



On stabilization at a soliton for generalized Korteweg–De Vries pure power equation for any power $p \in (1, 5)$

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ABSTRACT

We apply our idea, which we previously used in the analysis of the pure power NLS, consisting of splitting the virial inequality method into a large energy inequality combined with Kato smoothing, to the case of generalized Korteweg–De Vries pure power equations. We assume that a solution remains for all positive times very close to a soliton and then we prove an asymptotic stability result for $t \rightarrow +\infty$.

1. Introduction

We consider on the line $\mathbb{R}$ the generalized Korteweg–de Vries (gKdV) equation

$$\partial_t u = -\partial_x (\partial_x^2 u + f(u)) \quad (1.1)$$

where $f(u) = |u|^{p-1}u$ or $|u|^p$ for $1 < p < 5$. Formally, (1.1) is a Hamiltonian system with the Hamiltonian/Energy E and mass Q , where

$$E(u) = \frac{1}{2} \|\partial_x u\|_{L^2(\mathbb{R})}^2 - \int_{\mathbb{R}} F(u) dx, \quad \text{where } F(u) = \int_0^u f(s) ds \text{ and} \quad (1.2)$$

$$Q(u) = \frac{1}{2} \|u\|_{L^2(\mathbb{R})}^2. \quad (1.3)$$

By solutions of (1.1) we mean functions $u \in C^0([0, T^*), H^1(\mathbb{R}))$ satisfying the integral equation

$$u(t) = e^{-it\partial_x^3} u_0 - \int_0^t e^{-(t-s)\partial_x^3} \partial_x^3 (f(u(s))) ds \text{ for all } t \in [0, T^*), \quad (1.4)$$

for some $0 < T^* \leq \infty$.

Remark 1.1. Consider the operator $A = -\partial_x^3$ on $H^{-2}(\mathbb{R})$ with domain $D(A) = H^1(\mathbb{R})$. The graph norm of A is equivalent to the H^1 -norm. Moreover, since $u \in C^0([0, T^*), H^1(\mathbb{R}))$, the Sobolev chain rule yields $\partial_x(f(u)) \in C^0([0, T^*), L^2(\mathbb{R})) \subset C^0([0, T^*), H^{-2}(\mathbb{R}))$. Consequently, (1.4) is precisely the variation-of-constants formula in $H^{-2}(\mathbb{R})$ for

$$\partial_t u = Au - \partial_x f(u).$$

By [1, Corollary 4.1.8], it follows that $u \in C^1([0, T^*), H^{-2}(\mathbb{R}))$ and that (1.1) holds in $H^{-2}(\mathbb{R})$ for every $t \in [0, T^*)$. In particular, it holds in the distributional sense on $(0, T^*) \times \mathbb{R}$.

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If, in addition, $u \in C^0([0, T^*), H^3(\mathbb{R}))$, the same argument, now A on $L^2(\mathbb{R})$ and $D(A) = H^3(\mathbb{R})$, gives $u \in C^1([0, T^*), L^2(\mathbb{R}))$, and (1.1) holds in $L^2(\mathbb{R})$ for every $t \in [0, T^*)$.

Remark 1.2. In Appendix A we show conservation of $\mathbf{Q}$ (but not of $\mathbf{E}$) for the u in (1.4).

If $f \in C^3$, that is, if $p \in [3, 5)$ when $f(u) = |u|^{p-1}u$, or $p \in (3, 5) \cup \{2\}$ when $f(u) = |u|^p$, one can show that (1.1) is globally well-posed in H^1 with both $\mathbf{E}$ and $\mathbf{Q}$ conserved, by arguments similar to those in [2]. However, our focus in this paper is on small values of p , particularly $p \in (1, 2)$, and we will analyze the properties of the solution under the assumption of its existence. For example, when $p = \frac{3}{2}$, (1.1) is called the Schamel equation which appears in the study of ion-acoustic waves [3].

For local wellposedness results in a subset of H^1 , see [4,5]. Also, by following the argument of [6], one can construct a weak solution of (1.1) in the class $L^\infty(\mathbb{R}, H^1)$ with $\mathbf{E}$ and $\mathbf{Q}$ non-increasing.

The most important solutions of the gKdV Eq. (1.1), which are also central to this paper, are the soliton solutions $u(t, x) = \phi_c(x - ct - \gamma)$, where $c > 0$, $\gamma \in \mathbb{R}$ and

$$\phi_c(x) := c^{\frac{1}{p-1}} \phi(\sqrt{c}x), \quad \phi(x) = \left(\frac{p+1}{2} \right)^{\frac{1}{p-1}} \operatorname{sech}^{\frac{2}{p-1}} \left(\frac{p-1}{2} x \right).$$

It is well known, see e.g. [7–11], that solitons are trapped by $\mathbf{E}$ and $\mathbf{Q}$.

Proposition 1.3. For any $c > 0$, there exist $\delta_0 > 0$ and $C > 0$ such that for any $u \in C([0, T^*), H^1)$, satisfying $\mathbf{E}(u(t)) = \mathbf{E}(u(0))$, $\mathbf{Q}(u(t)) = \mathbf{Q}(u(0))$ for all $t \in (0, T^*)$ and

$$\delta := \inf_{\gamma \in \mathbb{R}} \|u(0) - \phi_c(\cdot - \gamma)\|_{H^1} \leq \delta_0, \quad (1.5)$$

will satisfy

$$\sup_{t \in [0, T^*)} \inf_{\gamma \in \mathbb{R}} \|u(t, \cdot) - \phi_c(\cdot - \gamma)\|_{H^1} \leq C\delta, \quad (1.6)$$

Remark 1.4. Proposition 1.3 is a purely variational trapping statement and does not use the evolution equation. This observation is relevant below because the proof of Theorem 1.5 uses energy conservation only to obtain the orbital bound (1.6); the latter may instead be assumed directly.

From Proposition 1.3, if we consider a solution of (1.4) satisfying the conservation of $\mathbf{E}$, which all solutions of (1.4) satisfy if $f \in C^3$, with initial data near a soliton, then the solution will always stay near the soliton as long as the solution exists. Here, recall $\mathbf{Q}$ is always conserved, see Proposition A.1. In this paper, we will study the behavior of such solutions.

Theorem 1.5. Let $a > 0$. Then for any $\epsilon > 0$ there exists a $\delta_0 > 0$ such that for any $0 < T^* \leq \infty$ and any solution $u \in C^0([0, T^*), H^1(\mathbb{R}))$ of (1.4) which satisfies $\mathbf{E}(u(t)) = \mathbf{E}(u_0)$ for all $t \in (0, T^*)$ and (1.5) with $c = 1$, there exist functions $c \in C^1([0, T^*), (1 - \epsilon, 1 + \epsilon))$, $\gamma \in C^1([0, T^*), \mathbb{R})$ and $v \in C^0([0, T^*), H^1(\mathbb{R}))$ such that

$$u(t) = \phi_{c(t)}(\cdot - \gamma(t)) + v(t, \cdot - \gamma(t)) \text{ with} \quad (1.7)$$

$$\int_0^{T^*} \|e^{-a|x|} v(t)\|_{H^1(\mathbb{R})}^2 dt < \epsilon. \quad (1.8)$$

Moreover, if $T^* = \infty$, there exists $c_+ > 0$ s.t.

$$\lim_{t \rightarrow +\infty} \|e^{-a|x|} v(t)\|_{L^2(\mathbb{R})} = 0 \text{ and} \quad (1.9)$$

$$\lim_{t \rightarrow +\infty} c(t) = c_+. \quad (1.10)$$

Remark 1.6. The restriction to $c = 1$ in Theorem 1.5 is only a normalization. More precisely, for every $c_0 > 0$, the conclusion of Theorem 1.5 remains valid with (1.5) imposed at $c = c_0$ and with $c(t) \in (c_0(1 - \epsilon), c_0(1 + \epsilon))$. Indeed, given a solution u of (1.1), define

$$\tilde{u}(s, y) := c_0^{-\frac{1}{p-1}} u\left(c_0^{-\frac{3}{2}} s, c_0^{-\frac{1}{2}} y\right).$$

Then, $\tilde{u}$ also solves (1.1). Moreover, $c_0^{-\frac{1}{p-1}} \phi_{c_0}\left(c_0^{-\frac{1}{2}} y\right) = \phi(y)$. Suppose that the decomposition given by Theorem 1.5 for $\tilde{u}$ is

$$\tilde{u}(s, y) = \phi_{\tilde{c}(s)}(y - \tilde{\gamma}(s)) + \tilde{v}(s, y - \tilde{\gamma}(s)).$$

Then the corresponding decomposition of u is obtained by setting

$$c(t) := c_0 \tilde{c}\left(c_0^{\frac{3}{2}} t\right), \quad \gamma(t) := c_0^{-\frac{1}{2}} \tilde{\gamma}\left(c_0^{\frac{3}{2}} t\right), \quad v(t, x) := c_0^{\frac{1}{p-1}} \tilde{v}\left(c_0^{\frac{3}{2}} t, c_0^{\frac{1}{2}} x\right).$$

In addition, with $\tilde{T}^* := c_0^{\frac{3}{2}} T^*$, we have

$$\int_0^{T^*} \|e^{-a|x|} v(t)\|_{H_x^1}^2 dt \leq C_{p, c_0} \int_0^{\tilde{T}^*} \left\| e^{-\frac{a}{\sqrt{c_0}} |y|} \tilde{v}(s) \right\|_{H_y^1}^2 ds,$$

where $C_{p,c_0} := \max \left\{ c_0^{\frac{2}{p-1}-2}, c_0^{\frac{2}{p-1}-1} \right\}$. Thus, applying [Theorem 1.5](#) to $\tilde{u}$ with the weight $a/\sqrt{c_0}$ and with a sufficiently smaller value of the accuracy parameter yields the assertion for the soliton ϕ_{c_0} .

Remark 1.7. In the proof of [Theorem 1.5](#), the conservation of E is used only through [Proposition 1.3](#). Thus, the assumption $E(u(t)) = E(u_0)$ may be replaced by the orbital bound [\(1.6\)](#).

We also comment on the energy-conservation assumption when $p \in (1, 2)$. If a solution satisfies $u \in C^0([0, T^*), H^3(\mathbb{R}))$, then the standard Hamiltonian calculation is justified and $E(u(t))$ is conserved. The same conclusion holds if, on every compact time interval, the solution is the strong $C_t^0 H_x^1$ -limit of smooth energy-conserving solutions, since E is continuous on $H^1(\mathbb{R})$.

On the other hand, local well-posedness in $H^1(\mathbb{R})$ is not presently known for $p \in (1, 2)$. To the best of our knowledge, neither a general energy-conservation result for all H^1 -solutions satisfying [\(1.4\)](#), nor a counterexample exhibiting loss of energy, is known. Local solutions have been constructed in a smaller class of weighted and more regular functions; see [\[4,5\]](#).

We prove [Theorem 1.5](#) for $p \in (1, 2)$. For $p \geq 2$, the same proof applies and can be simplified because the nonlinearity is more regular.

Our main result is inequality [\(1.8\)](#), which is a surrogate of the virial inequality in Lemma 2 in Martel and Merle [\[12\]](#). Thanks to the observation in [Remark 1.1](#), if $u \in C^0([0, \infty), H^3(\mathbb{R}))$ it is possible to follow basically verbatim the discussion in §3.1 in Martel and Merle [\[12\]](#) which starts from Step 2, and thanks to [Theorem 1.5](#) prove the following result.

Corollary 1.8. *For the solutions in [Theorem 1.5](#) which additionally satisfy $u \in C^0([0, \infty), H^3(\mathbb{R}))$ the following limit holds,*

$$\lim_{t \rightarrow +\infty} \|u(t, \cdot) - \phi_{c(t)}(\cdot - \gamma(t))\|_{H^1(\{x: x \geq \frac{t}{10}\})} = 0.$$

In the proof of [Theorem 1.5](#) we will assume $p \in (1, 2)$ and $T^* = \infty$. Since $H^1(\mathbb{R})$ is an algebra, for $p \geq 2$ essentially the same arguments taken verbatim of the case $p < 2$ hold. However, for $p \geq 2$ the proofs could be made simpler thanks to the fact that the nonlinearity $u \rightarrow f(u)$ is more regular. Even for some powers $p \geq 2$ our technique covers cases which appear not treated by the very general result in Martel and Merle [\[13\]](#), which requires well posedness of the Cauchy problem.

There is an important literature about asymptotic convergence to solitons for gKdV equations starting from Pego and Weinstein [\[14\]](#) who proved asymptotic stability of solitons in exponentially weighted spaces for the nonlinearity $f(u) = u^p$ with $p = 2, 3, 4$ and for $4 < p < 5$ in the case of u positive. Pego and Weinstein [\[14\]](#), using their analysis of the Evans functions in [\[15\]](#), showed also that the linearization, see [\(3.2\)](#), has no eigenvalues different from 0 for almost all exponents p , see [Proposition 3.1](#) below. Mizumachi [\[16\]](#) introduced weights different from Pego and Weinstein [\[15\]](#), part exponentials and part polynomials, useful for the analysis of multisolitons. The case with $f(u) = u^2$ is the integrable KdV for which soliton resolution, that is a quite complete classification of all solutions which live in a more restricted space than $H^1(\mathbb{R})$, can be obtained using the Inverse Scattering Transform, see for example Grunert and Teschl [\[17\]](#).

Other notable results are due to Martel and Merle [\[12,13,18,19\]](#). In particular Martel and Merle [\[12\]](#) contains a very direct proof of asymptotic stability for $p = 2, 3, 4$ based on the positivity, see [\[12, Proposition 3\]](#) with the proof in [\[18\]](#), of a quadratic forms which appears in the course of the virial inequality argument used to prove Lemma 2 in [\[12\]](#). Later Martel and Merle [\[13\]](#) generalized considerably their result focusing on a dual problem introduced by Martel [\[20\]](#). Further, for $p = 2$, Mizumachi and Tzvetkov [\[21\]](#) proved the asymptotic stability for L^2 -initial data by combining Pego-Weinstein and Martel-Merle (virial) methods, see also [\[22\]](#). However, while Martel and Merle [\[13\]](#), as well as Mizumachi and Tzvetkov, holds in contexts where local well posedness is true in $H^1(\mathbb{R})$ (resp. L^2), such local well posedness is not yet known for all our equations, especially when $p \in (1, 2)$, as discussed by Linares et al. [\[4\]](#). Following Martel and Merle, the series Kowalczyk et al. [\[23–26\]](#) splits the virial inequality argument in two parts, one involving first a high energy estimate and then, after a change of variables, a second, low energy, virial inequality. An approach along the lines of Kowalczyk et al. has been used for the asymptotic stability of the small amplitude ground states of cubic quintic NLS's by Martel [\[27,28\]](#) and by Rialland [\[29,30\]](#) for more general small perturbations of the cubic NLS. In the series of papers [\[31–33\]](#) devoted to the asymptotic stability of the ground states of the pure NLS in dimension 1, we replaced the second virial inequality with smoothing estimates. In the present paper we apply this idea to gKdV [\(1.1\)](#). We refer to [\[31–33\]](#) for a general discussion in the context of the NLS of the advantages of our method, which subsist also for gKdV [\(1.1\)](#), as we briefly explain now. Proving dispersion by means of Duhamel's formula, e.g. [\(1.4\)](#) or [\(7.12\)](#), is not easy or possible, especially for p is close to 1, when the nonlinearity becomes a long range perturbation of the linear equation. Hence we use virial inequalities, which are akin to the positive commutator method in [\[34, §B p.157\]](#) and are motivated by simple classical mechanics models, e.g. [\[35, §2.3 p. 65\]](#). But then the key issue, here as well as in work by Martel and Merle, is the $\|v\|_{\Sigma}^2$ term in [\(5.2\)](#). By a painstaking discussion on quadratic forms, Martel and Merle [\[18\]](#) insure that that term is not there, but only for some values of p . Alternatively, by the dual problem, in [\[13\]](#) they generalize considerably their result, but they require $f \in C^3$. Our approach is different. We first bound, with rather simple routine computations, high energies in terms of low energies, obtaining [Lemma 5.1](#). To bound of $\|v\|_{\Sigma} = \|\operatorname{sech}(\kappa x)v\|_{L^2(\mathbb{R})}$ for some $\kappa > 0$ we use Duhamel's formula, but only after we multiply the equation of v by a cutoff, which tames the nonlinearity, at the cost of a commutator term. This is similar to the second virial inequality method, see for example in [\[24\]](#). For related work on good Boussinesq equation see [\[36\]](#) and therein for further references; for the one dimensional Gross-Pitaevskii equation see [\[37\]](#). By using a Kato smoothing for the linearization, we do not need to devise the changes of coordinates of Martel and collaborators, which in some cases are rather subtle. Our method appears very robust and general.

To develop Kato smoothing for the linearized operator [\(3.2\)](#) we exploit the work by Pego and Weinstein [\[14,15\]](#) and by Mizumachi [\[16\]](#), with the latter inspired also by Miller [\[38\]](#) and by Murata [\[39\]](#). Kato smoothing is via Fourier transform in time reduced to

estimates on the resolvent which are obtained expressing the latter in terms of appropriate Jost functions. The theory of these Jost functions is very well known in the case of Schrödinger operators in the line, see [40], and is even simpler here. A generalization to higher order operators is in the monograph by Beals et al. [41]. However here we develop from the start the theory and the estimates on the Jost functions which we need. The integrability of the equation does not modify the structure of the problem, except for explicit formulas for the Jost functions in the integrable case, see [42, p. 6620]. Differently from the case of the cubic NLS, in the integrable KdV there is no additional resonance compared to the generic non integrable case. Indeed in [14,15] it is proved that the zeros λ of the Evans functions are for $\operatorname{Re} \lambda \geq 0$ always eigenvalues of the linearization.

Finally we remark that, since the work by Martel and Merle has been used as blueprint for a large variety of other problems also in higher dimension, we think that variations and somewhat different viewpoints might also prove useful, especially considering that the notion of smoothing introduced by Kato [43] has been extremely successful.

2. Notation

Notation 2.1. We will use the following miscellanea of notations and definitions.

1. In the course of the paper we will sometimes use the notation $\dot{u} = \partial_t u$ and $u' = \partial_x u$.
2. We will use the Japanese bracket $\langle x \rangle := \sqrt{1 + x^2}$.
3. In analogy to Kowalczyk et al. [24], we consider constants $A, B, A_1, c, \delta > 0$ satisfying

$$\log(\delta^{-1}) \gg \log(e^{-1}) \gg A \gg B^2 \gg B \gg 1 \text{ and } A_1 := B^{3/5}. \quad (2.1)$$

4. The notation $o_\varepsilon(1)$ means a constant with a parameter ε such that $o_\varepsilon(1) \xrightarrow{\varepsilon \rightarrow 0^+} 0$.
5. Following Pego and Weinstein [14] for $a \in \mathbb{R}$ we set

$$L_a^2 = \{v : e^{ax} v \in L^2(\mathbb{R})\} \text{ with } \|v\|_{L_a^2} = \|e^{ax} v\|_{L^2(\mathbb{R})}. \quad (2.2)$$

6. We also define (notice that $\tilde{\Sigma}$ has a natural structure as Hilbert space)

$$L^{p,s} = \{v : \langle x \rangle^s v \in L^p(\mathbb{R})\} \text{ with } \|v\|_{L^{p,s}} := \|\langle x \rangle^s v\|_{L^p(\mathbb{R})}, \text{ and} \quad (2.3)$$

$$\tilde{\Sigma} = \{v : \operatorname{sech}(\kappa x) v \in L^2(\mathbb{R})\} \text{ with } \|v\|_{\tilde{\Sigma}} := \|\operatorname{sech}(\kappa x) v\|_{L^2(\mathbb{R})}. \quad (2.4)$$

7. Given two Banach spaces X and Y we denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators from X to Y . We write $\mathcal{L}(X) := \mathcal{L}(X, X)$.
8. Following the framework in Kowalczyk et al. [24] we fix an even function $\chi \in C_c^\infty(\mathbb{R}, [0, 1])$ satisfying

$$1_{[-1,1]} \leq \chi \leq 1_{[-2,2]} \text{ and } x\chi'(x) \leq 0 \text{ and set } \chi_C := \chi(\cdot/C) \text{ for a } C > 0. \quad (2.5)$$

9. We consider a decreasing function $\vartheta_1 \in C^\infty(\mathbb{R}, [0, 1])$ with $\vartheta_1(x) = \begin{cases} 1, & \text{for } x \leq 1 \\ 0, & \text{for } x \geq 2 \end{cases}$, we set $\vartheta_2 = 1 - \vartheta_1$ and consider the partition of unity $1 = \vartheta_{1A_1} + \vartheta_{2A_1}$ where $\vartheta_{iA_1}(x) := \vartheta_i\left(\frac{x}{A_1}\right)$.

10. Following the framework in Kowalczyk et al. [24] and using the function χ in (2.5) we consider for $C > 0$ the function

$$\zeta_C(x) := \exp\left(-\frac{|x|}{C}(1 - \chi(x))\right) \text{ and } \varphi_C(x) := \int_0^x \zeta_C^2(y) dy. \quad (2.6)$$

We similarly define

$$\varphi_{iAA_1}(x) := \int_0^x \zeta_A^2(y) \vartheta_{iA_1}^2(y) dy. \quad (2.7)$$

11. For $x \in \mathbb{R}$ we set $x^\pm := \max(0, \pm x)$.
12. We will consider the Banach space

$$L_+^{1,1} := \{v : \langle x^+ \rangle v \in L^1\} \text{ with norm } \|v\|_{L_+^{1,1}} := \|\langle x^+ \rangle v\|_{L^1}. \quad (2.8)$$

13. We consider semi-norms

$$\|v\|_{\Sigma_{iAA_1}} := \left\| \left(\vartheta_{iA_1} \zeta_A v \right)' \right\|_{L^2(\mathbb{R})} + \left\| \vartheta_{iA_1} \zeta_A v \right\|_{L^2(\mathbb{R})} \text{ for } i = 1, 2. \quad (2.9)$$

For $v \in C^0(I, H^1(\mathbb{R}))$ with $I = [0, T)$, we set

$$\|v\|_{L^2(I, \Sigma_{iAA_1})} = \lim_{t \rightarrow T^-} \|v\|_{\Sigma_{iAA_1}} \|_{L^2(0,t)}. \quad (2.10)$$

14. By a simple computation we have

$$\begin{aligned}\partial_c \phi_c(x) &= \partial_c \left(c^{\frac{1}{p-1}} \phi(\sqrt{cx}) \right) = \frac{c^{\frac{1}{p-1}-1}}{p-1} \phi(\sqrt{cx}) + \frac{1}{2} c^{\frac{1}{p-1}-\frac{1}{2}} x \phi'(\sqrt{cx}) \\ &= \frac{c^{-1}}{p-1} \phi_c(x) + \frac{c^{-1}}{2} x \phi'_c(x).\end{aligned}\quad (2.11)$$

15. We set

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.\quad (2.12)$$

16. Give an operator T for us the resolvent is $R_T(z) = (T - z)^{-1}$.

3. Linearization and modulation

We set

$$L_{+c} := -\partial_x^2 + c - p\phi_c^{p-1} \quad (3.1)$$

$$\mathcal{L}_{[c]} = \partial_x L_{+c} \quad (3.2)$$

When $c = 1$, we will write simply $\mathcal{L}$ and L_+ and we will omit the $[c]$ also in the following functions which we take from Pego and Weinstein [14, Eq. (2.9)],

$$\xi_1[c](x) := \phi'_c(x), \quad \xi_2[c](x) := \partial_c \phi_c(x) \quad (3.3)$$

$$\eta_1[c](x) := \theta_1 \int_{-\infty}^x \partial_c \phi_c(y) dy + \theta_2 \phi_c(x), \quad \eta_2[c](x) := \theta_3 \phi_c(x) \quad (3.4)$$

where,

$$\theta_1 = -\theta_3 = \frac{1}{\mathbf{q}'(c)} \text{ and } \theta_2 = \frac{\|\partial_c \phi_c\|_{L^1}^2}{2(\mathbf{q}'(c))^2}, \text{ with } \mathbf{q}(c) := \mathbf{Q}(\phi_c) = \frac{1}{2} c^{\frac{5-p}{2(p-1)}} \|\phi\|_{L^2(\mathbb{R})}^2. \quad (3.5)$$

Notice here that $\mathbf{q}'(c) = \langle \partial_c \phi_c, \phi_c \rangle$ and that

$$\mathbf{q}'(1) = \langle \Lambda_p \phi, \phi \rangle \text{ where } \Lambda_p = \frac{1}{2} x \partial_x + \frac{1}{p-1}. \quad (3.6)$$

An elementary computation shows

$$\partial_c \left(c^{\frac{1}{p-1}} u(\sqrt{cx}) \right) = c^{\frac{2-p}{p-1}} (\Lambda_p u)(\sqrt{cx}). \quad (3.7)$$

For later use we record that

$$\begin{aligned}\int_{\mathbb{R}} \partial_c \phi_c(y) dy &= c^{\frac{2-p}{p-1}-\frac{1}{2}} \int_{\mathbb{R}} (\Lambda_p \phi)(y) dy = c^{\frac{5-3p}{2(p-1)}} \int_{\mathbb{R}} \left(\frac{1}{2} y \phi'(y) + \frac{1}{p-1} \phi(y) \right) dy \\ &= c^{\frac{5-3p}{2(p-1)}} \frac{3-p}{2(p-1)} \|\phi\|_{L^1(\mathbb{R})}.\end{aligned}\quad (3.8)$$

The functions in (3.3)-(3.4) satisfy, see [15],

$$\mathcal{L}_{[c]} \xi_1[c] = 0, \quad \mathcal{L}_{[c]} \xi_2[c] = -\xi_1[c], \quad (3.9)$$

$$\mathcal{L}_{[c]}^* \eta_2[c] = 0, \quad \mathcal{L}_{[c]}^* \eta_1[c] = -\eta_2[c], \quad (3.10)$$

$$\langle \xi_i[c], \eta_j[c] \rangle = \delta_{ij}. \quad (3.11)$$

Let $N_g(\mathcal{L}_{[c]}) := \bigcup_{n=1}^{+\infty} \ker \mathcal{L}_{[c]}^n$. Pego and Weinstein [14, Section 2] proved the following.

