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Jacobian Schemes of Conic-Line Arrangements and Eigenschemes

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Abstract. The Jacobian scheme of a reduced, singular projective plane curve is the zero-dimensional scheme, whose homogeneous ideal is generated by the partials of its defining polynomial. The degree of such a scheme is called the global Tjurina number and, if the curve is not a set of concurrent lines, some upper and lower bounds depending on the degree of the curve and the minimal degree of a Jacobian syzygy, have been given by A.A. du Plessis and C.T.C. Wall. In this paper, we give a complete geometric characterization of conic-line arrangements, with global Tjurina number attaining the upper bound. Furthermore, we characterize conic-line arrangements attaining the lower bound for the global Tjurina number, among all curves with a linear Jacobian syzygy. As an application, we characterize conic-line arrangements with Jacobian scheme equal to an eigenscheme of some ternary tensor, and we study the geometry of their polar maps.

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1. Introduction

The Jacobian scheme Σ_f of a reduced, singular projective plane curve $C = V(f) \subseteq \mathbb{P}^2$ is the zero-dimensional scheme, whose homogeneous ideal is generated by the partials of f. The degree of such a scheme is called the *global Tjurina number* $\tau(C)$ and it is equal to $(d-1)^2$, if C consists of concurrent lines, while in all other cases, a theorem by A.A. du Plessis and C.T.C. Wall

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in [22] determines the following bounds on $\tau(C)$ in terms of the minimal degree r of a syzygy between the three partials:

$$(d-1)(d-r-1) \leq \tau(C) \leq (d-1)(d-r-1) + r^2.$$

In particular, we have $\tau(C) \leq (d-1)(d-2)+1$.

The scheme structure of Jacobian schemes is, in general, not completely understood, even in the case of irreducible curves. For instance, a class of curves attaining the bound above for any $r \geq 2$ is given by some rational cuspidal curves, as shown in [15, Theorem 1.1]. The set of curves attaining the maximal bound or one less seems to be very rich, and examples having high genus and many branches have been given in [3, Theorem 3.9 and Theorem 3.11]. As far as we know, the only characterization result is given in the case of a linear Jacobian syzygy by A.A. du Plessis and C.T.C. Wall in [23, Proposition 1.1], which states that $r = 1$ and only if the curve admits a 1-dimensional symmetry, i.e. the curve admits a 1-dimensional algebraic subgroup of $PGL_2(\mathbb{C})$ as automorphism group.

In this paper, we focus on the case of conic-line arrangements with a linear Jacobian syzygy. Some results concerning conic-line arrangements with only nodes, tacnodes, and ordinary triple points are given in [12], but in degree $d \geq 5$ one has $r \geq 2$. The Jacobian syzygies and the global Tjurina number of conics arrangements of conics belonging to particular pencils can be found in [24, 26] and [9, 27].

The assumption $r = 1$ implies $(d-1)(d-2) \leq \tau(C) \leq (d-1)(d-2)+1$, and any curve attaining the upper bound is reducible by [23, Proposition 1.3]. Concerning the lower bound, there are irreducible curves, with a linear syzygy, satisfying $\tau(C) = (d-1)(d-2)$, like for instance some Sebastiani–Thom rational cuspidal curves (see [14, Proposition 2.11]).

We give a complete geometric characterization of the conic-line arrangements attaining the two bounds for $r = 1$. Concerning the upper bound, examples are given by line arrangements consisting of the union of $d-1$ concurrent lines and one general line, as described in [10, Proposition 4.7(5)]. Our first main result is the following (see Theorem 3.5):

Theorem A. *Let $C = V(f)$ be a conic-line arrangement in $\mathbb{P}^2$ of degree $d \geq 5$. Then, $\tau(C) = (d-1)(d-2)+1$ if and only if C is either:*

- *$\mathcal{L}$: a line arrangement with $d-1$ concurrent lines and a general line;*
- *$\mathcal{C}_1$: a union of conics belonging to a hyperosculating pencil, that is with base locus supported in a point;*
- *$\mathcal{CL}_1$: a union of conics belonging to a hyperosculating pencil and the tangent line in the hyperosculating point;*
- *$\mathcal{CL}_2$: the union of conics belonging to a bitangent pencil and a tangent line in one of the bitangency points;*
- *$\mathcal{CL}_3$: the union of conics belonging to a bitangent pencil and the two tangent lines in the bitangency points;*
- *$\mathcal{CL}_4$: the union of conics belonging to a bitangent pencil, one tangent line in a tangency point, and the line connecting the two tangency points;*

- $\mathcal{CL}_5$*: the union of conics belonging to a bitangent pencil, the two tangent lines in the bitangency points, and the line connecting the two tangency points.*

In the case $\tau(C) = (d-1)(d-2)$, the characterization of conic-line arrangements is given by the following result (see Theorem 3.6):

Theorem B. *Let $C = V(f)$ be a reduced conic-line arrangement in $\mathbb{P}^2$ of degree $d \geq 6$. Then, $\tau(C) = d^2 - 3d + 2$ if and only if C is either:*

- $\mathcal{C}_2$*: a conic arrangement given by the union of conics belonging to a bitangent pencil;*
- $\mathcal{CL}_6$*: a conic-line arrangement given by the union of conics belonging to a bitangent pencil and the line passing through the two bitangency points.*

As a consequence of our first result, we can determine the degree of the polar map of conic-line arrangements with quasihomogenous singularities, that is with total Milnor number $\mu(C)$ (see Definition 2.3) equal to the total Tjurina number. Indeed, recall that, when $C = V(f)$ is not a set of concurrent lines, the polar map ∇f associated with f defines a generically finite rational map $\nabla f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of degree $(d-1)^2 - \mu(C)$. Therefore, when $\mu(C) = \tau(C)$ and $\tau(C)$ is maximal, such a degree is minimal and equal to $d-2$. This occurs in cases $\mathcal{L}$ and $\mathcal{CL}_2$.

Another related problem is a Torelli-type question, posed by Dolgachev and Kapranov (see [16]), which asks whether the rank 2 vector bundle of logarithmic vector fields $\mathcal{T}\langle C\rangle$, given as the kernel of the map:

$$(\partial_x f, \partial_y f, \partial_z f) : \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}(1) \to \mathcal{J}_f(d),$$

with $\mathcal{J}_f$ the sheafyfied Jacobian ideal, determines uniquely the curve, see also [13] and [8]. In the cases under consideration this result does not hold.

Finally, we observe that in the case of maximal Tjurina number and linear syzygy given by three linearly independent forms, the Jacobian schemes Σ_f turn out to be also eigenschemes (for the definition see 2.6) of suitable partially symmetric tensors of order $d-1$. This happens in the cases $\mathcal{L}$ and $\mathcal{CL}_2$, and we can apply the results of [21] and [4]. In particular, such schemes arise also as zeroes of a section of the twisted tangent bundle $\mathcal{T}_{\mathbb{P}^2}(d-3)$, and we have that the blow-up $\mathrm{Bl}_{\Sigma_f}\mathbb{P}^2 \subset \mathbb{P}^2 \times \mathbb{P}^2$ is a complete intersection of the projective bundle $\mathbb{P}(\mathcal{T}_{\mathbb{P}^2})$ and a divisor of bidegree $(1, d-2)$, the possible contracted curves by the polar map are only lines, and as a consequence no subscheme of degree $k(d-1)$ is contained in a curve of degree k, for any $2 \leq k \leq d-2$. We further specify that in the line arrangement case $\mathcal{L}$, the contracted lines are precisely the components of $V(f)$, while in the conic-line arrangement, the only contracted line is the tangent line appearing in the configuration $V(f)$.

The study of singular curves with a minimal Jacobian syzygy of higher degree seem to be very involved and wild. We believe that the approach concerning the study of the fibers of the polar map deserves further investigations.

The techniques involved in our study rely on a result relating the syzygy module of a product of polynomials, with no common factor, given in [11, Theorem 5.1 and Corollary 5.3], and on the characterization of the Hilbert–Burch matrix in the case of particular linear Jacobian syzygies given in [5, Theorem 3.5]. The proofs of our main results are based on a careful analysis of the possible Jacobian syzygies of any subcurve of the considered curves.

The organization of this paper is the following: in the next section we recall definitions and preliminary results regarding Jacobian ideals, Jacobian sygyzies and eigenschemes of ternary tensors.

In Sect. 3 we determine the geometric classification of Jacobian schemes corresponding to Jacobian ideals with a linear syzygy.

Finally, in Sect. 4, we characterize conic-line arrangements with a Jacobian scheme, which is also an eigenscheme of some ternary tensor. For such curves we describe the degree $d-2$ generically finite polar map, characterizing its contracted curves.

