

STABLE DETERMINATION OF AN ANISOTROPIC INCLUSION IN THE SCHRÖDINGER EQUATION FROM LOCAL CAUCHY DATA

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ABSTRACT. We consider the inverse problem of determining an inclusion contained in a body for a Schrödinger type equation by means of local Cauchy data. Both the body and the inclusion are made by inhomogeneous and anisotropic materials. Under mild a priori assumptions on the unknown inclusion, we establish a logarithmic stability estimate in terms of the local Cauchy data. In view of possible applications, we also provide a stability estimate in terms of an ad-hoc misfit functional.

1. Introduction. The paper addresses the inverse boundary value problem of determining an inclusion D contained in a body Ω for a Schrödinger type equation by means of complete measurements on a portion Σ of the boundary. More precisely, we assume that Ω is a bounded domain in \mathbb{R}^n , $n \geq 3$ and D is an open set contained in Ω . Let both the body Ω and the inclusion D be made by different inhomogeneous and anisotropic materials. Consider a weak solution $u \in H^1(\Omega)$ to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla u) + qu = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \quad (1)$$

with

$$\sigma(x) = (a_b(x) + (a_D(x) - a_b(x))\chi_D(x))A(x), \quad (3)$$

and

$$q(x) = q_b(x) + (q_D(x) - q_b(x))\chi_D, \quad (4)$$

where a_b, q_b and a_D, q_D are the scalar parameters of the background body Ω and the inclusion D , respectively, χ_D is the characteristic function of D and $A(x)$ is a matrix-valued function. We denote by \mathcal{C}_D^Σ the set of all the possible Cauchy data $(u|_\Sigma, \sigma \nabla u \cdot \nu|_\Sigma)$ associated to the problem, where ν is the outer unit normal of Ω at Σ .

The inverse problem consists in the determination of D given \mathcal{C}_D^Σ .

For $q = 0$, namely when one deals with the conductivity equation, the direct problem is well-posed and one can define the so-called Dirichlet-to-Neumann map

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$$\begin{aligned}\Lambda_D : H_{00}^{\frac{1}{2}}(\Sigma) &\rightarrow H_{00}^{-\frac{1}{2}}(\Sigma) \\ f &\rightarrow \sigma \nabla u \cdot \nu|_{\Sigma},\end{aligned}\tag{5}$$

which, roughly speaking, assigns to each electric potential prescribed on Σ the corresponding current density. When no sign nor spectrum condition on q are assumed, the existence of such operator is not guaranteed.

Notice that the inverse problem we are discussing encompasses the classical inverse conducting problem for a special class of anisotropic conductivities σ of the form (3) as well as the reduced wave equation $\Delta u + k^2 c^{-2} u = 0$.

The prototype of this class of inverse problems is the determination of an inclusion in an isotropic electrostatic conductor by means of full boundary measurements of the electric potential and the current flux. The uniqueness issue for such an inverse problem was solved by Isakov in [25] by combining the Runge approximation theorem with the use of solutions with Green's function type singularities. The stability counterpart of Isakov's result has been tackled by Alessandrini and Di Cristo in [5] by providing a logarithmic stability estimate. Their argument is still based on singular solution method, whereas Runge approximation argument has been replaced by the quantitative unique continuation estimates, since they seem to be more suitable for stability purposes. This strategy has inspired a line of research in which some methods and results have been extended to more complicated equations and systems (see for instance [17], [18], [6], [31]). More recently, in the spirit of [9], where it has been observed that an improvement of the stability rate is possible for finite dimensional unknown conductivities, Lipschitz stability estimates for polygonal or polyhedral inclusions have been provided mainly in the context of the conductivity equation and the Helmholtz equation (see [14], [15], [13], [11]). Another class of problems which is a particular instance of our problem is the optical tomography, which is mostly studied in medical imaging to infer the properties of a tissue (see for instance [10]).

The purpose of our work is to prove the continuous dependence of D in the Hausdorff metric from the local Cauchy data via a modulus of continuity of logarithmic type. Although our strategy has been stimulated by the one introduced in [5], the more general context we are dealing with requires the development of new arguments and tools.

Let us summarize the main steps of our proof with the related issues.

- i) We introduce the space of local Cauchy data \mathcal{C}_D^Σ measured on the accessible portion Σ and their metric structure as a subspace of a Hilbert space. As in [4], we express the error on the boundary data in terms of the so-called distance (or aperture) between spaces of Cauchy data. We also wish to recall that when the local Dirichlet to Neumann map Λ_D^Σ exists, for instance when $q = 0$, the local Cauchy data is the graph of Λ_D^Σ and the distance between two sets of Cauchy data is equivalent to the norm of the difference of the corresponding local Dirichlet to Neumann maps. For two inclusions D_1 and D_2 , we consider the corresponding local Cauchy data sets $\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma$ collected on Σ , the distance $d(\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma)$ and, by using the Alessandrini identity argument, we establish the following inequality

$$\begin{aligned}&\left| \int_{\Omega} (\sigma_2 - \sigma_1) \nabla u_1 \cdot \nabla u_2 + \int_{\Omega} (q_1 - q_2) u_1 u_2 \right| \leq \\ &\leq d(\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma) \|(u_1, \sigma_1 \nabla u_1 \cdot \nu)\|_{\mathcal{H}} \|(\bar{u}_2, \sigma_2 \nabla \bar{u}_2 \cdot \nu)\|_{\mathcal{H}},\end{aligned}\tag{6}$$

(see Subsection 2.3). Here, u_i is a solution to (1) when $D = D_i$ in (3) and (4) for $i = 1, 2$. Let us also observe that, in order to prove the above estimate, the uniqueness of the solution to the Dirichlet problem (1) – (2) is not necessary.

- ii) Since the equation at hand might be in the eigenvalue regime, we need to construct and to estimate Green’s function with mixed type boundary conditions, namely Dirichlet type on a portion of the boundary and complex-valued Robin type in the remaining one. Such a boundary value problem with local complex Robin condition is well-posed as proved by Bamberger and Duong in [12]. We adapt the argument introduced in [4] to our equation, in which the principal part has a matrix-valued leading coefficient that might have a discontinuity across the boundary of the inclusion D . We overcome the leading term discontinuity issue by using a quite recent result of propagation of smallness due to Carstea and Wang [16] for a scalar second order elliptic equation in divergence form whose leading coefficients are Lipschitz continuous on two sides of a C^2 hypersurface that crosses the domain, but may have jumps across this hypersurface. We consider the above inequality for singular solutions $u_1(\cdot) = G_1(\cdot, y)$ and $u_2(\cdot) = G_2(\cdot, w)$ defined on a larger domain.

Focusing on the right hand side of (6), we introduce the function

$$f(y, w) = S_{D_1}(y, w) - S_{D_2}(y, w), \quad (7)$$

which is a solution of our underlying equation in the connected component \mathcal{G} of $\mathbb{R}^n \setminus (\overline{D_1} \cup \overline{D_2})$ which contains $\mathbb{R}^n \setminus \overline{\Omega}$. Moreover, in the case in which y, w are placed outside Ω , f is controlled in terms of $d(\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma)$. We propagate the smallness of f as y, w move inside Ω within \mathcal{G} . We wish to underline a delicate point of the proof which is the fact that we can only perform unique continuation estimates near points in a subset, say V , of the boundary of $D_1 \cup D_2$, that can be reached from \mathcal{G} in a quantitative form. This involves the use of chain of balls whose numbers are suitably bounded and whose radii must be bounded from below (see [8], [6] and also [32] for a related argument). In this respect, a crucial step is that under the a priori regularity assumptions on D_1, D_2 , we can prove that there exists a point $P \in \partial D_1 \cap V$ such that the Hausdorff distance between ∂D_1 and ∂D_2 is dominated by the distance $\text{dist}(P, D_2)$.

- iii) We show that when $y = w$ tends to a point P of $\partial D_1 \setminus \overline{D_2}$, $f(y, y)$ blows up. The combination of such a blow-up and the control of $f(y, y)$ in terms of $d(\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma)$ discussed above leads to the logarithmic estimate

$$d_H(\partial D_1, \partial D_2) \leq C |\log(d(\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma))|^{-\eta}. \quad (8)$$

The new features of anisotropic and inhomogeneous leading coefficient, as well as the additional zero-order term, require a careful analysis of the asymptotic behaviour of the singular solutions $G_i(\cdot, y)$ when the pole y approaches to the inclusion D_i , $i = 1, 2$ and serves as a tool to achieve the blow up estimate of $f(y, y)$.

We believe that the present study might be a theoretical building block for future Lipschitz stability result and corresponding numerical reconstruction procedure under the a priori assumption of polygonal or polyhedral inclusion. For this reason, we also choose to provide the following stability estimate

$$d_H(\partial D_1, \partial D_2) \leq C |\log \mathcal{J}(D_1, D_2)|^{-\eta}, \quad (9)$$

in the present context of a general inclusion with C^2 boundary and in terms of a misfit functional

$$\mathcal{J}(D_1, D_2) = \int_{D_y \times D_z} \left| \int_{\Sigma} [\sigma_1(x) \nabla G_1(x, y) \cdot \nu(x) G_2(x, z) - \sigma_2(x) \nabla G_2(x, z) \cdot \nu(x) G_1(x, y)] dS(x) \right|^2 dy dz, \quad (10)$$

where the D_y, D_z are suitably chosen sets compactly contained in $\mathbb{R}^n \setminus \bar{\Omega}$ (see Section 5 for a more precise definition). We expect that for polygonal or polyhedra inclusions, the logarithmic rate in (9) might be improved up to a Hölder type stability. As shown in [3] (see also [22]), the above mentioned Hölder estimate may be suitable to the numerical reconstructions and to the use of the singular solutions method. The use of a misfit functional of such kind suggests that the knowledge of the full Dirichlet to Neumann map or the full local Cauchy data set are not necessary, and that it suffices to sample them on Green's type functions with sources placed outside the physical domain.

The paper is organised as follows. In Section 2 we introduce the a priori assumptions, we define the local Cauchy data and state the main theorem. In Section 3 we introduce the geometric lemmas and we prove the main theorem. In Section 4 we introduce and prove technical propositions. In particular, we construct the Green function (Lemma 4.1) and we prove the upper bound for f (Proposition 3.4) and the lower bound for f (Proposition 3.5). In Section 5 we derive a stability result in terms of the misfit functional (10).

2. Main result.

2.1. Notation and definitions. Denote a point $x \in \mathbb{R}^n$ by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$, $n \geq 3$. Denote with $B_r(x) \subset \mathbb{R}^n$, $B'_r(x') \subset \mathbb{R}^{n-1}$ the open balls centred at x , x' respectively with radius r , with $Q_r(x)$ the cylinder

$$Q_r(x) = B'_r(x') \times (x_n - r, x_n + r).$$

Set $B_r = B_r(O)$, $Q_r = Q_r(O)$, denote with $\mathbb{R}_+^n = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$ the positive real half space, with $\mathbb{R}_-^n = \{(x', x_n) \in \mathbb{R}^n : x_n < 0\}$ the negative real half space, with $B_r^+ = B_r \cap \mathbb{R}_+^n$ the positive semisphere centred at the origin, with $B_r^- = B_r \cap \mathbb{R}_-^n$ the negative semisphere centred at the origin.

Definition 2.1 (C^2 regularity). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that a portion Σ of $\partial\Omega$ is of C^2 class with constants $r_0, L > 0$, if, for any point $P \in \Sigma$, there exists a rigid transformation of coordinates under which $P = O$ and

$$\Omega \cap Q_{r_0} = \{x \in Q_{r_0} : x_n > \varphi(x')\},$$

where $\varphi \in C^2(B'_{r_0})$ is such that

$$\varphi(O) = |\nabla \varphi(O)| = 0 \quad \text{and} \quad \|\varphi\|_{C^2(B'_{r_0})} \leq Lr_0.$$

Remark 2.2. The C^2 -norm in Definition 2.1 is normalized so that its terms are dimensionally homogeneous, i.e.

$$\|\varphi\|_{C^2(B'_{r_0})} = \sum_{i=0}^2 r_0^i \|\nabla^i \varphi\|_{L^\infty(B'_{r_0})}.$$

Definition 2.3. Let Ω be a domain of \mathbb{R}^n . We say that a portion Σ of the boundary of Ω is a flat portion of size r_0 if there exist a point $P \in \Sigma$ and a rigid transformation of coordinates under which $P = O$ and

$$\begin{aligned}\Sigma \cap Q_{\frac{r_0}{3}} &= \left\{ x \in Q_{\frac{r_0}{3}} : x_n = 0 \right\}, \\ \Omega \cap Q_{\frac{r_0}{3}} &= \left\{ x \in Q_{\frac{r_0}{3}} : x_n > 0 \right\}, \\ (\mathbb{R}^n \setminus \Omega) \cap Q_{\frac{r_0}{3}} &= \left\{ x \in Q_{\frac{r_0}{3}} : x_n < 0 \right\}.\end{aligned}$$

Definition 2.4. The *Hausdorff distance* between two bounded closed subsets D_1 and D_2 in \mathbb{R}^n is defined as

$$d_H(D_1, D_2) := \max \left\{ \max_{w \in D_2} \text{dist}(w, D_1), \max_{w \in D_1} \text{dist}(w, D_2) \right\}.$$

2.2. A priori information. In this section, we introduce the a priori information on the domain Ω , the inclusion D and the coefficients σ and q .

