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Lifting Theorems for Continuous Order-Preserving Functions and Continuous Multi-Utility

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Abstract: We present some lifting theorems for continuous order-preserving functions on locally and σ-compact Hausdorff preordered topological spaces. In particular, we show that a preorder on a locally and σ-compact Hausdorff topological space has a continuous multi-utility representation if, and only if, for every compact subspace, every continuous order-preserving function can be lifted to the entire space. Such a characterization is also presented by introducing a lifting property of $\precsim$-C-compatible continuous order-preserving functions on closed subspaces. The assumption of paracompactness is also used in connection to lifting conditions.

Keywords: order-preserving function; locally compact space; continuous multi-utility representation

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1. Introduction

The problem concerning the continuous representability of not necessarily total preorders is very interesting not only from a purely mathematical viewpoint, but also for its possible applications to economics and social sciences. General achievements concerning the existence of continuous utility representations were very recently presented by Bosi [1] in the case of nontotal preorders, and by Bosi and Zuanon [2] in the case of total preorders.

Several authors have presented contributions to this topic by following a direct approach, which is mainly based on the existence of a *separable system* or *decreasing scale* in a topological preordered space. Such a notion generalizes the concept of a *scale* in a topological space (see for example, Burgess and Fitzpatrick [3] and Gillman and Jerison [4]). In particular, Herden [5,6] proved very general results in this direction, but also other authors contributed to this field (see, for example, Herden and Pallack [7], Levin [8,9], Mehta [10,11] and Minguzzi [12]).

On the other hand, another possible approach to the existence of continuous representations of preorders by means of one real-valued function is based on *lifting theorems*, which concern the possibility of lifting a continuous (strictly) isotone function from a generic (typically closed or compact) subspace of the topological preordered space to the entire space.

Nachbin [13] generalized to *normally preordered topological spaces* the well-known *Tietze–Urysohn extension theorem* in normal spaces (see, for example, Engelking [14]), according to which it is always possible to lift a continuous real-valued function from a closed subspace of a normal space over the entire space.

Herden [15] was concerned with the possibility of lifting continuous order-preserving functions from (closed) subsets of preordered topological spaces. Further, Herden [16] generalized to arbitrary topological preordered spaces the aforementioned extension result

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proved by Nachbin and, as a consequence of his main result, characterized the possibility of lifting every bounded increasing continuous real-valued function from a closed subset of a preordered topological space to the whole space.

Mehta [17] studied a variant of Nachbin's lifting theorem. Recently, some results concerning the extensions of continuous increasing or order-preserving functions were presented by Evren and Hüsseinov [18].

In this paper, we present a lifting theorem for a *closed preorder* (i.e., a preorder which is closed with respect to the product topology), which guarantees the possibility of lifting a continuous real-valued order-preserving function defined on compact or closed subsets of a locally compact topological preordered space. The interest for closed preorders primarily arises in connection to the fact that the condition of being closed is necessary for the existence of a *continuous multi-utility representation*.

We recall that a preorder $\precsim$ is defined to admit a *continuous multi-utility representation* on a topological space (X, t) if there is a collection $\mathcal{F}$ of continuous increasing functions such that, for all points $x, y \in X$, we have that $x \precsim y$ if, and only if, $f(x) \leq f(y)$ for every $f \in \mathcal{F}$. It is worth noticing that continuous multi-utility representations, which were first introduced and investigated by Evren and Ok [19] (see also Bosi and Herden [20]), fully characterize the given closed preorder, while order-preserving functions only provide very particular continuous extensions by means of continuously representable total preorders (see, for example, Bosi and Herden [21]).

We concentrate our attention on locally and σ-compact Hausdorff spaces. We prove that a preorder on a locally and σ-compact Hausdorff space is closed (or equivalently, representable by a continuous multi-utility) if, and only if, for every compact subspace, every continuous order-preserving function can be lifted to the entire space. We further inaugurate the notion of a $\precsim$-*C-compatible* real-valued function on a topological preordered space $(X, \precsim, t)$, and we prove that a preorder on a locally and σ-compact Hausdorff space is closed if, and only if, for every closed subspace, every bounded, continuous and $\precsim$-C-compatible order-preserving function can be lifted to the entire space. Finally, we show that the assumption of σ-compactness cannot be avoided in such a characterization since the aforementioned lifting property from compact subspaces is equivalent to σ-compactness when the topological space is locally compact and paracompact.

