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On Small Energy Solutions of the Nonlinear Schrödinger Equation in 1D with a Generic Trapping Potential with a Single Eigenvalue

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Abstract: We prove in dimension $d = 1$ a result similar to a classical paper by Soffer and Weinstein, Jour. Diff. Eq. 98 (1992), improving it by encompassing for pure power nonlinearities the whole range of exponents $p > 1$. The proof is based on the virial inequality of Kowalczyk et al., J. Eur. Math. Soc. (JEMS) 24 (2022), with smoothing estimates as shown in Mizumachi J. Math. Kyoto Univ. 48 (2008).

Keywords: nonlinear Schrödinger equation; asymptotic stability; ground state

MSC: 35Q55



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1. Introduction

We consider the nonlinear Schrödinger equation (NLS) on the line

$$i\partial_t u = Hu + f(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \text{ with } u(0) = u_0 \in H^1(\mathbb{R}, \mathbb{C}) \text{ and } f(u) = g(|u|^2)u \quad (1)$$

where $H = -\partial_x^2 + V$ is a Schrödinger operator with $V \in C^1(\mathbb{R}, \mathbb{R})$, that is real valued, with $|V^{(k)}(x)| \lesssim e^{-\kappa_0|x|}$ for $k = 0, 1$ and some $\kappa_0 > 0$. We will assume that 0 is not a resonance for H , that is

$$Hu = 0 \text{ with } u \in L^\infty(\mathbb{R}) \implies u = 0. \quad (2)$$

It is well known that then H has a finite number of eigenvalues, that they are of dimension one, and that for the continuous spectrum we have $\sigma_c(H) = [0, +\infty)$. We will assume that H has exactly one single eigenvalue, which we denote by $-\lambda$, and we will assume $-\lambda < 0$. We will denote by $\varphi \in \ker(H + \lambda)$ a corresponding eigenfunction with $\|\varphi\|_{L^2} = 1$, $\varphi > 0$ everywhere, with $\varphi(x) \lesssim e^{-|x|\sqrt{\lambda}}$ [1].

We will consider the spectral decomposition

$$L^2(\mathbb{R}, \mathbb{C}) = \text{Span}\{\varphi\} \oplus L_c(H) \quad (3)$$

denoting by $Pu := \langle u, \varphi \rangle \varphi + \langle u, i\varphi \rangle i\varphi$ the corresponding projection on $\text{Span}\{\varphi\}$ and $P_c = 1 - P$ where

$$(u, v) = \int_{\mathbb{R}} u(x) \bar{v}(x) dx \text{ and } \langle u, v \rangle = \Re(u, v) \text{ for } u, v : \mathbb{R} \rightarrow \mathbb{C}, \quad (4)$$

where $\bar{v}$ is the complex conjugate of v . We will also use the following notation.

- Given a Banach space X , $v \in X$ and $\varepsilon > 0$, we set $D_X(v, \varepsilon) := \{x \in X \mid \|v - x\|_X < \varepsilon\}$.

- For $\gamma \in \mathbb{R}$, we set

$$L_\gamma^2 := \{u \in \mathcal{S}'(\mathbb{R}, \mathbb{C}) \mid \|u\|_{L_\gamma^2} := \|e^{\gamma|x|}u\|_{L^2} < \infty\}, \quad (5)$$

$$H_\gamma^1 := \{u \in \mathcal{S}'(\mathbb{R}, \mathbb{C}) \mid \|u\|_{H_\gamma^1} := \|e^{\gamma|x|}u\|_{H^1} < \infty\}. \quad (6)$$

- For $s \in \mathbb{R}$, we set and $\kappa \in (0, 1)$ fixed in terms of p and small enough, we consider

$$\|\eta\|_{L^{p,s}} := \|\langle x \rangle^s \eta\|_{L^p(\mathbb{R})} \text{ where } \langle x \rangle := \sqrt{1+x^2}, \quad (7)$$

$$\|\eta\|_{\Sigma_A} := \left\| \operatorname{sech}\left(\frac{2}{A}x\right) \eta' \right\|_{L^2(\mathbb{R})} + A^{-1} \left\| \operatorname{sech}\left(\frac{2}{A}x\right) \eta \right\|_{L^2(\mathbb{R})} \text{ and} \quad (8)$$

$$\|\eta\|_{\tilde{\Sigma}} := \|\operatorname{sech}(\kappa x) \eta\|_{L^2(\mathbb{R})}. \quad (9)$$

We assume that there is a $p > 1$ and a $C > 0$ such that, for $k = 0, 1, 2$, we have

$$|g^{(k)}(s)| \leq C|s|^{\frac{p-1}{2}-k} \text{ for all } s \in (0, 1]. \quad (10)$$

The following can be proved as in [2] (Proposition 1.2).

Proposition 1 (Bound states). *Let $p > 0$. Then there is an $a_0 > 0$, an $\varepsilon_0 > 0$, and a $C > 0$ such that there is a unique $Q[\cdot] \in C^1(D_{\mathbb{C}}(0, \varepsilon_0), H_{\gamma_0}^1)$ satisfying the gauge property*

$$Q[e^{i\theta}z] = e^{i\theta}Q[z], \quad (11)$$

such that there is an $E \in C([0, \varepsilon_0^2], \mathbb{R})$ such that

$$HQ[z] + g(|Q[z]|^2)Q[z] = E(|z|^2)Q[z], \quad (12)$$

and for $j = 1, 2$,

$$\|Q[z] - z\varphi\|_{H_{a_0}^1} \leq C|z|^p, \quad \|D_j Q[z] - i^{j-1}\varphi\|_{H_{a_0}^1} \leq C|z|^{p-1}, \quad |E(|z|^2) + \lambda| \leq C|z|^{p-1}. \quad (13)$$

Energy $\mathbf{E}$ and Mass $\mathbf{Q}$ are invariants of (1), where

$$\mathbf{E}(u) = \frac{1}{2} \langle Hu, u \rangle + \int_{\mathbb{R}} G(|u|^2) dx \text{ where } G(0) = 0 \text{ and } G'(s) \equiv g(s), \quad (14)$$

$$\mathbf{Q}(u) = \frac{1}{2} \|u\|_{L^2(\mathbb{R})}^2. \quad (15)$$

In this paper, we prove the following result. Near the origin $H^1(\mathbb{R})$, Problem (1) is globally well posed and there are constants $\delta_0 > 0$ and $C_0 > 0$ such that, for $\|u_0\|_{H^1} < \delta_0$, $\|u(t)\|_{H^1} \leq C_0\|u_0\|_{H^1}$ for all times (see Cazenave [3]).

