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PSPACE-completeness of the temporal logic of sub-intervals and suffixes

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ABSTRACT

In this paper, we prove PSPACE-completeness of the finite satisfiability and model checking problems for the fragment of Halpern and Shoham interval logic with modality ⟨E⟩, for the "suffix" relation on pairs of intervals, and modality ⟨D⟩, for the "sub-interval" relation, under the homogeneity assumption. The result significantly improves the EXPSPACE upper bound recently established for the same fragment, and proves the rather surprising fact that the complexity of the considered problems does not change when we add either the modality for suffixes (⟨E⟩) or, symmetrically, the modality for prefixes (⟨B⟩) to the logic of sub-intervals (featuring only ⟨D⟩).

1. Introduction

For a long time, in computer science, interval temporal logics (ITLs) have been considered an attractive, but impractical, alternative to standard point-based ones. On the one hand, they are a natural choice as a specification/representation language in a number of domains; on the other hand, the high undecidability of the satisfiability problem for the most well-known ITLs [1–5], such as Halpern's and Shoham's modal logic of time intervals (HS for short) [2] and Venema's CDT [5], discouraged their extensive use (but some restricted variants of them have been applied in formal verification and AI over the years [6–8]).

The present work finds its place in the framework of the logic HS, which features one modality for each of the 13 Allen's relations [9], apart from equality. In Table 1, we depict 6 Allen's relations for ordered pairs of intervals, together with the corresponding HS (existential) modalities; the other 7 relations are their inverses and the equality relation. The recent discovery of a significant number of expressive and computationally well-behaved fragments of HS changed the landscape of ITL research [10,11]. Meaningful examples are the logic $A\overline{A}$ of the temporal neighborhood [12] (the HS fragment with modalities for the *meets* relation and its inverse) and the logic D of (temporal) sub-intervals [13] (the HS fragment with modality ⟨D⟩ for the *contains* relation only) over dense orderings.

Model checking (MC) of (finite) Kripke structures against HS and its fragments has been investigated in a series of papers [14,15,6,16–19] and shown to be decidable. In this setting, each finite path of a Kripke structure is interpreted as an interval whose labeling satisfies the *homogeneity assumption* [20]: a proposition letter holds over an interval if and only

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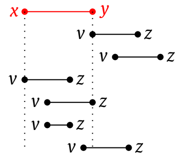
Table 1
Allen's relations and corresponding HS modalities.

Allen relation	HS	Definition w.r.t. interval structures	Example
MEETS	$\langle A \rangle$	$[x, y]\mathcal{R}_A[v, z] \iff y = v$	
BEFORE	$\langle L \rangle$	$[x, y]\mathcal{R}_L[v, z] \iff y < v$	
STARTED-BY	$\langle B \rangle$	$[x, y]\mathcal{R}_B[v, z] \iff x = v \wedge z < y$	
FINISHED-BY	$\langle E \rangle$	$[x, y]\mathcal{R}_E[v, z] \iff y = z \wedge x < v$	
CONTAINS	$\langle D \rangle$	$[x, y]\mathcal{R}_D[v, z] \iff [v, z] \subset [x, y]$	
OVERLAPS	$\langle O \rangle$	$[x, y]\mathcal{R}_O[v, z] \iff x < v < y < z$	



if it holds over all its constituent points (states). MC against full HS is at least Expspace-hard [15] and the only known upper bound is non-elementary [18,21].[1] The known complexity bounds of MC for full HS coincide with those of MC for the linear-time fragment BE of HS which features modalities $\langle B \rangle$ and $\langle E \rangle$ for prefixes and suffixes. These complexity bounds easily transfer to finite satisfiability, that is, satisfiability over finite linear orders, of BE under the homogeneity assumption. Whether or not these problems can be solved elementarily is a difficult open question. On the other hand, MC and finite satisfiability under the homogeneity assumption for all the fragments of BE are known to be elementarily decidable [25,14,26]. In particular, for the fragment D of BE (note that the *contains* relation $\mathcal{R}_D$ can be expressed as $\mathcal{R}_B \cup \mathcal{R}_E \cup \mathcal{R}_B \cdot \mathcal{R}_E$), these problems are known to be Pspace-complete [25].

In a recent contribution [26], we investigated finite satisfiability under the homogeneity assumption for the maximal fragment BD of BE that features modalities $\langle B \rangle$ and $\langle D \rangle$ (the other maximal fragment DE of BE with modalities $\langle D \rangle$ and $\langle E \rangle$ is completely symmetric, and thus all results for BD immediately transfer to it, and vice versa). The addition of modality $\langle B \rangle$ makes satisfiability checking for BD more complex than the one for D, as the two relations/modalities may interact in a non-trivial way. In [26] the Expspace membership of the problem has been proved by means of a purely model-theoretic argument, leaving open the issue of its exact complexity.

In this paper, we answer the question proving that, surprisingly, Pspace-completeness of D is not affected by the addition of either $\langle B \rangle$ or $\langle E \rangle$, and the MC problem for DE (and symmetrically BD) is Pspace-complete as well. In Fig. 1, we add these new MC results to the picture of known MC complexities, showing that they enrich the set of HS "tractable" fragments by two meaningful members. We propose an automata-theoretic approach for solving MC and finite satisfiability under the homogeneity assumption of DE and BD which non-trivially generalizes the one for D [25] and the well-known one for standard LTL [27]. In particular, some important aspects that were not well understood in [25] are generalized by an elegant algebraic framework, which allows us to solve in an asymptotically optimal way the considered problems for DE and BD. In addition, we prove that, over finite linear orders and under the homogeneity assumption, D is less expressive than BD and DE, which in turn are less expressive than BE (in [24], we show that, under the homogeneity assumption, BE and LTL over finite words have the same expressive power).

We conclude the introduction by recalling an interesting connection between the finite satisfiability problem for BE and its fragments, under the homogeneity assumption, and the non-emptiness problem for generalized $*$-free regular expressions [26]. The latter problem has been shown to be non-elementarily decidable by Stockmeyer in [28], and it can be easily proved to be equivalent to finite satisfiability for the interval temporal logic C of the chop modality [29,7,5] under the homogeneity assumption (the chop modality allows one to split the current interval in two parts and to state what is true over the first part and what over the second one). It can be shown that over finite linear orders and under the homogeneity assumption, BE (resp., its proper fragments BD and DE) is equivalent to the weakening of generalized $*$-free regular expressions where the concatenation operator is replaced by the weaker *prefix* and *suffix* ones (resp., *prefix* and *infix*, and *infix* and *suffix*) [26]. Note that the infix operator can be expressed in terms of the combination of the prefix and suffix operators. An immediate by-product of the results given in this paper is that the non-emptiness problem for $*$-free generalized regular expressions turns out to be elementarily decidable and, precisely, Pspace-complete if one makes use of the *suffix* (resp., *prefix*) operator and the *infix* operator instead of the concatenation operator in the expressions. As for the fragment with both the *prefix* and the *suffix* operators, we only know that its non-emptiness problem is (non-elementarily decidable and) Expspace-hard [15].

Structure of the paper In Section 2, we introduce some basic definitions which will be extensively used in the paper. Moreover, we recall the syntax and semantics of the logic BE and its fragments BD and DE in the setting of finite linear orders under the homogeneity assumption. In such a restricted setting, we address in Section 3 expressiveness issues and the encoding of the considered logics in fragments of generalized $*$-free regular expressions. In Section 4, we illustrate an asymptotically optimal automata-theoretic approach for solving satisfiability and model checking for DE-formulas and BD-formulas over finite linear orders under the homogeneity assumption. Finally, Section 5 outlines future research directions.

[1] An expressive comparison of MC for HS and standard point-based temporal logics LTL [22], CTL, and CTL* [23] can be found in [24].

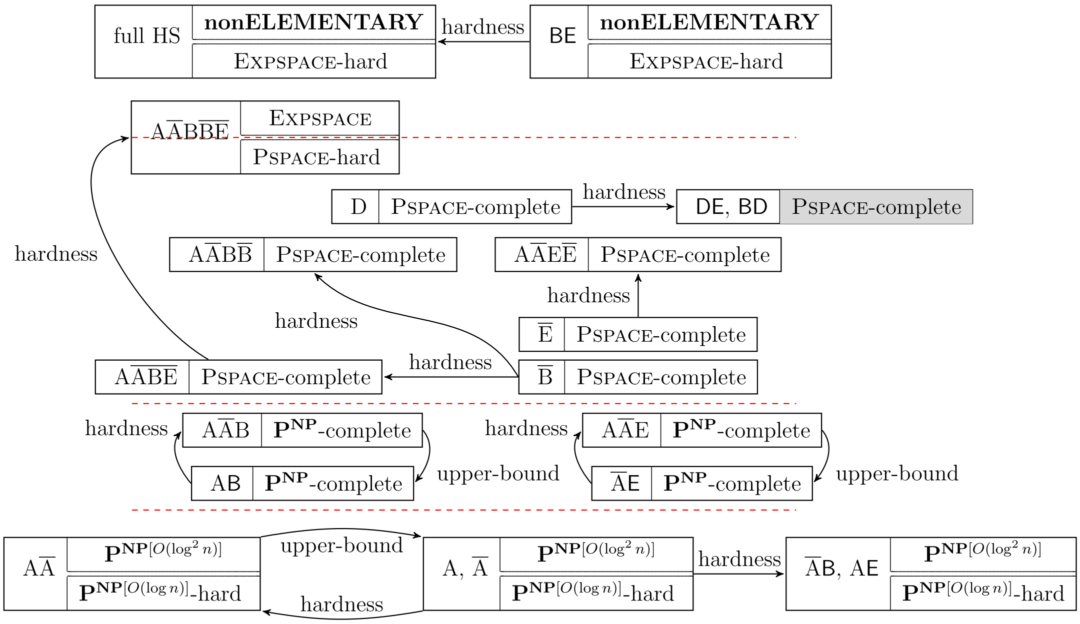


Fig. 1. Complexity of the MC problem for HS and its fragments.

2. Preliminaries

We fix the following notation. For a finite word (or sequence) w over some finite alphabet Σ, we denote by $|w|$ the length of w. Moreover, for all $0 \leq i < |w|$, $w[i]$ is the $(i+1)^{th}$ letter of w. Given two non-empty finite words w, w' over Σ, we write $w \cdot w'$ for the concatenation of w and w'. Moreover, if the last letter of w coincides with the first letter of w', we denote by $w \star w'$ the word $w \cdot w'[1] \dots w'[n-1]$, where $n = |w'|$ (i.e. the word obtained by concatenating w with the word obtained from w' by erasing the first letter). In particular, when $|w'| = 1$, then $w \star w' = w$.

Finite automata over finite words A nondeterministic finite automaton (NFA) is a tuple $\mathcal{N} = \langle \Sigma, Q, Q_0, \delta, F\rangle$, where Σ is a finite alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is the set of *initial* states, $\delta : Q \times \Sigma \to 2^Q$ is the transition function, and $F \subseteq Q$ is the set of *accepting* states. Given a finite word w over Σ, with $|w| = n$, a run of $\mathcal{N}$ over w is a finite sequence of states $q_0, \dots, q_n$ such that $q_0 \in Q_0$, and for all $0 \leq i < n$, $q_{i+1} \in \delta(q_i, w[i])$. The language $\mathcal{L}(\mathcal{N})$ accepted by $\mathcal{N}$ consists of the finite words w over Σ such that there is a run over w ending in some accepting state. A deterministic finite automaton (DFA) is an NFA $\mathcal{D} = \langle \Sigma, Q, \{q_0\}, \delta, F\rangle$ such that for all $(q, \sigma) \in Q \times \Sigma$, $\delta(q, \sigma)$ is a singleton.

Finite Kripke structures We fix a finite set $\mathcal{AP}$ of proposition letters which represent predicates over the states of the given system. A *finite Kripke structure* over $\mathcal{AP}$ is a tuple $\mathcal{K} = \langle W, s_0, E, \mu\rangle$, where W is a finite set of states, $s_0 \in W$ is the initial state, $E \subseteq W \times W$ is a relation between states, and $\mu : W \to 2^{\mathcal{AP}}$ is a labeling function assigning to each state the set of propositions that hold at it.

A *path of* $\mathcal{K}$ is a non-empty finite sequence of states $\pi = s_1 \cdots s_n$ such that (i) the first state s_1 coincides with the initial state s_0 of $\mathcal{K}$, and (ii) $(s_i, s_{i+1}) \in E$ for all $1 \leq i < n$. We extend the labeling μ to paths of $\mathcal{K}$ in the usual way: for a path $\pi = s_1 \dots s_n$, $\mu(\pi)$ denotes the word over $2^{\mathcal{AP}}$ of length n given by $\mu(s_1) \dots \mu(s_n)$. A *trace* of $\mathcal{K}$ is a non-empty finite word over $2^{\mathcal{AP}}$ of the form $\mu(\pi)$ for some path π of $\mathcal{K}$.

Example 2.1. In Fig. 2, we depict a finite Kripke structure $\mathcal{K}_{Sched}$ that models the behavior of a scheduler serving three processes which are continuously requesting the use of a common resource. In the initial state v_0 no process is served. In the states v_i, with $i \in \{1, 2, 3\}$, the i-th process is served (the proposition p_i holds in those states). The loop on v_i represents the use of the resource. A transition from the state v_i to u_i represents the unlock of the granted resource (the proposition $\overline{p_i}$ holds in that state). The scheduler cannot serve the same process twice in two successive rounds, and then v_i is not directly reachable from u_i. A transition from u_i to v_j with $j \neq i$, represents the fact that the j-th process has issued a request for the resource and is served.

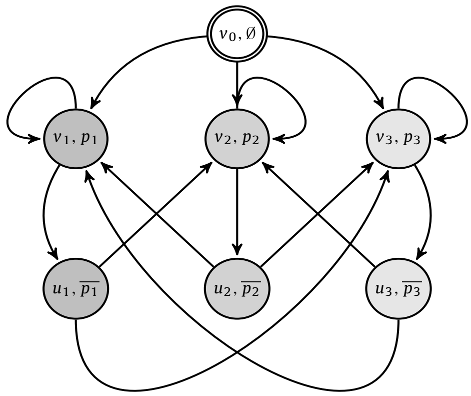


Fig. 2. The Kripke structure $\mathcal{K}_{Sched}$.

2.1. *The logics* DE *and* BD *under the homogeneity assumption*

In this section, we recall the logic BE of prefix and suffixes corresponding to the linear-time fragment of HS, and we focus our attention on the fragments DE and BD of BE interpreted over finite linear orders under the homogeneity assumption.