2. Preliminaries

This section contains the basic definitions and results on Jacobian ideals associated to reduced singular plane curves as well as on eigenschemes, and it lays the groundwork for the results in the later sections.

From now on, we fix the polynomial ring $R = \mathbb{C}[x, y, z]$ and we denote by $C = V(f)$ a reduced curve of degree d in the complex projective plane $\mathbb{P}^2 = \mathrm{Proj}(R)$ defined by a homogeneous polynomial $f \in R_d$.

2.1. Jacobian Ideal of a Reduced Curve

The Jacobian ideal J_f of a reduced singular plane curve $C = V(f)$ of degree d is definied as the homogeneous ideal in R generated by the 3 partial derivatives $\partial_x f$, $\partial_y f$ and $\partial_z f$. We denote by $\mathrm{Syz}(J_f)$ the graded R-module of all Jacobain relations for f, that is

$$\mathrm{Syz}(J_f) := \{(a, b, c) \in R^3 \mid a\partial_x f + b\partial_y f + c\partial_z f = 0\}.$$

We will denote by $\mathrm{Syz}(J_f)_t$ the homogeneous part of degree t of the graded R-module $\mathrm{Syz}(J_f)$; for any $t \geq 0$, we have that $\mathrm{Syz}(J_f)_t$ is a $\mathbb{C}$-vector space of finite dimension. The minimal degree of a Jacobian syzygy for f is the integer $\mathrm{mrd}(f)$ defined to be the smallest integer r such that there is a nontrivial relation $a\partial_x f + b\partial_y f + c\partial_z f = 0$ among the partial derivatives $\partial_x f$, $\partial_y f$ and $\partial_z f$ of f with coefficients $a, b, c \in R_r$. More precisely, we have:

$$\mathrm{mrd}(f) = min\{n \in \mathbb{N} \mid \mathrm{Syz}(J_f)_n \neq 0\}.$$

It is well-known that $\mathrm{mrd}(f) = 0$, i.e., the three partials $\partial_x f$, $\partial_y f$ and $\partial_z f$ are linearly dependent, if and only if C is a union of lines passing through one point $p \in \mathbb{P}^2$. Therefore, we will always assume that $\mathrm{mrd}(f) > 0$ and one of our goals will be to give a geometric classification of conic-line arrangements $C = V(f)$ of degree d with $\mathrm{mrd}(f) = 1$, see Theorems 3.5 and 3.6.

Definition 2.1. Let $C = V(f)$ be a reduced singular plane curve of degree d. We say that C is *free* if the graded R-module $\mathrm{Syz}(J_f)$ of all Jacobian relations for f is a free R-module, that is

$$\mathrm{Syz}(J_f) = R(-d_1) \oplus R(-d_2) \tag{2.1}$$

with $d_1 + d_2 = d - 1$. In this case (d_1, d_2) are called the *exponents* of C.

We say that C is *nearly free* if the minimal free resolution of $\mathrm{Syz}(J_f)$ looks like:

$$0 \longrightarrow R(-d-d_2) \longrightarrow R(1-d-d_1) \oplus R(1-d-d_2)^2 \longrightarrow \mathrm{Syz}(J_f) \longrightarrow 0 \tag{2.2}$$

with $d_1 \leq d_2$ and $d_1 + d_2 = d$.

Example 2.2. (1) The rational cuspidal quintic $C \subset \mathbb{P}^2$ of equation $C = V(f) = V(y^4z + x^5 + x^2y^3)$ is free. Indeed, $J_f = (5x^4 + 2xy^3, 3x^2y^2 + 4y^3z, y^4) \subset R$ and it has a minimal free R-resolution of the following type:

$$0 \longrightarrow R(-6)^2 \longrightarrow R(-4)^3 \longrightarrow J_f \longrightarrow 0.$$

We have $\mathrm{mrd}(f) = 2$, $\deg(J_f) = 12$ and C is free.

(2) The rational cuspidal quintic $C \subset \mathbb{P}^2$ of equation $C = V(f) = V(y^4z + x^5)$ is nearly free. Indeed, $J_f = (5x^4, 4y^3z, y^4) \subset R$ and it has a minimal free R-resolution of the following type:

$$0 \longrightarrow R(-9) \longrightarrow R(-5) \oplus R(-8)^2 \longrightarrow R(-4)^3 \longrightarrow J_f \longrightarrow 0.$$

We have $\mathrm{mrd}(f) = 1$, $\deg(J_f) = 12$ and C is not free but it is nearly free.

(3) The nodal quintic $C \subset \mathbb{P}^2$ of equation $C = V((x^2 + y^2 + z^2)(x^3 + y^3 + z^3)) = 0$ is neither free nor nearly free. Indeed $J_f = (5x^4 + 3x^2(y^2 + z^2) + 2x(y^3 + z^3), 2x^3y + 3x^2y^2 + 5y^4 + 3y^2z^2 + 2yz^3, 2x^3z + 2y^3z + 3x^2z^2 + 3y^2z^2 + 5z^4) \subset R$ and it has a minimal free R-resolution of the following type:

$$0 \longrightarrow R(-9) \oplus R(-10) \longrightarrow R(-7) \oplus R(-8)^3 \longrightarrow R(-4)^3 \longrightarrow J_f \longrightarrow 0:$$

We have $\mathrm{mrd}(f) = 3$ and $\deg(J_f) = 6$.

In general, the condition that a reduced singular curve $C = V(f)$ in $\mathbb{P}^2$ is free is equivalent to the Jacobian ideal J_f of f being arithmetically Cohen–Macaulay of codimension two; such ideals are completely described by the Hilbert–Burch theorem [17]: if $I = \langle g_1, \ldots, g_m \rangle \subset R$ is a Cohen–Macaulay ideal of codimension two, then I is defined by the maximal minors of the $(m+1) \times m$ matrix of the first syzygies of the ideal I. Combining this with Euler's formula for a homogeneous polynomial, we get that a free curve $C = V(f)$ in $\mathbb{P}^2$ has a very constrained structure: $f = \det(M)$ for a 3×3 matrix M, with one row consisting of the 3 variables, and the remaining 2 rows are the minimal first syzygies of J_f.

Free curves with some conditions are related with the total Tjurina number. We first recall some notions from singularity theory.

Let $C = V(f) \subset \mathbb{A}^2$ be a reduced, not necessarily irreducible, plane curve and fix a singular point $p \in C$. Let $\mathbb{C}\{x, y\}$ denote the ring of convergent power series.

Definition 2.3. The *Milnor number* of a reduced plane curve $C = V(f)$ at $(0,0) \in C$ is

$$\mu_{(0,0)}(C) = \dim \mathbb{C}\{x,y\}/\langle \partial_x f, \partial_y f \rangle.$$

The *Tjurina number* of a reduced plane curve $C = V(f)$ at $(0,0) \in C$ is

$$\tau_{(0,0)}(C) = \dim \mathbb{C}\{x,y\}/\langle \partial_x f, \partial_y f, f \rangle.$$

To define $\mu_p(C)$ and $\tau_p(C)$ for an arbitrary point p, translate p in the origin.

We clearly have $\tau_{(0,0)}(C) \leq \mu_{(0,0)}(C)$. For a projective plane curve $C = V(f) \subset \mathbb{P}^2$, it holds:

$$\tau(C) := \deg J_f = \sum_{P \in Sing(C)} \tau_p(C)$$

where J_f is the Jacobian ideal. We call to $\tau(C)$ the *total Tjurina number* of C.

A nice result of du Plessis and Wall gives upper and lower bounds for the total Tjurina number $\tau(C)$ of a reduced plane curve $C = V(f) \subset \mathbb{P}^2$ in terms of its degree d and the minimal degree $\operatorname{mrd}(f)$ of a syzygy of its Jacobian ideal J_f, and relates the freeness of a curve with $\tau(C)$. More precisely, we have:

Proposition 2.4. *Let $C = V(f)$ be a reduced singular plane curve of degree d and let $r := \operatorname{mrd}(f)$. Then, it holds:*

(1) the global Tjurina number satisfies

$$(d-1)(d-r-1) \leq \tau(C) \leq (d-1)(d-r-1) + r^2; \tag{2.3}$$

Moreover, if $\tau(C) = (d-1)(d-r-1) + r^2$, then the curve C is free, and such as condition is also sufficient if $d > 2r$;

(2) If, in addition, we have $2r + 1 > d$, then:

$$\tau(C) \leq (d-1)(d-r-1) + r^2 - (2r+1-d)(2r+2-d)/2 = \frac{d(d-1)}{2} - r^2 + r(d-2). \tag{2.4}$$

Proof. See [22, Theorem 3.2] and [7, Corollary 1.2]. □

Next, result will play an important role in next section. It relates the minimal degree $\operatorname{mrd}(f)$ of a syzygy of the Jacobian ideal of a reducible plane curve $C = C_1 \cup C_2 = V(f_1 f_2)$ with the minimal degrees $\operatorname{mrd}(f_1)$ and $\operatorname{mrd}(f_2)$ of a Jacobian syzygy of $C_1 = V(f_1)$ and $C_2 = V(f_2)$, respectively. Observe that the syzygy module can be identified with the module of derivations *killing* the polynomial g, that is the submodule $D_0(g)$ of the free R-module $D(g) = \{\delta = a\partial_x + b\partial_y + c\partial_z \;\; a,b,c \in R\}$ of $\mathbb{C}$-derivations of the polynomial ring R annihilated by g:

$$D_0(g) = \{\delta \in D(g) \;\; \delta g = 0\}.$$

In the case of a smooth curve $V(g)$, the syzygy module is trivial, so we will indeed consider the module $D_0(g)$ instead of $\operatorname{Syz}(J_g)$.

