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# Upper Semicontinuous Representations of Semiorders as Interval Orders

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**Abstract**

We characterize the upper semicontinuous representability of a semiorder $\prec$ as an interval order (namely, by a pair $(u, v)$ of upper semicontinuous real-valued functions) on a topological space with a countable basis of open sets, where one of the representing functions is a one-way utility for the characteristic weak order $\prec^0$ associated with the semiorder. Such a description generalizes the *upper semicontinuous threshold representation*. To this end, we introduce a suitable upper semicontinuity condition concerning a semiorder, namely *strict upper semicontinuity*. We further characterize the mere existence of an upper semicontinuous one-way utility for this characteristic weak order, with a view to the identification of maximal elements on compact metric spaces.

**Keywords:** semiorder; upper semicontinuous representations by means of two functions; traces; one-way utility

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## 1. Introduction

We recall that the order-theoretic model of an *interval order* is very important in decision theory, economics and psychology because interval orders $\prec$, on one hand, have nontransitive *indifference* $\sim$ (where the relation $\sim$ is defined, for all elements $x, y$, as $x \sim y$ being equivalent to the situation when neither $x \prec y$ nor $y \prec x$), and on the other hand, they can be represented in a very simple way by means of a pair of real functions $(u, v)$ (that is to say, for every $x, y$ in the *ground set* $X$, $x \prec y$ if and only if $v(x) < u(y)$).

In general, the indifference relation associated with a decision-maker's preferences may fail to exhibit transitivity, as illustrated by Luce's classical coffee-and-sugar example [1]. He proposed the following scenario:

"Find a subject who prefers a cup of coffee with one cube of sugar to one with five cubes (this should not be difficult). Now prepare 401 cups of coffee with $(1 + \frac{i}{100})x$ grams of sugar, for $i = 0, 1, \ldots, 400$, where $x$ is the weight of one cube of sugar. It is evident that the subject will be indifferent between cup $i$ and cup $i + 1$ for any $i$, but by choice the subject is not indifferent between $i = 0$ and $i = 400$."

The example above illustrates the fact that an agent may adopt intransitive choices even when evaluating highly similar alternatives. The original formalization of this phenomenon is based on the definition of a *semiorder*.

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"Let $S$ be a nonempty set and $P$ and $I$ two binary relations on $S$. The pair $(P, I)$ is said to be a semiorder on $S$ if, for every $a, b, c, d \in S$, the following axioms hold:

S1. Exactly one of $aPb$, $bPa$, or $aIb$ holds.
S2. For every $a \in S$, $aIa$ holds.
S3. If $(aPb \wedge bIc \wedge cPd)$, then $aPd$.
S4. If $(aPb \wedge bPc \wedge bId)$, then neither $aId$ nor $cId$ holds".

The behavior captured by these axioms has been repeatedly observed in psychological experiments. In particular, several studies indicate that an individual may fail to discriminate between alternatives unless the difference in the relevant stimulus exceeds a "just noticeable difference" [2]. Intransitive patterns have also been documented in the context of time preferences and multicriteria decision making [3]. To model such phenomena, Luce (1956) [1] introduced the notion of a semiorder, although the concept, together with that of an interval order, was first formulated by Wiener [4]. *Interval orders* generalize semiorders, and their representability has been deeply studied by Fishburn [5]. Various results concerning the real representations of interval orders were presented by Doignon et al. [6], Oloriz et al. [7] and Bosi et al. [8]. The case of continuous representations was examined, for example, by Chateauneuf [9], Bridges [10], Candeal et al. [11] and Bosi et al. [12]. On the other hand, the upper semicontinuous representability of interval orders has been less studied, despite the results presented by Bosi and Zuanon [13].

The reasons for interval order and semiorder frameworks arise from the need to represent situations involving intransitive indifference. Due to the fact that these structures more accurately reflect observed decision-making patterns, semiorders have been used to model consumer preferences (see e.g., [14,15]). Semiorder-type structures also arise in the study of intertemporal choice and multicriteria decision problems [3]. Given the presence of a threshold or just noticeable difference inherent to semiorders, Scott and Suppes (1958) [16] proposed to represent (via the so-called *Scott–Suppes representation*) these ordered structures through a utility function $u$ and a constant threshold $\delta$, thereby providing a numerical representation of semiordered preferences.

