How perturbations in the matrix of linear systems of ordinary differential equations propagate along solutions☆

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This paper addresses how perturbations in the matrix A propagate along the solution of the n-dimensional linear ordinary differential equation

$$\begin{cases} y'(t) = Ay(t), \ t \geq 0, \\ y(0) = y_0. \end{cases}$$

In other words, for fixed $t \geq 0$ and $y_0 \in \mathbb{R}^n$, we study the conditioning of the problem

$$A \mapsto e^{tA} y_0.$$

We also study the asymptotic behavior of the conditioning as $t \to +\infty$. The analysis is carried out for a normal matrix A.

1. Introduction

Consider the following linear Ordinary Differential Equation (ODE)

$$\begin{cases} y'(t) = Ay(t), \ t \geq 0, \\ y(0) = y_0, \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n \times n}$ and $y(t) = e^{tA} y_0$, $t \geq 0$, is the solution. Let $\| \cdot \|$ be a norm on $\mathbb{R}^n$ and let $|\!|\!| \cdot |\!|\!|$ be a norm on $\mathbb{R}^{n \times n}$. Suppose $A \neq 0$ and A perturbed to $\widetilde{A}$ with relative error ϵ given by

$$\epsilon = \frac{|\!|\!| \widetilde{A} - A |\!|\!|}{|\!|\!| A |\!|\!|}.$$

A perturbation in the matrix A results in a perturbation of the solution y of (1). The perturbed solution is $\widetilde{y}(t) = e^{t\widetilde{A}} y_0, \ t \geq 0$, with relative error

$$\xi(t) = \frac{\left\| e^{t\widetilde{A}} y_0 - e^{tA} y_0 \right\|}{\left\| e^{tA} y_0 \right\|}, \ t \geq 0.$$

The error $\xi(t)$ is defined for $y_0 \neq 0$ and it is the error propagated along the solution by the perturbation in the matrix A. We have $\xi(0) = 0$.

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Aim of this paper is to study how the error $\xi(t)$ is related to the error ϵ. In other words, we study the (relative) conditioning of the problem

$$A \mapsto e^{tA} y_0. \tag{2}$$

There are several papers in literature (see [1–7], and [8]) dealing with the conditioning of the problem

$$A \mapsto e^{tA}, \tag{3}$$

namely how a perturbation in A affects e^{tA}, not $e^{tA}y_0$. In these papers, the role of the initial value y_0 is not taken into account and the relative error

$$\frac{\left\|\left| e^{t\tilde{A}} - e^{tA} \right|\right\|}{\left\|\left| e^{tA} \right|\right\|},$$

rather then $\xi(t)$, is considered.

The conditioning of the problem (2), the topic of the present paper, has received less attention in literature. At the best of our knowledge there are only the two papers [9,10] on this topic. The paper [9], in order to analyze an algorithm for computing $e^{tA}Y_0$, where Y_0 is a matrix, considered the condition number (relevant to Frobenius norms) of the problem $(A, Y_0) \mapsto f(tA)Y_0$, where f is matrix function, and obtained a bound for it. The paper [10] presented algorithms for the computation of the condition number of the problem $(t, A, y_0) \mapsto f(tA)y_0$. On the other hand, the present paper does not consider the computational aspects but it is more theoretical, since it is interested to a qualitative analysis of the condition number of the problem (2): it studies how this condition number depends on the parameters time t and initial value y_0. Also the asymptotic behavior as $t \to +\infty$ is considered. Moreover, we also consider a condition number with direction of perturbation (which can be useful when one wants to analyze not the worst case but an average case, or when some information about the perturbation is known) and a condition number independent of y_0.

After the introduction of the condition numbers of the problem (2), defined for a general matrix A and general norms $\|\cdot\|$ on $\mathbb{R}^n$ and $\|\!|\cdot|\!\|$ on $\mathbb{R}^{n\times n}$, their analysis is done for A normal, $\|\cdot\| = \|\cdot\|_2$ and $\|\!|\cdot|\!\| = \|\cdot\|_2$. By assuming A normal, we can take advantage of the orthogonality of the eigenspaces and then give explicit expressions for the condition numbers when $\|\cdot\| = \|\cdot\|_2$ and $\|\!|\cdot|\!\| = \|\cdot\|_2$. Anyway, we remark that the class of the normal matrices includes the important families of the symmetric matrices and the shifted skew-symmetric matrices.

A qualitative theoretical analysis of the conditioning of the problem $y_0 \mapsto e^{tA}y_0$, for A normal and $\|\cdot\| = \|\cdot\|_2$, was accomplished in [11].

Here is the plan of the paper. In Section 2, we introduce the condition numbers of the problem (2). For A normal, $\|\cdot\| = \|\cdot\|_2$ and $\|\!|\cdot|\!\| = \|\cdot\|_2$, we analyze such condition numbers in Section 3 and we study their asymptotic behavior as $t \to +\infty$ in Section 4. Finally, in Section 5 we present some numerical tests and conclusions are given in Section 6.

2. The condition numbers

We specify the perturbed matrix as

$$\widetilde{A} = A + \epsilon \|\!|A|\!\| \widehat{B},$$

where $\widehat{B} \in \mathbb{R}^{n\times n}$ with $\|\!|\widehat{B}|\!\| = 1$ is the *direction of the perturbation*.

We define

$$K(t, A, y_0, \widehat{B}) := \lim_{\epsilon \to 0} \frac{\xi(t)}{\epsilon} \tag{4}$$

as the *condition number with direction of perturbation* $\widehat{B}$ of the problem (2).

The next Theorem gives an expression for such a condition number.

Theorem 2.1. *We have*

$$K\left(t, A, y_0, \widehat{B}\right) = \frac{\left\| L\left(t, A, \widehat{B}\right)\widehat{y}_0 \right\| \|\!|A|\!\|}{\left\| e^{tA}\widehat{y}_0 \right\|}, \tag{5}$$

where

$$L\left(t, A, \widehat{B}\right) = \int_0^t e^{(t-s)A}\widehat{B}e^{sA} ds$$

and

$$\widehat{y}_0 := \frac{y_0}{\|y_0\|}.$$

Proof. We have

$$e^{t\widetilde{A}}y_0 - e^{tA}y_0 = \left(e^{t(A+E)} - e^{tA}\right)y_0,$$

where

$$E = \epsilon |||A|||\widehat{B}.$$

Since (see [12])

$$e^{t(A+E)} - e^{tA} = L(t, A, E) + O(|||E|||^2),\ E \to 0,$$

where

$$E \mapsto L(t, A, E) = \int_0^t e^{(t-s)A}Ee^{sA}ds,\ \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n},$$

is the Frechét derivative of the map $A \mapsto e^{tA}$, we obtain

$$\xi(t) = \frac{\left\|L\left(t, A, \widehat{B}\right)y_0\right\| \, |||A|||}{\left\|e^{tA}y_0\right\|}\epsilon + O\left(\epsilon^2\right),\ \ \varepsilon \to 0,$$

and (5) follows since

$$\frac{\left\|L\left(t, A, \widehat{B}\right)y_0\right\|}{\left\|e^{tA}y_0\right\|} = \frac{\left\|L\left(t, A, \widehat{B}\right)\widehat{y}_0\right\|}{\left\|e^{tA}\widehat{y}_0\right\|}. \quad \blacksquare$$

We define

$$K(t, A, y_0) := \sup_{\substack{\widehat{B}\in\mathbb{R}^{n\times n}\\ |||\widehat{B}|||=1}} K\left(t, A, y_0, \widehat{B}\right) \tag{6}$$

as the *condition number* of the problem (2).