2. Basic Concepts

In the sequel, $\mathbb{N} = \{0, 1, 2, ..., , n, n+1, ...\}$ is the set of natural numbers, $\mathbb{R}$ the set of real numbers, and $[a, b]$ a non-degenerate (non-trivial) real interval. As usual, we denote by $[a, b[$, $]a, b]$ and $]a, b[$, respectively, the corresponding half-closed half-open, half-open half-closed and open real intervals, respectively. $[0, 1]$ is the real unit interval. For every set A, we abbreviate by $\mid A \mid$ the cardinality of A. $\Delta_A := \{(a, a) \mid a \in A\}$ is the diagonal of A.

Definition 1. *A* preorder $\precsim$ *on a set* X *is a binary relation on* X *satisfying reflexivity and transitivity. The pair* $(X, \precsim)$ *is referred to as a* preordered set.

The *strict part* $\prec$ of a preorder $\precsim$ on a set X is defined to be, for all $x, y \in X$,

$$x \prec y \Leftrightarrow (x \precsim y) \ and \ not(y \precsim x).$$

In the sequel, $(x, y) \in \prec$ will occasionally replace $x \prec y$.

Definition 2. *A real-valued function* f *on a preordered set* $(X, \precsim)$ *is defined to be*

(i) Increasing, *if, for all* $x, y \in X$,

$$x \precsim y \Rightarrow f(x) \leq f(y);$$

(ii) Order-preserving, *if f is increasing and, for all $x, y \in X$,*

$$x \prec y \Rightarrow f(x) < f(y).$$

An increasing (order-preserving) function is sometimes called an *isotone* (respectively, *strictly isotone*) function.

If $(X, \precsim, t)$ is a *preordered topological space*, C is a subset of X and h is a real-valued function on X, then $t_{|C}$, $\precsim_{|C}$ and $h_{|C}$ are the topology on C, which is induced by t, the restriction of $\precsim$ to C, and, respectively, the restriction of h to C.

$\overline{A}$ stands for the topological closure (with respect to t) of any subset A of X. In addition, t_{nat} will stand for the *natural topology* on $\mathbb{R}$.

Definition 3. *Let $\precsim$ be a preorder on a topological space (X, t). Then $\precsim$ is defined to be* closed *if $\precsim$ is a closed subset of $X \times X$ with the product topology $t \times t$.*

A closed preorder is referred to as a *continuous* preorder by some authors (see, for example, Evren and Ok [19]). For every closed preorder $\precsim$ on X and every point $x \in X$, we have that the sets

$$d(x) := \{y \in x \mid y \precsim x\}, \; i(x) := \{z \in x \mid x \precsim z\}$$

are both closed subsets of X. It is well known that a preorder $\precsim$ on X that has the property that, for every point $x \in X$ both sets $d(x)$ and $i(x)$ are closed, is not necessarily closed. However, if a preorder $\precsim$ is *total*, then the closedness of $\precsim$ is equivalent to the requirement according to which both sets $d(x)$ and $i(x)$ are closed. This latter property is sometimes referred to as *semiclosedness* (see, for example, Bosi and Herden [22]).

Definition 4. *Let $(X, \precsim)$ be a preordered set, and let $C \neq \emptyset$ be a subset of X. Then a real-valued function f on C is defined to be* $\precsim$-C-compatible *if, for every pair $(x, y) \in \prec$, the sets $\overline{f(C \cap d(x))}$ and $\overline{f(C \cap i(y))}$ are disjoint.*

Let $E^C_{\prec}$ be the family of all pairs $(x, y) \in \prec$ for which neither $C \cap d(x)$ nor $C \cap i(y)$ is empty. Let f be a real-valued function on X. For every pair $(x, y) \in E^C_{\prec}$, we set

$$s^C_x(f) := \sup f(C \cap d(x)), \; i^C_y(f) := \inf f(C \cap i(y)).$$

Then the following proposition, the simple proof of which may be omitted for the sake of brevity, somewhat characterizes real-valued $\precsim$-C-compatible functions.