As already outlined above, *semiorders* $\prec$ are particular interval orders which can admit a so called *threshold representation* (or *Scott–Suppes representation*). This is the case of a pair $(u, \delta)$, where $u$ is a function, $\delta$ is a positive real number and, for all $x, y$ in the ground set $X$, $x \prec y$ is equivalent to $u(x) + \delta < u(y)$. Deep contributions in this field were provided by Bouyssou and Pirlot [17,18], Candeal et al. [19,20] and Estevan [21], who was specifically concerned with the continuous case.

The crucial role played by semiorders in economics and psychology is due to the fact that they generalize *weak orders*, which can be fully represented by a single real-valued function and whose indifference relations are transitive. Therefore, semiorders are compatible with imperfect discrimination of preferences, which causes intransitivity of indifference when preferences are close.

It is known that the importance of semiorders also arises when considering so-called *undominated maximal elements* (see, e.g., Alcantud et al. [22]), namely, maximal elements for $\prec$ which are selected as maximal elements for some extension of $\prec$.

In the present work, we are primarily concerned with the upper semicontinuous representation of a semiorder $\prec$ as an interval order, and we further require that at least one of the representing function is a *one-way utility* for the characteristic weak order associated with $\prec$. This is the case of the weak order $\prec^0 = \prec^* \cup \prec^{**}$, where the *traces* $\prec^*$ and $\prec^{**}$ are the following typical binary relations:

$$x \prec^* y \text{ if and only if there exists } \xi \in X \text{ such that } x \prec \xi \precsim y,$$

$$x \prec^{**} y \text{ if and only if there exists } \eta \in X \text{ such that } x \precsim \eta \prec y.$$

Indeed, an asymmetric binary relation is an interval order if and only if both the traces are asymmetric, in which case they are both weak orders. Further, an interval order $\prec$ turns out to be a semiorder if and only if the binary relation $\prec^0$, defined as the union of $\prec^*$ and $\prec^{**}$, is asymmetric, and when this happens $\prec^0$ is necessarily a weak order.

The inspiration is due to the consideration that when we deal with a full description $(u, \delta)$ of a semiorder, $u$ is a one-way utility for $\prec^0$ (i.e., $x \prec^0 y$ implies that $u(x) < u(y)$ for all $x, y \in X$); that is to say that $u$ is a one-way utility for both $\prec^*$ and $\prec^{**}$.

The crucial problem is that, even when dealing with an upper semicontinuous interval order $\prec$ on a topological space $(X, \tau)$ (i.e., an interval order $\prec$ such that, for all $x \in X$, the *strict lower section* $L_{\prec}(x) = \{z \in X : z \prec x\}$ is open), while the trace $\prec^*$ is necessarily upper semicontinuous, this is not the case for $\prec^{**}$.

Therefore, an upper semicontinuity condition is needed which in some sense surrogates the upper semicontinuity of $\prec^{**}$. This is precisely our task. In fact, we characterize a pair $(u, v)$ of functions which are both upper semicontinuous and represent a semiorder $\prec$ on a topological space $(X, \tau)$ which is second countable (i.e., it has a countable basis of open sets), where at least $u$ is a one-way utility for $\prec^0$. We exploit a suitable upper semicontinuity condition concerning $\prec^0$, which guarantees the existence of an upper semicontinuous one-way utility.

Finally, since there are examples of semiorders which are not representable as interval orders, but nevertheless are such that there exists an upper semicontinuous one-way utility for $\prec^0$, we characterize specifically this case, without requiring upper semicontinuity of the semiorder. This is enough in order to identify maximal elements of a semiorder on a compact topological space by maximizing the one-way utility of $\prec^0$.

In this paper, we first introduce the concept of a *strictly upper semicontinuous* semiorder. Indeed, our strict upper semicontinuity is a necessary condition for the upper semicontinuous representability of a semiorder by means of a pair of upper semicontinuous functions. We then show that such condition is also sufficient for the existence of this kind of representation when the topological space is second countable. Finally, we identify a condition which characterizes the existence of a one-way upper semicontinuous utility for the characteristic weak order $\prec^0$ associated with a semiorder $\prec$.

In essence, this work integrates the theory of interval order representation with specific properties of semiorders related to their characteristic weak order, providing topological conditions (strict upper semicontinuity) that guarantee the existence of highly desirable upper semicontinuous representations (namely, upper semicontinuous weak utilities) on second countable topological spaces. The hypothesis of second countability, however, can be removed when we assume a priori the existence of a (not necessarily upper semicontinuous) representation $(u', v')$ of the semiorder as an interval order.