Proposition 3.1. *Let here $a \in (0, \sqrt{c/3})$ and $1 < p < 5$. The following hold.*

i *As an operator defined in $L^2(\mathbb{R})$ with domain $H^3(\mathbb{R})$ we have $\sigma(\mathcal{L}_{[c]}) = i\mathbb{R}$.*

ii *Both in $L^2(\mathbb{R})$ and $L_a^2(\mathbb{R})$*

$$\ker \mathcal{L}_{[c]} = \operatorname{Span}\{\xi_1[c]\}, \quad N_g(\mathcal{L}_{[c]}) = \operatorname{Span}\{\xi_1[c], \xi_2[c]\}.$$

iii *For $\mathcal{L}_{[c]}^* := -L_{+c} \partial_x$ in $L_{-a}^2(\mathbb{R})$ we have*

$$\ker \mathcal{L}_{[c]}^* = \operatorname{Span}\{\eta_2[c]\}, \quad N_g(\mathcal{L}_{[c]}^*) = \operatorname{Span}\{\eta_1[c], \eta_2[c]\}.$$

iv The eigenvalues of $\mathcal{L}_{[c]}$ in $L^2(\mathbb{R})$ and in $L^2_a(\mathbb{R})$ are the same.

v The subset $E \subseteq (1, 5)$ of p 's where $\mathcal{L}_{[c]}$ has a nonzero eigenvalue λ with $\operatorname{Re} \lambda \geq 0$ is a discrete set which does not contain 2 and 3.

□

In Section 10 we will prove the following.

Proposition 3.2. The set E , given above in Proposition 3.1, is empty.

Proof. For $p \in (1, 5)$ the absence of eigenvalues λ with $\operatorname{Re} \lambda > 0$ is well known. See, e.g. [44–46]. For the absence of eigenvalues $\lambda \in i\mathbb{R}$ with $\lambda \neq 0$, see Section 10 below. □

For $c_0, r \in \mathbb{R}_+ := (0, \infty)$ and for $D_{H^1(\mathbb{R})}(u, r) := \{v \in H^1(\mathbb{R}) \mid \|u - v\|_{H^1(\mathbb{R})} < r\}$, we set

$$\mathcal{U}(c_0, r) := \bigcup_{x_0 \in \mathbb{R}} e^{-x_0 \partial_x} D_{H^1(\mathbb{R})}(\phi_{c_0}, r).$$

Lemma 3.3 (Modulation). For any $c_0 > 0$ there exist $\delta_0 > 0$ and $B_0 > 0$ with $\delta_0 \ll B_0^{-3/2}$ such that for $\delta_0^{-1} \gg B \geq B_0$ and for the function ζ_B introduced in (2.6) there are $c, \gamma \in C^1(\mathcal{U}(c_0, \delta_0), \mathbb{R}_+ \times \mathbb{R})$ such that for any $u \in \mathcal{U}(c_0, \delta_0)$

$$v(u) := e^{\gamma(u) \partial_x} u - \phi_{c(u)} \text{ satisfies} \quad (3.12)$$

$$\langle v(u), \zeta_B \eta_1[c(u)] \rangle = \langle v(u), \eta_2[c(u)] \rangle = 0.$$

For any $c_1 > 0$ and $\gamma_0 \in \mathbb{R}$ we have the identities

$$c(\phi_{c_1}) = c_1, \gamma(\phi_{c_1}) = 0, \gamma(e^{-\gamma_0 \partial_x} u) = \gamma(u) + \gamma_0 \text{ and } c(e^{-\gamma_0 \partial_x} u) = c(u). \quad (3.13)$$

Furthermore there exists a constant $K > 0$ such that

$$B^{-\frac{1}{2}} |\gamma(u) - \gamma_0| + |c(u) - 1| \leq K \delta_0 \text{ for all } u \in e^{-\gamma_0 \partial_x} D_{H^1(\mathbb{R})}(\phi, \delta_0). \quad (3.14)$$

Proof. Set $F = (F_1, F_2)^\top$ with

$$F_i(u, \gamma, c) = \begin{cases} \langle u - e^{-\gamma \partial_x} \phi_c, e^{-\gamma \partial_x} (\zeta_B \eta_1[c]) \rangle, & \text{for } i = 1; \\ \langle u - e^{-\gamma \partial_x} \phi_c, e^{-\gamma \partial_x} \eta_2[c] \rangle & \text{for } i = 2. \end{cases}$$

We now introduce the variable

$$\tilde{\gamma} := B^{-\frac{1}{2}} (\gamma - \gamma_0) \quad (3.15)$$

and we set

$$G_1(u, \tilde{\gamma}, c) = B^{-\frac{1}{2}} F_1(u, B^{\frac{1}{2}} \tilde{\gamma} + \gamma_0, c) \text{ and } G_2(u, \tilde{\gamma}, c) = F_2(u, B^{\frac{1}{2}} \tilde{\gamma} + \gamma_0, c)$$

Then, from $\langle \xi_1[c], \zeta_B \eta_1[c] \rangle = \delta_{1i} + O(B^{-1})$, we have

$$\begin{aligned} \partial_{\tilde{\gamma}} G_1(u, \tilde{\gamma}, c) &= \langle \xi_1[c], \zeta_B \eta_1[c] \rangle - \langle u - e^{-\gamma \partial_x} \phi_c, e^{-\gamma \partial_x} (\zeta'_B \eta_1[c] + \zeta_B \eta'_1[c]) \rangle \\ &= 1 + O(B^{-1}) + O(\|u - e^{-\gamma \partial_x} \phi_c\|_{L^2(\mathbb{R})}) \end{aligned}$$

and, using (3.11),

$$\begin{aligned} \partial_{\tilde{\gamma}} G_2(u, \tilde{\gamma}, c) &= -B^{\frac{1}{2}} \langle u - e^{-\gamma \partial_x} \phi_c, e^{-\gamma \partial_x} \eta'_2[c] \rangle \\ &= O(B^{1/2} \|u - e^{-\gamma \partial_x} \phi_c\|_{L^2(\mathbb{R})}). \end{aligned}$$

From $\langle \xi_2[c], \zeta_B \eta_1[c] \rangle = O(B^{-1})$, we have

$$\begin{aligned} \partial_c G_1(u, \tilde{\gamma}, c) &= -B^{-\frac{1}{2}} \langle \xi_2[c], \zeta_B \eta_1[c] \rangle + B^{-\frac{1}{2}} \langle u - e^{-\gamma \partial_x} \phi_c, e^{-\gamma \partial_x} (\zeta_B \partial_c \eta_1[c]) \rangle \\ &= O(B^{-\frac{3}{2}}) + O(\|u - e^{-\gamma \partial_x} \phi_c\|_{L^2(\mathbb{R})}) \end{aligned}$$

and, using (3.11),

$$\begin{aligned} \partial_c G_2(u, \tilde{\gamma}, c) &= -\langle \xi_2[c], \eta_2[c] \rangle + \langle u - e^{-\gamma \partial_x} \phi_c, e^{-\gamma \partial_x} (\zeta_B \partial_c \eta_2[c]) \rangle \\ &= -1 + O(B^{-1}) + O(\|u - e^{-\gamma \partial_x} \phi_c\|_{L^2(\mathbb{R})}). \end{aligned}$$

So, for

$$\|u - e^{-\gamma \partial_x} \phi_c\|_{L^2(\mathbb{R})} \leq B^{-1} \quad (3.16)$$

we conclude

$$\frac{\partial G(u, \tilde{\gamma}, c)}{\partial(\tilde{\gamma}, c)} = \sigma_3 + O(B^{-1/2}) \text{ where } \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.17)$$

Using also (2.6), we have

$$\partial_u G_i(u, \tilde{\gamma}, c) = \begin{cases} B^{-\frac{1}{2}} e^{-\gamma \partial_x} (\zeta_B \eta_1[c]), & \text{if } i = 1 \text{ and} \\ e^{-\gamma \partial_x} \eta_2[c], & \text{if } i = 2. \end{cases}$$

Then $\|\partial_u G_i(u, \tilde{\gamma}, c)\|_{H^1} \lesssim 1$. The higher order derivatives of $G(u, \tilde{\gamma}, c)$ are similarly bounded with bounds which depend, continuously, only on c . Then for any c_0 there are $\delta = \delta(c_0) > 0$ with $\delta \ll B^{-3/2}$ and $K = K(c_0)$ such that for any γ_0 there is an implicit function

$$(c(\cdot), \tilde{\gamma}(\cdot)) \in C^1 \left(e^{-\gamma_0 \partial_x} D_{H^1(\mathbb{R})}(\phi_{c_0}, \delta), (c_0 - K\delta, c_0 + K\delta) \times (-K\delta, K\delta) \right)$$

with $c(e^{-\gamma_0 \partial_x} \phi_{c_0}) = c_0$ and $\tilde{\gamma}(e^{-\gamma_0 \partial_x} \phi_{c_0}) = 0$. Using (3.15) and back to the variable γ , we have with the same constants

$$(c(\cdot), \gamma(\cdot)) \in C^1 \left(e^{-\gamma_0 \partial_x} D_{H^1(\mathbb{R})}(\phi_{c_0}, \delta), (c_0 - K\delta, c_0 + K\delta) \times \left(\gamma_0 - KB^{\frac{1}{2}}\delta, \gamma_0 + KB^{\frac{1}{2}}\delta \right) \right) \quad (3.18)$$

with $c(e^{-\gamma_0 \partial_x} \phi_{c_0}) = c_0$ and $\gamma(e^{-\gamma_0 \partial_x} \phi_{c_0}) = \gamma_0$. Notice that

$$\|u - e^{-\gamma \partial_x} \phi_c\|_{L^2(\mathbb{R})} \leq \|u - e^{-\gamma_0 \partial_x} \phi_{c_0}\|_{L^2(\mathbb{R})} + \|e^{-\gamma_0 \partial_x} \phi_{c_0} - e^{-\gamma \partial_x} \phi_c\|_{L^2(\mathbb{R})} \leq C(K)B^{\frac{1}{2}}\delta \quad (3.19)$$

so that the choice $\delta \leq \frac{1}{C(K)B^{3/2}}$ yields (3.16). Here we have $F(u, \gamma(u), c(u)) = 0$ identically.

For any $y \in \mathbb{R}$ we have identically

$$F(e^{-y \partial_x} u, \gamma + y, c) = F(u, \gamma, c)$$

which differentiating in y at $y = 0$ implies

$$\partial_y F(u, \gamma, c) = \partial_u F(u, \gamma, c)u'.$$

Then by the chain rule, differentiating in y at $y = 0$ at a $u \in e^{-\gamma_0 \partial_x} D_{H^1(\mathbb{R})}(\phi_{c_0}, \delta(c_0))$ we obtain

$$0 = \partial_y F(e^{-y \partial_x} u, \gamma(e^{-y \partial_x} u), c(e^{-y \partial_x} u)) \Big|_{y=0} = -\partial_\gamma F(u, \gamma, c)(1 + d\delta(u)u') - \partial_c F(u, D, c)dc(u)u'$$

where $dc(u), d\gamma(u) \in \mathcal{L}(H^1(\mathbb{R}))$ are the Frechét differentials of the functions in (3.18). Then by (3.17), by taking $\delta(c_0) > 0$ small enough, we see that the last equalities can be true only for $dc(u)u' \equiv 0$ and $d\gamma(u)u' \equiv -1$ that is $\frac{d}{dy} \gamma(e^{-y \partial_x} u) \Big|_{y=0} = 1$ for all u , i.e. we obtain (3.13). Then we can extend $c, \gamma \in C^1(\mathcal{U}(c_0, \delta(c_0)), \mathbb{R}_+ \times \mathbb{R})$ by this equivariance property. Finally notice that the estimate in (3.14) is implicit in (3.18). $\square$

For $u \in C^0([0, +\infty), H^1(\mathbb{R}))$ the solution of (1.1) we have $u \in C^1([0, +\infty), H^{-2}(\mathbb{R}))$ and the function $(c(t), \gamma(t)) = (c(u(t)), \gamma(u(t))) \in C^1([0, +\infty), \mathbb{R}_+ \times \mathbb{R})$. Using Lemma 3.3 we can insert in (1.1) the ansatz $u = \phi_c(\cdot - \gamma) + v(\cdot - \gamma)$ obtaining

$$\begin{aligned} & \dot{v} - (\dot{\gamma} - c)v' - (\dot{\gamma} - c)\xi_1[c] + \dot{c}\xi_2[c] \\ &= \mathcal{L}_{[c]}v - \partial_x(f(\phi_c + v) - f(\phi_c) - f'(\phi_c)v). \end{aligned} \quad (3.20)$$

Notice that setting $\tilde{v}(t) := e^{t \partial_x^3} v(t)$ we have

$$e^{-t \partial_x^3} \tilde{v} = c v' - \partial_x(f(\phi_c + v) - f(\phi_c)) + (\dot{\gamma} - c)v' + (\dot{\gamma} - c)\xi_1[c] - \dot{c}\xi_2[c] \quad (3.21)$$

which yields

$$\tilde{v} \in C^0([0, +\infty), H^1(\mathbb{R})) \cap C^1([0, +\infty), L^2(\mathbb{R})). \quad (3.22)$$

The proof of Theorem 1.5 is mainly based on the following continuation argument, where the norms are defined in Notation 2.1.

Proposition 3.4. For any small $\epsilon > 0$ and for large constants A, B and $A_1 := B^{3/5}$ as in formula (2.1) there exists a $\delta_0 = \delta_0(\epsilon)$ s.t. if

$$\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})} + \|v\|_{L^2(I, \tilde{\Sigma})} + \|\dot{\gamma} - c\|_{L^2(I)} + \|\dot{c}\|_{L^2(I)} \leq \epsilon \quad (3.23)$$

holds for $I = [0, T]$ for some $T > 0$ and if the constant $C\delta$ in (1.6) satisfies $\delta \in (0, \delta_0)$ then in fact for $I = [0, T]$ inequality (3.23) holds for ϵ replaced by $\epsilon_\epsilon(1)\epsilon$.

We will split the proof of Proposition 3.4 in a number of partial results obtained assuming the hypotheses of Proposition 3.4. The first is the following, proved in Section 4.

Lemma 3.5. We have the estimates

$$\|\dot{c}\|_{L^2(I)} + \|\dot{\gamma} - c\|_{L^2(I)} \lesssim B^{-1} \left(\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})} \right). \quad (3.24)$$

In Section 5 we will prove the following.

Lemma 3.6 (Virial Inequality). *We have*

$$\|v\|_{L^2(I, \Sigma_{1AA_1})} \lesssim \sqrt{AB\delta} + \|v\|_{L^2(I, \tilde{\Sigma})} + A_1^{-1} \|v\|_{L^2(I, \Sigma_{2AA_1})} \quad \text{and} \quad (3.25)$$

$$\|v\|_{L^2(I, \Sigma_{2AA_1})} \lesssim \sqrt{AB\delta} + A_1^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})}. \quad (3.26)$$

Notice that $\sqrt{AB\delta} \ll A^{-1} \epsilon^2 = o_{B^{-1}}(1) \epsilon^2$ in (3.25) and similarly, from $A_1 = B^{3/5}$ below we have $A_1^{3/2} B^{-1} = B^{-\frac{1}{10}}$ and $B^{1/2} A_1^{-1} = B^{-\frac{1}{10}}$ where the following will be proved from Sections 6 to 9.

Lemma 3.7 (Smoothing Inequality). *We have*

$$\|v\|_{L^2(I, \tilde{\Sigma})} \lesssim A_1^{3/2} B^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})} + B^{1/2} \|v\|_{L^2(I, \Sigma_{2AA_1})} + o_{B^{-1}}(1) \epsilon. \quad (3.27)$$

Proof of Theorem 1.5. From Lemmas 3.6 and 3.7 and from (2.1) we have

$$\begin{aligned} \|v\|_{L^2(I, \tilde{\Sigma})} &\lesssim A_1^{3/2} B^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})} + B^{1/2} \|v\|_{L^2(I, \Sigma_{2AA_1})} + o_{B^{-1}}(1) \epsilon \\ &\lesssim A_1^{3/2} B^{-1} \|v\|_{L^2(I, \tilde{\Sigma})} + B^{1/2} A_1^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})} + A_1^{1/2} B^{-1} \|v\|_{L^2(I, \Sigma_{2AA_1})} + o_{B^{-1}}(1) \epsilon \end{aligned}$$

which by $A_1 = B^{3/5}$ and bootstrap yields

$$\|v\|_{L^2(I, \tilde{\Sigma})} \lesssim B^{-\frac{1}{10}} \left(\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})} \right) + o_{B^{-1}}(1) \epsilon$$

so that by (3.23) we obtain $\|v\|_{L^2(I, \tilde{\Sigma})} = o_{B^{-1}}(1) \epsilon$ which inserted in (3.25) yields $\|v\|_{L^2(I, \Sigma_{1AA_1})} = o_{B^{-1}}(1) \epsilon$ and in (3.26) yields $\|v\|_{L^2(I, \Sigma_{2AA_1})} = o_{B^{-1}}(1) \epsilon$. In turn these inserted in (3.24) yield $\|\dot{c}\|_{L^2(I)} + \|\dot{\gamma} - c\|_{L^2(I)} = o_{B^{-1}}(1) \epsilon$. This proves Proposition 3.4. By $u \in C^0([0, +\infty), H^1(\mathbb{R}))$ it is elementary that this implies that we can take $I = \mathbb{R}_+$ in all the above inequalities. By

$$|(e^{-a|x|} v)'| + |e^{-a|x|} v| \lesssim |(\zeta_A^{-1} e^{-a|x|} \zeta_A v)'| + |\zeta_A v| \lesssim |(\zeta_A v)'| + |\zeta_A v|$$

for $a > 1/A$, and inserting $1 = \partial_{1A_1} + \partial_{2A_1}$, we obtain $\|e^{-a|x|} v\|_{H^1(\mathbb{R})} \lesssim \|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}$ and so the inequalities in (3.25) for $I = \mathbb{R}_+$ implies (1.8). Then we can proceed like in [31, Proof of Theorem 1.2] considering

$$\mathbf{a}(t) := 2^{-1} \|e^{-a(x)} v(t)\|_{L^2(\mathbb{R})}^2 = 2^{-1} \langle e^{i\partial_x^3} e^{-2a(x)} e^{-i\partial_x^3} \tilde{v}(t), \tilde{v}(t) \rangle.$$

Taking time derivative we claim that $t \rightarrow \mathbf{a}(t)$ is an absolutely continuous function with

$$\dot{\mathbf{a}}(t) = \frac{1}{2} \langle [\partial_x^3, e^{-2a(x)}] v, v \rangle + \langle e^{-2a(x)} v(t), e^{-i\partial_x^3} \tilde{v}(t) \rangle.$$

This indeed is the case if $\tilde{v}$ is regular and can be obtained for general $\tilde{v}$ like in (3.22) taking a sequence of regular functions $\{\tilde{v}_n\}_{n \in \mathbb{N}}$ with $\tilde{v}_n \xrightarrow{n \rightarrow +\infty} \tilde{v}$ locally uniformly in the space in (3.22). Then we obtain

$$\begin{aligned} \dot{\mathbf{a}} &= \frac{1}{2} \langle [\partial_x^3, e^{-2a(x)}] v, v \rangle + \frac{1}{2} \dot{\gamma} \langle [e^{-2a(x)}, \partial_x] v, v \rangle + (\dot{\gamma} - c) \langle \xi_1[c], e^{-2a(x)} v \rangle \\ &\quad - \dot{c} \langle \xi_2[c], e^{-2a(x)} v \rangle + \langle f(\phi_c + v) - f(\phi_c), [\partial_x, e^{-2a(x)}] v + e^{-2a(x)} v' \rangle. \end{aligned}$$

We exploit the fact that when $u \in e^{-\gamma_0 \partial_x} D_{H^1(\mathbb{R})}(\phi, \delta)$

$$\begin{aligned} \|u\|_{H^1(\mathbb{R})} &= \|e^{\gamma(u) \partial_x} u - \phi_{c(u)}\|_{H^1(\mathbb{R})} \leq \|u - e^{-\gamma_0 \partial_x} \phi_{c(u)}\|_{H^1(\mathbb{R})} \\ &\quad + \|e^{(\gamma(u) - \gamma_0) \partial_x} \phi - \phi_{c(u)}\|_{H^1(\mathbb{R})} \lesssim \delta + |\gamma(u) - \gamma_0| + |c(u) - 1| \lesssim B^{\frac{1}{2}} \delta. \end{aligned} \quad (3.28)$$

Then by Lemma 3.5

$$|\dot{\mathbf{a}}| \lesssim |\dot{c}|^2 + |\dot{\gamma} - c|^2 + \|v\|_{H^1}^2 \lesssim B \delta^2$$

and since we already know from (1.8) that $\mathbf{a} \in L^1(\mathbb{R})$, we conclude that $\mathbf{a}(t) \xrightarrow{t \rightarrow +\infty} 0$.

Next we prove (1.10) by proceeding like in [31]. We can take $0 < a \leq \kappa/2$ such that we have the following, which will be used below,

$$e^{-2a(x)} \gtrsim \max\{\phi_{c(t)}, \phi_{c(t)}^{p-1}\} \text{ for all } t \geq 0. \quad (3.29)$$

Since $\mathbf{Q}(\phi_c) = \mathbf{q}(c)$ is strictly monotonic in c , it suffices to show that the $\mathbf{Q}(\phi_{c(t)})$ converges as $t \rightarrow \infty$. From the conservation of $\mathbf{Q}$, see Proposition A.1, the exponential decay of ϕ_c , (1.8) and (1.9), we have

$$\lim_{t \rightarrow \infty} (\mathbf{Q}(u_0) - \mathbf{Q}(\phi_{c(t)}) - \mathbf{Q}(v(t))) = 0. \quad (3.30)$$

Thus, our task is now to prove $\frac{d}{dt} \mathbf{Q}(v) \in L^1(\mathbb{R}_+)$, which is sufficient to show the convergence of $\mathbf{Q}(v)$. Like $\mathbf{a}(t)$ also $\mathbf{Q}(v(t))$ is absolutely continuous in t with derivative

$$\begin{aligned} \frac{d}{dt} \mathbf{Q}(v) &= \frac{d}{dt} \mathbf{Q}(\tilde{v}) = \langle v, e^{-i\partial_x^3} \dot{\tilde{v}} \rangle = \dot{\gamma} \langle v, v' \rangle + [(\dot{\gamma} - c) \langle v, \xi_1[c] \rangle - \dot{c} \langle v, \xi_2[c] \rangle] \\ &\quad + \langle v', f(\phi_c + v) - f(\phi_c) \rangle =: I + II \end{aligned}$$

where the canceled term is null and are hence neglected. Then (3.23) implies that $I \in L^1(\mathbb{R}_+)$. So the key term is II . We partition for $s \in (0, 1)$ the line as

$$\begin{aligned}\Omega_{1,t,s} &:= \{x \in \mathbb{R} \mid |s v(t, x)| \leq 2\phi_{c(t)}(x)\} \text{ and} \\ \Omega_{2,t,s} &:= \mathbb{R} \setminus \Omega_{1,t,s} = \{x \in \mathbb{R} \mid |s v(t, x)| > 2\phi_{c(t)}(x)\}.\end{aligned}\quad (3.31)$$

Then, we have

$$\begin{aligned}II(t) &= \sum_{j=1,2} \int_0^1 ds \int_{\Omega_{j,t,s}} (f'(\phi_{c(t)} + s v(t)) - f'(s v(t))) v(t) v'(t) dx \\ &=: II_1(t) + II_2(t).\end{aligned}$$

We have

$$|II_1(t)| \lesssim \int_{\mathbb{R}} \phi_{c(t)}^{p-1} |v(t) v'(t)| dx \lesssim \|e^{-a(x)} v(t)\|_{H^1(\mathbb{R})}^2 \in L^1(\mathbb{R}_+).$$

We have

$$II_2(t) = - \int_{[0,1]^2} d\tau ds \int_{\Omega_{2,t,s}} f''(\tau \phi_{c(t)} + s v(t)) \phi_{c(t)} v(t) v'(t) dx.$$

Then by $|s v(t, x)| > 2\phi_{c(t)}(x)$, we have the following, which yields (1.10),

$$\begin{aligned}|II_2(t)| &\lesssim \int_{[0,1]^2} d\tau ds \int_{\Omega_{2,t,s}} |s v(t)|^{p-2} \phi_{c(t)} |v(t) v'(t)| dx \\ &\lesssim \int_{\mathbb{R}} \phi_{c(t)}^{p-1} |v(t) v'(t)| dx \lesssim \|e^{-a(x)} v(t)\|_{H^1(\mathbb{R})}^2 \in L^1(\mathbb{R}_+).\end{aligned}$$

□

4. Bounds on the discrete modes: proof of Lemma 3.5

We will prove the following, which implies (3.24),

$$|\dot{c}| + |\dot{\gamma} - c| \lesssim B^{-1} \left(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}} \right). \quad (4.1)$$

We apply to Eq. (3.20) the inner product $\langle \cdot, \zeta_B \eta_1[c] \rangle$ obtaining, thanks to cancellations due to (3.10) and (3.12),

$$\begin{aligned}& -(\dot{\gamma} - c) [\langle \xi_1[c], \zeta_B \eta_1[c] \rangle + \langle v', \zeta_B \eta_1[c] \rangle] + \dot{c} [\langle \xi_2[c], \zeta_B \eta_1[c] \rangle - \langle v, \zeta_B \partial_c \eta_1[c] \rangle] \\ &= -\langle v, \zeta_B \eta_2[c] \rangle + \langle v, [L_{[c]}^*, \zeta_B] \eta_1[c] \rangle + \langle f(\phi_c + v) - f(\phi_c) - f'(\phi_c) v, (\zeta_B \eta_1[c])' \rangle \\ &=: A_0 + A_1 + A_2\end{aligned}\quad (4.2)$$

where we also used the following, due to (3.12),

$$-\langle \dot{v}, \zeta_B \eta_1[c] \rangle = \dot{c} \langle v, \zeta_B \partial_c \eta_1[c] \rangle.$$

