Monte Carlo Valuation of Future Annuity Contracts

Anna Rita Bacinello, Pietro Millossovich, and Fabio Viviano

Abstract In this paper we propose a methodology for valuing future annuity contracts based on the Least-Squares Monte Carlo approach. We adopt, as first step, a simplified computational framework where just one risk factor is taken into account. We give a brief description of the valuation procedure and provide some numerical illustrations. Furthermore, to test the efficiency of the proposed methodology, we compare our results with those obtained by applying a straightforward and time-consuming approach based on nested simulations.

Keywords LSMC · Life annuities · Longevity risk · Stochastic mortality

1 Introduction

Over the 20th century, due to health improvements and medical advances, it has become evident that people tend to live longer and longer. Indeed, the mortality of individuals over time has exhibited many stylized features. In particular, looking at the survival curve for most developed countries around the world, it is immediately clear that mortality levels are decreasing as time passes by, leading to an increase in

A. R. Bacinello · P. Millossovich · F. Viviano (✉)
Department of Business, Economics, Mathematics and Statistics 'B. de Finetti',
University of Trieste, Piazzale Europa 1, 34127 Trieste, Italy
e-mail: viviano.fabio@spes.uniud.it

A. R. Bacinello
e-mail: bacinel@units.it

P. Millossovich
e-mail: pietro.millossovich@deams.units.it

P. Millossovich
Faculty of Actuarial Science and Insurance, The Business School (Formerly Cass), City, University of London, 106 Bunhill Row, London EC1Y 8TZ, UK

F. Viviano
Department of Economics and Statistics, University of Udine, Via Tomadini 30/A,
33100 Udine, Italy

individual's life expectancy. As a consequence, life insurance companies and pension providers need to face the so-called longevity risk.

The actuarial literature has increasingly focused, in the last decades, on studying and proposing several methods for managing and evaluating this source of risk. The importance of modelling and transferring such a risk is argued in [2]. In particular, it is highlighted how the new longevity-linked capital market instruments could help in facilitating the development of annuity markets and hedging the long-term viability of retirement incomes. As a further consequence, we may recall the non-negligible impact on liabilities of insurers and pension plans, as studied in [11].

Recently, some attention has been devoted to the valuation of life annuity contracts issued at a distant future time. This problem has many sources of uncertainty, among which the most relevant are future interest rate and mortality levels. In this regard, [5, 7] suggest comonotonic approximations of the life annuity conditional expected present value. Moreover, [4, 6, 9] propose an approach based on a Taylor series approximation of the involved conditional expectation.

In this paper, we propose a simulation based method to evaluate the distribution of future annuity values. In particular, we aim at avoiding the straightforward approach based on nested simulations which is quite time-demanding, especially in a complex framework. The methodology described in what follows provides an application of the well-established Least-Squares Monte Carlo algorithm (LSMC), originally proposed by [10] for pricing American-type options. The most important advantage of this method is its flexibility to accommodate any type of Markov mortality model, and the possibility to extend it to more complicate frameworks without increasing the complexity of the involved computations.

The paper is structured as follows. In the next section we introduce the problem under scrutiny and describe our assumptions and the methodology used to solve it, in Sect. 3 we present a numerical example, and Sect. 4 contains some conclusions.

2 Problem and Methodology

The ever-increasing interest on adequately evaluating life insurance products or retirement incomes at a future time relates to the need of providing a reliable valuation of the cost of life expectancy, and to prevent somehow possible insolvency issues. In this paper, we aim at simulating the distribution of the value of an immediate life annuity contract issued to an individual aged $x + T$ at a future time horizon T.

We define the current value at the future time $T > 0$ of a unitary immediate annuity for an individual then aged $x + T$ as

$$a_{x+T}(T) = \sum_{i=1}^{+\infty} B(T, T+i)\, {}_i p_{x+T}\,(T)\,, \tag{1}$$

where $B(T, T+i)$ is the i-th years discount factor prevailing at time $T > 0$ and ${}_i p_y(T)$ is the i-th years survival probability for an individual aged y at time T.