Theorem 1. *There is an $a > 0$ such that, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, for $\|u_0\|_{H^1} < \delta$, the solution $u(t)$ of (1) can be written uniquely for all times as*

$$u(t) = Q[z(t)] + \eta(t) \text{ with } \eta(t) \in P_c H^1, \quad (16)$$

such that we have

$$|z(t)| + \|\eta(t)\|_{H^1} \leq \varepsilon \text{ for all } t \in [0, \infty), \quad (17)$$

$$\int_0^{+\infty} \|e^{-a\langle x \rangle} \eta\|_{H^1}^2 dt \leq \varepsilon^2 \quad (18)$$

$$\lim_{t \rightarrow +\infty} \|e^{-a\langle x \rangle} \eta(t)\|_{L^2(\mathbb{R})} = 0 \quad (19)$$

and there is a $r_+ \geq 0$ such that

$$\lim_{t \rightarrow +\infty} |z(t)| = r_+. \quad (20)$$

Remark 1. Theorem 1 implies that, for a δ that is sufficiently small, there is a function $\vartheta : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\lim_{t \rightarrow +\infty} u(t)e^{-i\vartheta(t)} = Q[r_+] \text{ in } L_{loc}^\infty(\mathbb{R}). \quad (21)$$

Equation (1) is one of the most classical nonlinear Hamiltonian PDEs. The Hamiltonian is the nonlinear functional E , defined in Formula (14), which is a nonlinear real-valued functional with domain in $H^1(\mathbb{R})$. The functional Q is invariant along the flow of (1) because of a symmetry of the system and is variously called in the literature “mass” or “charge.” The physical relevance of nonlinear Schrödinger equations is their ubiquitous appearance in mean field approximations and in multiscale expansions (see [4]). It is clear that, since (1) is nonlinear, we are not free to normalize the mass (or charge) of the solution. This makes (1) very different from linear Schrödinger equations, where it is possible to normalize a nonzero solution setting the mass to be equal to 1. This is not possible here. In fact, for an equation like (1), it is very hard to say what is the asymptotic behavior in time of solutions whose mass equals 1. In this paper, we are able to study only small solutions.

In Weder [5], there is a version of Theorem 1 for $p > 2$ but where the nonlinearity is multiplied by an appropriately decaying function in $x \rightarrow \infty$. The case of the cubic nonlinearity if $V(x) = -c_0\delta(x)$ was considered by Gong Chen [6] and by Masaki et al. [7], where $c_0 > 0$ and $\delta(x)$, the Dirac delta, concentrated in 0. We considered the cubic nonlinearity in [8] with a more general set of eigenvalues than the one considered here. In [8], the nonlinearity could be different but needs to be sufficiently regular, and the regularity depends on the ratio of the distances of the first excited state from the ground state and from 0. Notice that, in [8] (Assumption 1.13), there is an additional repulsivity hypothesis on a potential obtained after Darboux transformations, which is actually unnecessary. In fact, it is not required if we replace the second virial inequality in [8] with a smoothing estimate like the ones considered here.

Here we will focus only on the more delicate case $1 < p \leq 2$, where the nonlinearity is stronger. In [2], we considered a special case of Theorem 1 with $V(x) = -c_0\delta(x)$. We improve that result because, in [2], we proved a version of (20) only for $p > 2$.

Equation (1) was considered first for dimension $d = 3$ by Soffer and Weinstein [9,10] and Pillet and Wayne [11]. Of the subsequent literature up until 2020, we highlight Gustafson et al. [12] for the case $d = 3$ and the adaption of [12] for $d = 1, 2$ by Mizumachi [13], while we refer the reader to our survey for a long list of references on the asymptotic stability of ground states of the NLS up until 2020 [14]. Various papers have been written recently proving dispersive estimates of small solutions in the case when H has no eigenvalues (see [15–26]). The papers proving dispersive estimates (for a smaller class of solutions), which appear harder to prove than Theorem 1, require a certain degree of regularity for the nonlinearity. Notice that, in our $H^1(\mathbb{R})$ framework, all results must be invariant by translation in time, a condition satisfied by Theorem 1, so it is not possible to have a fixed rate of time decay $\|\eta(t)\|_{L^\infty(\mathbb{R})}$ for all our solutions. Li and Luhrmann [21] partially recover the result in [27] with methods that do not explicitly exploit the integrable structure of the cubic NLS.

As in [28], our proof is based on the ideas of Kowalczyk et al. [29–32] where dispersion is proved by means of two distinct virial estimates. As in [28], here we perform just the first, high energy, virial estimate, while we substitute the second, low energy, virial estimate by a Kato smoothing estimate. Very important to us has been the treatment of Kato smoothing by Mizumachi [13,33]. The methods of Kowalczyk and our particular use of smoothing estimates do not require much regularity of the nonlinearity and allow us to consider Equation (1) for any $p > 1$. For an earlier example of this for a different model, using both virial inequalities, see [34]. Our methods allow us to exploit the fact that, under our

hypotheses on the Schrödinger operator H , at small energy, radiation disperses to infinity more than small energy solutions of the constant coefficients linear Schrödinger equation. The two classes of solutions are different (see [35,36]). Notice that, for $H = -\partial_x^2$ and $p \geq 3$ by Ozawa [37], for any $\eta_+ \in L^{2,2}(\mathbb{R})$, the space defined in (7), and both for focusing and defocusing equations, there is a solution u of (1) with $\|\eta(t) - e^{iS(t,\cdot)} e^{it\partial_x^2} \eta_+\|_{L^2(\mathbb{R})} \leq O(t^{-\alpha})$ with $\alpha > 1/2$ and for an appropriate real phase $S(t, x)$, which is equal to 0 if $p > 3$. Since, for $|x| \leq 1$, we have $e^{it\partial_x^2} \eta_+(x) = \frac{1}{\sqrt{2it}} e^{i\frac{x^2}{4t}} \hat{\eta}_+(0) + O(t^{-\alpha})$ with $\alpha > 1/2$, in the generic case with $\hat{\eta}_+(0) \neq 0$, we conclude that $\|\eta\|_{L^2(\mathbb{R}_+, L^2(|x| \leq 1))} = +\infty$. The fact that here we have (18) is ultimately due to the fact that our H is generic and thus is better behaved in terms of the Kato smoothing than $-\partial_x^2$, which is non-generic and has a resonance at 0. The second virial inequality in various of the papers of Kowalczyk et al. and in [34] is also due to the fact that certain linearized operators are generic in the framework of these papers.

Turning to open problems, we do not know how to prove Theorem 1 for low p 's in dimension $d \geq 2$. Another interesting problem would be to improve [38] and prove a version of Theorem 1 on the line for a periodic potential $V = q_1 + q_2$ for q_1 and a small and rapidly decreasing q_2 to 0 at infinity. Unfortunately, our proof relies on the fact that the background is flat and V decays rapidly, whether exponentially as it is here or at a sufficiently rapid polynomial decay.

Notation 1. We will use the following miscellanea of notations and definitions.