By (3.12) we have

$$A_0 = \langle v, \eta_2[c] \rangle - \langle v, (1 - \zeta_B) \eta_2[c] \rangle = -\langle v, (1 - \zeta_B) \eta_2[c] \rangle$$

so that by the definition of $\eta_2[c] = \theta_3 \phi_c$ in (3.4) and using the partition of unity $1 = \theta_{1A_1} + \theta_{2A_1}$ we have

$$\begin{aligned}|A_0| &\leq \left\langle \frac{\zeta_B}{\zeta_A} \zeta_A (\theta_{1A_1} + \theta_{2A_1}) |v|, (\zeta_B^{-1} - 1) \eta_2[c] \right\rangle \leq (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}) \left\| \left(e^{\frac{|x|}{B}(1-\chi)} - 1 \right) \phi_c \right\|_{L^2} \\ &\lesssim (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}) \left\| \frac{|x|}{B} (1 - \chi) \zeta_B^{-1} \phi_c \right\|_{L^2} \lesssim B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}).\end{aligned}$$

With an elementary computation, we have

$$\begin{aligned}[L_{[c]}^*, \zeta_B] &= -L_{+c} \zeta_B' - [L_{+c}, \zeta_B] \partial_x = \zeta_B' (\partial_x^2 - c + f'(\phi_c)) + [\partial_x^2, \zeta_B'] + [\partial_x^2, \zeta_B] \partial_x \\ &= \zeta_B' (\partial_x^2 + f'(\phi_c)) - c \zeta_B' + 2\zeta_B'' \partial_x + \zeta_B''' + (2\zeta_B' \partial_x + \zeta_B'') \partial_x.\end{aligned}$$

Then we write

$$\begin{aligned}A_1 &= \langle v, \zeta_B' (\partial_x^2 + f'(\phi_c)) \eta_1[c] + (2\zeta_B' \partial_x + 3\zeta_B'') \eta_1'[c] \rangle \\ &\quad + \langle v, \zeta_B''' \eta_1[c] - c \zeta_B' \eta_1[c] \rangle =: A_{11} + A_{12} + A_{13}.\end{aligned}$$

Then, using again $1 = \vartheta_{1A_1} + \vartheta_{2A_1}$, we have

$$\begin{aligned} |A_{11}| &\lesssim B^{-1} \sum_{i=1,2} \|\zeta_A \vartheta_{iA_1} v\|_{L^2} \left(\|\zeta_A^{-1} (\partial_x^2 + f'(\phi_c)) \eta_j[c]\|_{L^2} + B \|\zeta_A^{-1} (2\zeta_B' \partial_x + 3\zeta_B'') \eta_j'[c]\|_{L^2} \right) \\ &\lesssim B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \end{aligned}$$

By $|\zeta_B'''| \lesssim B^{-3} \zeta_B \lesssim B^{-3} \zeta_{2B}^2$ we have

$$|A_{12}| \lesssim B^{-3} \|\zeta_{2B}\|_{L^2} \sum_{i=1,2} \left\| \frac{\zeta_{2B}}{\zeta_A} \zeta_A \vartheta_{iA_1} v \right\|_{L^2} \lesssim B^{-\frac{5}{2}} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}).$$

Next we focus on A_{13} . From (2.6) we have

$$\langle v, \zeta_B' \eta_1[c] \rangle = -\frac{1}{B} \langle v, \zeta_B \frac{x}{|x|} \eta_1[c] \rangle + \frac{1}{B} \langle v, \zeta_B \left(\frac{x}{|x|} \chi + |x| \chi' \right) \eta_1[c] \rangle.$$

Then by supp $\chi \subseteq [-2, 2]$ we have like above

$$\left| B^{-1} \langle v, \zeta_B \left(\frac{x}{|x|} \chi + |x| \chi' \right) \eta_1[c] \rangle \right| \lesssim B^{-1} \|v\|_{\Sigma_{1AA_1}}$$

while we write

$$\frac{1}{B} \langle v, \zeta_B \frac{x}{|x|} \eta_1[c] \rangle = -\frac{2}{B} \langle v, \zeta_B \eta_1[c] \rangle_{L^2(\mathbb{R}_-)} + \underline{B^{-1} \langle v, \zeta_B \eta_1[c] \rangle}$$

where the canceled term is null by (3.12). Since $\eta_1[c] \xrightarrow{x \rightarrow -\infty} 0$ exponentially fast, then

$$\begin{aligned} |B^{-1} \langle v, \zeta_B \eta_1[c] \rangle_{L^2(\mathbb{R}_-)}| &\lesssim B^{-1} \left(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}} \right) \|\zeta_A^{-1} \zeta_B \eta_1[c]\|_{L^2(\mathbb{R}_-)} \\ &\lesssim B^{-1} \left(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}} \right). \end{aligned}$$

Hence we have proved

$$|A_{13}| \lesssim B^{-1} \left(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}} \right).$$

Turning to A_2 , for $s \in (0, 1)$ we set

$$\begin{aligned} \Omega_{1,t,s} &:= \{x \in \mathbb{R} \mid 2|sv(t, x)| \leq \phi_{c(t)}(x)\} \text{ and} \\ \Omega_{2,t,s} &:= \mathbb{R} \setminus \Omega_{1,t,s} = \{x \in \mathbb{R} \mid 2|sv(t, x)| > \phi_{c(t)}(x)\}. \end{aligned} \tag{4.3}$$

Then we have

$$A_2 = \sum_{i=1,2} \int_0^1 ds \int_{\Omega_{i,t,s}} (f'(\phi_c + sv) - f'(\phi_c)) v (\zeta_B \eta_1[c])' dx =: \sum_{i=1,2} A_{2i}.$$

We have

$$|A_{21}| \lesssim \int_0^1 ds \int_{\Omega_{1,t,s}} \phi_c^{p-1} |v| |\zeta_B' \eta_1[c]| dx + \int_0^1 ds \int_{\Omega_{1,t,s}} \phi_c^{p-2} s |v|^2 |\zeta_B \eta_1'[c]| dx =: \sum_{i=1,2} A_{21i}$$

Then from $|\zeta_B'| \lesssim B^{-1} \zeta_B$, $\|\eta_1[c]\|_{L^\infty(\mathbb{R})} \lesssim 1$ and $\|\phi_c^{p-1} \zeta_B \zeta_A^{-1}\|_{L^2(\mathbb{R})} \lesssim 1$, we have

$$A_{211} \lesssim B^{-1} \int_{\mathbb{R}} \phi_c^{p-1} |\eta_1[c]| |\zeta_B \zeta_A^{-1} \zeta_A (\vartheta_{1A_1} + \vartheta_{2A_1})| v |dx| \lesssim B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}})$$

and from $\|\eta_1'[c]\|_{L^2(\mathbb{R})} \lesssim 1$ and, in $\Omega_{1,t,s}$, from $\phi_c^{p-2} |sv| = \phi_c^{p-2} |sv|^{\frac{3-p}{2} + \frac{p-1}{2}} \leq \phi_c^{\frac{p-1}{2}} |sv|^{\frac{p-1}{2}}$, we have

$$\begin{aligned} A_{212} &\lesssim \int_{\mathbb{R}} \phi_c^{\frac{p-1}{2}} |v|^{\frac{p-1}{2}+1} \zeta_B (|\partial_c \phi_c| + |\phi_c'|) dx \\ &\lesssim \|v\|_{L^\infty(\mathbb{R})}^{\frac{p-1}{2}} \int_{\mathbb{R}} \phi_c^{\frac{p-1}{2}} (|\partial_c \phi_c| + |\phi_c'|) \zeta_B \zeta_A^{-1} \zeta_A (\vartheta_{1A_1} + \vartheta_{2A_1}) |v| dx \\ &\lesssim B^{\frac{p-1}{4}} \delta^{\frac{p-1}{2}} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}) \end{aligned}$$

where we used $\|v\|_{L^\infty(\mathbb{R})} \lesssim B^{\frac{1}{2}} \delta$ from (3.28). Summing up, we conclude that

$$|A_{21}| \lesssim B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}).$$

Next we turn to

$$|A_{22}| \lesssim \int_{\mathbb{R}} |v|^p |\zeta_B' \eta_1[c]| dx + \int_{\mathbb{R}} |v|^p |\zeta_B \eta_1'[c]| dx =: A_{221} + A_{222}.$$

Then, using again $\|v\|_{L^\infty(\mathbb{R})} \lesssim B^{1/2}\delta$, we have

$$A_{222} \lesssim B^{\frac{p-1}{2}} \delta^{p-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}})$$

and

$$\begin{aligned} A_{221} &\lesssim \delta^{p-1} B^{\frac{p-1}{2}-1} \|\zeta_A^{-1} \zeta_B\|_{L^2} \|\zeta_A (\partial_{1A_1} + \partial_{2A_1}) v\|_{L^2} \\ &\lesssim \delta^{p-1} B^{\frac{p-1}{2}-\frac{1}{2}} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \end{aligned}$$

Collecting the estimates, taking δ sufficiently small so that $B^{\frac{p-1}{4}} \delta^{\frac{p-1}{2}} \leq B^{-1}$, we have

$$|A_0 + A_1 + A_2| \lesssim B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}).$$

Therefore, since

$$|\langle \xi_1[c], \zeta_B \eta_1[c] \rangle - 1| + |\langle \xi_2[c], \zeta_B \eta_1 \rangle| \lesssim B^{-1}$$

and

$$\begin{aligned} \langle v', \zeta_B \eta_1[c] \rangle &= -\langle v, \zeta'_B \eta_1[c] \rangle - \langle v, \zeta_B (\eta_1[c])' \rangle = O(\|v\|_{L^2}) = O(B^{\frac{1}{2}}\delta) \text{ and} \\ \langle v, \zeta_B \partial_c \eta_1[c] \rangle &= O(B^{\frac{1}{2}}\|v\|_{L^2}) = O(B\delta), \end{aligned} \quad (4.4)$$

we have

$$|\dot{\gamma} - c| \lesssim B^{-1} |\dot{c}| + B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \quad (4.5)$$

Next, we apply to Eq. (3.20) the inner product $\langle \cdot, \eta_2[c] \rangle$, obtaining

$$\begin{aligned} (\dot{\gamma} - c) \langle v, \eta_2[c] \rangle' + \dot{c} [1 - \langle v, \partial_c \eta_2[c] \rangle] \\ = \langle f(\phi_c + v) - f(\phi_c) - f'(\phi_c), \eta_2[c] \rangle' =: \tilde{A}_2. \end{aligned} \quad (4.6)$$

Proceeding as above, using (3.28), we have $|\tilde{A}_2| \lesssim B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}})$, where we have used $B^{1/2}\delta \leq B^{-1}$. Thus, we have

$$|\dot{c}| \lesssim B^{1/2}\delta |\dot{\gamma} - c| + B^{-1} (\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \quad (4.7)$$

Combining this with (4.5) and by bootstrap we obtain (4.1). $\square$

5. The virial inequality: proof of Lemma 3.6

Using the notation in (2.7) we set

$$I_i := 2^{-1} \langle v, \varphi_{iAA_1} v \rangle \text{ for } i = 1, 2. \quad (5.1)$$

Lemma 5.1. *There exists a fixed constant $C > 0$ s.t.*

$$\begin{aligned} \|v\|_{\Sigma_{1AA_1}}^2 &\lesssim C \left[\dot{I}_1 + |\dot{\gamma} - c|^2 + |\dot{c}|^2 + \|v\|_{\Sigma}^2 + A_1^{-2} \|v\|_{\Sigma_{2AA_1}}^2 \right] \text{ and} \\ \|v\|_{\Sigma_{2AA_1}}^2 &\lesssim C \left[\dot{I}_2 + A_1^{-2} \|v\|_{\Sigma_{1AA_1}}^2 \right]. \end{aligned} \quad (5.2)$$

Proof. Like in Section 3 we have $t \rightarrow I_i$ absolutely continuous with

$$\begin{aligned} \dot{I}_i &= \frac{d}{dt} 2^{-1} \langle e^{-i\partial_x^3} \tilde{v}, \varphi_{iAA_1} e^{-i\partial_x^3} \tilde{v} \rangle = 2^{-1} \langle v, [\partial_x^3, \varphi_{iAA_1}] v \rangle + \langle e^{-i\partial_x^3} \tilde{v}, \varphi_{iAA_1} e^{-i\partial_x^3} \tilde{v} \rangle \\ &= \langle v, (2^{-1} [\partial_x^3, \varphi_{iAA_1}] + c \varphi_{iAA_1} \partial_x) v \rangle - \langle (f(\phi_c + v) - f(\phi_c))', \varphi_{iAA_1} v \rangle \\ &\quad + (\dot{\gamma} - c) \langle v', \varphi_{iAA_1} v \rangle + (\dot{\gamma} - c) \langle \xi_1[c], \varphi_{iAA_1} v \rangle - \dot{c} \langle \xi_2[c], \varphi_{iAA_1} v \rangle =: \sum_{k=1}^5 B_k^{(i)}. \end{aligned} \quad (5.3)$$

Only for this proof, we set $\omega^{(i)} := \zeta_A \partial_{iA_1} v$. We have

$$B_1^{(i)} = \langle v'', \zeta_A^2 \partial_{iAA_1}^2 v \rangle - \frac{1}{2} \langle \zeta_A \partial_{iA_1} v', \zeta_A \partial_{iA_1} v' \rangle - \frac{c}{2} \|\omega^{(i)}\|_{L^2}^2, \quad (5.4)$$

with

$$\langle v'', \zeta_A^2 \partial_{iAA_1}^2 v \rangle - \frac{1}{2} \langle \zeta_A \partial_{iA_1} v', \zeta_A \partial_{iA_1} v' \rangle = \frac{1}{2} \langle (\zeta_A^2 \partial_{iAA_1}^2)'' v, v^2 \rangle - \frac{3}{2} \langle \zeta_A \partial_{iA_1} v', \zeta_A \partial_{iA_1} v' \rangle$$

where

$$\begin{aligned} \langle \zeta_A \vartheta_{iA_1} v', \zeta_A \vartheta_{iA_1} v' \rangle &= \langle (\zeta_A \vartheta_{iA_1} v)' - (\zeta_A \vartheta_{iA_1})' v, (\zeta_A \vartheta_{iA_1} v)' - (\zeta_A \vartheta_{iA_1})' v \rangle \\ &= \|(\omega^{(i)})'\|_{L^2}^2 - 2\langle (\zeta_A \vartheta_{iA_1})' v, (\zeta_A \vartheta_{iA_1})' v + \zeta_A \vartheta_{iA_1} v' \rangle + \langle (\zeta_A \vartheta_{iA_1})'^2, v^2 \rangle \\ &= \|(\omega^{(i)})'\|_{L^2}^2 - \langle ((\zeta_A \vartheta_{iA_1})')^2, v^2 \rangle + \langle ((\zeta_A \vartheta_{iA_1})' \zeta_A \vartheta_{iA_1})', v^2 \rangle \\ &= \|(\omega^{(i)})'\|_{L^2}^2 + \langle (\zeta_A \vartheta_{iA_1})'' \zeta_A \vartheta_{iA_1}, v^2 \rangle. \end{aligned}$$

Then we conclude the following,

$$B_1^{(i)} = -\frac{3}{2} \|(\omega^{(i)})'\|_{L^2}^2 - \frac{c}{2} \|\omega^{(i)}\|_{L^2}^2 + B_{11}^{(i)} \text{ with } B_{11}^{(i)} := \langle ((\zeta_A \vartheta_{iA_1})')^2 - \frac{1}{2} (\zeta_A \vartheta_{iA_1})'' \zeta_A \vartheta_{iA_1}, v^2 \rangle.$$

We have

$$\begin{aligned} B_{11}^{(i)} &= \langle (\zeta_A')^2 - \frac{1}{2} \zeta_A \zeta_A'' \rangle \vartheta_{iA_1}^2, v^2 \rangle + \langle \zeta_A' \vartheta_{iA_1}' v, \zeta_A \vartheta_{iA_1} v \rangle \\ &\quad + \langle (\vartheta_{iA_1}')^2 - \frac{1}{2} \vartheta_{iA_1}'' \vartheta_{iA_1}, \zeta_A^2 v^2 \rangle =: \sum_{k=1}^3 B_{11k}^{(i)}. \end{aligned}$$

From $|\zeta_A^{(k)}| \lesssim A^{-k} \zeta_A$ we have

$$|B_{111}^{(i)}| \lesssim A^{-2} \|\omega^{(i)}\|_{L^2}^2$$

while from $A \gg A_1$

$$\begin{aligned} |B_{112}^{(i)} + B_{113}^{(i)}| &\lesssim A^{-1} \left(\|\omega^{(1)}\|_{L^2}^2 + \|\omega^{(2)}\|_{L^2}^2 \right) + A_1^{-2} \|v\|_{L^2(A_1, 2A_1)}^2 \\ &\lesssim A_1^{-2} \left(\|\omega^{(1)}\|_{L^2}^2 + \|\omega^{(2)}\|_{L^2}^2 \right). \end{aligned} \quad (5.5)$$

Moving to $B_2^{(i)}$, we claim that for any fixed small constant $\varepsilon_1 > 0$

$$|B_2^{(1)}| \lesssim \varepsilon_1 \|v\|_{\Sigma_{1AA_1}}^2 + A_1^{-100} \|v\|_{\Sigma_{2AA_1}}^2 + \varepsilon_1^{-1} \|v\|_{\Sigma}^2 \text{ and} \quad (5.6)$$

$$|B_2^{(2)}| \lesssim A_1^{-100} \left(\|v\|_{\Sigma_{1AA_1}}^2 + \|v\|_{\Sigma_{2AA_1}}^2 \right). \quad (5.7)$$

To prove (5.6) and (5.7) we write

$$B_2^{(i)} = -\langle (f(\phi_c + v) - f(\phi_c) - f(v))', \varphi_{iAA_1} v \rangle - \langle (f(v))', \varphi_{iAA_1} v \rangle =: B_{21}^{(i)} + B_{22}^{(i)}.$$

From

$$B_{22}^{(i)} = -\langle (f(v))', \varphi_{iAA_1} v \rangle = \langle f(v) v', \varphi_{iAA_1} \rangle + \langle f(v) v, \zeta_A^2 \vartheta_{iA_1}^2 \rangle = -\langle F(v) - f(v) v, \zeta_A^2 \vartheta_{iA_1}^2 \rangle$$

using (3.28) we obtain

$$|B_{22}^{(i)}| \lesssim B^{\frac{p-1}{2}} \delta^{p-1} \|v\|_{\Sigma_{iAA_1}}^2. \quad (5.8)$$

Setting

$$\begin{aligned} \Omega_{1,t} &:= \{x \in \mathbb{R} \mid 2|v(t, x)| \leq \phi_{c(t)}(x)\} \text{ and} \\ \Omega_{2,t} &:= \mathbb{R} \setminus \Omega_{1,t} = \{x \in \mathbb{R} \mid 2|v(t, x)| > \phi_{c(t)}(x)\} \end{aligned} \quad (5.9)$$

we split

$$B_{21}^{(i)} = \sum_{k=1,2} \langle f(\phi_c + v) - f(\phi_c) - f(v), (\varphi_{iAA_1} v)' \rangle_{L^2(\Omega_{k,t})} =: \sum_{k=1,2} B_{21k}^{(i)}.$$

We further split

$$B_{212}^{(i)} = -\langle f(\phi_c), (\varphi_{iAA_1} v)' \rangle_{L^2(\Omega_{2,t})} + \int_0^1 \langle f'(v + s\phi_c) \phi_c, (\varphi_{iAA_1} v)' \rangle_{L^2(\Omega_{2,t})} ds =: B_{2121}^{(i)} + B_{2122}^{(i)}.$$

By $\phi_c^p \lesssim \phi_c^{\frac{p-1}{2}} |v|^{\frac{p-1}{2}} |v|$ in $\Omega_{2,t}$ and by (3.28), we have

$$\begin{aligned} |B_{2121}^{(i)}| &\lesssim \int_{\mathbb{R}} \phi_c^{\frac{p-1}{2}} |v|^{\frac{p-1}{2}+1} \zeta_A \zeta_A^{-1} \left| \varphi_{iAA_1} \zeta_A^{-1} (\zeta_A v)' + \zeta_A v (\varphi_{iAA_1} \zeta_A^{-1})' \right| dx \\ &\lesssim B^{\frac{p-1}{4}} \delta^{\frac{p-1}{2}} \|\zeta_A v\|_{H^1}^2 \lesssim B^{\frac{p-1}{4}} \delta^{\frac{p-1}{2}} (\|v\|_{\Sigma_{1AA_1}}^2 + \|v\|_{\Sigma_{2AA_1}}^2). \end{aligned}$$

By $\phi_c \lesssim |v|^{1-\frac{p-1}{2}} \phi_c^{\frac{p-1}{2}}$ in Ω_{2A_1} and by $1 = \vartheta_{1A_1} + \vartheta_{2A_1}$, we get the following,

$$\begin{aligned} |B_{2122}^{(i)}| &\lesssim \int_{\mathbb{R}} |v|^{p-1} \phi_c \left| \varphi_{iAA_1} \zeta_A^{-1} (\zeta_A v)' + \zeta_A v (\varphi_{iAA_1} \zeta_A^{-1})' \right| dx \\ &\lesssim B^{\frac{p-1}{4}} \delta^{\frac{p-1}{2}} \int_{\mathbb{R}} |v| \zeta_A \zeta_A^{-1} \phi_c^{\frac{p-1}{2}} \left| \varphi_{iAA_1} \zeta_A^{-1} (\zeta_A v)' + \zeta_A v (\varphi_{iAA_1} \zeta_A^{-1})' \right| dx \\ &\lesssim B^{\frac{p-1}{4}} \delta^{\frac{p-1}{2}} \|(\vartheta_{1A_1} + \vartheta_{2A_1}) \zeta_A v\|_{H^1}^2 \lesssim B^{\frac{p-1}{4}} \delta^{\frac{p-1}{2}} (\|v\|_{\Sigma_{1AA_1}}^2 + \|v\|_{\Sigma_{2AA_1}}^2), \end{aligned}$$

where we used (3.28). Next we split

$$\begin{aligned} B_{211}^{(i)} &= \langle f'(\phi_c) v, (\varphi_{iAA_1} v)' \rangle_{L^2(\Omega_{1,r})} + \int_0^1 \langle [f'(\phi_c + sv) - f'(\phi_c)] v, (\varphi_{iAA_1} v)' \rangle_{L^2(\Omega_{1,r})} ds \\ &\quad - \int_0^1 \langle f'(sv) v, (\varphi_{iAA_1} v)' \rangle_{L^2(\Omega_{1,r})} ds =: \sum_{k=1}^3 B_{211k}^{(i)}. \end{aligned}$$

We have

$$\begin{aligned} |B_{2113}^{(i)}| &\lesssim \langle |v|^p, |\varphi_{iAA_1} v' + \zeta_A^2 \vartheta_{iA_1}^2 v| \rangle_{L^2(\Omega_{1,r})} \lesssim \langle \phi_c^{\frac{p-1}{2}} |v|^{\frac{p-1}{2}} |v|, |\zeta_A^{-1} \varphi_{iAA_1} \zeta_A v' + \zeta_A^2 \vartheta_{iA_1}^2 v| \rangle_{L^2(\Omega_{1,r})} \\ &\lesssim \|v\|_{L^\infty(\mathbb{R})}^{\frac{p-1}{2}} \|\zeta_A v\|_{H^1}^2 \lesssim B^{\frac{p-1}{2}} \delta^{\frac{p-1}{2}} (\|v\|_{\Sigma_{1AA_1}}^2 + \|v\|_{\Sigma_{2AA_1}}^2). \end{aligned} \quad (5.10)$$

By concavity $|f'(\phi_c + sv) - f'(\phi_c)| \leq |f'(sv)|$ (in the proof $1 < p \leq 2$, Remark 1.7) and as in (5.10)

$$|B_{2112}^{(i)}| \lesssim B^{\frac{p-1}{2}} \delta^{\frac{p-1}{2}} (\|v\|_{\Sigma_{1AA_1}}^2 + \|v\|_{\Sigma_{2AA_1}}^2).$$

With $B_{2111}^{(i)}$ we make a clear distinction between the two cases $i = 1, 2$. For $i = 1$, for a preassigned fixed small constant $\varepsilon_1 > 0$, we have the following, which completes the proof of (5.6),

$$\begin{aligned} |B_{2111}^{(1)}| &= \left| \langle f'(\phi_c) v, (\vartheta_{1A_1} + \vartheta_{2A_1}) \varphi_{1AA_1} v' + \zeta_A^2 \vartheta_{1A_1}^2 v \rangle_{L^2(\Omega_{1,r})} \right| \\ &\lesssim \varepsilon_1 \|v\|_{\Sigma_{1AA_1}}^2 + A_1^{-100} \|v\|_{\Sigma_{2AA_1}}^2 + \varepsilon_1^{-1} \|v\|_{\Sigma}^2 \end{aligned} \quad (5.11)$$

where we exploit the exponential decay at infinity of $f'(\phi_c)$ and

$$\text{supp } \varphi_{2AA_1} \subseteq \text{supp } \vartheta_{2A_1} \subseteq [A_1, +\infty). \quad (5.12)$$

In the $i = 2$ case once again by (5.12) we obtain the following, which yields (5.7),

$$\begin{aligned} |B_{2111}^{(2)}| &= \left| \langle f'(\phi_c) v, (\vartheta_{1A_1} + \vartheta_{2A_1}) \varphi_{2AA_1} v' + \zeta_A^2 \vartheta_{2A_1}^2 v \rangle_{L^2(\Omega_{1,r} \cap \{x \geq A_1\})} \right| \\ &\lesssim A_1^{-100} (\|v\|_{\Sigma}^2 + \|v\|_{\Sigma_{1AA_1}}^2 + \|v\|_{\Sigma_{2AA_1}}^2) \lesssim A_1^{-100} (\|v\|_{\Sigma_{1AA_1}}^2 + \|v\|_{\Sigma_{2AA_1}}^2). \end{aligned} \quad (5.13)$$

We examine the remaining terms in (5.3). By (3.28) and (4.1) we have

$$|B_3^{(i)}| = 2^{-1} |\dot{\gamma} - c| \|v\|_{\Sigma_{iAA_1}}^2 \lesssim \delta \|v\|_{\Sigma_{iAA_1}}^2.$$

Finally, exploiting (5.12) and the exponential decay at infinity of $\xi_k[c]$ for $k = 1, 2$, we have

$$|B_4^{(i)}| + |B_5^{(i)}| \lesssim A_1^{-100(i-1)} (|\dot{\gamma} - c|^2 + |\dot{c}|^2 + \|v\|_{\Sigma}^2).$$

Bootstrapping gives the following and with (4.1) yields (5.2) proving Lemma 5.1,

$$\begin{aligned} \|v\|_{\Sigma_{1AA_1}}^2 &\lesssim \tilde{C} \left[\dot{I}_1 + \|v\|_{\Sigma}^2 + |\dot{\gamma} - c|^2 + |\dot{c}|^2 + A_1^{-2} \|v\|_{\Sigma_{2AA_1}}^2 \right] \text{ and} \\ \|v\|_{\Sigma_{2AA_1}}^2 &\lesssim \tilde{C} \left[\dot{I}_2 + A_1^{-2} (|\dot{\gamma} - c|^2 + |\dot{c}|^2) + A_1^{-2} \|v\|_{\Sigma_{1AA_1}}^2 \right]. \end{aligned}$$

□

Proof of Lemma 3.6. We integrate in the time interval I the inequalities in (5.2) and we use

$$|I_i(t)| \leq \|\varphi_{iAA_1}\|_{L^\infty(\mathbb{R})} \|v(t)\|_{L^2(\mathbb{R})}^2 \lesssim AB\delta^2 \text{ for } i = 1, 2.$$

By Lemmas 3.5 and 5.1, for $A_1^{10} = B^2 \gg B$ the following ensues, proving Lemma 3.6,

$$\begin{aligned} \|v\|_{L^2(I, \Sigma_{1AA_1})} &\lesssim \sqrt{AB\delta} + \|v\|_{L^2(I, \Sigma)} + A_1^{-1} \|v\|_{L^2(I, \Sigma_{2AA_1})} \text{ and} \\ \|v\|_{L^2(I, \Sigma_{2AA_1})} &\lesssim \sqrt{AB\delta} + A_1^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})}. \end{aligned}$$

□

To complete the proof of Proposition 3.4 we are left with the proof of Lemma 3.7, which will take Sections 6–9. For this we will first state some linear dispersive and smoothing estimates for the linearization operator in (3.2). We will then use this linear theory to prove Lemma 3.7. Finally, we will prove the linear estimates.