Proposition 1. *Let f be a real-valued function on a preordered set $(X, \precsim)$, and let C be a subset of X. The following conditions, concerning a real-valued function f on C, are equivalent:*

(i) *f is $\precsim$-C-compatible;*

(ii) *$s^C_x(f) < i^C_y(f)$ for every pair $(x, y) \in E^C_{\prec}$;*

(iii) *For every pair $(x, y) \in E^C_{\prec}$, the following implication holds:*

$$f^{-1}(]-\infty, s^C_x(f)]) \cup f^{-1}([i^C_y(f), \infty[) = C \Rightarrow s^C_x(f) < i^C_y(f).$$

Definition 5. *Let $(P, \precsim)$ be a preordered set. Then a subset A of P is said to be* decreasing *if $x \in A$ and $y \precsim x$ imply that $y \in A$. Dually, the notion of an* increasing *subset B of P is expressed.*

Definition 6. *A preorder* $\precsim$ *on* (X,t) *is said to be* representable by continuous multi-utility *if, for some family* $\mathcal{F}$ *of continuous increasing real-valued functions* f *on* $(X,\precsim,t)$*, the following equivalence is valid for all* $x \in X$ *and all* $y \in Y$*:*

$$x \precsim y \Leftrightarrow \forall f \in \mathcal{F}\,(f(x) \leq f(y)).$$

Definition 7. *A preordered topological space* $(X,\precsim,t)$ *is defined to be* normally preordered *if, for every pair* (F_0,F_1) *of disjoint closed sets,* F_0 *being decreasing and* F_1 *being increasing, there exists a pair of disjoint open sets* (A_0,A_1)*, where* A_0 *is decreasing and contains* F_0*, and* A_1 *is increasing and contains* F_1*.*

Nachbin [13], Theorem 2 on page 36, proved the following generalization to continuous increasing functions of the *Tietze–Urysohn extension theorem* (see, for example, Engelking [14], Theorem 2.1.8).

Theorem 1 (Nachbin [13])**.** *Consider a normally preordered topological space* $(X,\precsim,t)$*, and let* f *be a bounded, continuous and increasing real-valued function defined on some closed subset* C *of* X*. The function* f *can be extended to* X *in such a way that the resulting extension is bounded, continuous and increasing on* $(X,\precsim,t)$ *if, and only if, for every pair of real numbers* $r < r'$*, the smallest closed decreasing subset* $A(r)$ *of* X *that contains the set* A_r *of all points* $x \in C$ *such that* $f(x) \leq r$*, and the smallest closed increasing subset* $B(r')$ *of* X *that contains the set* $B_{r'}$ *of all points* $y \in C$ *such that* $r' \leq f(y)$*, are disjoint.*

Bosi and Herden [22] used the following definition.

Definition 8. *A preordered topological space* $(X,\precsim,t)$ *is defined to be* strongly normally preordered *if, for every pair* (A,B) *of disjoint closed subsets of* X *with not*$(x \succsim y)$ *for every pair* $(x,y) \in A \times B$*, there exists a pair* (U,V) *of disjoint open subsets of* X*, where* U *is decreasing and contains* A*, and* V *is increasing and contains* V*.*

It is immediate to check that a strongly normally preordered topological space is normally preordered.

3. The Lifting Theorems

We are going to show the validity of a general lifting theorem concerning continuous order-preserving functions on compact and, respectively, closed subspaces of a topological space satisfying particular conditions of compactness. In order to prove our theorem, characterizing closed preorders in terms of a lifting property on locally and σ-compact (a topological space (X,t) is said to be *locally compact* if every point in X has an open neighborhood whose closure is compact, and (X,t) is said to be σ*-compact* if it is a union of countably many compact subsets) Hausdorff topological spaces, we need to use Theorem 1 presented by Evren and Ok [19] and to prove two lemmas, together with a resulting proposition.