We have

$$K(t, A, y_0) = \frac{\|\mathcal{L}(t, A, y_0)\| \, |||A|||}{\|e^{tA}y_0\|},$$

where $\mathcal{L}(t, A, y_0) : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ is the linear operator given by

$$\mathcal{L}(t, A, y_0)E = L(t, A, E)y_0,\ \ E \in \mathbb{R}^{n\times n},$$

and $\|\mathcal{L}(t, A, y_0)\|$ is the operator norm relevant to the norms $|||\cdot|||$ on $\mathbb{R}^{n\times n}$ and $\|\cdot\|$ on $\mathbb{R}^n$. Observe that

$$L(t, A, E)y_0 = (y_0^T \otimes I_n)\text{vec}(L(t, A, E)) = (y_0^T \otimes I_n)M(t, A)\text{vec}(E),$$

where $\otimes$ is the Kronecker product, the operator vec stacks the columns of a matrix in a column vector and $M(t, A) \in \mathbb{R}^{n^2\times n^2}$ is the matrix corresponding to the linear operator

$$\mathbb{R}^{n^2} \to \mathbb{R}^{n^2},\ \ \text{vec}(E) \mapsto \text{vec}(L(t, A, E)).$$

So, we get

$$\|\mathcal{L}(t, A, y_0)\| = \|(y_0^T \otimes I_n)M(t, A)\|_2 \ \text{ if } \|\cdot\| = \|\cdot\|_2 \text{ and } |||\cdot||| = \|\cdot\|_F$$

and

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\|(y_0^T \otimes I_n)M(t, A)\|_2 \le \|\mathcal{L}(t, A, y_0)\| \le \sqrt{n}\|(y_0^T \otimes I_n)M(t, A)\|_2 \\
&\|(y_0^T \otimes I_n)M(t, A)\|_1 \le \|\mathcal{L}(t, A, y_0)\| \le n\|(y_0^T \otimes I_n)M(t, A)\|_1 \\
&\quad \text{if } \|\cdot\| = \|\cdot\|_1 \text{ and } |||\cdot||| = \|\cdot\|_1.
\end{aligned} \tag{7}$$

The condition number (6) corresponds to the standard definition of condition number of a problem (see [13]) and it is the condition number considered in the papers [9,10]. It is an analog of the matrix exponential condition number in [5, pag. 9], which is given for the problem (3). The paper [10] used (7) for estimating the condition number.

Finally, we define

$$K(t, A) := \sup_{\substack{y_0\in\mathbb{R}^n\\ y_0\neq 0}} K(t, A, y_0) \tag{8}$$

as the *condition number independent of* y_0 of the problem (2).

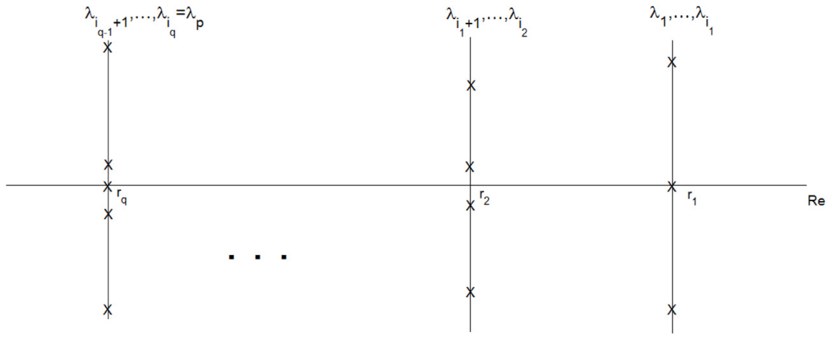


Fig. 1. Spectrum of A partitioned by decreasing real parts.

3. Analysis for A normal

From now on, we consider A normal, $\|\cdot\| = \|\cdot\|_2$ and $|||\cdot||| = \|\cdot\|_2$. We write the condition numbers $K(\cdot)$ defined in the previous section as $K_2(\cdot)$.

Let $\lambda_1, \ldots, \lambda_p$ be the distinct eigenvalues of A. We partition the spectrum $\Lambda := \{\lambda_1, \ldots \lambda_p\}$ of A (see Fig. 1) in the subsets

$$\Lambda_j := \{\lambda_{i_{j-1}+1}, \lambda_{i_{j-1}+2}, \ldots, \lambda_{i_j}\}, \quad j = 1, \ldots, q,$$

where $0 = i_0 < i_1 < \cdots < i_q = p$, by decreasing real parts: we have

$$\mathrm{Re}(\lambda_{i_{j-1}+1}) = \mathrm{Re}(\lambda_{i_{j-1}+2}) = \cdots = \mathrm{Re}(\lambda_{i_j}) = r_j, \ j = 1, \ldots, q,$$

with

$$r_1 > r_2 > \cdots > r_q.$$

For $\lambda_i \in \Lambda$, let P_i be the orthogonal projection on the eigenspace of λ_i. For $j = 1, \ldots, q$, let

$$Q_j = \sum_{\lambda_i \in \Lambda_j} P_i.$$

3.1. *The condition number $K_2\left(t, A, y_0, \widehat{B}\right)$ with direction of perturbation*

The next theorem provides an expression for $K_2\left(t, A, y_0, \widehat{B}\right)$.

Theorem 3.1. *We have*

$$K_2\left(t, A, y_0, \widehat{B}\right) = \frac{\sqrt{\sum_{j=1}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\left(\int_0^t e^{-sA}\widehat{B}e^{sA}ds\right)\widehat{y}_0\right\|_2\right)^2}}{\sqrt{\sum_{j=1}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}}\|A\|_2 \tag{9}$$

and for the numerator in (9) we have

$$\sqrt{\sum_{j=1}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\left(\int_0^t e^{-sA}\widehat{B}e^{sA}ds\right)\widehat{y}_0\right\|_2\right)^2} = \left\|\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0\right\|_2, \tag{10}$$

where, for $\lambda_i \in \Lambda_j$, with $j \in \{1, \ldots, q\}$, and $\lambda_k \in \Lambda$,

$$C(t, \lambda_i, \lambda_k) := e^{(r_j-r_1)t}\int_0^t e^{(\lambda_k-\lambda_i)s}ds. \tag{11}$$

Proof. By recalling (5), we write

$$K_2\left(t, A, y_0, \widehat{B}\right) = \frac{\left\| e^{tA} \int_0^t e^{-sA}\widehat{B}e^{sA} ds \widehat{y}_0 \right\|_2}{\left\| e^{tA}\widehat{y}_0 \right\|_2} \|A\|_2 .$$

Since A is normal, we have, for $u \in \mathbb{R}^n$,

$$\|e^{tA}u\|_2 = \left\| \sum_{\lambda_i \in \Lambda} e^{\lambda_i t} P_i u \right\|_2 = \sqrt{\sum_{\lambda_i \in \Lambda} \left(|e^{\lambda_i t}| \, \|P_i u\|_2 \right)^2} = \sqrt{\sum_{j=1}^{q} \left(e^{r_j t} \left\| Q_j u \right\|_2 \right)^2} .$$

Thus

$$\left\| e^{tA} \int_0^t e^{-sA}\widehat{B}e^{sA} ds \widehat{y}_0 \right\|_2 = \sqrt{\sum_{j=1}^{q} \left(e^{r_j t} \left\| Q_j \left(\int_0^t e^{-sA}\widehat{B}e^{sA} ds \right) \widehat{y}_0 \right\|_2 \right)^2}$$

and

$$\left\| e^{tA}\widehat{y}_0 \right\|_2 = \sqrt{\sum_{j=1}^{q} \left(e^{r_j t} \left\| Q_j \widehat{y}_0 \right\|_2 \right)^2}$$

and then we obtain (9).

By the orthogonality of the projections Q_j, $j = 1, \ldots, q$, we get

$$\sqrt{\sum_{j=1}^{q} \left(e^{(r_j - r_1)t} \left\| Q_j \int_0^t e^{-sA}\widehat{B}e^{sA} ds \widehat{y}_0 \right\|_2 \right)^2}$$
$$= \left\| \sum_{j=1}^{q} e^{(r_j - r_1)t} Q_j \int_0^t e^{-sA}\widehat{B}e^{sA} ds \widehat{y}_0 \right\|_2 .$$

Now, by decomposing the matrices e^{-sA} and e^{sA} as

$$e^{-sA} = \sum_{i=1}^{p} e^{-\lambda_i s} P_i \quad \text{and} \quad e^{sA} = \sum_{k=1}^{p} e^{\lambda_k s} P_k,$$

we obtain (10). ■

The next proposition concerns the functions $C(t, \lambda_i, \lambda_k)$ defined in (11).

