2002) 025020, arXiv:hep-th/0203020. 14

[38] M. Billò, V. Gonçalves, E. Lauria, and M. Meineri, "Defects in conformal field theory," *JHEP* **04** (2016) 091, arXiv:1601.02883 [hep-th]. 19, 24

[39] J. D. Qualls, "Lectures on Conformal Field Theory," arXiv:1511.04074 [hep-th]. 19

[40] W. Nahm, "Supersymmetries and their Representations," *Nucl. Phys. B* **135** (1978) 149. 20

[41] L. Eberhardt, "Superconformal symmetry and representations," *J. Phys. A* **54** no. 6, (2021) 063002, arXiv:2006.13280 [hep-th]. 21

[42] P. Liendo, L. Rastelli, and B. C. van Rees, "The Bootstrap Program for Boundary CFT_d," *JHEP* **07** (2013) 113, arXiv:1210.4258 [hep-th]. 24

[43] D. Mazáč, L. Rastelli, and X. Zhou, "An analytic approach to BCFT$_d$," *JHEP* **12** (2019) 004, arXiv:1812.09314 [hep-th]. 24

[44] G. Festuccia and N. Seiberg, "Rigid Supersymmetric Theories in Curved Superspace," *JHEP* **06** (2011) 114, arXiv:1105.0689 [hep-th]. 25

[45] K. Hosomichi, "$\mathcal{N} = 2$ SUSY gauge theories on S^4," *J. Phys. A* **50** no. 44, (2017) 443010, arXiv:1608.02962 [hep-th]. 26, 30, 37

[46] J. Gomis and N. Ishtiaque, "Kähler potential and ambiguities in 4d $\mathcal{N} = 2$ SCFTs," *JHEP* **04** (2015) 169, arXiv:1409.5325 [hep-th]. 35, 68, 69, 71

[47] E. Gerchkovitz, J. Gomis, and Z. Komargodski, "Sphere Partition Functions and the Zamolodchikov Metric," *JHEP* **11** (2014) 001, arXiv:1405.7271 [hep-th]. 35, 68, 70, 76

[48] J. Gomis, P.-S. Hsin, Z. Komargodski, A. Schwimmer, N. Seiberg, and S. Theisen, "Anomalies, Conformal Manifolds, and Spheres," *JHEP* **03** (2016) 022, arXiv:1509.08511 [hep-th]. 35, 76

[49] D. Z. Freedman and S. S. Pufu, "The holography of F-maximization," *JHEP* **03** (2014) 135, arXiv:1302.7310 [hep-th]. 37, 41

[50] A. Dabholkar, N. Drukker, and J. Gomes, "Localization in supergravity and quantum AdS_4/CFT_3 holography," *JHEP* **10** (2014) 090, arXiv:1406.0505 [hep-th]. 37, 48, 56

[51] D. Carmi, L. Di Pietro, and S. Komatsu, "A Study of Quantum Field Theories in AdS at Finite Coupling," *JHEP* **01** (2019) 200, arXiv:1810.04185 [hep-th]. 38

[52] M. Bianchi, D. Z. Freedman, and K. Skenderis, "Holographic renormalization," *Nucl. Phys. B* **631** (2002) 159–194, arXiv:hep-th/0112119. 40

[53] S. de Haro, S. N. Solodukhin, and K. Skenderis, "Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence," *Commun. Math. Phys.* **217** (2001) 595–622, arXiv:hep-th/0002230. 40

[54] E. Witten, "Topological Quantum Field Theory," *Commun. Math. Phys.* **117** (1988) 353. 47

[55] N. A. Nekrasov, "Seiberg-Witten prepotential from instanton counting," *Adv. Theor. Math. Phys.* **7** no. 5, (2003) 831–864, `arXiv:hep-th/0206161`. 47, 48, 55, 57, 61

[56] E. Gava, K. S. Narain, M. N. Muteeb, and V. I. Giraldo-Rivera, "$N = 2$ gauge theories on the hemisphere HS^4," *Nucl. Phys. B* **920** (2017) 256–297, `arXiv:1611.04804 [hep-th]`. 48, 60, 126, 130

[57] M. Dedushenko, "Gluing II: boundary localization and gluing formulas," *Lett. Math. Phys.* **111** no. 1, (2021) 18, `arXiv:1807.04278 [hep-th]`. 48, 60

[58] S. G. Avery and B. U. W. Schwab, "Noether's second theorem and Ward identities for gauge symmetries," *JHEP* **02** (2016) 031, `arXiv:1510.07038 [hep-th]`. 52

[59] B. Assel, D. Martelli, S. Murthy, and D. Yokoyama, "Localization of supersymmetric field theories on non-compact hyperbolic three-manifolds," *JHEP* **03** (2017) 095, `arXiv:1609.08071 [hep-th]`. 56, 59, 138