i) *Domain.* The set Ω is a bounded domain in \mathbb{R}^n such that

$$\partial\Omega \text{ is of Lipschitz class with constants } r_0, L, \quad (11)$$

$$|\Omega| \leq Nr_0^n, \quad (12)$$

where r_0, L, N are given positive constants, $n \geq 3$.

ii) *Open portion of the boundary.*

$$\Sigma \subset \partial\Omega \text{ is a non-empty, flat open portion,} \quad (13)$$

iii) *Inclusion.* Let D be a connected subset of Ω such that

$$D \subset\subset \Omega \quad \text{and} \quad \text{dist}(\partial D, \partial\Omega) \geq \delta_0 > 0, \quad (14)$$

$$\partial D \text{ is of } C^2 \text{ class with constants } r_0, L, \quad (15)$$

$$\Omega \setminus \bar{D} \text{ is connected.} \quad (16)$$

iv) *Parameters.*

The coefficient $\sigma \in L^\infty(\Omega, \text{Sym}_n)$ has the following structure

$$\sigma(x) = (a_b(x) + (a_D(x) - a_b(x))\chi_D(x))A(x), \quad (17)$$

where the scalar functions a_b, a_D are in $C^{0,1}(\bar{\Omega})$. Moreover, there exist $\bar{\gamma} > 1$, $\eta_0 > 0$ such that

$$\bar{\gamma}^{-1} \leq a_b(x), a_D(x) \leq \bar{\gamma}, \quad \text{for } x \in \Omega, \quad (18)$$

$$(a_D(x) - a_b(x))^2 \geq \eta_0^2 > 0, \quad \text{for } x \in \Omega. \quad (19)$$

The real $n \times n$ matrix-valued function $A(x)$ is a symmetric Lipschitz continuous function such that there exists $\bar{A} > 0$ for which

$$\|A\|_{C^{0,1}(\Omega)} \leq \bar{A}. \quad (20)$$

The matrix-valued function σ satisfies the uniform ellipticity condition, i.e. there exists a constant $\bar{\lambda} > 1$ such that

$$\bar{\lambda}^{-1}|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq \bar{\lambda}|\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n. \quad (21)$$

The scalar function q has the form

$$q(x) = q_b(x) + (q_D(x) - q_b(x))\chi_D(x), \quad (22)$$

where the functions q_b, q_D are in $L^\infty(\bar{\Omega})$. Moreover, there exists $\bar{\gamma} > 0$ such that

$$\|q\|_{L^\infty(\Omega)} \leq \bar{\gamma}. \quad (23)$$

The set $\{n, N, r_0, L, \bar{A}, \gamma^-, \bar{\lambda}, \delta_0\}$ is called the *a priori data*.

2.3. Local Cauchy data and the main result. Consider a domain Ω , an inclusion D satisfying (11)-(12) and (14)-(16) respectively. Let Σ be the accessible portion of $\partial\Omega$ where the boundary measurements are taken. A Cauchy data set is a collection of boundary data measurements associated with an inclusion. Before giving the formal definition of the notion of local Cauchy data, we introduce suitable trace spaces.

Let

$$H_{co}^{\frac{1}{2}}(\Sigma) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) : \text{supp}(f) \subset \Sigma \right\}$$

be the trace space that contains all the trace functions which are compactly supported in Σ . Denote with $H_{00}^{\frac{1}{2}}(\Sigma)$ its closure under the norm $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega)}$. Similarly, let

$$H_{co}^{\frac{1}{2}}(\partial\Omega \setminus \bar{\Sigma}) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) : \text{supp}(f) \subset \partial\Omega \setminus \bar{\Sigma} \right\},$$

and denote with $H_{00}^{\frac{1}{2}}(\partial\Omega \setminus \bar{\Sigma})$ its closure under the norm $\|\cdot\|_{H^{\frac{1}{2}}(\partial\Omega)}$. Let $H^{-\frac{1}{2}}(\partial\Omega)$ be the dual space of $H_{co}^{\frac{1}{2}}(\partial\Omega)$.

Definition 2.5. The Cauchy data set on Σ associated with the inclusion D is defined as the set

$$\begin{aligned} \mathcal{C}_D^\Sigma(\Omega) = & \left\{ (f, g) \in H_{00}^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega) : \exists u \in H^1(\Omega) \text{ weak solution to} \right. \\ & \text{div}(\sigma \nabla u) + qu = 0 \quad \text{in } \Omega, \\ & \left. u|_{\partial\Omega} = f, \quad \sigma \nabla u \cdot \nu|_{\partial\Omega} = g \right\}. \end{aligned} \quad (24)$$

Let

$$H_{00}^{-\frac{1}{2}}(\partial\Omega \setminus \bar{\Sigma}) = \left\{ \psi \in H^{-\frac{1}{2}}(\partial\Omega) : \langle \psi, \varphi \rangle = 0, \quad \forall \varphi \in H_{00}^{\frac{1}{2}}(\Sigma) \right\},$$

where $\langle \psi, \varphi \rangle$ represents the duality between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$ based on the inner product on $L^2(\partial\Omega)$

$$\langle \psi, \varphi \rangle = \int_{\partial\Omega} \psi(x) \cdot \overline{\varphi(x)} \, dx.$$

Define $H^{\frac{1}{2}}(\partial\Omega)|_\Sigma$ and $H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma$ as the restrictions to Σ of the trace spaces $H^{\frac{1}{2}}(\partial\Omega)$ and $H^{-\frac{1}{2}}(\partial\Omega)$ respectively. These trace spaces can be equivalently defined as quotient spaces via the following relation:

$$\varphi \sim \psi \quad \iff \quad \varphi - \psi \in H_{00}^{\frac{1}{2}}(\partial\Omega \setminus \bar{\Sigma}),$$

so that

$$H^{\frac{1}{2}}(\partial\Omega)|_\Sigma = H^{\frac{1}{2}}(\partial\Omega) / \sim = H^{\frac{1}{2}}(\partial\Omega) / H_{00}^{\frac{1}{2}}(\partial\Omega \setminus \bar{\Sigma}).$$

Similarly,

$$H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma = H^{-\frac{1}{2}}(\partial\Omega) / H_{00}^{-\frac{1}{2}}(\partial\Omega \setminus \bar{\Sigma}).$$

Definition 2.6. The local Cauchy data on Σ associated with the inclusion D whose first component vanishes on $\partial\Omega \setminus \Sigma$ is defined as

$$\begin{aligned} \mathcal{C}_D^\Sigma(\Sigma) = & \left\{ (f, g) \in H_{00}^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma : \exists u \in H^1(\Omega) \text{ weak solution to} \right. \\ & \text{div}(\sigma \nabla u) + qu = 0 \quad \text{in } \Omega, \\ & u = f \quad \text{on } \partial\Omega, \\ & \left. \langle \sigma \nabla u \cdot \nu|_{\partial\Omega}, \varphi \rangle = \langle g, \varphi \rangle, \quad \forall \varphi \in H_{00}^{\frac{1}{2}}(\Sigma) \right\}. \end{aligned}$$

Notice that $\mathcal{C}_D^\Sigma(\Sigma)$ is a subspace of the product space $H_{00}^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma$.

Set $\mathcal{H} := H_{00}^{\frac{1}{2}}(\Sigma) \times H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma$. Notice that \mathcal{H} is a Hilbert space with norm

$$\|(f, g)\|_{\mathcal{H}} = \left(\|f\|_{H_{00}^{\frac{1}{2}}(\Sigma)}^2 + \|g\|_{H^{-\frac{1}{2}}(\partial\Omega)|_\Sigma}^2 \right)^{\frac{1}{2}}, \quad (f, g) \in \mathcal{H}.$$

Let D_1, D_2 be two inclusions satisfying (14)-(16). In order to simplify the notation, set $\mathcal{C}_i = \mathcal{C}_{D_i}^\Sigma(\Sigma)$ for $i = 1, 2$ which denote the local Cauchy data associated with the corresponding inclusions.

As in [3], we find convenient to introduce the notion of *distance* (or *aperture*) between closed subspaces \mathcal{F} and \mathcal{G} of a Hilbert space \mathcal{H} as the quantity

$$d(\mathcal{F}, \mathcal{G}) = \max \left\{ \sup_{h \in \mathcal{G}, h \neq 0} \inf_{k \in \mathcal{F}} \frac{\|h - k\|}{\|h\|}, \sup_{k \in \mathcal{F}, k \neq 0} \inf_{h \in \mathcal{G}} \frac{\|h - k\|}{\|k\|} \right\}.$$

As a reference for the related theory and applications see [26], [27]. If $d(\mathcal{F}, \mathcal{G}) < 1$, it is known that the two quantities in the maximum coincide (see [27, Corollary 2.13]), therefore we can assume that

$$d(\mathcal{F}, \mathcal{G}) = \sup_{h \in \mathcal{G}, h \neq 0} \inf_{k \in \mathcal{F}} \frac{\|h - k\|}{\|h\|}. \quad (25)$$

Choose $\mathcal{F} = \mathcal{C}_1$ and $\mathcal{G} = \mathcal{C}_2$, since we consider the case in which $d(\mathcal{C}_1, \mathcal{C}_2) < 1$, then by (25) the distance between two local Cauchy data has the form

$$d(\mathcal{C}_1, \mathcal{C}_2) = \sup_{(f_1, g_1) \in \mathcal{C}_1 \setminus \{(0,0)\}} \inf_{(f_2, g_2) \in \mathcal{C}_2} \frac{\|(f_1, g_1) - (f_2, g_2)\|_{\mathcal{H}}}{\|(f_1, g_1)\|_{\mathcal{H}}}. \quad (26)$$

Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a non decreasing, positive function such that

$$\omega(t) \leq |\ln t|^{-\eta} \quad \text{for } t \in (0, 1), \quad \omega(t) \rightarrow 0 \quad \text{as } t \rightarrow 0^+, \quad (27)$$

where η is a positive constant.

Theorem 2.7. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain satisfying (11)-(12) and let D_1, D_2 be two inclusions of Ω satisfying (14)-(16). Let σ_1, σ_2 be the anisotropic coefficients satisfying (17)-(21) and let q_1, q_2 be the coefficients of the zeroth order term satisfying (22)-(23). Let Σ be a non-empty open portion of $\partial\Omega$. Let $\mathcal{C}_1, \mathcal{C}_2$ be the local Cauchy data corresponding to the inclusions D_1, D_2 , respectively. For $\epsilon \in (0, 1)$, if $d(\mathcal{C}_1, \mathcal{C}_2) < \epsilon$, then

$$d_H(\partial D_1, \partial D_2) \leq C\omega(\epsilon), \quad (28)$$

where C is a positive constant depending on the a priori data only and ω is defined in (27).

3. Proof of theorem 2.7. To begin with, in the spirit of [5], we introduce the so-called *modified distance*. After that, we introduce the singular solutions and state Proposition 3.4 and Proposition 3.5 which allow us to upper-bound the function f defined in (42) in terms of the distance between two Cauchy data sets and to lower bound f in terms of the geometric quantities related with our problem. In the last part, we prove Theorem 2.7.

3.1. Metric lemmas. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying (11)-(12) and let D_1, D_2 be two inclusions contained in Ω satisfying (14)-(16). Let σ_1, σ_2 be the anisotropic coefficients satisfying (17)-(21), and q_1, q_2 satisfying (22)-(23).

Denote with \mathcal{G} the connected component of $\Omega \setminus (D_1 \cup D_2)$ whose boundary contains $\partial\Omega$. Set $\Omega_D = \Omega \setminus \mathcal{G}$.

Definition 3.1. The modified distance between two closed subsets D_1 and D_2 of \mathbb{R}^n is the number

$$d_\mu(D_1, D_2) = \max \left\{ \max_{x \in \partial D_1 \cap \partial \Omega_D} \text{dist}(x, D_2), \max_{x \in \partial D_2 \cap \partial \Omega_D} \text{dist}(x, D_1) \right\}. \quad (29)$$

The introduction of another map that quantifies the distance between two inclusions D_1, D_2 is justified by the fact that the point in which the Hausdorff distance is attained may lay on $\partial D_1 \cup \partial D_2$ but not on $\partial \Omega_D$. This is an obstruction in view of the application of the propagation of smallness argument, since in order to reach that point from $\partial\Omega$ we have to cross $\partial D_1 \cup \partial D_2$.

In general, the modified distance is not a distance and it does not bound the Hausdorff distance from above (see [2] for a counterexample). Under our assumptions on the inclusions, the following lemma guarantees that in our case d_μ dominates $d_{\mathcal{H}}$ (for the proof see [5, Proposition 3.3]).

Lemma 3.2. *Let Ω, D_1, D_2 be, respectively, a bounded domain satisfying (11)-(12) and two inclusions satisfying (14)-(16). Then there is a positive constant c which depends only on the a priori data such that*

$$d_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c d_\mu(D_1, D_2). \quad (30)$$

Let us remark now that for simplicity we assume that there exists a point on $\partial D_1 \cap \partial \Omega_D$ which realizes the modified distance. We denote that quantity simply as d_μ .

In order to apply the quantitative estimates for unique continuation, based on the iterated application of the three-spheres inequality, it is important to control the radii of these balls from below and to avoid the case in which points of $\partial \Omega_D$ are not reachable by such a chain of balls. Hence, we find convenient to introduce here ideas first presented in [8] and then applied by [6] in the elasticity case.