6. Dispersion and Kato smoothing for the linearized equation

We will need the dispersive estimates for the linearized operator in (3.2) due to Pego and Weinstein [14]. We consider the operators

$$P_{[c]} := \sum_{i=1,2} \xi_i[c] \langle \eta_i[c], \cdot \rangle \text{ and } Q_{[c]} = 1 - P_{[c]}. \quad (6.1)$$

Here $c = 1$ and we will omit the $[c]$. These are the operators associated to the spectral decomposition

$$L_a^2 = N_g(\mathcal{L}) \oplus N_g^\perp(\mathcal{L}^*) \quad (6.2)$$

where the perpendicularity is with $N_g(\mathcal{L}^*) \subseteq L_{-a}^2$ in terms of the L^2 inner product in (1.3). Then we quote Pego and Weinstein [14, Theorem 4.2] to which we refer for the proof, where here $c = 1$. Notice that the following follows for all $p \in (1, 5)$ by Proposition 3.2.

Theorem 6.1. For $a \in (0, \sqrt{1/3})$ and $p \in (1, 5)$ there are constants $C(a, p) > 0$ and $b(a, p) > 0$ s.t.

$$\|e^{t\mathcal{L}} Q v_0\|_{L_a^2} \leq C(a, p) \langle t \rangle^{-1/2} e^{-b(a, p)t} \|Q v_0\|_{L_a^2} \text{ for all } v_0 \in L_a^2. \quad (6.3)$$

□

Besides the above dispersive estimate, we will need the following version of the Kato smoothing, which we will prove in this paper. Notice that in (6.4) we use in the left hand side the weighted spaces in (2.3) while in (6.5) we have the standard $L^1(\mathbb{R})$ space. Later we will need to examine in detail why in (6.4) and (6.8) instead of $L^1(\mathbb{R})$ we need the space $L_+^{1,1}$ defined in (2.8).

Proposition 6.2. For any $p \in (1, 5)$ there is a constant $C(p) > 0$ such that

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} Q f(s) ds \right\|_{L^2(\mathbb{R}_+, \tilde{\Sigma})} \leq C(p) \|f\|_{L_+^{1,1}(\mathbb{R}, L^2(\mathbb{R}_+))} \text{ for all } f \in L_+^{1,1}(\mathbb{R}, L^2(\mathbb{R}_+)) \text{ and} \quad (6.4)$$

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} Q \partial_x f(s) ds \right\|_{L^2(\mathbb{R}_+, \tilde{\Sigma})} \leq C(p) \|f\|_{L^1(\mathbb{R}, L^2(\mathbb{R}_+))} \text{ for all } f \in L^1(\mathbb{R}, L^2(\mathbb{R}_+)). \quad (6.5)$$

Proof. The two estimates are proved similarly. Pego and Weinstein [14, p. 328] show that $e^{t\mathcal{L}}$ for $t \geq 0$ is a C_0 semigroup in $L^2(\mathbb{R})$ and also in the L_a^2 in Theorem 6.1. Then, see [47, p. 240],

$$R_{\mathcal{L}}(z) Q w_0 = - \int_0^{+\infty} e^{-zt} e^{t\mathcal{L}} Q w_0 dt \text{ for any } \operatorname{Re} z > -b(a, p), \quad (6.6)$$

gives the resolvent in L_a^2 . By Theorem 6.1 and by Plancherel, see [34, XIII.7, Lemma 1], we have

$$\frac{1}{2\pi} \|R_{\mathcal{L}}(i\lambda) Q w_0\|_{L^2(\mathbb{R}, L_a^2)} = \|e^{t\mathcal{L}} Q w_0\|_{L^2(\mathbb{R}_+, L_a^2)} \leq C(a, p) \|w_0\|_{L_a^2}$$

for some $C(a, p) > 0$. Since for $a \in (0, \kappa)$ we have $L_a^2 \subset \tilde{\Sigma}$ we conclude also

$$\frac{1}{2\pi} \|R_{\mathcal{L}}(i\lambda) Q w_0\|_{L^2(\mathbb{R}, \tilde{\Sigma})} = \|e^{t\mathcal{L}} Q w_0\|_{L^2(\mathbb{R}_+, \tilde{\Sigma})} \leq \|e^{t\mathcal{L}} Q w_0\|_{L^2(\mathbb{R}_+, L_a^2)} \leq C(a, p) \|w_0\|_{L_a^2}.$$

For $f \in C_c^\infty([0, +\infty))$ we have

$$\begin{aligned} \int_{\mathbb{R}} dt e^{-i\lambda t} \int_0^t e^{(t-s)\mathcal{L}} Q f(s) ds &= \int_0^{+\infty} ds e^{-i\lambda s} \int_0^{+\infty} e^{-i\lambda t} e^{t\mathcal{L}} Q f(s) dt \\ &= - \int_0^{+\infty} e^{-i\lambda s} R_{\mathcal{L}}(i\lambda) Q f(s) ds = -R_{\mathcal{L}}(i\lambda) Q \hat{f}(\lambda) \end{aligned}$$

where, in the first equality, the exchange of integrals is allowed by the absolute integrability in (t, s) , thanks to (6.3) and the compact support of f . Then by Plancherel

$$\begin{aligned} 2\pi \left\| \int_0^t e^{(t-s)\mathcal{L}} Q f(s) ds \right\|_{L^2(\mathbb{R}_+, \tilde{\Sigma})} &= \|R_{\mathcal{L}}(i\lambda) Q \hat{f}(\lambda)\|_{L^2(\mathbb{R}, \tilde{\Sigma})} \\ &\leq \|R_{\mathcal{L}}(i\lambda) Q\|_{L^\infty(\mathbb{R}, \mathcal{L}(L_+^{1,1}(\mathbb{R}, \tilde{\Sigma}))} \|f\|_{L_+^{1,1}(\mathbb{R}, L^2(\mathbb{R}_+))}. \end{aligned} \quad (6.7)$$

The proof of (6.4) is completed by means of Proposition 6.3, see below. The proof of (6.5) is similar.

□

The following is proved in Section 9 using the information on the Jost functions derived in Section 8. We will examine in considerable detail the reason why in (6.8) we need to use the space $L_+^{1,1}(\mathbb{R})$ instead of the larger space $L^1(\mathbb{R})$.

Proposition 6.3. For any $p \in (1, 5)$ there is a constant $C(p) > 0$ such that

$$\sup_{\lambda \in \mathbb{R} \setminus \{0\}} \|R_{\mathcal{L}}(i\lambda) Q\|_{\mathcal{L}(L_+^{1,1}(\mathbb{R}, \tilde{\Sigma}))} \leq C(p) \text{ and} \quad (6.8)$$

$$\sup_{\lambda \in \mathbb{R} \setminus \{0\}} \|R_{\mathcal{L}}(i\lambda) Q \partial_x\|_{\mathcal{L}(L^1(\mathbb{R}, \tilde{\Sigma}))} \leq C(p). \quad (6.9)$$

7. Bounding $\|v\|_{\Sigma}$ by Kato smoothing

We prove Lemma 3.7 assuming Proposition 6.3. Multiplying Eq. (3.20) by ζ_B we obtain

$$\begin{aligned}\partial_t(\zeta_B v) &= \mathcal{L}(\zeta_B v) + [\zeta_B, \mathcal{L}]v \\ &\quad + (\dot{\gamma} - c)\zeta_B v' + (\dot{\gamma} - c)\zeta_B \xi_1[c] + \dot{c}\zeta_B \xi_2[c] \\ &\quad + \zeta_B(\mathcal{L}_{[c]} - \mathcal{L})v - \zeta_B \partial_x(f(\phi_c + v) - f(\phi_c) - f'(\phi_c)v).\end{aligned}\quad (7.1)$$

Next we consider the spectral decomposition

$$\zeta_B v = P\zeta_B v + w \text{ where } w := Q\zeta_B v. \quad (7.2)$$

The first and preliminary observation is the following.

Lemma 7.1. *We have*

$$\|P\zeta_B v\|_{L^2} \lesssim B^{-1}(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \quad (7.3)$$

Proof.

We have the following,

$$P\zeta_B v = P_{[c]}\zeta_B v + (P - P_{[c]})\zeta_B v.$$

Then, since $\langle \eta_1[c], \zeta_B v \rangle = 0$ by (3.12), we obtain

$$P_{[c]}\zeta_B v = \xi_2[c]\langle \eta_2[c], \zeta_B v \rangle = \xi_2[c]\langle \eta_2[c], (\zeta_B - 1)(\vartheta_{1A_1} + \vartheta_{2A_1})v \rangle = O\left(B^{-1}(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}})\right).$$

Next we focus on

$$(P_{[c]} - P)\zeta_B v = \sum_{i=1,2} [(\xi_i[c] - \xi_i)\langle \eta_i[c], \zeta_B v \rangle + \xi_i\langle \eta_i[c] - \eta_i, \zeta_B v \rangle].$$

Here (3.12) implies $\langle \eta_1[c], \zeta_B v \rangle = 0$ and

$$|\langle \eta_2[c], \zeta_B v \rangle| \lesssim B^{-1}\|\zeta_A v\|_{L^2} = B^{-1}\|\zeta_A(\vartheta_{1A_1} + \vartheta_{2A_1})v\|_{L^2} \leq B^{-1}(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \quad (7.4)$$

It is elementary that

$$\|\xi_2\langle \eta_2[c] - \eta_2, \zeta_B v \rangle\|_{L^2} \lesssim |\langle \eta_2[c] - \eta_2, \zeta_B v \rangle| \lesssim |c - 1|\|\zeta_B v\|_{L^2} \lesssim \delta(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \quad (7.5)$$

We have the following, where we exploit (3.12),

$$\|\xi_1\langle \eta_1[c] - \eta_1, \zeta_B v \rangle\|_{L^2} \lesssim |\langle \eta_1[c] - \eta_1, \zeta_B v \rangle| = |\langle \eta_1, \zeta_B v \rangle|. \quad (7.6)$$

Recall by (3.4) that

$$\langle \eta_1, \zeta_B v \rangle = \theta_1(1)\left\langle \int_{-\infty}^x \partial_c \phi_c(y) dy, \zeta_B v \right\rangle + \frac{\theta_2(1)}{\theta_3(1)}\langle \eta_2, \zeta_B v \rangle =: I + II.$$

Then like in (7.4) and (7.5)

$$|II| \lesssim |\langle \eta_2[c], \zeta_B v \rangle| + |\langle \eta_2 - \eta_2[c], \zeta_B v \rangle| \lesssim B^{-1}(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}). \quad (7.7)$$

We have

$$\begin{aligned}I &= \langle \theta_1(1) \int_{-\infty}^x \Lambda_p \phi(y) dy - \frac{\theta_1(1) \int_{\mathbb{R}} \Lambda_p \phi(y) dy}{\theta_1(c) \int_{\mathbb{R}} \partial_c \phi_c(y) dy} \theta_1(c) \int_{-\infty}^x \partial_c \phi_c(y) dy, \zeta_B v \rangle \\ &= \theta_1(1) \left\langle \int_{-\infty}^x \Lambda_p \phi(y) dy - \frac{\int_{\mathbb{R}} \Lambda_p \phi(y) dy}{c^{\frac{2-p}{p-1} - \frac{1}{2}} \int_{\mathbb{R}} \Lambda_p \phi(y) dy} c^{\frac{2-p}{p-1}} \int_{-\infty}^x \Lambda_p \phi(\sqrt{c}y) dy, \zeta_B v \right\rangle \\ &= \theta_1(1) \left\langle \int_{\sqrt{cx}}^x \Lambda_p \phi(y) dy, \zeta_B v \right\rangle.\end{aligned}$$

Since now $|\Lambda_p \phi(x)| \lesssim e^{-\frac{|x|}{2}}$ then for $x > 0$ and for $c > 1$

$$\left| \int_{\sqrt{cx}}^x \Lambda_p \phi(y) dy \right| \lesssim 2e^{-\frac{x}{2}} \left(1 - e^{-\frac{\sqrt{c}-1}{2}x} \right) \leq 2e^{-\frac{x}{2}} \frac{\sqrt{c}-1}{2} x$$

and proceeding similarly there is a $K > 0$ such that for c close enough to 1 we have

$$\left| \int_{\sqrt{cx}}^x \Lambda_p \phi(y) dy \right| \leq K e^{-\frac{|x|}{4}} |c - 1| \text{ for all } x \in \mathbb{R} \quad (7.8)$$

where we exploit also the fact that the left hand side is even in x . From (7.8) we conclude

$$|I| \lesssim \delta(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}})$$

which, together with (7.7), yields the following which completes the proof of Lemma 7.1,

$$|\langle \eta_1, \zeta_B v \rangle| \lesssim \delta(\|v\|_{\Sigma_{1AA_1}} + \|v\|_{\Sigma_{2AA_1}}).$$

□

We now focus on the w in (7.2). Applying the projection Q in Section 6 (case $c = 1$) to (7.1) gives

$$\begin{aligned} \dot{w} &= \mathcal{L}w + Q[\zeta_B, \mathcal{L}]v \\ &+ (\dot{\gamma} - c)Q\zeta_B v' + (\dot{\gamma} - c)Q\zeta_B \xi_1[c] + \dot{c} Q\zeta_B \xi_2[c] \end{aligned} \quad (7.9)$$

$$+ Q\zeta_B(\mathcal{L}_{[c]} - \mathcal{L})v \quad (7.10)$$

$$- Q\zeta_B \partial_x (f(\phi_c + v) - f(\phi_c) - f'(\phi_c)v). \quad (7.11)$$

We next consider

$$w = e^{t\mathcal{L}}w(0) + \int_0^t e^{(t-s)\mathcal{L}}Q[\zeta_B, \mathcal{L}]v ds + \int_0^t e^{(t-s)\mathcal{L}} \text{lines (7.9)–(7.11)} ds. \quad (7.12)$$

Lemma 7.2. *We have*

$$\|e^{t\mathcal{L}}w(0)\|_{L^2(\mathbb{R}_+, \tilde{\Sigma})} \leq \sqrt{\delta}. \quad (7.13)$$

Proof. By Theorem 6.1 we have

$$\begin{aligned} \|e^{t\mathcal{L}}w(0)\|_{\tilde{\Sigma}} &\leq \|e^{t\mathcal{L}}w(0)\|_{L^2_{\frac{1}{B}}} \lesssim C(1/B)\langle t \rangle^{-1/2} e^{-b(1/B,p)t} \|w(0)\|_{L^2_{\frac{1}{B}}} \\ &\leq C(1/B)\langle t \rangle^{-1/2} e^{-b(1/B,p)t} \left(\|e^{\frac{x}{B}} \zeta_B v(0)\|_{L^2} + \|e^{\frac{x}{B}} P\zeta_B v(0)\|_{L^2} \right) \end{aligned}$$

From (3.28), we have

$$\|e^{\frac{x}{B}} \zeta_B v(0)\|_{L^2} \lesssim \|v(0)\|_{L^2} \lesssim B^{1/2} \delta$$

and by (7.3) similarly

$$\|e^{\frac{x}{B}} P\zeta_B v(0)\|_{L^2} \lesssim \sum_{i=1}^2 |\langle \eta_i, \zeta_B v(0) \rangle| \lesssim B^{1/2} \|v(0)\|_{\tilde{\Sigma}} \lesssim B\delta.$$

Hence the following whose $L^2(\mathbb{R})$ norm in time, for δ very small in terms of B , yields (7.13)

$$\|e^{t\mathcal{L}}w(0)\|_{\tilde{\Sigma}} \lesssim C(1/B)\langle t \rangle^{-1/2} e^{-b(1/B,p)t} B^{1/2} \delta.$$

□

In Lemmas 7.3–7.6, we will use (6.4) with $L^{1,1}$ instead of $L_+^{1,1}$, which obviously holds from $\|\cdot\|_{L_+^{1,1}} \leq \|\cdot\|_{L^{1,1}}$. The only place we need $L_+^{1,1}$ instead of $L^{1,1}$ will be Lemma 7.7.

Lemma 7.3. *We have*

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} \text{line (7.9)} ds \right\|_{L^2(I, \tilde{\Sigma})} \lesssim \|\dot{\gamma} - c\|_{L^2(I)} + \|\dot{c}\|_{L^2(I)}. \quad (7.14)$$

Proof. Applying Proposition 6.2 we conclude immediately

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\mathcal{L}} (\dot{\gamma} - c) Q\zeta_B \xi_1[c] ds \right\|_{L^2(I, \tilde{\Sigma})} &\lesssim \|(\dot{\gamma} - c) \zeta_B \xi_1[c]\|_{L^{1,1}(\mathbb{R}, L^2(I))} \\ &\lesssim \|(\dot{\gamma} - c) \xi_1[c]\|_{L^{2,2}(\mathbb{R}, L^2(I))} = \|(\dot{\gamma} - c) \xi_1[c]\|_{L^2(I, L^{2,2}(\mathbb{R}))} \\ &\lesssim \|(\dot{\gamma} - c)\|_{L^2(I)} \|\xi_1[c]\|_{L^\infty(I, L^{2,2}(\mathbb{R}))} \lesssim \|(\dot{\gamma} - c)\|_{L^2(I)} \end{aligned}$$

and similarly

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} \dot{c} Q\zeta_B \xi_2[c] ds \right\|_{L^2(I, \tilde{\Sigma})} \lesssim \|\dot{c}\|_{L^2(I)}.$$

Similarly applying Proposition 6.2 and (3.28) for the remaining term we have

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\mathcal{L}} (\dot{\gamma} - c) Q \zeta_B v' ds \right\|_{L^2(I, \tilde{\Sigma})} \lesssim \|(\dot{\gamma} - c) \zeta_B v'\|_{L^{1,1}(\mathbb{R}, L^2(I))} \\ & \lesssim \|\zeta_B\|_{L^{2,1}(\mathbb{R})} \|(\dot{\gamma} - c) v'\|_{L^2(\mathbb{R}, L^2(I))} \lesssim B^{\frac{3}{2}} \|(\dot{\gamma} - c)\|_{L^2(I)} \|v'\|_{L^\infty(I, L^2(\mathbb{R}))} \\ & \lesssim B^2 \delta \|(\dot{\gamma} - c)\|_{L^2(I)}. \end{aligned}$$

□

Lemma 7.4. We have

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} \text{line (7.10)} ds \right\|_{L^2(I, \tilde{\Sigma})} \lesssim \sqrt{\delta} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}). \quad (7.15)$$

Proof. We use again Proposition 6.2 to obtain

$$\begin{aligned} & \left\| \int_0^t e^{(t-s)\mathcal{L}} Q \zeta_B (\mathcal{L}_{[c]} - \mathcal{L}) v ds \right\|_{L^2(I, \tilde{\Sigma})} \\ & \lesssim \|\zeta_B (c - 1) v'\|_{L^{1,1}(\mathbb{R}, L^2(I))} + \|\zeta_B \partial_x (\phi^{p-1} - \phi_c^{p-1}) v\|_{L^{1,1}(\mathbb{R}, L^2(I))}. \end{aligned}$$

From (3.14), we have

$$\begin{aligned} \|c - 1\|_{L^\infty(\mathbb{R})} \|\zeta_B v'\|_{L^{1,1}(\mathbb{R}, L^2(I))} & \lesssim \delta \|\zeta_{2B}\|_{L^{2,1}(\mathbb{R})} \|\zeta_{2B} v'\|_{L^2(\mathbb{R}, L^2(I))} \\ & \lesssim B^{3/2} \delta \left\| \frac{\zeta_{2B}}{\zeta_A} \zeta_A (\vartheta_{1A_1} + \vartheta_{2A_1}) v' \right\|_{L^2(I, L^2(\mathbb{R}))} \\ & \lesssim B^{3/2} \delta (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}) \end{aligned}$$

and similarly, it is possible to prove the following, which yields the desired inequality in (7.15)

$$\|\zeta_B \partial_x (\phi^{p-1} - \phi_c^{p-1}) v\|_{L^{1,1}(\mathbb{R}, L^2(I))} \lesssim \delta (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}).$$

□

Next we have the following. Recall, see Remark 1.7, that in the proof we are assuming $1 < p < 2$.