Proposition 3.1. *Let $j, l \in \{1, \ldots, q\}$, let $\lambda_i \in \Lambda_j$ and let $\lambda_k \in \Lambda_l$. Let*

$$\lambda_i = r_j + \sqrt{-1}\omega_i \quad \text{and} \quad \lambda_k = r_l + \sqrt{-1}\omega_k$$

be the cartesian forms of the complex numbers λ_i and λ_k, where $\sqrt{-1}$ denotes the imaginary unit.

If $j \leq l$, then

$$|C(t, \lambda_i, \lambda_k)| \leq e^{(r_j - r_1)t} t.$$

If $j \geq l$, then

$$|C(t, \lambda_i, \lambda_k)| \leq e^{(r_l - r_1)t} t.$$

If $\lambda_i \neq \lambda_k$, then

$$C(t, \lambda_i, \lambda_k) = \frac{e^{(r_l - r_1)t} e^{\sqrt{-1}(\omega_k - \omega_i)t} - e^{(r_j - r_1)t}}{\lambda_k - \lambda_i}.$$

If $\lambda_i = \lambda_k$, then

$$C(t, \lambda_i, \lambda_k) = e^{(r_j - r_1)t} t.$$

Proof. We have

$$\left| \int_0^t e^{(\lambda_k - \lambda_i)s} ds \right| \leq \int_0^t e^{(r_l - r_j)s} ds$$

and then

$$|C(t,\lambda_i,\lambda_k)| \le e^{(r_j-r_1)t}\int_0^t e^{(r_l-r_j)s}ds \le e^{(r_j-r_1)t}t$$

for $j \le l$ and

$$|C(t,\lambda_i,\lambda_k)| \le e^{(r_j-r_1)t}\int_0^t e^{(r_l-r_j)s}ds \le e^{(r_j-r_1)t}e^{(r_l-r_j)t}t \le e^{(r_l-r_1)t}t.$$

for $j \ge l$.

If $\lambda_i \neq \lambda_k$, we have

$$\int_0^t e^{(\lambda_k-\lambda_i)s}ds = \frac{e^{(\lambda_k-\lambda_i)t}-1}{\lambda_k-\lambda_i}$$

and then

$$\begin{aligned}
C(t,\lambda_i,\lambda_k) &= e^{(r_j-r_1)t}\frac{e^{(\lambda_k-\lambda_i)t}-1}{\lambda_k-\lambda_i} \\
&= e^{(r_j-r_1)t}\frac{e^{(r_l-r_j)t}e^{\sqrt{-1}(\omega_k-\omega_i)t}-1}{\lambda_k-\lambda_i} \\
&= \frac{e^{(r_l-r_1)t}e^{\sqrt{-1}(\omega_k-\omega_i)t}-e^{(r_j-r_1)t}}{\lambda_k-\lambda_i}.
\end{aligned}$$

If $\lambda_i = \lambda_k$, we have

$$\int_0^t e^{(\lambda_k-\lambda_i)s}ds = t$$

and then

$$C(t,\lambda_i,\lambda_k) = e^{(r_j-r_1)t}t. \quad \blacksquare$$

Remark 3.1. Let $j, l \in \{1,\ldots,q\}$, let $\lambda_i \in \Lambda_j$ and let $\lambda_k \in \Lambda_l$. The previous proposition shows that:

- if $j > 1$ and $l > 1$, then $C(t,\lambda_i,\lambda_k)$ vanishes as $t \to +\infty$;
- if ($j = 1$ or $l = 1$) and $\lambda_i \neq \lambda_k$, then $C(t,\lambda_i,\lambda_k)$ is a bounded function of $t \ge 0$ and it does not vanish as $t \to +\infty$;
- if ($j = 1$ or $l = 1$) and $\lambda_i = \lambda_k$, i.e. $j = l = 1$ and $\lambda_i = \lambda_k$, then $C(t,\lambda_i,\lambda_k) = t$.

3.2. The condition number $K_2(t,A,y_0)$

The next theorem gives lower and upper bounds for $K_2(t,A,y_0)$.

Theorem 3.2. *We have the lower bounds*

$$K_2(t,A,y_0) \ge \|A\|_2\, t \tag{12}$$

and

$$K_2(t,A,y_0) \ge \frac{\max_{\lambda_i,\lambda_k\in\Lambda}|D(t,\lambda_i,\lambda_k)|\,\|P_k\widehat{y}_0\|_2}{\sqrt{\sum_{j=1}^q\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}}\,\|A\|_2\,, \tag{13}$$

where

$$D(t,\lambda_i,\lambda_k) := \begin{cases} C(t,\lambda_i,\lambda_k) & \text{if } \lambda_k \text{ is real} \\ \frac{\sqrt{2}}{2}\left(C(t,\lambda_i,\lambda_k)+C(t,\lambda_i,\overline{\lambda_k})\right) & \text{if } \lambda_k \text{ is not real.}\end{cases}$$

Here $\overline{\lambda_k}$ denotes the complex conjugate of λ_k.

Moreover, we have the upper bound

$$K_2(t,A,y_0) \le \frac{\sqrt{\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda}|C(t,\lambda_i,\lambda_k)|^2\,\|P_k\widehat{y}_0\|^2}}{\sqrt{\sum_{j=1}^q\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}}\,\|A\|_2\,. \tag{14}$$

Proof. The first lower bound (12) follows by putting $\widehat{B} = I$ in (5) or (9).

Now, we prove the second lower bound (13). We show that

$$\sup_{\substack{\widehat{B}\in\mathbb{R}^{n\times n}\\ \|\widehat{B}\|_2=1}} \left\| \sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0 \right\|_2 \geq \max_{\lambda_a,\lambda_b\in\Lambda} |D(t,\lambda_a,\lambda_b)|\,\|P_b\widehat{y}_0\|_2 \tag{15}$$

holds for (9)–(10). Fix $\lambda_a, \lambda_b \in \Lambda$ with $P_b y_0 \neq 0$. We consider the four cases:

A λ_a and λ_b are real;
B λ_a is not real and λ_b is real;
C λ_a is real and λ_b is not real;
D λ_a and λ_b are not real.

When λ_a is not real, let $\lambda_{\overline{a}}$, where $\overline{a} \in \{1,\ldots,p\}\setminus\{a\}$, be the eigenvalue which is the complex conjugate of λ_a. Similarly, when λ_b is not real, let $\lambda_{\overline{b}}$, where $\overline{b} \in \{1,\ldots,p\}\setminus\{b\}$, be the eigenvalue which is the complex conjugate of λ_b.

In the case A, consider a unit vector $\widehat{v}\in\mathbb{R}^n$ (i.e. $\|\widehat{v}\|_2 = 1$) such that $P_a\widehat{v} = \widehat{v}$ and consider the direction of perturbation

$$\widehat{B} = \widehat{v}\left(\frac{P_b\widehat{y}_0}{\|P_b\widehat{y}_0\|_2}\right)^H.$$

For $\lambda_k \in \Lambda$, we have

$$\begin{aligned}\widehat{B}P_k\widehat{y}_0 &= \widehat{v}\left(\frac{P_b\widehat{y}_0}{\|P_b\widehat{y}_0\|_2}\right)^H P_k\widehat{y}_0 = \frac{1}{\|P_b\widehat{y}_0\|_2}\left((P_b\widehat{y}_0)^H P_k\widehat{y}_0\right)\widehat{v}\\ &= \begin{cases} 0 \text{ if } k\neq b\\ \|P_b\widehat{y}_0\|_2\,\widehat{v} \text{ if } k=b.\end{cases}\end{aligned}$$

Thus

$$\begin{aligned}\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0 &= \|P_b\widehat{y}_0\|_2 \sum_{\lambda_i\in\Lambda} C(t,\lambda_i,\lambda_b)P_i\widehat{v}\\ &= \|P_b\widehat{y}_0\|_2\, C(t,\lambda_a,\lambda_b)\widehat{v}\end{aligned}$$

and then

$$\begin{aligned}\left\| \sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0 \right\|_2 &= \|P_b\widehat{y}_0\|_2\,|C(t,\lambda_a,\lambda_b)|\\ &= \|P_b\widehat{y}_0\|_2\,|D(t,\lambda_a,\lambda_b)|\,.\end{aligned}$$