Theorem 2.5. *Let $C_i = V(f_i)$ for $i = 1, 2$ be two reduced curves in $\mathbb{P}^2$ without common irreducible components. Set $d_i = \deg f_i$ and $r_i = \mathrm{mrd}(f_i)$ for $i = 1, 2$. Let $C = V(f_1 f_2)$ be the union of C_1 and C_2, let $d = d_1 + d_2 = \deg f$ and $r = \mathrm{mrd}(f)$. Then it holds:*

(1) *If $\delta_1 \in D_0(f_1)$, then*

$$\delta = f_2\delta_1 - \frac{1}{d}\delta_1(f_2)\ E \in D_0(f),$$

where $E = x\partial_x + y\partial_y + z\partial_z$ denotes the Euler derivation;

(2) *$D_0(f) \subset D_0(f_1) \cap D_0(f_2)$; more precisely, for $\delta \neq 0$, one has $\delta \in D_0(f)$ if and only if δ can be written in a unique way in the form $\delta = h\ E + \delta_1 = -h\ E + \delta_2$, where h is a suitable homogeneous polynomial and $\delta_j \in D_0(f_j)$ are non-zero for $j = 1, 2$.*

(3) *In particular, we have*

$$\max(r_1, r_2) \leq r \leq \min(r_1 + d_2, r_2 + d_1).$$

and r is the minimal integer t such that either $D_0(f_1)_t \cap D_0(f_2)_t \neq 0$ or $D_0(f_1)_t + D_0(f_2)_t$ contains a non-zero multiple of the Euler derivation E.

Proof. See [11, Theorem 5.1] and [11, Corollary 5.3]. □

2.2. Eigenschemes in $\mathbb{P}^2$

Since we shall investigate whether a Jacobian scheme is also an eigenscheme of some tensor, we conclude the preliminary section with the basic definitions and results concerning tensor eigenschemes.

There are several notions of eigenvectors and eigenvalues for tensors, as introduced independently in [20] and [25]. Here we focus our attention on the algebraic–geometric point of view. We choose a basis for $\mathbb{C}^3$, we identify a partially symmetric tensor T with a triple of homogeneous polynomials of degree $d-2$ and we describe the eigenpoint of a tensor T algebraically by the vanishing of the minors of a homogeneous matrix. More precisely, we have:

Definition 2.6. Let $T = (g_1, g_2, g_3) \in (Sym^{d-2}\mathbb{C}^3)^{\oplus 3}$ be a partially symmetric tensor. The *eigenscheme* of T is the closed subscheme $E(T) \subset \mathbb{P}^2$ defined by the 2×2 minors of the homogeneous matrix:

$$M = \begin{pmatrix} x & y & z \\ g_1 & g_2 & g_3 \end{pmatrix}. \tag{2.5}$$

If T is general, then $E(T)$ is a 0-dimensional scheme (see, for instance, [1]). Moreover, by the Hochster–Eagon Theorem [19], the coordinate ring $R/I(E(T))$ is a Cohen–Macaulay ring, and as a consequence, the homogeneous ideal $I(E(T))$ is saturated. Hence $E(T)$ is a standard determinantal scheme. When the tensor T is symmetric, i.e., there is a homogeneous polynomial f and $g_0 = \partial_x f$, $g_1 = \partial_y f$ and $g_2 = \partial_z f$, we denote its eigenscheme by $E(f)$.

It is worthwhile to point out that in the case of a symmetric tensor corresponding to some homogeneous polynomial f, the eigenpoints are the fixed points of the polar map

$$\nabla f = (\partial_x f, \partial_y f, \partial_z f) : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$$

of f.

Example 2.7. We consider the Fermat cubic $V(f)$ with $f = x^3 + y^3 + z^3 \in \mathbb{C}[x, y, z]$. The eigenscheme $E(f)$ is the 0-dimensional subscheme of $\mathbb{P}^2$ of length 7 defined by the maximal minors of

$$M = \begin{pmatrix} x & y & z \\ x^2 & y^2 & z^2 \end{pmatrix}.$$

Therefore, $E(f) = \{(1, 0, 0,), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)\}$.

If we fix an integer $d \geq 2$ and $T = (g_1, g_2, g_3) \in (Sym^{d-2}\mathbb{C}^3)^{\oplus 3}$ is a general partially symmetric tensor, then it holds (see [17, Theorem A2.10]):

(1) $E(T)$ is a reduced 0-dimensional scheme of length $d^2 - 3d + 3$.
(2) The homogeneous ideal $I(E(T)) \subset R$ has a minimal free R-resolution

$$0 \longrightarrow R(-2d + 3) \oplus R(-d) \longrightarrow R(-d + 1)^3 \longrightarrow I(E(T)) \longrightarrow 0.$$

The two conditions above are not sufficient for a planar 0-dimensional subscheme to be an eigenscheme, and a characterization is given by the following result (see [2, Proposition 5.2].

Proposition 2.8. *Let Z be a 0-dimensional subscheme of $\mathbb{P}^2$ of degree $d^2 - 3d+3$. Then Z is the eigenscheme of a tensor if and only if its Hilbert–Burch matrix has the form*

$$\begin{pmatrix} L_1 & G_1 \\ L_2 & G_2 \\ L_3 & G_3 \end{pmatrix},$$

where L_1, L_2, L_3 are linearly independent linear forms.

3. Conic-Line Arrangements with a Linear Jacobian Syzygy

We start this section with two series of examples of reduced conic-line arrangements. All these examples will play an important role since, as we will see, they are the only examples of reduced conic-line arrangements C in $\mathbb{P}^2$, whose Jacobian ideal has a linear syzygy.

In what follows we shall use the result [5, Theorem 3.5], due to R. O. Buchweitz and A. Conca, which determines the Hilbert–Burch matrix of Jacobian schemes admitting a linear syzygy of the type (ax, by, cz) for some coefficients $a, b, c \in \mathbb{C}$. For completeness, we recall its statement:

Theorem 3.1. (Buchweitz–Conca) *Let K be a field of characteristic zero and $f \in K[x, y, z]$ a reduced polynomial of degree d in three variables such that f is contained in the ideal of its partial derivatives. Assume further that* *there is a triple (a, b, c) of elements of K that are not all zero, such that $ax\ \partial_x f + by\ \partial_y f + cz\ \partial_z f = 0$.*

We then have the following possibilities, up to renaming the variables:

(1) If $abc \neq 0$, then f is a free divisor with Hilbert–Burch matrix

$$\begin{pmatrix} ax & \left(\frac{1}{c} - \frac{1}{b}\right)\ (d+2)^{-1}\partial_{yz} f \\ by & \left(\frac{1}{a} - \frac{1}{c}\right)\ (d+2)^{-1}\partial_{xz} f \\ cz & \left(\frac{1}{b} - \frac{1}{a}\right)\ (d+2)^{-1}\partial_{xy} f \end{pmatrix}, \tag{3.1}$$

where $\partial_{**} f$ denotes the corresponding second order derivative of f.

(2) If $a = 0$, but $bc \neq 0$, then f is a free divisor if, and only if, $\partial_x f \in (y, z)$. If that condition is verified and $\partial_x f = yg + zh$, then $\frac{\partial f_y}{cz} = \frac{-\partial_z f}{by}$ is an element of $K[x, y, z]$, and a Hilbert–Burch matrix is given by

$$\begin{pmatrix} 0 & \partial_y f/cz \\ by & -h/c \\ cz & g/b \end{pmatrix}. \tag{3.2}$$

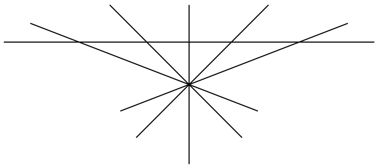
(3) If $a = b = 0$, then f is independent of z and, so, being the suspension of a reduced plane curve, is a free divisor.