Theorem 2 (Evren and Ok [19])**.** *Every closed preorder* $\precsim$ *on a locally and* σ*-compact Hausdorff space* (X,t) *is representable by a continuous multi-utility.*

Lemma 1. *A preordered locally and* σ*-compact Hausdorff space* $(X,\precsim,t)$ *is strongly normally preordered provided that the preorder* $\precsim$ *is closed.*

Proof. Consider two disjoint closed subsets A and B of X, with the property that, for every pair $(x,y) \in A \times B$, *not*$(x \succsim y)$. By Theorem 2, $\precsim$ is representable by a continuous multi-utility $\mathcal{F}$. Therefore, for every pair $(x,y) \in A \times B$, there is some continuous increasing function $f_{xy} \in \mathcal{F}$, $f_{xy} : (X \precsim, t) \to ([0,1], \leq_{|[0,1]}, t_{nat|[0,1]})$, with the property that $f_{xy}(x) = 0$ and $f_{xy}(y) = 1$ (see Evren and Ok [18], Remark 3). It follows that $A \times B \subset \bigcup_{(x,y) \in A \times B} f_{xy}^{-1}\left(\left[0, \frac{1}{2}\right[\right) \times f_{xy}^{-1}\left(\left]\frac{1}{2}, 1\right]\right)$. Since a σ-compact (Hausdorff) space is Lindelöf (a topological space (X, t) is said to be Lindelöf if every open cover of X has a countable subcover), and since, in addition, finite products of locally and σ-compact Hausdorff spaces are locally and σ-compact Hausdorff spaces, it follows that there is a countable collection $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ of pairs $(x_n, y_n) \in A \times B$ such that $A \times B \subset \bigcup_{(x_n, y_n)} f_{x_n y_n}^{-1}\left(\left[0, \frac{1}{2}\right[\right) \times f_{x_n y_n}^{-1}\left(\left]\frac{1}{2}, 1\right]\right)$. Let $F := \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} f_{x_n y_n}$. Then $U := F^{-1}\left(\left[0, \frac{1}{2}\right[\right)$ and $V := F^{-1}\left(\left]\frac{1}{2}, 1\right]\right)$ are two disjoint open subsets of X, with the additional property that U is decreasing and contains A, and V is increasing and contains B. □

Let us now show that a continuous increasing function on a closed subset of a strongly normally preordered topological space can be lifted to the entire space in order for it to remain continuous and increasing.

Lemma 2. *Let $(X \precsim, t)$ be a strongly normally preordered space. Then the following property holds:*

"If C is any closed subset of X, and $f : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$ is any continuous increasing function, then $F_{|C} = f$ for some continuous increasing function $F : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$".

Proof. By the above Theorem 1, it suffices to show that, for every pair $r < r'$ of real numbers, the smallest closed decreasing subset $A(r)$ of X that includes the set A_r of all points $x \in C$ such that $f(x) \leq r$, and the smallest closed increasing subset $B(r')$ of X that includes the set $B_{r'}$ of all points $y \in C$ such that $r' \leq f(y)$ are disjoint. Let, therefore, real numbers $r < r'$ be arbitrarily chosen. Since $r < r'$ and since f is continuous and increasing, A_r and $B_{r'}$ are disjoint closed subsets of X such that $not(x \succsim y)$ for every pair $(x, y) \in A_r \times B_{r'}$. Hence, the assumption that $(X \precsim, t)$ is a strongly normally preordered space implies that there exist disjoint open decreasing and increasing subsets U, respectively, V of X, such that $A_r \subset U$ and $B_{r'} \subset V$. It follows that $X \setminus V$ is a closed decreasing subset of X such that $A_r \subset X \setminus V$. This means, in particular, that $A(r) \subset X \setminus V$. With help of the observations that $A(r)$ is decreasing, V is increasing, $A(r) \cap V = \emptyset$ and $B_{r'} \subset V$, we may conclude that $A(r) \cap B_{r'} = \emptyset$ and that $not(x \succsim y)$ for all pairs $(x, y) \in A(r) \times B_{r'}$. Therefore, there exist disjoint open decreasing and increasing subsets H, respectively, W of X, such that $A(r) \subset H$ and $B(r') \subset W$. These inclusions imply that $X \setminus H$ is a closed increasing subset of X that includes $B_{r'}$. It, thus, follows that $B(r') \subset X \setminus H$. Therefore, we have that $A(r) \cap B(r') = \emptyset$. □

Needless to say, we can put together Lemma 1 and Lemma 2 so that the following proposition holds true.