Let $\mathbb{S} = \langle S, < \rangle$ be a linear order over the nonempty set $S \neq \emptyset$, and $\leq$ be the reflexive closure of $<$. Given $x, y \in S$ such that $x \leq y$, we denote by $[x, y]$ the (closed) *interval* over S given by the set of elements $z \in S$ such that $x \leq z$ and $z \leq y$. We denote the set of all intervals over $\mathbb{S}$ by $\mathbb{I}(\mathbb{S})$. We focus our attention on three Allen's relations over intervals:

1. the *proper prefix (or started-by)* relation $\mathcal{R}_B$ defined as follows: $[x, y]\mathcal{R}_B[x', y']$ if $x = x'$ and $y' < y$,
2. the *proper sub-interval (or contains)* relation $\mathcal{R}_D$ defined as follows: $[x, y]\mathcal{R}_D[x', y']$ if $x' \geq x$, $y' \leq y$, and $[x, y] \neq [x', y']$ (the proper subset relation over intervals), and
3. the *proper suffix (or finished-by)* relation $\mathcal{R}_E$ defined as: $[x, y]\mathcal{R}_E[x', y']$ if $x < x'$ and $y' = y$.

The temporal logic BE consists of a finite set $\mathcal{AP}$ of proposition letters, the logical connectives $\neg$ and $\vee$, and the existential temporal modalities for Allen's relations $\mathcal{R}_B$, $\mathcal{R}_D$, and $\mathcal{R}_E$. Formally, BE formulas φ are defined by the following abstract syntax:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle \mathrm{B} \rangle\, \varphi \mid \langle \mathrm{D} \rangle\, \varphi \mid \langle \mathrm{E} \rangle\, \varphi$$

where $\top$ is for 'true', $p \in \mathcal{AP}$, and $\langle \mathrm{B} \rangle$ (resp, $\langle \mathrm{D} \rangle$, resp., $\langle \mathrm{E} \rangle$) is the existential temporal modality for the Allen's relation $\mathcal{R}_B$ (resp., $\mathcal{R}_D$, resp., $\mathcal{R}_E$). We also exploit the conjunction connective $\wedge$ and the implication connective $\rightarrow$ as abbreviations, and for any temporal modality $\langle \mathrm{X} \rangle$, with $X \in \{B, D, E\}$, the dual universal modality $[\mathrm{X}]$ defined as: $[\mathrm{X}]\,\psi := \neg\,\langle \mathrm{X} \rangle\,\neg\psi$. The size $|\varphi|$ of a formula φ is the number of distinct sub-formulas of φ. We focus our attention on the fragments DE (logic of sub-intervals and suffixes) and BD (logic of sub-intervals and prefixes) of BE obtained by disallowing the temporal modalities for the Allen's relations $\mathcal{R}_B$ and $\mathcal{R}_E$, respectively. We also consider the fragments B, D, and E defined in the obvious way.

The semantics of the logic BE is given in terms of *interval models*. An interval model $\mathcal{M}$ is a pair $\langle \mathbb{I}(\mathbb{S}), \mathcal{V} \rangle$, where $\mathcal{V} : \mathcal{AP} \mapsto 2^{\mathbb{I}(\mathbb{S})}$ is a *valuation function* that assigns to every proposition letter p the set of intervals $\mathcal{V}(p)$ over which p holds. Given an interval model $\mathcal{M} = \langle \mathbb{I}(\mathbb{S}), \mathcal{V} \rangle$, an interval $[x, y] \in \mathbb{I}(\mathbb{S})$, and a formula φ, the *satisfaction relation* $\mathcal{M}, [x, y] \models \varphi$, meaning that φ holds over the interval $[x, y]$ of $\mathcal{M}$, is inductively defined as follows:

- $\mathcal{M}, [x, y] \models \top$;
- for every proposition letter $p \in \mathcal{AP}$, $\mathcal{M}, [x, y] \models p$ if $[x, y] \in \mathcal{V}(p)$;
- $\mathcal{M}, [x, y] \models \neg\varphi$ if $\mathcal{M}, [x, y] \not\models \varphi$;
- $\mathcal{M}, [x, y] \models \varphi_1 \vee \varphi_2$ if $\mathcal{M}, [x, y] \models \varphi_1$ or $\mathcal{M}, [x, y] \models \varphi_2$;
- $\mathcal{M}, [x, y] \models \langle \mathrm{X} \rangle\, \varphi$ for $\mathrm{X} \in \{\mathrm{B}, \mathrm{D}, \mathrm{E}\}$ if there is an interval $[x', y'] \in \mathbb{I}(\mathbb{S})$ such that $[x, y]\mathcal{R}_X[x', y']$ and $\mathcal{M}, [x', y'] \models \varphi$.

A BE-formula is *satisfiable* if it holds over some interval of an interval model. In this paper, we restrict our attention to the finite satisfiability problems, that is, satisfiability over the class of finite linear orders, for the fragments BD and DE. The problems are known to be undecidable [4] in the general case, but decidability can be recovered by restricting to the class of *homogeneous* interval models [18]. Formally, an interval model $\mathcal{M} = \langle \mathbb{I}(\mathbb{S}), \mathcal{V} \rangle$ is *homogeneous* if for every interval

$[x, y] \in \mathbb{I}(\mathbb{S})$ and every $p \in \mathcal{AP}$, it holds that $[x, y] \in \mathcal{V}(p)$ if and only if $[x', x'] \in \mathcal{V}(p)$ for every $x' \in [x, y]$. Intuitively, this means that p holds at an interval $[x, y]$ iff p holds at each singleton sub-interval of $[x, y]$. Hence, the valuation function $\mathcal{V}$ is uniquely determined by its restriction to singleton intervals. Note that one could give a different notion of homogeneity (dual with respect to the previous one) where a proposition p holds at an interval $[x, y]$ iff p holds at some singleton interval of $[x, y]$ (hence, a proposition p does *not* hold at an interval $[x, y]$ iff p does *not* hold at each singleton interval of $[x, y]$). In this paper, we adopt the standard notion of homogeneity for interval models [18].

We observe that homogeneous interval models over finite linear orders correspond to non-empty finite words over $2^{\mathcal{AP}}$. In particular, each non-empty finite word w over $2^{\mathcal{AP}}$ induces the homogeneous interval model $\mathcal{M}(w) = \langle \mathbb{I}(\mathbb{S}), \mathcal{V} \rangle$ over the finite linear order induced by w defined as follows:

- $\mathbb{S} = \langle \{0, \ldots, |w| - 1\}, < \rangle$, and
- for every interval $[i, j]$ of $\mathbb{S}$ (note that $0 \le i \le j < |w|$) and $p \in \mathcal{AP}$, $[i, j] \in \mathcal{V}(p)$ if and only if $p \in w[h]$ for all $h \in [i, j]$.

Any fragment $\mathcal{F}$ of BE interpreted over homogeneous models is denoted by $\mathcal{F}_{\mathcal{H}om}$. A non-empty finite word w over $2^{\mathcal{AP}}$ *satisfies an* $\mathcal{F}_{\mathcal{H}om}$ *formula* φ, denoted by $w \models \varphi$, if $\mathcal{M}(w), [0, |w| - 1] \models \varphi$. A finite Kripke structure $\mathcal{K}$ over $\mathcal{AP}$ is *a model of* φ, written $\mathcal{K} \models \varphi$, if each trace w of $\mathcal{K}$ satisfies φ. We also consider the *model checking problem* against $\mathsf{DE}_{\mathcal{H}om}$ (resp., $\mathsf{BE}_{\mathcal{H}om}$) that is the problem of deciding for a given finite Kripke structure $\mathcal{K}$ and a $\mathsf{DE}_{\mathcal{H}om}$ (resp., $\mathsf{BD}_{\mathcal{H}om}$) formula φ, whether $\mathcal{K} \models \varphi$.

Example 2.2. We now give an example of properties expressible both in $\mathsf{BE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ to be checked over the Kripke structure $\mathcal{K}_{Sched}$ of Example 2.1. We start by defining a formula $Activity_i$ with $i \in \{1, 2, 3\}$ which precisely characterizes a subpath of $\mathcal{K}_{Sched}$ corresponding with the use and unlock of the shared resource by the i-th process. $Activity_i$ is a formula in $\mathsf{BD}_{\mathcal{H}om}$ defined as follows: $Activity_i := \neg p_i \wedge [B] p_i$. The formula ensures that the path underlying the interval has the form $v_i^+ \cdot u_i$ (all the proper prefixes satisfy p_i but the whole interval does not). $\mathcal{K}_{Sched}$ satisfies the property that any two activities of a process are interleaved with at least an activity of a different process. Such a property can be expressed in $\mathsf{BE}_{\mathcal{H}om}$ as follows:

$$\bigwedge_{i \in \{1,2,3\}} [D]\big((\langle \mathrm{B} \rangle\, Activity_i \wedge \langle \mathrm{E} \rangle\, Activity_i) \rightarrow \langle \mathrm{D} \rangle \bigvee_{j \in \{1,2,3\},\, j \neq i} Activity_j\big)$$

Any subpath of a path starting and ending with an activity of the i-th process has an internal activity of another process. The property expressed by the $\mathsf{BE}_{\mathcal{H}om}$ formula above can be expressed also in the fragment $\mathsf{BD}_{\mathcal{H}om}$ with a small adjustment as follows:

$$\bigwedge_{i \in \{1,2,3\}} [D]\big((\langle \mathrm{B} \rangle\, Activity_i \wedge \langle \mathrm{D} \rangle (\langle \mathrm{B} \rangle \bigvee_{j \in \{1,2,3\}} \overline{p}_j \wedge \langle \mathrm{D} \rangle\, Activity_i)) \rightarrow$$
$$\langle \mathrm{D} \rangle \bigvee_{j \in \{1,2,3\},\, j \neq i} Activity_j\big)$$

Notice that the formula $\langle \mathrm{B} \rangle\, Activity_i \wedge \langle \mathrm{D} \rangle (\langle \mathrm{B} \rangle \bigvee_{j \in \{1,2,3\}} \overline{p}_j \wedge \langle \mathrm{D} \rangle\, Activity_i)$ ensures that there are two instances of $Activity_i$, the second one occurring after a singleton interval where $\bigvee_{j \in \{1,2,3\}} \overline{p}_j$ holds.

Example 2.3. In order to illustrate the succinctness of the logics $\mathsf{D}_{\mathcal{H}om}$, $\mathsf{BD}_{\mathcal{H}om}$, and $\mathsf{DE}_{\mathcal{H}om}$, we consider a combinatorial requirement. For each $n \geq 1$, let $\mathcal{AP}_n = \{p_1, \ldots, p_n, q_1, \ldots, q_n\}$. The property that "there is a proper infix such for each $i \in [1, n]$, *exclusively* either p_i holds at some position of the infix, or q_i holds at some position of the infix" can be expressed by the following $\mathsf{D}_{\mathcal{H}om}$ formula ψ_n.[2]

$$\psi_n := \langle \mathrm{D} \rangle \bigwedge_{i=1}^{n} ((\langle \mathrm{D} \rangle\, p_i \wedge [D] \neg q_i) \vee (\langle \mathrm{D} \rangle\, q_i \wedge [D] \neg p_i))$$

Evidently, the previous requirement can be expressed in standard LTL over finite words. However, we conjecture that there is no LTL formula equivalent to ψ_n of size polynomial in n.

3. Expressiveness of $\mathsf{BE}_{\mathcal{H}om}$ and its fragments over finite linear orders

In this section, we compare the expressiveness of the logics $\mathsf{DE}_{\mathcal{H}om}$, $\mathsf{BD}_{\mathcal{H}om}$, and $\mathsf{BE}_{\mathcal{H}om}$ over finite linear orders (see Subsection 3.1), and we show that these logics can be characterized by fragments of generalized $*$-free regular expressions (see Subsection 3.2).

[2] A proper infix v of a word w is a non-empty word such that $w = w' \cdot v \cdot w''$ for some words w' and w'' such that $w' \cdot w''$ is non-empty.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two formalisms interpreted over non-empty finite words over $2^{\mathcal{AP}}$. Hence, a specification ψ of $\mathcal{F}_1$ (resp., $\mathcal{F}_2$) denotes a language $\mathcal{L}(\psi)$ of non-empty finite words over $2^{\mathcal{AP}}$. Given $\psi_1 \in \mathcal{F}_1$ and $\psi_2 \in \mathcal{F}_2$, ψ_1 and ψ_2 are *equivalent* if $\mathcal{L}(\psi_1) = \mathcal{L}(\psi_2)$. We say that $\mathcal{F}_1$ *is subsumed by* $\mathcal{F}_2$, denoted $\mathcal{F}_1 \preceq_f \mathcal{F}_2$, if for each $\mathcal{F}_1$ specification there is an equivalent $\mathcal{F}_2$ specification. $\mathcal{F}_1$ and $\mathcal{F}_2$ *have the same expressiveness* (resp., *are expressively incomparable*) if $\mathcal{F}_1 \preceq_f \mathcal{F}_2$ and $\mathcal{F}_2 \preceq_f \mathcal{F}_1$ (resp., $\mathcal{F}_1 \not\preceq_f \mathcal{F}_2$ and $\mathcal{F}_2 \not\preceq_f \mathcal{F}_1$). Finally, $\mathcal{F}_1$ *is less expressive than* $\mathcal{F}_2$, denoted by $\mathcal{F}_1 \prec_f \mathcal{F}_2$, if $\mathcal{F}_1 \preceq_f \mathcal{F}_2$ and $\mathcal{F}_2 \not\preceq_f \mathcal{F}_1$.

3.1. Expressiveness comparison between $\mathsf{BE}_{\mathcal{H}om}$ fragments

We compare the expressiveness of the various syntactical fragments of $\mathsf{BE}_{\mathcal{H}om}$ ($\mathsf{BE}_{\mathcal{H}om}$ included) interpreted over finite linear orders. It is known [24] that $\mathsf{BE}_{\mathcal{H}om}$ has the same expressiveness as standard LTL over finite words. Here, we show that over finite words, the fragment $\mathsf{D}_{\mathcal{H}om}$ is less expressive than the fragments $\mathsf{BD}_{\mathcal{H}om}$ and $\mathsf{DE}_{\mathcal{H}om}$ which in turn are less expressive than $\mathsf{BE}_{\mathcal{H}om}$ or, equivalently, LTL. In particular, we establish the following result.

Theorem 3.1. *Over finite linear orders, the following holds:*

1. *there exists an* $\mathsf{E}_{\mathcal{H}om}$ *formula which cannot be expressed in* $\mathsf{BD}_{\mathcal{H}om}$*;*
2. *there exists a* $\mathsf{B}_{\mathcal{H}om}$ *formula which cannot be expressed in* $\mathsf{DE}_{\mathcal{H}om}$*.*

Hence, $\mathsf{D}_{\mathcal{H}om} \prec_f \mathsf{BD}_{\mathcal{H}om} \prec_f \mathsf{BE}_{\mathcal{H}om}$, $\mathsf{D}_{\mathcal{H}om} \prec_f \mathsf{DE}_{\mathcal{H}om} \prec_f \mathsf{BE}_{\mathcal{H}om}$*, and* $\mathsf{BD}_{\mathcal{H}om}$ *and* $\mathsf{DE}_{\mathcal{H}om}$ *are expressively incomparable.*

In the following, we provide a proof of Theorem 3.1. Let $\mathcal{AP} = \{p\}$. For the proof of Theorem 3.1(1), we consider the $\mathsf{E}_{\mathcal{H}om}$ formula φ_E over $\mathcal{AP}$ defined as follows:

$$\varphi_E := \langle \mathrm{E} \rangle\, p$$

which asserts the existence of a proper suffix where p holds. Symmetrically, for the proof of Theorem 3.1(2), we consider the $\mathsf{B}_{\mathcal{H}om}$ formula φ_B over $\mathcal{AP}$ given by:

$$\varphi_B := \langle \mathrm{B} \rangle\, p$$

which requires the existence of a proper prefix where p holds. We prove that over finite linear orders (i) no $\mathsf{BD}_{\mathcal{H}om}$ formula is equivalent to φ_E, and (ii) no $\mathsf{DE}_{\mathcal{H}om}$ formula is equivalent to φ_B. Hence, Theorem 3.1 follows. Here, we focus on the result for φ_E (the result for φ_B being similar).