The paper is structured as follows. Section 2 contains the necessary definitions and the preliminary results. Section 3 presents the results concerning the upper semicontinuous representability of a semiorder by means of a pair of real functions. Section 4 contains a characterization of the existence of a one-way upper semicontinuous utility for the characteristic weak order. Section 5 closes the paper with a summary of the results presented and an outline of future directions of research.

## 2. Notation and Preliminary Results

The mathematical concepts in this section, related to both topology and order, are found, for example, in Bridges and Mehta [23].

Let $R$ be a binary relation on a nonempty set $X$. Then, for any two points $x, y \in X$, we interpret $xRy$ as "$x$ is less preferred than $y$ according to the binary relation $R$".

**Definition 1.** *The lower section $L_R(x)$ of any element $x \in X$, corresponding to the binary relation $R$ on $X$, is*

$$L_R(x) = \{z \in X : zRx\}.$$

**Definition 2.** *A set $D \subset X$ is defined to be $R$-decreasing if $L_R(x) \subset X$ for all $x \in X$.*

From now on, an *asymmetric* binary relation $R$ will be denoted by $\prec$.

**Definition 3.** *Let $\prec$ be an asymmetric binary relation on a set $X$. Then $\prec$ is said to be:*

1. *A weak order if it is negatively transitive, i.e., for all $x, y, z \in X$,*

$$x \prec y \text{ implies that } (x \prec z) \textit{ or } (z \prec y);$$

2. *An interval order if, for all $x, y, z, w \in X$,*

$$(x \prec z) \textit{ and } (y \prec w) \text{ implies that } (x \prec w) \textit{ or } (y \prec z);$$

3. *A semiorder if it is an interval order and, for all $x, y, z, w \in X$,*

$$(x \prec y) \textit{ and } (y \prec z) \text{ implies that } (x \prec w) \textit{ or } (w \prec z).$$

It is clear that a weak order is a semiorder.

**Definition 4.** *The symmetric complement of an interval order $\prec$ is denoted by $\precsim$:*

$$x \precsim y \text{ if and only if } not(y \prec x).$$

**Definition 5.** *The indifference relation of an interval order $\prec$ is denoted by $\sim$:*

$$x \sim y \Leftrightarrow (x \precsim y) \textit{ and } (y \precsim x).$$

**Definition 6.** *For every interval order $\prec$ on a set $X$, it is possible to associate its traces, $\prec^*$ and $\prec^{**}$, defined as follows, for every $x, y \in X$:*

1. $x \prec^* y \Leftrightarrow \exists \xi \in X : x \prec \xi \precsim y$;
2. $x \prec^{**} y \Leftrightarrow \exists \eta \in X : x \precsim \eta \prec y$.

**Remark 1.** *It is known (see Fishburn [5]) that each trace of an interval order turns out to be a weak order.*

**Definition 7.** *If $\prec$ is an interval order on a set $X$, then define the following binary relation $\prec^0$ on $X$:*

$$\prec^0 = \prec^* \cup \prec^{**}.$$

**Remark 2.** *The binary relation $\prec^0$ is a weak order, provided that $\prec$ is a semiorder, so that its associated symmetric complement*

$$\precsim^0 = \precsim^* \cap \precsim^{**}$$

*is a total preorder (that is to say, $\precsim^0$ is transitive and* total*). This result is very well known and it can be found, for example, in Fishburn [5] and Bosi and Isler [24]. Clearly, this always happens when $\prec$ is an interval order such that $\prec^* = \prec^{**}$.*

For the reader's convenience, we recall the following result and we refer to Bosi and Isler ([24], Proposition 3).

**Proposition 1.** *Consider an interval order $\prec$ on a set $X$. Then the following conditions are equivalent:*

1. $\prec^0 = \prec^* \cup \prec^{**}$ *is asymmetric;*
2. $\prec^0$ *is a weak order;*
3. $\prec$ *is a semiorder.*

**Remark 3.** *It is clear that if $\prec$ is a semiorder on a set $X$, then we have that, for every $x \in X$,*

$$L_{\prec^0}(x) = L_{\prec^*}(x) \cup L_{\prec^{**}}(x).$$

We are now going to present a new definition of an *internally isolated* subset of a set $X$.

**Definition 8.** *If $\prec$ is an interval order on a set $X$, then a subset $A$ of $X$ is defined to be internally isolated if $x \sim y$ for all points $x, y \in A$.*

Let us now consider a characterization of a semiorder.