Lemma 7.5. We have

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} \text{line (7.11)} ds \right\|_{L^2(I, \tilde{\Sigma})} \lesssim \delta^{\frac{p-1}{2}} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}). \quad (7.16)$$

Proof. By Proposition 6.2 the left hand side of (7.16) is up to a fixed constant bounded above by

$A_1 + A_2$ where

$$A_1 = \left\| \zeta_B (f'(\phi_c + v) - f'(\phi_c) - f''(\phi_c)v) \phi'_c \right\|_{L^{1,1}(\mathbb{R}, L^2(I))} \text{ and}$$

$$A_2 = \left\| \zeta_B (f'(\phi_c + v) - f'(\phi_c)) v' \right\|_{L^{1,1}(\mathbb{R}, L^2(I))}.$$

We have

$$\begin{aligned} A_1 & \leq \left\| \zeta_B (f'(\phi_c + v) - f'(\phi_c) - f''(\phi_c)v) \phi'_c \right\|_{L^{1,1} L^2(\{|v| \ll \phi_c\})} \\ & \quad + \left\| \zeta_B (f'(\phi_c + v) - f'(\phi_c) - f''(\phi_c)v) \phi'_c \right\|_{L^{1,1} L^2(\{|v| \gtrsim \phi_c\})} =: A_{11} + A_{12}. \end{aligned}$$

By $\zeta_{2B} = \sqrt{\zeta_B}$, (2.1), (2.9) and (3.28) we have

$$\begin{aligned} A_{11} & \leq \int_0^1 \left\| \zeta_B (f''(\phi_c + sv) - f''(\phi_c)) v \phi'_c \right\|_{L^{1,1} L^2(\{|v| \ll \phi_c\})} ds \\ & \leq \int_{[0,1]^2} \left\| \zeta_B f'''(\phi_c + \tau sv) v^2 \phi'_c \right\|_{L^{1,1} L^2(\{|v| \ll \phi_c\})} s ds d\tau \\ & \lesssim \left\| \zeta_B \phi_c^{p-3} v^2 \phi'_c \right\|_{L^{1,1} L^2(\{|v| \ll \phi_c\})} \lesssim \|\zeta_B |v|^p\|_{L^{1,1}(\mathbb{R}, L^2(I))} \\ & \leq \|\zeta_{2B}\|_{L^{2,1}(\mathbb{R})} \|\zeta_{2B} |v|^p\|_{L^2(\mathbb{R}, L^2(I))} \lesssim B^{3/2} \|\zeta_{2B} |v|^p\|_{L^2(I, L^2(\mathbb{R}))} \\ & \leq B^{3/2} \|v\|_{L^\infty(I \times \mathbb{R})}^{p-1} \|\zeta_{2B} v\|_{L^2(I, L^2(\mathbb{R}))} \lesssim B^{\frac{p}{2}+1} \delta^{p-1} \left\| \frac{\zeta_{2B}}{\zeta_A} \zeta_A (\vartheta_{1A_1} + \vartheta_{2A_1}) v \right\|_{L^2(I, L^2(\mathbb{R}))} \end{aligned}$$

$$\leq B^{\frac{p}{2}+1} \delta^{p-1} \left\| \zeta_A (\vartheta_{1A_1} + \vartheta_{2A_1}) v \right\|_{L^2(\mathbb{R})} \leq B^{\frac{p}{2}+1} \delta^{p-1} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}),$$

where we used $|\phi_c| \lesssim \phi_c$ and here we are using $p < 3$. Similarly

$$A_{12} \lesssim \|\zeta_B |v|^p\|_{L^{1,1}(\mathbb{R}, L^2(I))} \lesssim B^{\frac{p}{2}+1} \delta^{p-1} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}).$$

We have

$$\begin{aligned} A_2 &\leq \left\| \zeta_B (f'(\phi_c + v) - f'(\phi_c)) v' \right\|_{L^{1,1} L^2(\{|v| \ll \phi_c\})} \\ &\quad + \left\| \zeta_B (f'(\phi_c + v) - f'(\phi_c)) v' \right\|_{L^{1,1} L^2(\{|v| \gtrsim \phi_c\})} =: A_{21} + A_{22}. \end{aligned}$$

We have, proceeding like above in the last inequality,

$$\begin{aligned} A_{21} &\lesssim \int_0^1 \left\| \zeta_B f''(\phi_c + sv) v v' \right\|_{L^{1,1} L^2(\{|v| \ll \phi_c\})} ds \lesssim \left\| \zeta_B \phi_c^{p-2} v v' \right\|_{L^{1,1} L^2(\{|v| \ll \phi_c\})} \\ &\lesssim \left\| \zeta_B |v|^{p-1} v' \right\|_{L^{1,1}(\mathbb{R}, L^2(I))} \lesssim \|\zeta_{2B}\|_{L^{2,1}(\mathbb{R})} \left\| \frac{\zeta_{2B}}{\zeta_A} \zeta_A (\vartheta_{1A_1} + \vartheta_{2A_1}) |v|^{p-1} v' \right\|_{L^2(\mathbb{R}, L^2(I))} \\ &\lesssim B^{3/2} \|v\|_{L^\infty(I \times \mathbb{R})}^{p-1} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}) \\ &\lesssim B^{\frac{p}{2}+1} \delta^{p-1} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}) \end{aligned}$$

and the following, which with the above inequalities and with (3.28) yields immediately (7.16)

$$A_{22} \lesssim \|\zeta_B |v|^{p-1} v'\|_{L^{1,1}(\mathbb{R}, L^2(I))} \lesssim B^{\frac{p}{2}+1} \delta^{p-1} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}).$$

□

The most delicate contributor in the formula (7.9)–(7.11) is the commutator. We have

$$[\zeta_B, \mathcal{L}]v = 3\zeta'_B v'' - c\zeta'_B v + \zeta'_B f'(\phi_c)v + 3\zeta''_B v' + \zeta'''_B v. \quad (7.17)$$

The troublesome term is $-c\zeta'_B v$, which we set aside for the moment.

Lemma 7.6. *We have*

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} Q([\zeta_B, \mathcal{L}] + c\zeta'_B v) ds \right\|_{L^2(I, \tilde{\Sigma})} \lesssim B^{-1/2} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}). \quad (7.18)$$

Proof. We omit initially the term $3\zeta''_B v''$ in (7.17) proceeding like above by Proposition 6.2

$$\begin{aligned} &\left\| \int_0^t e^{(t-s)\mathcal{L}} Q(\zeta'_B f'(\phi_c)v + 3\zeta''_B v' + \zeta'''_B v) ds \right\|_{L^2(I, \tilde{\Sigma})} \\ &\lesssim \|\zeta'_B f'(\phi_c)v + 3\zeta''_B v' + \zeta'''_B v\|_{L^{1,1}(\mathbb{R}, L^2(I))} \lesssim B^{-1/2} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}). \end{aligned}$$

Writing $\zeta'_B v'' = \partial_x(\zeta'_B v') - \zeta''_B v'$, using for the first time also (6.5) we obtain

$$\begin{aligned} &\left\| \int_0^t e^{(t-s)\mathcal{L}} Q(\partial_x(\zeta'_B v') - \zeta''_B v') ds \right\|_{L^2(I, \tilde{\Sigma})} \\ &\lesssim \|\zeta'_B v'\|_{L^1(\mathbb{R}, L^2(I))} + \|\zeta''_B v'\|_{L^{1,1}(\mathbb{R}, L^2(I))} \lesssim B^{-1/2} (\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})}). \end{aligned}$$

□

Lemma 7.7. *We have*

$$\left\| \int_0^t e^{(t-s)\mathcal{L}} Q\zeta'_B v ds \right\|_{L^2(I, \tilde{\Sigma})} \lesssim A_1^{3/2} B^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})} + B^{1/2} \|v\|_{L^2(I, \Sigma_{2AA_1})}. \quad (7.19)$$

Proof. Split $v = \vartheta_{1A_1} v + \vartheta_{2A_1} v$. Then (6.4) gives

$$\begin{aligned} \left\| \int_0^t e^{(t-s)\mathcal{L}} Q(\zeta'_B v)(s) ds \right\|_{L^2(I, \tilde{\Sigma})} &\lesssim \|\langle x^+ \rangle \zeta'_B \vartheta_{1A_1} v\|_{L^1_x L^2_t} + \|\langle x^+ \rangle \zeta'_B \vartheta_{2A_1} v\|_{L^1_x L^2_t} \\ &\lesssim \|\langle x^+ \rangle \zeta'_B 1_{(-\infty, 2A_1]} \zeta_A^{-1}\|_{L^2_x} \|\vartheta_{1A_1} v\|_{L^2(I, \Sigma_A)} \\ &\quad + \|\langle x^+ \rangle \zeta'_B 1_{[A_1, \infty)} \zeta_A^{-1}\|_{L^2_x} \|\vartheta_{2A_1} v\|_{L^2(I, \Sigma_A)}. \end{aligned} \quad (7.20)$$

Since $|\zeta'_B| \lesssim B^{-1}$,

$$\|\langle x^+ \rangle \zeta'_B 1_{(-\infty, 2A_1]} \zeta_A^{-1}\|_{L^2_x} \lesssim B^{-1/2} + A_1^{3/2} B^{-1} \lesssim A_1^{3/2} B^{-1}, \quad (7.21)$$

where we have used $A_1 = B^{3/5}$. For the second term, we have

$$\|\langle x^+ \rangle \zeta_B' 1_{[A_1, \infty)} \zeta_A^{-1}\|_{L_x^2}^2 = \int_{A_1}^{\infty} \langle x \rangle^2 |\zeta_B'(x)|^2 \zeta_A(x)^{-2} dx. \quad (7.22)$$

Moreover, for $x \geq 0$, $\langle x \rangle |\zeta_B'(x)| \lesssim e^{-x/(2B)}$, and hence

$$\|\langle x^+ \rangle \zeta_B' \vartheta_{2A_1} \zeta_A^{-1}\|_{L_x^2}^2 \lesssim \int_{A_1}^{\infty} e^{-x/B+2x/A} dx \lesssim B e^{-cA_1/B} \lesssim B. \quad (7.23)$$

Therefore

$$\|\langle x^+ \rangle \zeta_B' \vartheta_{2A_1} \zeta_A^{-1}\|_{L_x^2} \lesssim B^{1/2}. \quad (7.24)$$

Combining (7.20), (7.21) and (7.24), we have the conclusion. $\square$

Proof of Lemma 3.7. From Lemmas 7.2–7.7 and 3.5 we have

$$\|w\|_{L^2(I, \tilde{\Sigma})} \lesssim A_1^{3/2} B^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})} + B^{1/2} \|v\|_{L^2(I, \Sigma_{2AA_1})} + \sqrt{\delta}.$$

Using (2.1), decomposition (7.2) and inequality (7.3), we have the following

$$\begin{aligned} \|v\|_{L^2(I, \tilde{\Sigma})} &\leq \|(1 - \zeta_B)v\|_{L^2(I, \tilde{\Sigma})} + \|\zeta_B v\|_{L^2(I, \tilde{\Sigma})} \\ &\lesssim B^{-1} \left(\|v\|_{L^2(I, \Sigma_{1AA_1})} + \|v\|_{L^2(I, \Sigma_{2AA_1})} \right) + \|P \zeta_B v\|_{L^2(I, \tilde{\Sigma})} + \|w\|_{L^2(I, \tilde{\Sigma})} \\ &\lesssim A_1^{3/2} B^{-1} \|v\|_{L^2(I, \Sigma_{1AA_1})} + B^{1/2} \|v\|_{L^2(I, \Sigma_{2AA_1})} + \sqrt{\delta}. \end{aligned}$$

$\square$

We completed the proof of Theorem 1.5 up to Proposition 3.2, proved in Sections 10, 6.3 proved in the next two Sections 8 and 9.

8. The Jost functions

We look for appropriate solutions of

$$\mathcal{L}_{[c]} \mu = \lambda \mu \text{ for } \lambda \in \mathbb{C}. \quad (8.1)$$

We will consider only case $c = 1$ dropping the subscript $[c]$. The third order Eq. (8.1) has characteristic equation

$$\mu^3 - \mu + \lambda = 0. \quad (8.2)$$

We summarize the properties of the solutions of (8.2).

Lemma 8.1. *There exist three analytic roots of (8.2) $\mu_j : \Omega := \left\{ \lambda \in \mathbb{C} : |\operatorname{Re} \lambda| < \frac{1}{3\sqrt{3}} \right\} \rightarrow \mathbb{C}$, ($j = 1, 2, 3$), which are uniquely determined by*

$$\mu_1(0) = -1, \quad \mu_2(0) = 0, \quad \mu_3(0) = 1.$$

The relations between the roots and the coefficients of (8.2) are

$$\mu_1 + \mu_2 + \mu_3 = 0, \quad (8.3)$$

$$\mu_1 \mu_2 + \mu_1 \mu_3 + \mu_2 \mu_3 = -1, \quad (8.4)$$

$$\mu_1 \mu_2 \mu_3 = -\lambda. \quad (8.5)$$

For $0 < \operatorname{Re} \lambda < \frac{1}{3\sqrt{3}}$,

$$\operatorname{Re} \mu_1 < 0 < \operatorname{Re} \mu_2 < \operatorname{Re} \mu_3. \quad (8.6)$$

Moreover, we can continuously extend μ_j on $\overline{\Omega}$ and there exists $\kappa > 0$ such that

$$\operatorname{Re} \mu_1(\lambda) \leq -\kappa, \quad (8.7)$$

$$\operatorname{Re}(\mu_2(\lambda) - \mu_1(\lambda)) \geq \kappa, \quad (8.8)$$

$$\operatorname{Re}(\mu_3(\lambda) - \mu_2(\lambda)) \geq \kappa \quad (8.9)$$

for all $\lambda \in \overline{\Omega}$.

For every $k \in \mathbb{R}$, if $\lambda = ik(k^2 + 1)$, then

$$\mu_1(\lambda) = -\sqrt{1 + \frac{3}{4}k^2} - \frac{ik}{2}, \quad \mu_2(\lambda) = ik, \quad \mu_3(\lambda) = \sqrt{1 + \frac{3}{4}k^2} - \frac{ik}{2}. \quad (8.10)$$

Define

$$a_j(\lambda) := \prod_{\substack{1 \leq \ell \leq 3 \\ \ell \neq j}} (\mu_j(\lambda) - \mu_\ell(\lambda))^{-1}, \quad j = 1, 2, 3. \quad (8.11)$$

Then, for every $n \in \mathbb{N} \cup \{0\}$,

$$|\partial_\lambda^n \mu_j(\lambda)| \lesssim_n \langle \lambda \rangle^{\frac{1}{3}-n}, \quad |\mu_j(\lambda) - \mu_\ell(\lambda)| \gtrsim \langle \lambda \rangle^{\frac{1}{3}}, \quad j \neq \ell, \quad (8.12)$$

$$|\partial_\lambda^n a_j(\lambda)| \lesssim_n \langle \lambda \rangle^{-\frac{2}{3}-n}, \quad |\partial_\lambda^n (a_j(\lambda) \mu_j(\lambda))| \lesssim_n \langle \lambda \rangle^{-\frac{1}{3}-n} \quad (8.13)$$

for all $\lambda \in \Omega$ and $j = 1, 2, 3$.

Proof. The discriminant is $4 - 27\lambda^2$, and hence the equation has no multiple root on $\overline{\Omega}$, which is simply connected. Thus, the analytic roots with the stated normalization follow from analytic perturbation theory; see [34, Theorem XII.2].

For $\lambda = ik(k^2 + 1)$, substituting $\mu_2 = ik$ into (8.2) and factoring the remaining quadratic factor gives (8.10). The ordering (8.6), the large $|\lambda|$ behavior of the roots, and the corresponding separation estimates are standard; see [14, Section 2]. Combining the large $|\lambda|$ estimates with (8.10) and compactness on bounded subsets of Ω yields (8.7) and the second estimate in (8.12). The remaining estimates follow by repeatedly differentiating (8.2) and the definition of a_j . $\square$

In the following, we fix $\varepsilon \in (0, \frac{1}{3\sqrt{3}})$.

Given three scalar functions y_j with $j = 1, 2, 3$ we consider the Wronskian

$$W[y_1, y_2, y_3] := \det \begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}. \quad (8.14)$$

For the functions $y_j = e^{\mu_j(\lambda)x}$ we set

$$W_0(\lambda) := W[y_1, y_2, y_3] = \prod_{1 \leq i < j \leq 3} (\mu_i - \mu_j). \quad (8.15)$$

By the variation of parameter formulas the solutions of the nonhomogeneous equation

$$y''' - y' + \lambda y = F \quad (8.16)$$

are

$$y = \sum_{j=1,2,3} a_j e^{\mu_j x} \int e^{-\mu_j x} F(x) dx, \quad (8.17)$$

where a_j is given in (8.11). A simple computation, see [14, Lemma 4.3], yields the identity

$$\sum_{j=1,2,3} a_j \equiv 0. \quad (8.18)$$

The solutions of (8.1) are of the form (8.17) for $F = -(f'(\phi_c)y)'$. Since

$$-a_j e^{\mu_j x} \int e^{-\mu_j x} (f'(\phi)y)' dx = -a_j f'(\phi)y - a_j \mu_j e^{\mu_j x} \int e^{-\mu_j x} f'(\phi)y dx,$$

summing on j and using (8.18) we have

$$y = - \sum_{j=1,2,3} a_j \mu_j e^{\mu_j x} \int e^{-\mu_j x} f'(\phi)y dx. \quad (8.19)$$

The first Jost function of (8.1) is provided by the following, proved like [40, Lemma 1 p. 130].

Lemma 8.2. *There exists a solution $f_1(x, \lambda) = e^{\mu_1 x} m_1(x, \lambda)$ of (8.1) such that for any $\alpha, \beta \geq 0$ there is a constant $C_{\alpha\beta}$ such that for all $\lambda \in \mathbb{C}$ such that $|\operatorname{Re} \lambda| \leq \varepsilon$ and $x \in \mathbb{R}$*

$$|\partial_\lambda^\beta \partial_x^\alpha (m_1(x, \lambda) - 1)| \leq C_{\alpha,\beta} \langle \lambda \rangle^{-\frac{1}{3}-\beta} \int_x^{+\infty} f'(\phi(x')) dx'. \quad (8.20)$$

Proof. We seek a solution to

$$f_1(x, \lambda) = e^{\mu_1 x} + \sum_{j=1,2,3} a_j \mu_j e^{\mu_j x} \int_x^{+\infty} e^{-\mu_j y} f'(\phi(y)) f_1(y, \lambda) dy$$

or equivalently

$$m_1(x, \lambda) = 1 + \int_x^{+\infty} q(x-y, \lambda) f'(\phi(y)) m_1(y, \lambda) dy \quad (8.21)$$

where

$$q(x, \lambda) = \sum_{j=1,2,3} a_j \mu_j e^{(\mu_j - \mu_1)x}.$$

The solution of (8.21) can be expressed like in Deift and Trubowitz [40, Lemma 1] as

$$m_1(x, \lambda) = \sum_{n=0}^{\infty} m_1(x, \lambda)_{[n]}, \quad (8.22)$$

where

$$\begin{aligned} m_1(x, \lambda)_{[0]} &= 1, \\ m_1(x, \lambda)_{[n+1]} &= \int_x^{+\infty} q(x-y, \lambda) f'(\phi(y)) m_1(y, \lambda)_{[n]} dy. \end{aligned} \quad (8.23)$$

By (8.12) and (8.17) there exists a constant $C_* > 0$ such that

$$\sup_{|\operatorname{Re} \lambda| \leq \epsilon} \sum_{j=1,2,3} |a_j \mu_j| \leq C_* \langle \lambda \rangle^{-\frac{1}{3}}. \quad (8.24)$$

Then like in [40, Lemma 1] we obtain

$$\begin{aligned} |m_1(x, \lambda)_{[n]}| &\leq C_*^n \langle \lambda \rangle^{-\frac{n}{3}} \int_{x \leq x_1 \leq \dots \leq x_n} |f'(\phi(x_1))| \dots |f'(\phi(x_n))| dx_n \dots dx_1 \\ &= \frac{1}{n!} C_*^n \langle \lambda \rangle^{-\frac{n}{3}} \left(\int_x^{\infty} |f'(\phi(y))| dy \right)^n \end{aligned}$$

which yields (8.20) for $\alpha = \beta = 0$. Differentiating (8.21) and integrating by parts we obtain

$$\begin{aligned} \partial_x m_1(x, \lambda) &= -q(0, \lambda) f'(\phi(x)) m_1(x, \lambda) + \int_x^{+\infty} \partial_x q(x-y, \lambda) f'(\phi(y)) m_1(y, \lambda) dy \\ &= m'_1(x, \lambda)_{[0]} + \int_x^{+\infty} q(x-y, \lambda) f'(\phi(y)) \partial_y m_1(y, \lambda) dy \text{ where} \\ m'_1(x, \lambda)_{[0]} &:= \int_x^{\infty} q(x-y, \lambda) f''(\phi(y)) \phi'(y) m_1(y, \lambda) dy. \end{aligned} \quad (8.25)$$

This can be solved as (8.22) as a sum

$$\partial_x m_1(x, \lambda) = \sum_{n=0}^{\infty} m'_1(x, \lambda)_{[n]}$$

where for $n \geq 0$

$$m'_1(x, \lambda)_{[n+1]} = \int_x^{+\infty} q(x-y, \lambda) f'(\phi(y)) m'_1(y, \lambda)_{[n]} dy.$$

Then

$$|m'_1(x, \lambda)_{[0]}| \leq \|m_1(\cdot, \lambda)\|_{L^\infty(\mathbb{R})} C_* \langle \lambda \rangle^{-\frac{1}{3}} \int_x^{+\infty} |f''(\phi(y)) \phi'(y)| dy$$

and for $n \geq 1$ and using that there exists $C_0 > 0$ such that $|f''(\phi(y)) \phi'(y)| \leq C_0 |f'(\phi(y))|$ we have

$$\begin{aligned} |m'_1(x, \lambda)_{[n]}| &\leq C_*^n \langle \lambda \rangle^{-\frac{n}{3}} \|m'_1(\cdot, \lambda)_{[0]}\|_{L^\infty} \int_{x \leq x_1 \leq \dots \leq x_n} |f'(\phi(x_1))| \dots |f'(\phi(x_n))| dx_n \dots dx_1 \\ &= \frac{1}{n!} C_0 C_*^{n+1} \langle \lambda \rangle^{-\frac{n+1}{3}} \left(\int_x^{\infty} |f'(\phi(y))| dy \right)^{n+1}. \end{aligned}$$

Then we obtain (8.20) for $\alpha = 1$ and $\beta = 0$. Applying ∂_λ to (8.21) we obtain

$$\begin{aligned} \partial_\lambda m_1(x, \lambda) &= \int_x^{\infty} \partial_\lambda q(x-y, \lambda) f'(\phi(y)) m_1(y, \lambda) dy \\ &\quad + \int_x^{\infty} q(x-y, \lambda) f'(\phi(y)) \partial_\lambda m_1(y, \lambda) dy. \end{aligned}$$

Then

$$\partial_\lambda m_1(x, \lambda) = \sum_{n=0}^{\infty} \dot{m}_1(x, \lambda)_{[n]}$$

where

$$\begin{aligned}\dot{m}_1(x, \lambda)_{[0]} &= \int_x^\infty \partial_\lambda q(\lambda, x-y) f'(\phi(y)) m_1(y, \lambda) dy \text{ and} \\ \dot{m}_1(x, \lambda)_{[n+1]} &= \int_x^\infty q(x-y, \lambda) f'(\phi(y)) \dot{m}_1(y, \lambda)_{[n]} dy.\end{aligned}$$

We have for some constant $C_1 > 0$

$$\begin{aligned}|\dot{m}_1(x, \lambda)_{[0]}| &\leq |\partial_\lambda(a_1 \mu_1)| \|m_1(\cdot, \lambda)\|_{L^\infty(\mathbb{R})} \int_x^\infty f'(\phi(y)) dy \\ &+ \|m_1(\cdot, \lambda)\|_{L^\infty(\mathbb{R})} \sum_{j=2,3} \int_x^\infty \left| a_j \mu_j \partial_\lambda \left(e^{-(\mu_j - \mu_1)(x-y)} \right) \right| f'(\phi(y)) dy \\ &\leq C_1 \langle \lambda \rangle^{-\frac{4}{3}} \int_x^\infty f'(\phi(y)) dy\end{aligned}$$

where we used (8.12), (8.13) and

$$\begin{aligned}\left| (x-y) e^{(\mu_j - \mu_1)(x-y)} \partial_\lambda (\mu_j - \mu_1) \right| &\lesssim \left| (\mu_j - \mu_1)(x-y) e^{(\mu_j - \mu_1)(x-y)} \frac{\partial_\lambda (\mu_j - \mu_1)}{\mu_j - \mu_1} \right| \\ &\lesssim \left| \frac{\partial_\lambda (\mu_j - \mu_1)}{\mu_j - \mu_1} \right| \lesssim \langle \lambda \rangle^{-1} \text{ and for all } y > x.\end{aligned}\quad (8.26)$$

For $n \geq 1$ we have for a constant $C_2 > 0$

$$\begin{aligned}|\dot{m}_1(x, \lambda)_{[n]}| &\leq C_2 \langle \lambda \rangle^{-\frac{4}{3}} C_*^n \langle \lambda \rangle^{-\frac{n}{3}} \int_{x \leq x_1 \leq \dots \leq x_n} |f'(\phi(x_1))| \dots |f'(\phi(x_n))| dx_n \dots dx_1 \\ &= \frac{1}{n!} C_2 \langle \lambda \rangle^{-\frac{4}{3}} C_*^n \langle \lambda \rangle^{-\frac{n}{3}} \left(\int_x^\infty |f'(\phi(y))| dy \right)^n.\end{aligned}$$

We hence get case $\alpha = 0$ and $\beta = 1$. For the general case we proceed by induction. Differentiating (8.21) and integrating by parts we obtain

$$\partial_x^\alpha \partial_\lambda^\beta m_1(x, \lambda) = \tilde{m}_1(x, \lambda)_{[0]} + \int_x^{+\infty} q(x-y, \lambda) f'(\phi(y)) \partial_y^\alpha \partial_\lambda^\beta m_1(y, \lambda) dy$$

with the first term on the right of the form

$$\tilde{m}_1(x, \lambda)_{[0]} = \sum a_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} \int_x^{+\infty} \partial_\lambda^{\beta_1} q(x-y, \lambda) \partial_y^{\alpha_1} (f'(\phi(y))) \partial_y^{\alpha_2} \partial_\lambda^{\beta_2} m_1(y, \lambda) dy, \quad (8.27)$$

where $\alpha_1 + \alpha_2 = \alpha$, $\beta_1 + \beta_2 = \beta$ and $\alpha_2 + \beta_2 < \alpha + \beta$ and where the $a_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}$ are fixed constants. Formula (8.27) can be obtained in elementary fashion by induction on α proceeding like for (8.25). We have like above by induction

$$\left| \partial_\lambda^{\beta_1} q_1(x-y, \lambda) \partial_y^{\alpha_1} (f'(\phi(y))) \partial_y^{\alpha_2} \partial_\lambda^{\beta_2} m_1(y, \lambda) \right| \lesssim \langle \lambda \rangle^{-\frac{1}{3}-\beta} |\partial_y^{\alpha_1} (f'(\phi(y)))|$$

so that we get the following which, fed in the Volterra equation resolution mechanism, yields (8.20)

$$|\tilde{m}_1(x, \lambda)_{[0]}| \leq C_{\alpha\beta}' \langle \lambda \rangle^{-\frac{1}{3}-\beta} \sum_{j=0}^{\alpha} \int_x^{+\infty} |\partial_y^j (f'(\phi(y)))| dy,$$

where we remark that this last integral can be bounded in terms of the integral in (8.20). $\square$

The second Jost function of (8.1) is obtained similarly.

Lemma 8.3. *There exists a solution $f_3(x, \lambda) = e^{\mu_3 x} m_3(x, \lambda)$ of (8.1) such that for any $\alpha, \beta \geq 0$ there is a constant $C_{\alpha\beta}$ such that for all $\lambda \in \mathbb{C}$ such that $|\operatorname{Re} \lambda| \leq \varepsilon$ and $x \in \mathbb{R}$*

$$\left| \partial_x^\alpha \partial_\lambda^\beta (m_3(x, \lambda) - 1) \right| \leq C_{\alpha\beta} \langle \lambda \rangle^{-\frac{1}{3}-\beta} \int_{-\infty}^x f'(\phi(x')) dx'. \quad (8.28)$$

Proof. The proof is similar to that of Lemma 8.2. We only remark that here the integral equation is

$$m_3(x, \lambda) = 1 - \sum_{j=1,2,3} a_j \mu_j \int_{-\infty}^x e^{(\mu_j - \mu_3)(x-y)} f'(\phi(y)) m_3(y, \lambda) dy. \quad (8.29)$$

$\square$

Observe that for $\lambda \in i\mathbb{R}$ we have that $f_1(-x, \lambda)$ solves Eq. (8.1) with λ replaced by $-\lambda$. Then, by the asymptotics, we conclude

$$f_1(-x, \lambda) = f_3(x, -\lambda) \text{ for all } x \in \mathbb{R} \text{ and } \lambda \in i\mathbb{R}. \quad (8.30)$$

A third Jost function of (8.1) is the following one. As we will see later, there is no as good control globally for $x \in \mathbb{R}$ of this function.