In order to prove that the $\mathsf{E}_{\mathcal{H}om}$-formula φ_E is not expressible in $\mathsf{BD}_{\mathcal{H}om}$ over finite linear orders, a standard approach would be to exhibit two non-empty finite words w and w' over $2^{\{p\}}$ that φ_E can *distinguish* (i.e., one word satisfies φ_E and the other one not), and prove that no $\mathsf{BD}_{\mathcal{H}om}$ formula can distinguish the two words w and w' (note that in this case w and w' need to have the same length since for each $n \geq 1$, one can easily define a $\mathsf{BD}_{\mathcal{H}om}$ formula characterizing the finite words of length n).

However, we do not know whether the previous approach is applicable to the formula φ_E. Thus, we use a different technique. We define two families $(w_n)_{n\geq 1}$ and $(w'_n)_{n\geq 1}$ of non-empty finite words over $2^{\{p\}}$ such that:

- φ_E distinguishes between w_n and w'_n for each $n \geq 1$, and
- for every $\mathsf{BD}_{\mathcal{H}om}$ formula ψ, there is $n \geq 1$ such that ψ does not distinguish between w_n and w'_n.

Before defining the words w_n and w'_n for each $n \geq 1$, we first give some preliminary definitions.

For a $\mathsf{BD}_{\mathcal{H}om}$ formula ψ over $\mathcal{AP} = \{p\}$, the *joint nesting depth* $d(\psi)$ of ψ is the nesting depth of all the temporal modalities in ψ. Formally, (i) $d(p) = 0$, (ii) $d(\neg\psi) := d(\psi)$, (iii) $d(\psi_1 \vee \psi_2) = \max(d(\psi_1), d(\psi_2))$, and (iv) $d(\langle \mathrm{X} \rangle\, \psi) = d(\psi) + 1$ for each $X \in \{B, D\}$.

For each $h \geq 0$, we introduce an equivalence relation $\equiv_h$ on the non-empty finite words on $2^{\{p\}}$. Intuitively, if $w \equiv_h w'$ (we also say that *w and w' are h-equivalent*), then no $\mathsf{BD}_{\mathcal{H}om}$ formula having joint nesting depth at most h can distinguish w and w'. The equivalence relation $\equiv_h$ is defined by induction on $h \geq 0$ as follows:
Case $h = 0$**.** $w \equiv_0 w'$ if $w \models p \Leftrightarrow w' \models p$.
Case $h > 0$**.** $w \equiv_h w'$ if $w \equiv_0 w'$ and the following two properties hold:

- *Forward h-rule*: for each proper prefix (resp.: proper infix) ν of w, there is a proper prefix (resp.: proper infix) ν' of w' such that $\nu \equiv_{h-1} \nu'$.
- *Backward h-rule*: for each proper prefix (resp.: proper infix) ν' of w', there is a proper prefix (resp.: proper infix) ν of w such that $\nu \equiv_{h-1} \nu'$.

By the semantics of $\mathsf{BD}_{\mathcal{H}om}$ and by using a straightforward induction on h, we deduce the following.

Proposition 3.1. *Let $h \geq 0$ and $w \equiv_h w'$. Then, for each $\mathsf{BD}_{\mathcal{H}om}$ formula ψ such that $d(\psi) \leq h$, it holds that $w \models \psi$* if and only if *$w' \models \psi$.*

For each $n \geq 1$, let w_n and w'_n be the non-empty finite words over $2^{\{p\}}$ defined as follows:

$$w_n = (\emptyset\{p\})^{n+2} \text{ and } w'_n = w_n \cdot \emptyset.$$

Note that for each $n \geq 1$, the property of having a suffix where p holds is satisfied by w_n but not by w'_n. Hence, the $\mathsf{E}_{\mathcal{H}om}$ formula φ_E distinguishes w_n and w'_n for each $n \geq 1$.

Lemma 3.1. *For each $n \geq 1$, $w_n \models \varphi_E$ and $w'_n \not\models \varphi_E$.*

The intuition in the definition of w_n and w'_n is that they preserve the following invariants for each $n \geq 1$.

Lemma 3.2 *(Invariants for w_n and w'_n). Let $n \geq 1$. Then, the words w_n and w'_n satisfy the following two properties:*

- *Forward invariant: for each proper prefix (resp.: proper infix) ν of w_n, ν is a proper prefix (resp.: proper infix) of w'_n too.*
- *Backward invariant: for each proper prefix (resp.: proper infix) ν of w'_n:*
  - *either ν is a proper prefix (resp.: proper infix) of w_n,*
  - *or ν is of the form $\nu_1 \cdot (\nu_2)^{n+1} \cdot \nu_3$ with ν_2 being non-empty and $\nu_1 \cdot (\nu_2)^n \cdot \nu_3$ is a proper prefix (resp.: proper infix) of w_n.*

Proof. Let $n \geq 1$. Recall that $w_n = (\emptyset\{p\})^{n+2}$ and $w'_n = w_n \cdot \emptyset$. Thus, since w_n is a proper prefix of w'_n, the forward invariant property directly follows.
Next, we show that the backward invariant property holds. Let ν be a proper prefix (resp.: proper infix) of w'_n. First, assume that ν is a proper prefix of w'_n. If ν is not a proper prefix of w_n, then $\nu = w_n$. Thus, since $w_n = (\emptyset\{p\})^{n+2}$ and $(\emptyset\{p\})^{n+1}$ is a proper prefix of w_n, by setting $\nu_1 = \nu_2 = \emptyset\{p\}$ and $\nu_3 = \varepsilon$ (ε is the empty word), we obtain that $\nu = \nu_1 \cdot (\nu_2)^{n+1} \cdot \nu_3$ and $\nu_1 \cdot (\nu_2)^n \cdot \nu_3$ is a proper prefix of w_n, and the result follows.
Now, assume that ν is a proper infix of w'_n. By construction, ν is of the form $\nu_1 \cdot (\emptyset\{p\})^k \cdot \nu_3$, where $0 \leq k \leq n+1$, ν_1 is a (possibly empty) suffix of $\emptyset\{p\}$, and either $\nu_3 = \emptyset$ or $\nu_3 = \varepsilon$. If $k < n+1$, then $\nu_1 \cdot (\emptyset\{p\})^k \cdot \nu_3$ is also a proper infix of w_n. Otherwise, $k = n+1$ and by construction, $\nu_1 \cdot (\emptyset\{p\})^n \cdot \nu_3$ is a proper infix of w_n. Thus, by setting $\nu_2 = \emptyset\{p\}$, the result follows. □

We now show that for all $n \geq 1$, the words w_n and w'_n are n-equivalent. Since w_n and w'_n are clearly 0-equivalent, the result directly follows from Lemma 3.2 and the following Lemma 3.3.

Lemma 3.3. *Let ν_1, ν_2, and ν_3 be finite words over $2^{\{p\}}$ such that ν_2 is not empty. Then, for all $n \geq 1$, $\nu_1 \cdot (\nu_2)^n \cdot \nu_3 \equiv_{n-1} \nu_1 \cdot (\nu_2)^{n+1} \cdot \nu_3$.*

Proof. Let ν_1, ν_2, and ν_3, and $n \geq 1$ as in the statement of the lemma. The proof is by induction on $n \geq 1$. For the base case ($n = 1$), the result is trivial. For the induction step ($n > 1$), we have to prove that the forward and backward $(n-1)$-rules are satisfied. We focus on the backward $(n-1)$-rule (the treatment of the forward $(n-1)$-rule is similar). Let ν be a proper prefix (resp.: a proper infix) of $\nu_1 \cdot (\nu_2)^{n+1} \cdot \nu_3$. We need to show that there is a proper prefix (resp.: a proper infix) ν' of $\nu_1 \cdot (\nu_2)^n \cdot \nu_3$ such that ν and ν' are $(n-2)$-equivalent. By construction, one of the following three cases occurs:

- ν is a proper prefix (resp.: a proper infix) of $\nu_1 \cdot (\nu_2)^n \cdot \nu_3$ too. Hence, in this case, the result trivially follows.
- ν is of the form $\nu'_1 \cdot (\nu_2)^n \cdot \nu'_3$ and $\nu'_1 \cdot (\nu_2)^{n-1} \cdot \nu'_3$ is a proper prefix (resp.: proper infix) of $\nu_1 \cdot (\nu_2)^n \cdot \nu_3$. Hence, by the induction hypothesis on n, the result directly follows.
- ν is of the form $\nu'_1 \cdot (\nu_2)^{n+1} \cdot \nu'_3$ with $|\nu'_1 \cdot \nu'_3| < |\nu_1 \cdot \nu_3|$ and $\nu'_1 \cdot (\nu_2)^n \cdot \nu'_3$ is a proper prefix (resp.: proper infix) of $\nu_1 \cdot (\nu_2)^n \cdot \nu_3$. By a straightforward double induction on $|\nu_1 \cdot \nu_3|$, it follows that $\nu'_1 \cdot (\nu_2)^{n+1} \cdot \nu'_3$ and $\nu'_1 \cdot (\nu_2)^n \cdot \nu'_3$ are $(n-1)$-equivalent, hence, $(n-2)$-equivalent as well, and the result follows. □

Proof of Theorem 3.1(1). By Lemmata 3.2 and 3.3, for each $n \geq 1$, the words w_n and w'_n are n-equivalent. By Proposition 3.1, it follows that for each $\mathsf{BD}_{\mathcal{H}om}$-formula ψ, ψ cannot distinguish the words w_n and w'_n for all $n \geq d(\psi)$. Thus, by Lemma 3.1, the $\mathsf{E}_{\mathcal{H}om}$-formula φ_E cannot be expressed in $\mathsf{BD}_{\mathcal{H}om}$. Hence, Theorem 3.1(1) directly follows.

3.2. Fragments of generalized $*$-free regular expressions induced by $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ over finite linear orders

In this section, we give characterizations of the logic $\mathsf{BE}_{\mathcal{H}om}$ and its less expressive fragments $\mathsf{D}_{\mathcal{H}om}$, $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ over finite linear orders in terms of subclasses of *generalized $*$-free regular expressions* [28]. These subclasses are obtained by replacing the concatenation operator with weaker versions which represent the operator counterparts of the Allen's relations $\mathcal{R}_B$, $\mathcal{R}_D$, and $\mathcal{R}_E$. For each of the considered logics $\mathcal{F}$, we show that there are linear-time translations from $\mathcal{F}$-formulas into equivalent expressions of the corresponding subclass of generalized $*$-free regular expressions ($\mathcal{F}$-expressions for short), and vice versa. In Section 4, we will show that finite satisfiability of both $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ are PSPACE-complete. As a consequence, we obtain that non-emptiness of $\mathsf{DE}_{\mathcal{H}om}$-expressions and $\mathsf{BD}_{\mathcal{H}om}$-expressions are both PSPACE-complete. We think that this result is of independent interest since it is well-known that non-emptiness of generalized $*$-free regular expressions is already non-elementary hard [28].

We recall that LTL over finite words characterizes the class of languages defined by generalized $*$-free regular expressions [30]. Moreover, it has been proved that LTL over finite words has the same expressiveness as $\mathsf{BE}_{\mathcal{H}om}$ [24]. Hence, we also deduce that $\mathsf{BE}_{\mathcal{H}om}$-expressions are expressively complete for generalized $*$-free regular expressions.

For the given finite set $\mathcal{AP}$ of proposition letters, let Σ be the finite alphabet given by $2^{\mathcal{AP}}$. *Generalized $*$-free regular expressions* (hereafter, simply called *general expressions*) e over the alphabet $\Sigma = 2^{\mathcal{AP}}$ are inductively defined as follows:

$$e ::= \emptyset \mid a \mid [p] \mid \neg e \mid e + e \mid \ e \cdot e$$

where $a \in \Sigma$ and $p \in \mathcal{AP}$. We exclude the empty word ϵ from the syntax as it makes the correspondence between restricted expressions and $\mathsf{BE}_{\mathcal{H}om}$ fragments more direct (such a simplification is quite common in the literature). Moreover, the non-standard atomic expression $[p]$ with $p \in \mathcal{AP}$ captures the non-empty finite words over $2^{\mathcal{AP}}$ such that p holds at each position. The term $[p]$ does not add expressive power (it can be removed by a singly exponential blowup in the cardinality of $\mathcal{AP}$) but it is useful for ensuring a linear-time translation from formulas of a $\mathsf{BE}_{\mathcal{H}om}$ fragment into equivalent expressions of the corresponding subclass of general expressions. Note that under the assumption that $\mathcal{AP}$ is fixed, then the terms $[p]$ can be removed with a constant blowup.

A general expression e defines the language $\mathcal{L}(e) \subseteq \Sigma^+$, which is inductively defined as follows:

- $\mathcal{L}(\emptyset) = \emptyset$;
- $\mathcal{L}(a) = \{a\}$, for every $a \in \Sigma$;
- $\mathcal{L}([p]) = \{w \in \Sigma^+ \mid p \in w[i] \text{ for all } 0 \leq i < |w|\}$, for every $p \in \mathcal{AP}$;
- $\mathcal{L}(\neg e) = \Sigma^+ \setminus \mathcal{L}(e)$;
- $\mathcal{L}(e_1 + e_2) = \mathcal{L}(e_1) \cup \mathcal{L}(e_2)$;
- $\mathcal{L}(e_1 \cdot e_2) = \{w_1 w_2 \mid w_1 \in \mathcal{L}(e_1), w_2 \in \mathcal{L}(e_2)\}$.

In [28], Stockmeyer proves that the problem of deciding non-emptiness of $\mathcal{L}(e)$, for a given general expression e, is non-elementary hard.

The fragment of general expressions we are considering, called *prefix/suffix expressions* or *$\mathsf{BE}_{\mathcal{H}om}$-expressions*, replaces the concatenation operator $\cdot$ by the three unary operators Pre (*prefix*), Inf (*infix*), and Suf (*suffix*), representing the operator counterparts of the Allen's relations $\mathcal{R}_B$, $\mathcal{R}_D$, and $\mathcal{R}_E$, respectively. The subclass of $\mathsf{BE}_{\mathcal{H}om}$-expressions e is defined by the following syntax:

$$e ::= \emptyset \mid a \mid [p] \mid \neg e \mid e + e \mid \ \mathrm{Pre}(e) \mid \ \mathrm{Suf}(e) \mid \mathrm{Inf}(e)$$

where $a \in \Sigma$, $p \in \mathcal{AP}$, $\mathrm{Pre}(e)$ and $\mathrm{Suf}(e)$ are, respectively, a shorthand for $e \cdot (\neg\emptyset)$ and $(\neg\emptyset) \cdot e$, while $\mathrm{Inf}(e)$ is a shorthand for $\mathrm{Pre}(e) + \mathrm{Suf}(e) + \mathrm{Pre}(\mathrm{Suf}(e))$.