**Proposition 2.** *If $\prec$ is an interval order on a set $X$, then the following statements are equivalent:*

1. $\prec$ *is a semiorder.*
2. *The following statement is true for all points $x, y \in X$:*

$$x \sim y \Rightarrow \quad \begin{array}{l} (L_{\prec}(x) \subset L_{\prec}(y)) \text{ and } (L_{\prec}(y) \setminus L_{\prec}(x) \text{ is internally isolated}) \text{ or} \\ (L_{\prec}(y) \subset L_{\prec}(x)) \text{ and } (L_{\prec}(x) \setminus L_{\prec}(y) \text{ is internally isolated}). \end{array}$$

**Proof.** From Rabinovitch ([25], Theorem 2), we know that an asymmetric binary relation is an interval order if and only if $\{L_{\prec}(x)\}$ is *linearly ordered by set inclusion* (i.e., for all $x, y \in X$, either $L_{\prec}(x) \subset L_{\prec}(y)$ or $L_{\prec}(y) \subset L_{\prec}(x)$). Therefore, it only remains to show that an interval order $\prec$ is a semiorder if and only if the following property holds for all $x, y \in X$:

$$(*) \ \ x \sim y \Rightarrow (L_{\prec}(y) \setminus L_{\prec}(x) \text{ is internally isolated}) \ \text{ or } (L_{\prec}(x) \setminus L_{\prec}(y) \text{ is internally isolated}).$$

In order to prove that $1 \Rightarrow 2$, assume, by contraposition, that property $(*)$ does not hold, and therefore there exist $x, y \in X$ with $x \sim y$ and at the same time $L_{\prec}(x) \subsetneqq L_{\prec}(y)$ and $L_{\prec}(y) \setminus L_{\prec}(x)$ are not internally isolated. Then there exist $z, w \in L_{\prec}(y) \setminus L_{\prec}(x)$ such that, for example, $z \prec w$. Therefore, we have that $x \precsim z \prec w \prec y$. This is in contradiction with $x \sim y$, which implies that $\prec$ cannot be a semiorder, proving the first implication. The other implication can be proved in a perfectly analogous way. So the proof is complete. □

**Definition 9.** *Consider an interval order $\prec$ on a topological space $(X, \tau)$. Then $\prec$ is defined to be upper semicontinuous if we have that, for all points $x \in X$, $L_{\prec}(x)$ is an open set.*

**Remark 4.** *The trace $\prec^*$ associated with an interval order $\prec$ is always upper semicontinuous, provided that $\prec$ is upper semicontinuous. This is due to the fact that, for all points $x \in X$,*

$$L_{\prec^*}(x) = \bigcup_{\{\xi \in X, \xi \precsim x\}} L_{\prec}(\xi).$$

*This fact is not valid for the trace $\prec^{**}$, even in the semiorder case. Indeed, consider the following example.*

**Example 1.** *Define $X = [1,2] \cup [\sqrt{5}, \sqrt{6}]$ as a subset of a real line with the natural induced topology. Then define on X the following semiorder $\prec$:*

$$x \prec y \Leftrightarrow x^2 + 1 < y^2 \quad (x, y \in X).$$

*We take that $(u, 1)$ is a representation of $\prec$ and $u$ is (upper semi-) continuous when we put $u(x) = x^2$ for all $x \in X$. On the other hand, the trace $\prec^{**}$ is not upper semicontinuous. To prove this fact, consider for example that*

$$L_{\prec^{**}}(\sqrt{6}) = [1,2] \cup \{\sqrt{5}\}$$

*is not an open set. Notice that $x \prec \sqrt{6}$ for all $x \in [1,2]$, $\sqrt{5} \prec^{**} \sqrt{6}$ since $\sqrt{5} \precsim 2 \prec \sqrt{6}$, but for no $\sqrt{5} < x < \sqrt{6}$ we have that $x \prec^{**} \sqrt{6}$. Indeed, if this fact was true, then there would exist a point $\eta \in X$ such that $5 < x^2 \leq \eta^2 + 1 < 6$. Further, it is easily seen that $L_{\prec^*}(\sqrt{6}) = [1,2]$, so that, in the end, $L_{\prec^0}(\sqrt{6}) = L_{\prec^*}(\sqrt{6}) \cup L_{\prec^{**}}(\sqrt{6}) = L_{\prec^{**}}(\sqrt{6}) = [1,2] \cup \{\sqrt{5}\}$ is not an open set.*

**Definition 10.** *A real function u on a set X endowed with an asymmetric binary relation $\prec$ is defined as follows:*

1. *As a utility for $\prec$ if*

$$x \prec y \text{ is equivalent to } u(x) < u(y) \quad \text{for every } x, y \in X;$$

2. *As a one-way utility for $\prec$ if*

$$x \prec y \text{ implies that } u(x) < u(y) \quad \text{for every } x, y \in X.$$

**Remark 5.** *Clearly, if a utility function u for $\prec$ exists, then $\prec$ is a weak order. In addition, if u is upper semicontinuous, then $\prec$ is also upper semicontinuous.*

The following proposition is an easy consequence of Proposition 1.