In the case B, consider a unit vector $\widehat{v}\in\mathbb{R}^n$ such that $(P_a+P_{\overline{a}})\,\widehat{v} = \widehat{v}$ and consider the direction of perturbation

$$\widehat{B} = \widehat{v}\left(\frac{P_b\widehat{y}_0}{\|P_b\widehat{y}_0\|_2}\right)^H.$$

We have

$$\begin{aligned}\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0 &= \|P_b\widehat{y}_0\|_2 \sum_{\lambda_i\in\Lambda} C(t,\lambda_i,\lambda_b)P_i\widehat{v}\\ &= \|P_b\widehat{y}_0\|_2\,(C(t,\lambda_a,\lambda_b)P_a\widehat{v} + C(t,\lambda_{\overline{a}},\lambda_b)P_{\overline{a}}\widehat{v})\end{aligned}$$

and then

$$\begin{aligned}&\left\| \sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0 \right\|_2\\ &= \|P_b\widehat{y}_0\|_2\,\|C(t,\lambda_a,\lambda_b)P_a\widehat{v} + C(t,\lambda_{\overline{a}},\lambda_b)P_{\overline{a}}\widehat{v}\|_2\\ &= \|P_b\widehat{y}_0\|_2\sqrt{|C(t,\lambda_a,\lambda_b)|^2\,\|P_a\widehat{v}\|_2^2 + |C(t,\lambda_{\overline{a}},\lambda_b)|^2\,\|P_{\overline{a}}\widehat{v}\|_2^2}\\ &= \|P_b\widehat{y}_0\|_2\,|C(t,\lambda_a,\lambda_b)| = \|P_b\widehat{y}_0\|_2\,|D(t,\lambda_a,\lambda_b)|\,,\end{aligned}$$

where the second last $=$ follows since $C(t,\lambda_a,\lambda_b)$ and $C(t,\lambda_{\overline{a}},\lambda_b)$ are complex conjugate.

In the case C, consider a unit vector $\widehat{v}\in\mathbb{R}^n$ such that $P_a\widehat{v} = \widehat{v}$ and consider the direction of perturbation

$$\widehat{B} = \widehat{v}\left(\frac{\left(P_b+P_{\overline{b}}\right)\widehat{y}_0}{\left\|\left(P_b+P_{\overline{b}}\right)\widehat{y}_0\right\|_2}\right)^H.$$

For $\lambda_k \in \Lambda$, we have

$$\widehat{B}P_k\widehat{y}_0 = \begin{cases} 0 \text{ if } k \neq b \text{ and } k \neq \overline{b} \\ \frac{\|P_b\widehat{y}_0\|_2^2}{\|(P_b+P_{\overline{b}})\widehat{y}_0\|_2}\widehat{v} \text{ if } k = b \\ \frac{\|P_{\overline{b}}\widehat{y}_0\|_2^2}{\|(P_b+P_{\overline{b}})\widehat{y}_0\|_2}\widehat{v} \text{ if } k = \overline{b}. \end{cases}$$

Since

$$\|P_b\widehat{y}_0\|_2^2 + \left\|P_{\overline{b}}\widehat{y}_0\right\|_2^2 = \left\|\left(P_b + P_{\overline{b}}\right)\widehat{y}_0\right\|_2^2 \quad \text{and} \quad \|P_b\widehat{y}_0\|_2 = \left\|P_{\overline{b}}\widehat{y}_0\right\|_2,$$

we get

$$\widehat{B}P_k\widehat{y}_0 = \begin{cases} 0 \text{ if } k \neq b \text{ and } k \neq \overline{b} \\ \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|\,\widehat{v} \text{ if } k = b \text{ or } k = \overline{b}. \end{cases}$$

Thus

$$\begin{aligned} &\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0 \\ &= \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|\sum_{\lambda_i\in\Lambda}\left(C(t,\lambda_i,\lambda_b)+C(t,\lambda_i,\lambda_{\overline{b}})\right)P_i\widehat{v} \\ &= \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|\left(C(t,\lambda_a,\lambda_b)+C(t,\lambda_a,\lambda_{\overline{b}})\right)\widehat{v} \end{aligned}$$

and then

$$\begin{aligned} \left\|\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0\right\|_2 &= \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|\left|C(t,\lambda_a,\lambda_b)+C(t,\lambda_a,\lambda_{\overline{b}})\right| \\ &= \|P_b\widehat{y}_0\|_2\left|D(t,\lambda_a,\lambda_b)\right|. \end{aligned}$$

In the case D, consider a unit vector $\widehat{v} \in \mathbb{R}^n$ such that $(P_a + P_{\overline{a}})\,\widehat{v} = \widehat{v}$ and consider the direction of perturbation

$$\widehat{B} = \widehat{v}\left(\frac{\left(P_b + P_{\overline{b}}\right)\widehat{y}_0}{\left\|\left(P_b + P_{\overline{b}}\right)\widehat{y}_0\right\|_2}\right)^H.$$

We have

$$\begin{aligned} &\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0 \\ &= \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|_2\sum_{\lambda_i\in\Lambda}\left(C(t,\lambda_i,\lambda_b)+C(t,\lambda_i,\lambda_{\overline{b}})\right)P_i\widehat{v} \\ &= \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|_2 \\ &\quad \left(\left(C(t,\lambda_a,\lambda_b)+C(t,\lambda_a,\lambda_{\overline{b}})\right)P_a\widehat{v} + \left(C(t,\lambda_{\overline{a}},\lambda_b)+C(t,\lambda_{\overline{a}},\lambda_{\overline{b}})\right)P_{\overline{a}}\widehat{v}\right) \end{aligned}$$

and then

$$\begin{aligned} &\left\|\sum_{\lambda_i\in\Lambda}\sum_{\lambda_k\in\Lambda} C(t,\lambda_i,\lambda_k)P_i\widehat{B}P_k\widehat{y}_0\right\|_2 \\ &= \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|_2 \\ &\quad \cdot\sqrt{\left|C(t,\lambda_a,\lambda_b)+C(t,\lambda_a,\lambda_{\overline{b}})\right|^2\|P_a\widehat{v}\|_2^2 + \left|C(t,\lambda_{\overline{a}},\lambda_b)+C(t,\lambda_{\overline{a}},\lambda_{\overline{b}})\right|^2\|P_{\overline{a}}\widehat{v}\|_2^2} \\ &= \frac{\sqrt{2}}{2}\|P_b\widehat{y}_0\|_2\left|C(t,\lambda_a,\lambda_b)+C(t,\lambda_a,\lambda_{\overline{b}})\right| \\ &= \|P_b\widehat{y}_0\|_2\left|D(t,\lambda_a,\lambda_b)\right|. \end{aligned}$$

where the second last $=$ follows since $C(t, \lambda_a, \lambda_b) + C(t, \lambda_a, \lambda_{\overline{b}})$ and $C(t, \lambda_{\overline{a}}, \lambda_b) + C(t, \lambda_{\overline{a}}, \lambda_{\overline{b}})$ are complex conjugate.

Now, (15) and then the lower bound (13) follow.

The upper bound (14) follows by observing that

$$\left\| \sum_{\lambda_i \in \Lambda} \sum_{\lambda_k \in \Lambda} C(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2 = \left\| \sum_{\lambda_i \in \Lambda} P_i \widehat{B} \left(\sum_{\lambda_k \in \Lambda} C(t, \lambda_i, \lambda_k) P_k \widehat{y}_0 \right) \right\|_2$$

$$= \sqrt{\sum_{\lambda_i \in \Lambda} \left\| P_i \widehat{B} \left(\sum_{\lambda_k \in \Lambda} C(t, \lambda_i, \lambda_k) P_k \widehat{y}_0 \right) \right\|_2^2} \le \sqrt{\sum_{\lambda_i \in \Lambda} \left\| \sum_{\lambda_k \in \Lambda} C(t, \lambda_i, \lambda_k) P_k \widehat{y}_0 \right\|_2^2}$$

$$= \sqrt{\sum_{\lambda_i \in \Lambda} \sum_{\lambda_k \in \Lambda} |C(t, \lambda_i, \lambda_k)|^2 \, \|P_k \widehat{y}_0\|^2}. \quad \blacksquare$$

Let j^* be the minimum index $j \in \{1, \ldots, q\}$ such that $Q_j y_0 \neq 0$. The next theorem gives neater bounds for $K_2(t, A, y_0)$.