An *infix/suffix expression* or $\mathsf{DE}_{\mathcal{H}om}$-expression restricts a prefix/suffix expression by disallowing the prefix operator Pre. Similarly, an *infix/prefix expression* or $\mathsf{BD}_{\mathcal{H}om}$-expression is a $\mathsf{BE}_{\mathcal{H}om}$-expression which does not use the suffix operator Suf, and an *infix expression* or $\mathsf{D}_{\mathcal{H}om}$-expression is a $\mathsf{BE}_{\mathcal{H}om}$-expression where both the suffix and prefix operators Suf and Pre are disallowed.

For each of the considered logics $\mathcal{F}$, there are natural linear-time translations from $\mathcal{F}$-formulas into equivalent $\mathcal{F}$-expressions, and vice versa.

Proposition 3.2. *Let $\mathcal{F} \in \{\mathsf{D}_{\mathcal{H}om}, \mathsf{BD}_{\mathcal{H}om}, \mathsf{DE}_{\mathcal{H}om}, \mathsf{BE}_{\mathcal{H}om}\}$. Then:*

- *for each $\mathcal{F}$-formula ψ, one can construct in linear time an equivalent $\mathcal{F}$-expression e_ψ;*
- *for each $\mathcal{F}$-expression e, one can construct in linear time an equivalent $\mathcal{F}$-formula ψ_e.*

Proof. The equivalence proof is based on a standard approach. We assume that $\mathcal{F}$ is $\mathsf{BE}_{\mathcal{H}om}$. The proofs for the other logics are similar. For the translation from $\mathsf{BE}_{\mathcal{H}om}$ formulas to $\mathsf{BE}_{\mathcal{H}om}$ expressions, let ψ be a $\mathsf{BE}_{\mathcal{H}om}$ formula over $\mathcal{AP}$. The formula ψ can be mapped in linear time into an equivalent $\mathsf{BE}_{\mathcal{H}om}$-expression e_ψ over $\Sigma = 2^{\mathcal{AP}}$ by applying the usual constructions for negation and disjunction. For proposition letters and the temporal operators, we have the following three rules:

- $e_p := [p]$ for each $p \in AP$;
- $e_{\langle B \rangle\, \psi} := \mathrm{Pre}(e_\psi)$;

- $e_{\langle \mathrm{E} \rangle\, \psi} := \mathrm{Suf}(e_\psi)$.

For the converse translation, let e be a $\mathsf{BE}_{\mathcal{H}om}$-expression over $\Sigma = 2^{\mathcal{AP}}$. We construct in linear time an equivalent $\mathsf{BE}_{\mathcal{H}om}$-formula ψ_e over $\mathcal{AP}$ by applying the usual constructions for empty language, negation, and union, plus the following five rules:

- $\psi_a := \bigwedge_{p \in a} p \wedge \bigwedge_{p \in \mathcal{AP} \setminus a} \neg p \wedge \neg \langle \mathrm{D} \rangle \top$ for each $a \in \Sigma$;
- $\psi_{[p]} := p$ for each $p \in \mathcal{AP}$;
- $\psi_{\mathrm{Pre}(e)} := \langle \mathrm{B} \rangle\, \psi_e$;
- $\psi_{\mathrm{Suf}(e)} := \langle \mathrm{E} \rangle\, \psi_e$;
- $\psi_{\mathrm{Inf}(e)} := \langle \mathrm{D} \rangle\, \psi_e$. $\square$

In Section 4, we will show that finite satisfiability of both $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ are PSPACE-complete (see Theorem 4.2). In particular, given a $\mathsf{DE}_{\mathcal{H}om}$ (resp.: $\mathsf{BD}_{\mathcal{H}om}$) formula on can construct in singly exponential time a DFA (resp., NFA) accepting the non-empty finite words which are models of the formula. Thus since finite satisfiability for the logic $\mathsf{D}_{\mathcal{H}om}$ is known to be PSPACE-complete [25], by Proposition 3.2, we deduce the following corollary.

Corollary 3.1. *For each $\mathcal{F} \in \{\mathsf{D}_{\mathcal{H}om}, \mathsf{BD}_{\mathcal{H}om}, \mathsf{DE}_{\mathcal{H}om}\}$, non-emptiness of $\mathcal{F}$-expressions is* PSPACE*-complete.*

It is worth noting that there is a simple and compositional automata-theoretic approach for the logics $\mathsf{BE}_{\mathcal{H}om}$ and its fragments which exploits as an intermediate step the considered subclasses of generalized $*$-free regular expressions. Indeed, the class of languages accepted by NFA is closed under Boolean operations, prefix, infix, and suffix operations. The inconvenience of this approach is that each complementation step in the compositional translation introduces a singly exponential blowup. Hence, the resulting automaton equivalent to the given formula has a non-elementary size. In Section 4, we will show that for the fragments $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ of $\mathsf{BE}_{\mathcal{H}om}$, this non-elementary blowup can be avoided by using a sophisticated automata construction.

It is well known that LTL over finite words characterizes the class of languages defined by generalized $*$-free regular expressions [30]. Since over finite words, LTL and $\mathsf{BE}_{\mathcal{H}om}$ have the same expressiveness [24], prefix/suffix expressions and generalized $*$-free regular expressions have the same expressiveness as well. Thus, by Theorem 3.1, we obtain the following expressiveness results.

Corollary 3.2. *$\mathsf{BE}_{\mathcal{H}om}$-expressions are expressively complete for the class of generalized $*$-free regular expressions. Moreover, $\mathsf{D}_{\mathcal{H}om}$-expressions are less expressive than both $\mathsf{DE}_{\mathcal{H}om}$-expressions and $\mathsf{BD}_{\mathcal{H}om}$-expressions, which in turn are less expressive than generalized $*$-free regular expressions.*

4. Satisfiability and model checking of $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ over finite linear orders

In this section, we provide an automata-theoretic approach for solving satisfiability and model checking for $\mathsf{DE}_{\mathcal{H}om}$-formulas and $\mathsf{BD}_{\mathcal{H}om}$-formulas over finite linear orders. We focus on the logic $\mathsf{DE}_{\mathcal{H}om}$. In Subsection 4.3, we will extend the results for $\mathsf{DE}_{\mathcal{H}om}$ in order to take into account the logic $\mathsf{BD}_{\mathcal{H}om}$ as well.

The proposed approach for $\mathsf{DE}_{\mathcal{H}om}$ generalizes in a non-trivial way the classical automata construction [27] for standard LTL over finite words based on the notion of Hintikka sequences. Given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ and a non-empty finite word w over $2^{\mathcal{AP}}$, we associate to each interval $[i, j]$ of w a maximal propositionally consistent set of formulas (φ*-atom*) in the syntactical closure $\mathrm{CL}(\varphi)$ of φ which, intuitively, represents the set of formulas in $\mathrm{CL}(\varphi)$ which hold at the interval $[i, j]$. According to this intuition, the word w satisfies φ iff φ is contained in the φ-atom associated with the interval $[0, |w|-1]$.

The syntactical definition of φ-atom locally captures the semantics of the Boolean connectives. In order to capture the semantics of the temporal modalities and the homogeneity assumption, we define syntactical 'semi-local' rules which allow one to:

(i) specify in a functional way the atom associated to a non-singleton interval I in terms of the atoms associated to the two proper maximal sub-intervals of I (i.e., the maximal proper prefix of I and the maximal proper suffix of I);
(ii) enforce 'termination' conditions on the atoms associated with singleton intervals of w by requiring that these atoms do not contain temporal requirements $\langle \mathrm{X} \rangle\, \psi \in \mathrm{CL}(\varphi)$.

Note that a naïve approach in the construction of an automaton for the formula φ, would be to keep track in the state of the automaton of the φ-atom associated with the interval $[0, i]$ of positions of the given word w read so far. However, this approach does not work since, as observed before, the φ-atom of the 'next' interval $[0, i+1]$ depends on both the φ-atom of $[0, i]$ (the maximal proper prefix of $[0, i+1]$) and the φ-atom of $[1, i+1]$ (the maximal proper suffix of $[0, i+1]$). Thus, a more sophisticated approach is required. In particular, for each position $0 \leq i < |w|$ of the given word w, let w_i be the prefix of w corresponding to the interval $[0, i]$. For such a prefix w_i, we consider the sequence of φ-atoms, called *row*, associated with the suffixes of w_i (corresponding to the intervals of the form $[j, i]$ for $0 \leq j \leq i$) ordered for increasing values of the length (note that in the automata-theoretic approach for LTL, the notion of row collapses to the atom associated with the current position i of the given finite word).

The previous syntactical rules (i) and (ii) guarantee monotonicity properties on the atoms of a row and the existence of a *functional relation* that given the row of a proper prefix w_i of w associated with position $0 \leq i < |w| - 1$, and the uniquely determined atom of the singleton interval $[i+1, i+1]$ of w, provides the row for the prefix of w leading to position $i+1$ (see Subsection 4.1).

As a main technical step (see Subsection 4.2), by exploiting the monotonicity of rows, we deduce for the given $\mathsf{DE}_{\mathcal{H}om}$-formula φ, the existence of an equivalence relation on the set of rows of exponential-size index satisfying three fundamental properties:

(i) the equivalence relation preserves the set of atoms visited by a row and their relative ordering along the row;
(ii) each equivalence class has a minimal representative whose length is polynomial in the size of the given formula;
(iii) the functional relation crucially preserves the equivalence between rows: this means that given two *equivalent* rows ρ and ρ' and a φ-atom A, the application of the functional relation to the two pairs (ρ, A) and (ρ', A) produces two rows which are equivalent as well.

The previous three properties lead to the construction in singly exponential time of a DFA whose states are the set of *minimal rows* and accepting the non-empty finite words over $2^{\mathcal{AP}}$ which satisfy the given formula (see Subsection 4.3). More in detail, at the current position i of the given word w, the DFA keeps track in its state of the minimal representative ρ in the equivalence class of the row associated with the prefix $[0, i]$. On reading the next input position $i+1$, starting from ρ and the uniquely determined atom of the singleton interval $[i+1, i+1]$ of w, the automaton can compute (thanks to the properties of the functional relation) the minimal representative ρ' in the equivalence class of the row associated with the prefix $[0, i+1]$. The row ρ' represents the next state in the run of the automaton. The run is accepting if the last atom of the last state contains the formula φ.

We now proceed with the technical details.

We first introduce some basic definitions and notation which will be extensively used in the following. Given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ, we define the *closure of* φ, denoted by $\mathrm{CL}(\varphi)$, as the set of all sub-formulas ψ of φ and of their negations $\neg\psi$ (we identify $\neg\neg\psi$ with ψ). A φ*-atom* A is a subset of $\mathrm{CL}(\varphi)$ satisfying the following requirements:

- for every $\psi \in \mathrm{CL}(\varphi)$, $\psi \in A$ if and only if $\neg\psi \notin A$, and
- for every $\psi_1 \vee \psi_2 \in \mathrm{CL}(\varphi)$, $\psi_1 \vee \psi_2 \in A$ if and only if $\psi_1 \in A$ or $\psi_2 \in A$.

We denote by $\mathcal{A}_\varphi$ the set of all φ-atoms. Its cardinality is clearly bounded by $2^{|\varphi|}$. We now consider non-empty finite words over $2^{\mathcal{AP}}$ equipped with a mapping assigning to each interval a φ-atom.

Definition 4.1 *(φ-word structures and fulfilling φ-word structures).* Let φ be a $\mathsf{DE}_{\mathcal{H}om}$-formula. A φ*-word structure* $\mathcal{W}$ is a pair $\mathcal{W} = (w, \mathcal{L})$ consisting of a non-empty finite word over $2^{\mathcal{AP}}$ and a mapping $\mathcal{L}$ assigning to each interval of w (i.e., an interval in the homogeneous interval model $\mathcal{M}(w)$) a φ-atom such that for each position $0 \leq i < |w|$, $\mathcal{L}([i, i]) \cap \mathcal{AP} = w[i]$. The φ-word structure $\mathcal{W} = (w, \mathcal{L})$ is *fulfilling* if for each interval I of w (we also say that I is an interval of $\mathcal{W}$) and for each $\psi \in \mathrm{CL}(\varphi)$, it holds that $\psi \in \mathcal{L}(I)$ if and only if $\mathcal{M}(w), I \models \psi$.

Evidently, for each non-empty finite word w over $2^{\mathcal{AP}}$, there exists a unique fulfilling φ-word structure associated with w. Let $\mathcal{W} = (w, \mathcal{L})$ be a φ-word structure. For each interval $[i, j]$ of $\mathcal{W}$, we write $\mathcal{L}(i, j)$ to mean $\mathcal{L}([i, j])$. For each $0 \leq i < |w|$, the *i-row of* $\mathcal{W}$ is the sequence ρ_i of φ-atoms given by

$$\rho_i := \mathcal{L}(i, i) \cdot \mathcal{L}(i-1, i) \cdots \mathcal{L}(0, i)$$

Hence, the i-row ρ_i is the sequence of atoms labeling the suffixes of the prefix of w until position i ordered for increasing values of their length (see Fig. 3 for a graphical hint).

4.1. Characterization of fulfilling φ-word structures

In this section, given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ, we provide a characterization of fulfilling φ-word structures $\mathcal{W}$ in terms of a 'syntactical' functional relation between adjacent $\mathcal{W}$-rows.

For a φ-atom A and $X \in \{D, E\}$, we consider the following sets:

- $Req_X(A) := \{\psi \in \mathrm{CL}(\varphi) : \langle X \rangle\, \psi \in A\}$ (*temporal requests* of A);
- $Obs_X(A) := \{\psi \in A : \langle X \rangle\, \psi \in \mathrm{CL}(\varphi)\}$ (*observables* of A).

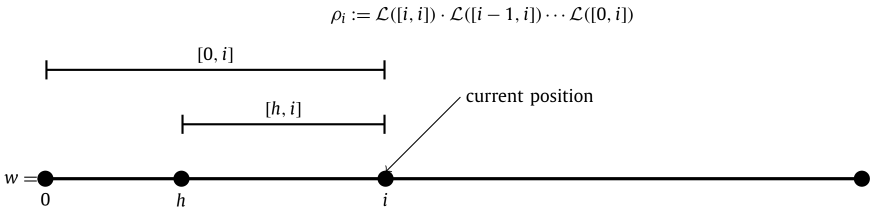


Fig. 3. The i-row ρ_i of a φ-word structure $(w, \mathcal{L})$.