1. As in the theory of Kowalczyk et al. [31], we consider constants $A, B, \epsilon, \delta > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(\epsilon^{-1}) \gg A \gg B^2 \gg B \gg 1. \quad (22)$$

Here we will take $A \sim B^3$ (see Section 6 below), but in fact $A \sim B^n$ for any $n > 2$ would make no difference.

2. The notation $o_\epsilon(1)$ means a constant with a parameter ϵ such that

$$o_\epsilon(1) \xrightarrow{\epsilon \rightarrow 0^+} 0. \quad (23)$$

3. Given two Banach spaces X and Y , we denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators from X to Y . We write $\mathcal{L}(X) := \mathcal{L}(X, X)$.
4. We have the following elementary formulas:

$$Df(u)X = \left. \frac{d}{dt} f(u + tX) \right|_{t=0} = g(|u|^2)X + 2g'(|u|^2)u \langle u, X \rangle_{\mathbb{C}}. \quad (24)$$

5. Following the framework in Kowalczyk et al. [31], we fix an even function $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ satisfying

$$1_{[-1,1]} \leq \chi \leq 1_{[-2,2]} \text{ and } x\chi'(x) \leq 0 \text{ and set } \chi_C := \chi(\cdot/C) \text{ for a } C > 0. \quad (25)$$

6. We set

$$\mathbb{C}_\pm := \{z \in \mathbb{C} : \pm \Im z > 0\}. \quad (26)$$

2. Notation and Coordinates

We have the following ansatz, which is an elementary consequence of the Implicit Function Theorem (see [2] (Lemma 2.1)).

Lemma 1. There is a $c_0 > 0$ and a $C > 0$ such that, for all $u \in H^1$ with $\|u\|_{H^1} < c_0$, there is a unique pair $(z, \eta) \in \mathbb{C} \times P_c H^1$ such that

$$u = Q[z] + \eta \text{ with } |z| + \|\eta\|_{H^1} \leq C\|u\|_{H^1}. \quad (27)$$

The map $u \rightarrow (z, \eta)$ is in $C^1(D_{H^1}(0, c_0), \mathbb{C} \times H^1)$.

The proof of Theorem 1 is mainly based on the following continuation argument.

Proposition 2. *There is a $\delta_0 = \delta_0(\epsilon)$ s.t. if*

$$\|\eta\|_{L^2(I, \Sigma_A)} + \|\eta\|_{L^2(I, \tilde{\Sigma})} + \|\dot{z} + iEz\|_{L^2(I)} \leq \epsilon \quad (28)$$

holds for $I = [0, T]$ for some $T > 0$ and for $\delta \in (0, \delta_0)$, then, for $I = [0, T]$, Inequality (28) holds for ϵ replaced by $o_\epsilon(1)\epsilon$.

Notice that this implies that the result is true for $I = \mathbb{R}_+$. We will split the proof of Proposition 2 in a number of partial results obtained assuming the hypotheses of Proposition 2.

Proposition 3. *We have*

$$\|\dot{z} + iE(|z|^2)z\|_{L^2(I)} \lesssim \delta^{p-1}\epsilon, \quad (29)$$

$$\|\dot{z}\|_{L^\infty(I)} \lesssim \delta. \quad (30)$$

Proposition 4 (Virial Inequality). *We have*

$$\|\eta\|_{L^2(I, \Sigma_A)} \lesssim A\delta + \|\dot{z} + iE(|z|^2)z\|_{L^2(I)} + \|\eta\|_{L^2(I, \tilde{\Sigma})} + \epsilon^2. \quad (31)$$

Notice that $A\delta \ll A^{-1}\epsilon^2 = o_{B^{-1}}(1)\epsilon^2$ in (31).

Proposition 5 (Smoothing Inequality). *We have*

$$\|\eta\|_{L^2(I, \tilde{\Sigma})} \lesssim o_{B^{-1}}(1)\epsilon. \quad (32)$$

Proof of Theorem 1. It is straightforward that Propositions 3–5 imply Proposition 2 and thus the fact that we can take $I = \mathbb{R}_+$ in all the above inequalities. This in particular implies (18).

We next focus on the limit (19). We first rewrite our equation: entering the ansatz (27) in (1) and using (12), we obtain

$$i\dot{\eta} + iD_z Q[z](\dot{z} + iEz) = H\eta + (f(Q[z] + \eta) - f(Q[z])). \quad (33)$$

Then we can proceed as in [28] considering

$$\mathbf{b}(t) := 2^{-1} \|e^{-\gamma\langle x \rangle} \eta(t)\|_{L^2(\mathbb{R})}^2$$

and by obvious cancellations and orbital stability, obtaining

$$\begin{aligned} \dot{\mathbf{b}} &= \left\langle \left[e^{-2\gamma\langle x \rangle}, i\partial_x^2 \right] \eta, \eta \right\rangle - \left\langle D_z Q[z](\dot{z} + iEz), e^{-2\gamma\langle x \rangle} \eta \right\rangle \\ &\quad + \left\langle e^{-\gamma\langle x \rangle} i(f(Q[z] + \eta) - f(Q[z])), e^{-\gamma\langle x \rangle} \eta \right\rangle = O(\delta^2) \text{ for all times.} \end{aligned} \quad (34)$$

Since we already know from (18) that $\mathbf{b} \in L^1(\mathbb{R})$, we conclude that $\mathbf{b}(t) \xrightarrow{t \rightarrow +\infty} 0$. Notice that the integration by parts in (34) can be made rigorous considering that if $u_0 \in H^2(\mathbb{R})$ by the well known regularity result by Kato (see [3]), we have $\eta \in C^0(\mathbb{R}, H^2(\mathbb{R}))$, and the above argument is correct. By a standard density argument, the result can be extended to $u_0 \in H^1(\mathbb{R})$.

We prove (20). Here, notice that, by orbital stability, we can take $a > 0$ such that we have the following, which will be used below,

$$e^{-2a\langle x \rangle} |z| \gtrsim \max\{|Q[z]|, |Q[z]|^{2p-1}\} \text{ for all } t \in \mathbb{R}. \quad (35)$$

Since $\mathbf{Q}(Q[z])$ is strictly monotonic in $|z|$, it suffices to show that $\mathbf{Q}(Q[z(t)])$ converges to $t \rightarrow \infty$. From the conservation of $\mathbf{Q}$, the exponential decay of $\phi[\omega, v, z]$, (19), and (18), we have

$$\lim_{t \rightarrow \infty} (\mathbf{Q}(u_0) - \mathbf{Q}(Q[z(t)]) - \mathbf{Q}(\eta(t))) = 0. \quad (36)$$

Thus, our task is now to prove $\frac{d}{dt} \mathbf{Q}(\eta) \in L^1$, which is sufficient to show the convergence of $\mathbf{Q}(\eta)$. Now, from (33), we have

$$\begin{aligned} \frac{d}{dt} \mathbf{Q}(\eta) &= \langle \eta, \dot{\eta} \rangle = -\langle \eta, D_z Q[z](\dot{z} + iEz) \rangle + \langle \eta, i(f(Q[z] + \eta) - f(Q[z])) \rangle \\ &= I + II. \end{aligned}$$