We focus now on conic-line arrangements. The first series of examples corresponds to reduced conic-line arrangements $C = V(f)$ of degree d with $\mathrm{mrd}(f) = 1$ and maximal Tjurina number $\tau(C) = (d-1)(d-2) + 1 = d^2 - 3d + 3$.

Example 3.2.

(1) We fix an integer $d \geq 3$. Let $\mathcal{L}$ be a line arrangement with $d-1$ lines through a point p, and one other line in general position. Without loss of generality we can assume that $p = (0:0:1)$ and that the general line is $V(x)$, so that the equation of the line arrangement $\mathcal{L}$ is given by

$$\mathcal{L}:\ z\prod_{i=1}^{d-1}(a_i x + b_i y) = 0$$



with $(a_i : b_i) \neq (a_j : b_j)$ for $i \neq j$. It is simple to determine a linear syzygy between the three partials of f, by observing that $\partial_z f = \prod_{i=1}^{d-1}(a_i x + b_i y)$. Therefore, we have

$$f = z\ \partial_z f.$$

On the other hand, by Euler formula we also have $f = \frac{1}{d}(x\ \partial_x f + y\partial_y f + z\ \partial_z f)$, hence we get the identity

$$x\ \partial_x f + y\ \partial_y f + (1-d)z\ \partial_z f = 0.$$

Therefore, according to Theorem 3.1, (1), the Hilbert–Burch matrix of J_f is given by

$$\begin{pmatrix} x & \frac{1}{(1-d)}\,\partial_{yz}f \\ y & \frac{1}{(d-1)}\partial_{xz}f \\ (1-d)z & 0 \end{pmatrix}, \tag{3.3}$$

a minimal free R-resolution of J_f is given by:

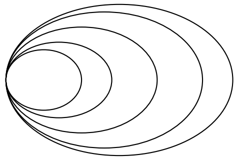
$$0 \longrightarrow R(-d) \oplus R(-2d+3) \longrightarrow R(-d+1)^3 \longrightarrow J_f \longrightarrow 0,$$

and $\mathcal{L}$ is free with exponents $(1, d-2)$ and global Tjurina number $d^2 - 3d + 3$.

It is worthwhile to point out that the 3 entries $(x, y, (1-d)z) \in \mathrm{Syz}(J_f)_1$ of the linear syzygy are linearly independent.

(2) We fix an even integer $d = 2m \geq 4$. Let $\mathcal{C}_1$ be a conic arrangement with m conics $C_1, \ldots, C_m$ such that there exists a point $p \in \mathbb{P}^2$, for all i, j, $1 \leq i < j \leq m$, satisfying $C_i \cap C_j = \{p\}$, and the intersection point p is a singularity A_7 for $C_i \cup C_j$. In other words, the m conics belong to a hyperosculating pencil; such a curve is called an *even Płoski curve* in [6, Definition 1.7]. Without loss of generality, we can assume that $p = (0 : 0 : 1)$ and the equation of the conic arrangement $\mathcal{C}_1$ is given by

$$\mathcal{C}_1 : \; f = \prod_{i=1}^{m}(x^2 + a_i(xz + y^2)) = 0,$$



with $a_i \neq 0$ and $a_i \neq a_j$ for $i \neq j$. The reduced plane curve $\mathcal{C}_1$ has degree $d = 2m$. A linear Jacobian syzygy can be determined by observing that $(0, x, -2y)$ is a linear syzygy of the Jacobian ideal of $f_i := x^2 + a_i(xz + y^2)$, for any $i = 1, \ldots, m$, so we deduce that $\mathrm{Syz}(J_f)_1$ is also generated by $(0, x, -2y)$.

Moreover, we claim that $\mathcal{C}_1$ is free with exponents $(1, d-2)$, so that the global Tjurina number is $d^2 - 3d + 3$.

To prove the claim, observe that since $r = 1$, the conic arrangement $\mathcal{C}_1$ is either free or nearly free, and by (2.1) and (2.2), it is free if and only if J_f admits a sygyzy of degree $d-2$, which is not proportional to $(0, x, -2y)$. Let us prove the latter fact by induction on the number m of conics.

If $m = 2$, a degree 2 syzygy, which is not proportional to $(0, x, -2y)$, is given by

$$\begin{aligned}(-(a_1+a_2)x^2 - 2a_1a_2(y^2+xz), a_1a_2yz, 4x^2 + 2(a_1+a_2)y^2 \\ +3(a_1+a_2)xz + 2a_1a_2z^2).\end{aligned}$$

Now assume that $m \geq 3$, and that any conic arrangement of $m-1$ conics belonging to a hyperosculating pencil admits a syzygy of degree $2m-4$, not proportional to $(0, x, -2y)$. Let $f = \prod_{i=1}^{m}(x^2 + a_i(xz + y^2))$, and set

$$f_1 = \prod_{i=1}^{m-1} (x^2 + a_i(xz + y^2)), \qquad f_2 = x^2 + a_m(xz + y^2).$$

By induction hypothesis J_{f_1} admits a syzygy $\delta_1 \in \mathrm{Syz}(J_{f_1})_{2\,m-4}$, with $\delta_1 \notin \langle(0, x, -2y)\rangle$, and by Theorem 2.5, (1), we have

$$\delta = f_2\delta_1 - \frac{1}{2m}(\delta_1 \cdot \nabla f_2)\ E \in \mathrm{Syz}(J_f)_{2m-2},$$

where we set $\delta_1 \cdot \nabla f_2 = h_1\partial_x f_2 + h_2\partial_y f_2 + h_3\partial_z f_2$, and $E = (x, y, z)$ is the Euler relation. As observed in the proof of such a Theorem (see [11, Theorem 5.1]), since $\delta_1 \neq 0$, it is also $\delta \neq 0$. Finally, we claim that $\delta \notin \langle(0, x, -2y)\rangle$. Indeed, if $\delta_1 \cdot \nabla f_2 = 0$, we have $\delta = f_2\delta_1$ and since $\delta_1 \notin \langle(0, x, -2y)\rangle$ by induction hypothesis, the claim follows. If $\delta_1 \cdot \nabla f_2 \neq 0$, we see that

$$\begin{aligned} \delta \cdot \nabla f_1 &= f_2\delta_1 \cdot \nabla f_1 - \frac{1}{2m}(\delta_1 \cdot \nabla f_2)\ E \cdot \nabla f_1 \\ &= -\frac{2m-2}{2m}(\delta_1 \cdot \nabla f_2) f_1 \neq 0. \end{aligned}$$

On the other hand, if we had $\delta = h\ (0, x, -2y)$ for some polynomial h, we would have

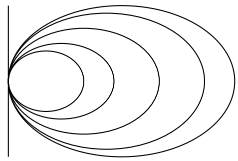
$$\delta \cdot \nabla f_1 = h\ (0, x, -2y) \cdot \nabla f_1 = 0,$$

as $(0, x, -2y) \in \mathrm{Syz}(J_{f_1})$.

It is important to point out that in this case the linear syzygy of J_f has only 2 linearly independent entries, and that such an example is not of the type considered in Buchweitz–Conca Theorem 3.1.

(3) We fix an odd integer $d = 2m + 1 \geq 5$. Let $\mathcal{CL}_1$ be a conic-line arrangement with m conics $C_1, \ldots, C_m$ and a line ℓ such that there exists a point $p \in \mathbb{P}^2$, ℓ is a common tangent line to all C_i's, $C_i \cap C_j = \{p\}$ and the intersection point p is a singularity A_7 for $C_i \cup C_j$, for all i, j, $1 \leq i < j \leq m$. In other words, the conics belong to a hypersculating pencil; such a curve is called an *odd Płoski curve* in [6, Definition 1.7]. Without loss of generality we can assume that $p = (0 : 0 : 1)$, the line $\ell = V(x)$ and the equation of the conic-line arrangement $\mathcal{CL}_1$ is given by

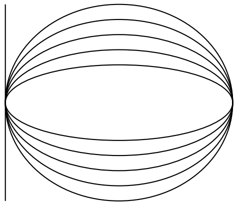
$$\mathcal{CL}_1 :\ f = x\prod_{i=1}^{m}(x^2 + a_i(xz + y^2)) = 0$$



with $a_i \neq 0$ and $a_i \neq a_j$ for $i \neq j$, and $a_i \neq a_j$ if $1 \neq j$. The reduced plane curve $\mathcal{CL}_1$ has degree $d = 2m + 1$; by using the same argument as in the previous example, it is not difficult to see that $\mathcal{CL}_1$ is free with exponents $(1, d-2)$ and global Tjurina number $d^2 - 3d + 3$. The linear syzygy of $\mathrm{Syz}(J_f)_1$ is generated by $(0, x, -2y)$. Therefore, again the linear syzygy of the Jacobian ideal of f has only two independent entries, and is not of the type considered in Buchweitz–Conca Theorem 3.1.