Lemma 8.4. *There exists a Jost function $\tilde{f}_2(x, \lambda) = e^{\mu_2 x} \tilde{m}_2(x, \lambda)$ and $x_0 \in \mathbb{R}$ such that for all $\lambda \in \mathbb{C}$ such that $|\operatorname{Re} \lambda| \leq \varepsilon$ we have*

$$\|\tilde{m}_2(\cdot, \lambda)\|_{L^\infty((-\infty, x_0))} \leq 2 \text{ and} \quad (8.31)$$

$$\lim_{x \rightarrow -\infty} \tilde{m}_2(x, \lambda) = 1. \quad (8.32)$$

There are constants $C_{\alpha\beta} = C_{\alpha\beta}(c, x_0)$ such that for $\alpha + \beta \geq 1$

$$\|\partial_x^\alpha \partial_\lambda^\beta \tilde{m}_2(\cdot, \lambda)\|_{L^\infty((-\infty, x_0))} \leq C_{\alpha,\beta} \langle \lambda \rangle^{-\frac{1}{3}-\beta}. \quad (8.33)$$

Finally, there are constants $C_{\alpha\beta} = C_{\alpha\beta}(x_0)$ such that for $x \leq x_0$ and for $|\operatorname{Re} \lambda| \leq \varepsilon$,

$$|\partial_x^\alpha \partial_\lambda^\beta (\tilde{m}_2(x, \lambda) - 1)| \leq C_{\alpha,\beta} \langle \lambda \rangle^{-\frac{1}{3}-\beta} e^{-(p-1)|x|}. \quad (8.34)$$

Proof. For $x_0 \in \mathbb{R}$ consider the equation

$$\tilde{m}_2(x, \lambda) = 1 + T \tilde{m}_2(\cdot, \lambda), \quad (8.35)$$

where

$$\begin{aligned} Tg(x) := & -a_1 \mu_1 \int_{-\infty}^x e^{(\mu_1 - \mu_2)(x-y)} f'(\phi(y)) g(y) dy \\ & - a_2 \mu_2 \int_{-\infty}^x f'(\phi(y)) g(y) dy + a_3 \mu_3 \int_x^{x_0} e^{(\mu_3 - \mu_2)(x-y)} f'(\phi(y)) g(y) dy. \end{aligned}$$

Then, from Lemma 8.1, there exists a $x_1 \in \mathbb{R}$ such that for $x_0 \leq x_1$ we have that

$$\|T\|_{\mathcal{L}(L^\infty((-\infty, x_0)))} \leq 1/2$$

for all $|\operatorname{Re} \lambda| \leq \varepsilon$. Then $\|(1 - T)^{-1}\|_{\mathcal{L}(L^\infty((-\infty, x_0)))} \leq 2$ yielding (8.31). The limit (8.32) is obtained using dominated convergence. Differentiating (8.35) we obtain

$$(1 - T) \partial_x^\alpha \partial_\lambda^\beta \tilde{m}_2(x, \lambda) = A_1 + A_2 \text{ where}$$

$$A_1 = - \sum a_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} \int_{-\infty}^x \partial_\lambda^{\beta_1} (a_1 \mu_1 e^{(\mu_1 - \mu_2)(x-y)} + a_2 \mu_2) \partial_y^{\alpha_1} (f'(\phi(y)) \partial_y^{\alpha_2} \partial_\lambda^{\beta_2} \tilde{m}_2(y, \lambda) dy$$

$$A_2 = \sum a_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} \int_x^{x_0} \partial_\lambda^{\beta_1} (a_3 \mu_3 e^{(\mu_3 - \mu_2)(x-y)}) \partial_y^{\alpha_1} (f'(\phi(y)) \partial_y^{\alpha_2} \partial_\lambda^{\beta_2} \tilde{m}_2(y, \lambda) dy$$

like in Lemma 8.2. Since A_1 and A_2 are bounded above by the right hand side of (8.33), inverting $1 - T$ we obtain (8.33). Finally, the last statement follows in an elementary way from

$$\begin{aligned} |Tg(x)| \leq & \langle \lambda \rangle^{-\frac{1}{3}} \|g\|_{L^\infty((-\infty, x_0))} \left[\int_{-\infty}^x e^{(p-1)y} dy \right. \\ & \left. + e^{-\operatorname{Re}(\mu_3 - \mu_2) \frac{|x|}{2}} \int_{-\infty}^{x_0} e^{(p-1)y} dy + e^{-\frac{(p-1)}{2}|x|} \int_x^{\frac{x}{2}} e^{-\operatorname{Re}(\mu_3 - \mu_2)|x-y|} dy \right]. \end{aligned}$$

□

We will need also additional solutions.

Lemma 8.5.

1. *There exists $F_1(x, \lambda) = e^{\mu_1 x} M_1(x, \lambda)$ and $x_1 \in \mathbb{R}$ such that for any $\alpha, \beta \geq 0$ there is a constant $C_{\alpha\beta}$ such that for all $x \leq x_1$ and $|\operatorname{Re} \lambda| \leq \varepsilon$*

$$|\partial_\lambda^\beta \partial_x^\alpha (M_1(x, \lambda) - 1)| \leq C_{\alpha,\beta} \langle \lambda \rangle^{-\frac{1}{3}-\beta} e^{-(p-1)|x|} \text{ and } \lim_{x \rightarrow -\infty} M_1(x, \lambda) = 1.$$

2. *There exists $F_2(x, \lambda) = e^{\mu_2 x} M_2(x, \lambda)$ and $x_2 \in \mathbb{R}$ such that for any $\alpha, \beta \geq 0$ there is a constant $C_{\alpha\beta}$ such that for all $x \geq x_2$ and $|\operatorname{Re} \lambda| \leq \varepsilon$*

$$|\partial_\lambda^\beta \partial_x^\alpha (M_2(x, \lambda) - 1)| \leq C_{\alpha,\beta} \langle \lambda \rangle^{-\frac{1}{3}-\beta} e^{-(p-1)|x|} \text{ and } \lim_{x \rightarrow +\infty} M_2(x, \lambda) = 1.$$

3. *There exists $F_3(x, \lambda) = e^{\mu_3 x} M_3(x, \lambda)$ and $x_3 \in \mathbb{R}$ such that for any $\alpha, \beta \geq 0$ there is a constant $C_{\alpha\beta}$ such that for all $x \geq x_3$ and $|\operatorname{Re} \lambda| \leq \varepsilon$*

$$|\partial_\lambda^\beta \partial_x^\alpha (M_3(x, \lambda) - 1)| \leq C_{\alpha,\beta} \langle \lambda \rangle^{-\frac{1}{3}-\beta} e^{-(p-1)|x|} \text{ and } \lim_{x \rightarrow +\infty} M_3(x, \lambda) = 1.$$

Proof. Here $M_1(x, \lambda)$ is obtained solving

$$M_1(x, \lambda) = 1 + T M_1(\cdot, \lambda)$$

where

$$Tg(x) := -a_1\mu_1 \int_{-\infty}^x f'(\phi(y))g(y)dy + a_2\mu_2 \int_x^{x_1} e^{(\mu_2-\mu_1)(x-y)} f'(\phi(y))g(y)dy + a_3\mu_3 \int_x^{x_1} e^{(\mu_3-\mu_1)(x-y)} f'(\phi(y))g(y)dy.$$

Like in Lemma 8.4 there exists a $x_0 \in \mathbb{R}$ such that $\|T\|_{\mathcal{L}(L^\infty((-\infty, x_1)))} \leq 1/2$ for $x_1 \leq x_0$. This gives the estimate for $(\alpha, \beta) = (0, 0)$ and dominated convergence yields the limit. Proceeding by induction

$$(1-T)\partial_x^\alpha \partial_\lambda^\beta M_1(x, \lambda) = A_1 + A_2 \text{ where}$$

$$A_1 = -\sum a_{\beta_1\beta_2}^{\alpha_1\alpha_2} \int_{-\infty}^x \partial_\lambda^{\beta_1} (a_1\mu_1) \partial_y^{\alpha_1} (f'(\phi(y)) \partial_y^{\alpha_2} \partial_\lambda^{\beta_2} M_1(y, \lambda) dy$$

$$A_2 = -\sum_{j=2}^3 \sum a_{\beta_1\beta_2}^{\alpha_1\alpha_2} \int_x^{x_1} \partial_\lambda^{\beta_1} (a_j\mu_j e^{(\mu_j-\mu_1)(x-y)}) \partial_y^{\alpha_1} (f'(\phi(y)) \partial_y^{\alpha_2} \partial_\lambda^{\beta_2} M_1(y, \lambda) dy$$

like in Lemma 8.2. Then using (8.26) by induction

$$|A_j| \lesssim \langle \lambda \rangle^{-2/3-\beta} \int_{-\infty}^x e^{-|y|} dy \text{ for } j = 1, 2.$$

The statement for $F_3(x, \lambda)$ is obtained similarly. Finally $M_2(x, \lambda)$ is obtained solving

$$M_2(x, \lambda) = 1 + \tilde{T}M_2(\cdot, \lambda)$$

where

$$\tilde{T}g(x) := a_2\mu_2 \int_x^{+\infty} f'(\phi(y))g(y)dy - a_1\mu_1 \int_{x_2}^x e^{(\mu_1-\mu_2)(x-y)} f'(\phi(y))g(y)dy + a_3\mu_3 \int_x^{+\infty} e^{(\mu_3-\mu_2)(x-y)} f'(\phi(y))g(y)dy.$$

Then again there exists a $\tilde{x}_0 \in \mathbb{R}$ such that $\|\tilde{T}\|_{\mathcal{L}(L^\infty((x_2, +\infty)))} \leq 1/2$ for $x_2 \geq \tilde{x}_0$. This gives the estimate for $(\alpha, \beta) = (0, 0)$ and dominated convergence yields the limit. The other estimates are obtained by induction as above.

□

The following shows that $\tilde{f}_2(x, \lambda)$ does not satisfy good bounds: we will remedy this later.

Lemma 8.6. For any pair (α, β) and any λ with $0 \leq \operatorname{Re} \lambda \leq \varepsilon$ there is a $C_{\alpha\beta}(\lambda)$ which is locally bounded in λ such that we have

$$|\partial_x^\alpha \partial_\lambda^\beta \tilde{m}_2(x, \lambda)| \leq C_{\alpha\beta}(\lambda) (1 + e^{\operatorname{Re}(\mu_3-\mu_2)x}) \text{ for all } x \in \mathbb{R}. \quad (8.36)$$

Proof. First of all, there exists some x_0 such that for $x \leq x_0$ we have the estimates in Lemma 8.4. Next we observe that $\tilde{f}_2(\cdot, \lambda) \in \operatorname{Span}\{f_1(\cdot, \lambda), F_2(\cdot, \lambda), F_3(\cdot, \lambda)\}$ since the latter three functions are, by their asymptotic behavior at $+\infty$, linearly independent and so they generate the vector space of all the solutions of (8.1). Hence

$$\tilde{f}_2(x, \lambda) = A_1(\lambda)f_1(x, \lambda) + A_2(\lambda)F_2(x, \lambda) + A_3(\lambda)F_3(x, \lambda)$$

for the $A_j(\lambda)$ three analytic functions in λ . The estimate (8.36) is then true for some x_1 and for $x \geq x_1$ by Lemmas 8.2 and 8.5. Finally for $x_0 \leq x \leq x_1$ the estimates hold for (8.36) by smooth dependence of solutions of (8.1) on λ .

□

We set now

$$b_{ij}(x, \lambda) := f_i(x, \lambda)f_j'(x, \lambda) - f_i'(x, \lambda)f_j(x, \lambda) \text{ and} \quad (8.37)$$

$$\tilde{b}_{j2}(x, \lambda) := f_j(x, \lambda)\tilde{f}_2'(x, \lambda) - f_j'(x, \lambda)\tilde{f}_2(x, \lambda) \text{ for } j = 1, 3 \quad (8.38)$$

and $\tilde{b}_{2j}(x, \lambda) := -\tilde{b}_{j2}(x, \lambda)$. For $Y = Y(x, \lambda) := (f_j(x, \lambda), f_j'(x, \lambda), f_j''(x, \lambda))^T$ for $j = 1$ or $j = 3$ then

$$\frac{d}{dx}Y = AY \text{ for } A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\lambda - (f'(\phi))' & 1 - f'(\phi) & 0 \end{pmatrix}. \quad (8.39)$$

Consider $Z = (z_1, z_2, z_3)^T$ a solution of the dual system $\frac{d}{dx}Z = -A^T Z$. Then we have $\mathcal{L}^* z_3 = \lambda z_3$. Notice that this observation implies that

$$(\mathcal{L}^* - \lambda)b_{ij}(\cdot, \lambda) = 0 \text{ for all } i, j. \quad (8.40)$$

Pego and Weinstein [15] define the Evans function $D(\lambda) = Y^T(x, \lambda)Z(x, \lambda)$ for solutions Y of (8.39) and Z like above chosen so that

$$Y_+^T(\lambda)Z_-(\lambda) = 1 \text{ where } Y_+(\lambda) = \lim_{x \rightarrow +\infty} e^{-\mu_1(\lambda)x} Y(x, \lambda) \text{ and } Z_-(\lambda) = \lim_{x \rightarrow -\infty} e^{\mu_1(\lambda)x} Z(x, \lambda).$$

In Pego and Weinstein [14,15] it is shown that $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda \geq 0$ is an eigenvalue if and only if $D(\lambda) = 0$. We can apply Pego and Weinstein [15, Corollary 1.18] and conclude $D(\lambda) \xrightarrow{\lambda \rightarrow \infty} 1$. From Pego and Weinstein [15], in particular p. 73, we have

$$D(0) = D'(0) = 0 \text{ and } D''(0) = \left(\frac{p+1}{2}\right)^{-\frac{2}{p-1}} \langle \Lambda_p \phi, \phi \rangle. \quad (8.41)$$

From Pego and Weinstein [15, Formula (1.28)]

$$\lim_{x \rightarrow -\infty} e^{-\mu_1 x} Y(x, \lambda) = D(\lambda) Y_+(\lambda). \quad (8.42)$$

Lemma 8.7. *We have, for the Λ_p in (3.6),*

$$f_1(x, 0) = f_3(-x, 0) = -\beta^{-1}(p) \phi'(x) \text{ where } \beta(p) := (2(p+1))^{\frac{1}{p-1}}, \quad (8.43)$$

$$\partial_\lambda f_1(x, 0) = \beta^{-1}(p) \left(\frac{1}{p-1} \phi' + \Lambda_p \phi \right) \text{ and} \quad (8.44)$$

$$\partial_\lambda f_3(x, 0) = \beta^{-1}(p) \left(\frac{1}{p-1} \phi' - \Lambda_p \phi \right). \quad (8.45)$$

Proof. We have $f_1(\cdot, 0) \in \operatorname{Span}\{\phi', F_3(\cdot, 0), M_2(\cdot, 0)\}$. We have $f_1(x, 0) = e^{-x}(1 + o(1))$ for $x \rightarrow +\infty$ by Lemma 8.2. On the other hand $\phi'(x) = -\phi(x) \tanh\left(\frac{p-1}{2}x\right)$ and for $x \rightarrow +\infty$

$$\begin{aligned} \phi'(x) &= -\left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left(\cosh\left(\frac{p-1}{2}x\right) \right)^{-\frac{2}{p-1}} \tanh\left(\frac{p-1}{2}x\right) \\ &\sim -\left(\frac{p+1}{2}\right)^{\frac{1}{p-1}} \left(2^{-1} \exp\left(\frac{p-1}{2}x\right) \right)^{-\frac{2}{p-1}} = -(2(p+1))^{\frac{1}{p-1}} e^{-x}. \end{aligned}$$

Hence the asymptotic behaviors of $M_2(\cdot, 0)$, Lemma 8.4, and $F_3(\cdot, 0)$, Lemma 8.5, and the following,

$$e^x (f_1(x, 0) + \beta^{-1}(p) \phi'(x)) \xrightarrow{x \rightarrow +\infty} 0,$$

yield the formula for $f_1(x, 0)$. From (8.30)

$$f_3(x, 0) = f_1(-x, 0) = -\beta^{-1}(p) \phi'(-x) = \beta^{-1}(p) \phi'(x)$$

we conclude (8.43). Next we have

$$\mathcal{L} \partial_\lambda f_1(\cdot, 0) = f_1(\cdot, 0) = -\beta^{-1}(p) \phi'.$$

By (3.9) we have $\mathcal{L} \Lambda_p \phi = -\phi'$ and so

$$G(x) := \partial_\lambda f_1(\cdot, 0) - \beta^{-1}(p) \Lambda_p \phi \in \operatorname{Span}\{\phi', F_3(\cdot, 0), M_2(\cdot, 0)\}.$$

Since $G(x) \xrightarrow{x \rightarrow +\infty} 0$ by Lemma 8.2, necessarily $G(x) = \alpha \beta^{-1}(p) \phi'$ for some $\alpha \in \mathbb{R}$ and so

$$\partial_\lambda f_1(\cdot, 0) = \alpha \beta^{-1}(p) \phi' + \beta^{-1}(p) \Lambda_p \phi.$$

We obtain α matching asymptotic behaviors in the last formula. By Lemma 8.2, for $x \rightarrow +\infty$

$$\partial_\lambda f_1(x, 0) = e^{-x} (\partial_\lambda \mu_1(0)x + o(1)) = e^{-x} \left(-\frac{1}{2}x + o(1) \right) \text{ by } \partial_\lambda \mu_1(0) = -\frac{1}{2}, \quad (8.46)$$

with the last equality obtained by elementary implicit differentiation in (8.2). For $x \rightarrow +\infty$

$$\alpha \beta^{-1}(p) \phi'(x) + \beta^{-1}(p) \Lambda_p \phi(x) = e^{-x} \left(-\frac{1}{2}x - \alpha + \frac{1}{p-1} + o(1) \right)$$

which to match with (8.46) requires $\alpha = \frac{1}{p-1}$ giving (8.44). Using (8.30) and (8.44) we obtain

$$\begin{aligned} \partial_\lambda f_3(x, 0) &= -\partial_\lambda f_1(-x, 0) = -\beta^{-1}(p) \left(\frac{1}{p-1} \phi'(-x) + \Lambda_p \phi(-x) \right) \\ &= \beta^{-1}(p) \left(\frac{1}{p-1} \phi'(x) - \Lambda_p \phi(x) \right). \end{aligned}$$

□

Corollary 8.8. *The following facts are true.*

- i We have $\partial_\lambda^a b_{13}(x, 0) = 0$ for $a = 0, 1$.
- ii For any $n, m \geq 0$ there is a constant $C_{n,m}$ such that for all $\lambda \in \mathbb{C}$ such that $0 \leq \operatorname{Re} \lambda \leq \varepsilon$ and $x \in \mathbb{R}$

$$\left| \partial_x^n \partial_\lambda^m (e^{\mu_2(\lambda)x} b_{13}(x, \lambda)) \right| \leq C_{n,m} \langle \lambda \rangle^{\frac{1}{3}-m}. \quad (8.47)$$

iii There exists a constant $b_0 \in \mathbb{C}$ such that $\partial_\lambda^2 b_{13}(x, 0) = 2D''(0) + b_0\phi(x)$

Proof. Starting from i, case $a = 0$ is obvious from (8.43) while

$$\partial_\lambda b_{13}(x, 0) = \partial_\lambda f_1(x, 0)f_3'(x, 0) + f_1(x, 0)\partial_\lambda f_3'(x, 0) - \partial_\lambda f_1'(x, 0)f_3(x, 0) - f_1'(x, 0)\partial_\lambda f_3(x, 0) =$$

$$\frac{1}{\beta^2(p)} \left[\left(\frac{1}{p-1} \phi' + \Lambda_p \phi \right) \phi'' - \phi' \left(\frac{1}{p-1} \phi' - \Lambda_p \phi \right)' - \left(\frac{1}{p-1} \phi' + \Lambda_p \phi \right)' \phi' + \phi'' \left(\frac{1}{p-1} \phi' - \Lambda_p \phi \right) \right] = 0.$$

By elementary computation and using (8.3) we obtain

$$b_{13}(x, \lambda) = e^{\mu_1(\lambda)x} m_1(x, \lambda) (e^{\mu_3(\lambda)x} m_3(x, \lambda))' - (e^{\mu_1(\lambda)x} m_1(x, \lambda))' e^{\mu_3(\lambda)x} m_3(x, \lambda)$$

$$= e^{-\mu_2(\lambda)x} ((\mu_3(\lambda) - \mu_1(\lambda)) m_1(x, \lambda) m_3(x, \lambda) + m_1(x, \lambda) m_3'(x, \lambda) - m_1'(x, \lambda) m_3(x, \lambda)).$$

Hence from Lemmas 8.1–8.3, we have the following, proving ii,

$$\left| \partial_x^n \partial_\lambda^m (e^{\mu_2(\lambda)x} b_{13}(x, \lambda)) \right| =$$

$$\left| \partial_x^n \partial_\lambda^m ((\mu_3(\lambda) - \mu_1(\lambda)) m_1(x, \lambda) m_3(x, \lambda) + m_1(x, \lambda) m_3'(x, \lambda) - m_1'(x, \lambda) m_3(x, \lambda)) \right| \leq C_{n,m} \langle \lambda \rangle^{\frac{1}{3}-m}.$$

Differentiating (8.40) and using Claim i, it is easy to conclude that $\mathcal{L}^* \partial_\lambda^2 b_{13}(\cdot, 0) = 0$. Then

$$\partial_\lambda^2 b_{13}(\cdot, 0) \in \text{Span}\{1, \phi, g\}$$

where g is an unbounded solution of $\mathcal{L}^* u = 0$, but since, by Claim i and (8.47), $\partial_\lambda^2 b_{13}(\cdot, 0) \in L^\infty(\mathbb{R})$ there are two constants such that $\partial_\lambda^2 b_{13}(\cdot, 0) = a_0 + b_0\phi$. Here we show the explicit value $a_0 = 2D''(0) \neq 0$, as this is the source of some difficulty later. Obviously we have $a_0 = \lim_{x \rightarrow -\infty} \partial_\lambda^2 b_{13}(x, 0)$. We have

$$\partial_\lambda^2 b_{13}(x, 0) = 2a(x) + b(x) \text{ with } a(x) = \partial_\lambda^2 m_1(x, 0) m_3(x, 0) \text{ and}$$

$$b(x) = 4\partial_\lambda m_1(x, 0) \partial_\lambda m_3(x, 0) + 2m_1(x, 0) \partial_\lambda^2 m_3(x, 0)$$

$$+ \partial_\lambda^2 (m_1(x, \lambda) m_3'(x, \lambda) - m_1'(x, \lambda) m_3(x, \lambda))|_{\lambda=0}$$

where we used i and $\mu_3(0) - \mu_1(0) = 2$. Now $m_3(x, 0) \xrightarrow{x \rightarrow -\infty} 1$ by (8.28). We have

$$m_1(x, \lambda) - 1 = \sum_{j=2,3} a_j \mu_j \int_x^{+\infty} e^{(\mu_j - \mu_1)(x-y)} f'(\phi(y)) m_1(y, \lambda) dy$$

$$- a_1 \mu_1 \int_x^{+\infty} f'(\phi(y)) m_1(y, \lambda) dy =: A_1(x, \lambda) + A_2(x, \lambda).$$

It is easy to see that always $\partial_\lambda^m \partial_\lambda^n A_1(x, \lambda) \xrightarrow{x \rightarrow -\infty} 0$ and by (8.42)

$$\lim_{x \rightarrow -\infty} \partial_\lambda^n A_2(x, \lambda) = -\partial_\lambda^n \left(a_1 \mu_1 \int_{\mathbb{R}} f'(\phi(y)) m_1(y, \lambda) dy \right) = \partial_\lambda^n (D(\lambda) - 1)$$

So we have proved $a(x) \xrightarrow{x \rightarrow -\infty} 2D''(0)$. Finally we prove $b(x) \xrightarrow{x \rightarrow -\infty} 0$. For most of the terms in $b(x)$ this follows from (8.28). The only possibly critical term is the following, which gives the proof,

$$\partial_\lambda^2 \partial_x m_1(x, \lambda) = \partial_\lambda^2 \partial_x A_1(x, \lambda) + \partial_\lambda^2 (a_1 \mu_1 f'(\phi(x)) m_1(x, \lambda)) \xrightarrow{x \rightarrow -\infty} 0.$$

□

Lemma 8.9. For $j = 1$ and 3 and using the notation in (8.38), we have

$$\tilde{b}_{j2}(x, 0) = (-1)^{\frac{j-1}{2}} \beta^{-1}(p) \langle \Lambda_p \phi, \phi \rangle \eta_2 \text{ and} \quad (8.48)$$

$$\partial_\lambda \tilde{b}_{j2}(x, 0) = (-1)^{\frac{j-1}{2}} \beta^{-1}(p) \langle \Lambda_p \phi, \phi \rangle \eta_1 + \alpha_j \eta_2 \quad (8.49)$$

$$\text{with } \alpha_1 = -\alpha_3 - \frac{2}{(p-1)\beta(p)\theta_3} \text{ and } \alpha_3 = \frac{1 + \frac{1}{p-1} + \frac{\theta_2}{\theta_1}}{\beta(p)\theta_3}. \quad (8.50)$$