As previously mentioned, a φ-atom A associated with an interval I represents the set of formulas in $\mathsf{CL}(\varphi)$ which hold at I. A temporal request associated with the φ-atom is a formula ψ in the closure of φ which must hold in a suitable subinterval of I: a suffix of I if $\langle \mathrm{E}\rangle\, \psi \in A$ or an infix of I if $\langle \mathrm{E}\rangle\, \psi \in A$. Dually, if $\psi \in A$ and $\langle \mathrm{E}\rangle\, \psi \in \mathsf{CL}(\varphi)$ (resp., $\langle \mathrm{D}\rangle\, \psi \in \mathsf{CL}(\varphi)$) meaning that ψ is an observable of A, then the formula $\langle \mathrm{E}\rangle\, \psi$ holds in every superinterval I' of I such that I is a suffix (resp., an infix) of I'.

Example 4.1. With reference to the Kripke structure $\mathcal{K}_{Sched}$ of Example 2.1, let us consider the interval I having $u_2 v_1 v_1 v_1 u_1$ as underlying path. Let $Activity_i^E$ be the $\mathsf{DE}_{\mathcal{H}om}$-formula defined as $\langle \mathrm{E}\rangle\, \overline{p_i} \wedge \neg \langle \mathrm{D}\rangle (\neg \langle \mathrm{E}\rangle\, \overline{p_i} \wedge \neg p_i)$ characterizing a subpath of $\mathcal{K}_{Sched}$ corresponding with the use and unlock of the shared resource by the i-th process (i.e. the counterpart of the $\mathsf{BD}_{\mathcal{H}om}$-formula $Activity_i$ defined in Example 2.2). If we consider the $\mathsf{DE}_{\mathcal{H}om}$-formula $\varphi := \langle \mathrm{D}\rangle\, Activity_1^E$ we have:

1. The φ-atom A_I associated with interval I is the set $\{\varphi, \neg Activity_1^E, \neg p_1, \neg\overline{p}_1, \langle \mathrm{E}\rangle\, \overline{p}_1, \langle \mathrm{D}\rangle(\neg \langle \mathrm{E}\rangle\, \overline{p_1} \wedge \neg p_1)\}$;
2. $Req_D(A_I) = \{Activity_1^E, \neg \langle \mathrm{E}\rangle\, \overline{p_1} \wedge \neg p_1\}$ and $Req_E(A_I) = \{\overline{p_1}\}$;
3. $Obs_D(A_I) = Obs_E(A_I) = \emptyset$.

Considering the maximal suffix I_E of I with underlying path $v_1 v_1 v_1 u_1$ we have:

1. The φ-atom A_{I_E} associated with interval I_E is the set $\{\varphi, Activity_1^E, \neg p_1, \neg\overline{p}_1, \langle \mathrm{E}\rangle\, \overline{p}_1, \neg \langle \mathrm{D}\rangle(\neg \langle \mathrm{E}\rangle\, \overline{p_1} \wedge \neg p_1)\}$;
2. $Req_D(A_{I_E}) = \{Activity_1^E\}$ and $Req_E(A_{I_E}) = \{\overline{p_1}\}$;
3. $Obs_D(A_{I_E}) = \{Activity_1^E\}$ and $Obs_E(A_{I_E}) = \emptyset$.

Considering the maximal prefix I_B of I with underlying path $u_2 v_1 v_1 v_1$ we have:

1. The φ-atom A_{I_B} associated with interval I_B is the set $\{\neg\varphi, \neg Activity_1^E, \neg p_1, \neg\overline{p}_1, \neg \langle \mathrm{E}\rangle\, \overline{p}_1, \langle \mathrm{D}\rangle(\neg \langle \mathrm{E}\rangle\, \overline{p_1} \wedge \neg p_1)\}$;
2. $Req_D(A_{I_B}) = \{\neg \langle \mathrm{E}\rangle\, \overline{p_1} \wedge \neg p_1\}$ and $Req_E(A_{I_B}) = \emptyset$;
3. $Obs_D(A_{I_B}) = \{\neg \langle \mathrm{E}\rangle\, \overline{p_1} \wedge \neg p_1\}$ and $Obs_E(A_{I_B}) = \emptyset$.

The following proposition states that, once the proposition letters of a φ-atom A and its temporal requests have been fixed, A gets unambiguously determined.

Proposition 4.1. *Let φ be a* DE*-formula. Given a set $R_D \subseteq \{\psi \mid \langle \mathrm{D}\rangle\, \psi \in \mathsf{CL}(\varphi)\}$, a set $R_E \subseteq \{\psi \mid \langle \mathrm{E}\rangle\, \psi \in \mathsf{CL}(\varphi)\}$, and a set $P \subseteq \mathsf{CL}(\varphi) \cap \mathcal{AP}$, there exists a unique φ-atom A that satisfies $Req_D(A) = R_D$, $Req_E(A) = R_E$, and $A \cap \mathcal{AP} = P$.*

Proof. First, notice that the notion of φ-atom imposes requirements only on the Boolean connectives. Hence, the existence of a φ-atom A that satisfies $Req_D(A) = R_D$, $Req_E(A) = R_E$, and $A \cap \mathcal{AP} = P$ is ensured. Now, let A and A' be two φ-atoms such that $Req_D(A) = Req_D(A')$, $Req_E(A) = Req_E(A')$, and $A \cap \mathcal{AP} = A' \cap \mathcal{AP}$. It remains to show that $A = A'$, i.e., for each $\psi \in \mathsf{CL}(\varphi)$, $\psi \in A$ iff $\psi \in A'$. The proof is by induction on the structure of $\psi \in \mathsf{CL}(\varphi)$. If ψ is an atomic proposition or a formula of the form $\langle \mathrm{X}\rangle\, \psi'$ for some $X \in \{D, E\}$, the result directly follows from the hypothesis. Otherwise, ψ is either of the form $\psi_1 \vee \psi_2$ or of the form $\neg\psi'$. For these cases, the result directly follows from the induction hypothesis and the definition of φ-atom. □

Now, we show how the φ-atom associated with a non-singleton interval I can be functionally derived from the φ-atoms associated with the maximal proper prefix I_B and the maximal proper suffix I_E of I.

Definition 4.2. Let A_B and A_E be two φ-atoms. We denote by $succ_\varphi(A_B, A_E)$ the unique φ-atom A whose propositions, D-temporal requests and E-temporal requests satisfy:

(i) $A \cap \mathcal{AP} = A_B \cap A_E \cap \mathcal{AP}$,
(ii) $Req_D(A) = Req_D(A_B) \cup Obs_D(A_B) \cup Req_D(A_E) \cup Obs_D(A_E)$, and

(iii) $Req_E(A) = Req_E(A_E) \cup Obs_E(A_E)$.

Notice that the first point of Definition 4.2 is an immediate consequence of the homogeneity assumption: a proposition p holds at a non-singleton interval I iff it holds at the maximal proper sub-intervals of I (i.e., the maximal proper prefix and the maximal proper suffix of I). Since all the possible proper infixes of a non-singleton interval I are the union of the infixes of the maximal proper prefix I_B of I and the infixes of the maximal proper suffix I_E of I, the D-temporal requests in the φ-atom A of I (second point of Definition 4.2) are obtained from the D-temporal requests and the D-observables of the φ-atoms A_B and A_E of I_B and I_E, respectively. As for the third point, the E-temporal requests in the φ-atom A of I correspond to the E-temporal requests and the E-observables of the φ-atom A_E of I_E. With reference to Example 4.1, notice that $A_I = succ_\varphi(A_{I_B}, A_{I_E})$.

Definition 4.2 can be exploited to label a fulfilling φ-word structure $\mathcal{W}$, namely, to determine the φ-atoms labeling all the intervals $[i, j]$ of $\mathcal{W}$, starting from the singleton ones. The idea is the following: if two φ-atoms A_B and A_E label respectively the greatest proper prefix $[i, j-1]$ and the greatest proper suffix $[i+1, j]$ of the same non-singleton interval $[i, j]$, then the atom A labeling interval $[i, j]$ is unique, and it is precisely the one given by $succ_\varphi(A_B, A_E)$.

Lemma 4.1. *Let $\mathcal{W} = (w, \mathcal{L})$ be a φ-word structure. Then, $\mathcal{W}$ is fulfilling* if and only if *for each interval $[i, j]$ of $\mathcal{W}$, it holds that:*

- *if $i < j$, then $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$;*
- *if $i = j$, then $Req_D(\mathcal{L}(i, j)) = \emptyset$ and $Req_E(\mathcal{L}(i, j)) = \emptyset$.*

Proof. ($\Rightarrow$) Assume that $\mathcal{W}$ is fulfilling. Hence, for each interval $[i, j]$ of $\mathcal{W}$, $\mathcal{L}(i, j)$ is the set of formulas $\psi \in \mathrm{CL}(\varphi)$ such that $\mathcal{M}(w), [i, j] \models \psi$ (recall that $\mathcal{M}(w)$ is the homogeneous interval model associated with the word w). Thus, if $i = j$, then $Req_D(\mathcal{L}(i, j)) = \emptyset$ and $Req_E(\mathcal{L}(i, j)) = \emptyset$. Otherwise, $i < j$ and being $\mathcal{M}(w)$ homogeneous, we have that $\mathcal{L}(i, j) \cap \mathcal{AP} = \mathcal{L}(i, j-1) \cap \mathcal{L}(i+1, j) \cap \mathcal{AP}$. Moreover, by the semantics of DE, the following holds:

- for each $\langle \mathrm{D} \rangle\, \psi \in \mathrm{CL}(\varphi)$, $\langle \mathrm{D} \rangle\, \psi \in \mathcal{L}(i, j)$ if and only if $\langle \mathrm{D} \rangle\, \psi \in \mathcal{L}(i, j-1)$, or $\psi \in \mathcal{L}(i, j-1)$, or $\langle \mathrm{D} \rangle\, \psi \in \mathcal{L}(i+1, j)$, or $\psi \in \mathcal{L}(i+1, j)$;
- for each $\langle \mathrm{E} \rangle\, \psi \in \mathrm{CL}(\varphi)$, $\langle \mathrm{E} \rangle\, \psi \in \mathcal{L}(i, j)$ if and only if $\langle \mathrm{E} \rangle\, \psi \in \mathcal{L}(i+1, j)$ or $\psi \in \mathcal{L}(i+1, j)$.

This means that $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$, and the result follows.

($\Leftarrow$) Assume that for every interval $[i, j]$ of $\mathcal{W}$, we have:

- if $i < j$, then $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$;
- if $i = j$, then $Req_D(\mathcal{L}(i, j)) = \emptyset$ and $Req_E(\mathcal{L}(i, j)) = \emptyset$.

We need to prove that $\mathcal{W}$ is fulfilling. Let $[i, j]$ be an interval of $\mathcal{W}$ and $\psi \in \mathrm{CL}(\varphi)$. We prove by induction on the structure of ψ that $\psi \in \mathcal{L}(i, j)$ if and only if $\mathcal{M}(w), [i, j] \models \psi$. Hence, the result follows.

- $\psi \in \mathcal{AP}$: we have to show that $\mathcal{L}(i, j) \cap \mathcal{AP} = \bigcap_{h \in [i,j]} \mathcal{L}(h, h) \cap \mathcal{AP}$. The proof is by a double induction on $j - i \geq 0$. If $i = j$, the property trivially holds. Let us assume now that $j - i > 0$. Since $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$, by Condition (i) of Definition 4.2 and the induction hypothesis, we obtain that $\mathcal{L}(i, j) \cap \mathcal{AP} = \bigcap_{h \in [i+1,j]} \mathcal{L}(h, h) \cap \bigcap_{h \in [i,j-1]} \mathcal{L}(h, h) \cap \mathcal{AP}$. Hence, the result directly follows.
- $\psi = \neg \psi_1$ or $\psi = \psi_1 \vee \psi_2$: for these cases, the result directly follows from the induction hypothesis and the definition of φ-atom (recall that $\mathcal{L}(i, j)$ is a φ-atom).
- $\psi = \langle \mathrm{D} \rangle\, \psi_1$ or $\psi = \langle \mathrm{E} \rangle\, \psi_1$: the proof is by a double induction on $j - i \geq 0$. If $i = j$, then $\mathcal{M}(w), [i, j] \not\models \psi$, $Req_D(\mathcal{L}(i, j)) = \emptyset$, and $Req_E(\mathcal{L}(i, j)) = \emptyset$. Hence, the result follows. Now, assume that $j - i > 0$. First, let us consider the case where $\psi = \langle \mathrm{D} \rangle\, \psi_1$. Since $\mathcal{L}(i, j) = succ_\varphi(\mathcal{L}(i, j-1), \mathcal{L}(i+1, j))$, by Condition (ii) of Definition 4.2 and the induction hypothesis, we have that

  $\langle \mathrm{D} \rangle\, \psi_1 \in \mathcal{L}(i, j) \;\Leftrightarrow$
  either $\mathcal{M}(w), [i, j-1] \models \psi_1 \vee \langle \mathrm{D} \rangle\, \psi_1$ *or*
  $\mathcal{M}(w), [i+1, j] \models \psi_1 \vee \langle \mathrm{D} \rangle\, \psi_1 \;\Leftrightarrow$
  $\mathcal{M}(w), [i, j] \models \langle \mathrm{D} \rangle\, \psi_1$.

  Now, let us consider the case where $\psi = \langle \mathrm{E} \rangle\, \psi_1$. By Condition (iii) of Definition 4.2 and the induction hypothesis, we have that

$\langle \mathrm{E} \rangle\, \psi_1 \in \mathcal{L}(i, j) \;\Leftrightarrow$
either $\mathcal{M}(w), [i+1, j] \models \langle \mathrm{E} \rangle\, \psi_1$ *or*
$\mathcal{M}(w), [i+1, j] \models \psi_1 \;\Leftrightarrow$
$\mathcal{M}(w), [i, j] \models \langle \mathrm{E} \rangle\, \psi_1$,

and the result follows. □

We now introduce the abstract notion of *φ-rows*, finite sequences of φ-atoms satisfying 'syntactical' adjacency requirements which capture the behavior of $\mathcal{W}$-rows in fulfilling φ-word structures $\mathcal{W}$. Intuitively, a φ-row is a sequence of φ-atoms for an increasing sequence of suffixes of an interval.

Definition 4.3 *(φ-row).* A non-empty finite sequence ρ of φ-atoms is a *φ-row* if, for all $0 \leq i < |\rho| - 1$, the following holds:

- $(\rho[i] \cap \mathcal{AP}) \supseteq (\rho[i+1] \cap \mathcal{AP})$,
- $Req_D(\rho[i+1]) \supseteq Req_D(\rho[i]) \cup Obs_D(\rho[i])$ and
- $Req_E(\rho[i+1]) = Req_E(\rho[i]) \cup Obs_E(\rho[i])$.

The φ-row ρ is *initialized* if $Req_D(\rho[0]) = \emptyset$ and $Req_E(\rho[0]) = \emptyset$.