(4) We fix an odd integer $d = 2m+1 \geq 5$. Let $\mathcal{CL}_2$ be a conic-line arrangement with m conics $C_1, \ldots, C_m$ and a line ℓ such that there exist two points $p, q \in \mathbb{P}^2$ such that $C_i \cap C_j = \{p, q\}$ and the two intersection points p, q are tacnodes for $C_i \cup C_j$, for all i, j, $1 \leq i < j \leq m$, and ℓ is a common tangent line to all C_i's at p. Without loss of generality we can assume that $p = (0 : 0 : 1)$, $q = (1 : 0 : 0)$, $\ell = V(x)$, so that the equation of the conic-line arrangement $\mathcal{CL}_2$ is given by

$$\mathcal{CL}_2 : \; f = x \prod_{i=1}^{m} (xz + a_i y^2) = 0$$



with $a_i \neq 0$ for all i, $1 \leq i \leq m$, and $a_i \neq a_j$ if $1 \neq j$. The reduced singular plane curve $\mathcal{CL}_2$ has degree $d = 2m+1$. We claim that $Syz(J_f)_1 = \langle ((d-1)x, -y, -(d+1)z) \rangle$. Indeed, set $q_i(x, y, z) = xz + a_i y^2$. We have

$$\partial_x f = \prod_{i=1}^{m} q_i + xz \left(\sum_{j=1}^{m} \prod_{i=1, i \neq j}^{m} q_i \right), \; \partial_y f = 2xy \left(\sum_{j=1}^{m} a_j \prod_{i=1, i \neq j}^{m} q_i \right),$$

$$\partial_z f = x^2 \left(\sum_{j=1}^{m} \prod_{i=1, i \neq j}^{m} q_i \right),$$

which, in particular, gives

$$f = x \, \partial_x f - z \, \partial_z f.$$

Thus by the Euler identity we get

$$(d-1)x \, \partial_x f - y \, \partial_y f - (d+1)z \, \partial_z f = 0.$$

Therefore, the linear syzygy of the Jacobian ideal of f has again 3 linear independent entries.

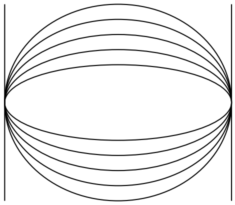
Therefore, according to Theorem 3.1, (1), the curve $\mathcal{CL}_2$ is free with global Tjurina number $d^2 - 3d + 3$, and the Hilbert–Burch matrix of J_f is

$$\begin{pmatrix} (d-1)x & \frac{1}{(d+1)} \, \partial_{yz} f \\ -y & \frac{2}{(d^2-1)} \partial_{xz} f \\ -(d+1)z & -\frac{1}{(d-1)} \partial_{xy} f \end{pmatrix}. \tag{3.4}$$

(5) The following three examples are of the type given in [5, Example 3.7], and they all correspond to free curves of exponents $(1, d-2)$ and $\mathrm{Syz}(J_f)_1$ generated by $(x, 0, -z)$.

- Let $d = 2m+2 \geq 6$ and let $\mathcal{CL}_3$ be a conic-line arrangement of degree d with m conics $C_1, \ldots, C_m$ and two lines ℓ_1 and ℓ_2 such that there exist two points $p, q \in \mathbb{P}^2$ satisfying $C_i \cap C_j = \{p, q\}$ and the two intersection points p, q are tacnodes for $C_i \cup C_j$, for all i, j, $1 \leq i < j \leq m$, ℓ_1 is a common tangent line to all C_i's at p, ℓ_2 is a common tangent line to all C_i's at q. Without loss of generality we can assume that $p = (0 : 0 : 1)$, $q = (1 : 0 : 0)$, $\ell_1 = V(x)$, $\ell_2 = V(z)$, so that the equation of the conic-line arrangement $\mathcal{CL}_3$ is given by

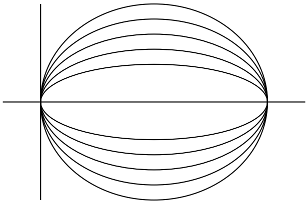
$$\mathcal{CL}_3 : \ f = xz \prod_{i=1}^{m} (xz + a_i y^2)) = 0$$



with $a_i \neq 0$ for all i, $1 \leq i \leq m$, and $a_i \neq a_j$ if $1 \neq j$.

- Let $d = 2m + 2 \geq 6$ and let $\mathcal{CL}_4$ be a conic-line arrangement of degree d with m conics $C_1, \ldots, C_m$ and two lines ℓ_1 and ℓ_2 such that there exist two points $p, q \in \mathbb{P}^2$ satisfying $C_i \cap C_j = \{p, q\}$ and the two intersection points p, q are tacnodes for $C_i \cup C_j$, for all i, j, $1 \leq i < j \leq m$, ℓ_1 is a common tangent line to all C_i's at p, ℓ_2 is the line joining p and q. Without loss of generality we can assume that $p = (0 : 0 : 1)$, $q = (1 : 0 : 0)$, $\ell_1 = V(x)$, $\ell_2 = V(y)$, so that the equation of the conic-line arrangement $\mathcal{CL}_4$ is given by

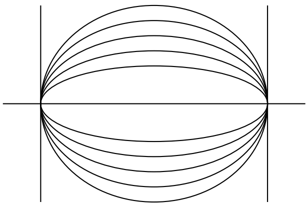
$$\mathcal{CL}_4 : \ f = xy \prod_{i=1}^{m} (xz + a_i y^2)) = 0$$



with $a_i \neq 0$ for all i, $1 \leq i \leq m$, and $a_i \neq a_j$ if $1 \neq j$.

- Let $d = 2m + 3 \geq 6$ and let $\mathcal{CL}_5$ be a conic-line arrangement of degree d with m conics $C_1, \ldots, C_m$ and three lines ℓ_1, ℓ_2 and ℓ_3 such that there exist two points $p, q \in \mathbb{P}^2$ satisfying $C_i \cap C_j = \{p, q\}$ and the two intersection points p, q are tacnodes for $C_i \cup C_j$, for all i, j, $1 \leq i < j \leq m$, ℓ_1 is a common tangent line to all C_i's at p, ℓ_2 is a common tangent line to all C_i's at q, ell_3 is the line joining p and q. Without loss of generality we can assume that $p = (0 : 0 : 1)$, $q = (1 : 0 : 0)$, $\ell_1 = V(x)$, $\ell_2 = V(z)$ and $\ell_3 = V(y)$, so that the equation of the conic-line arrangement $\mathcal{CL}_5$ is given by

$$\mathcal{CL}_5 : \ f = xyz \prod_{i=1}^{m} (xz + a_i y^2)) = 0$$



with $a_i \neq 0$ for all i, $1 \leq i \leq m$, and $a_i \neq a_j$ if $1 \neq j$.

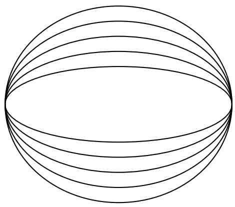
Next series of example corresponds to reduced nearly free plane curves $C = V(f)$ of degree d with $\mathrm{mrd}(f) = 1$ and $\tau(C) = d^2 - 3d + 2$.

Example 3.3. By [5, Example 3.7], next examples have the property that $\mathrm{Syz}(J_f)_1$ is generated by $(x, 0, -z)$ and they are not free. Since $r = 1$, we have $\tau(C) = (d-1)(d-2)$ and they are nearly free by Lemma 3.4.

(1) Let $\mathcal{C}_2$ be a conic arrangement with m conics $C_1, \ldots, C_m$ such that there exist two points $p, q \in \mathbb{P}^2$ satisfying $C_i \cap C_j = \{p, q\}$ and the two intersection points p, q are tacnodes for $C_i \cup C_j$, for all i, j, $1 \leq i < j \leq m$.

Without loss of generality we can assume that $p = (0 : 0 : 1)$, $q = (1 : 0 : 0)$, so that the equation of the conic arrangement is

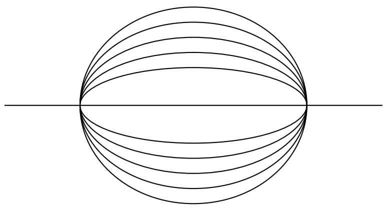
$$\mathcal{C}_2 : \ f = \prod_{i=1}^{m} (xz + a_i y^2) = 0$$



with $a_i \neq 0$ for all i, $1 \leq i \leq m$, and $a_i \neq a_j$ if $1 \neq j$.