Theorem 3.3. *We have*

$$K_2(t, A, y_0) \ge \max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \bigcup_{j=j^*}^q \Lambda_j}} |D(t, \lambda_i, \lambda_k)| \, \|P_k \widehat{y}_0\|_2 \, \|A\|_2 \, e^{\left(r_1 - r_{j^*}\right)t}$$

and

$$K_2(t, A, y_0) \le \frac{\sqrt{|\Lambda|}}{\left\| Q_{j^*} \widehat{y}_0 \right\|_2} \|A\|_2 \, e^{(r_1 - r_{j^*})t} t.$$

In the generic situation $j^* = 1$ for y_0, we have

$$\|A\|_2 \, t \le K_2(t, A, y_0) \le \frac{\sqrt{|\Lambda|}}{\|Q_1 \widehat{y}_0\|_2} \|A\|_2 \, t.$$

Proof. By the lower bound (13), we obtain

$$K_2(t, A, y_0) \ge \frac{\max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \bigcup_{j=j^*}^q \Lambda_j}} |D(t, \lambda_i, \lambda_k)| \, \|P_k \widehat{y}_0\|_2}{\sqrt{\sum_{j=j^*}^q \left(e^{(r_j - r_1)t} \left\| Q_j \widehat{y}_0 \right\|_2 \right)^2}} \|A\|_2$$

$$\ge e^{\left(r_1 - r_{j^*}\right)t} \max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \bigcup_{j=j^*}^q \Lambda_j}} |D(t, \lambda_i, \lambda_k)| \, \|P_k \widehat{y}_0\|_2 \, \|A\|_2 \, .$$

By the upper bound (14) and

$$|C(t, \lambda_i, \lambda_k)| \le t \text{ for all } \lambda_i, \lambda_k \in \Lambda \tag{16}$$

(see Proposition 3.1), we obtain

$$K_2(t, A, y_0) \le \frac{\sqrt{\sum_{\lambda_i \in \Lambda} \sum_{\lambda_k \in \Lambda} t^2 \, \|P_k \widehat{y}_0\|^2}}{\sqrt{\sum_{j=1}^q \left(e^{(r_j - r_1)t} \left\| Q_j \widehat{y}_0 \right\|_2 \right)^2}} \|A\|_2$$

$$= \frac{\sqrt{|\Lambda|}}{\sqrt{\sum_{j=1}^q \left(e^{(r_j - r_1)t} \left\| Q_j \widehat{y}_0 \right\|_2 \right)^2}} \|A\|_2 \, t$$

$$\le \frac{\sqrt{|\Lambda|}}{\left\| Q_{j^*} \widehat{y}_0 \right\|_2} \|A\|_2 \, e^{(r_1 - r_{j^*})t} t.$$

For $j^* = 1$ use the lower bound (12). ■

The previous theorem shows a linear growth in t of $K_2(t, A, y_0)$ for $j^* = 1$ and an exponential growth in t of $K_2(t, A, y_0)$ for $j^* > 1$ (observe that

$$\max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \bigcup_{j=j^*}^q \Lambda_j}} |D(t, \lambda_i, \lambda_k)| \, \|P_k \widehat{y}_0\|_2$$

does not vanish as $t \to +\infty$: remind Remark 3.1).

Remark 3.2. In the situation $j^* > 1$, $K_2(t, A, y_0)$ can be arbitrarily larger than the lower bound (12), due to the exponential growth in t of the condition number. For $q > 1$, $K_2(t, A, y_0)$ can be arbitrarily larger than the lower bound (12) also in the situation $j^* = 1$.

In fact, for $q > 1$, the lower bound (13) gives

$$K_2(t, A, y_0) \geq \frac{\max\limits_{\substack{\lambda_i \in \Lambda_1 \\ \lambda_k \in \Lambda \setminus \Lambda_1}} |D(t, \lambda_i, \lambda_k)| \, \|P_k \widehat{y}_0\|_2}{\sqrt{\sum_{j=1}^{q} \left(e^{(r_j - r_1)t} \left\|Q_j \widehat{y}_0\right\|_2\right)^2}} \|A\|_2$$

and the right-hand side of this inequality is a continuous function of $\|Q_1 \widehat{y}_0\|_2$, whose value for $\|Q_1 \widehat{y}_0\|_2 = 0$ is not smaller than

$$e^{(r_1 - r_2)t} \max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \Lambda \setminus \Lambda_1}} |D(t, \lambda_i, \lambda_k)| \, \|P_k \widehat{y}_0\|_2 \, \|A\|_2 \,.$$

Hence, fixed $t \geq 0$, for any $c \in (0, 1)$ we have

$$\overline{K}_2(t, A, y_0) \geq c e^{(r_1 - r_2)t} \max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \Lambda \setminus \Lambda_1}} |D(t, \lambda_i, \lambda_k)| \, \|P_k \widehat{y}_0\|_2 \, \|A\|_2$$

for $\|Q_1 \widehat{y}_0\|_2$ sufficiently small.

This proves what follows. Consider y_0 with fixed projections $P_k \widehat{y}_0$, $\lambda_k \in \Lambda \setminus \Lambda_1$. For any $M > 1$, there exists $t \geq 0$ such that

$$\frac{K_2(t, A, y_0)}{\|A\|_2 \, t} \geq M$$

for $\|Q_1 \widehat{y}_0\|_2$ sufficiently small.

The next results concern the case $q = 1$, namely the case of shifted skew-symmetric matrices.

Theorem 3.4. *If $q = 1$, i.e. A is a shifted skew-symmetric matrix, then*

$$K_2(t, A, y_0) = \|A\|_2 \, t.$$

Proof. For A shifted skew-symmetric, i.e. $A = \alpha I + S$ for some $\alpha \in \mathbb{R}$ and $S \in \mathbb{R}^{n \times n}$ skew-symmetric, by (9) we get

$$K_2(t, A, y_0, \widehat{B}) = \left\| \int_0^t e^{-sA} \widehat{B} e^{sA} ds \widehat{y}_0 \right\|_2 \|A\|_2 \leq \left\| \int_0^t e^{-sA} \widehat{B} e^{sA} ds \right\|_2 \|A\|_2 \,.$$

Now,

$$\int_0^t e^{-sA} \widehat{B} e^{sA} ds = \int_0^t e^{-s\alpha} e^{-sS} \widehat{B} e^{s\alpha} e^{sS} ds = \int_0^t e^{-sS} \widehat{B} e^{sS} ds$$

and then

$$\left\| \int_0^t e^{-sA} \widehat{B} e^{sA} ds \right\|_2 \leq \int_0^t \left\| e^{-sS} \right\|_2 \left\| \widehat{B} \right\|_2 \left\| e^{sS} \right\|_2 ds = t$$

since e^{-sA} and e^{sA} are orthogonal matrices. Thus

$$K_2(t, A, y_0, \widehat{B}) \leq \|A\|_2 \, t.$$

We conclude that

$$K_2(t, A, y_0) \leq \|A\|_2 \, t.$$

The thesis follows by recalling the lower bound (12). ■

When y_0 stays in the rightmost eigenspace, we have the same situation of the case $q = 1$, namely the condition number is equal to $\|A\|_2 t$.

Theorem 3.5. *If $Q_1 y_0 = y_0$, then*

$$K_2(t, A, y_0) = \|A\|_2 \, t.$$

Proof. Assume $Q_1 y_0 = y_0$.

In our discussion we are assuming that A is a normal real matrix, but (9)–(10) also holds when A is a normal complex matrix.