By the bound of the 1st and the 3rd term of (28), we have $I \in L^1(\mathbb{R}_+)$. To show $II \in L^1(\mathbb{R}_+)$, we partition for $s \in (0, 1)$ the line where x exists as

$$\begin{aligned} \Omega_{1,t,s} &= \{x \in \mathbb{R} \mid |s\eta(t, x)| \leq 2|Q[z(t)]|\} \text{ and} \\ \Omega_{2,t,s} &= \mathbb{R} \setminus \Omega_{1,t,s} = \{x \in \mathbb{R} \mid |s\eta(t, x)| > 2|Q[z(t)]|\}, \end{aligned}$$

Then, we have

$$\begin{aligned} II(t) &= \sum_{j=1,2} \int_0^1 ds \int_{\Omega_{j,t,s}} \langle iDf(Q[z(t)] + s\eta(t))\eta(t), \eta(t) \rangle_{\mathbb{C}} dx \\ &=: II_1(t) + II_2(t). \end{aligned}$$

For II_1 , by (10) and (24), we have

$$\begin{aligned} |II_1(t)| &\lesssim \int_0^1 ds \int_{\Omega_{1,t,s}} |\langle iDf(Q[z(t)] + s\eta(t))\eta(t), \eta(t) \rangle_{\mathbb{C}}| dx \\ &\lesssim \int_0^1 ds \int_{\Omega_{1,t,s}} |Q[z(t)] + s\eta(t)|^{p-1} |\eta(t)|^2 dx \lesssim \int_{\mathbb{R}} |Q[z(t)]|^{p-1} |\eta(t)|^2 dx \\ &\lesssim \|e^{-\gamma\langle x \rangle} \eta(t)\|_{L^2(\mathbb{R})}^2 \in L^1(\mathbb{R}_+). \end{aligned}$$

Turning to II_2 , by exploiting $\langle Df(s\eta(t))\eta(t), i\eta(t) \rangle_{\mathbb{C}} \equiv 0$, which can be easily checked from (24), we write

$$\begin{aligned} II_2(t) &= - \int_{[0,1]} ds \int_{\Omega_{2,t,s}} \langle Df(Q[z(t)] + s\eta(t))\eta(t) - Df(s\eta(t))\eta(t), i\eta(t) \rangle_{\mathbb{C}} dx \\ &= - \int_{[0,1]^2} d\tau ds \int_{\Omega_{2,t,s}} \left\langle D^2 f(\tau Q[z(t)] + s\eta(t))(Q[z(t)], \eta(t)), i\eta(t) \right\rangle_{\mathbb{C}} dx. \end{aligned}$$

Then by $|s\eta(t, x)| > 2|Q[z(t)]|$, and $p \in (1, 2]$, we obtain the following, which completes the proof of $II \in L^1(\mathbb{R}_+)$ and (20):

$$\begin{aligned} |II_2(t)| &\lesssim \int_{[0,1]^2} d\tau ds \int_{\Omega_{2,t,s}} |s\eta(t)|^{p-2} |Q[z(t)]| |\eta(t)|^2 dx \\ &\lesssim \int_{[0,1]} ds s^{p-2} \int_{\Omega_{2,t,s}} |\eta(t)|^{p-2} |Q[z(t)]|^{p-1} |\eta(t)|^{2-p} |\eta(t)|^2 dx \\ &\lesssim \|e^{-a\langle x \rangle} \eta(t)\|_{L^2(\mathbb{R})}^2 \in L^1(\mathbb{R}_+). \end{aligned}$$

□

3. Proof of Proposition 3

Proposition 3 is an immediate consequence of (28) and the following lemma.

Lemma 2. *We have the following estimate:*

$$|\dot{z} + iE(|z|^2)z| \lesssim \delta^{p-1} \|\eta\|_{\tilde{\Sigma}}. \quad (37)$$

Proof. Applying $\langle \cdot, \Theta\varphi \rangle$ with $\Theta = 1, i$ to (33) we have the following, where the cancelled terms are null by Lemma 1,

$$\begin{aligned} & \langle i\dot{\eta}, \Theta\varphi \rangle + \langle iD_z Q[z](\dot{z} + iEz), \Theta\varphi \rangle = \langle H\eta, \Theta\varphi \rangle \\ & - \langle f(Q[z] + \eta) - f(Q[z]) - Df(Q[z])\eta, \Theta\varphi \rangle - \langle Df(Q[z])\eta, \Theta\varphi \rangle. \end{aligned}$$

We have

$$|\langle Df(Q[z])\eta, \Theta\varphi \rangle| \lesssim |z|^{p-1} \|\eta\|_{\tilde{\Sigma}}. \quad (38)$$

Next, we claim

$$|\langle f(Q[z] + \eta) - f(Q[z]) - Df(Q[z])\eta, \Theta\varphi \rangle| \lesssim \|\eta\|_{\tilde{\Sigma}}^p. \quad (39)$$

We set for $s \in (0, 1)$

$$\begin{aligned} \Omega'_{1,t,s} &= \{x \in \mathbb{R} \mid 2|s\eta(t, x)| \leq |Q[z(t)]|\} \text{ and} \\ \Omega'_{2,t,s} &= \mathbb{R} \setminus \Omega'_{1,t,s} = \{x \in \mathbb{R} \mid 2|s\eta(t, x)| > |Q[z(t)]|\} \end{aligned}$$

and split

$$\begin{aligned} & \langle f(Q[z] + \eta) - f(Q[z]) - Df(Q[z])\eta, \Theta\varphi \rangle = I_1(t) + I_2(t) \text{ for} \\ & I_j(t) = \int_0^1 ds \int_{\Omega'_{j,t,s}} \langle [Df(Q[z] + s\eta) - Df(Q[z])]\eta, \Theta\varphi \rangle_{\mathbb{C}} dx. \end{aligned}$$

We then have

$$I_1(t) = \int_{[0,1]^2} d\tau ds s \int_{\Omega'_{1,t,s}} \langle D^2 f(Q[z] + s\tau\eta)(\eta, \eta), \Theta\varphi \rangle_{\mathbb{C}} dx$$

with

$$|I_1(t)| \lesssim \int_{[0,1]^2} d\tau ds \int_{\Omega'_{1,t,s}} \varphi |Q[z]|^{p-2} |\eta|^{2-p+p} dx \lesssim \|\eta\|_{\tilde{\Sigma}}^p.$$

We have the following, which completes the proof of (39) and with (38) yields (37):

$$|I_2(t)| \lesssim \int_{[0,1]} ds \int_{\Omega'_{2,t,s}} \varphi |\eta|^p dx \lesssim \|\eta\|_{\tilde{\Sigma}}^p.$$

□

4. High Energies: Proof of Proposition 4

Following the framework in Kowalczyk et al. [31] and using the function χ in (25) we consider the function

$$\zeta_A(x) := \exp\left(-\frac{|x|}{A}(1 - \chi(x))\right) \text{ and } \varphi_A(x) := \int_0^x \zeta_A^2(y) dy \quad (40)$$

and the vector field

$$S_A := \varphi'_A + 2\varphi_A \partial_x. \quad (41)$$

Next we introduce

$$\mathcal{I}_A := 2^{-1} \langle i\eta, S_A \eta \rangle. \quad (42)$$

Notice that $|\mathcal{I}_A(t)| \lesssim A\delta^2$ for any $t \in \mathbb{R}$.