(2) Let $\mathcal{CL}_6$ be a conic-line arrangement with m conics $C_1, \ldots, C_m$ and a line ℓ such that there exist two points $p, q \in \mathbb{P}^2$ such that $C_i \cap C_j = \{p, q\}$ and the two intersection points p, q are tacnodes for $C_i \cup C_j$, for all i, j, $1 \leq i < j \leq m$, and ℓ is the line through p and q. We can assume that $p = (0 : 0 : 1)$, $q = (1 : 0 : 0)$, $\ell = V(y)$ and the equation of the conic-line arrangement is

$$\mathcal{CL}_6 : \ f = y \prod_{i=1}^{m} (xz + a_i y^2) = 0$$



with $a_i \neq 0$ for all i, $1 \leq i \leq m$, and $a_i \neq a_j$ if $i \neq j$.

Our next goal is to establish a geometric characterization of all reduced conic-line arrangements $C = V(f)$ in $\mathbb{P}^2$ of degree $d \geq 3$, whose Jacobian ideal J_f has a linear syzygy, i.e., $\mathrm{mrd}(f) = 1$.

Lemma 3.4. *Let $C = V(f)$ be a reduced singular curve in $\mathbb{P}^2$ of degree $d \geq 3$ and let $J_f = (f_x, f_y, f_z)$ be its Jacobian ideal. Assume that* $\mathrm{mrd}(f) = 1$. *Then $d^2 - 3d + 2 \leq \tau(C) \leq d^2 - 3d + 3$. Moreover, if $\tau(C) = d^2 - 3d + 3$ (resp. $\tau(C) = d^2 - 3d + 2$) then C is free (nearly free).*

Proof. By Proposition 2.4 we have $d^2 - 3d + 2 \leq \tau(C) \leq d^2 - 3d + 3$. By [7, Theorem 1.2] (see also [18, Proposition 26]), if $\tau(C) = d^2 - 3d + 3$ (respectively $d^2 - 3d + 2$) then C is free (nearly free). $\square$

3.1. Free Conic-Line Arrangements with a Linear Jacobian Syzygy

Our first goal is to classify free conic-line arrangements $C = V(f)$ of degree d in $\mathbb{P}^2$ with $r = \mathrm{mrd}(f) = 1$; such curves have maximum Tjurina number $\tau(C) = d^2 - 3d + 3$. We have

Theorem 3.5. *Let $C = V(f)$ be a conic-line arrangement in $\mathbb{P}^2$ of degree $d \geq 5$, which is not a set of concurrent lines. Then, $\tau(C) = d^2 - 3d + 3$ if and only if C is either a line arrangement $\mathcal{L}$ as in example 3.2(1), or a conic arrangement $\mathcal{C}_1$ as in example 3.2(2), or a conic-line arrangement $\mathcal{CL}_1$, $\mathcal{CL}_2$, $\mathcal{CL}_3$, $\mathcal{CL}_4$ or $\mathcal{CL}_5$ as in examples 3.2(3)–(7).*

Proof. All conic-line arrangements $C \subset \mathbb{P}^2$ described in examples 3.2(1)–(7) have total Tjurina number $\tau(C) = d^2 - 3d + 3$ and $\mathrm{mdr}(f) = 1$. Let us prove the converse.

We claim that the hypothesis $\tau(C) = d^2 - 3d + 3$, $d \geq 5$ and Proposition 2.4 imply that $r = \mathrm{mrd}(f) = 1$. Indeed, observe that the inequality 2.4 is never satisfied for $d \geq 5$, hence we have

$$1 \leq r \leq \frac{d-1}{2}.$$

Moreover, the inequality $d^2 - 3d + 3 \leq (d-1)(d-r-1) + r^2$ of 2.3 holds if and only if $r \leq 1$ or $r \geq d-2$. Since $\frac{d-1}{2} < d-2$ if $d \geq 5$, we only have $r \leq 1$, and as C is not a set of concurrent lines, the case $r = 0$ is excluded.

To perform the further analysis, we distinguish several cases:

<ins>Case 1:</ins> C is a line arrangement. By [10, Proposition 4.7(5)], C is the union of $d-1$ lines through a point p, and one other line in general position.

<ins>Case 2:</ins> Let $C = \cup_{i=1}^m C_i \ : \ f = \prod_{i=1}^m f_i = 0$ be a conic arrangement. By Theorem 2.5, (3), whenever we extract the union $C' = C_1 \cup C_2$ of two conics, the relative $r' = \mathrm{mrd}(C') = 1$, so by the classification given in [11, Proposition 5.5], the only possible cases are: either $|C_1 \cap C_2| = 2$ and the two intersection points are two tacnodes for C', or $|C_1 \cap C_2| = 1$ and the singular point is an A_7 singularity (the two conics are hyperosculating). Since in the first case the total Tjurina number is $\tau = 6$, it does not occur. This settles the case $d = 4$. Assume now $d \geq 6$.

Claim: Whenever we extract the union $C_{i_1} \cup C_{i_2} \cup C_{i_3}$ of three conics, they belong to the same pencil, so they are either bitangent, or they are hyperosculating.

Proof of the Claim. Indeed, assume first that $C_{i_1} \cap C_{i_2} = \{p, q\}$ and p, q are two tacnodes for $C_{i_1} \cup C_{i_2}$. Without loss of generality we can assume that

$$C_{i_1} = V(f_{i_1}) = V(xz + a_{i_1} y^2), \qquad C_{i_2} = V(f_{i_2}) = V(xz + a_{i_2} y^2)$$

with $a_{i_1}, a_{i_2} \in \mathbb{C}$. Therefore, $D_0(f_{i_1})_1 = D_0(f_{i_2})_1 \cong \mathrm{Syz}(f_{i_1} f_{i_2})_1 = \langle (x, 0, -z) \rangle$. By hypothesis $\mathrm{mrd}(f_{i_1} f_{i_2} f_{i_3}) = 1$. Therefore, applying Theorem 2.5, we have

$$D_0(f_{i_3})_1 \cap \langle (x, 0, -z) \rangle \neq 0, \text{ or } \alpha(x, 0, -z) + \beta(x, y, z) \in D_0(f_{i_3})_1.$$

A straightforward computation shows that necessarily $C_{i_3} = V(f_{i_3}) = V(\lambda xz + \mu y^2)$.

Assume now that $C_{i_1} \cap C_{i_2} = \{p\}$ and p is a singularity A_7 for $C_{i_1} \cup C_{i_2}$. Without loss of generality we can assume

$$C_{i_1} = V(f_{i_1}) = V(x^2 + a_{i_1}(xz + y^2)), \quad C_{i_2} = V(f_{i_2}) = V(x^2 + a_{i_2}(xz + y^2))$$

with $a_{i_1}, a_{i_2} \in \mathbb{C}$. Therefore, $D_0(f_{i_1})_1 = D_0(f_{i_2})_1 \cong \mathrm{Syz}(f_{i_1} f_{i_2})_1 = \langle (0, x, -2y) \rangle$. By hypothesis $\mathrm{mrd}(f_{i_1} f_{i_2} f_{i_3}) = 1$, hence by Theorem 2.5, we have in this case

$$D_0(f_{i_3})_1 \cap \langle (0, x, -2y)) \rangle \neq 0, \text{ or } \alpha(0, x, -2y) + \beta(x, y, z) \in D_0(f_{i_3})_1.$$

A direct computation shows that in this case necessarily $C_{i_3} = V(f_{i_3}) = V(\lambda x^2 + \mu(xz + y^2))$.

It follows from the claim that the irreducible components of $C = \cup_{i=1}^m C_i$ belong either to a bitangent pencil of conics or to a hyperosculating pencil of conics. In the first case we have $\tau(C) = (2m)^2 - 3(2m) + 2$ (see Example 3.3(1)) and in the second case it is $\tau(C) = (2m)^2 - 3(2m) + 3$ (see Example 3.2(2)), which proves what we want.

Case 3: Let $C = \cup_{i=1}^m C_i \bigcup \cup_{j=1}^s L_j \,:\, f = \prod_{i=1}^m f_i \cdot \prod_{j=1}^s \ell_j = 0$ be a conic-line arrangement. By Theorem 2.5, the conic arrangement $\cup_{i=1}^m C_i$ satisfies $\mathrm{mrd}(\prod_{i=1}^m f_i) = 1$. By the above discussion, the components of $\cup_{i=1}^m C_i$ belong either to a hyperosculating pencil or to a bitangent pencil of conics. We analyze this two cases separately.