Let O denote the origin in \mathbb{R}^n , ν be a unit vector, $h > 0$ and $\theta \in (0, \frac{\pi}{2})$ be an angle. We recall that the closed truncated cone with vertex at O , axis along the direction ν , height h and aperture 2θ is given by

$$C(O, \nu, h, \theta) = \{x \in \mathbb{R}^n : |x - (x \cdot \nu)\nu| \leq |x| \sin \theta, 0 \leq x \cdot \nu \leq h\}. \quad (31)$$

In our case, for $d, R > 0$ such that $R < d$, fixed a point $Q = -de_n$, we consider the closed cone

$$C \left(O, -e_n, \frac{d^2 - R^2}{d}, \arcsin \frac{R}{d} \right)$$

whose oblique sides are tangent to the sphere $\partial B_R(O)$.

Let us pick a point $P \in \partial D_1 \cap \partial \Omega_D$ and let ν be the outer unit normal to ∂D_1 at P . For a suitable $d > 0$, let $[P + d\nu, P]$ be the segment contained in $\mathbb{R}^n \setminus \bar{\Omega}_D$.

For $P_0 \in \mathbb{R}^n \setminus \bar{\Omega}$, let $\gamma : [0, 1] \rightarrow \mathbb{R}^n \setminus \bar{\Omega}_D$ be the path contained in $\mathbb{R}^n \setminus \bar{\Omega}_D$ such that $\gamma(0) = P_0$ and $\gamma(1) = P + d\nu$. Consider the following neighbourhood of $\gamma \cup [P + d\nu, P] \setminus \{P\}$ formed by a tubular neighbourhood of γ attached to a cone with vertex at P and axis along ν ,

$$V(\gamma) = \left[\bigcup_{S \in \gamma} B_R(S) \right] \cup C \left(P, \nu, \frac{d^2 - R^2}{d}, \arcsin \frac{R}{d} \right). \quad (32)$$

Notice that $V(\gamma)$ will depend on the parameters d, R , that will be chosen step by step. The following result guarantees the application of the three-sphere inequality along the tubular neighbourhood contained in $\mathbb{R}^n \setminus \bar{\Omega}_D$ centred at any point of $\partial D_1 \cap \partial \Omega_D$ (see [6, section 11] for a proof).

Lemma 3.3. *There exist positive constants \bar{d}, c_1 where $\frac{\bar{d}}{r_0}, c_1$ depend on L and there exists a point $P \in \partial D_1$ satisfying*

$$c_1 d_\mu \leq \text{dist}(P, D_2), \quad (33)$$

and such that, for any $P_0 \in B_{\frac{r_0}{16}}(P_0)$, where $B_{\frac{r_0}{16}}(P_0) \subset \subset \mathbb{R}^n \setminus \bar{\Omega}_D$, there exists a path $\gamma \subset \mathbb{R}^n \setminus \bar{\Omega}_D$ joining P_0 to $P + \bar{d}\nu$ where ν is the outer unit normal to D_1 at P such that, if we choose the coordinate system in which P coincides with the origin and $\nu = -e_n$, then

$$V(\gamma) \subset \mathbb{R}^n \setminus \Omega_D, \quad (34)$$

where $V(\gamma)$ is the tubular neighbourhood introduced in (32), provided that $R = \frac{\bar{d}}{\sqrt{1+L_0^2}}$, where $L_0 > 0$ depends only on L .

3.2. Proof of theorem 2.7. Before proving the theorem, fix a point on the surface Σ so that, up to a rigid transformation, it coincides with the origin. We define D_0 as the bounded domain with Lipschitz constants $r_0 > 0, L > 0$ of the form

$$D_0 = \{x \in (\mathbb{R}^n \setminus \bar{\Omega}) \cap B_{r_0} : |x_i| < r_0, i = 1, \dots, n-1, -r_0 < x_n < 0\}, \quad (35)$$

such that $\partial D_0 \cap \partial \Omega \subset \subset \Sigma$. We define the augmented domain Ω_0 as the set

$$\Omega_0 = \overline{\Omega \cup D_0}^\circ. \quad (36)$$

It turns out that Ω_0 is of Lipschitz class with constants r_0 and \tilde{L} , where \tilde{L} depends on L only. Moreover, let $\Sigma_0 \subset \partial D_0$ be a nonempty portion of the boundary of D_0 of the form

$$\Sigma_0 = \{x \in \Omega_0 : |x_i| \leq r_0, x_n = -r_0\},$$

such that $\Sigma_0 \cap \partial \Omega = \emptyset$.

Consider D_1, D_2 the two inclusions of Ω satisfying (14)-(16). Without loss of generality, we can extend the corresponding coefficients $\sigma_1, \sigma_2, q_1, q_2$ to the augmented domain Ω_0 by setting their value equal to the identity matrix on D_0 , so that they are of the form

$$\begin{aligned} \sigma_i(x) &= \left(a_i(x) + (a_{D_i}(x) - a_b(x))\chi_{D_i}(x) \right) A(x), \quad \text{for any } x \in \Omega, \\ \sigma_i|_{D_0} &= I, \quad q_i|_{D_0} = 1. \end{aligned}$$

We denote with the same symbol the extended coefficients σ_i and q_i for $i = 1, 2$.

By now, let us drop the pedix i and consider the coefficients σ, q associated with a generic inclusion D . Denote with G the Green function associated with the

operator $\operatorname{div}(\sigma(\cdot)\nabla\cdot) + q(\cdot)$ on the augmented domain Ω_0 . For every $y \in D_0$, $G(\cdot, y)$ is the distributional solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma(\cdot)\nabla G(\cdot, y)) + q(\cdot)G(\cdot, y) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot)\nabla G(\cdot, y) \cdot \nu(\cdot) + iG(\cdot, y) = 0 & \text{on } \Sigma_0, \end{cases} \quad (37)$$

where $\delta(\cdot - y)$ is the Dirac distribution centred at y and ν is the outer unit normal at Σ_0 . The following property holds true for the Green's functions (see Lemma 4.1):

$$0 < |G(x, y)| < C|x - y|^{2-n} \quad \text{for any } x \neq y, \quad (38)$$

where C is a positive constant depending on $\bar{\lambda}$ and n .

Let us now consider G_1 and G_2 the Green functions solutions to (37) associated to the inclusions D_1 and D_2 respectively. Fix k in (37), then multiply the first equation by $G_j(\cdot, y)$ for $j \neq k$ and integrate by parts on Ω . Repeat the same procedure interchanging the role of k and j . Subtracting the two quantities obtained leads to the following equality

$$\begin{aligned} & \int_{\Sigma} \left[\sigma_1(x)\nabla G_1(x, y) \cdot \nu(x) G_2(x, z) - \sigma_2(x)\nabla G_2(x, z) \cdot \nu(x) G_1(x, y) \right] dS(x) = \\ & = \int_{\Omega} (\sigma_1(x) - \sigma_2(x))\nabla G_1(x, y) \cdot \nabla G_2(x, z) dx + \\ & + \int_{\Omega} (q_2(x) - q_1(x))G_1(x, y) G_2(x, z) dx. \end{aligned} \quad (39)$$

Define

$$\begin{aligned} S_1(y, z) &= \int_{D_1} (a_{D_1}(x) - a_b(x))A(x)\nabla G_1(x, y) \cdot \nabla G_2(x, z) dx - \\ & - \int_{D_1} (q_{D_1}(x) - q_b(x))G_1(x, y) G_2(x, z) dx. \end{aligned} \quad (40)$$

$$\begin{aligned} S_2(y, z) &= \int_{D_2} (a_{D_2}(x) - a_b(x))A(x)\nabla G_1(x, y) \cdot \nabla G_2(x, z) dx - \\ & - \int_{D_2} (q_{D_2}(x) - q_b(x))G_1(x, y) G_2(x, z) dx. \end{aligned} \quad (41)$$

Set

$$f(y, z) = S_1(y, z) - S_2(y, z). \quad (42)$$

As a premise to the main result, we state the Proposition 3.4 and Proposition 3.5 that allow us to determine an upper bound of f in terms of the Cauchy data $d(\mathcal{C}_1, \mathcal{C}_2)$ and a lower bound of f .

Proposition 3.4. *Let D_1, D_2 be two inclusions of Ω satisfying (14)-(16). Let $\mathcal{C}_1, \mathcal{C}_2$ be the local Cauchy data associated with the inclusions D_1, D_2 , respectively. Under the notation of Lemma 3.3, let*

$$y = h\nu(O),$$

where

$$0 < h \leq \bar{d} \left(1 - \frac{\sin \theta_0}{4} \right) \text{ for } \theta_0 = \arctan \frac{1}{L},$$

and $\nu(O)$ is the outer unit normal of D_1 at O . For $\epsilon \in (0, 1)$, if $d(\mathcal{C}_1, \mathcal{C}_2) < \epsilon$, it follows that

$$|f(y, y)| \leq c_1 \frac{\epsilon^{Bh^F}}{h^A}, \quad (43)$$

where $A, B, F, c_1 > 0$ constants that depend on the a priori data only.

Proposition 3.5. *Under the same assumptions as in Proposition 3.4 and Lemma 3.3, there exist $c_2, c_3 > 0, \bar{h} \in (0, \frac{1}{2})$ that depend on the a priori data only such that*

$$|f(y, y)| \geq c_2 h^{2-n} - c_3 (\text{dist}(O, D_2) - h)^{2-2n} \quad \text{for } 0 < h < \bar{h}\bar{r}_2, \quad (44)$$

where $y = h\nu(O)$ with $\bar{r}_2 \in (0, \min\{\frac{r_0}{\sqrt{1+L^2}} \min\{1, L\}, \text{dist}(O, D_2)\})$.

Proof of Theorem 2.7. Let $O \in \partial D_1$ be the point of Lemma 3.3 such that (33) is satisfied and w.l.o.g. assume that it coincides with the origin. Choose

$$y_h = h\nu(O) \quad \text{for } 0 < h < \bar{h}\bar{r}_2.$$

Combining the upper bound (43) and the lower bound (44) at y_h , it follows that

$$c_2 h^{2-n} - c_3 (\text{dist}(O, D_2) - h)^{2-2n} \leq c_1 \frac{\epsilon^{Bh^F}}{h^A},$$

where c_1, c_2, c_3, A, B, F are the constants appearing in the Propositions 3.4 and 3.5 which depends on the a priori data only.

Let $\epsilon_1 \in (0, 1)$ be such that $\exp(-B|\ln \epsilon_1|^{\frac{1}{2}}) = \frac{1}{2}$. We distinguish between two cases.

- a) Assume that $\epsilon \in (0, \epsilon_1)$. Define $h = h(\epsilon) = \min\{|\ln \epsilon|^{-\frac{1}{2F}}, \text{dist}(O, D_2)\}$. If $\text{dist}(O, D_2) \leq |\ln \epsilon|^{-\frac{1}{2F}}$ then from Lemma 3.2 and Lemma 3.3 the thesis follows straightforwardly. If $\text{dist}(O, D_2) \geq |\ln \epsilon|^{-\frac{1}{2F}}$, then $h = |\ln \epsilon|^{-\frac{1}{2F}}$ so that

$$c_4 (\text{dist}(O, D_2) - h)^{2(1-n)} \geq c_5 (1 - \epsilon^{Bh^F} h^{\tilde{A}}) h^{2-n}, \quad (45)$$

where $\tilde{A} = 1 - A$. Since

$$\epsilon^{Bh^F} h^{\tilde{A}} \leq \exp(-B|\ln \epsilon|^{\frac{1}{2}}),$$

then

$$(\text{dist}(O, D_2) - h)^{2(1-n)} \geq c_6 h^{2-n},$$

and since $h = |\ln \epsilon|^{-\frac{1}{2F}}$,

$$\text{dist}(O, D_2) \leq c_7 |\ln \epsilon|^{-\eta} \quad \text{for } \eta = \frac{n-2}{4F(n-1)}.$$

- b) Assume that $\epsilon \in [\epsilon_1, 1)$, then, since $\text{dist}(O, D_2) \leq \text{diam}(\Omega)$, it follows that

$$\text{dist}(O, D_2) \leq \text{diam}(\Omega) \frac{|\ln \epsilon|^{-\frac{1}{2F}}}{|\ln \epsilon_1|^{-\frac{1}{2F}}}.$$

Finally, applying Lemma 3.2 and Lemma 3.3, the thesis follows. \square

4. Proofs of technical propositions. In this section, we construct the Green functions associated with (37) and we sketch the proofs of the Proposition 3.4 and Proposition 3.5 stated in section 3.1. In order to simplify the notation, we prefer to drop the pedix $i = 1, 2$ for all the quantities related to the two inclusions, so that the following statements hold true for a generic inclusion D satisfying (14)-(16), coefficients σ satisfying (17)-(21) and q satisfying (22)-(23).

4.1. The construction of the green function.

Lemma 4.1. *For any $\sigma \in L^\infty(\Omega_0, \text{Sym}_n)$ that satisfies the uniform ellipticity condition, for any $q \in L^\infty(\Omega_0)$, $y \in \Omega_0$ there exists a unique distributional solution $G(\cdot, y)$ of the boundary value problem*

$$\begin{cases} \operatorname{div}(\sigma(\cdot)\nabla G(\cdot, y)) + q(\cdot)G(\cdot, y) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot)\nabla G(\cdot, y) \cdot \nu(\cdot) + iG(\cdot, y) = 0 & \text{on } \Sigma_0, \end{cases} \quad (46)$$

such that for any $x, y \in \Omega_0, x \neq y$, there exists a constant C depending on the a priori data such that

$$0 < |G(x, y)| \leq C|x - y|^{2-n}. \quad (47)$$

Proof. Our proof is based on the reasoning introduced in [4, Proposition 3.1]. We find more convenient to divide the proof into three steps: in the first step we prove the well-posedness of the problem, in the second step we construct the Green function and in the final step we prove the symmetry of the Green function.