Lemma 3. *There is a fixed constant $C > 0$ s.t. for an arbitrary small number*

$$\|\eta\|_{\Sigma_A}^2 \leq C \left[\dot{\mathcal{I}}_A + \|\eta\|_{\Sigma}^2 + |\dot{z} + iEz|^2 \right]. \quad (43)$$

Proof. From (33), we obtain

$$\begin{aligned} \dot{\mathcal{I}}_A &= -\langle \dot{\eta}, iS_A \eta \rangle = -\langle \partial_x^2 \eta, S_A \eta \rangle + \langle V\eta, S_A \eta \rangle \\ &\quad - \langle f(Q[z] + \eta) - f(Q[z]), S_A \eta \rangle + O(|\dot{z} + iEz| \|\eta\|_{\Sigma}). \end{aligned}$$

From Kowalczyk et al. [31], we have

$$-\langle \partial_x^2 \eta, S_A \eta \rangle \geq 2\|(\zeta_A \eta)'\|_{L^2}^2 - \frac{C}{A} \|\eta\|_{\Sigma}^2. \quad (44)$$

It is clear that

$$|\langle V\eta, S_A \eta \rangle| = 2^{-1} |\langle [S_A, V]\eta, \eta \rangle| \lesssim \|\eta\|_{\Sigma}^2. \quad (45)$$

We write

$$\langle f(Q[z] + \eta) - f(Q[z]) - f(\eta), S_A \eta \rangle + \langle f(\eta), S_A \eta \rangle =: B_1 + B_2.$$

Then

$$|B_2| = |\langle f(\eta) \bar{\eta} - G(|\eta|^2), \zeta_A^2 \rangle| \lesssim \int_{\mathbb{R}} |\eta|^{p+1} \zeta_A^2 dx$$

and we use the crucial estimate by Kowalczyk et al. [31]:

$$\int_{\mathbb{R}} |\eta|^{p+1} \zeta_A^2 dx \lesssim A^2 \|\eta\|_{L^\infty(\mathbb{R})}^{p-1} \|(\zeta_A \eta)'\|_{L^2(\mathbb{R})}^2 \lesssim A^{-1} \|(\zeta_A \eta)'\|_{L^2(\mathbb{R})}^2. \quad (46)$$

We claim

$$|B_1| \lesssim \|\eta\|_{\Sigma}^2 + \delta_2 \|(\zeta_A \eta)'\|_{L^2}^2. \quad (47)$$

We consider

$$\begin{aligned} \Omega_{1,t} &= \{x \in \mathbb{R} \mid |\eta(t, x)| \leq 2|Q[z(t)]|\} \text{ and} \\ \Omega_{2,t} &= \mathbb{R} \setminus \Omega_{1,t} = \{x \in \mathbb{R} \mid |\eta(t, x)| > 2|Q[z(t)]|\}, \end{aligned} \quad (48)$$

and split accordingly

$$B_1 = \sum_{j=1,2} \langle f(Q[z] + \eta) - f(Q[z]) - f(\eta), S_A \eta \rangle_{L^2(\Omega_{j,t})} =: B_{11} + B_{12}.$$

We have

$$\begin{aligned} |B_{11}| &\leq \int_0^1 \int_{\Omega_{1,t}} |\langle (Df(Q[z] + s\eta) - Df(s\eta))\eta, S_A\eta \rangle_{\mathbb{C}}| dx ds \\ &\lesssim \int_{\Omega_{1,t}} \zeta_A^2 |Q[z]|^{p-1} |\eta|^2 dx + \int_{\Omega_{1,t}} \zeta_A^{-1} |\varphi_A| |Q[z]|^{p-1} |\eta| |(\zeta_A\eta)' - \zeta'_A\eta| dx =: B_{111} + B_{112}. \end{aligned}$$

Then

$$B_{111} \lesssim \|\eta\|_{\Sigma}^2$$

and, for a fixed small and preassigned $\delta_2 > 0$,

$$B_{112} \lesssim \int_{\Omega_{1,t}} \zeta_A^{-1} |\varphi_A| |Q[z]|^{p-1} |\eta| \left(|(\zeta_A\eta)'| + \frac{1}{A} \zeta_A |\eta| \right) dx \lesssim \|\eta\|_{\Sigma}^2 + \delta_2 \|(\zeta_A\eta)'\|_{L^2}^2.$$

We have

$$\begin{aligned} |B_{12}| &\leq \int_0^1 \int_{\Omega_{2,t}} |\langle (Df(sQ[z] + \eta) - Df(sQ[z]))Q[z], S_A\eta \rangle_{\mathbb{C}}| dx ds \\ &\lesssim \int_{\Omega_{2,t}} \zeta_A^2 |\eta|^p |Q[z]| dx + \int_{\Omega_{2,t}} \zeta_A^{-1} |\varphi_A| |\eta|^{p-1} |Q[z]| |(\zeta_A\eta)' - \zeta'_A\eta| dx =: B_{121} + B_{122}. \end{aligned}$$

We have

$$B_{121} \lesssim \int_{\Omega_{2,t}} \zeta_A^2 |\eta|^p |Q[z]|^{2-p+p-1} dx \leq \int_{\mathbb{R}} |\eta|^2 |Q[z]|^{p-1} dx \lesssim \|\eta\|_{\Sigma}^2.$$

We have similarly, for a fixed small and preassigned $\delta_2 > 0$ and completing the proof of (47),

$$B_{122} \lesssim \int_{\Omega_{2,t}} \zeta_A^{-1} |\varphi_A| |\eta| |Q[z]|^{p-1} \left(|(\zeta_A\eta)'| + \frac{1}{A} \zeta_A |\eta| \right) dx \lesssim \|\eta\|_{\Sigma}^2 + \delta_2 \|(\zeta_A\eta)'\|_{L^2}^2.$$

□

Proof of Proposition 4. Integrating inequality (3), we obtain (31). □

5. A Review of Kato Smoothing

This section is mainly inspired by Mizumachi [13] and is based also on material in [8,39]. Since 0 is not a resonance of H , by Lemma 1, p. 130 and Theorem 1 [1,5] (Formula (2.45)), we have the following result about Jost functions.