So, now, we consider the case where A is a normal complex matrix with a unique complex eigenvalue λ_1 as rightmost eigenvalue. Since $Q_1 y_0 = P_1 y_0 = y_0$, we have, for the numerator (10) in the right-hand side of (9),

$$\left\| \sum_{\lambda_i \in \Lambda} \sum_{\lambda_k \in \Lambda} C(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2 = \left\| \sum_{\lambda_i \in \Lambda} C(t, \lambda_i, \lambda_1) P_i \widehat{B} \widehat{y}_0 \right\|_2$$
$$= \sqrt{\sum_{\lambda_i \in \Lambda} |C(t, \lambda_i, \lambda_1)|^2 \left\| P_i \widehat{B} \widehat{y}_0 \right\|_2^2} \le t \left\| \widehat{B} \widehat{y}_0 \right\|_2 \le t$$

by recalling (16). Thus, since the denominator in the right-hand side of (9) is 1, we obtain

$$K_2(t, A, y_0, \widehat{B}) \le \|A\|_2 \, t. \tag{17}$$

Now, we pass to consider the case where A is a normal real matrix. Fix a direction of perturbation $\widehat{B}$. For any $\varepsilon > 0$, there exists a normal complex matrix A_ε such that A_ε has a unique complex eigenvalue λ_1 as rightmost eigenvalue,

$$\left| K_2(t, A, y_0, \widehat{B}) - K_2(t, A_\varepsilon, y_0, \widehat{B}) \right| \le \varepsilon$$

and

$$\|A - A_\varepsilon\|_2 \le \varepsilon.$$

Thus

$$\begin{aligned} K_2(t, A, y_0, \widehat{B}) &= K_2(t, A, y_0, \widehat{B}) - K_2(t, A_\varepsilon, y_0, \widehat{B}) + K_2(t, A_\varepsilon, y_0, \widehat{B}) \\ &\le \varepsilon + \|A_\varepsilon\|_2 t \qquad \text{by (17)} \\ &\le \varepsilon + \varepsilon t + \|A\|_2 t. \end{aligned}$$

Since ε is arbitrarily small, we obtain

$$K_2(t, A, y_0, \widehat{B}) \le \|A\|_2 t.$$

By using the lower bound (12), $K_2(t, A, y_0) = \|A\|_2 t$ follows. This is also true when A is a normal complex matrix. ■

Observe that now Theorem 3.4 becomes a corollary of Theorem 3.5.

3.3. The condition number $K_2(t, A)$ independent of y_0

The next theorem gives lower and upper bounds for $K_2(t, A)$.

Theorem 3.6. *We have the lower bound*

$$K_2(t, A) \ge \max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \Lambda_q}} |D(t, \lambda_i, \lambda_k)| \, \|A\|_2 \, e^{(r_1 - r_q)t}. \tag{18}$$

Moreover, we have the upper bound

$$K_2(t, A) \le \sqrt{\max_{\lambda_k \in \Lambda} \sum_{\lambda_i \in \Lambda} |C(t, \lambda_i, \lambda_k)|^2} \, \|A\|_2 \, e^{(r_1 - r_q)t}. \tag{19}$$

Proof. First, we prove the lower bound. For any $\lambda_k \in \Lambda_q$, consider $y_0 \neq 0$ such that $P_k y_0 = y_0$. By (13) we have

$$K_2(t, A) \ge K_2(t, A, y_0) \ge \frac{\max_{\lambda_i \in \Lambda} |D(t, \lambda_i, \lambda_k)|}{e^{(r_q - r_1)t}} \|A\|_2 .$$

Now, we prove the upper bound. For the numerator in (14), we have

$$\sqrt{\sum_{\lambda_i \in \Lambda} \sum_{\lambda_k \in \Lambda} |C(t, \lambda_i, \lambda_k)|^2 \, \|P_k \widehat{y}_0\|^2} \le \sqrt{\max_{\lambda_k \in \Lambda} \sum_{\lambda_i \in \Lambda} |C(t, \lambda_i, \lambda_k)|^2}$$

and for the denominator we have

$$\sqrt{\sum_{j=1}^{q} \left(e^{(r_j - r_1)t} \left\| Q_j \widehat{y}_0 \right\|_2 \right)^2} \ge e^{(r_q - r_1)t}. \quad ■$$

The previous theorem shows that $K_2(t, A)$ grows exponentially in t for $q > 1$. Since (16) holds, the upper bound (19) gives this other neater upper bound.

Theorem 3.7. *We have*

$$K_2(t, A) \le \sqrt{|\Lambda|}\ \ \|A\|_2\, e^{(r_q - r_1)t}\, t.$$

4. Asymptotic analysis

In this section, we study the asymptotic behavior of the three condition numbers $K_2(t, A, y_0, \widehat{B})$, $K_2(t, A, y_0)$ and $K_2(t, A)$, as $t \to +\infty$.

We use the following notations.

- Let j^* be the minimum index $j \in \{1, \ldots, q\}$ such that $Q_j y_0 \neq 0$. This index has been already introduced in Section 3, just before Theorem 3.3.
- Let j^{**} be the minimum index $j \in \{1, \ldots, q\}$ such that $Q_j \widehat{B} P_k y_0 \neq 0$ for some $k \in \{1, \ldots, p\}$.
- For $\lambda_i \in \Lambda_j$ and $\lambda_k \in \Lambda_l$, where $j, l \in \{1, \ldots, q\}$ with $j < l$, let

$$C_\infty(\lambda_i, \lambda_k) := \frac{1}{\lambda_i - \lambda_k}.$$

- For $\lambda_i \in \Lambda_j$ and $\lambda_k \in \Lambda_l$, where $j, l \in \{1, \ldots, q\}$ with $j > l$, let

$$C_\infty(t, \lambda_i, \lambda_k) := \frac{e^{\sqrt{-1}(\omega_k - \omega_i)t}}{\lambda_k - \lambda_i},$$

  where ω_i and ω_k are the imaginary parts of λ_i and λ_k, respectively, and $\sqrt{-1}$ is the imaginary unit.
- For $\lambda_i \in \Lambda_j$ and $\lambda_k \in \Lambda_l$, where $j, l \in \{1, \ldots, q\}$ with $j < l$, let

$$D_\infty(\lambda_i, \lambda_k) := \begin{cases} C_\infty(\lambda_i, \lambda_k) & \text{if } \lambda_k \text{ is real} \\ \frac{\sqrt{2}}{2}\left(C_\infty(\lambda_i, \lambda_k) + C_\infty\left(\lambda_i, \overline{\lambda_k}\right)\right) & \text{if } \lambda_k \text{ is not real.} \end{cases}$$

- $f(t) \sim g(t),\ t \to +\infty$, stands for

$$\lim_{t \to +\infty} \frac{f(t)}{g(t)} = 1.$$

- $f(t) \lesssim g(t),\ t \to +\infty$, stands for

$$f(t) \le h(t), \quad \text{for } t \text{ sufficiently large,}$$

  and

$$h(t) \sim g(t),\ t \to +\infty,$$

  for some function $h(t)$. Similarly, $f(t) \gtrsim g(t),\ t \to +\infty$, stands for

$$f(t) \ge h(t), \quad \text{for } t \text{ sufficiently large,}$$

  and

$$h(t) \sim g(t),\ t \to +\infty,$$

  for some function $h(t)$.

The next proposition, which is a trivial consequence of Proposition 3.1, describes the asymptotic behavior, as $t \to +\infty$, of the functions $C(t, \lambda_i, \lambda_k)$ defined in (11).

Proposition 4.1. *Let $j, l \in \{1, \ldots, q\}$, let $\lambda_i \in \Lambda_j$ and let $\lambda_k \in \Lambda_l$.*

If $j < l$, then

$$C(t, \lambda_i, \lambda_k) \sim e^{(r_j - r_1)t} C_\infty(\lambda_i, \lambda_k),\ t \to +\infty.$$

If $j > l$, then

$$C(t, \lambda_i, \lambda_k) \sim e^{(r_l - r_1)t} C_\infty(t, \lambda_i, \lambda_k),\ t \to +\infty.$$

If $j = l$ and $\lambda_i \neq \lambda_k$, then

$$C(t, \lambda_i, \lambda_k) = e^{(r_j - r_1)t} \frac{e^{\sqrt{-1}(\omega_k - \omega_i)t} - 1}{\lambda_k - \lambda_i},\ t \ge 0.$$

If $j = l$ and $\lambda_i = \lambda_k$, then

$$C(t, \lambda_i, \lambda_k) = e^{(r_j - r_1)t}\, t,\ t \ge 0.$$

4.1. Asymptotic analysis of the condition number $K_2(t, A, y_0, \widehat{B})$ with direction of perturbation

The next theorem describes the asymptotic behavior of $K_2(t, A, y_0, \widehat{B})$, as $t \to +\infty$.