First step (well-posedness) For $f \in L^2(\Omega_0)$, consider the following mixed boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla v) + qv = f, & \text{in } \Omega_0, \\ v = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot)\nabla v(\cdot) \cdot \nu(\cdot) + iv(\cdot) = 0, & \text{on } \Sigma_0. \end{cases} \quad (48)$$

Our goal is to find a solution $v \in H^1(\Omega_0)$ in the weak sense. Consider the adjoint mixed boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla w) + qw = f, & \text{in } \Omega_0, \\ w = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot)\nabla w(\cdot) \cdot \nu(\cdot) - iw(\cdot) = 0, & \text{on } \Sigma_0. \end{cases} \quad (49)$$

The Fredholm alternative tells us that existence of a solution to (48) implies uniqueness to (49) and viceversa (see [21, Theorem 4, §6]). We prove uniqueness for both boundary value problems. Consider the homogeneous problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) + qu = 0, & \text{in } \Omega_0, \\ u = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot)\nabla u(\cdot) \cdot \nu(\cdot) \pm iu(\cdot) = 0 & \text{on } \Sigma_0. \end{cases} \quad (50)$$

Assume that $u \in H^1(\Omega_0)$. If we multiply (50) by \bar{u} and integrate on Ω_0 , by the Green's identity it follows that

$$\int_{\Omega_0} \sigma(x)\nabla u(x) \cdot \nabla \bar{u}(x) \, dx - \int_{\Omega_0} q(x)|u(x)|^2 \, dx \pm i \int_{\Sigma_0} |u(x)|^2 \, dx = 0. \quad (51)$$

Therefore, $u = 0$ on Σ_0 . Hence, u satisfies the Cauchy problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) + qu = 0, & \text{in } \Omega_0, \\ u = 0, & \text{on } \partial\Omega_0. \end{cases} \quad (52)$$

so that $u = 0$ in Ω_0 . In conclusion, we have proved existence and uniqueness for (48). It remains to prove stability. For this purpose, consider $v \in H^1(\Omega_0)$ the weak solution to (48), then by the weak formulation the following identities hold:

$$\int_{\Sigma_0} |v|^2 = -\Im\left(\int_{\Omega_0} f\bar{v}\right), \quad (53)$$

$$\int_{\Omega_0} \sigma(x) \nabla v(x) \cdot \nabla \bar{v}(x) \, dx = -\Re \left(\int_{\Omega_0} f \bar{v} \right) + \int_{\Omega_0} q(x) |v(x)|^2 \, dx. \quad (54)$$

Define the following quantities:

$$\epsilon^2 = \int_{\Sigma_0} |v|^2 + \int_{\Sigma_0} \sigma(x) \nabla v(x) \cdot \nu(x) \bar{v}(x) \, dx,$$

$$\eta = \|f\|_{L^2(\Omega_0)}, \quad \delta = \|v\|_{L^2(\Omega_0)}, \quad E = \|\nabla v\|_{L^2(\Omega_0)}.$$

From the Schwarz inequality and (53), it follows that

$$\int_{\Sigma_0} |v|^2 \leq \eta \delta, \quad (55)$$

and combined with the impedance condition,

$$\epsilon^2 \leq 2\eta \delta. \quad (56)$$

From (53), we derive

$$E^2 \leq \eta \delta + \|q\|_{L^\infty(\Omega_0)}^2 \delta^2. \quad (57)$$

Our claim is that there exists a positive constant which depends on the a priori data such that

$$E^2 \leq C\eta^2. \quad (58)$$

We distinguish between two cases.

- If $\delta^2 \leq \eta^2$, then the claim follows from (57).
- If $\delta^2 \geq \eta^2$, we recall a quantitative estimate of unique continuation due to Carstea and Wang [16, Theorem 5.3], which is a generalization of [7, Theorem 1.9], so that

$$\delta^2 \leq (E^2 + \epsilon^2 + \eta^2) \omega \left(\frac{\epsilon^2 + \eta^2}{E^2 + \epsilon^2 + \eta^2} \right), \quad (59)$$

where $\omega(t) \leq C |\ln t|^{-\mu}$ for $t \in (0, 1)$, $\omega(t) \rightarrow 0$ for $t \rightarrow 0^+$ and $C > 0, \mu \in (0, 1)$ positive constants depending on the a priori data only. From (56) and (57),

$$\begin{aligned} \delta^2 &\leq (\eta\delta + \|q\|_{L^\infty(\Omega_0)}\delta^2 + 2\eta\delta + \eta^2) \omega \left(\frac{\epsilon^2 + \eta^2}{E^2 + \epsilon^2 + \eta^2} \right) \\ &\leq (4\delta^2 + \|q\|_{L^\infty(\Omega_0)}\delta^2) \omega \left(\frac{\epsilon^2 + \eta^2}{E^2 + \epsilon^2 + \eta^2} \right). \end{aligned}$$

Multiplying by δ^2 leads to

$$1 \leq (4 + \|q\|_{L^\infty(\Omega_0)}) \omega \left(\frac{\epsilon^2 + \eta^2}{E^2 + \epsilon^2 + \eta^2} \right).$$

Inverting with respect to ω leads to

$$\omega^{-1} \left(\frac{1}{4 + \|q\|_{L^\infty(\Omega_0)}} \right) \leq \frac{\epsilon^2 + \eta^2}{E^2 + \epsilon^2 + \eta^2}.$$

Set $C = \omega^{-1} \left(\frac{1}{4 + \|q\|_{L^\infty(\Omega_0)}} \right)$, it follows that

$$CE^2 \leq C(E^2 + \epsilon^2 + \eta^2) \leq \epsilon^2 + \eta^2 \leq 2\eta\delta + \eta^2 \leq 3\eta^2,$$

so that the claim follows.

Finally, applying Poincarè inequality, it follows that

$$\|v\|_{H^1(\Omega_0)} \leq C\|f\|_{L^2(\Omega_0)}.$$

Second step (construction of the Green function) Fix $y \in \Omega_0$ and let $\tilde{G}(\cdot, y)$ be the weak solution to the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma(\cdot)\nabla\tilde{G}(\cdot, y)) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ \tilde{G}(\cdot, y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot)\nabla\tilde{G}(\cdot, y) \cdot \nu(\cdot) + i\tilde{G}(\cdot, y) = 0 & \text{on } \Sigma_0. \end{cases} \quad (60)$$

From [29], $\tilde{G}(\cdot, y)$ satisfies the following properties:

$$\tilde{G}(x, y) = \tilde{G}(y, x), \quad (61)$$

and

$$|\tilde{G}(x, y)| \leq C|x - y|^{2-n} \quad \text{for any } x \neq y, x, y \in \Omega_0. \quad (62)$$

Fix $J = \lfloor \frac{n-1}{2} \rfloor$, for $x \in \Omega_0$, $x \neq y$, define

$$\begin{cases} R_0(x, y) = \tilde{G}(x, y) \\ R_j(x, y) = \int_{\Omega_0} q(z)\tilde{G}(x, z)R_{j-1}(z, y) \, dz \quad \text{for } j = 1, \dots, J. \end{cases}$$

The distribution $R_j(\cdot, y)$ is a weak solution to the boundary value problem

$$\begin{cases} \operatorname{div}_x(\sigma(x)\nabla_x R_j(x, y)) = -q(x)R_{j-1}(x, y) & \text{for } x \in \Omega_0, \\ R_j(x, y) = 0 & \text{for } x \in \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(x)\nabla_x R_j(x, y) \cdot \nu(x) + iR_j(x, y) = 0 & \text{for } x \in \Sigma_0, \end{cases}$$

for $j = 1, \dots, J$. From [30, Chapter 2] one can show that

$$|R_j(x, y)| \leq C|x - y|^{2j+2-n}, \quad \text{for every } j = 0, 1, \dots, J-1. \quad (63)$$

For $j = J$, one has to distinguish between two cases:

- for n even,

$$|R_J(x, y)| \leq C(|\ln|x - y|| + 1), \quad (64)$$

- for n odd,

$$|R_J(x, y)| \leq C, \quad (65)$$

where in both cases C is a positive constant which depends on the a priori data only. In either cases,

$$\|R_J(\cdot, y)\|_{L^p(\Omega_0)} \leq C, \quad \text{for } 1 \leq p < \infty.$$

Define the distribution $R_{J+1}(\cdot, y)$, for $y \in \Omega_0$ as the weak solution to the boundary value problem

$$\begin{cases} \operatorname{div}_x(\sigma(x)\nabla_x R_{J+1}(x, y)) + q(x)R_{J+1}(x, y) = -q(x)R_J(x, y) & \text{for } x \in \Omega_0, \\ R_{J+1}(x, y) = 0 & \text{for } x \in \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(x)\nabla_x R_{J+1}(x, y) \cdot \nu(x) + iR_{J+1}(x, y) = 0 & \text{for } x \in \Sigma_0. \end{cases}$$

It follows that $\|R_{J+1}(\cdot, y)\|_{H^1(\Omega_0)} \leq C$ where C is a positive constant and by interior regularity estimates

$$|R_{J+1}(x, y)| \leq C \quad \text{for } x \neq y, x, y \in \Omega_0. \quad (66)$$

Define

$$G(x, y) = \tilde{G}(x, y) + \sum_{j=1}^{J+1} R_j(x, y). \quad (67)$$

For $y \in \Omega_0$, $G(\cdot, y)$ is a distributional solution to the boundary value problem (46) so that G is the Green's function that we were looking for.

Third step (Symmetry of the Green function). Let $f, g \in C_0^\infty(\Omega_0)$. Let $u \in H^1(\Omega_0)$ be a weak solution to

$$\begin{cases} \operatorname{div}(\sigma \nabla u) + q u = f, & \text{in } \Omega_0, \\ u = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot) \nabla u(\cdot) \cdot \nu(\cdot) + i u(\cdot) = 0, & \text{on } \Sigma_0. \end{cases} \quad (68)$$

Let $v \in H^1(\Omega_0)$ be a weak solution to

$$\begin{cases} \operatorname{div}(\sigma \nabla v) + q v = g, & \text{in } \Omega_0, \\ v = 0, & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(\cdot) \nabla v(\cdot) \cdot \nu(\cdot) + i v(\cdot) = 0, & \text{on } \Sigma_0. \end{cases} \quad (69)$$

Let $G(\cdot, y)$ be the Green function solution to (46). The weak solution u of (68) can be written as

$$u(x) = \int_{\Omega_0} G(x, y) f(y) \, dy,$$

and similarly,

$$v(x) = \int_{\Omega_0} G(x, y) g(y) \, dy.$$

Hence, by the Green's identity, it follows that

$$\int_{\Omega_0} u(x) g(x) \, dx = \int_{\Omega_0} f(x) v(x) \, dx.$$

Hence,

$$\int_{\Omega_0} \left[\int_{\Omega_0} G(x, y) f(y) \, dy \right] g(x) \, dx = \int_{\Omega_0} \left[\int_{\Omega_0} G(y, x) g(x) \, dx \right] f(y) \, dy. \quad (70)$$

By Fubini's theorem and the arbitrariness of f and g , it follows that

$$G(x, y) = G(y, x) \quad \text{for any } x, y \in \Omega_0. \quad \square$$

4.2. Upper bound for the function f . Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be a bounded domain satisfying (11)-(12) and let D_1, D_2 be two inclusions of Ω satisfying (14)-(16). Let σ_1, σ_2 be the anisotropic coefficients satisfying (17)-(21) and let q_1, q_2 be the coefficients of the zeroth order term satisfying (22)-(23). Before proving Proposition 3.4, let us introduce some useful formulas.