Proposition 6 (Jost functions). *For any $k \in \overline{\mathbb{C}}_+$, there are functions $f_{\pm}(x, k) = e^{\pm ikx} m_{\pm}(x, k)$ which solve $Hu = k^2u$ with*

$$\lim_{x \rightarrow +\infty} m_+(x, k) = 1 = \lim_{x \rightarrow -\infty} m_-(x, k).$$

These functions, for $x^{\pm} := \max\{0, \pm x\}$, satisfy

$$|m_{\pm}(x, k) - 1| \leq C_1 \langle x^{\mp} \rangle \langle k \rangle^{-1} \left| \int_x^{\pm\infty} \langle y \rangle |V(y)| dy \right|. \quad (49)$$

For the Wronskian, we have $[f_+(x, k), f_-(x, k)] = \frac{2ik}{T(k)}$, where $T(k) = \alpha k(1 + o(1))$ near $k = 0$ for some $\alpha \in \mathbb{R}$ and $T(k) = 1 + O(1/k)$ for $k \rightarrow \infty$ and $T \in C^0(\mathbb{R})$.

For $k \in \mathbb{C}_+$, we have the following formula for the integral kernel of the resolvent $R_H(k^2)$ of H , for $x < y$, with an analogous formula for $x > y$:

$$R_H(k^2)(x, y) = \begin{cases} \frac{T(k)}{2ik} f_-(x, k) f_+(y, k) = \frac{T(k)}{2ik} e^{ik(x-y)} m_-(x, k) m_+(y, k) & \text{for } x < y \\ \frac{\overline{T(k)}}{2ik} f_+(x, k) f_+(-, k) = \frac{\overline{T(k)}}{2ik} e^{-ik(x-y)} m_+(x, k) m_-(y, k) & \text{for } x > y. \end{cases} \quad (50)$$

Let now $\lambda \geq 0$ and set $k = \sqrt{\lambda} \geq 0$. Then the Formula (50) makes sense yielding the kernel $R_H^+(\lambda)(x, y)$ of an operator, which is denoted by $R_H^+(\lambda)$. We can define similarly Jost functions for $\Im k < 0$, which are of the form $f_{\pm}(x, \bar{k})$. In particular, $R_H^-(k^2)(x, y) = \overline{R_H^+(k^2)}(x, y)$ for $k \geq 0$.

We have the following.

Lemma 4. For any $s > 3/2$ and $\tau > 1/2$, there is a constant $C_{s\tau} > 0$ such that for any $\lambda \geq 0$, the following limit exists for

$$\lim_{a \rightarrow 0^+} R_H(\lambda \pm ia) = R_H^{\pm}(\lambda) \text{ in } \mathcal{L}(L^{2\tau}(\mathbb{R}), L^{2,-s}(\mathbb{R})) \text{ and we have the bound} \quad (51)$$

$$\sup_{\lambda, a \in \mathbb{R}_+} \|R_H(\lambda \pm ia)\|_{\mathcal{L}(L^{2,\tau}(\mathbb{R}), L^{2,-s}(\mathbb{R}))} \leq C_{s\tau}. \quad (52)$$

Proof. The result is standard, and we sketch it for completeness. It is enough to consider the $+$ sign. For any $\lambda \geq 0$, $a > 0$ (small), $x < y$, and $k_a = \sqrt{\lambda + ia}$, we have

$$\langle x \rangle^{-s} |R_H(\lambda + ia)(x, y)| \langle y \rangle^{-\tau} \lesssim \langle x \rangle^{-s} (1 + x^+ + y^-) \langle y \rangle^{-s} \lesssim \begin{cases} \langle x \rangle^{-s+1} \langle y \rangle^{-\tau} & \text{if } y > x > 0; \\ \langle x \rangle^{-s} \langle y \rangle^{-\tau+1} & \text{if } x < y < 0. \\ \langle x \rangle^{-s} \langle y \rangle^{-\tau} & \text{otherwise.} \end{cases} \quad (53)$$

We have an analogous estimate for $x > y$. It is then elementary to show that the latter kernel $\langle x \rangle^{-s} R_H(\lambda + ia)(x, y) \langle y \rangle^{-\tau} \in L^2(\mathbb{R} \times \mathbb{R})$, with a norm independent from $a > 0$ and $\lambda \geq 0$. Indeed, for $z = \lambda \pm ia$, it is enough to consider

$$\begin{aligned} & \int_{\mathbb{R}} dx \langle x \rangle^{-2s} \int_{\mathbb{R}} |R_H(x, y, z)|^2 \langle y \rangle^{-2\tau} dy = \int_{\mathbb{R}} dx \langle x \rangle^{-2s} \int_{-\infty}^x |R_H(x, y, z)|^2 \langle y \rangle^{-2\tau} dy \\ & + \int_{\mathbb{R}} dx \langle x \rangle^{-2s} \int_x^{+\infty} |R_H(x, y, z)|^2 \langle y \rangle^{-2\tau} dy. \end{aligned} \quad (54)$$

The second term on the right hand side is bounded using (53) by

$$\begin{aligned} & \int_{x < y} \langle x \rangle^{-2s} \langle y \rangle^{-2\tau} (1 + x^+ + y^-)^2 dx dy \leq \int_{0 < x < y} \langle x \rangle^{-2s+2} \langle y \rangle^{-2\tau} dx dy \\ & + \int_{x < y < 0} \langle x \rangle^{-2s} \langle y \rangle^{-2\tau+2} dx dy + \int_{x < 0 < y} \langle x \rangle^{-2s} \langle y \rangle^{-2\tau} dx dy =: \sum_{j=1}^3 I_j. \end{aligned}$$

Then

$$I_1 \leq \int_{\mathbb{R}} \langle x \rangle^{-2s+2} dx \int_{\mathbb{R}} \langle y \rangle^{-2\tau} dy =: I_4 < \infty \text{ for } s > 3/2 \text{ and } \tau > 1/2.$$

Similarly $I_j < I_4$ for $j = 2, 3$. Similar estimates independent from $\lambda > 0$ and $a > 0$ hold for the term in the first line on the right hand side of (54). This gives us also (52). The exact same bound holds for the $L^2(\mathbb{R} \times \mathbb{R})$ norm of $\langle x \rangle^{-s} R_H^+(\lambda)(x, y) \langle y \rangle^{-\tau}$. Furthermore,

$$\lim_{a \rightarrow 0^+} \langle x \rangle^{-s} R_H(\lambda + ia)(x, y) \langle y \rangle^{-\tau} = \langle x \rangle^{-s} R_H^+(\lambda)(x, y) \langle y \rangle^{-\tau} \text{ for all } x, y$$

and by Lebesgue's dominated convergence, the above convergence holds in $L^2(\mathbb{R} \times \mathbb{R})$. This yields the limit (51) for any $\lambda \geq 0$. $\square$

Taking $s = \tau$, we have the following.