Let us first assume that $\cup_{i=1}^m C_i = V(\prod_{i=1}^m (x^2 + a_i(xz + y^2)))$ belong to a hyperosculating pencil of conics with hyperosculating point $p = (0 : 0 : 1)$. Therefore, $\mathrm{Syz}(\prod_{i=1}^m f_i)_1 = \langle (0, x, -2y) \rangle$. We look for a line $L = V(\ell) = V(ax + by + cz)$ such that $\mathrm{mrd}((ax + by + cz) \prod_{i=1}^m (x^2 + a_i(xz + y^2)) = 1$. By Theorem 2.5, (3), if we set

$$f_1 = ax + by + cz, \quad f_2 = \prod_{i=1}^m (x^2 + a_i(xz + y^2)),$$

we have that either $D_0(f_1)_1 \cap D_0(f_2)_1 \neq 0$, or $\alpha(x\partial_x + y\partial_y + z\partial_z) \in D_0(f_1)_1 + D_0(f_2)_1$ for some nonzero constant α. As to the line $V(f_1)$, if $a \neq 0$, we have

$$D_0(f_1)_1 = \{L_1(-b\partial_x + a\partial_y) + L_2(-c\partial_x + a\partial_z) \;\; L_1, L_2 \in R_1\},$$

while $D_0(f_2) = \langle x\partial_y - 2y\partial_z \rangle$.

The condition $D_0(f_1)_1 \cap D_0(f_2)_1 \neq 0$ is never satisfied, while the condition $\alpha(x\partial_x + y\partial_y + z\partial_z) = L_1(-b\partial_x + a\partial_y) + L_2(-c\partial_x + a\partial_z) + \beta x\partial_y - 2\beta y\partial_z$ for $\alpha \neq 0$ has as a unique solution $b = c = 0$. It follows that

$$\ell = x.$$

The case $a = 0$ can be treated similarly and it never occurs.

Next it is possible to check in a similar way that $h = x(ax + by + cz) \prod_{i=1}^m (x^2 + a_i(xz + y^2))$ has a linear Jacobian syzygy if and only if $b = c = 0$, but this would give rise to a non reduced polynomial.

So, C is as in example 3.2.

Let us now assume that $\cup_{i=1}^m C_i = V(\prod_{i=1}^m (xz + a_{i_1} y^2))$ belongs to a pencil of conics, all of them bitangent at $\{p = (1 : 0 : 0), q = (0 : 0 : 1)\}$, so that we have the linear syzygy $(x, 0, -z)$. Arguing as above we can determine the lines that we can add to this conic arrangement in such a way that the new conic-line arrangement has Jacobian ideal with a linear syzygy. It turns out that we have only six possibilities:

$$x\prod_{i=1}^{m}(xz+a_{i_1}y^2),\quad y\prod_{i=1}^{m}(xz+a_{i_1}y^2),\quad z\prod_{i=1}^{m}(xz+a_{i_1}y^2),$$
$$xz\prod_{i=1}^{m}(xz+a_{i_1}y^2),\quad yz\prod_{i=1}^{m}(xz+a_{i_1}y^2),\quad xyz\prod_{i=1}^{m}(xz+a_{i_1}y^2).$$

The case $y\prod_{i=1}^{m}(xz+a_{i_1}y^2)=0$ is not free by Theorem 3.1 (2). By observing that the first and the third case are projectively equivalent, this concludes the proof. $\square$

3.2. Nearly Free Conic-Line Arrangements with a Linear Jacobian Syzygy

In this subsection, we classify conic-line arrangements $C=V(f)$ of degree d in $\mathbb{P}^2$ with $r=\mathrm{mrd}(f)=1$ and minimal Tjurina number $\tau(C)=d^2-3d+2$.

Theorem 3.6. *Let $C=V(f)$ be a conic-line arrangement in $\mathbb{P}^2$ of degree $d\geq 6$. Then, $\tau(C)=d^2-3d+2$ if and only if C is either a conic arrangement $\mathcal{C}_2$ as in example 3.3(1), or a conic-line arrangement $\mathcal{CL}_6$ as in example 3.3(2).*

Proof. All conic-line arrangements $C\subset\mathbb{P}^2$ described in examples 3.3(1)–(2) have total Tjurina number $\tau(C)=d^2-3d+2$.

Let us prove the converse. The hypothesis $\tau(C)=d^2-3d+2$, $d\geq 6$ and Proposition 2.4 imply that $r=\mathrm{mrd}(f)=1$.

Indeed, the inequality 2.4 is never satisfied if $d\geq 6$, hence we have

$$1\leq r\leq \frac{d-1}{2}.$$

Moreover, the inequality $d^2-3d+2\leq(d-1)(d-r-1)+r^2$ of 2.3 holds if and only if $r\leq\frac{d-1}{2}-\frac{\sqrt{d^2-6*d+5}}{2}$ or $r\geq\frac{d-1}{2}+\frac{\sqrt{d^2-6*d+5}}{2}$. Since $\frac{d-1}{2}<d-2$ if $d\geq 4$, and since $\frac{d-1}{2}-\frac{\sqrt{d^2-6*d+5}}{2}<2$ if $d\geq 6$, we get $r=1$.

By [15, Proposition 4.3], there are no line arrangements with $\mathrm{mrd}(f)=1$ and $\tau(C)=d^2-3d+2$. Therefore, we only have two possibilities: either C is a conic arrangement, or C is a conic-line arrangement. Arguing as in the proof of Theorem 3.5 is it possible to conclude. $\square$

4. Applications

As application of the previous results we obtain the main result of this paper, namely, we determine when the Jacobian ideal of a conic-line arrangement is the ideal of an eigenscheme. More precisely, we have:

Theorem 4.1. *Let $C=V(f)$ be a conic-line arrangement in $\mathbb{P}^2$ of degree $d\geq 5$. The Jacobian ideal J_f of f is the ideal of an eigenscheme $E(T)$ if and only if C is either a line arrangement $\mathcal{L}$, or a conic-line arrangement $\mathcal{CL}_2$.*

Proof. Let $C = V(f)$ be a line arrangement (resp. conic-line arrangement) as described in the statement of the theorem. We have seen in example 3.2(1) (resp. example 3.2(4)) that the Jacobian ideal J_f of C has is defined by the maximal minors of the matrix

$$\begin{pmatrix} (d-1)x & g_0 \\ -y & g_1 \\ -(1+d)z & g_2 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} x & h_0 \\ y & h_1 \\ (1-d)z & h_2 \end{pmatrix}.$$

Equivalently, the Jacobian ideal is generated by the minors of the matrix

$$\begin{pmatrix} x & \frac{1}{d-1}g_0 \\ y & -g_1 \\ z & -\frac{1}{d+1}g_2 \end{pmatrix}, \quad \text{resp.} \quad \begin{pmatrix} x & h_0 \\ y & h_1 \\ z & \frac{1}{d-1}h_2 \end{pmatrix}.$$

By definition we have $J_f = I(E(T))$, where $T = (\frac{1}{d-1}g_0, -g_1, -\frac{1}{d+1}g_2) \in (Sym^{d-1}\mathbb{C}^3)^{\oplus(3)}$, resp. $T = (h_0, h_1, \frac{1}{d-1}h_2) \in (Sym^{d-1}\mathbb{C}^3)^{\oplus(3)}$ are partially symmetric tensors.

Let us prove the converse. Assume that there is a partially symmetric tensor $T = (g_1, g_2, g_3) \in (Sym^{d-1}\mathbb{C}^3)^{\oplus(3)}$ such that $I(E(T)) = J_f$. This implies that C is free, $\tau(C) = d^2 - 3d + 3$, $\mathrm{mrd}(f) = 1$ and that $\mathrm{Syz}(J_f)_1$ is generated by three linearly independent linear forms. Example 3.2 together with Theorem 3.5 proves what we want. □

Remark 4.2. There are examples of reduced plane curves $C = V(f) \subset \mathbb{P}^2$ whose Jacobian ideal J_f is the ideal of an eigenscheme $E(T)$ and they are not conic-line arrangements. For instance, $f = y(x^3 - y^2z)$.

Remark 4.3. In particular, the geometry of the Jacobian scheme of a line arrangement of type $\mathcal{L}$ or a conic-line arrangement $\mathcal{CL}_2$ is completely described by [4, Theorem 5.5 and Remark 5.8]. We observe that the cited result concerns only reduced eigenschemes, but it is not difficult to see, that it can be extended to all non reduced zero-dimensional eigenschemes.