Let $u_j \in H^1(\Omega)$ with $j = 1, 2$ be a weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma_j \nabla u_j) + q_j u_j = 0 & \text{in } \Omega, \\ u_j|_{\partial\Omega} \in H_{00}^{\frac{1}{2}}(\Sigma). \end{cases} \quad (71)$$

Integrating by parts (71) leads to the following identity,

$$\int_{\Omega} (\sigma_2 - \sigma_1) \nabla u_1 \cdot \nabla u_2 + \int_{\Omega} (q_1 - q_2) u_1 u_2 = \langle \sigma_2 \nabla \bar{u}_2 \cdot \nu|_{\partial\Omega}, u_1 \rangle - \langle \sigma_1 \nabla u_1 \cdot \nu|_{\partial\Omega}, \bar{u}_2 \rangle. \quad (72)$$

Let v_j for $j = 1, 2$ with $v_j \in H^1(\Omega)$ be weak solution to $\operatorname{div}(\sigma_j \nabla v_j) + q_j v_j = 0$ in Ω .

$$\langle \sigma_j \nabla v_j \cdot \nu|_{\partial\Omega}, \bar{u}_j \rangle - \langle \sigma_j \nabla \bar{u}_j \cdot \nu|_{\partial\Omega}, v_j \rangle = 0 \quad \text{for } j = 1, 2. \quad (73)$$

Moreover, from (72) and (73), one can show that

$$\begin{aligned} & \int_{\Omega} (\sigma_2 - \sigma_1) \nabla u_1 \cdot \nabla u_2 + \int_{\Omega} (q_1 - q_2) u_1 u_2 = \\ & = \langle \sigma_2 \nabla \bar{u}_2 \cdot \nu |_{\partial\Omega}, (u_1 - v_2) \rangle - \langle \sigma_1 \nabla u_1 \cdot \nu |_{\partial\Omega} - \sigma_2 \nabla v_2 \cdot \nu |_{\partial\Omega}, \bar{u}_2 \rangle. \end{aligned} \quad (74)$$

Finally, (74) and the the Cauchy-Schwarz inequality allow us to bound $f(y, y)$ with $d(\mathcal{C}_1, \mathcal{C}_2)$ as

$$\begin{aligned} & \left| \int_{\Omega} (\sigma_2 - \sigma_1) \nabla u_1 \cdot \nabla u_2 + \int_{\Omega} (q_1 - q_2) u_1 u_2 \right| \leq \\ & \leq d(\bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2) \|(u_1, \sigma_1 \nabla u_1 \cdot \nu)\|_{\mathcal{H}} \|(\bar{u}_2, \sigma_2 \nabla \bar{u}_2 \cdot \nu)\|_{\mathcal{H}}. \end{aligned} \quad (75)$$

We introduce the asymptotic estimates for the gradient of the Green function G that will be used in the proof of Theorem 2.7.

Proposition 4.2. *Let Ω, D be, respectively, a bounded domain satisfying (11)-(12) and an inclusion satisfying (14)-(16). Then there exists a positive constant C_1 that depends on the a priori data only such that*

$$|\nabla_x G(x, y)| \leq C_1 |x - y|^{1-n}, \quad (76)$$

for any $x, y \in \mathbb{R}^3$.

Proof of Proposition 4.2. See [5, Proposition 3.4]. \square

Proof of Proposition 3.4. Fix a point $\bar{y} \in D_0$ such that $\text{dist}(\bar{y}, \partial\Omega) \geq \tilde{c}r_0$, for $0 < \tilde{c} < 1$ suitable constant. For any $\bar{w} \in \mathbb{R}^n \setminus \bar{\Omega}_D$, $f(\bar{y}, \cdot)$ is a weak solution to

$$\text{div}_w (a_b(\cdot)A(\cdot)\nabla_w f(\bar{y}, \cdot)) + q_b(\cdot)f(\bar{y}, \cdot) = 0 \quad \text{in } \mathbb{R}^n \setminus \bar{\Omega}_D. \quad (77)$$

First, choose $\bar{w} \in D_0$ such that $\text{dist}(\bar{w}, \partial\Omega) \geq \tilde{c}r_0$, for $0 < \tilde{c} < 1$. By (75),

$$|f(\bar{y}, \bar{w})| \leq d(\bar{\mathcal{C}}_1, \bar{\mathcal{C}}_2) \|(G_1(\cdot, \bar{y}), \sigma_1 \nabla G_1(\cdot, \bar{y}) \cdot \nu)\|_{\mathcal{H}} \|(\bar{G}_2(\cdot, \bar{w}), \sigma_2 \nabla \bar{G}_2(\cdot, \bar{w}) \cdot \nu)\|_{\mathcal{H}}.$$

Since by Proposition 4.2 and the definition of the norm on \mathcal{H} ,

$$\|(G_1(\cdot, \bar{y}), \sigma_1 \nabla G_1(\cdot, \bar{y}) \cdot \nu)\|_{\mathcal{H}} \leq \left(\|G_1(\cdot, \bar{y})\|_{H_{00}^{\frac{1}{2}}(\Sigma)}^2 + C \|\nabla G_1(\cdot, \bar{y}) \cdot \nu\|_{H^{-\frac{1}{2}}(\partial\Omega)|_{\Sigma}}^2 \right)^{\frac{1}{2}},$$

for C suitable constant depending on \bar{A} and $\bar{\gamma}$, it follows that

$$|f(\bar{y}, \bar{w})| \leq C_1 \epsilon. \quad (78)$$

where C_1 is a positive constant depending on the a priori data only.

Now let $\bar{w} \in \Omega^{r_0} \setminus \bar{\Omega}_D$, where $\Omega^{r_0} = \{x \in \mathbb{R}^3 : \text{dist}(x, \partial\Omega) < r_0\}$ and let $\bar{y} \in D_0$. By Proposition 4.2 and since $|x - \bar{y}| \geq r_0$,

$$\begin{aligned} |f(\bar{y}, \bar{w})| & \leq C \sum_{j=1}^2 \int_{D_j} |x - \bar{y}|^{1-n} |x - \bar{w}|^{1-n} dx \\ & \leq C \sum_{j=1}^2 \int_{D_j} |x - \bar{w}|^{1-n} dx. \end{aligned}$$

Choosing $\tilde{R} = \text{diam}(\Omega) + r_0 \leq Cr_0$, where C is a constant which depends only on L . Hence, $\Omega \subset B_{\tilde{R}}(\bar{w})$ and for $j = 1, 2$,

$$\int_{D_j} |x - \bar{w}|^{1-n} dx \leq \int_{B_{\tilde{R}}(\bar{w})} |x - \bar{w}|^{1-n} dx \leq C.$$

The next step consists in the determination of an estimate for $f(\bar{y}, w)$ when $w \in \mathcal{G}$. For $h > 0$, define

$$(\mathcal{G})^h = \{x \in \Omega_D^c : \text{dist}(x, \partial\Omega_D) \geq h\}.$$

For $w \in (\mathcal{G})^h$, by Proposition 4.2,

$$|S_1(\bar{y}, w)| \leq C \int_{D_1} |x - \bar{y}|^{1-n} |x - w|^{1-n} dx \leq Ch^{1-n}.$$

Similarly, $|S_2(\bar{y}, w)| \leq Ch^{1-n}$, so that

$$|f(\bar{y}, w)| \leq Ch^{1-n}. \quad (79)$$

We proceed with a quantitative estimate of propagation of smallness for the function f with respect to the second variable. Let O be the point of Lemma 3.3 and assume to be in a coordinate system in which O coincides with the origin and set $y_h = h\nu(O)$. The goal is to propagate (78) inside \mathcal{G} up to y_h .

In order to do it, fix $\bar{y}, w \in D_0$ such that $\text{dist}(\bar{y}, \partial\Omega) \geq r_0$ and $\text{dist}(w, \partial\Omega) \geq r_0$. By Lemma 3.3 we know that there exists a curve $\gamma \subset (\bar{\Omega}^{r_0} \cup D_0) \setminus \bar{\Omega}_D$ joining w to the point $Q = \bar{d}\nu(O)$, where $\nu(O) = -e_n$, such that $V(\gamma) \subset \mathbb{R}^3 \setminus \Omega_D$ with $R = \frac{\bar{d}}{\sqrt{1+L^2}}$ and $\theta_0 = \arcsin \frac{R}{\bar{d}}$.

Notice that, since $f(\bar{y}, \cdot)$ is a weak solution to (77), one can apply the three sphere inequality in the ball $B_{r_0}(\bar{x})$, where the point $\bar{x} \in D_0$ is such that $\text{dist}(\bar{x}, \partial\Omega) = \frac{3r_0}{4}$. Choose $r = \frac{r_0}{4}$, then, for radii $r, 3r, 4r$, the following estimate holds,

$$\|f(\bar{y}, \cdot)\|_{L^\infty(B_{3r}(\bar{x}))} \leq C \|f(\bar{y}, \cdot)\|_{L^\infty(B_r(\bar{x}))}^\tau \|f(\bar{y}, \cdot)\|_{L^\infty(B_{4r}(\bar{x}))}^{1-\tau},$$

where

$$\tau = \frac{\ln \frac{4\bar{\lambda}}{3}}{\ln \frac{4\bar{\lambda}}{3} + c \ln \frac{3}{\bar{\lambda}}},$$

$0 < \tau < 1$ and $C > 0$ depends on $\bar{\lambda}, L, r_0$.

Consider w, Q and the curve γ as above. We select a finite number of points on γ as follows. Set $\phi_1 = w$. Then

1. if $|\phi_{j-1} - Q| > r$, then set $\phi_j = \gamma(t_j)$ where $t_j = \max\{t : |\gamma(t) - \phi_{j-1}| = r\}$;
2. otherwise, set $s = j$, $\phi_s = Q$ and stop the process.

By iterating the three sphere inequality along the chain of balls centred at ϕ_j for $j = 1, \dots, s$, and assuming that $s \leq S$ where S depends on n only, one derives that for any r_1 with $0 < r_1 < r$,

$$\|f(\bar{y}, \cdot)\|_{L^\infty(B_{r_1}(Q))} \leq C \|f(\bar{y}, \cdot)\|_{L^\infty(B_{\frac{r_1}{2}}(w))}^\tau \|f(\bar{y}, \cdot)\|_{L^\infty(\mathcal{G})}^{1-\tau^s}. \quad (80)$$

By (78), (79) and (80),

$$\|f(\bar{y}, \cdot)\|_{L^\infty(B_{r_1}(Q))} \leq C \epsilon^{\tau^S} (h^{1-n})^{1-\tau^S}. \quad (81)$$

The goal is to propagate the smallness from Q to y_h . Consider the truncated cone $C(O, -e_n, d, \theta_0)$ where $d = \frac{\bar{d}^2 - R^2}{\bar{d}}$. Define

$$\lambda_1 = \min \left\{ \frac{d}{1 + \sin \theta_0}, \frac{d}{3 \sin \theta_0} \right\}, \quad \tilde{\theta}_0 = \arcsin \left(\frac{\sin \theta_0}{8} \right),$$

$$w_1 = O + \lambda_1 \nu(O), \quad \rho_1 = \lambda_1 \sin \theta_1, \quad a = \frac{1 - \sin \theta_1}{1 + \sin \theta_1},$$

so that $B_{\rho_1}(w_1) \subset C(O, \nu(O), d, \theta_1)$ and $B_{4\rho_1}(w_1) \subset C(O, \nu(O), d, \tilde{\theta}_0)$. Since $\rho_1 < \frac{r_0}{2}$, one can apply (81) in the cone $C(O, -e_n, d, \theta_1)$ over a chain of balls of shrinking

radii $\rho_k = a\rho_{k-1}$ centred at points $w_k = O + \lambda_k\nu(O)$ with $\lambda_k = a\lambda_{k-1}$. Denote by $d(k) = |w_k - O| - \rho_k$, then $d(k) = a^{k-1}d(1)$. We consider $h \leq d(1)$ and define a natural number $k(h)$ as the smallest positive integer such that $d(k(h)) \leq h$. Hence

$$\frac{\left| \ln \frac{h}{d(1)} \right|}{|\ln a|} \leq k(h) - 1 \leq \frac{\left| \ln \frac{h}{d(1)} \right|}{|\ln a|} + 1.$$

By iterating the three-sphere inequality over the chain of balls $B_{\rho_1}(w_1), \dots, B_{\rho_{k(h)}}(w_{k(h)})$, one derives

$$\|f(\bar{y}, \cdot)\|_{L^\infty(B_{\rho_{k(h)}}(w_{k(h)}))} \leq c(h^{1-n})^{A''} \epsilon^{\beta\tau^{k(h)-1}}, \quad (82)$$

where $\beta = \tau^S$ and $A'' = 1 - \beta$.

Consider now $f(y, w)$ as a function of y . Notice that for any $w \in \mathbb{R}^n \setminus \Omega_D$, $f(\cdot, w)$ is a weak solution to

$$\operatorname{div}_y(a_b(\cdot)A(\cdot)\nabla_y f(\cdot, w)) + q_b(\cdot)f(\cdot, w) = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega_D.$$

For any $y, w \in \mathcal{G}^h$, by Proposition 4.2,

$$|S_1(y, w)| \leq c \int_{D_1} |x - y|^{1-n} |x - z|^{1-n} dx \leq ch^{2(1-n)}.$$

Similarly, $|S_2(y, w)| \leq ch^{2(1-n)}$, so that

$$|f(y, w)| \leq ch^{2(1-n)} \quad \text{for any } y, w \in (\mathcal{G})^h.$$

Now, for $y \in D_0$ such that $\operatorname{dist}(y, \partial\Omega) \geq \tilde{c}r_0$, for $w \in (\mathcal{G})^h$ and by (82),

$$|f(y, w)| \leq c(h^{1-n})^{A''} \epsilon^{\beta\tau^{k(h)-1}},$$

where A'', β are defined as above. Fix $w \in \mathcal{G}$ such that $\operatorname{dist}(w, \Omega_D) = h$, $\bar{y} \in D_0$ such that $\operatorname{dist}(\bar{y}, \partial\Omega) \geq \frac{3r_0}{2}$, then for $\bar{r} = \frac{r_0}{2}$, $3\bar{r}$, $4\bar{r}$ and $y_1 = w_1$ defined as above, by an iterated application of the three sphere inequality one derives

$$\begin{aligned} \|f(\cdot, w)\|_{L^\infty(B_{\bar{r}}(y_1))} &\leq c \|f(\cdot, w)\|_{L^\infty(B_{\bar{r}}(\bar{y}))}^{\tau^s} \|f(\cdot, w)\|_{L^\infty(\mathcal{G})}^{1-\tau^s} \\ &\leq c(h^{2-2n})^{A''} \epsilon^{\beta^2\tau^{k(h)-1}}, \end{aligned}$$

where $A' = 1 - \beta + A''\tau^s$, $\beta = \tau^S$. Once more we apply the three sphere inequality inside the cone of vertex O over a chain of balls with shrinking radii as above so that

$$\|f(\cdot, w)\|_{L^\infty(B_{\rho_{k(h)}}(y_{k(h)}))} \leq c(h^{A'})^{1-\tau^{k(h)-1}} (\epsilon^{\beta^2\tau^{k(h)-1}})^{\tau^{k(h)-1}}.$$

Now, if we choose $y = w = y_h$, one derives

$$|f(y_h, y_h)| \leq ch^{-A} (\epsilon^{\beta^2\tau^{k(h)-1}})^{\tau^{k(h)-1}},$$

where $A = -(2 - 2n)A'(1 - \tau^{k(h)-1}) > 0$. Since $k(h) \leq c|\ln h| = -c \ln h$, then

$$\tau^{k(h)} = e^{-c \ln h \ln \tau} = h^{-c \ln \tau} = h^F, \quad \text{where } F = c|\ln h|.$$

In conclusion,

$$|f(y, y)| \leq c_1 h^{-A} \epsilon^{\beta^2\tau^{2(k(h)-1)}} = c_1 h^{-A} \epsilon^{\beta^2\tau^{2(k(h)-1)} \ln \epsilon} = c_1 h^{-A} \epsilon^{Bh^F},$$

where $B = \beta^2$. \square

4.3. Lower bound for the function f . In view of the proof Proposition 3.5, we introduce the asymptotic estimates for the Green's functions which are solutions of (37) with respect to two auxiliary families of Green's functions.