**Proposition 3.** *If $\prec$ is an interval order on a set X, and there exists a one-way utility u for $\prec^0$, then $\prec$ is a semiorder.*

**Proof.** Consider that $\prec^0$ is asymmetric when it has a one-way utility. Then apply Proposition 1. □

**Remark 6.** *The binary relation $\prec^{**}$ on $X = [1,2] \cup [\sqrt{5}, \sqrt{6}]$ introduced in Example 1, that is,*

$$x \prec^{**} y \Leftrightarrow \exists \eta \in X : x^2 \leq \eta^2 + 1 < y^2$$

*is an example of a weak order which is not upper semicontinuous. Hence, $\prec^{**}$ has no upper semicontinuous utility representation. However, it is worthwhile noticing that there exists a (upper semi-) continuous one-way utility for $\prec^{**}$, namely the function $u(x) = x^2$. Clearly, because of the definition of $\prec$, u is also a one-way utility for the weak order $\prec^*$, and therefore for $\prec^0 = \prec^* \cup \prec^{**}$.*

**Definition 11.** *If $(u, v)$ is a pair of real functions on a set $X$ with $u \leq v$, then we say that $(u, v)$ represents an interval order $\prec$ if*

$$x \prec y \text{ is equivalent to } v(x) < u(y) \quad \text{for every } x, y \in X.$$

**Remark 7.** *It can be immediately checked that if $\prec$ is represented by a pair $(u, v)$ of real functions, then we have the following:*

1. *$v$ is a one-way utility for the trace $\prec^*$;*
2. *$u$ is a one-way utility for the trace $\prec^{**}$.*

Aiming to characterize the representability of a semiorder as an interval order, by means of a pair $(u, v)$ of upper semicontinous real-valued functions, we now introduce our new definition of *strictly upper semicontinuous semiorder* on a topological space.

**Definition 12.** *A semiorder $\prec$ on a topological space $(X, \tau)$ is defined to be strictly upper semicontinuous if the following statements hold true:*

1. *$\prec$ is upper semicontinuous.*
2. *There is a function $L^0_{\prec^0} : X \to \tau$ such that the following conditions are true for all points $x \in X$:*
   - *(a) $x \notin L^0_{\prec^0}(x)$;*
   - *(b) $L^0_{\prec^0}(x)$ is both $\prec^*$-decreasing and $\prec^{**}$-decreasing;*
   - *(c) $L^0_{\prec^0}(x) \supset L_{\prec^0}(x)$;*
   - *(d) $\{L^0_{\prec^0}(x)\}$ is linearly ordered by the inclusion relation $\subset$.*

**Definition 13.** *A semiorder $\prec$ on a set $X$ is said to admit a* threshold representation *$(u, \delta)$, where $u$ is a real-valued function on $X$ and $\delta$ is a positive real number, if*

$$x \prec y \text{ implies that } u(x) + \delta < u(y) \quad \text{for every } x, y \in X.$$

**Remark 8.** *When a threshold representation $(u, \delta)$ exists for an interval order $\prec$, then $\prec$ is a semiorder. Indeed, in this case $u$ is a one-way utility for both the traces $\prec^*$ and $\prec^{**}$, and therefore it is a one-way utility for $\prec^0$ too.*

The previous definition of a strictly upper semicontinuous semiorder is justified by the proposition below.

**Proposition 4.** *If $\prec$ is a semiorder on a topological space $(X, \tau)$, which admits an upper semicontinuous threshold representation $(u, \delta)$, then $\prec$ is strictly upper semicontinuous.*

**Proof.** We have that $\prec$ is upper semicontinuous, since

$$L_{\prec}(x) = (u + \delta)^{-1}(] - \infty, u(x)[)$$

is an open subset of $X$. Indeed, $u + \delta$ is upper semicontinuous. Further, by Remark 8, we have that $u$ is an upper semicontinuous one-way utility for $\prec^0$. At this point, we define, for all $x \in X$,

$$L^0_{\prec^0}(x) = u^{-1}(] - \infty, u(x)[).$$

Notice that the family $\{L^0_{\prec^0}(x)\}$ verifies conditions (a) to (d) of Definition 12. In this way, the proof is complete. □

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A real function $u$ on a topological space $(X, \tau)$ is defined to be upper semicontinuous if, for every $x \in X$ and for every $\alpha \in \mathbb{R}$ such that $u(x) < \alpha$, there exists an open set $O_x$ containing $x$ such that $u(z) < \alpha$ for every $z \in O_x$ (i.e., when $u^{-1}(]-\infty, \alpha[) = \{z \in X : u(z) < \alpha\}$ is an open set for every $\alpha \in \mathbb{R}$, or equivalently it belongs to $\tau$).