Theorem 4.1. *If $j^{**} < j^*$, then*

$$K_2(t, A, y_0, \widehat{B}) = \frac{\left\| \sum_{\lambda_i \in \Lambda_{j^{**}}} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C_\infty(\lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2}{\left\| Q_{j^*} \widehat{y}_0 \right\|_2} \|A\|_2 \, e^{(r_{j^{**}} - r_{j^*})t} + o(e^{(r_{j^{**}} - r_{j^*})t}), \quad t \to +\infty.$$

If $j^{**} = j^*$, then

$$K_2(t, A, y_0, \widehat{B}) = \frac{\left\| \sum_{\lambda_i \in \Lambda_{j^*}} P_i \widehat{B} P_i \widehat{y}_0 \right\|_2}{\left\| Q_{j^*} \widehat{y}_0 \right\|_2} \|A\|_2 \, t + o(t), \quad t \to +\infty.$$

If $j^{**} > j^*$, then

$$K_2(t, A, y_0, \widehat{B}) = \frac{\left\| \sum_{j=j^{**}}^{q} \sum_{\lambda_i \in \Lambda_j} \sum_{\lambda_k \in \Lambda_{j^*}} C_\infty(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2}{\| Q_{j^*} \widehat{y}_0 \|_2} \|A\|_2 + o(1), \quad t \to +\infty. \tag{20}$$

Proof. We write (9)–(10) as

$$K_2(t, A, y_0, B) = \frac{\left\| \sum_{j=j^{**}}^{q} \sum_{\lambda_i \in \Lambda_j} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2}{\sqrt{\sum_{j=j^*}^{q} \left(e^{(r_j - r_1)t} \left\| Q_j \widehat{y}_0 \right\|_2 \right)^2}} \|A\|_2 \, . \tag{21}$$

Consider the numerator in (21). If $j^{**} < j^*$, then, by Proposition 4.1, the major contributory terms $C(t, \lambda_i, \lambda_k)$ as $t \to +\infty$ are obtained for $j = j^{**}$ and then

$$\left\| \sum_{j=j^{**}}^{q} \sum_{\lambda_i \in \Lambda_j} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2 = e^{(r_{j^{**}} - r_1)t} \left\| \sum_{\lambda_i \in \Lambda_{j^{**}}} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C_\infty(\lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2 + o(e^{(r_{j^{**}} - r_{j^*})t}), \quad t \to +\infty.$$

If $j^{**} = j^*$, then the major contributory terms $C(t, \lambda_i, \lambda_k)$ as $t \to +\infty$ are obtained for $j = l = j^*$ and $\lambda_i = \lambda_k$ and then

$$\left\| \sum_{j=j^{**}}^{q} \sum_{\lambda_i \in \Lambda_j} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2 = e^{(r_{j^*} - r_1)t} t \left\| \sum_{\lambda_i \in \Lambda_{j^*}} P_i \widehat{B} P_i \widehat{y}_0 \right\|_2 + o\left(e^{(r_{j^*} - r_1)t} t \right), \quad t \to +\infty.$$

If $j^{**} > j^*$, the major contributory terms $C(t, \lambda_i, \lambda_k)$ as $t \to +\infty$ are obtained for $l = j^*$ and then

$$\left\| \sum_{j=j^{**}}^{q} \sum_{\lambda_i \in \Lambda_j} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2 = e^{(r_{j^*} - r_1)t} \left\| \sum_{j=j^{**}}^{q} \sum_{\lambda_i \in \Lambda_j} \sum_{\lambda_k \in \Lambda_{j^*}} C_\infty(t, \lambda_i, \lambda_k) P_i \widehat{B} P_k \widehat{y}_0 \right\|_2 + o\left(e^{(r_{j^*} - r_1)t} \right) \quad t \to +\infty.$$

Consider the denominator in (21). The major contributory term as $t \to +\infty$ is $e^{(r_{j^*}-r_1)t} \left\| Q_{j^*}\widehat{y}_0 \right\|_2$ and then

$$\sqrt{\sum_{j=j^*}^{q} \left(e^{(r_j-r_1)t} \left\| Q_j\widehat{y}_0 \right\|_2\right)^2} \sim e^{(r_{j^*}-r_1)t} \left\| Q_{j^*}\widehat{y}_0 \right\|_2, \quad t \to +\infty.$$

Now, the theorem follows. ■

Remark 4.1. Observe that the generic situation for the initial value y_0 and the direction of the perturbation $\widehat{B}$ is $j^* = 1$, $j^{**} = 1$ and

$$\sum_{\lambda_i \in \Lambda_1} P_i\widehat{B}P_i\widehat{y}_0 \neq 0,$$

where we have

$$K_2(t, A, y_0, \widehat{B}) \sim \frac{\left\| \sum_{\lambda_i \in \Lambda_1} P_i\widehat{B}P_i\widehat{y}_0 \right\|_2}{\|Q_1\widehat{y}_0\|_2} \|A\|_2\, t, \quad t \to +\infty.$$

In the non-generic situation $j^* > 1$ or $j^{**} > 1$, the previous theorem shows that:

- if $j^{**} < j^*$ and
$$\sum_{\lambda_i \in \Lambda_{j^{**}}} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C_\infty\left(\lambda_i, \lambda_k\right) P_i\widehat{B}P_k\widehat{y}_0 \neq 0,$$
then $K_2(t, A, y_0, \widehat{B})$ grows exponentially in t as $t \to +\infty$:
$$K_2(t, A, y_0, \widehat{B}) \sim \frac{\left\| \sum_{\lambda_i \in \Lambda_{j^{**}}} \sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} C_\infty\left(\lambda_i, \lambda_k\right) P_i\widehat{B}P_k\widehat{y}_0 \right\|_2}{\left\| Q_{j^*}\widehat{y}_0 \right\|_2} \|A\|_2\, e^{(r_{j^{**}}-r_{j^*})t}$$
$$t \to +\infty;$$
- if $j^{**} = j^*$ and
$$\sum_{\lambda_i \in \Lambda_{j^*}} P_i\widehat{B}P_i\widehat{y}_0 \neq 0,$$
then $K_2(t, A, y_0, \widehat{B})$ grows linearly in t as $t \to +\infty$:
$$K_2(t, A, y_0, \widehat{B}) \sim \frac{\left\| \sum_{\lambda_i \in \Lambda_{j^*}} P_i\widehat{B}P_i\widehat{y}_0 \right\|_2}{\left\| Q_{j^*}\widehat{y}_0 \right\|_2} \|A\|_2\, t, \quad t \to +\infty;$$
- if $j^{**} > j^*$, then $K_2(t, A, y_0, \widehat{B})$ oscillates (due to the terms $C_\infty(t, \lambda_i, \lambda_k)$ in (20)), but it remains bounded as $t \to +\infty$.

4.2. Asymptotic analysis of the condition number $K_2(t, A, y_0)$

The next theorem describes the asymptotic behavior of $K_2(t, A, y_0)$, as $t \to +\infty$.