First, let $P \in \partial D_1 \cap \partial \Omega_D$ be the point of Lemma 3.3. Up to a rigid transformation, we can assume that P coincides with the origin O and

$$D_1 \cap Q_{\frac{r_0}{3}} = \left\{ x \in Q_{\frac{r_0}{3}} : x_n \geq \varphi(x') \right\},$$

where $\varphi \in C^2(B'_{\frac{r_0}{3}})$. In the remaining part of this section we drop out the pedix of the inclusion to improve readability of the argument.

Following the lines of [9, Theorem 4.2], we introduce a change of coordinates which flattens the boundary near O . Let $\tau \in C^\infty(\mathbb{R})$ such that $0 \leq \tau(s) \leq 1$, $\tau(s) = 1$ for $s \in (-1, 1)$ and $\tau(s) = 0$ for $s \in \mathbb{R} \setminus (-2, 2)$ and $|\tau'(s)| \leq 2$ for any $s \in \mathbb{R}$. Set

$$r_1 = \frac{r_0}{3} \min \left\{ \frac{1}{2}(8L)^{-1}, \frac{1}{4} \right\}.$$

The following change of coordinates

$$\xi = \phi(x) = \begin{cases} \xi' = x' \\ \xi_n = x_n - \varphi(x')\tau\left(\frac{|x'|}{r_1}\right)\tau\left(\frac{x_n}{r_1}\right) \end{cases}$$

is a $C^{1,1}$ diffeomorphism of \mathbb{R}^n into itself and allows us to flatten locally the boundary of the inclusion. In what follows, we keep the notation with x , since the exponent appearing in the asymptotic estimates does not depend on the change of coordinates.

Set

$$\sigma_0(x) = (a^- + (a^+ - a^-)\chi^+)A, \quad q_0(x) = q_b(0) + (q_D(0) - q_b(0))\chi^+(x), \quad (83)$$

where

$$a^- = a_b(0), \quad a^+ = a_D(0), \quad A = A(0), \quad \chi^+ = \chi_{\mathbb{R}_+^n}.$$

For $y \in D_0$, let $G_0(\cdot, y)$ be the weak solution to

$$\begin{cases} \operatorname{div}(\sigma_0(x)\nabla G_0(x, y)) + q_0(x)G_0(x, y) = -\delta(x - y) & \text{for } x \in \Omega_0, \\ G_0(x, y) = 0 & \text{for } x \in \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_0(x)\nabla G_0(x, y) \cdot \nu(x) + iG_0(x, y) = 0 & \text{for } x \in \Sigma_0. \end{cases}$$

Let Γ denote the fundamental solution associated with the Laplacian operator defined on \mathbb{R}^n . Let H be the fundamental solution to

$$\operatorname{div}(\sigma_0(\cdot)\nabla H(\cdot, y)) = -\delta(\cdot - y) \quad \text{in } \mathbb{R}^n.$$

Recalling [22], H has the following expression

$$H(x, y) = |J| \begin{cases} \frac{1}{a^+}\Gamma(Jx, Jy) + \frac{a^+ - a^-}{a^+(a^+ + a^-)}\Gamma(Jx, Jy^*) & \text{if } x_n, y_n > 0, \\ \frac{2}{a^- + a^+}\Gamma(Jx, Jy) & \text{if } x_n \cdot y_n < 0, \\ \frac{1}{a^-}\Gamma(Jx, Jy) + \frac{a^- - a^+}{a^-(a^+ + a^-)}\Gamma(Jx, Jy^*) & \text{if } x_n, y_n < 0, \end{cases} \quad (84)$$

where $y^* = (y_1, \dots, y_{n-1}, -y_n)$, $J = \sqrt{A(0)^{-1}}$ and $|J| = \det(\sqrt{A(0)^{-1}})$.

We introduce the asymptotic estimates for the Green's functions G with respect to H .

Proposition 4.3. *Under the same assumptions as in Proposition 4.2, there exist positive constants C_2, C_3 and $\theta_1 \in (0, 1)$ that depend on the a priori data only such that*

$$|G(x, y) - H(x, y)| \leq C_2|x - y|^{3-n}, \quad (85)$$

$$|\nabla_x G(x, y) - \nabla_x H(x, y)| \leq C_3|x - y|^{1-n+\theta_1}, \quad (86)$$

for every $x \in D \cap B_r$ and $y = h\nu(O)$ where $r \in (0, \frac{r_0}{2} \min\{(8L)^{-1}, \frac{1}{4}\})$ and $h \in (0, \frac{r_0}{4} \min\{(8L)^{-1}, \frac{1}{4}\})$.

Proof of Proposition 4.3. Notice that for x, y as in the assumptions,

$$|G(x, y) - H(x, y)| \leq |G(x, y) - G_0(x, y)| + |G_0(x, y) - H(x, y)|.$$

Hence we can split the proof of Proposition 4.3 into two claims.

Claim 4.4. *There exist positive constants C_4, C_5 and $\theta_1 \in (0, 1)$ that depend on the a priori data only such that*

$$|G(x, y) - G_0(x, y)| \leq C_4|x - y|^{3-n}, \quad (87)$$

$$|\nabla_x G(x, y) - \nabla_x G_0(x, y)| \leq C_5|x - y|^{1-n+\theta_1}, \quad (88)$$

for every $x \in D \cap B_r$ and $y = h\nu(O)$ where $r \in (0, \frac{r_0}{2} \min\{(8L)^{-1}, 1\})$ and $h \in (0, \frac{r_0}{4} \min\{(8L)^{-1}, 1\})$.

Proof of Claim 4.4. We follow the lines of the proof of [5, Proposition 3.4]. For simplicity, we consider a generic inclusion D with jump coefficients σ, q . Fix $P \in \partial D$ so that under a suitable transformation of coordinates, $P = O$. Let G denote the Green function associated with the elliptic operator $\operatorname{div}(\sigma(\cdot)\nabla\cdot) + q(\cdot)$ so that for any $y \in \Omega_0$, $G(\cdot, y)$ is a distributional solution to the following boundary value problem

$$\begin{cases} \operatorname{div}(\sigma(x)\nabla G(x, y)) + q(x)G(x, y) = -\delta(x - y) & \text{for } x \in \Omega_0, \\ G(x, y) = 0 & \text{for } x \in \partial\Omega_0 \setminus \Sigma_0, \\ \sigma(x)\nabla G(x, y) \cdot \nu(x) + iG(x, y) = 0 & \text{for } x \in \Sigma_0. \end{cases} \quad (89)$$

For $O \in \partial D$, let σ_0, q_0 be as in (83). For $y \in \Omega_0$, let $G_0(\cdot, y)$ be the Green's function which is a distributional solution to the following auxiliary boundary value problem

$$\begin{cases} \operatorname{div}(\sigma_0(x)\nabla G_0(x, y)) + q_0(x)G_0(x, y) = -\delta(x - y) & \text{for } x \in \Omega_0, \\ G_0(x, y) = 0 & \text{for } x \in \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_0(x)\nabla G_0(x, y) \cdot \nu(x) + iG_0(x, y) = 0 & \text{for } x \in \Sigma_0. \end{cases} \quad (90)$$

Define

$$R(x, y) = G(x, y) - G_0(x, y). \quad (91)$$

Subtracting the first equation of (90) to (89), it follows that $R(x, y)$ is a weak solution in Ω_0 to the equation

$$\begin{aligned} \operatorname{div}(\sigma(\cdot)\nabla R(\cdot, y)) + q(\cdot)R(\cdot, y) &= -\operatorname{div}((\sigma(\cdot) - \sigma_0(\cdot))\nabla G_0(\cdot, y)) + \\ &+ (q_0(\cdot) - q(\cdot))G_0(\cdot, y), \quad \text{in } \Omega_0 \end{aligned} \quad (92)$$

with boundary conditions

$$\begin{cases} R(\cdot, y) = 0 & \text{in } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_0(\cdot)\nabla_x R(\cdot, y) \cdot \nu(\cdot) + iR(\cdot, y) = (\sigma_0(\cdot) - \sigma(\cdot))\nabla_x G_0(\cdot, y) \cdot \nu(\cdot) & \text{in } \Sigma_0. \end{cases}$$

Then the following representation formula holds

$$\begin{aligned}
-R(x, y) &= \int_{\Omega_0} (\sigma(z) - \sigma_0(z)) \nabla_z G(z, x) \cdot \nabla_z G_0(z, y) \, dz + \\
&\quad + \int_{\Omega_0} (q(z) - q_0(z)) G(z, x) G_0(z, y) \, dz + \\
&\quad + \int_{\Sigma_0} [\sigma_0(z) \nabla_z G_0(z, y) \cdot \nu G(z, x) - \sigma(z) \nabla_z G(z, x) \cdot \nu G_0(z, y)] \, dS(y).
\end{aligned} \tag{93}$$

The boundary integrals are bounded (for instance by the Schwarz inequality and the trace estimates). The second volume integral in (93) is less singular than the first volume integral, so that we find convenient to study the first volume integral. Let us split the domain of integration into the union of the subdomains $\Omega \cap Q_{\bar{r}_0}$ and $\Omega \setminus Q_{\bar{r}_0}$ for $\bar{r}_0 = \frac{r_0}{4} \min\{(8L)^{-1}, 1\}$. For $x \in \Omega \cap Q_{\bar{r}_0}$,

$$|\sigma(z) - \sigma_0(z)| \leq C|z|.$$

Then we can apply the same argument of [17, Proposition 4.1] and conclude that

$$|R(x, y)| \leq C_4 |x - y|^{3-n}, \tag{94}$$

where C_4 is a positive constant which depends only on the a priori data.

Regarding the estimate for the gradient of the residual R , recalling that the boundary of D is C^2 and hence $C^{1,1}$, for $x \in D \subset B_r$, we consider a cube $Q \subset B_{\frac{r}{4}}^+$ of side $\frac{cr_0}{4}$, $c \in (0, 1)$ so that $y \notin Q$ and $x \in \partial Q$. By [1, Lemma 3.2], the following interpolation formula holds

$$\|\nabla R(\cdot, y)\|_{L^\infty(Q)} \leq C \|R(\cdot, y)\|_{L^\infty(Q)}^{\frac{1}{2}} |\nabla_x R(\cdot, y)|_{1,Q}^{\frac{1}{2}}, \tag{95}$$

where C depends on L only. For $y = h\nu(O)$ for h as in the Proposition statement, from the piecewise Hölder continuity of $\nabla_x G(x, y)$ and $\nabla_x G_0(x, y)$ (see [28, Theorem 16.2]),

$$|\nabla_x G(\cdot, y)|_{1,Q}, |\nabla_x G_0(\cdot, y)|_{1,Q} \leq Ch^{-n}.$$

Therefore,

$$|\nabla_x R(\cdot, y)|_{1,Q} \leq Ch^{-n}, \tag{96}$$

and collecting (94), (95) and (96), it follows that

$$|\nabla_x R(x, y)| \leq C_5 |x - y|^{1-n+\theta_1},$$

where $\theta_1 = \frac{1}{2}$. □

Claim 4.5. *There exist positive constants C_6, C_7 that depend on the a priori data only such that*

$$|G_0(x, y) - H(x, y)| \leq C_6 |x - y|^{4-n}, \tag{97}$$

$$|\nabla_x G_0(x, y) - \nabla_x H(x, y)| \leq C_7 |x - y|^{2-n}, \tag{98}$$

for every $x \in D \cap B_r$ and $y = h\nu(O)$ where $r \in (0, \frac{r_0}{2} \min\{(8L)^{-1}, 1\})$ and $h \in (0, \frac{r_0}{4} \min\{(8L)^{-1}, 1\})$.