Proof. The space of the solutions of equation $\mathcal{L}^*u = 0$ coincides with $\text{Span}\{1, \eta_2, g\}$ for a function g that explodes exponentially as $x \rightarrow \pm\infty$. By (8.40) we have $\mathcal{L}^*\tilde{b}_{32}(\cdot, \lambda) = -\lambda\tilde{b}_{32}(\cdot, \lambda)$. By $\tilde{b}_{32}(x, 0) \xrightarrow{x \rightarrow -\infty} 0$ we can conclude that there exists an $\alpha \in \mathbb{R}$ such that $\tilde{b}_{32}(\cdot, 0) = \alpha\beta^{-1}(p)\phi$. We find α by matching the asymptotic behaviors: for $x \rightarrow -\infty$ we have

$$\begin{aligned}\tilde{b}_{32}(x, 0) &= -f'_3(x, 0)(1 + o(1)) = -e^x(1 + o(1)) \text{ and} \\ \alpha\beta^{-1}(p)\phi(x) &= \alpha e^x(1 + o(1))\end{aligned}$$

obtaining $\alpha = -1$. Formula (8.48) for $j = 3$ follows from (3.4). Case $j = 1$ follows from $f_3(x, 0) = -f_1(x, 0)$. Differentiating in λ the equation of $\tilde{b}_{j2}(\cdot, \lambda)$ and using (8.48) we obtain

$$\mathcal{L}^*\partial_\lambda\tilde{b}_{j2}(x, 0) = -\tilde{b}_{j2}(x, 0) = -(-1)^{\frac{j-1}{2}}\beta^{-1}(p)\langle\Lambda_p\phi, \phi\rangle\eta_2(x).$$

By (3.10) we obtain

$$\mathcal{L}^*\left(\partial_\lambda\tilde{b}_{j2}(\cdot, 0) + (-1)^{\frac{j-1}{2}}\beta^{-1}(p)\langle\Lambda_p\phi, \phi\rangle\eta_1\right) = 0. \quad (8.51)$$

Since, by (3.4), (8.34) and (8.38) and by Lemma 8.7 the function in (8.51) decays to 0 for $x \rightarrow -\infty$, there exists an $\alpha_j \in \mathbb{R}$ such that

$$\partial_\lambda\tilde{b}_{j2}(\cdot, 0) + (-1)^{\frac{j-1}{2}}\beta^{-1}(p)\langle\Lambda_p\phi, \phi\rangle\eta_1 = \alpha_j\eta_2. \quad (8.52)$$

We find α_j by matching the asymptotic behaviors for $x \rightarrow -\infty$. By (3.4) and (3.6)

$$\begin{aligned}\eta_1(x) &= \frac{\theta_1}{2} \int_{-\infty}^x y\phi'(y)dy + \theta_1 \frac{1}{p-1} \int_{-\infty}^x \phi(y)dy + \theta_2\phi(x) \\ &= \frac{\theta_1}{2}x\phi(x) + \left[\theta_1\left(\frac{1}{p-1} - \frac{1}{2}\right) + \theta_2\right]\phi(x) + o(\phi(x)) \text{ for } x \rightarrow -\infty.\end{aligned} \quad (8.53)$$

By (8.3), (8.28) and (8.34) we have

$$\tilde{b}_{32}(x, \lambda) = e^{-\mu_1(\lambda)x}(\mu_2(\lambda) - \mu_3(\lambda)) + o\left(\frac{e^{-\mu_1(\lambda)x}}{x}\right) \text{ for } x \rightarrow -\infty$$

so that

$$\partial_\lambda\tilde{b}_{32}(x, 0) = e^x\left(\frac{3}{2} - \frac{x}{2} + o(1)\right) \text{ for } x \rightarrow -\infty$$

Matching the asymptotic terms in (8.52), after a cancellation of the two dominating terms in the left hand side, we obtain the following,

$$\alpha_3\beta(p)\theta_3 = \frac{3}{2} + \beta^{-1}(p)\frac{1}{\theta_1}\left[\theta_1\left(\frac{1}{p-1} - \frac{1}{2}\right) + \theta_2\right]\beta(p) = 1 + \frac{1}{p-1} + \frac{\theta_2}{\theta_1}.$$

By (8.34) and Lemma 8.7 we have, using $\mu'_2(0) = 1$,

$$\begin{aligned}\partial_\lambda\tilde{b}_{12}(x, 0) &= \partial_\lambda f_1(x, 0)\tilde{f}_2(x, 0) + f_1(x, 0)\partial_\lambda\tilde{f}_2(x, 0) - \partial_\lambda f'_1(x, 0)\tilde{f}_2(x, 0) - f'_1(x, 0)\partial_\lambda\tilde{f}_2(x, 0) \\ &= o(e^x) - e^x\mu'_2(0) - \left(\frac{2}{p-1}e^x + \frac{x}{2}e^x\right)' + e^x\mu'_2(0)x \\ &= \frac{1}{2}xe^x - \left(\frac{3}{2} + \frac{2}{p-1}\right)e^x + o(e^x) \text{ for } x \rightarrow -\infty.\end{aligned}$$

By matching the asymptotic behaviors for $x \rightarrow -\infty$ in (8.52) we obtain

$$\alpha_1\beta(p)\theta_3 = -\left(\frac{3}{2} + \frac{2}{p-1}\right) - \frac{\theta_2}{\theta_1} - \left(\frac{1}{p-1} - \frac{1}{2}\right) = -1 - \frac{3}{p-1} - \frac{\theta_2}{\theta_1}.$$

□

In general $\tilde{f}_2$ increases exponentially as $x \rightarrow \infty$ even if $\text{Re } \lambda = 0$. In the following, we modify $\tilde{f}_2$ to be globally bounded. We first examine the solutions of $\mathcal{L}u = 0$.

Lemma 8.10. *The space of solutions of $\mathcal{L}u = 0$ is spanned by $\{\phi', h_2, h_3\}$, where h_2, h_3 are even functions satisfying $\lim_{x \rightarrow -\infty} h_2(x) = 1$ and $\lim_{x \rightarrow \infty} h_3(x) = \infty$. In particular, there exists no solution of $\mathcal{L}u = 0$ satisfying $\lim_{x \rightarrow -\infty} u(x) = 1$ and $\lim_{x \rightarrow \infty} |u(x)| = \infty$.*

Proof. Since $\mathcal{L}u = 0$ is an 3rd order ordinal differential equation, the solutions are determined by $u(0)$, $u'(0)$ and $u''(0)$. Since ϕ' solves $\mathcal{L}u = 0$ and $\phi'(0) = \phi'''(0) = 0$ since ϕ is an even function, it suffices to consider the remaining 2 dimensional space of solutions with $u'(0) = 0$. Now, since $\mathcal{L}u = 0$ is equivalent to $L_+u = c$ for some $c \in \mathbb{R}$ and L_+ preserves the parity, we see that solutions of $\mathcal{L}u = 0$ with $u'(0) = 0$ are even. From Lemma 8.5, there exist solutions of $\mathcal{L}u = 0$ satisfying $\lim_{x \rightarrow +\infty} u(x) = 1$ and solutions satisfying $\lim_{x \rightarrow +\infty} u(x) = \infty$ (recall $\mu_3(0) = 1$), yielding the desired h_2 and h_3 . Finally, if $\lim_{x \rightarrow +\infty} u(x) = \infty$, then u has to have h_3 component. However, this will imply $\lim_{x \rightarrow -\infty} u(x) = \infty$. □

Now, since $\tilde{f}_2(x, \lambda), f_3(x, \lambda) \in \text{Span}\{f_1(x, \lambda), F_2(x, \lambda), F_3(x, \lambda)\}$, we can write

$$\tilde{f}_2(x, \lambda) = c_{21}(\lambda)f_1(x, \lambda) + c_{22}(\lambda)F_2(x, \lambda) + c_{23}(\lambda)F_3(x, \lambda), \quad (8.54)$$

$$f_3(x, \lambda) = c_{31}(\lambda)f_1(x, \lambda) + c_{32}(\lambda)F_2(x, \lambda) + c_{33}(\lambda)F_3(x, \lambda). \quad (8.55)$$

Lemma 8.11. The $c_{jk}(\lambda)$ ($j = 2, 3, k = 1, 2, 3$) in (8.54)–(8.55) are analytic in λ . Furthermore, we have $c_{33}(\lambda) = D(-\lambda)$ and $c_{23}(0) = 0$.

Proof. Set c_{1j} by $F_1(x, \lambda) = c_{11}(\lambda)f_1(x, \lambda) + c_{12}(\lambda)F_2(x, \lambda) + c_{13}F_3(x, \lambda)$. Then, the 3×3 matrix $C = (c_{jk})_{1 \leq j, k \leq 3}$ becomes invertible and since it can be determined by the values of $F_j(0, \lambda)$ ($j = 1, 2, 3$), $f_1(0, \lambda)$, $\tilde{f}_2(0, \lambda)$ and $f_3(0, \lambda)$ which are all analytic, we have the analyticity of c_{jk} .

Next, by the symmetry in (8.30) we have $m_3(x, \lambda) = m_1(-x, -\lambda)$ for $\lambda \in i\mathbb{R}$ so that by the limit (8.42) we have

$$c_{33}(\lambda) = \lim_{x \rightarrow +\infty} m_3(x, \lambda) = \lim_{x \rightarrow -\infty} m_1(x, -\lambda) = D(-\lambda). \quad (8.56)$$

Finally, suppose $c_{23}(0) \neq 0$. Then, $\tilde{f}_2(x, 0) = \tilde{m}_2(x, 0)$ will satisfy $\lim_{x \rightarrow -\infty} \tilde{f}_2(x, 0) = 1$ and $\lim_{x \rightarrow -\infty} |\tilde{f}_2(x, 0)| = \infty$. Since $\tilde{f}_2(x, 0)$ solves $\mathcal{L}u = 0$ this contradicts with Lemma 8.10. Therefore, we have $c_{23}(0) = 0$. $\square$

Now, we set

$$c_0(\lambda) := \frac{\lambda c_{23}(\lambda)}{D(-\lambda)}. \quad (8.57)$$

Notice that from Lemma 8.11 and (8.41), c_0 is analytic in the neighborhood of 0. Set

$$f_2(x, \lambda) := \lambda \tilde{f}_2(x, \lambda) - c_0(\lambda)f_3(x, \lambda). \quad (8.58)$$

Then, from the choice of $c_0(\lambda)$, we have $f_2(x, \lambda) \in \text{Span}\{f_1(x, \lambda), F_2(x, \lambda)\}$. Indeed, from (8.54), (8.55), (8.57) and Lemma 8.11,

$$\begin{aligned} f_2(x, \lambda) &= (\lambda c_{21}(\lambda) - c_0(\lambda)c_{31}(\lambda))f_1(x, \lambda) + (\lambda c_{22}(\lambda) - c_0(\lambda)c_{32}(\lambda))F_2(x, \lambda) \\ &\quad + (\lambda c_{23}(\lambda) - c_0(\lambda)D(-\lambda))F_3(x, \lambda). \end{aligned}$$

The new Jost function f_2 satisfies better estimates than $\tilde{f}_2(x, \lambda)$.

Lemma 8.12. For any preassigned $a_0 > 0$ there are constants $C_{a\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\lambda^\beta m_2(x, \lambda)| \leq C_{a\beta} \text{ for all } x \in \mathbb{R} \text{ and all } \lambda \in (-ia_0, ia_0). \quad (8.59)$$

Proof. We know from Lemma 8.4 that (8.59) holds for $x \leq 0$. For $x \geq 0$ it follows from $f_2(\cdot, \lambda) \in \text{Span}\{f_1(\cdot, \lambda), F_2(\cdot, \lambda)\}$ and from Lemmas 8.2 and 8.5. $\square$

Consider now the Wronskian, see (8.14),

$$W(\lambda) := W[f_1(y, \lambda), f_2(y, \lambda), f_3(y, \lambda)]. \quad (8.60)$$

We claim that

$$W(\lambda) = \lambda D(\lambda)W_0(\lambda). \quad (8.61)$$

To prove (8.61) notice that

$$W(\lambda) = \lambda W[f_1(y, \lambda), \tilde{f}_2(y, \lambda), f_3(y, \lambda)].$$

For

$$Y(x, \lambda) := (f_1(x, \lambda), f_1'(x, \lambda), f_1''(x, \lambda))^T \text{ and } Z(x, \lambda) := (z_1, z_2, z_3)^T \text{ with}$$

$$z_1 = \tilde{f}_2'(x, \lambda)f_3''(x, \lambda) - \tilde{f}_2''(x, \lambda)f_3'(x, \lambda),$$

$$z_2 = \tilde{f}_2''(x, \lambda)f_3(x, \lambda) - \tilde{f}_2(x, \lambda)f_3''(x, \lambda) \text{ and}$$

$$z_3 = \tilde{f}_2(x, \lambda)f_3'(x, \lambda) - \tilde{f}_2'(x, \lambda)f_3(x, \lambda),$$

then Y solves (8.39) and Z solves the dual system $\frac{d}{dx}Z = -A^T Z$ and we have

$$W[f_1(x, \lambda), \tilde{f}_2(x, \lambda), f_3(x, \lambda)] = Y^T(x, \lambda)Z(x, \lambda). \quad (8.62)$$

By Lemmas 8.2–8.4 we have

$$e^{-\mu_1 x} Y(x, \lambda) \xrightarrow{x \rightarrow +\infty} Y_+(\lambda) = (1, \mu_1, \mu_1^2)^T \text{ and}$$

$$e^{\mu_1 x} Z(x, \lambda) \xrightarrow{x \rightarrow -\infty} Z_-(\lambda) = (\mu_2 \mu_3 (\mu_3 - \mu_2), \mu_2^2 - \mu_3^2, \mu_3 - \mu_2)^T$$

with, by an elementary computation which uses (8.3) and (8.15),

$$Y_+^T(\lambda)Z_-(\lambda) = W_0(\lambda).$$

Then we obtain the following which, with (8.62), yields (8.61),

$$Y^T(x, \lambda)Z(x, \lambda) = D(\lambda)W_0(\lambda).$$

Lemma 8.13. For $R(\lambda) = R_{\mathcal{L}}(\lambda)$ for $\varepsilon \geq \operatorname{Re} \lambda \geq 0$ with $\lambda \neq 0$ we have

$$R(\lambda)g = R_1(\lambda)g + R_2(\lambda)g \text{ where} \quad (8.63)$$

$$R_1(\lambda)g(x) = -f_2(x, \lambda) \int_x^{+\infty} \frac{g(y)b_{13}(y, \lambda)}{W(\lambda)} dy \text{ and} \quad (8.64)$$

$$R_2(\lambda)g = R_{23}(\lambda)g + R_{21}(\lambda)g \text{ with} \quad (8.65)$$

$$R_{23}(\lambda)g(x) = -f_1(x, \lambda) \int_{-\infty}^x \frac{g(y)b_{23}(y, \lambda)}{W(\lambda)} dy, \quad R_{21}(\lambda)g(x) = f_3(x, \lambda) \int_x^{+\infty} \frac{g(y)b_{12}(y, \lambda)}{W(\lambda)} dy,$$

Proof. First of all, by the variation of parameters formula the right hand side in (8.63), which is well defined in the range $\varepsilon \geq \operatorname{Re} \lambda \geq 0$ for some $\varepsilon > 0$, solves the equation $(\mathcal{L} - \lambda)u = g$. Furthermore, by (8.6) and by Lemmas 8.2, 8.3 and 8.12 it defines a bounded operator in $L^2(\mathbb{R})$ when $\varepsilon \geq \operatorname{Re} \lambda > 0$. By continuity in $\lambda = z$ in (6.6), it holds also for $\operatorname{Re} \lambda \geq 0$ with $\lambda \neq 0$. $\square$

Notice that (8.63)–(8.65) are equivalent to

$$R(\lambda)g = \tilde{R}_1(\lambda)g + \tilde{R}_2(\lambda)g \text{ where} \quad (8.66)$$

$$\tilde{R}_1(\lambda)g(x) = -\tilde{f}_2(x, \lambda) \int_x^{+\infty} \frac{g(y)b_{13}(y, \lambda)}{D(\lambda)W_0(\lambda)} dy \text{ and} \quad (8.67)$$

$$\tilde{R}_2(\lambda)g = \tilde{R}_{23}(\lambda)g + \tilde{R}_{21}(\lambda)g \text{ with} \quad (8.68)$$

$$\tilde{R}_{23}(\lambda)g(x) = -f_1(x, \lambda) \int_{-\infty}^x \frac{g(y)\tilde{b}_{23}(y, \lambda)}{D(\lambda)W_0(\lambda)} dy \text{ and } \tilde{R}_{21}(\lambda)g(x) = f_3(x, \lambda) \int_x^{+\infty} \frac{g(y)\tilde{b}_{12}(y, \lambda)}{D(\lambda)W_0(\lambda)} dy.$$

9. Proof of Proposition 6.3

We start by first proving the inequality (6.8). We distinguish between λ close to 0 and away from 0. We start with the small λ case.

Lemma 9.1. There exists an $\varepsilon_4 > 0$ and a $C(p) > 0$ such that

$$\|R(i\tau)g\|_{\tilde{\Sigma}} \leq C(p)\|g\|_{L^{1,1}_+(\mathbb{R})} \text{ for all } g \in \ker P \text{ and } \tau \in (-\varepsilon_4, \varepsilon_4) \setminus \{0\}. \quad (9.1)$$

Proof. For $\lambda = i\tau$, using (8.63) and (8.66) we set

$$R(\lambda)g = \frac{N(\lambda)g}{W(\lambda)} = \frac{\tilde{N}(\lambda)g}{D(\lambda)W_0(\lambda)},$$

where $N(\lambda) = \lambda\tilde{N}(\lambda)$ can be defined using (8.64)–(8.65) resp. (8.67)–(8.68). Next we consider

$$\begin{aligned} R^{(0)}(\lambda)g &:= \frac{1}{D(\lambda)W_0(\lambda)} \sum_{i=0,1} \lambda^i \partial_x^i \tilde{N}(0)g \\ &= \frac{1}{D(\lambda)W_0(\lambda)} \left(-f_1(x, 0) \int_{-\infty}^x g(y)\tilde{b}_{23}(y, 0)dy + f_3(x, 0) \int_x^{+\infty} g(y)\tilde{b}_{12}(y, 0)dy \right) \\ &\quad + \frac{\lambda}{D(\lambda)W_0(\lambda)} \left(-\partial_\lambda f_1(x, 0) \int_{-\infty}^x g(y)\tilde{b}_{23}(y, 0)dy + \partial_\lambda f_3(x, 0) \int_x^{+\infty} g(y)\tilde{b}_{12}(y, 0)dy \right) \\ &\quad + \frac{\lambda}{D(\lambda)W_0(\lambda)} \left(-f_1(x, 0) \int_{-\infty}^x g(y)\partial_\lambda \tilde{b}_{23}(y, 0)dy + f_3(x, 0) \int_x^{+\infty} g(y)\partial_\lambda \tilde{b}_{12}(y, 0)dy \right) \\ &=: \frac{1}{D(\lambda)W_0(\lambda)} A_0 + \frac{\lambda}{D(\lambda)W_0(\lambda)} A_{11} + \frac{\lambda}{D(\lambda)W_0(\lambda)} A_{12} \end{aligned}$$

with the contribution of $\tilde{R}_1(\lambda)$ null by Corollary 8.8. We claim that, since $g \in \ker P$, we have $R^{(0)}(\lambda)g = 0$. From (8.43) and (8.48) we have

$$A_0 = \beta^{-1}(p)f_3(x, 0)\langle \Lambda_p \phi, \phi \rangle \langle g, \eta_2 \rangle = 0 \text{ by } g \in \ker P.$$

We have

$$\begin{aligned} A_{11} &= -\frac{\beta^{-2}(p)}{\theta_3} \left(\frac{1}{p-1} \phi' + \Lambda_p \phi \right) \int_{-\infty}^x g(y)\eta_2(y)dy + \frac{\beta^{-2}(p)}{\theta_3} \left(\frac{1}{p-1} \phi' - \Lambda_p \phi \right) \int_x^{+\infty} g(y)\eta_2(y)dy \\ &= \cancel{\frac{\beta^{-2}(p)}{\theta_3} \Lambda_p \phi \langle g, \eta_2 \rangle} - \frac{\beta^{-2}(p)}{\theta_3} \frac{\phi'}{p-1} \int_{-\infty}^x g(y)\eta_2(y)dy + \frac{\beta^{-2}(p)}{\theta_3} \frac{\phi'}{p-1} \int_x^{+\infty} g(y)\eta_2(y)dy, \end{aligned}$$

where the cancelled term is null by $g \in \ker P$. Similar cancellations yield

$$A_{12} = \frac{\beta^{-2}(p)}{\theta_3} \phi' \langle g, \eta_1 \rangle - \beta^{-1}(p) \alpha_3 \phi' \int_{-\infty}^x g(y) \eta_2(y) dy \\ - \left(\alpha_3 + \frac{2}{(p-1)\beta(p)\theta_3} \right) \beta^{-1}(p) \phi' \int_x^{+\infty} g(y) \eta_2(y) dy = - \frac{2}{(p-1)\beta^2(p)\theta_3} \phi' \int_x^{+\infty} g(y) \eta_2(y) dy.$$

Then we obtain the following, which proves $R^{(0)}(\lambda)g = 0$ for $g \in \ker P$,

$$A_{11} + A_{12} = - \frac{\beta^{-2}(p)}{\theta_3} \frac{\phi'}{p-1} \langle g, \eta_2 \rangle = 0.$$

Since

$$R^{(0)}(\lambda)g = \frac{1}{\lambda D(\lambda)W_0(\lambda)} \sum_{i=0}^2 \frac{\lambda^i}{i!} \partial_z^i N(0)g = 0$$

we have for $R^{(1)}(\lambda)g = R(\lambda)g - R^{(0)}(\lambda)g$, by the Taylor expansion formula we have

$$R^{(1)}(\lambda)g = R_1^{(1)}(\lambda)g + R_{21}^{(1)}(\lambda)g + R_{23}^{(1)}(\lambda)g \text{ where} \quad (9.2)$$

$$R_{21}^{(1)}(\lambda)g = \frac{2^{-1}}{\lambda D(\lambda)W_0(\lambda)} \int_0^\lambda dz (\lambda - z)^2 \int_x^{+\infty} g(y) \partial_z^3 (f_3(x, z) b_{12}(y, z)) dy, \quad (9.3)$$

$$R_{23}^{(1)}(\lambda)g = - \frac{2^{-1}}{\lambda D(\lambda)W_0(\lambda)} \int_0^\lambda dz (\lambda - z)^2 \int_{-\infty}^x g(y) \partial_z^3 (f_1(x, z) b_{23}(y, z)) dy \text{ and} \quad (9.4)$$

$$R_1^{(1)}(\lambda)g = - \frac{1}{\lambda D(\lambda)W_0(\lambda)} \int_x^{+\infty} g(y) \left(f_2(x, \lambda) b_{13}(y, \lambda) - \frac{\lambda^2}{2} f_2(x, 0) \partial_\lambda^2 b_{13}(y, 0) \right) dy. \quad (9.5)$$

We examine each of these three terms. We claim that

$$\left| R_{21}^{(1)}(\lambda)g(x) \right| \lesssim \int_x^{+\infty} \langle x - y \rangle^3 e^{-|x-y|} |g(y)| dy \quad (9.6)$$

which in turn, by the convolution Young's inequality, implies

$$\|R_{21}^{(1)}(\lambda)g\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \lesssim \|g\|_{L^1(\mathbb{R})}. \quad (9.7)$$

To obtain (9.6) notice that

$$\partial_z^3 (f_3(x, z) b_{12}(y, z)) = \partial_z^3 [e^{\mu_3(z)(x-y)} m_3(x, z) \gamma(y, z)] \text{ where} \\ \gamma(y, z) = (\mu_2(\lambda) - \mu_1(\lambda)) m_1(y, z) m_2(y, z) + m_1(x, z) m_2'(y, z) - m_1'(y, z) m_2(y, z).$$

Since by Lemmas 8.2, 8.3 and 8.12 for $y \geq x$ for appropriate constants $C_{nm\ell}$ we have

$$\left| \partial_x^n \partial_y^m \partial_z^\ell [m_3(x, z) \gamma(y, z)] \right| \leq C_{nm\ell}$$

if, as in our case, z is close to 0, by $D''(0) \neq 0$ and by $\operatorname{Re} \mu_3(z) \geq 1$, see (8.10), we obtain

$$|\partial_z^3 (f_3(x, z) b_{12}(y, z))| \lesssim \langle x - y \rangle^3 e^{-|x-y|},$$

yielding (9.6). Since with the same proof we obtain also

$$\left| R_{23}^{(1)}(\lambda)g(x) \right| \lesssim \int_{-\infty}^x \langle x - y \rangle^3 e^{-|x-y|} |g(y)| dy \quad (9.8)$$

we conclude also

$$\|R_{23}^{(1)}(\lambda)g\|_{L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})} \lesssim \|g\|_{L^1(\mathbb{R})}. \quad (9.9)$$

Next we examine $R_1^{(1)}(\lambda)g$, which is subtler. Since $\partial_\lambda^2 b_{13}(y, 0) = \partial_\lambda^2 (e^{\mu_2 y} b_{13})(y, 0)$ by Corollary 8.8, we can write

$$f_2(x, \lambda) b_{13}(y, \lambda) - \frac{\lambda^2}{2} f_2(x, 0) \partial_\lambda^2 b_{13}(y, 0) \\ = f_2(x, \lambda) e^{-\mu_2(\lambda)y} (e^{\mu_2(\lambda)y} b_{13}(y, \lambda)) - \frac{\lambda^2}{2} f_2(x, 0) \partial_\lambda^2 (e^{\mu_2 y} b_{13})(y, 0) \\ = e^{-\mu_2(\lambda)y} \left(f_2(x, \lambda) e^{\mu_2(\lambda)y} b_{13}(y, \lambda) - \frac{\lambda^2}{2} f_2(x, 0) \partial_\lambda^2 (e^{\mu_2 y} b_{13})(y, 0) \right) \\ + \frac{\lambda^2}{2} (e^{-\mu_2(\lambda)y} - 1) f_2(x, 0) \partial_\lambda^2 b_{13}(y, 0) \\ = e^{-\mu_2(\lambda)y} \int_0^\lambda \frac{(\lambda - z)^2}{2} \partial_z^3 (e^{\mu_2(z)x} m_2(x, z) e^{\mu_2(z)y} b_{13}(y, z)) dz$$

$$+ \frac{\lambda^2}{2} (e^{-\mu_2(\lambda)y} - 1) f_2(x, 0) \partial_\lambda^2 b_{13}(y, 0) =: A(\lambda, x, y) + B(\lambda, x, y).$$