## 3. Upper Semicontinuous Representations of Semiorders

We first characterize the existence of a pair of upper semicontinuous functions representing a semiorder on a second countable topological space, such that one of these functions is a one-way utility for the characteristic weak order associated with the given semiorder.

For that reason, we present the following lemma.

**Lemma 1.** *If $\prec$ is strictly upper semicontinuous semiorder on a second countable topological space, then there is a pair $(u', v')$ of real functions which represents $\prec$ as an interval order.*

**Proof.** Consider that, under our assumptions, the following conditions are valid for all $x, y \in X$:

$$(^*)\ x \prec y \Rightarrow L^0_{\prec^0}(x) \subsetneqq L_{\prec}(y),$$

$$(^{**})\ y \precsim x \Rightarrow L_{\prec}(y) \subset L^0_{\prec^0}(x).$$

To prove the first property, notice that $L_{\prec^0}(x) \subset L_{\prec}(y)$ whenever $x \prec y$. Actually, it must be that $L^0_{\prec^0}(x) \subsetneqq L_{\prec}(y)$, because otherwise there is a point $z \in X$ with $x \prec y \precsim z \in L^0_{\prec^0}(x)$, which is contradictory since $L^0_{\prec^0}(x)$ is $\prec^*$-decreasing, and $x \notin L^0_{\prec^0}(x)$. To prove the second property, just consider that $y \precsim x$ entails $L_{\prec}(y) \subset L_{\prec^*}(x) \subset L^0_{\prec^0}(x)$. Now, define functions $u'$ and $v'$ in the following manner, for all $x \in X$:

$$u'(x) = \begin{cases} \displaystyle\sum_{k \in \mathbb{N}^+, B_k \subset L_{\prec}(x)} \frac{1}{2^k} & \text{if } L_{\prec}(x) \neq \emptyset \\ 0 & \text{if } L_{\prec}(x) = \emptyset \end{cases},$$

$$v'(x) = \begin{cases} \displaystyle\sum_{k \in \mathbb{N}^+, B_k \subset L^0_{\prec^0}(x)} \frac{1}{2^k} & \text{if } L^0_{\prec^0}(x) \neq \emptyset \\ 0 & \text{if } L^0_{\prec^0}(x) = \emptyset \end{cases}.$$

At this point, we show that $(u', v')$ is a representation of $\prec$ as an interval order. If $x \prec y$, then the fact that $v'(x) < u'(y)$ is an immediate consequence of the above property $(^*)$. If $y \precsim x$, as an immediate consequence of the above property $(^{**})$, we have that $u'(y) \leq v'(x)$ is . Therefore, the pair $(u', v')$ is a representation of the semiorder $\prec$, and clearly $v'$ is a one-way utility for $\prec^0$. □

**Theorem 1.** *Let $\prec$ be a semiorder on a second countable topological space $(X, \tau)$. The following statements are equivalent:*

1. *There is an upper semicontinous representation $(u, v)$ of $\prec$, with $u$ being a one-way utility for $\prec^0$;*
2. *$\prec$ is strictly upper semicontinuous.*

**Proof.** $1 \Rightarrow 2$. Assume that there is a representation $(u, v)$ of $\prec$, with $u$ exhibiting the above property. Then $\prec$ is upper semicontinuous. Now define, for all points $x \in X$,

$$L^0_{\prec^0}(x) = u^{-1}(]-\infty, u(x)[).$$

The collection $\{L^0_{\prec^0}(x)\}$ satisfies properties (a) to (d) of Definition 12. Therefore, we have that $\prec$ is strictly upper semicontinuous.