Proof of Claim 4.5. We follow the argument in [17, Proposition 4.2]. Let $y, z \in \Omega_0$ and let D be an inclusion in Ω satisfying (14)-(16). Recall that $G_0(\cdot, y)$ is the weak solution to

$$\begin{cases} \operatorname{div}(\sigma_0(x)\nabla G_0(x, y)) + q_0(x)G_0(x, y) = -\delta(x - y) & \text{for } x \in \Omega_0, \\ G_0(x, y) = 0 & \text{for } x \in \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_0(x)\nabla G_0(x, y) \cdot \nu(x) + iG_0(x, y) = 0 & \text{for } x \in \Sigma_0. \end{cases}$$

and $H(\cdot, y)$ is the fundamental solution to

$$\operatorname{div}_x(\sigma_0(\cdot)\nabla_x H(\cdot, y)) = -\delta(\cdot - y) \quad \text{in } \mathbb{R}^n. \quad (99)$$

The residual function

$$R(x, y) = G_0(x, y) - H(x, y),$$

is a weak solution to the equation

$$\begin{cases} \operatorname{div}_x(\sigma_0(\cdot)\nabla_x R(\cdot, y)) = -q_0(\cdot)G_0(\cdot, y) & \text{in } \Omega_0, \\ R(\cdot, y) = -H(\cdot, y) & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_0(\cdot)\nabla R(\cdot, y) \cdot \nu(\cdot) + iR(\cdot, y) = -\sigma_0(\cdot)\nabla H(\cdot, y) \cdot \nu(\cdot) - iH(\cdot, y) & \text{on } \Sigma_0. \end{cases}$$

Its representation formula is

$$\begin{aligned} -R(x, y) &= - \int_{\Omega_0} q_0(z)G_0(z, x)H(z, y) \, dx + \\ &\quad + \int_{\partial\Omega_0} \sigma_0(z) [\nabla_z H(z, x) \cdot \nu H(z, y) - \nabla_z G_0(z, x) \cdot \nu H(z, y)] \, dS(z) + \\ &\quad + \int_{\partial\Omega_0} \sigma_0(z) [\nabla_z H(z, y) \cdot \nu G_0(z, x) - \nabla_z H(z, y) \cdot \nu H(z, x)] \, dS(z). \end{aligned} \quad (100)$$

The surface integral can be easily bounded from above using Cauchy-Schwarz inequality by a constant that depends on the a priori data only. Regarding the volume integral, by (38) it follows that

$$\begin{aligned} \left| \int_{\Omega} q_0(z)G_0(z, x)H(z, y) \, dx \right| &\leq \|q_0\|_{L^\infty(\Omega)} \int_{\Omega} |G_0(z, x)||H(z, y)| \, dz \\ &\leq C \int_{\Omega} |z - x|^{2-n}|z - y|^{2-n} \, dz. \end{aligned}$$

Set $\tilde{r} = |x - y|$ and let $N \in \mathbb{N}$ be such that $B_{\frac{\tilde{r}}{N}}(x) \cap B_{\frac{\tilde{r}}{N}}(y) = \emptyset$. Let $\mathcal{O} = \Omega \setminus (B_{\frac{\tilde{r}}{N}}(x) \cup B_{\frac{\tilde{r}}{N}}(y))$ and split the integral over the domain Ω as the sum of three integrals over the subdomains $B_{\frac{\tilde{r}}{N}}(x)$, $B_{\frac{\tilde{r}}{N}}(y)$ and \mathcal{O} . Our goal is to estimate $\int_{B_{\frac{\tilde{r}}{N}}(y)} |z - x|^{2-n}|z - y|^{2-n} \, dz$.

For $z \in B_{\frac{\tilde{r}}{N}}(y)$, by the triangular inequality it follows that $|x - z| \geq |x - y| - |y - z| \geq \frac{|x - y|}{\tilde{c}}$ for a suitable constant \tilde{c} , then

$$\int_{B_{\frac{\tilde{r}}{N}}(y)} |z - x|^{2-n}|z - y|^{2-n} \, dz \leq c|x - y|^{2-n} \int_{B_{\frac{\tilde{r}}{N}}(y)} |z - y|^{2-n} \, dz \leq c|x - y|^{4-n}.$$

Similarly,

$$\int_{B_{\frac{\tilde{r}}{N}}(x)} |z - x|^{2-n}|z - y|^{2-n} \, dz \leq c|x - y|^{4-n}.$$

Then, for $z \in \mathcal{O}$, since $|x - z| \geq \frac{|z-y|}{N}$, it follows that

$$\int_{\mathcal{O}} |z-x|^{2-n} |z-y|^{2-n} dz \leq c \int_{\mathcal{O}} |z-y|^{4-2n} dz \leq c \int_{\Omega \setminus B_{\frac{\bar{r}}{N}}(y)} |z-y|^{4-2n} dz \leq c|x-y|^{4-n},$$

where the constants c appearing in the inequalities depend on the a priori data only. In conclusion, we have proved that

$$|R(x, y)| \leq C_6|x-y|^{4-n}, \quad (101)$$

where C_6 depends on the a priori data only.

The next quantity that we wish to estimate is the gradient of R . By a similar argument as in Proposition 4.3, we pick a cube Q such that $Q \subset B_{\frac{r}{4}}^+$ of side $\frac{cr}{4}$ with $c \in (0, 1)$ such that $x \in \partial Q$. By [1, Lemma 3.2], the following interpolation formula holds

$$\|\nabla R(\cdot, y)\|_{L^\infty(Q)} \leq C\|R(\cdot, y)\|_{L^\infty(Q)}^{\frac{1}{2}}|\nabla_x R(\cdot, y)|_{1,Q}^{\frac{1}{2}}, \quad (102)$$

where C depends on L only. Since G_0 and H are Hölder continuous, the following estimates hold

$$|\nabla_x G_0(\cdot, y)|_{1,Q} \leq c|x-y|^{-n} \quad \text{and} \quad |\nabla_x H(\cdot, y)|_{1,Q} \leq c|x-y|^{-n},$$

where c depends on L only. By (95) and (101),

$$\|\nabla_x R(\cdot, y)\|_{L^\infty(Q)} \leq C_7|x-y|^{2-n},$$

where C_7 depends on the a priori data only. \square

Collecting the results obtained by the two claims, the asymptotic estimates for the Green function follow. \square

Proof of Proposition 3.5. The proof follows the lines of [5, Proposition 3.5] and [6, Theorem 6.5]. Let $O \in \partial D_1$ be the point of Lemma 3.3, let $y = h\nu(O)$ where $\nu(O)$ is the outer unit normal of D_1 at O . Recall the definition of S_1 as

$$\begin{aligned} S_1(y, y) &= \int_{D_1} (a_{D_1}(x) - a_0(x))A(x)\nabla_x G_1(x, y) \cdot \nabla_x G_2(x, y) dx - \\ &\quad - \int_{D_1} (q_{D_1}(x) - q_0(x))G_1(x, y) G_2(x, y) dx. \end{aligned} \quad (103)$$

Set $\bar{r}_2 = \min \left\{ \text{dist}(O, D_2), \frac{r_0}{12\sqrt{1+L^2}} \cdot \min\{1, L\} \right\}$ and fix $r \in (0, \bar{r}_2)$.

Then (103) can be rewritten as follows:

$$\begin{aligned} S_1(y, y) &= \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x H_1(x, y) \cdot \nabla_x H_2(x, y) dx + \\ &\quad + \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x H_1(x, y) \cdot \nabla_x (G_2(x, y) - H_2(x, y)) dx + \\ &\quad + \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x (G_1(x, y) \\ &\quad - H_1(x, y)) \cdot \nabla_x (G_2(x, y) - H_2(x, y)) dx + \end{aligned}$$

$$\begin{aligned}
& + \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x)) A(x) \nabla_x (G_1(x, y) - H_1(x, y)) \cdot \nabla_x H_2(x, y) \, dx + \\
& + \int_{D_1 \setminus B_r(O)} (a_{D_1}(x) - a_b(x)) A(x) \nabla_x G_1(x, y) \cdot \nabla_x G_2(x, y) \, dx - \\
& - \int_{D_1 \cap B_r(O)} (q_{D_1}(x) - q_b(x)) H_1(x, y) H_2(x, y) \, dx - \\
& - \int_{D_1 \cap B_r(O)} (q_{D_1}(x) - q_b(x)) H_1(x, y) (G_2(x, y) - H_2(x, y)) \, dx - \\
& - \int_{D_1 \cap B_r(O)} (q_{D_1}(x) - q_b(x)) (G_1(x, y) - H_1(x, y)) (G_2(x, y) - H_2(x, y)) \, dx - \\
& - \int_{D_1 \cap B_r(O)} (q_{D_1}(x) - q_b(x)) (G_1(x, y) - H_1(x, y)) H_2(x, y) \, dx - \\
& - \int_{D_1 \setminus B_r(O)} (q_{D_1}(x) - q_b(x)) G_1(x, y) G_2(x, y) \, dx.
\end{aligned} \tag{104}$$

Set

$$\begin{aligned}
I_1 &= \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x)) A(x) \nabla_x H_1(x, y) \cdot \nabla_x H_2(x, y) \, dx, \\
R_1 &= \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x)) A(x) \nabla_x H_1(x, y) \cdot \nabla_x (G_2(x, y) - H_2(x, y)) \, dx + \\
& + \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x)) A(x) \nabla_x (G_1(x, y) \\
& - H_1(x, y)) \cdot \nabla_x (G_2(x, y) - H_2(x, y)) \, dx, \\
R_2 &= \int_{D_1 \cap B_r(O)} (a_{D_1}(x) - a_b(x)) A(x) \nabla_x (G_1(x, y) - H_1(x, y)) \cdot \nabla_x H_2(x, y) \, dx, \\
R_3 &= \int_{D_1 \setminus B_r(O)} (a_{D_1}(x) - a_b(x)) A(x) \nabla_x G_1(x, y) \cdot \nabla_x G_2(x, y) \, dx.
\end{aligned} \tag{105}$$

Hence,

$$|S_1(y, y)| \geq |I_1| - |R_1| - |R_2| - |R_3|.$$

For the term I_1 , one can simply notice that

$$H_1(x, y) = \tilde{c} \Gamma(Jx, Jy), \quad \text{and} \quad H_2(x, y) = \tilde{c} \Gamma(Jx, Jy),$$

where $J = \sqrt{A(O)^{-1}}$ and \tilde{c} is a constant that depends only on a^+, a^- . Hence, by the uniform ellipticity condition and the lower bound (19),

$$|I_1| \geq c \int_{D_1 \cap B_r(O)} |x - y|^{2-2n} \, dx \geq cr^{2-n} \geq ch^{2-n}.$$

Regarding the term R_2 , by Proposition 4.3 we know that

$$|\nabla_x G_1(x, y) - \nabla_x H_1(x, y)| \leq C|x - y|^{1-n+\theta_2},$$

so that

$$|R_2| \leq \tilde{c} \int_{D_1 \cap B_r(O)} |x - y|^{2-2n+\theta_2} \, dx \leq ch^{2-n+\theta_2}.$$

The term R_3 can be bounded in terms of a constant depending on the a priori data only, since $x \neq y$.

It remains to estimate the term R_1 . One of the issues is that, by our choice of r , there are no asymptotic estimates for the term $\nabla_x(G_2(x, y) - H_2(x, y))$, but we can solve this problem by applying the following trick. Recalling Lemma 4.1, one has that G_2 has the form

$$G_2(x, y) = \tilde{G}_2(x, y) + \sum_{j=1}^{J+1} R_j(x, y),$$

where \tilde{G}_2 is a weak solution to

$$\begin{cases} \operatorname{div}(\sigma_2(\cdot)\nabla\tilde{G}_2(\cdot, y)) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ \tilde{G}_2(\cdot, y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_2(\cdot)\nabla\tilde{G}_2(\cdot, y) \cdot \nu(\cdot) + i\tilde{G}_2(\cdot, y) = 0 & \text{on } \Sigma_0. \end{cases} \quad (106)$$

Hence,

$$\begin{aligned} |\nabla_x(G_2(x, y) - H_2(x, y))| &\leq |\nabla_x(\tilde{G}_2(x, y) - H_2(x, y))| \\ &\quad + \sum_{j=1}^{J+1} |\nabla_x R_j(x, y)|. \end{aligned}$$

Since

$$|\nabla_x R_j(x, y)| \leq c|x - y|^{2j+1-n},$$

for any $j = 1, \dots, J - 1$, one can infer that

$$\sum_{j=1}^{J+1} |\nabla_x R_j(x, y)| \leq \sum_{j=1}^{J+1} (d_\mu - h)^{2j+1-n} \leq c(d_\mu - h)^{2-n},$$

where $d_\mu > 0$. Regarding the other term, let us first consider a change of variable Φ as in [9, Theorem 4.2] that allows us to flatten the boundary of Ω_D near the point O . Consider $\tilde{G}_{2,0}(\cdot, y)$ as the Green function which is weak solution to

$$\begin{cases} \operatorname{div}(\sigma_{2,0}(\cdot)\nabla\tilde{G}_{2,0}(\cdot, y)) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ \tilde{G}_{2,0}(\cdot, y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_{2,0}(\cdot)\nabla\tilde{G}_{2,0}(\cdot, y) \cdot \nu + i\tilde{G}_{2,0}(\cdot, y) = 0 & \text{on } \Sigma_0, \end{cases} \quad (107)$$

where

$$\sigma_{2,0}(x) = (a_b(0) + (a_{D_2}(0) - a_b(0)\chi_+(x))A(0).$$

Hence,

$$\begin{aligned} |\nabla_x(\tilde{G}_2(x, y) - H_2(x, y))| &\leq |\nabla_x(\tilde{G}_2(x, y) - \tilde{G}_{2,0}(x, y))| + \\ &\quad + |\nabla_x(\tilde{G}_{2,0}(x, y) - H_2(x, y))|. \end{aligned} \quad (108)$$