Proposition 2. *Let $(X, \precsim, t)$ be a preordered locally and σ-compact Hausdorff space, the preorder $\precsim$ of which is closed. Then the following property holds:*

"If C is any closed subset of X, and $f : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$ is any continuous increasing function, then $F_{|C} = f$ for some continuous increasing function $F : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$".

The following theorem characterizes the existence of a continuous multi-utility representation for a preorder on a locally and σ-compact Hausdorff space in terms of lifting properties from closed and, respectively, compact subspaces.

Theorem 3. *Consider a preordered locally and σ-compact Hausdorff space $(X, \precsim, t)$. Then the following conditions are equivalent:*

(i) *$\precsim$ is representable by a continuous multi-utility;*
(ii) *If $\precsim$ is any closed preorder on (X, t), then the following property is verified:*

> *"If C is any closed subset of X, and $f : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$ is any bounded, continuous, order-preserving and $\precsim$-C-compatible function, then $F_{|C} = f$ for some continuous order-preserving function $F : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$";*

(iii) *If $\precsim$ is any closed preorder on (X, t), then the following property is verified:*

> *"If C is any compact subset of X, and $f : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$ is any continuous order-preserving function, then $F_{|C} = f$ for some continuous order-preserving function $F : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$".*

Proof. (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii): We prove jointly the two implications for the sake of convenience. Since $\precsim$ is representable by a continuous multi-utility, we have that $\precsim$ is closed by Bosi and Herden [22], Proposition 2.1. We consider a subset C of X and a continuous order-preserving function $f : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$. In order to verify that f can be lifted to a continuous order-preserving function $F : (X, \precsim, t_{nat}) \to (\mathbb{R}, \leq, t_{nat})$, according to other suitable assumptions suggested by the consideration of the implication we want to prove, we arbitrarily choose a pair $(x, y) \in \prec$, and we define $C_{xy} := C \cup \{x, y\}$. We proceed by showing that f can be extended to a continuous order-preserving function $f_{xy} : (C_{xy}, \precsim_{|C_{xy}}, t_{|C_{xy}}) \to (\mathbb{R}, \leq, t_{nat})$. Therefore, we distinguish between the following four cases:

Case 1: $C \cap d(x) = \emptyset$ and $C \cap i(y) = \emptyset$.
Case 2: $C \cap d(x) \neq \emptyset$ and $C \cap i(y) = \emptyset$.
Case 3: $C \cap d(x) = \emptyset$ and $C \cap i(y) \neq \emptyset$.
Case 4: $C \cap d(x) \neq \emptyset$ and $C \cap i(y) \neq \emptyset$.

The only case that needs particular reflection is the case that both sets $C \cap d(x)$ and $C \cap i(y)$ are not empty (i.e., the pair $(x, y) \in E^C_{\prec}$). In this case it, clearly, suffices to prove that $s^C_x(f)$ and $i^C_y(f)$ exist and that $s^C_x(f) < i^C_y(f)$. Indeed, having proved the existence of $s^C_x(f)$ and $i^C_y(f)$ as well as the strong inequality $s^C_x(f) < i^C_y(f)$, we may assume that $x \notin C$ or $y \notin C$. In this situation, the inequality $s^C_x(f) < i^C_y(f)$ allows us to set $f_{xy}(x) := s^C_x(f) + \frac{i^C_y(f) - s^C_x(f)}{4}$ if $x \notin C$ and $y \in C$ or $f_{xy}(y) := i^C_y(f) - \frac{i^C_y(f) - s^C_x(f)}{4}$ if $x \in C$ and $y \notin C$ or $f_{xy}(x) := s^C_x(f) + \frac{i^C_y(f) - s^C_x(f)}{4}$ and $f_{xy}(y) := i^C_y(f) - \frac{i^C_y(f) - s^C_x(f)}{4}$ if $x \notin C$ and $y \notin C$. It, thus, remains to verify that $s^C_x(f)$ and $i^C_y(f)$ exist and that the strong inequality $s^C_x(f) < i^C_y(f)$ holds.