Intuitively, if $\rho[i]$ and $\rho[i+1]$ are the φ-atoms associated with two *adjacent* suffixes J_i and J_{i+1}, respectively, of an interval I such that J_i is contained in J_{i+1} (hence, J_i is also the maximal proper suffix of J_{i+1}), then the first point in Definition 4.3 asserts that each proposition holding at J_{i+1} holds at J_i too. Moreover, being J_i a proper infix of J_{i+1}, the D-temporal requests in the φ-atom $\rho[i+1]$ of J_{i+1} include the D-temporal requests and the D-observables of the φ-atom $\rho[i]$ of J_i (second point in Definition 4.3). Furthermore, since J_i is the maximal proper suffix of J_{i+1}, the E-temporal requests in $\rho[i+1]$ correspond to the E-temporal requests and the E-observables of $\rho[i]$ (third point in Definition 4.3). Finally, notice that the initialization requirement in Definition 4.3 ensures that the first atom in the φ-row ρ is associated with a singleton interval.

We denote by $\mathcal{R}ows_\varphi$ the set of all possible φ-rows. We observe that the sequence of atoms along a φ-row $A_0 \cdots A_n$ has a monotonic behavior and the number of distinct occurring atoms is linearly bounded by the size of φ. Indeed, by Definition 4.3, a φ-row ρ presents three monotonic sequences:

(i) the decreasing sequence of atomic propositions $(A_0 \cap \mathcal{AP}) \supseteq (A_1 \cap \mathcal{AP}) \supseteq \ldots \supseteq (A_n \cap \mathcal{AP})$;
(ii) the two increasing sequences of temporal requests $Req_D(A_0) \subseteq Req_D(A_1) \subseteq \ldots \subseteq Req_D(A_n)$ and $Req_E(A_0) \subseteq Req_E(A_1) \subseteq \ldots \subseteq Req_E(A_n)$.

The number of distinct elements in each sequence is bounded by $|\varphi|$ (w.l.o.g., we assume that $|\mathcal{AP}| \leq |\varphi|$, i.e. we can consider only the propositional letters actually occurring in φ). Since a set of temporal requests and a set of proposition letters uniquely determine a φ-atom (Proposition 4.1), any φ-row may feature at most $3|\varphi|$ distinct atoms, i.e., $n \leq 3|\varphi|$. Since in a fulfilling φ-word structure there are no temporal requests in the atoms labeling the singleton intervals, by Definition 4.2 and Lemma 4.1, we obtain the following result.

Lemma 4.2. *The following statements hold:*

1. *The number of distinct atoms in a φ-row $\rho = A_0 \cdots A_n$ is at most $3|\varphi|$. Moreover, for all $0 \leq i < j < |\rho|$, if $A_i = A_j$, then $A_k = A_i$ for all $k \in [i, j]$.*
2. *Each $\mathcal{W}$-row of a fulfilling φ-word structure $\mathcal{W}$ is an initialized φ-row.*

We now generalize the successor function $succ_\varphi$ to φ-rows. Given a φ-row ρ and a φ-atom A, $succ_\varphi(\rho, A)$ returns the φ-row of length $|\rho| + 1$ whose first atom is A and the other atoms are obtained by a component-wise application of $succ_\varphi$ starting from A and the first atom of ρ.

Definition 4.4. Given a φ-atom A and a φ-row ρ with $|\rho| = n$, the *A-successor of ρ*, denoted by $succ_\varphi(\rho, A)$, is the sequence $B_0 \cdots B_n$ of φ-atoms inductively defined as follows:

- $B_0 = A$ and
- $B_{i+1} = succ_\varphi(\rho[i], B_i)$ for all $0 \leq i < n$.

Intuitively, if ρ is the i-row

$$\rho = \mathcal{L}(i, i) \cdot \mathcal{L}(i-1, i) \cdots \mathcal{L}(0, i)$$

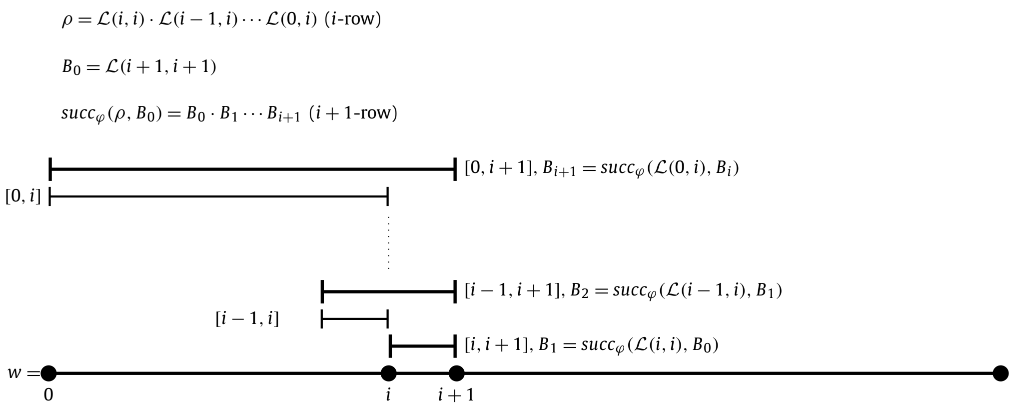


Fig. 4. Functional relation between adjacent rows in fulfilling φ-word structures.

of a fulfilling φ-word structure $\mathcal{W} = (w, \mathcal{L})$ for some $0 \le i < |w| - 1$, and A is the φ-atom associated with the singleton interval $[i+1, i+1]$ (i.e., $A = \mathcal{L}(i+1, i+1)$), then the A-successor $succ_\varphi(\rho, A)$ of ρ represents the $(i+1)$-row of $\mathcal{W}$, i.e.

$$succ_\varphi(\rho, A) = \mathcal{L}(i+1, i+1) \cdot \mathcal{L}(i, i+1) \cdots \mathcal{L}(0, i+1).$$

Indeed, by Lemma 4.1, the φ-atom associated with a non-singleton interval I is given by $succ_\varphi(A_B, A_E)$, where A_B and A_E are the φ-atoms associated with the maximal proper prefix of I and the maximal proper suffix of I, respectively. In particular, for all $j \in [0, i]$, $\mathcal{L}(j, i+1) = succ_\varphi(\mathcal{L}(j, i), \mathcal{L}(j+1, i+1))$ (see Fig. 4 for a graphical intuition). From Definitions 4.2 and 4.4, we can easily derive the following lemma.

Lemma 4.3. *Let ρ be a φ-row and A be a φ-atom. Then, $succ_\varphi(\rho, A)$ is a φ-row. Moreover, if ρ is of the form $\rho = \rho_1 \cdot \rho_2$, then $succ_\varphi(\rho, A) = succ_\varphi(\rho_1, A) \star succ_\varphi(\rho_2, A_1)$, where A_1 is the last φ-atom of $succ_\varphi(\rho_1, A)$.*

By Lemma 4.1, consecutive rows in fulfilling φ-word structures respect the successor function. In particular, by Lemmata 4.1 and 4.2, we obtain the following characterization result.

Corollary 4.1 *(Characterization of fulfilling φ-word structures). Let $\mathcal{W} = (w, \mathcal{L})$ be a φ-word structure such that for all $0 \le i < |w|$, $Req_D(\mathcal{L}(i, i)) = \emptyset$ and $Req_E(\mathcal{L}(i, i)) = \emptyset$. Then, $\mathcal{W}$ is fulfilling* if and only if, *for each $0 \le j < |w| - 1$, $\rho_{j+1} = succ_\varphi(\rho_j, \rho_{j+1}[0])$, where ρ_i is the i-row of $\mathcal{W}$ for all $0 \le i < |w|$.*

4.2. Finite abstractions of rows

We describe now the core of the automata-theoretic approach we exploit to solve the satisfiability and model checking problems for $\mathsf{DE}_{\mathcal{H}om}$.

Given a $\mathsf{DE}_{\mathcal{H}om}$ formula φ, we introduce an equivalence relation $\sim_\varphi$ of finite index over the infinite set $\mathcal{R}ows_\varphi$ of φ-rows whose number of equivalence classes is singly exponential in the size of φ and such that each equivalence class has a representative whose length is polynomial in the size of φ. The equivalence relation preserves (i) the set of atoms visited by a φ-row and their relative ordering along the row, and (ii) the property of a φ-row to be initialized. Moreover, as a crucial result we show that the successor function preserves the equivalence between φ-rows.

In the following, we denote by $N_{D,\varphi}$ the number of D-temporal requests in $\mathrm{CL}(\varphi)$ plus one, i.e., $|\{\psi \mid \langle \mathrm{D} \rangle\, \psi \in \mathrm{CL}(\varphi)\}| + 1$, and by $N_{E,\varphi}$ the number of E-temporal requests in $\mathrm{CL}(\varphi)$ plus one, i.e., $|\{\psi \mid \langle \mathrm{E} \rangle\, \psi \in \mathrm{CL}(\varphi)\}| + 1$. Note that $1 \le N_{D,\varphi} \le |\varphi|$ and $1 \le N_{E,\varphi} \le |\varphi|$.

The equivalence relation $\sim_\varphi$ is based on the monotonicity properties of φ-rows: along a φ-row, the sets of atomic propositions form a decreasing sequence, while the sets of D-temporal requests and the sets of E-temporal requests form two increasing sequences. Thus, a φ-row can be factorized into a concatenation of segments such that the number of segments is linearly bounded in the size of φ, and along a segment (called *uniform φ-row*), the φ-atoms have the same propositional letters and the same D-temporal requests (*uniform factorization*). Note that in a φ-row, the E-temporal requests of a non-first atom are completely specified by the previous atom along the row. Thus, two *uniform* φ-rows which have the same first atom and the same length coincide. Two uniform φ-rows ρ and ρ' are then defined to be equivalent when they have the same first atom A and their lengths either (i) coincide or (ii) are both greater than a parameter depending on the number of D-temporal requests in A and the overall number $N_{E,\varphi}$ of E-temporal requests in $\mathrm{CL}(\varphi)$ (such a parameter is called *A-rank*). In the second case, we show that the two uniform φ-rows ρ and ρ' are similar being of the form $\rho'' \cdot B^k$ and $\rho'' \cdot B^h$, respectively, for some φ-atom B, where $h, k \ge 1$ and $|\rho''| = N_{E,\varphi}$. Two arbitrary φ-rows are then defined to be equivalent when their factorizations are similar: i.e., they have the same number of segments and corresponding segments are equivalent uniform φ-rows.

We now formally define the equivalence relation $\sim_\varphi$ over $\mathcal{R}ows_\varphi$.

Definition 4.5 *(Uniform φ-rows).* A φ-row ρ is *uniform* if for all $0 \leq i < |\rho| - 1$, $(\rho[i] \cap \mathcal{AP}) = (\rho[i+1] \cap \mathcal{AP})$ and $Req_D(\rho[i+1]) = Req_D(\rho[i])$.

Thus, in a uniform φ-row ρ, all the atoms occurring in ρ have the same propositional letters and the same D-temporal requests. We represent an arbitrary φ-row ρ in the form $\rho = \rho_1 \cdots \rho_k$ (*uniform factorization of* ρ) where $\rho_1, \ldots, \rho_k$ are uniform φ-rows and $\rho_i \cdot \rho_{i+1}[0]$ is not uniform for all $1 \leq i < k$. Intuitively, the uniform factorization of a φ-row ρ is the unique factorization of ρ consisting of maximal uniform sub-rows of ρ. By the monotonicity properties of a φ-row ρ (see Definition 4.3 and Lemma 4.2(1)), the number k of uniform segments in the uniform factorization of ρ is linearly bounded in the size of φ.

Lemma 4.4. *The following statements hold:*

1. *Let ρ be a φ-row with uniform factorization $\rho_1 \cdots \rho_k$. Then, k is at most $3|\varphi|$.*
2. *Let ρ be a uniform φ-row such that $|\rho| > N_{E,\varphi}$. Then, ρ is of the form $\rho = \rho' \cdot B^m$ where $|\rho'| = N_{E,\varphi}$, $m \geq 1$, and B is the last atom of ρ'.*
3. *Given a φ-atom A and an integer $n \geq 1$, there is at most one uniform φ-row ρ such that $\rho[0] = A$ and $|\rho| = n$.*

Proof. By Lemma 4.2(1), the number k of segments in the uniform factorization of the φ-row ρ is at most the number of distinct atoms in ρ. Hence, k is at most $3|\varphi|$, and Property (1) directly follows.

As for Property (2), let ρ be a uniform φ-row such that $|\rho| > N_{E,\varphi}$. By definitions of φ-row and uniform φ-row, for all $0 \leq i < |\rho| - 1$, *either* $Req_E(\rho[i]) \subset Req_E(\rho[i+1])$ *or* $\rho[j] = \rho[i]$ for all $i \leq j < |\rho|$. Thus, since for a φ-atom A, $0 \leq |Req_E(A)| < N_{E,\varphi}$, we obtain that ρ is of the form $\rho = \rho' \cdot B^m$ where $|\rho'| = N_{E,\varphi}$, $m \geq 1$, and B is the last atom of ρ'.

As for Property (3), it suffices to observe that in a uniform φ-row all the atoms have the same propositional letters and the same D-temporal requests. Thus, since the E-temporal requests of a non-first atom in a φ-row are completely specified by the previous atom along the row (third point in Definition 4.3), the result follows. $\square$

We now introduce the notion of *rank* of a φ-atom. We first define the *D-rank of a φ-atom A*, written $rank_D(A)$, as $N_{D,\varphi} - |Req_D(A)|$. Clearly, $1 \leq rank_D(A) \leq |\varphi|$. The *rank of a φ-atom A*, written $rank(A)$, is defined as $rank_D(A) \cdot N_{E,\varphi}$ (i.e. the product of the D-rank of A with the increment of the overall number of suffix temporal requests in φ). Clearly, $rank(A) \geq N_{E,\varphi}$ and $1 \leq rank(A) \leq |\varphi|^2$. In particular, by Definition 4.3, for every φ-row $\rho = A_0 \cdots A_n$, we have $rank_D(A_0) \geq \ldots \geq rank_D(A_n)$ and $rank(A_0) \geq \ldots \geq rank(A_n)$. Moreover, in a uniform φ-row ρ, all the atoms occurring in ρ have the same rank, and we write $rank(\rho)$ for such a rank. Similarly, the notation $rank_D(\rho)$ for a uniform φ-row ρ refers to the common D-rank of the atoms occurring in ρ.

Definition 4.6 *(Equivalence relation $\sim_\varphi$).* Given two *uniform* φ-rows ρ and ρ', we say that ρ and ρ' are *equivalent*, written $\rho \sim_\varphi \rho'$, if the following conditions hold:

- $\rho[0] = \rho'[0]$ (hence $rank(\rho) = rank(\rho')$), and
- either $|\rho| = |\rho'|$ or both $|\rho|$ and $|\rho'|$ are strictly greater than $rank(\rho)$.

Two arbitrary φ-rows ρ and ρ' with uniform factorizations $\rho_1 \cdots \rho_k$ and $\rho'_1 \cdots \rho'_{k'}$, respectively, are *equivalent*, written $\rho \sim_\varphi \rho'$, if $k = k'$ and $\rho_i \sim_\varphi \rho'_i$ for all $i \in [1, k]$. A *minimal φ-row* is a φ-row whose uniform factorization $\rho_1 \cdots \rho_k$ is such that $|\rho_i| \in [1, rank(\rho_i) + 1]$, for each $1 \leq i \leq k$.