Regarding the second term on the right-hand side of (108), first notice that $(\tilde{G}_{2,0} - H_2)(\cdot, y)$ is a weak solution to

$$\begin{cases} \operatorname{div}_x(\sigma_{2,0}(\cdot)\nabla(\tilde{G}_{2,0}(\cdot, y) - H_2(\cdot, y))) = 0 & \text{in } B_r(O), \\ \left| (\tilde{G}_{2,0}(\cdot, y) - H_2(\cdot, y)) \right|_{\partial B_r(O)} \leq cr^{2-n}, \end{cases} \quad (109)$$

so that by the Maximum Principle one has

$$|\tilde{G}_{2,0}(x, y) - H_2(x, y)| \leq cr^{2-n}.$$

Hence, by interior gradient estimates (see for instance [23]), it follows that

$$|\nabla_x(\tilde{G}_{2,0}(x, y) - H_2(x, y))| \leq cr^{1-n}. \quad (110)$$

For the first term on the right-hand side of (108), define

$$\tilde{R}_2(x, y) = \tilde{G}_2(x, y) - \tilde{G}_{2,0}(x, y).$$

One can notice that $\tilde{R}_2(\cdot, y)$ is a weak solution to

$$\begin{cases} \operatorname{div}(\sigma_2(\cdot)\nabla\tilde{R}_2(\cdot, y)) = -\operatorname{div}((\sigma_2(\cdot) - \sigma_{2,0}(\cdot))\nabla\tilde{G}_{2,0}(\cdot, y)) & \text{in } \Omega_0, \\ \tilde{R}_2(\cdot, y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_2(\cdot)\nabla\tilde{R}_2(\cdot, y) \cdot \nu + i\tilde{R}_2(\cdot, y) = -(\sigma_2(\cdot) - \sigma_{2,0}(\cdot))\nabla\tilde{G}_{2,0}(\cdot, y) \cdot \nu & \text{on } \Sigma_0. \end{cases}$$

By the representation formula, the remainder has the form

$$\begin{aligned} -\tilde{R}_2(x, y) &= \int_{\Omega_0} (\sigma_2(z) - \sigma_{2,0}(z))\nabla_z\tilde{G}_2(z, x) \cdot \nabla_z\tilde{G}_{2,0}(z, y) \, dz + \\ &\quad + \int_{\partial\Omega_0} \sigma_{2,0}(z)\nabla_z\tilde{G}_{2,0}(z, y) \cdot \nu \left[\tilde{G}_2(z, x) - \tilde{G}_{2,0}(z, x) \right] \, dS(z) + \\ &\quad + \int_{\partial\Omega_0} \sigma_2(z)\nabla_z \left[\tilde{G}_2(z, x) - \tilde{G}_{2,0}(z, x) \right] \cdot \nu \tilde{G}_{2,0}(z, y) \, dS(z). \end{aligned} \quad (111)$$

The integral over $\partial\Omega_0$ are bounded from above by a positive constant that depends on the a priori data only. In order to estimate the volume integral, first notice that

$$|\sigma_2(z) - \sigma_{2,0}(z)| \leq C|z|,$$

where C is a positive constant depending only on a priori data.

Hence, by Proposition 4.2,

$$\begin{aligned} &\left| \int_{\Omega_0} (\sigma_2(z) - \sigma_{2,0}(z))\nabla_z\tilde{G}_2(z, x) \cdot \nabla_z\tilde{G}_{2,0}(z, y) \, dz \right| \leq \\ &\leq c \int_{\Omega_0} |z| |z - x|^{1-n} |z - y|^{1-n} \, dz, \end{aligned} \quad (112)$$

where c is a positive constant depending on the a priori data only. Set $\tilde{h} = |x - y|$ and define

$$I_1 = \int_{B_{4\tilde{h}}} |z| |z - x|^{1-n} |z - y|^{1-n} \, dz, \quad (113)$$

$$I_2 = \int_{\mathbb{R}^n \setminus B_{4\tilde{h}}} |z| |z - x|^{1-n} |z - y|^{1-n} \, dz, \quad (114)$$

so that

$$|\tilde{R}_2(x, y)| \leq c(I_1 + I_2). \quad (115)$$

First, let us estimate I_1 . Set $z = \tilde{h}w$, $t = \frac{x}{\tilde{h}}$ and $s = \frac{y}{\tilde{h}}$, then

$$\begin{aligned} I_1 &= \int_{B_4} \tilde{h}|w| |\tilde{h}(w - t)|^{1-n} |\tilde{h}(w - s)|^{1-n} \tilde{h} \, dw \\ &= 4\tilde{h}^{3-n} \int_{B_4} |w - t|^{1-n} |w - s|^{1-n} \, dw \\ &\leq c\tilde{h}^{3-n}, \end{aligned}$$

since $\int_{B_4} |w - t|^{1-n} |w - s|^{1-n} \, dw \leq c$ (see [30, Chapter 2, section 11]). Hence,

$$I_1 \leq c(h - \operatorname{dist}(O, D_2))^{3-n}. \quad (116)$$

Regarding the integral I_2 , notice that since $y = h\nu(O) = -he_n$ in a suitable coordinate system, we might choose h so that

$$|y| = -h \leq |x - y| = \tilde{h},$$

and

$$|x| \leq |x - y| + |y| \leq 2\tilde{h}.$$

For any $z \in \mathbb{R}^n \setminus B_{4\tilde{h}}$, since $|z| > 4\tilde{h}$, it follows that

$$\frac{3}{4}|z| \leq |z - y| \quad \text{and} \quad \frac{1}{2}|z| \leq |z - x|.$$

Therefore,

$$I_2 \leq \left(\frac{8}{3}\right)^{1-n} \int_{\mathbb{R}^n \setminus B_{4\tilde{h}}} |z|^{3-2n} \, dz \leq c\tilde{h}^{3-n} \leq c(h - \text{dist}(O, D_2))^{3-n}. \quad (117)$$

By (116) and (117), we can conclude that

$$|\tilde{R}_2(x, y)| \leq c|x - y|^{3-n}. \quad (118)$$

At this point, in order to determine an upper bound for $\nabla_x \tilde{R}_2$. Consider a cube $Q \subset D_1 \cap B_r(O)$. Since $\tilde{G}_2(\cdot, y)$ and $\tilde{G}_{2,0}(\cdot, y)$ are Hölder continuous, it follows that

$$|\nabla \tilde{R}_2(x, y)|_{\alpha, Q} \leq c|x - y|^{-n}.$$

By the known inequality,

$$\|\nabla \tilde{R}_2(\cdot, y)\|_{L^\infty(Q)} \leq \|\tilde{R}_2(\cdot, y)\|_{L^\infty(Q)}^{\frac{1}{2}} |\nabla \tilde{R}_2(\cdot, y)|_{1, Q}^{\frac{1}{2}},$$

by (118) it follows that

$$|\nabla \tilde{R}_2(x, y)| \leq c|x - y|^{1-n+\theta_3} \quad \text{where} \quad \theta_3 = \frac{1}{2}. \quad (119)$$

Collecting (108), (110) and (119) together, we obtain

$$|\nabla_x(\tilde{G}_2(x, y) - H_2(x, y))| \leq ch^{1-n+\theta_3}. \quad (120)$$

In conclusion, the lower bound of S_1 is given by

$$|S_1(y, y)| \geq ch^{2-n}.$$

Regarding the estimate for S_2 , from Proposition 4.2 it follows that

$$|S_2(y, y)| \leq C \int_{D_2} |x - y|^{1-n} |x - y|^{1-n} \, dx \leq C(\text{dist}(O, D_2) - h)^{2(1-n)}.$$

In conclusion,

$$\begin{aligned} |f(y, y)| &= |S_1(y, y) - S_2(y, y)| \geq |S_1(y, y)| - |S_2(y, y)| \\ &\geq c_2 h^{2-n} - c_3 (\text{dist}(O, D_2) - h)^{2(1-n)}, \end{aligned} \quad (121)$$

for suitable $c_2, c_3 > 0$ constants depending on the a priori data only. \square

5. The misfit functional. In this section we introduce a stability estimate in terms of the misfit functional that is defined in (123).

Let Ω , D_1 , D_2 be, respectively, a bounded domain satisfying (11)-(12) and two inclusions satisfying (14)-(16). Let $\sigma_1, \sigma_2, q_1, q_2$ be the jump coefficients that correspond to the two inclusions. Let G_i be the Green functions associated to the operator $\operatorname{div}(\sigma_i(\cdot)\nabla\cdot) + q_i$ for $i = 1, 2$ so that for $y \in D_0$, $G_i(\cdot, y)$ is a distributional solution to the boundary value problem (37). Pick $D_y, D_z \subset\subset D_0$ suitable Lipschitz domains whose intersection is empty. For $(y, z) \in D_0 \times D_0$, define

$$S_{\mathcal{U}_0}(y, z) = \int_{\Sigma} \left[\sigma_1(x) \nabla G_1(x, y) \cdot \nu(x) G_2(x, z) - \sigma_2(x) \nabla G_2(x, z) \cdot \nu(x) G_1(x, y) \right] dS(x), \quad (122)$$

where Σ is the open portion of the boundary of Ω where the measurements are performed.

The *misfit functional* is defined as

$$\mathcal{J}(D_1, D_2) = \int_{D_y \times D_z} |S_{\mathcal{U}_0}(y, z)|^2 dy dz. \quad (123)$$

where $\mathcal{J} : L^\infty(\Omega_0) \times L^\infty(\Omega_0) \rightarrow \mathbb{R}$ encodes the error that occur when one approximates the boundary data induced by σ_1 and q_1 by the one induced by σ_2 and q_2 .

5.1. The stability estimate. Let $\omega : [0, +\infty) \rightarrow [0, +\infty)$ be a non-decreasing function such that for any $t \in (0, 1)$, $\omega(t) \leq C \cdot |\ln t|^{-\eta}$, where $\eta \in (0, 1)$ is a suitable constant.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying (11)-(12) and let D_1, D_2 be two inclusions of C^2 class contained in Ω satisfying (14)-(16). Let σ_1 and σ_2 be the anisotropic conductivities satisfying (17)-(21) and let q_1 and q_2 be the coefficients of the zero order term satisfying (22)-(23). Let Σ be a non-empty open portion of $\partial\Omega$. For any $\epsilon \in (0, 1)$, if $\mathcal{J}(D_1, D_2) < \epsilon$, then*

$$d_H(\partial D_1, \partial D_2) \leq \omega(\epsilon), \quad (124)$$

where $C > 0$ is a constant that depends on the a priori data only.

The proof of Theorem 5.1 follows the lines of the proof of Theorem 2.7, but instead of Proposition 3.5 we need a modified version, Proposition 5.2. Before stating it, notice that by the Green's identity and (39), (122) can be rewritten as

$$S_{\mathcal{U}_0}(y, z) = \int_{\Omega} (\sigma_2(x) - \sigma_1(x)) \nabla G_1(x, y) \cdot \nabla G_2(x, z) + \int_{\Omega} (q_1(x) - q_2(x)) G_1(x, y) G_2(x, z) dx. \quad (125)$$

Hence, recalling (40), (41) and (42), we have that

$$S_{\mathcal{U}_0}(y, z) = f(y, z).$$

Proposition 5.2. *Under the same assumptions of Theorem 5.1, for $\epsilon \in (0, 1)$, if $\mathcal{J}(D_1, D_2) < \epsilon$, then*

$$|f(y, y)| \leq C_1 \frac{\epsilon^{Bh^F}}{h^A},$$

where $A \in (0, 1)$, C_1, B, F are positive constants that depend on the a priori data only, $y = h\nu(O)$ where

$$0 < h \leq \bar{d} \left(1 - \frac{\sin \theta_0}{4} \right) \text{ for } \theta_0 = \arctan \frac{1}{L}.$$

Proof of Proposition 5.2. As in the previous proofs, we drop the indices and consider a generic inclusion D . Fix $\bar{y} \in D_0$, then $f(\bar{y}, \cdot)$ is a weak solution to

$$\operatorname{div}_z(\sigma(\cdot)\nabla_z f(\bar{y}, \cdot)) + q(\cdot)f(\bar{y}, \cdot) = 0, \quad \text{in } \Omega_D^c.$$

Since in this case $f(\bar{y}, z) = S_{U_0}(\bar{y}, z)$, by [22, Equation (3.23)] we have

$$\max_{z \in (D_0)_r} S_{U_0}(\bar{y}, z) \leq c(\mathcal{J}(D_1, D_2))^{\frac{1}{2}},$$

where $(D_0)_r = \{x \in D_0 : \operatorname{dist}(x, \partial D_0) > r\}$ for $r > 0$ and the constant c depends on the a priori data only. Then

$$f(\bar{y}, z) \leq c\varepsilon. \tag{126}$$

The remaining part of the proof follows the line of the proof of Proposition 3.4. \square

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