$2 \Rightarrow 1$. Consider a strictly upper semicontinuous semiorder $\prec$ on a second countable topological space $(X, \tau)$. Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}^+}$ be a countable basis of $\tau$. Indicate by $\tau^0$ the topology induced by the collection $\{L^0_{\prec^0}(x)\}$ in Definition 12. Observe that, since $\tau$ is second countable, $\tau^0$ is itself second countable as soon as $\{L^0_{\prec^0}(x)\}$ is nested (see, e.g., Bosi and Sbaiz ([26], Proof of Corollary 5)), and this is the case according to the definition of strict upper semicontinuity. Therefore, since it is clear that $\tau^0$ is a subtopology of $\tau$ (i.e., $\tau^0 \subset \tau$), it suffices to show that there exists a representation $(u, v)$ with the indicated properties, such that $u$ and $v$ are upper semicontinuous with respect to $\tau^0$.

From conditions (a) to (d) in Definition 12, there is an upper semicontinuous one-way utility function $u'$ for $\prec^0$. Indeed, consider any countable basis

$$\{L^0_{\prec^0}(x_n) : n \in \mathbb{N}^+\}$$

of the topology $\tau^0$, and define, for every $n \in \mathbb{N}^+$,

$$u'_n(x) = \begin{cases} 0 & \text{if } x \in L^0_{\prec^0}(x_n) \\ 1 & \text{if } x \in L^0_{\prec^0}(x_n) \end{cases} .$$

We have that the function

$$u'(x) = \sum_{n=1}^{\infty} 2^{-n} u'_n(x)$$

is an upper semicontinuous one-way utility for $\prec^0$. Indeed, if $x \prec^0 y$, $u'_n(x) \leq u'_n(y)$ for every $n \in \mathbb{N}^+$ and in addition, since $L^0_{\prec^0}(x) \subsetneqq L^0_{\prec^0}(y)$, there is $n \in \mathbb{N}^+$ with $L^0_{\prec^0}(x) \subsetneqq L^0_{\prec^0}(x_n) \subset L^0_{\prec^0}(y)$, so that actually $u'(x) < u'(y)$.

Since $\prec$ is representable by a pair of real-valued functions by Lemma 1, we have that $\prec$ is *separable as an interval order*, i.e., there is a countable subset $\mathcal{D} = \{d_n : n \in \mathbb{N}^+\}$ such that for every $x \prec y$ there exists $n \in \mathbb{N}^+$ such that

$$x \prec d_n \precsim^{**} y.$$

This result can be found, for example, in Bosi et al. [8]. Without loss of generality, the *order-dense set* $\mathcal{D}$ can be considered as satisfying the following property:

$(^{***})$ For every $x \prec y$, there exists $n \in \mathbb{N}^+$ with $x \prec d_n \precsim^{**} y$ and $u'(d_n) \leq u'(y)$.

This is clear if $x \prec d_n \prec^{**} y$ for some $n \in \mathbb{N}^+$, since the pair $(u', v')$ represents $\prec$. Otherwise, if there does not exist the $\min\{u'(z) : z \sim^{**} d_n\}$, we include in $\mathcal{D}$, together with $d_n$, any sequence $\{d_n^{(k)}\}_{k \in \mathbb{N}^+}$ such that

$$\lim_{k \to +\infty} u'(d_n^{(k)}) = \inf\{u'(z) : z \sim^{**} d_n\}$$

in such a way the above property $(^{***})$ can be guaranteed.

At this point, we define, for all $n \in \mathbb{N}^+$, the following upper semicontinuous functions $u_n$ and $v_n$:

$$u_n(x) = \begin{cases} 0 & \text{if } u'(x) < u'(d_n) \\ 1 & \text{if } u'(x) \geq u'(d_n) \end{cases}, \quad v_n(x) = \begin{cases} 0 & \text{if } x \in L_{\prec}(d_n) \\ 1 & \text{if } x \notin L_{\prec}(d_n) \end{cases},$$

where $u'$ is an upper semicontinuous one-way utility for $\prec^0$, whose existence has been shown above. Finally, define two functions $u, v : X \to [0,1]$ as follows, for every $x \in X$:

$$u(x) = \sum_{n=1}^{\infty} 2^{-n} u_n(x), \quad v(x) = \sum_{n=1}^{\infty} 2^{-n} v_n(x).$$

It is easily seen that $(u, v)$ is a pair of upper semicontinuous functions on $(X, \tau)$. Further, $u$ is a one-way utility for $\prec^0$ since this is the case for $u'$. Finally, the proof that the pair $(u, v)$ represents $\prec$ as an interval order is contained in the proof of Theorem 3.1 in Bosi and Zuanon [13]. This consideration completes the proof. □

The proof of the following corollaries are immediate and they are left to the reader.