The factors $N_{E,\varphi}$ and $rank_D(A)$ in the definition of the rank $rank(A) = rank_D(A) \cdot N_{E,\varphi}$ of a φ-atom A are used to handle the E-temporal requests and the D-temporal requests, respectively. In particular, in moving from a *uniform* φ-row ρ to a successor φ-row $\rho' = succ_\varphi(\rho, A)$, we will show that the length of each non-last segment in the uniform factorization of ρ' is at most $|N_{E,\varphi}|$. Moreover, since the D-rank decreases along a φ-row and $rank_D(\rho'[0]) \leq rank_D(\rho)$, we will show that if $|\rho| > rank(\rho)$, than the last segment ρ_s in the uniform factorization of ρ' satisfies $|\rho_s| > rank(\rho_s)$. This result (see Lemma 4.8(1)) is crucial for ensuring that the successor function $succ_\varphi$ preserves the equivalence between φ-rows. It is worth noting that the E-temporal requests of the φ-row ρ' are not related to the φ-row ρ. Thus, we use the unique integer $N_{E,\varphi}$, which is independent of the specific φ-atom, in order to take into account the E-temporal requests.

Since for a φ-atom A, $rank(A) \geq N_{E,\varphi}$, by Lemma 4.4, we easily deduce the following result.

Proposition 4.2 *(Equivalent uniform φ-rows). Let ρ and ρ' be two uniform φ-rows. If ρ and ρ' are equivalent, then* either $\rho = \rho'$, or *ρ and ρ' are of the form $\rho'' \cdot B^m$ and $\rho'' \cdot B^k$, respectively, where $|\rho''| = N_{E,\varphi}$, $m, k \geq 1$, and B is the last atom of ρ''. Moreover, there exists a unique minimal φ-row which is equivalent to ρ.*

Proof. We focus on the second part of Proposition 4.2 (the proof of the first part being similar). Let ρ be a uniform φ-row. We distinguish two cases:

- $|\rho| \leq rank(\rho)$. Hence, ρ is minimal. By Lemma 4.4(3), two equivalent uniform φ-rows which have the same length coincide. Hence, by Definition 4.6 and since $|\rho| \leq rank(\rho)$, the equivalence class of ρ is a singleton and the result follows.
- $|\rho| > rank(\rho)$. By Definition 4.6, the prefix ρ_{min} of ρ of length $rank(\rho)+1$ is a minimal φ-row equivalent to ρ. Moreover, by Lemma 4.4(3), it easily follows that ρ_{min} is the unique minimal φ-row equivalent to ρ, and the result follows. □

By Definition 4.6, Lemma 4.4 and Proposition 4.2, the number of minimal φ-rows is finite and each equivalence class of $\sim_\varphi$ contains a unique minimal φ-row. Thus, the equivalence relation $\sim_\varphi$ has finite index coinciding with the number of minimal φ-rows. This number is roughly bounded by the number of all the possible uniform factorizations of the form $\rho_1 \cdots \rho_k$ where $k \leq 3|\varphi|$ and for all $i \in [1, k]$, $|\rho_i|$ ranges from 1 to $|\varphi|^2$ and ρ_i is the unique uniform φ-row of length $|\rho_i|$ having as first atom $\rho_i[0]$. Since the number of possible φ-atoms is $2^{|\varphi|}$, the number of distinct equivalence classes of $\sim_\varphi$ is bounded by $(2^{|\varphi|} \cdot |\varphi|^2)^{3|\varphi|} \leq 2^{9|\varphi|^2}$, which is exponential in the length of the input formula φ. Moreover, each minimal φ-row has length at most $3|\varphi|^3$ (this is because, as explained before, the number of segments in a uniform factorization is at most $3|\varphi|$, and the length of a segment in the uniform factorization of a minimal φ-row is at most $|\varphi|^2$). Hence, we obtain the following result.

Lemma 4.5. *Each equivalence class of $\sim_\varphi$ contains a unique minimal φ-row. The length of a minimal φ-row is at most $3|\varphi|^3$, and the number of minimal φ-rows is at most $2^{9|\varphi|^2}$.*

We observe that if we replace a segment (sub-row) of a φ-row by an equivalent one, we obtain a φ-row which is equivalent to the original one.

Lemma 4.6. *Let $\rho_1, \rho_1', \rho_2, \rho_2'$ be φ-rows such that $\rho_1 \sim_\varphi \rho_1'$ and $\rho_2 \sim_\varphi \rho_2'$. If $\rho_1 \star \rho_2$ and $\rho_1' \star \rho_2'$ are defined, then $\rho_1 \star \rho_2 \sim_\varphi \rho_1' \star \rho_2'$.*

Proof. We consider the case where ρ_1 and ρ_2 are uniform, hence, ρ_1' and ρ_2' are uniform as well. The general case easily follows from the considered case. By hypothesis $\rho_1 \star \rho_2$ and $\rho_1' \star \rho_2'$ are defined. This entails that $\rho_1 \star \rho_2$ and $\rho_1' \star \rho_2'$ are uniform as well. Thus since $\rho_1 \sim_\varphi \rho_1'$ and $\rho_2 \sim_\varphi \rho_2'$, by Definition 4.6, we obtain that $\rho_1 \star \rho_2$ and $\rho_1' \star \rho_2'$ have the same first atom A and indicated by m (resp.: m') the length of $\rho_1 \star \rho_2$ (resp.: $\rho_1' \star \rho_2'$), it holds that either $m = m'$, or both $m > rank(A)$ and $m' > rank(A)$. Hence, the result follows. □

We now show that the successor function $succ_\varphi$ on φ-rows preserves the equivalence of φ-rows: this means that given two equivalent φ-rows ρ and ρ' and a φ-atom A, the application of the functional $succ_\varphi$ to the two pairs (ρ, A) and (ρ', A) produces two φ-rows which are equivalent as well. We first show (Lemma 4.8) that the result holds for uniform φ-rows, and then we generalize Lemma 4.8 to arbitrary φ-rows (Lemma 4.9). In order to prove Lemma 4.8, we need a preliminary technical result that considers uniform φ-rows of the form B^m for some φ-atom B.

Lemma 4.7. *Let A and B be two φ-atoms such that $rank_D(succ_\varphi(B, A)) = rank_D(B) - h$ for some $h \geq 0$ (note that $h < rank_D(B)$). Given $m > (rank_D(B) - h) \cdot N_{E,\varphi}$, if B^m is a φ-row, then the φ-row $succ_\varphi(B^m, A)$ is of the form $A \cdot \rho_1 \cdots \rho_k$ for some $k \geq 1$ such that*

- *$\rho_1, \ldots, \rho_k$ are uniform φ-rows,*
- *$rank_D(\rho_i) > rank_D(\rho_{i+1})$ for each $1 \leq i < k$, and*
- *$|\rho_k| > rank(\rho_k)$.*

Proof. Let $rank_D(succ_\varphi(B, A)) = rank_D(B) - h$ for some $0 \leq h < rank_D(B)$, $m > (rank_D(B) - h) \cdot N_{E,\varphi}$, and ρ be the φ-row of length $m + 1$ given by $succ_\varphi(B^m, A)$. Since $\rho[0] = A$ and $\rho[i+1] = succ_\varphi(B, \rho[i])$ for all $i \in [0, m-1]$, by Definition 4.2, for all $i \in [1, m]$, the following holds:

- $\rho[i] \cap \mathcal{AP} = B \cap A \cap \mathcal{AP}$;
- $rank_D(\rho[i-1]) \geq rank_D(\rho[i])$ and $Req_E(\rho[i-1]) \subseteq Req_E(\rho[i])$;
- if $i < m$ and $\rho[i] = \rho[i+1]$, then $\rho[j] = \rho[i]$ for all $j \geq i$.

Since by hypothesis $rank_D(\rho[1]) = rank_D(B) - h$, we easily deduce that $\rho = succ_\varphi(B^m, A)$ is of the form

$$A \cdot \rho_1 \cdots \rho_k$$

for some $k \geq 1$ such that $\rho_1, \ldots, \rho_k$ are uniform φ-rows and

- $rank_D(\rho_1) = rank_D(B) - h$,
- $rank_D(\rho_i) > rank_D(\rho_{i+1})$ for each $1 \leq i < k$, and
- $|\rho_i| \leq N_{E,\varphi}$ for each $1 \leq i < k$.

It remains to show that $|\rho_k| > rank(\rho_k)$. By the previous points, we have that $rank_D(B) - h = rank_D(\rho_1) > \ldots > rank_D(\rho_k)$. Hence, $rank_D(B) - h \geq rank_D(\rho_k) + k - 1$. Since $m > (rank_D(B) - h) \cdot N_{E,\varphi}$ and $|\rho_i| \leq N_{E,\varphi}$ for each $1 \leq i < k$, we obtain $|\rho_k| = m - \sum_{i=1}^{i=k-1} |\rho_i| \geq m - \sum_{i=1}^{i=k-1} N_{E,\varphi} > (rank_D(B) - h) \cdot N_{E,\varphi} - (k-1)N_{E,\varphi} \geq rank_D(\rho_k) \cdot N_{E,\varphi} = rank(\rho_k)$. □

By exploiting Lemma 4.7, we can prove the following result.

Lemma 4.8. *Let A be a φ-atom. Then, the following statements hold:*

1. *Let ρ be a uniform φ-row such that $|\rho| > rank(\rho)$. Then, the φ-row $succ_\varphi(\rho, A)$ is of the form $A \cdot \rho_1 \cdots \rho_k$ for some $k \geq 1$ such that $\rho_1, \ldots, \rho_k$ are uniform φ-rows and $|\rho_k| > rank(\rho_k)$.*
2. *Let ρ and ρ' be two uniform φ-rows such that $\rho \sim_\varphi \rho'$. Then, $succ_\varphi(\rho, A) \sim_\varphi succ_\varphi(\rho', A)$.*

Proof. *Property (1).* Let A be a φ-atom and ρ be a uniform φ-row such that $|\rho| > rank(\rho)$. We need to show that the length $|\rho_L|$ of the last segment ρ_L in the uniform factorization of $succ_\varphi(\rho, A)$ satisfies $|\rho_L| > rank(\rho_L)$. Since $|\rho| > rank(\rho)$ and $rank(\rho) \geq N_{E,\varphi}$, by Lemma 4.4(2), ρ is of the form $\rho = \rho_1 \cdot B^m$ where $m \geq 1$, $|\rho_1| = N_{E,\varphi}$ and B is the last atom of ρ_1. Let ρ' be the φ-row given by $succ_\varphi(\rho, A)$. Then ρ' can be written in the form

$$\rho' = (A \cdot \rho'_1) \star succ_\varphi(B^m, B')$$

where $A \cdot \rho'_1 = succ_\varphi(\rho_1, A)$ and B' is the last atom of ρ'_1. In particular, $|\rho'_1| = N_{E,\varphi}$. Let $B'' = succ_\varphi(B, B')$. By Definition 4.2, we have that $rank_D(B'') \leq rank_D(\rho'_1[0]) \leq rank_D(\rho)$. We distinguish two cases:

- $rank_D(B'') = rank_D(\rho)$. In this case, we have that all the atoms in $\rho'_1 \cdot B''$ have the same D-temporal requests. Moreover, since ρ is uniform, by Definition 4.2, all the atoms in $\rho'_1 \cdot B''$ have the same propositional letters. Hence, $\rho'_1 \cdot B''$ is a uniform φ-row. Since $|\rho'_1| = N_{E,\varphi}$, by Lemma 4.4(2), B'' coincides with the last atom B' of ρ'_1. Thus, $B' = succ_\varphi(B, B')$ and $\rho' = A \cdot \rho'_1 \cdot (B')^m$ where $\rho'_1 \cdot (B')^m$ is a uniform φ-row having the same length and the same rank as ρ. Thus, since $|\rho| > rank(\rho)$, the result in this case holds.
- $rank_D(B'') < rank_D(\rho) = rank_D(B)$. We have that $m = |\rho| - N_{E,\varphi} > rank(\rho) - N_{E,\varphi} = (rank_D(B) - 1) \cdot N_{E,\varphi} \geq rank(B'')$. Since $B'' = succ_\varphi(B, B')$, by Lemma 4.7, the length $|\rho_L|$ of the last segment ρ_L in the uniform factorization of $succ_\varphi(B^m, B')$ satisfies $|\rho_L| > rank(\rho_L)$, and the result follows.

Property (2). Let A be a φ-atom and ρ and ρ' be two uniform φ-rows such that $\rho \sim_\varphi \rho'$. We need to show that $succ_\varphi(\rho, A) \sim_\varphi succ_\varphi(\rho', A)$. By hypothesis and Definition 4.6, there are two cases:

- $\rho[0] = \rho'[0]$ and $|\rho| = |\rho'|$. Since ρ and ρ' are uniform, by Lemma 4.4(3), $\rho = \rho'$, and the result obviously follows.
- $\rho[0] = \rho'[0]$, $|\rho| \neq |\rho'|$, $|\rho| > rank(\rho)$ and $|\rho'| > rank(\rho')$. Assume that $|\rho| < |\rho'|$ (the case where $|\rho'| < |\rho|$ being similar). Since ρ and ρ' are uniform and $\rho[0] = \rho'[0]$, it holds that $rank(\rho) = rank(\rho')$. Moreover, $|\rho| > rank(\rho) \geq N_{E,\varphi}$. Applying Lemma 4.4(2) and Lemma 4.4(3), we deduce that ρ is of the form $\rho = \rho_1 \cdot B^2$ and $\rho' = \rho_1 \cdot B^{k+2}$ where B is a φ-atom and $k = |\rho'| - |\rho|$. By Property (1) of Lemma 4.8 the last uniform segment ρ_L of $succ_\varphi(\rho, A)$ satisfies $|\rho_L| > rank(\rho_L) \geq N_{E,\varphi}$. Thus, by Lemma 4.4(2), $succ_\varphi(\rho, A)$ is of the form $\rho' \cdot (B')^2$ for a φ-atom B' such that $B' = succ_\varphi(B, B')$. Since $succ_\varphi(\rho', A) = (\rho' \cdot (B')^2) \star succ_\varphi(B^k, B')$, we obtain that $succ_\varphi(\rho', A) = \rho' \cdot (B')^{2+k}$. Thus, since the last uniform segment ρ_L in $\rho' \cdot (B')^2$ satisfies $|\rho_L| > rank(\rho_L)$, we deduce that $succ_\varphi(\rho, A)$ and $succ_\varphi(\rho', A)$ are equivalent. □

Finally, by applying Lemma 4.6, we generalize Lemma 4.8 to arbitrary φ-rows.

Lemma 4.9. *Let A be a φ-atom and ρ and ρ' be two φ-rows such that $\rho \sim_\varphi \rho'$. Then, for the φ-rows $succ_\varphi(\rho, A)$ and $succ_\varphi(\rho', A)$, it holds that $succ_\varphi(\rho, A) \sim_\varphi succ_\varphi(\rho', A)$.*

Proof. The proof is by induction on the number $N(\rho)$ of distinct uniform segments in the uniform factorization of ρ. Since ρ and ρ' are equivalent, we have that $N(\rho') = N(\rho)$.