The quantities $B(T, T+i)$ and ${}_i p_y(T)$ appearing in (1) are both random variables at time 0 (today), and consequently also $a_{x+T}(T)$ is random. More precisely, these variables are expectations conditional on the information available at time T.

To evaluate these conditional expectations we need models for describing the stochastic evolution of both interest and mortality rates. Under some circumstances, some closed form formulae for computing them are available, for instance when affine processes are used (see [1]), but in general this is not guaranteed. As previously mentioned, a straightforward approach would rely on a simulation within simulation procedure, also known as nested simulations; however, since it is quite computationally time-consuming, we are going to propose an application of the LSMC method.

2.1 Framework: Stochastic Mortality Dynamics

Although we have just mentioned that there are at least two sources of uncertainty affecting the value of an annuity, in this paper we assume a constant risk-free rate and adopt a stochastic model only for projecting future mortality levels. To this end, we use the Poisson version of one of the most significant and widely applied stochastic mortality models, i.e. the Lee-Carter model (see [8]). Hence, we assume that the number of deaths at age x and calendar year t, $D_{x;t}$, is Poisson distributed with parameter $E_{x;t} m_{x;t}$, where $E_{x;t}$ and $m_{x;t}$ denote the central exposure and the central death rate, respectively. Moreover, according to [8], we assume that the force of mortality is constant over each year of age and calendar year and equal to the corresponding central death rate $m_{x;t}$, modelled as

$$\log m_{x;t} = \alpha_x + \beta_x \kappa_t,$$

where α_x, β_x are age specific parameters and κ_t is a period index dictating the decrease over time in $m_{x;t}$.

Therefore, by exploiting the fact that κ_t is usually modelled as a Markov process, and typically as a random walk with drift, we have:

$${}_i p_{x+T} = \mathbb{E}\left[\exp\left\{-\left(m_{x+T;T} + \cdots + m_{x+T+i-1;T+i-1}\right)\right\} \mid \kappa_T\right],$$

and, within this framework, we can rewrite (1) as

$$a_{x+T}(T) = \mathbb{E}\left[\sum_{i=1}^{\omega-T-x} \exp\left\{-\left(ir + m_{x+T;T} + \cdots + m_{x+T+i-1;T+i-1}\right)\right\} \mid \kappa_T\right], \tag{2}$$

where ω is the ultimate age and r the constant interest rate.

2.2 Valuation Procedure

The previously introduced framework does not produce a closed form formula for (2), as typically the central death rates have a lognormal distribution so each exponent in (2) involves the sum of lognormal variables. Hence, a possible strategy is to evaluate the involved conditional expectation through simulation based methods.

A straightforward approach would rely on a nested simulations procedure. This strategy requires first to simulate all relevant risk factors up to time T (outer scenarios); then, for each simulated time T value of such factors, one would need to simulate forward starting from that particular value (inner simulations), and finally compute conditional expectations by averaging across all inner simulations. It follows that this method can be computationally expensive, in particular when several annuity values (at different times and/or ages) are needed.

Therefore, in order to reduce the computational complexity, we propose an alternative methodology based on the LSMC approach and, to check the accuracy of the results, we compare them with those obtained through nested simulations, so that the latter acts as benchmark for evaluating the efficiency and the accuracy of the LSMC procedure (see [3]).

The LSMC approach involves two main steps: firstly, we need to perform simulations of future mortality patterns; then, we use regression across the simulated trajectories in order to obtain estimates of future annuity values. In this way, the conditional expectation is evaluated through regression taking into account the information available at time T (i.e. the simulated values of the time index parameter κ_T exploited as predictor). Moreover, this method allows to obtain an estimate of the probability distribution of annuity values at future time horizon T for individuals aged $x + T$ at that date. Finally, a single set of simulations, without increasing the computational demand, can be used for different ages and time horizons.

3 A Numerical Example

In this section, we provide an example based on an immediate life annuity issued to an individual aged 40 at different future time horizons $T \in \{10, 20, 30, 40\}$. In order to simulate future mortality patterns, we fit the Poisson Lee-Carter model to the Italian male population data over the period 1965–2014 and range of ages 0–90, obtained through the Human Mortality Database. Further, we assume that year 2014 corresponds to the evaluation time 0 (today). The risk-free rate is set at the (constant) level $r = 0.03$. Moreover, we simulate 10000 different trajectories of future mortality and, in the nested simulation approach, we further simulate another 10000 paths starting from each value generated at time T; in total this amounts to 100 millions inner simulations. Regarding the basis functions, we use polynomials with degree $p = 4$.