Let us now concentrate on the implication (i) $\Rightarrow$ (ii). In this case, since C is a closed subset of X and (X, t) is a Hausdorff space, we may conclude that $C_{xy} := C \cup \{x, y\}$ is a closed subset of X. In addition, besides the assumption that f is continuous and order-preserving, we have that f is bounded and $\precsim$-C-compatible. Using the fact that f is bounded and continuous on C closed, and that $d(x)$ and $i(y)$ are closed due to the closedness of $\precsim$, we have that, actually, $s^C_x(f) := \max f(C \cap d(x))$ and $i^C_y(f) := \min f(C \cap i(y))$, and $s^C_x(f) < i^C_y(f)$ for every pair $(x, y) \in E^C_{\prec}$ by condition (ii) of Proposition 1.

Let us now consider the implication (i) $\Rightarrow$ (iii). Since C is a compact subset of X and (X, t) is a Hausdorff space, we may conclude that $C_{xy} := C \cup \{x, y\}$ is a compact subset of X. Well, the compactness of C implies that there exist points $v \in C \cap d(x)$ and $z \in C \cap i(y)$ such that $f(v) = s^C_x(f)$ and $f(z) = i^C_y(f)$. Since f is order-preserving, we thus may conclude that $f(v) = s^C_x(f) < f(z) = i^C_y(f)$. Let us abbreviate the above considerations by $(^*)$. Since C_{xy} is compact, there exist real numbers $a < b$ such that $f_{xy}(C_{xy}) \subset [a, b]$. Applying $(^*)$, it follows that the real numbers $a < b$ can be chosen in such a way that $f_{xy} \subset [a, b]$ for all pairs $(x, y) \in \prec$.

For both implications, Proposition 2 now implies that every function f_{xy} can be lifted to a continuous increasing function $F_{xy} : (X, \precsim, t) \to ([a, b], \leq_{|[a,b]}, t_{|[a,b]})$. In particular, we may conclude that, for every pair $(x, y) \in \prec$, there exists a real number $\epsilon_{xy} \in]a, b[$ such that

$(x,y) \in F_{xy}^{-1}([a, \epsilon_{xy}[) \times F_{xy}^{-1}(]\epsilon_{xy}, b])$. Hence, $\prec \subset \bigcup_{(x,y) \in \prec} F_{xy}^{-1}([a, \epsilon_{xy}[) \times F_{xy}^{-1}(]\epsilon_{xy}, b])$. Now, we may apply the results on the Lindelöf property of (X,t) that already have been quoted in the proof of Lemma 1 in order to conclude that there exists a countable family $\{(x_n, y_n)\}_{n \in \mathbb{N}}$ of pairs $(x_n, y_n) \in \prec$ such that the inclusion $\prec \subset \bigcup_{(x_n,y_n) \in \prec} F_{x_n y_n}^{-1}([a, \epsilon_{x_n y_n}[) \times F_{x_n y_n}^{-1}(]\epsilon_{x_n y_n}, b])$ holds. Hence, we set $F := \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} F_{x_n y_n}$. The definition of F allows to conclude that F is a continuous order-preserving real-valued function such that $F_{|C} = f$. In this way, we have proven both implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (i). Consider a preorder $\precsim$ on (X,t) satisfying property (ii). In order to show that $\precsim$ is representable by a continuous multi-utility, consider any pair $(x,y) \in X \times X$ with $not(x \succsim y)$. It suffices to verify that $f_{xy}(x) < f_{xy}(y)$ for some continuous increasing real-valued function f_{xy} on $(X \precsim, t)$. Therefore, we set $C := \{x, y\}$. Clearly, C is a closed subset of X. Furthermore, the function $g_{xy} : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$ defined by $g_{xy}(x) := 0$ and $g_{xy}(y) := 1$ is a bounded, continuous, order-preserving and $\precsim$-C-compatible function on $(C, \precsim_{|C}, t_{|C})$. Hence, there exists a continuous order-preserving function $f_{xy} : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$ such that $f_{xy}(x) = 0$ and $f_{xy}(y) = 1$, which implies that $\precsim$ is representable by a continuous multi-utility.