Base step $N(\rho) = N(\rho') = 1$, i.e. ρ and ρ' are uniform. In this case, the result directly follows from Lemma 4.8.

Inductive step $N(\rho) = N(\rho') > 1$. Hence, since $\rho \sim_\varphi \rho'$, ρ (resp.: ρ') can be written in the form $\rho = \rho_1 \cdot \rho_2$ (resp.: $\rho' = \rho'_1 \cdot \rho'_2$) such that $\rho_1 \sim_\varphi \rho'_1$, $\rho_2 \sim_\varphi \rho'_2$, $N(\rho_1) = N(\rho'_1) < N(\rho) = N(\rho')$, and $N(\rho_2) = N(\rho'_2) < N(\rho) = N(\rho')$. Let A_1 (resp.: A'_1) be the last atom in $succ_\varphi(\rho_1, A)$ (resp.: $succ_\varphi(\rho'_1, A)$). By the inductive hypothesis, $succ_\varphi(\rho_1, A) \sim_\varphi succ_\varphi(\rho'_1, A)$, $A_1 = A'_1$, and $succ_\varphi(\rho_2, A_1) \sim_\varphi succ_\varphi(\rho'_2, A'_1)$ (note that by Lemma 4.4, two equivalent φ-rows have the same last atom). By Lemma 4.3, it follows that $succ_\varphi(\rho, A) = succ_\varphi(\rho_1, A) \star succ_\varphi(\rho_2, A_1)$ and $succ_\varphi(\rho', A) = succ_\varphi(\rho'_1, A) \star succ_\varphi(\rho'_2, A'_1)$. Thus, by applying Lemma 4.6, we obtain that $succ_\varphi(\rho, A) \sim_\varphi succ_\varphi(\rho', A)$, and the assertion is proved. □

4.3. Optimal upper bounds for $\mathsf{DE}_{\mathcal{H}om}$ satisfiability and model-checking

In this subsection, by exploiting Corollary 4.1 and Lemma 4.9, we devise an asymptotical optimal automaton-theoretic approach for satisfiability and model checking of $\mathsf{DE}_{\mathcal{H}om}$ over finite linear orders. Given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ, we show that it is possible to construct a deterministic finite automaton (DFA) $\mathcal{D}_\varphi$ over the alphabet $2^{\mathcal{AP}}$, whose set of states is the set of initialized minimal φ-rows and which accepts the non-empty finite words over $2^{\mathcal{AP}}$ which satisfy formula φ. At the end of this subsection, we show how the proposed automata-theoretic approach for the logic $\mathsf{DE}_{\mathcal{H}om}$ over finite linear orders can be adapted in order to handle the logic $\mathsf{BD}_{\mathcal{H}om}$ as well.

Definition 4.7. Let ρ be a minimal φ-row and A be an atom. We denote by $succ^{min}_\varphi(\rho, A)$ the unique minimal φ-row in the equivalence class of $\sim_\varphi$ containing $succ_\varphi(\rho, A)$. Moreover, for a set $P \subseteq \mathcal{AP}$ of proposition letters, we denote by $A(P)$ the unique φ-atom such that $A(P) \cap \mathcal{AP} = P$, $Req_D(A(P)) = \emptyset$, and $Req_E(A(P)) = \emptyset$.

We consider the DFA $\mathcal{D}_\varphi = \langle 2^{\mathcal{AP}}, \mathcal{R}ows^{min}_\varphi \cup \{q_0\}, \{q_0\}, \delta, F\rangle$, associated with the formula φ, which is defined as follows:

- $\mathcal{R}ows^{min}_\varphi$ is the set of initialized minimal φ-rows;
- $\delta(q_0, P) = A(P)$ for all $P \in 2^{\mathcal{AP}}$;
- $\delta(\rho, P) = succ^{min}_\varphi(\rho, A(P))$ for all $P \in 2^{\mathcal{AP}}$ and $\rho \in \mathcal{R}ows^{min}_\varphi$;
- F is the set of φ-rows $\rho \in \mathcal{R}ows^{min}_\varphi$ such that $\varphi \in \rho[n-1]$, with $n = |\rho|$.

We can now state the main technical result of this paper.

Theorem 4.1. *Given a $\mathsf{DE}_{\mathcal{H}om}$-formula φ, the DFA $\mathcal{D}_\varphi$ accepts all and only the non-empty finite words over $2^{\mathcal{AP}}$ which satisfy φ.*

Proof. Let w be a non-empty finite word over $2^{\mathcal{AP}}$ and $n = |w| - 1$. We show that for the homogeneous interval model $\mathcal{M}(w)$, $\mathcal{M}(w), [0, n] \models \varphi$ if and only if $w \in \mathcal{L}(\mathcal{D}_\varphi)$.

($\Rightarrow$) Assume that $\mathcal{M}(w), [0, n] \models \varphi$. Let $\mathcal{W} = (w, \mathcal{L})$ be the unique fulfilling φ-word structure associated with the word w and for all $i \in [0, n]$, let ρ_i be the initialized φ-row corresponding to the i-row of $\mathcal{W}$. By hypothesis $\varphi \in \rho_n[n]$, and by construction $|\rho_0| = 1$ and $\rho_i[0] = A(w[i])$ for all $i \in [0, n]$. Moreover, by Corollary 4.1, $\rho_{i+1} = succ_\varphi(\rho_i, \rho_{i+1}[0])$ for all $i \in [0, n-1]$. For each $i \in [0, n]$, let ρ^{min}_i be the unique minimal φ-row in the equivalence class $[\rho_i]_{\sim_\varphi}$. Note that $\rho^{min}_0 = \rho_0$, the last atom of ρ^{min}_n contains φ, ρ^{min}_i is initialized and $\rho^{min}_i[0] = \rho_i[0] = A(w[i])$ for all $i \in [0, n]$. By applying Lemma 4.9, $succ_\varphi(\rho^{min}_i, \rho_{i+1}[0])$ is equivalent to $\rho_{i+1} = succ_\varphi(\rho_i, \rho_{i+1}[0])$ for all $i \in [0, n-1]$. Hence, by the definition of $succ^{min}_\varphi$, we obtain that $\rho^{min}_{i+1} = succ^{min}_\varphi(\rho^{min}_i, A(w[i+1]))$ for all $i \in [0, n-1]$. By Definition 4.7, it follows that there is an accepting run of $\mathcal{D}_\varphi$ over w, i.e. $w \in \mathcal{L}(\mathcal{D}_\varphi)$.

($\Leftarrow$) Let us assume that w is accepted by $\mathcal{D}_\varphi$. By Definition 4.7, there exist $n+1$ initialized minimal φ-rows $\rho^{min}_0, \ldots, \rho^{min}_n$ such that $\rho^{min}_0 = A(w[0])$, φ belongs to the last atom of ρ^{min}_n, $\rho^{min}_i[0] = A(w[i])$ for all $i \in [0, n]$, and $\rho^{min}_{i+1} = succ^{min}_\varphi(\rho^{min}_i, \rho^{min}_{i+1}[0])$ for all $i \in [0, n-1]$. Let $\rho_0, \ldots, \rho_n$ be the sequence of φ-rows defined as follows: $\rho_0 = \rho^{min}_0$ and $\rho_{i+1} = succ_\varphi(\rho_i, \rho^{min}_{i+1}[0])$ for all $i \in [0, n-1]$. By Lemma 4.9, we have $\rho_i \sim_\varphi \rho^{min}_i$ for all $i \in [0, n]$. Hence, ρ_i is initialized for all $i \in [0, n]$, and $\varphi \in \rho_n[n]$. Let us define the φ-word structure $\mathcal{W} = (w, \mathcal{L})$ where $\mathcal{L}(i, j) = \rho_j[j-i]$ for every $0 \leq i \leq j \leq n$. By Corollary 4.1, $\mathcal{W}$ is fulfilling. Thus, since $\varphi \in \mathcal{L}(0, n)$, we obtain that $\mathcal{M}(w), [0, n] \models \varphi$ and the result follows. □

By Theorem 4.1, satisfiability of a $\mathsf{DE}_{\mathcal{H}om}$-formula φ reduces to checking non-emptiness of the DFA $\mathcal{D}_\varphi$ in Definition 4.7 whose number of states is singly exponential in the size of φ (Lemma 4.5).

Let us consider now the model-checking problem. Given a finite Kripke structure $\mathcal{K}$, we check that $\mathcal{K}$ is a model of φ by applying the standard model-checking approach which considers the synchronous product $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ of $\mathcal{K}$ with the automaton associated with the negation of the formula φ. The NFA $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ accepts all and only the traces of $\mathcal{K}$ which violate the property expressed by the $\mathsf{DE}_{\mathcal{H}om}$-formula φ. Hence, $\mathcal{K} \not\models \varphi$ if and only if the language accepted by $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ is not empty. Note that the number of states in $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ is linear in the number of $\mathcal{K}$-states and singly exponential in the size of φ. Moreover, the automata $\mathcal{D}_\varphi$ and $\mathcal{K} \times \mathcal{D}_{\neg\varphi}$ can be constructed 'on the fly'. This is because, given a φ-row ρ, one can compute in polynomial time the minimal φ-row in the equivalence class of ρ. Moreover, given a minimal row ρ and an atom A, one can compute in polynomial time the row $succ_{\varphi}(\rho, A)$. Hence, given a state ρ of $\mathcal{D}_{\neg\varphi}$ (i.e., an initialized minimal φ-row) and $P \subseteq \mathcal{AP}$, one can compute in polynomial time the next state $\delta(\rho, P)$ (where δ is the transition function of $\mathcal{D}_{\neg\varphi}$). Thus, since non-emptiness of NFA is in NLogspace, the complexity classes NPspace =Pspace and NLogspace are closed under complement, and finite satisfiability and model checking against the fragment $\mathsf{D}_{\mathcal{H}om}$ are known to be Pspace-complete [25], we obtain the following result.

Theorem 4.2. *Finite satisfiability and model checking for $\mathsf{DE}_{\mathcal{H}om}$-formulas are both* Pspace*-complete. Moreover, for $\mathsf{DE}_{\mathcal{H}om}$-formulas of fixed size, model checking is in* NLogspace*.*

Note that the proposed automata-based algorithms for model checking and satisfiability of $\mathsf{DE}_{\mathcal{H}om}$ would also work for nondeterministic automata, which are in general exponentially more succinct than deterministic automata. However, the Pspace-hardness of the considered problems (Theorem 4.2) imply that there is probably no nondeterministic automaton of polynomial size recognizing the models of a given $\mathsf{DE}_{\mathcal{H}om}$ formula. This is in contrast to standard LTL over finite words, where nondeterministic automata can be exponentially smaller than deterministic automata for the same formula. In particular, the translation of LTL formulas over finite words into equivalent deterministic automata requires in general a double exponential blowup (a double exponential lower bound for this translation is given in [31]).

As for the logic $\mathsf{BD}_{\mathcal{H}om}$ over finite linear orders, we can state results similar to those provided in Theorem 4.2. Let $\mathsf{DE}(\varphi)$ be the $\mathsf{DE}_{\mathcal{H}om}$ formula obtained from a $\mathsf{BD}_{\mathcal{H}om}$ formula φ by replacing each occurrence of modality $\langle \mathrm{B} \rangle$ with $\langle \mathrm{E} \rangle$. For each non-empty finite word w over $2^{\mathcal{AP}}$, $w \models \varphi$ iff $w^R \models \mathsf{DE}(\varphi)$ (w^R is the reverse of w). Hence, the automaton $\mathcal{N}_{\varphi}$ accepting the models w of φ corresponds to the 'reverse' of the DFA $\mathcal{D}_{\mathsf{DE}(\varphi)}$ of Definition 4.7 associated with $\mathsf{DE}(\varphi)$.

Note that the automaton $\mathcal{N}_{\varphi}$ has the same states as $\mathcal{D}_{\mathsf{DE}(\varphi)}$ but it is not deterministic. On the other hand, $\mathcal{N}_{\varphi}$ is deterministic in the *backward-direction*. Thus, for the $\mathsf{DE}_{\mathcal{H}om}$ formulas, the associated automata are deterministic in the *forward-direction* but non-deterministic in the *backward-direction*. Dually, for the $\mathsf{BD}_{\mathcal{H}om}$ formulas, the associated automata are deterministic in the *backward-direction* but non-deterministic in the *forward-direction*.

Corollary 4.2. *Finite satisfiability and model checking for $\mathsf{BD}_{\mathcal{H}om}$-formulas are both* Pspace*-complete. Moreover, for $\mathsf{BD}_{\mathcal{H}om}$-formulas of fixed size, model checking is in* NLogspace*.*

5. Concluding remarks

In this paper, we have proved that even though the addition of either $\langle \mathrm{B} \rangle$ or $\langle \mathrm{E} \rangle$ modality to the interval logic D increases the expressiveness of the resulting interval logic, surprisingly, such an addition does not affect the complexity of satisfiability and Model Checking problems which are proved to be Pspace-complete.

Pspace-completeness of the satisfiability and Model Checking problems for $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ are particularly interesting when compared with known results for $\mathsf{BE}_{\mathcal{H}om}$, where the latter includes $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ as proper fragments and, apparently, is quite close to $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$. The complexity of Model Checking for $\mathsf{BE}_{\mathcal{H}om}$ is still unknown: the problem is at least Expspace-hard [15], while the only known upper bound is nonelementary [18]. Whether or not this problem can be solved elementarily is a difficult open question. The exact complexity of finite satisfiability for $\mathsf{BE}_{\mathcal{H}om}$ is also an open issue: the same upper/lower bounds can be shown to hold by linear-time reductions to/from the Model Checking problem. Since $\mathsf{DE}_{\mathcal{H}om}$ and $\mathsf{BD}_{\mathcal{H}om}$ are the most significant fragments of $\mathsf{BE}_{\mathcal{H}om}$, the results proved in this paper provide a better insight into such open questions. Note that like the fragment $\mathsf{DE}_{\mathcal{H}om}$, for a $\mathsf{BE}_{\mathcal{H}om}$ formula φ, we can give similar definitions for the notions of φ-atom, φ-row and functional $succ_{\varphi}$ with the difference that D-temporal requests are replaced with B-temporal requests. However, the main difficulty in generalizing the considered approach for $\mathsf{DE}_{\mathcal{H}om}$ to $\mathsf{BE}_{\mathcal{H}om}$ is that for $\mathsf{BE}_{\mathcal{H}om}$ formulas φ, φ-rows have very weak monotonicity properties: in particular, B-temporal requests have no monotonic behavior along a φ-row. It is an intriguing open question whether it is possible to define a finite abstraction on the set of φ-rows for a $\mathsf{BE}_{\mathcal{H}om}$ formula φ such that the functional $succ_{\varphi}$ preserves such an abstraction.

Another issue left open is the extension of the considered framework in order to take into account infinite intervals too (or, equivalently, infinite paths in a Kripke structure). A generalization of the proposed approach to the infinite word setting for getting an elementary upper bound (in particular, a Pspace upper bound) does not seem trivial. In particular, note that in this context, the symmetry between prefixes and suffixes is broken. While prefixes are always finite, suffixes can be either finite or infinite depending on whether the given interval is finite or infinite.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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