Table 1 Distribution of annuity values at time horizon T for individuals aged 40 in year $2014 + T$

		Mean	Std dev	Skewness	Kurtosis	10th perc.	90th perc.
$T = 10$	MC	24.42	0.11	−0.08	2.94	24.27	24.56
	LSMC	24.42	0.11	−0.06	2.91	24.27	24.56
$T = 20$	MC	24.78	0.14	−0.14	3.04	24.60	24.96
	LSMC	24.78	0.14	−0.11	3.02	24.61	24.96
$T = 30$	MC	25.11	0.16	−0.20	3.05	24.91	25.30
	LSMC	25.11	0.15	−0.19	3.04	24.91	25.30
$T = 40$	MC	25.40	0.16	−0.23	3.13	25.20	25.60
	LSMC	25.40	0.16	−0.24	3.05	25.20	25.60

Table 1 reports some statistics of the distributions of future annuity values obtained through the two valuation algorithms. Looking at the results, it immediately turns out that, as the time horizon T increases, the distribution changes. Specifically, its mean increases, which is quite reasonable and in line with the ever-increasing life expectancy registered in the last decades. In addition, its standard deviation increases as well, which implies a more dispersed distribution. This result also seems to be reasonable due to the higher uncertainty caused by the longer time horizon. Furthermore, it seems that the distributions tend to be increasingly left-skewed, which implies a longer left tail, hence the distribution is concentrated on the right tail (i.e. higher values of the annuity contract). Finally, we see that the kurtosis increases, meaning that we recognize a heavier tailed distribution, hence a greater propensity to result in extreme annuity values with respect to the Gaussian case.

Concerning the validation procedure, we can see from Table 1 that the LSMC approach provides quite accurate estimates. Moreover, the reliability of the proposed approach is evidenced by the fact that the obtained distribution overlaps substantially with the one produced through nested simulations; this is also confirmed by the Kolmogorov–Smirnov test (KS, see Table 2). In addition, we have constructed the Q-Q plots by considering the distribution obtained through nested simulations as the theoretical one, and these graphs, once again, confirm the goodness of the proposed method in approaching this kind of problem.[1]

Finally, for a more comprehensive analysis, we checked whether the LSMC approach tends to over- or under-estimate the quantities of interest. In this regard, in Table 2 we report the frequency with which the LSMC estimates lie inside the 95% confidence interval obtained through the nested simulation procedure or outside (on the left or on the right, respectively). Looking at this result, we can see that most of the time the LSMC method provides an estimate which lies within that interval. However, even if there is a small signal of under-estimation effect which could be due to biases in the regression, we can assess the goodness of the proposed method.

[1] We do not report the distribution graphs and the Q-Q plots for space considerations.

Table 2 Frequency of hitting the confidence intervals (left table) and KS Test (right table)

	Left (%)	Inside (%)	Right (%)	KS stat. Value	p-value
$T = 10$	8.09	87.68	4.23	0.0059	0.9950
$T = 20$	12.57	80.43	7.00	0.0094	0.7689
$T = 30$	5.35	87.50	7.15	0.0036	0.9996
$T = 40$	12.67	83.47	3.86	0.0053	0.9990

4 Conclusions

In this paper, we faced the problem of approximating future annuity values. We proposed a methodology based on the LSMC approach which turns out to be quite accurate. Our results highlight the need of developing reliable actuarial models able to capture the source of risk arising from longevity. This is not a negligible aspect, especially for solvency purposes. This paper can be extended in several directions, by assuming more complicated valuation frameworks (e.g., stochastic interest rates), or by dealing with other types of life annuity contracts such as variable annuities, equity-indexed products, or by implementing de-risking strategies for pension plans, e.g. Buy-Ins and Buy-Outs, which require an accurate valuation of annuities.

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