Specifically, we have that if $k \in \{2, \dots, d-1\}$ then no subscheme of degree kd of Σ_f lies on a curve of degree k. Moreover, the class of $S = \mathrm{Bl}_{\Sigma_f}\mathbb{P}^2$ in the Chow ring $A(\mathbb{P}^2 \times \mathbb{P}^2)$ can be determined. By choosing L_1 and L_2 as generators of the Picard groups of the two factors, and by setting $p_i : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ to be the two projections, we have that the two divisors $h_1 = p_1^\star L_1$ and $h_2 = p_2^\star L_2$ are generators for $A(\mathbb{P}^2 \times \mathbb{P}^2)$. Then it is simple to check that the class of S in $A(\mathbb{P}^2 \times \mathbb{P}^2)$ is given by

$$[S] = (d-1)h_1^2 + dh_1h_2 + h_2^2,$$

and, by taking into account the Hilbert-Burch matrices given in (3.3) and (3.4), the surface S turns out to be the complete intersection of the two divisors $T \sim h_1 + h_1$ and $D \sim (d-2)h_1 + h_2$ given by

$$T = V(p_0x + p_1y + (1-d)p_2z), \quad D = V(p_0\partial_{yz}f + p_1\partial_{xz}f)$$

in case $\mathcal{L}$, respectively

$$\begin{aligned} T &= V((d-1)p_0x - p_1y - (d+1)p_2z), \\ D &= V((d-1)\ p_0\partial_{yz}f + 2p_1\partial_{xz}f - (d+1)p_2\partial_{xy}f), \end{aligned}$$

in case $\mathcal{CL}_2$, where $((x:y:z),(p_0:p_1:p_2)) \in \mathbb{P}^2 \times \mathbb{P}^2$.

Finally, we observe that by [21] or [1, Lemma 5.6], every planar eigenscheme is the zero locus of section $s \in H^0(\mathcal{T}_{\mathbb{P}^2}(d-2))$, where $\mathcal{T}_{\mathbb{P}^2}$ denotes the tangent bundle of $\mathbb{P}^2$.

Next we shall study the polar map associated with $\mathcal{L}$ and $\mathcal{CL}_2$ arrangements. Observe that since we are concerned with curves of maximal total Tjurina number and quasihomogeneous singularities, the degree of the generically finite polar map is $(d-1)^2 - \mu(C) = d-2$.

Remark 4.4. We can apply the argument used in the proof of [4, Theorem 5.5] and we see that the possible contracted curves by the polar map associated with line arrangements of type $\mathcal{L}$ or conic-line arrangements $\mathcal{CL}_2$ are only lines.

Indeed, we observe that for any $p = (p_0 : p_1 : p_2) \notin \Sigma_f$, the point $\nabla f(p)$ is the intersection point of the two distinct lines:

$$\nabla f(p) : \begin{cases} p_0 x + p_1 y + (1-d) p_2 z = 0 \\ \partial_{yz} f(p) x + \partial_{xz} f(p) y = 0. \end{cases}$$

As a consequence, the fiber of ∇f over any point $q = (q_0 : q_1 : q_2) \in \mathbb{P}^2$ is given by the zero locus of

$$\begin{cases} q_0 x + q_1 y + (1-d) q_2 z = 0 \\ q_0 \partial_{yz} f + q_1 \partial_{xz} f = 0. \end{cases}$$

Since the first equations represents a line for any choice of $q \in \mathbb{P}^2$, the claim follows.

We shall see in the next result that the presence of contracted lines is indeed always confirmed for $\mathcal{L}$ and $\mathcal{CL}_2$ arrangements. Recall that the critical locus of the polar map is given by the hessian curve, and it consists of the contracted curves and the ramification points for the polar map.

Proposition 4.5. *Let $C = V(f)$ be a conic-line arrangement in $\mathbb{P}^2$ of degree d such that the Jacobian ideal J_f of f is the ideal of an eigenscheme $E(T)$.*

Then, in case $\mathcal{L}$, the critical locus of ∇f is given by an arrangement of $3(d-2)$ lines of the same type of $\mathcal{L}$, it contains $\mathcal{L}$ and the contracted lines by ∇f are precisely the lines of $\mathcal{L}$.

In case $\mathcal{CL}_2$, the critical locus contains the tangent line ℓ, and it is the only contracted line.

Proof. It is classically known that all the lines of a line arrangement are contained in the hessian curve.

Now we verify that the residual curve to $\mathcal{L}$ in $\mathrm{Hess}(f)$ consists of $3(d-2)-d = 2d-6$ concurrent lines through $O = (0:0:1)$ and that such residual lines are not contracted by ∇f.

The first claim follows by writing the hessian matrix explicitly:

$$\mathrm{Hess}(f) = \begin{pmatrix} \partial_{xx} f & \partial_{xy} f & \partial_{xz} f \\ \partial_{xy} f & \partial_{yy} f & \partial_{yz} f \\ \partial_{xz} f & \partial_{yz} f & 0 \end{pmatrix}.$$

Since both $\partial_{xz} f$ and $\partial_{yz} f$ are polynomials in x and y only, by developing the determinant $h(f) = \det \operatorname{Hess}(f)$ with respect to the last row we see that $\frac{h(f)}{f}$ is a polynomial in x and y.

Moreover, as the polar map is given by

$$\nabla f = \left(x_2 \left(\sum_{i=1}^{d-1} a_i \prod_{j\neq i, j=1}^{d-1} (a_j x + b_j y) \right), z \left(\sum_{i=1}^{d-1} b_i \prod_{j\neq i, j=1}^{d-1} (a_j x + b_j y) \right), \right.$$
$$\left. \prod_{i=1}^{d-1} (a_i x + b_i y) \right),$$

we see that the line $z = 0$ is contracted to the point $(0 : 0 : 1)$ and the lines $a_i x + b_i y = 0$ to the points $(a_i : b_i : 0)$.

Finally, to prove that there are no other contracted lines, we recall that the Hilbert-Burch matrix of J_f is given by (3.1), and that ∇f is given by its 2×2 minors. It follows that for any $p = (p_0 : p_1 : p_2) \notin \Sigma_f$, the point $\nabla f(p)$ is the intersection point of the two distinct lines

$$\nabla f(p) : \begin{cases} p_0 x + p_1 y + (1-d) p_2 z = 0 \\ \partial_{yz} f(p) x + \partial_{xz} f(p) y = 0. \end{cases}$$

As a consequence, the fiber of ∇f over a point $q = (q_0 : q_1 : q_2) \in \mathbb{P}^2$ is given by the zero locus of

$$\begin{cases} q_0 x + q_1 y + (1-d) q_2 z = 0 \\ q_0 \partial_{yz} f + q_1 \partial_{xz} f = 0. \end{cases}$$

In particular, a ramification point appears in a fiber if and only if the set of $d-2$ concurrent lines through $(0 : 0 : 1)$ given by the equation $q_0 \partial_{yz} f + q_1 \partial_{xz} f = 0$ contains a (non reduced) double line, so the question is to determine the non reduced elements of the pencil $q_0 \partial_{yz} f + q_1 \partial_{xz} f$. But the latter can be seen as a pencil of divisors in $\mathbb{P}^1$, and precisely the Jacobian pencil of the polynomial $\partial_2 f$. If the factors of f are general, the polynomial $\partial_2 f \in \mathbb{C}[x, y]_{d-1}$ is general too. Therefore, the ramification points of the polar map $\nabla \partial_2 f$ are given by its hessian.

We finally treat the case of an $\mathcal{CL}_2$ arrangement. It is well-known that any linear component of a plane curve is contained in the Hessian curve. Moreover, if $f = x \prod_{i=1}^{m} (xz + a_i y^2)$, the polar map is given by

$$\nabla f = \left(\prod_{i=1}^{m} q_i + xz \left(\sum_{j=1}^{m} \prod_{i=1, i\neq j}^{m} q_i \right), 2xy \left(\sum_{j=1}^{m} a_j \prod_{i=1, i\neq j}^{m} q_i \right), \right.$$
$$\left. x^2 \left(\sum_{j=1}^{m} \prod_{i=1, i\neq j}^{m} q_i \right) \right),$$

where we set $q_i = xz + a_i y^2$; we see that the line $V(x)$ is contracted to a point. To see that there are no other contracted lines, we observe that such a line should contain a subscheme of degree at least $d-1$ in the Jacobian scheme; the only possible candidates are the tangent line in the second osculating point of the conics, that is the line $V(z)$, or the line $V(y)$ connecting the two singular points; but we can directly check that these cases do not occur. $\square$

We conclude by observing that the geometry of the polar map seems to encode some information concerning the topological type of the singularities of a given curve, so we believe that it deserves further investigations.

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