Lemma 5 (Kato smoothing). *For any $s > 3/2$, there is a constant $c_s > 0$ such that*

$$\|e^{-itH}P_c f\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \leq c_s \|f\|_{L^2(\mathbb{R})};$$

Proof. We sketch this well known result for completeness. Obtain $g(t, x) \in \mathcal{S}(\mathbb{R} \times \mathbb{R})$ with $g(t) = P_c g(t)$. By the limiting absorption principle, taking f such that $f = P_c f$

$$\left(e^{-itH} f, g \right)_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} = \frac{1}{2\pi i} \int_0^{+\infty} ((R_H^+(\lambda) - R_H^-(\lambda))f, \widehat{g}(\lambda))_{L^2(\mathbb{R}_x)} d\lambda.$$

Then, from Fubini and Plancherel, we have

$$\begin{aligned} \left| \left(e^{-itH} f, g \right)_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right| &\leq (2\pi)^{-\frac{1}{2}} \|(R_H^+(\lambda) - R_H^-(\lambda))f\|_{L_x^{2,-s} L_\lambda^2} \|g\|_{L_x^{2,s} L_t^2} \\ &= (2\pi)^{-\frac{1}{2}} \|(R_H^+(\lambda) - R_H^-(\lambda))f\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \|g\|_{L^2(\mathbb{R}, L^{2,s}(\mathbb{R}))}. \end{aligned} \quad (55)$$

By the Fathou lemma,

$$\|(R_H^+(\lambda) - R_H^-(\lambda))f\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \leq \liminf_{a \rightarrow 0^+} \|(R_H(\lambda + ia) - R_H(\lambda - ia))f\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))}.$$

Then we repeat an argument in [40] (Lemma 5.5). For $\Im \mu > 0$, set $K(\mu)$ as the positive square root of $(2\pi i)^{-1}[R_H(\mu) - R_H(\bar{\mu})] = \pi^{-1}(\Im \mu)R_H(\bar{\mu})R_H(\mu)$. Then

$$\begin{aligned} \int_{\mathbb{R}_+} \|(R_H^+(\lambda) - R_H^-(\lambda))f\|_{L^{2,-s}(\mathbb{R})}^2 d\lambda &\leq \liminf_{a \rightarrow 0^+} \int_{\mathbb{R}_+} \|(R_H(\lambda + ia) - R_H(\lambda - ia))f\|_{L^{2,-s}(\mathbb{R})}^2 d\lambda \\ &= 4\pi^2 \liminf_{a \rightarrow 0^+} \int_{\mathbb{R}_+} \|\langle x \rangle^{-s} K(\lambda + ia) K(\lambda + ia) f\|_{L^2(\mathbb{R})}^2 d\lambda \\ &= 4\pi^2 \liminf_{a \rightarrow 0^+} \int_{\mathbb{R}_+} \|K(\lambda + ia) \langle x \rangle^{-s} K(\lambda + ia) f\|_{L^2(\mathbb{R})}^2 d\lambda \\ &= -2i\pi \liminf_{a \rightarrow 0^+} \int_{\mathbb{R}_+} (\langle x \rangle^{-s} (R_H(\lambda + ia) - R_H(\lambda - ia)) \langle x \rangle^{-s} K(\lambda + ia) f, K(\lambda + ia) f) d\lambda \\ &\leq 4\pi C_{ss} \liminf_{a \rightarrow 0^+} \int_{\mathbb{R}_+} \|K(\lambda + ia) f\|_{L^2(\mathbb{R})}^2 d\lambda \\ &= 4\pi C_{ss} \left(\lim_{a \rightarrow 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}_+} (R_H(\lambda + ia) - R_H(\lambda - ia)) f d\lambda, f \right) = 4\pi C_{ss} \|f\|_{L^2(\mathbb{R})}^2. \end{aligned}$$

Then, by (55), we obtain the following, which yields the lemma:

$$\left| \left(e^{-itH} f, g \right)_{L^2(\mathbb{R}_t \times \mathbb{R}_x)} \right| \leq c_s \|f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R}, L^{2,s}(\mathbb{R}))}.$$

$\square$

The following smoothing stems from Mizumachi [13].

Lemma 6. *For any $s > 3/2$ and $\tau > 1/2$, there is a constant $c_{s\tau} > 0$ such that*

$$\left\| \int_0^t e^{-i(t-s)H} P_c(H) g(s) ds \right\|_{L^2(\mathbb{R}_+, L^{2,-s}(\mathbb{R}))} \leq c_{s\tau} \|g\|_{L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R}))}. \quad (56)$$

Proof. For $g \equiv P_c(H)g$ supported in $t \geq 0$, by Plancherel, taking the inverse Fourier transform in t , we have

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-s)H} g(s) ds \right\|_{L^2(\mathbb{R}_+, L^{2,-s}(\mathbb{R}))} = \frac{1}{\sqrt{2\pi}} \|R_H^+(\lambda) g^\vee(\lambda)\|_{L^2(\mathbb{R}_\lambda, L^{2,-s}(\mathbb{R}))} \\ & \leq \frac{1}{\sqrt{2\pi}} \sup_{\lambda \in \mathbb{R}} \|R_H^+(\lambda)\|_{L^{2,\tau} \rightarrow L^{2,-s}} \|g^\vee\|_{L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R}))} \lesssim \|g^\vee\|_{L^2(\mathbb{R}, L^{2,\tau}(\mathbb{R}))} = \|g\|_{L^2(\mathbb{R}_+, L^{2,\tau}(\mathbb{R}))}. \end{aligned}$$

□

6. Low Energies: Proof of Proposition 5

Setting $v := \chi_B \eta$ for B as in (22) and then $w = P_c v$ from (33), we have

$$i\dot{w} = Hw + P_c(\chi_B'' + 2\chi_B' \partial_x) \eta - iP_c \chi_B D_z Q[z](\dot{z} + iEz) + P_c \chi_B (f(Q[z] + \eta) - f(Q[z])). \quad (57)$$

Then we have the following, using the spaces in (7).