(iii) $\Rightarrow$ (i). Consider a preorder $\precsim$ on (X,t) satisfying property (iii). We proceed as in the proof of the previous implication, by considering any pair $(x,y) \in X \times X$ with $not(x \succsim y)$. Then, we set $C := \{x, y\}$. Clearly, C is a compact subset of X. Furthermore, the function $g_{xy} : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$ defined by $g_{xy}(x) := 0$ and $g_{xy}(y) := 1$ is a continuous order-preserving real-valued function on $(C, \precsim_{|C}, t_{|C})$. Hence, there exists a continuous order-preserving function $f_{xy} : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$ such that $f_{xy}(x) = 0$ and $f_{xy}(y) = 1$, which implies that $\precsim$ is representable by a continuous multi-utility. This observation finishes the proof of the implication and, thus, of the theorem. □

It seems that the postulate of (X,t) being σ-compact cannot be avoided in Theorem 3. Indeed, the following restrictive theorem that is based upon the additional assumption (X,t) to be *paracompact* holds (a topological space (X,t) is said to be *paracompact* if it is Hausdorff and every open cover of X has a locally finite open refinement).

Theorem 4. *Let (X,t) be a locally compact paracompact topological space. Then the following assertions are equivalent:*

(i) *(X,t) is σ-compact;*

(ii) *If $\precsim$ is any closed preorder on (X,t), then the following property is verified:*
"If C is any compact subset of X, and $f : (C, \precsim_{|C}, t_{|C}) \to (\mathbb{R}, \leq, t_{nat})$ is any continuous order-preserving function, then $F_{|C} = f$ for some continuous order-preserving function $F : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$".

Proof. (i) $\Rightarrow$ (ii). This implication follows from the implication "(i) $\Rightarrow$ (ii)" of Theorem 3.

(ii) $\Rightarrow$ (i). This implication is based upon the well-known result that a locally compact topological space is paracompact if and only if it is the direct sum of locally and σ-compact topological spaces (cf., for instance, Grotemeyer [23], Satz 97). In order to prove the validity of assertion (i) it, therefore, suffices to show that (X,t) must be the direct sum of countably many locally and σ-compact topological spaces. Indeed, let us assume, in contrast, that (X,t) is the direct sum of uncountably many locally and σ-compact topological spaces. Then we may assume these summands to be indexed by the ordinal numbers α that are strictly smaller than some uncountable cardinal number κ, i.e., we may assume X to be given in the form $X = \bigoplus_{\alpha < \kappa} X_\alpha$, where each summand X_α ($\alpha < \kappa$) is a locally and σ-compact topological space. Therefore, we consider the binary relation $\precsim$ on X that is defined by setting

$$\precsim := \{(x, y) \in X \times X \mid \text{ there exist ordinal numbers } \alpha \leq \beta < \kappa \text{ such that } x \in X_\alpha \text{ and } y \in X_\beta\}.$$

Obviously, $\precsim$ is a closed (continuous) total preorder on (X, t). In addition, since $\precsim$ contains uncountable well-ordered sub-chains, there cannot exist any compact subset C of X and any continuous order-preserving function $f : (C, \precsim_{|C}, t_{|C}) \rightarrow (\mathbb{R}, \leq, t_{nat})$ for which there exists a continuous order-preserving function $F : (X, \precsim, t) \rightarrow (\mathbb{R}, \leq, t_{nat})$ such that $F_{|C} = f$. □

Consider that the proof of Theorem 4 demonstrates that the assumption that (X, t) is paracompact does not mean a great loss of generality.

4. Conclusions

A lifting theorem was presented for continuous order-preserving functions on locally and σ-compact Hausdorff preordered topological spaces. In particular, we showed that, on such spaces, a preorder is closed (or equivalently, representable by a continuous multi-utility) if, and only if, for every compact subspace, every continuous order-preserving function can be lifted to the entire space. A lifting property from closed sets was also introduced in such spaces for a bounded, continuous, order-preserving and $\precsim$-C-compatible function. We showed that the assumption of σ-compactness cannot be avoided in such a characterization since the aforementioned lifting property is equivalent to σ-compactness when the topological space is locally compact and paracompact.

These theorems are helpful in order to provide necessary and sufficient conditions on a topology on a preordered set, according to which every closed preorder is representable by a continuous multi-utility. The corresponding more general results will be presented in a future paper.

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