[60] J. R. David, E. Gava, R. K. Gupta, and K. Narain, "Boundary conditions and localization on AdS. Part I," *JHEP* **09** (2018) 063, `arXiv:1802.00427 [hep-th]`. 56

[61] A. González Lezcano, I. Jeon, and A. Ray, "Supersymmetric localization: $\mathcal{N} = (2)$ theories on S^2 and AdS_2," *JHEP* **07** (2023) 056, `arXiv:2302.10370 [hep-th]`. [Erratum: JHEP 09, 003 (2023)]. 59, 138

[62] D. L. Jafferis, "The Exact Superconformal R-Symmetry Extremizes Z," *JHEP* **05** (2012) 159, `arXiv:1012.3210 [hep-th]`. 61, 64, 86

[63] D. Gaiotto, "Boundary F-maximization," `arXiv:1403.8052 [hep-th]`. 63, 64, 78, 82, 124

[64] H. Casini, I. Salazar Landea, and G. Torroba, "Irreversibility, QNEC, and defects," *JHEP* **07** (2023) 004, `arXiv:2303.16935 [hep-th]`. 64

[65] A. B. Zamolodchikov, "Irreversibility of the Flux of the Renormalization Group in a 2D Field Theory," *JETP Lett.* **43** (1986) 730–732. 76

[66] N. Seiberg, "Observations on the Moduli Space of Superconformal Field Theories," *Nucl. Phys. B* **303** (1988) 286–304. 76

[67] D. Kutasov, "Geometry on the Space of Conformal Field Theories and Contact Terms," *Phys. Lett. B* **220** (1989) 153–158. 76

[68] K. Papadodimas, "Topological Anti-Topological Fusion in Four-Dimensional Superconformal Field Theories," *JHEP* **08** (2010) 118, `arXiv:0910.4963 [hep-th]`. 76, 78

[69] V. Niarchos and K. Papadodimas, "On the Kähler-Hodge structure of superconformal manifolds," `arXiv:2112.11425 [hep-th]`. 76

[70] C. P. Bachas, I. Brunner, M. R. Douglas, and L. Rastelli, "Calabi's diastasis as interface entropy," *Phys. Rev. D* **90** no. 4, (2014) 045004, `arXiv:1311.2202 [hep-th]`. 77, 78

[71] E. Calabi, "Isometric imbedding of complex manifolds," *Annals of Mathematics* **58** no. 1, (1953) 1–23. `http://www.jstor.org/stable/1969817`. 77

[72] K. Goto and T. Okuda, "Interface entropy in four dimensions as Calabi's diastasis on the conformal manifold," *JHEP* **11** (2018) 122, `arXiv:1805.09981 [hep-th]`. 78

[73] K. Goto, L. Nagano, T. Nishioka, and T. Okuda, "Janus interface entropy and Calabi's diastasis in four-dimensional $\mathcal{N} = 2$ superconformal field theories," *JHEP* **08** no. 08, (2020) 048, `arXiv:2005.10833 [hep-th]`. 78

[74] S. Cecotti and C. Vafa, "Topological antitopological fusion," *Nucl. Phys. B* **367** (1991) 359–461. 78

[75] M. Baggio, V. Niarchos, and K. Papadodimas, "On exact correlation functions in SU(N) $\mathcal{N} = 2$ superconformal QCD," *JHEP* **11** (2015) 198, `arXiv:1508.03077 [hep-th]`. 78

[76] A. Kapustin and M. Tikhonov, "Abelian duality, walls and boundary conditions in diverse dimensions," *JHEP* **11** (2009) 006, `arXiv:0904.0840 [hep-th]`. 79

[77] L. Di Pietro, D. Gaiotto, E. Lauria, and J. Wu, "3d Abelian Gauge Theories at the Boundary," *JHEP* **05** (2019) 091, `arXiv:1902.09567 [hep-th]`. 79

[78] J. Wu, "Conformal Boundary for the Four Dimensional Free Gauge Theory," Master's thesis, Perimeter Institute, 2017. 86, 91

[79] R. Kumar Gupta, C. P. Herzog, and I. Jeon, "Duality and Transport for Supersymmetric Graphene from the Hemisphere Partition Function," *JHEP* **05** (2020) 023, arXiv:1912.09225 [hep-th]. 86, 89, 91, 96

[80] A. Kapustin, B. Willett, and I. Yaakov, "Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter," *JHEP* **03** (2010) 089, arXiv:0909.4559 [hep-th]. 86

[81] S. Benvenuti and S. Pasquetti, "3d $\mathcal{N} = 2$ mirror symmetry, pq-webs and monopole superpotentials," *JHEP* **08** (2016) 136, arXiv:1605.02675 [hep-th]. 91

[82] L. D. Faddeev, R. M. Kashaev, and A. Y. Volkov, "Strongly coupled quantum discrete Liouville theory. 1. Algebraic approach and duality," *Commun. Math. Phys.* **219** (2001) 199–219, arXiv:hep-th/0006156. 91, 96