**Corollary 1.** *Consider a semiorder $\prec$ on topological space $(X, \tau)$, and assume that $\prec$ admits a representation $(u', v')$ . The following statements are equivalent:*

1. *There is an upper semicontinuous representation $(u, v)$ of $\prec$, with u as a one-way utility for $\prec^0$;*
2. *$\prec$ is strictly upper semicontinuous.*

**Corollary 2.** *Let $\prec$ be a semiorder on a second countable topological space $(X, \tau)$, and assume that $\prec^* = \prec^{**}$. Then the following conditions are equivalent:*

1. *There exists a representation $(u, v)$ of $\prec$ as an interval order, such that u and v are both weak utilities for the weak order $\prec^0$;*
2. *$\prec$ is strictly upper semicontinuous.*

An analysis of the proof of Theorem 1 allows us to immediately establish the following more general theorem, whose statement no longer requires the second countability assumption. Indeed, in the proof of the implication 2 ⇒ 1 of Theorem 1, we substantially only use second countability in order to force the existence of a representation $(u', v')$ of the semiorder. The straightforward proof is omitted.

**Theorem 2.** *Let $\prec$ be a semiorder on a topological space $(X, \tau)$, and assume that $\prec$ is representable as an interval order by a pair of real-valued functions $(u', v')$. Then the following conditions are equivalent:*

1. *There exists an upper semicontinous representation $(u, v)$ of $\prec$ as an interval order, such that u is a one-way utility for the weak order $\prec^0$;*
2. *$\prec$ is strictly upper semicontinuous.*

**Remark 9.** *Notice that Theorem 2 is quite general and adheres with the literature concerning the existence of upper semicontinuous utility functions representing weak orders on arbitrary topological spaces.*

## 4. Upper Semicontinuous Representations of the Characteristic Weak Order Associated with a Semiorder

In this section, we characterize the existence of an upper semicontinuous one-way utility for the characteristic weak order $\prec^0$ associated with a semiorder $\prec$ on a second countable topological space.

The following example shows that there are cases of semiorders on second countable topological spaces which are not representable as interval orders, but nevertheless are such that there is a (upper semi-) continuous one-way utility for the characteristic weak order $\prec^0$.

**Example 2.** *Let $\prec$ be the asymmetric binary relation on $\mathbb{R}$ defined as follows for all $x, y \in \mathbb{R}$:*

$$x \prec y \Leftrightarrow x + 1 \leq y.$$

*Then, we have that $\prec$ is a semiorder which is not representable by means of a pair of functions (see Bosi et al. [8] (Remark 4)). Nevertheless, the identity function $i_{\mathbb{R}}$ on $\mathbb{R}$ is a (upper semi-) continuous one-way utility for the characteristic weak order $\prec^0$, since it is a one-way utility for both the weak orders $\prec^*$ and $\prec^{**}$.*

**Theorem 3.** *Let $\prec$ be a semiorder on a second countable topological space $(X, \tau)$. Then the following conditions are equivalent:*

1. *There exists an upper semicontinuous one-way utility $u$ for the weak order $\prec^0$;*
2. *Condition 2 of Definition 12 is verified.*

**Proof.** $1 \Rightarrow 2$. If $u$ is an upper semicontinuous one-way utility for $\prec^0$, we just define, for every $x \in X$, $L^0_{\prec^0}(x) = u^{-1}(]-\infty, u(x)[)$ in order to immediately verify that the mapping $L^0_{\prec^0} : X \to \tau$ verifies condition 2 of Definition 12.

$2 \Rightarrow 1$. This part of the proof has been already presented at the beginning of the proof in the implication "$2 \Rightarrow 1$" of Theorem 1. This consideration completes the proof. □

## 5. Conclusions

In this paper we have characterized the existence of a representation $(u, v)$ of a semiorder $\prec$ as an interval order in such a way that the function $u$ is a one-way utility for the characteristic weak order $\prec^0$ associated with $\prec$. We have been actually concerned with the case of semiorders on second countable topological spaces in order to guarantee the representability only using continuity conditions. The representation studied here is interesting since optimization of a one-way utility for $\prec^0$ leads to maximal elements of $\prec$. This is obviously the case of compact metric spaces.

As we have shown, the assumption of second countability can be removed, as regards the characterization of the existence of an upper semicontinuous representation of a semiorder as an interval order, by requiring the existence of a (*not necessarily* upper semicontinuous) representation $(u', v')$.

A natural development of our research will be characterizing the existence of an upper semicontinuous threshold representation. Another development of our study could be that of considering, on suitable topological spaces, semiorders induced by uniform random points (see Biró and Boone [27]).

The continuous case is very interesting but it presents more difficulties, and it will be studied in a future paper.

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