The $A(\lambda, x, y)$ term gives a contribution to (9.5) because

$$\frac{1}{|\lambda D(\lambda) W_0(\lambda)|} \int_x^{+\infty} |g(y) A(\lambda, x, y)| dy \lesssim (1 + |x|)^3 \int_x^{+\infty} |g(y)| dy$$

from the fact that

$$|A(\lambda, x, y)| \lesssim (1 + |x|)^3 \int_0^{|\lambda|} (|\lambda| + \sigma)^2 d\sigma \sim |\lambda|^3 (1 + |x|)^3$$

by $|\partial_z^n (m_2(x, z) e^{\mu_2(\lambda)y} b_{13}(y, z))| \lesssim 1$ by (8.47) and (8.59). Next, Corollary 8.8 and (8.58) give

$$\begin{aligned} B(\lambda, x, y) &= -D''(0) c_0(0) \lambda^2 (e^{-\mu_2(\lambda)y} - 1) f_3(x, 0) \\ &\quad - b_0 c_0(0) \frac{\lambda^2}{2} (e^{-\mu_2(\lambda)y} - 1) f_3(x, 0) \phi(y) =: B_1(\lambda, x, y) + B_2(\lambda, x, y). \end{aligned} \quad (9.10)$$

The second term is easy to treat because

$$\begin{aligned} \frac{1}{|\lambda D(\lambda) W_0(\lambda)|} \int_x^{+\infty} |g(y) B_2(\lambda, x, y)| dy &\lesssim \frac{|\lambda \mu_2(\lambda)|}{|D(\lambda) W_0(\lambda)|} |\phi'(x)| \int_x^{+\infty} |g(y)| \langle y \rangle \phi(y) dy \\ &\lesssim |\phi'(x)| \int_x^{+\infty} |g(y)| dy. \end{aligned}$$

Let us set now

$$\tilde{\phi}(x) := \frac{\int_{-\infty}^x \phi(y) dy}{\int_{\mathbb{R}} \phi(y) dy} \quad (9.11)$$

and let us consider the term

$$\hat{R}_1^{(1)}(\lambda) g := -D''(0) c_0(0) \frac{\lambda}{2 D(\lambda) W_0(\lambda)} f_3(x, 0) \int_x^{+\infty} g(y) (e^{-\mu_2(\lambda)y} - 1) dy. \quad (9.12)$$

Then we can write

$$\begin{aligned} \hat{R}_1^{(1)}(\lambda) g &= -D''(0) c_0(0) \frac{\lambda}{2 D(\lambda) W_0(\lambda)} f_3(x, 0) \int_x^{+\infty} g(y) (e^{-\mu_2(\lambda)y} - 1) (1 - \tilde{\phi}(y)) dy \\ &\quad + D''(0) c_0(0) \frac{\lambda}{2 D(\lambda) W_0(\lambda)} f_3(x, 0) \int_{-\infty}^x g(y) (e^{-\mu_2(\lambda)y} - 1) \tilde{\phi}(y) dy \\ &\quad - D''(0) c_0(0) \frac{\lambda}{2 D(\lambda) W_0(\lambda)} f_3(x, 0) \int_{\mathbb{R}} g(y) (e^{-\mu_2(\lambda)y} - 1) \tilde{\phi}(y) dy = \sum_{j=1}^3 \hat{R}_1^{(1j)}(\lambda) g. \end{aligned} \quad (9.13)$$

We have, recalling $y^\pm := \max(0, \pm y)$,

$$|\hat{R}_1^{(11)}(\lambda) g(x)| \lesssim \frac{|\mu_2(\lambda)|}{|\lambda|} |\phi'(x)| \int_x^{+\infty} |g(y)| \langle y \rangle O(e^{-y^+}) dy \lesssim e^{-|x|} \int_x^{+\infty} |g(y)| dy.$$

Similarly

$$|\hat{R}_1^{(12)}(\lambda) g(x)| \lesssim \frac{|\mu_2(\lambda)|}{|\lambda|} |\phi'(x)| \int_{-\infty}^x |g(y)| \langle y \rangle O(e^{-y^-}) dy \lesssim e^{-|x|} \int_{-\infty}^x |g(y)| dy.$$

So far we have proved that

$$\left\| \left(R(i\tau) - \hat{R}_1^{(13)}(i\tau) \right) g \right\|_{\Sigma} \leq C_0 \|g\|_{L^1(\mathbb{R})} \text{ for all } g \in \ker P \text{ and } \tau \in (-\varepsilon_4, \varepsilon_4) \setminus \{0\}. \quad (9.14)$$

We finally have the following, which completes the proof of (9.1),

$$\left\| \hat{R}_1^{(13)}(i\tau) g \right\|_{\Sigma} \lesssim \|\phi'\|_{\Sigma} \frac{|\mu_2(i\tau)|}{|\tau|} \int_{\mathbb{R}} \langle y \rangle |g(y)| \tilde{\phi}(y) dy \lesssim \|g\|_{L^1(\mathbb{R}_-)} + \|g\|_{L^{1,1}(\mathbb{R}_+)} \sim \|g\|_{L_+^{1,1}}.$$

□

Having discussed what happens for λ close to 0, we consider λ away from 0.

Lemma 9.2. *There exists a $C(p, \varepsilon_4) > 0$ such that*

$$\sup_{\tau \in \mathbb{R} \setminus (-\varepsilon_4, \varepsilon_4)} \|R_{\mathcal{L}}(i\tau)\|_{\mathcal{L}(L^1(\mathbb{R}), \Sigma)} \leq C(p, \varepsilon_4). \quad (9.15)$$

Proof. We have for $R_0(i\tau) = R_{-\partial_x^3 + \partial_x}(i\tau + 0)$,

$$R(i\tau) = (1 - R_0(i\tau)\partial_x f'(\phi))^{-1} R_0(i\tau).$$

Here $R_0(i\tau)\partial_x f'(\phi)$ is a compact operator in $L^\infty(\mathbb{R})$. Recall, see [16, formula (A.4)], that $R_0(i\tau)$ is a convolution with the function

$$R_0(i\tau, x) = \begin{cases} a_1(i\tau)e^{\mu_1(i\tau)x}, & \text{if } x \geq 0 \text{ and} \\ -a_2(i\tau)e^{\mu_2(i\tau)x} - a_3(i\tau)e^{\mu_3(i\tau)x}, & \text{if } x \leq 0. \end{cases}$$

Notice that thanks to (8.18) we have $R_0(i\tau)\partial_x = \partial_x R_0(i\tau)$ with the latter having kernel

$$\tilde{R}_0(i\tau, x) = \begin{cases} \mu_1(i\tau)a_1(i\tau)e^{\mu_1(i\tau)x}, & \text{if } x \geq 0 \text{ and} \\ -\mu_2(i\tau)a_2(i\tau)e^{\mu_2(i\tau)x} - \mu_3(i\tau)a_3(i\tau)e^{\mu_3(i\tau)x}, & \text{if } x \leq 0. \end{cases}$$

So if $g = R_0(i\tau)\partial_x f'(\phi)g$ we have $g \in L_a^2(\mathbb{R})$ for $0 < a < 3^{-1/2}$. But then g is an eigenfunction in $L_a^2(\mathbb{R})$ with eigenvalue $i\tau$ with $\tau \in \mathbb{R} \setminus (-\varepsilon_4, \varepsilon_4)$, not possible by Propositions 3.1 and 3.2. Furthermore from (8.24), which is relating to the smoothing properties of the group of $-\partial_x^3 + \partial_x$, we have

$$\|R_0(i\tau)\partial_x f'(\phi)\|_{\mathcal{L}(L^\infty(\mathbb{R}))} + \|R_0(i\tau)\|_{\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))} + \|\partial_x R_0(i\tau)\|_{\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))} \xrightarrow{\tau \rightarrow \infty} 0. \quad (9.16)$$

Then from the immersion $L^\infty(\mathbb{R}) \hookrightarrow \tilde{\Sigma}$ we obtain

$$\begin{aligned} & \sup_{\tau \in \mathbb{R} \setminus (-\varepsilon_4, \varepsilon_4)} \|R_{\mathcal{L}}(i\tau)\|_{\mathcal{L}(L^1(\mathbb{R}), \tilde{\Sigma})} \\ & \lesssim \sup_{\tau \in \mathbb{R} \setminus (-\varepsilon_4, \varepsilon_4)} \left\| (1 - R_0(i\tau)\partial_x f'(\phi))^{-1} \right\|_{\mathcal{L}(L^\infty(\mathbb{R}))} \sup_{\tau \in \mathbb{R}} \|R_0(i\tau)\|_{\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))} < +\infty. \end{aligned}$$

□

The above two lemmas prove inequality (6.8).

Proof of inequality (6.9). The proof is easier than the proof of (6.8). In higher energies we use

$$\begin{aligned} & \sup_{\tau \in \mathbb{R} \setminus (-\varepsilon_4, \varepsilon_4)} \|R_{\mathcal{L}}(i\tau)\partial_x\|_{\mathcal{L}(L^1(\mathbb{R}), \tilde{\Sigma})} \\ & \lesssim \sup_{\tau \in \mathbb{R} \setminus (-\varepsilon_4, \varepsilon_4)} \left\| (1 - R_0(i\tau)\partial_x f'(\phi))^{-1} \right\|_{\mathcal{L}(L^\infty(\mathbb{R}))} \sup_{\tau \in \mathbb{R}} \|\partial_x R_0(i\tau)\|_{\mathcal{L}(L^1(\mathbb{R}), L^\infty(\mathbb{R}))} < +\infty. \end{aligned}$$

For $\tau \in (-\varepsilon_4, \varepsilon_4) \setminus \{0\}$ we proceed like in Lemma 9.1. We have for $\lambda = i\tau$

$$(R_{23}^{(0)}(\lambda) + R_{21}^{(0)}(\lambda)) \circ Q \circ \partial_x = 0,$$

while we recall $R_1^{(0)}(\lambda) \equiv 0$. Then, see (9.2), we have

$$R^{(1)}(\lambda)Qg' = R^{(1)}(\lambda)g' + \sum_{j=1}^2 \langle g, \eta_j' \rangle R^{(1)}(\lambda)\xi_j$$

where obviously $|\langle g, \eta_j' \rangle| \lesssim \|g\|_{L^1(\mathbb{R})}$ by (3.3) and by (9.6) and (9.8)

$$\|R_{21}^{(1)}(\lambda)\xi_j\|_{L^\infty(\mathbb{R})} + \|R_{23}^{(1)}(\lambda)\xi_j\|_{L^\infty(\mathbb{R})} \lesssim \|\xi_j\|_{L^1(\mathbb{R})}.$$

Using the operator in (9.12), we have

$$\begin{aligned} \|R_1^{(1)}(\lambda)\xi_j\|_{L^\infty(\mathbb{R})} & \leq \| (R_1^{(1)}(\lambda) - \hat{R}_1^{(1)}(\lambda)) \xi_j \|_{L^\infty(\mathbb{R})} + \|\hat{R}_1^{(1)}(\lambda)\xi_j\|_{L^\infty(\mathbb{R})} \\ & \lesssim \|\xi_j\|_{L^1(\mathbb{R})} + \|\hat{R}_1^{(1)}(\lambda)\xi_j\|_{L^\infty(\mathbb{R})} \text{ where} \end{aligned}$$

$$\begin{aligned} |\hat{R}_1^{(1)}(\lambda)\xi_j| & \lesssim \frac{1}{|\lambda|} |f_3(x, 0)| \int_x^{+\infty} |\xi_j(y)(e^{-\mu_2(\lambda)y} - 1)| dy \\ & \lesssim \frac{|\mu_2(\lambda)|}{|\lambda|} |\phi'(x)| \int_{\mathbb{R}} |y\xi_j(y)| dy \lesssim 1. \end{aligned}$$

Next, integrating by parts and using the identity

$$-f_3(x, z)b_{12}(x, z) - f_1(x, z)b_{23}(x, z) + f_2(x, z)b_{13}(x, z) \equiv 0$$

which eliminates a boundary term, we have

$$R^{(1)}(\lambda)g' = \frac{2^{-1}}{\lambda D(\lambda)W_0(\lambda)} \int_0^\lambda dz (\lambda - z)^2 \int_{-\infty}^x g(y) \partial_z^3 (f_1(x, z) \partial_y b_{23}(y, z)) dy$$

$$\begin{aligned}
& + \frac{2^{-1}}{\lambda D(\lambda) W_0(\lambda)} \int_0^\lambda dz (\lambda - z)^2 \int_x^{+\infty} g(y) \partial_z^3 (f_2(x, z) \partial_y b_{13}(y, z)) dy \\
& - \frac{1}{\lambda D(\lambda) W_0(\lambda)} \int_x^{+\infty} g(y) \tilde{\gamma}(x, y, \lambda) dy =: \sum_{j=1}^3 \mathcal{R}_j(\lambda) g \text{ where} \\
& \tilde{\gamma}(x, y, \lambda) = f_2(x, \lambda) \partial_y b_{13}(y, \lambda) - \frac{\lambda^2}{2} f_2(x, 0) \partial_y \partial_\lambda^2 b_{13}(y, 0).
\end{aligned}$$

By the arguments applied to $R_{21}^{(1)}(\lambda)$ and $R_{23}^{(1)}(\lambda)$ it is easy to show that

$$\|\mathcal{R}_j(\lambda)g\|_{\tilde{\Sigma}} \lesssim \|g\|_{L^1(\mathbb{R})} \text{ for } j = 2, 3.$$

As we show now the term $\mathcal{R}_1(\lambda)g$ is simpler than $R_1^{(1)}(\lambda)g$. We write

$$\begin{aligned}
\tilde{\gamma}(x, y, \lambda) &= -\mu_2(\lambda) e^{-\mu_2(\lambda)y} \left(f_2(x, \lambda) e^{\mu_2(\lambda)y} b_{13}(y, \lambda) - \frac{\lambda^2}{2} f_2(x, 0) \partial_\lambda^2 b_{13}(y, 0) \right) \\
&+ e^{-\mu_2(\lambda)y} f_2(x, \lambda) \partial_y (e^{\mu_2(\lambda)y} b_{13}(y, \lambda)) - \frac{\lambda^2}{2} f_2(x, 0) \partial_y \partial_z^2 (e^{\mu_2(z)y} b_{13}(y, z))|_{z=0} \\
&- \mu_2(\lambda) e^{-\mu_2(\lambda)y} \frac{\lambda^2}{2} f_2(x, 0) \partial_\lambda^2 b_{13}(y, 0) =: \sum_{j=1}^3 \tilde{\gamma}_j(x, y, \lambda)
\end{aligned} \tag{9.17}$$

where we use $\partial_y^n \partial_z^2 (e^{\mu_2(z)y} b_{13}(y, z))|_{z=0} = \partial_y^n \partial_\lambda^2 b_{13}(y, 0)$ for any n by [Corollary 8.8](#). Then

$$\begin{aligned}
& \left| \frac{1}{\lambda D(\lambda) W_0(\lambda)} \int_x^{+\infty} g(y) \tilde{\gamma}_1(x, y, \lambda) dy \right| \\
& \lesssim \frac{1}{|\lambda|^2} \int_0^{|\lambda|} |\lambda|^2 \int_x^{+\infty} dy |g(y)| \left| \partial_z^3 (e^{\mu(z)2x} m_2(x, z) e^{\mu_2(\lambda)y} b_{13}(y, z)) \right| dz \lesssim |\lambda| \langle x \rangle^3 \int_x^{+\infty} |g(y)| dy
\end{aligned}$$

by [\(8.20\)](#) and [\(8.47\)](#). For the same reasons

$$\begin{aligned}
& \left| \frac{1}{\lambda D(\lambda) W_0(\lambda)} \int_x^{+\infty} g(y) \tilde{\gamma}_2(x, y, \lambda) dy \right| \\
& \lesssim \frac{1}{|\lambda|^3} \int_0^{|\lambda|} |\lambda|^2 \int_x^{+\infty} dy |g(y)| \left| \partial_z^3 (e^{\mu(z)2x} m_2(x, z) \partial_y (e^{\mu_2(\lambda)y} b_{13}(y, z))) \right| dz \\
& \lesssim \langle x \rangle^3 \int_x^{+\infty} |g(y)| dy.
\end{aligned}$$

Finally, thanks to the $\mu_2(\lambda)$ factor appearing in [\(9.17\)](#) and conspicuously absent in [\(9.10\)](#), we have

$$\begin{aligned}
& \left| \frac{1}{\lambda D(\lambda) W_0(\lambda)} \int_x^{+\infty} g(y) \tilde{\gamma}_3(x, y, \lambda) dy \right| \\
& \lesssim \frac{|\mu_2(\lambda) \lambda^2|}{|\lambda|^3} |\phi'(x)| \int_x^{+\infty} dy |g(y) (2D''(0) + b_0 \phi(y))| \lesssim |\phi'(x)| \int_x^{+\infty} |g(y)| dy.
\end{aligned}$$

The above guarantee the following thus completing the proof of [\(6.9\)](#) and thus of [Proposition 6.3](#),

$$\|R^{(1)}(\lambda)g'\|_{\tilde{\Sigma}} \leq C(p) \|g\|_{L^1(\mathbb{R})} \text{ for all } g \in L^1(\mathbb{R}) \text{ and all } \lambda \in i(-\varepsilon_4, \varepsilon_4) \setminus \{0\}.$$

□

10. Proof of [Proposition 3.2](#)

In this section, we prove what is left of [Proposition 3.2](#), that is that $\mathcal{L}_{[c]}$ has no nonzero purely imaginary eigenvalue. First of all, it is elementary to see by scaling that it suffices to consider case $c = 1$. Next, we observe the following.

Lemma 10.1. *Let $\lambda > 0$. Then $i\lambda$ is an eigenvalue of $\mathcal{L}$ if and only if it is an eigenvalue of $L_+ \partial_x$.*

Remark 10.2. Since if $i\lambda$ ($\lambda \in \mathbb{R}$) is an eigenvalue of $\mathcal{L}$ with an eigenfunction ψ , then $-i\lambda$ is also an eigenvalue with the eigenfunction $\bar{\psi}$. Thus, it suffices to show the nonexistence of an embedded eigenvalue $i\lambda$ with $\lambda > 0$.

Proof. Let $i\lambda$ be an eigenvalue with associated eigenfunction φ . From the results in [Section 8](#), φ and its derivatives decay exponentially. Now, since φ satisfies $\partial_x L_+ \varphi = i\lambda \varphi$, setting $\psi = L_+ \varphi$, we have

$$L_+ \partial_x \psi = L_+ (\partial_x L_+ \varphi) = i\lambda L_+ \varphi = i\lambda \psi.$$

Thus, ψ is an eigenvalue of $L_+ \partial_x$ associated to $i\lambda$. Next, if ψ satisfies $L_+ \partial_x \psi = i\lambda \psi$, then $\varphi := \partial_x \psi$ satisfies

$$\partial_x L_+ \varphi = \partial_x L_+ \partial_x \psi = i\lambda \partial_x \psi = i\lambda \varphi.$$

Therefore, we have the conclusion. $\square$

We prove the non-existence of non-zero purely imaginary eigenvalue of $L_+ \partial_x$ by using the virial functional in [13].

Proposition 10.3. $\mathcal{L}$ has no eigenvalue in $i\mathbb{R} \setminus \{0\}$.

Proof. From Lemma 10.1, it suffices to show that $L_+ \partial_x$ does not have an eigenvalue in $i\mathbb{R} \setminus \{0\}$. Assume $i\lambda$ is an eigenvalue of $L_+ \partial_x$ and let ψ be the corresponding eigenfunction. We set

$$\mu(x) = -\phi'(x)/\phi(x) = \frac{p-1}{2} \tanh\left(\frac{p-1}{2}x\right).$$

Then, we have

$$\operatorname{Re} \int \overline{\psi}(L_+ \psi') \mu \, dx = \operatorname{Re} i\lambda \int |\psi|^2 \mu \, dx = 0.$$

Now, setting $w = \psi/\phi$, we have

$$\begin{aligned} \operatorname{Re} \int \overline{\psi}(L_+ \psi') \mu \, dx &= \operatorname{Re} \int \psi' \overline{L_+(\psi \mu)} \, dx \\ &= \operatorname{Re} \int (w' \phi + w \phi') (2w' \phi'' + w'' \phi') \, dx \\ &= \operatorname{Re} \int (2|w'|^2 \phi \phi'' + w' \overline{w''} \phi \phi' + 2w \overline{w'} \phi' \phi'' + w \overline{w''} (\phi')^2) \, dx \\ &= \operatorname{Re} \int (2|w'|^2 \phi \phi'' - \frac{1}{2} |w'|^2 (\phi \phi')' - (w \overline{w'})' (\phi')^2 + w \overline{w''} (\phi')^2) \, dx \\ &= \operatorname{Re} \int (2|w'|^2 \phi \phi'' - \frac{1}{2} |w'|^2 (\phi')^2 - \frac{1}{2} |w'|^2 \phi \phi'' - |w'|^2 (\phi')^2) \, dx \\ &= -\frac{3}{2} \int |w'|^2 (-\phi \phi'' + (\phi')^2) \, dx. \end{aligned}$$

From $\phi'' = \phi - \phi^p$ and $(\phi')^2 = \phi^2 - \frac{2}{p+1} \phi^{p+1}$, we have

$$-\phi \phi'' + (\phi')^2 = \frac{p-1}{p+1} \phi^{p+1}.$$

Therefore, we have

$$0 = -\frac{3(p-1)}{2(p+1)} \int_{\mathbb{R}} \left| \left(\frac{\psi(x)}{\phi(x)} \right)' \right|^2 \phi(x)^{p+1} \, dx. \quad (10.1)$$

This shows that there exists C s.t. $\psi(x) = C\phi(x)$. However, this will imply that ψ is an eigenfunction of 0 and contradicts our assumption. $\square$

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Data availability

No data was used for the research described in the article.

Appendix A. Conservation of Q

In this section, we prove the conservation of $\mathbf{Q}$, which is well-known when $f \in C^3$ but seems to need verification for small p . For the proof, we follow Ozawa's argument [48].

Proposition A.1. Let $T^* > 0$ and $u \in C([0, T^*), H^1(\mathbb{R}))$ be a solution of (1.4). Then, we have $\mathbf{Q}(u(t)) = \mathbf{Q}(u_0)$ for all $t \geq 0$.

Proof. We have

$$\begin{aligned} \|u(t)\|_{L^2}^2 &= \|e^{i\partial_x^3} u(t)\|_{L^2}^2 \\ &= \|u_0 - \int_0^t e^{is\partial_x^3} \partial_x (f(u(s))) \, ds\|_{L^2}^2 \end{aligned}$$

$$\begin{aligned}
&= \|u_0\|_{L^2}^2 - 2 \int_0^t \langle u_0, e^{s\partial_x^3} \partial_x (f(u(s))) \rangle ds \\
&\quad + 2 \int_0^t \langle e^{s_1\partial_x^3} \partial_x (f(u(s_1))), \int_0^{s_1} e^{s_2\partial_x^3} \partial_x (f(u(s_2))) ds_2 \rangle ds_1,
\end{aligned} \tag{A.1}$$

where for the last line, we have used

$$\begin{aligned}
&\int_{[0,t]^2} \langle e^{s_1\partial_x^3} \partial_x (f(u(s_1))), e^{s_2\partial_x^3} \partial_x (f(u(s_2))) \rangle ds_1 ds_2 \\
&= \left(\int_{(s_1,s_2) \in [0,t], s_1 \geq s_2} + \int_{(s_1,s_2) \in [0,t], s_2 \geq s_1} \right) \langle e^{s_1\partial_x^3} \partial_x (f(u(s_1))), e^{s_2\partial_x^3} \partial_x (f(u(s_2))) \rangle ds_1 ds_2 \\
&= 2 \int_{(s_1,s_2) \in [0,t], s_1 \geq s_2} \langle \partial_x (f(u(s_1))), e^{s_2\partial_x^3} \partial_x (f(u(s_2))) \rangle ds_1 ds_2.
\end{aligned}$$

Now, from (1.4), the last term in (A.1) can be rewritten as

$$\begin{aligned}
&2 \int_0^t \langle e^{s_1\partial_x^3} \partial_x (f(u(s_1))), \int_0^{s_1} e^{s_2\partial_x^3} \partial_x (f(u(s_2))) ds_2 \rangle ds_1 \\
&= 2 \int_0^t \langle e^{s_1\partial_x^3} \partial_x (f(u(s_1))), u_0 - e^{s_1\partial_x^3} u(s_1) \rangle ds_1 \\
&= 2 \int_0^t \langle u_0, e^{s_1\partial_x^3} \partial_x (f(u(s_1))) \rangle ds_1 - 2 \int_0^t \langle \partial_x (f(u(s_1))), u(s_1) \rangle ds_1.
\end{aligned} \tag{A.2}$$

Finally, since $\partial_x(f(u))u = \partial_x g$, where $g = \frac{p}{p+1}|u|^{p+1}$ in the case $f(u) = |u|^{p-1}u$ and $g = \frac{p}{p+1}|u|^p u$ in the case $f(u) = |u|^p$. Thus, we see that the last term in (A.2) vanishes. Therefore, we have the conclusion. $\square$

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