[83] N. Seiberg and E. Witten, "Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory," *Nucl. Phys. B* **426** (1994) 19–52, arXiv:hep-th/9407087. [Erratum: Nucl.Phys.B 430, 485–486 (1994)]. 102

[84] A. Hanany and E. Witten, "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics," *Nucl. Phys. B* **492** (1997) 152–190, arXiv:hep-th/9611230. 111, 119, 143, 145

[85] N. Halmagyi, C. Romelsberger, and N. P. Warner, "Inherited duality and quiver gauge theory," *Adv. Theor. Math. Phys.* **10** no. 2, (2006) 159–179, arXiv:hep-th/0406143. 111

[86] A. Strominger, "Massless black holes and conifolds in string theory," *Nucl. Phys. B* **451** (1995) 96–108, arXiv:hep-th/9504090. 113

[87] B. R. Greene, D. R. Morrison, and A. Strominger, "Black hole condensation and the unification of string vacua," *Nucl. Phys. B* **451** (1995) 109–120, arXiv:hep-th/9504145. 113

[88] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov, and C. Vafa, "F theory, geometric engineering and N=1 dualities," *Nucl. Phys. B* **505** (1997) 153–164, arXiv:hep-th/9612052. 113

[89] C. Vafa and B. Zwiebach, "N=1 dualities of SO and USp gauge theories and T duality of string theory," *Nucl. Phys. B* **506** (1997) 143–156, `arXiv:hep-th/9701015`. 113

[90] H. Ooguri and C. Vafa, "Geometry of N=1 dualities in four-dimensions," *Nucl. Phys. B* **500** (1997) 62–74, `arXiv:hep-th/9702180`. 113, 117

[91] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz, and C. Vafa, "A Geometric unification of dualities," *Nucl. Phys. B* **628** (2002) 3–78, `arXiv:hep-th/0110028`. 114, 118

[92] E. Witten, "Branes, Instantons, And Taub-NUT Spaces," *JHEP* **06** (2009) 067, `arXiv:0902.0948 [hep-th]`. 114, 115

[93] M. Bertolini, "Four lectures on the gauge / gravity correspondence," *Int. J. Mod. Phys. A* **18** (2003) 5647–5712, `arXiv:hep-th/0303160`. 117

[94] C. V. Johnson, *D-Branes*. Cambridge Monographs on Mathematical Physics. 2023. 117, 118

[95] S. Hellerman, D. Orlando, S. Reffert, and M. Watanabe, "On the CFT Operator Spectrum at Large Global Charge," *JHEP* **12** (2015) 071, `arXiv:1505.01537 [hep-th]`. 123

[96] A. Grassi and C. Iossa, "Matrix models for extremal and integrated correlators of higher rank," `arXiv:2408.07391 [hep-th]`. 123

[97] M. Beccaria, M. Billò, M. Frau, A. Lerda, and A. Pini, "Exact results in a $\mathcal{N} = 2$ superconformal gauge theory at strong coupling," *JHEP* **07** (2021) 185, `arXiv:2105.15113 [hep-th]`. 123

[98] M. Billo, M. Frau, A. Lerda, A. Pini, and P. Vallarino, "Localization vs holography in 4d$\mathcal{N} = 2$ quiver theories," *JHEP* **10** (2022) 020, `arXiv:2207.08846 [hep-th]`. 123

[99] D. He and C. F. Uhlemann, "Solving $\mathcal{N} = 4$ SYM BCFT matrix models at large N," `arXiv:2409.13016 [hep-th]`. 123

[100] A. Brown, F. Galvagno, and C. Wen, "Exact results for giant graviton four-point correlators," *JHEP* **07** (2024) 049, `arXiv:2403.17263 [hep-th]`. 123

[101] N. Dorey, V. V. Khoze, and M. P. Mattis, "Multi instanton calculus in N=2 supersymmetric gauge theory. 2. Coupling to matter," *Phys. Rev. D* **54** (1996) 7832–7848, arXiv:hep-th/9607202. 124

[102] T. W. Grimm, A. Klemm, M. Marino, and M. Weiss, "Direct Integration of the Topological String," *JHEP* **08** (2007) 058, arXiv:hep-th/0702187. 124

[103] S. M. Chester, "Bootstrapping 4d $\mathcal{N} = 2$ gauge theories: the case of SQCD," *JHEP* **01** (2023) 107, arXiv:2205.12978 [hep-th]. 124

[104] N. Seiberg, Y. Tachikawa, and K. Yonekura, "Anomalies of Duality Groups and Extended Conformal Manifolds," *PTEP* **2018** no. 7, (2018) 073B04, arXiv:1803.07366 [hep-th]. 124

[105] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007. 143, 145