Lemma 7. For $s > 3/2$, we have

$$\|w\|_{L^2(I, L^{2,-s}(\mathbb{R}))} \leq o_{B^{-1}}(1)\epsilon. \quad (58)$$

Proof. By Lemma 5, for the first inequality and using $w = P_c w$, we have

$$\|e^{itH} w(0)\|_{L^2(\mathbb{R}, L^{2,-s}(\mathbb{R}))} \lesssim \|w(0)\|_{L^2(\mathbb{R})} \lesssim \|\eta(0)\|_{L^2(\mathbb{R})} \lesssim \delta (= o_{B^{-1}}(1)\epsilon).$$

Using Proposition 1 and Lemmas 2 and 6, taking $\tau \in (1/2, 1)$, we obtain

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-t')H} P_c \chi_B D_z Q[z](\dot{z} + iEz) dt' \right\|_{L^2(I, L^{2,-s}(\mathbb{R}))} \lesssim \|\dot{z} + iEz\|_{L^2(I)} \|D_z Q[z]\|_{L^\infty(I, L^{2,\tau}(\mathbb{R}))} \\ & \lesssim \delta^{p-1} \|\eta\|_{L^2(I, \tilde{\Sigma})} = o_{B^{-1}}(1)\epsilon. \end{aligned}$$

By Lemma 6,

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-t')H} P_c (\chi_B'' + 2\chi_B' \partial_x) \eta dt' \right\|_{L^2(I, L^{2,-s}(\mathbb{R}))} \\ & \lesssim \|(2\chi_B' \partial_x + \chi_B'') \eta\|_{L^2(I, L^{2,\tau}(\mathbb{R}))} \lesssim B^{\tau-1} \|\operatorname{sech}\left(\frac{2}{A}x\right) \eta'\|_{L^2(I, L^2(\mathbb{R}))} \\ & + B^{\tau-2} \|1_{B \leq |x| \leq 2B} \operatorname{sech}\left(\frac{2}{A}x\right) \eta\|_{L^2(I, L^2(\mathbb{R}))} \lesssim B^{\tau-1} \|\eta\|_{L^2(I, \Sigma_A)} \\ & + B^{\tau-1} \left(\left\| \left(\operatorname{sech}\left(\frac{2}{A}x\right) \eta \right)' \right\|_{L^2(I, L^2(\mathbb{R}))} + \|\eta\|_{L^2(I, \tilde{\Sigma})} \right) = o_{B^{-1}}(1)\epsilon, \end{aligned}$$

where we used $\tau \in (1/2, 1)$ and (see Merle and Raphael [41] (Appendix C)),

$$\|u\|_{L^2(|x| \leq 2B)} \lesssim B \left(\|u'\|_{L^2(\mathbb{R})} + \|u\|_{\tilde{\Sigma}} \right).$$

Next we consider

$$\begin{aligned} & \left\| \int_0^t e^{-i(t-t')H} P_c \chi_B (f(Q[z] + \eta) - f(Q[z])) dt' \right\|_{L^2(I, L^{2,-s}(\mathbb{R}))} \leq I_1 + I_2 \text{ where} \\ & I_1 = \left\| \int_0^t e^{-i(t-t')H} P_c \chi_B f(\eta) dt' \right\|_{L^2(I, L^{2,-s}(\mathbb{R}))} \text{ and} \\ & I_2 = \left\| \int_0^t e^{-i(t-t')H} P_c \chi_B (f(Q[z] + \eta) - f(Q[z]) - f(\eta)) dt' \right\|_{L^2(I, L^{2,-s}(\mathbb{R}))}. \end{aligned}$$

We have, by (22),

$$\begin{aligned} I_1 &\lesssim \|\chi_B f(\eta)\|_{L^2(I, L^{2,\tau}(\mathbb{R}))} \lesssim B^\tau \|\eta\|_{L^\infty(I, L^2(\mathbb{R}))}^{p-1} A A^{-1} \|\operatorname{sech}\left(\frac{2}{A}x\right)\eta\|_{L^2(I, L^2(\mathbb{R}))} \\ &\lesssim \delta^{p-1} B^\tau A \|\eta\|_{L^2(I, \Sigma_A)} = o_{B^{-1}}(1)\epsilon. \end{aligned}$$

We have

$$I_2^2 \lesssim B^{2\tau} \int_{I \times \mathbb{R}} \chi_B^2 \left| \int_0^1 [Df(Q[z] + \sigma\eta) - Df(\sigma\eta)] \eta \right|^2 dx dt.$$

Then, by $|Df(Q[z] + \sigma\eta) - Df(\sigma\eta)| \leq |Df(Q[z] + \sigma\eta)| + |Df(\sigma\eta)| \lesssim |Q[z]|^{p-1} + |\eta|^{p-1}$,

$$\begin{aligned} I_2^2 &\lesssim B^{2\tau} \int_{I \times \mathbb{R}} \chi_B^2 \left(|z|^{2p-2} + |\eta|^{2p-2} \right) |\eta|^2 dx dt \\ &\lesssim B^{2\tau} \left(|z|^{2p-2} + \|\eta\|_{L^\infty(I, L^\infty(\mathbb{R}))}^{2p-2} \right) A A^{-1} \|\operatorname{sech}\left(\frac{2x}{A}\right)\eta\|_{L^2(I, L^2(\mathbb{R}))} \lesssim B^{2\tau} A \delta^{p-1} \|\eta\|_{L^2(I, \Sigma_A)}. \end{aligned}$$

□

So we conclude the following, which completes the proof of (58):

$$I_1 + I_2 = o_{B^{-1}}(1)\epsilon.$$

Proof of Proposition 5. We have, by $P\eta = 0$,

$$v = \chi_B \eta = P_c v + P \chi_B \eta = w - \varphi \langle (1 - \chi_B) \varphi, \eta \rangle - i \varphi \langle i(1 - \chi_B) \varphi, \eta \rangle$$

where $\varphi(x) \lesssim e^{-\sqrt{\lambda}|x|}$ yields the pointwise estimate

$$|\varphi \langle (1 - \chi_B) \varphi, \eta \rangle + i \varphi \langle i(1 - \chi_B) \varphi, \eta \rangle| \lesssim e^{-\sqrt{\lambda}B} \|\eta\|_{\tilde{\Sigma}}.$$

This and (58) yield $\|v\|_{L^2(I, \tilde{\Sigma})} \lesssim o_{B^{-1}}(1)\epsilon$. Next, from $v = \chi_B \eta$, and thanks to the relation $A \sim B^3$ set in (22), we have

$$\begin{aligned} \|\eta\|_{\tilde{\Sigma}} &\lesssim \|v\|_{\tilde{\Sigma}} + \|(1 - \chi_B)\eta\|_{\tilde{\Sigma}} \lesssim \|v\|_{\tilde{\Sigma}} + A^{-2} \|\operatorname{sech}\left(\frac{2}{A}x\right)\eta\|_{L^2} \\ &\lesssim \|v\|_{\tilde{\Sigma}} + A^{-1} \|\eta\|_{\Sigma_A} \end{aligned} \quad (59)$$

Thus, by (31), we obtain the following, which implies (32):

$$\begin{aligned} \|\eta\|_{L^2(I, \tilde{\Sigma})} &\lesssim \|v\|_{L^2(I, \tilde{\Sigma})} + A^{-1} \|\eta\|_{L^2(I, \Sigma_A)} \\ &\lesssim o_{B^{-1}}(1)\epsilon + A^{-1} \left(\|\dot{z} + iEz\|_{L^2(I)} + \|\eta\|_{L^2(I, \tilde{\Sigma})} \right) \lesssim o_{B^{-1}}(1)\epsilon + A^{-1} \|\eta\|_{L^2(I, \tilde{\Sigma})}. \end{aligned}$$

□

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