Theorem 4.2. *If $j^* = 1$, we have*

$$K_2(t, A, y_0) \sim \|A\|_2\, t, \quad t \to +\infty. \tag{22}$$

If $j^* > 1$, we have the asymptotic lower bound

$$K_2(t, A, y_0) \gtrsim \frac{\max\limits_{\substack{\lambda_i \in \Lambda_1 \\ \lambda_k \in \bigcup_{j=j^*}^{q} \Lambda_j}} \left|D_\infty\left(\lambda_i, \lambda_k\right)\right| \|P_k y_0\|_2}{\left\| Q_{j^*}\widehat{y}_0 \right\|_2} \|A\|_2\, e^{\left(r_1 - r_{j^*}\right)t}$$
$$t \to +\infty \tag{23}$$

and the asymptotic upper bound

$$K_2(t, A, y_0) \lesssim \frac{\sqrt{\sum_{l=j^*}^{q} \sum_{\lambda_k \in \Lambda_l} \left(\sum_{\lambda_i \in \Lambda_1} \left|C_\infty\left(\lambda_i, \lambda_k\right)\right|^2\right) \|P_k\widehat{y}_0\|^2}}{\left\| Q_{j^*}\widehat{y}_0 \right\|_2} \|A\|_2\, e^{\left(r_1 - r_{j^*}\right)t}$$
$$t \to +\infty. \tag{24}$$

Proof. We write the right-hand side of the upper bound (14) as

$$\frac{\sqrt{\sum_{j=1}^{q}\sum_{\lambda_i\in\Lambda_j}\sum_{l=j^*}^{q}\sum_{\lambda_k\in\Lambda}|C(t,\lambda_i,\lambda_k)|^2\,\|P_k\widehat{y}_0\|^2}}{\sqrt{\sum_{j=j^*}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}}\,\|A\|_2\,. \tag{25}$$

Suppose $j^* = 1$. The major contributory terms $C(t,\lambda_i,\lambda_k)$ as $t \to +\infty$ in the numerator of (25) are obtained for $j = l = 1$ and $\lambda_i = \lambda_k$ and then

$$\sqrt{\sum_{j=1}^{q}\sum_{\lambda_i\in\Lambda_j}\sum_{l=j^*}^{q}\sum_{\lambda_k\in\Lambda_l}|C(t,\lambda_i,\lambda_k)|^2\,\|P_k\widehat{y}_0\|_2^2}$$
$$\sim t\,\|Q_1\widehat{y}_0\|_2\,,\ \ t\to+\infty.$$

The major contributory term in the denominator of (25) is $\|Q_1\widehat{y}_0\|_2$ and then

$$\sqrt{\sum_{j=1}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}\sim\|Q_1\widehat{y}_0\|_2\,,\ \ t\to+\infty.$$

Thus

$$K_2(t,A,y_0)\lesssim\|A\|_2\,t,\ \ t\to+\infty.$$

Now, (22) follows by the lower bound (12).

Suppose $j^* > 1$. The major contributory terms $C(t,\lambda_i,\lambda_k)$ as $t \to +\infty$ in the numerator of (25) are obtained for $j = 1$ and then

$$\sqrt{\sum_{j=1}^{q}\sum_{\lambda_i\in\Lambda_j}\sum_{l=j^*}^{q}\sum_{\lambda_k\in\Lambda_l}|C(t,\lambda_i,\lambda_k)|^2\,\|P_k\widehat{y}_0\|_2^2}$$
$$\sim\sqrt{\sum_{l=j^*}^{q}\sum_{\lambda_k\in\Lambda_l}\left(\sum_{\lambda_i\in\Lambda_1}|C_\infty(\lambda_i,\lambda_k)|^2\right)\|P_k\widehat{y}_0\|^2},\ \ t\to+\infty.$$

The major contributory term in the numerator of (refeqrefeqref) is $e^{(r_{j^*}-r_1)t}\left\|Q_{j^*}\widehat{y}_0\right\|_2$ and then

$$\sqrt{\sum_{j=1}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}\sim e^{(r_{j^*}-r_1)t}\left\|Q_{j^*}\widehat{y}_0\right\|_2\,,\ \ t\to+\infty.$$

Now, the asymptotic upper bound (24) follows.

Finally, we prove the asymptotic lower bound (23). By the lower bound (13), we have

$$K_2(t,A,y_0)\geq\frac{\max\limits_{\substack{\lambda_i\in\Lambda\\ \lambda_k\in\bigcup_{j=j^*}^{q}\Lambda_j}}|D\,(t,\lambda_i,\lambda_k)|\,\|P_k\widehat{y}_0\|_2}{\sqrt{\sum_{j=1}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}}\,\|A\|_2\,.$$

By Proposition 4.1, we obtain

$$\max_{\substack{\lambda_i\in\Lambda\\ \lambda_k\in\bigcup_{j=j^*}^{q}\Lambda_j}}|D\,(t,\lambda_i,\lambda_k)|\,\|P_k\widehat{y}_0\|_2\sim\max_{\substack{\lambda_i\in\Lambda_1\\ \lambda_k\in\bigcup_{j=j^*}^{q}\Lambda_j}}|D_\infty\,(\lambda_i,\lambda_k)|\,\|P_k\widehat{y}_0\|_2\,,\ \ t\to+\infty.$$

Thus

$$\frac{\max\limits_{\substack{\lambda_i\in\Lambda\\ \lambda_k\in\bigcup_{j=j^*}^{q}\Lambda_j}}|D\,(t,\lambda_i,\lambda_k)|\,\|P_k\widehat{y}_0\|_2}{\sqrt{\sum_{j=1}^{q}\left(e^{(r_j-r_1)t}\left\|Q_j\widehat{y}_0\right\|_2\right)^2}}\,\|A\|_2$$
$$\sim\frac{\max\limits_{\substack{\lambda_i\in\Lambda_1\\ \lambda_k\in\bigcup_{j=j^*}^{q}\Lambda_j}}|D_\infty\,(\lambda_i,\lambda_k)|\,\|P_k\widehat{y}_0\|_2}{e^{\left(r_{j^*}-r_1\right)t}\left\|Q_{j^*}\widehat{y}_0\right\|_2}\,\|A\|_2\,,\ \ t\to+\infty,$$

and the asymptotic lower bound follows. ■

Remark 4.2. The generic situation for the initial value y_0 is $j^* = 1$, where we have the asymptotic behavior (22).

It is interesting to observe that, for the problem (3), the condition number relevant to the norm $\|\!|\cdot|\!\| = \|\cdot\|_2$ on $\mathbb{R}^{n\times n}$ is $\|A\|_2\, t$ in case of a normal matrix A (see [4]). So, asymptotically as $t \to +\infty$, the condition numbers of the problems (2) and (3) are equal for a normal matrix in the generic situation $j^* = 1$ for y_0.

Remark 4.3. In the non-generic situation $j^* > 1$ for y_0, the previous theorem says that

$$K_2(t, A, y_0) = O\left(e^{\left(r_1 - r_{j^*}\right)t}\right), \quad t \to +\infty$$

$$\frac{1}{K_2(t, A, y_0)} = O\left(e^{-\left(r_1 - r_{j^*}\right)t}\right), \quad t \to +\infty.$$

We also have

$$\log K_2(t, A, y_0) \sim \left(r_1 - r_{j^*}\right) t, \quad t \to +\infty.$$

4.3. Asymptotic analysis of the condition number $K_2(t, A)$ independent of y_0

The next theorem describes the asymptotic behavior of $K_2(t, A)$, as $t \to +\infty$.

Theorem 4.3. *We have the asymptotic lower bound*

$$K_2(t, A) \gtrsim \max_{\substack{\lambda_i \in \Lambda_1 \\ \lambda_k \in \Lambda_q}} |D_\infty(\lambda_i, \lambda_k)|\, \|A\|_2\, e^{\left(r_1 - r_q\right)t}, \quad t \to +\infty,$$

and the asymptotic upper bound

$$K_2(t, A) \lesssim \|A\|_2\, e^{\left(r_1 - r_q\right)t}\, t, \quad t \to +\infty.$$

Proof. By the lower bound (18), we have

$$\begin{aligned} K_2(t, A) &\geq \max_{\substack{\lambda_i \in \Lambda \\ \lambda_k \in \Lambda_q}} |D(t, \lambda_i, \lambda_k)|\, \|A\|_2\, e^{\left(r_1 - r_q\right)t} \\ &\sim \max_{\substack{\lambda_i \in \Lambda_1 \\ \lambda_k \in \Lambda_q}} |D_\infty(\lambda_i, \lambda_k)|\, \|A\|_2\, e^{\left(r_1 - r_q\right)t}, \quad t \to +\infty. \end{aligned}$$

By the upper bound (19), we have

$$\begin{aligned} K_2(t, A) &\leq \sqrt{\max_{\lambda_k \in \Lambda} \sum_{\lambda_i \in \Lambda} |C(t, \lambda_i, \lambda_k)|^2}\, \|A\|_2\, e^{\left(r_1 - r_q\right)t} \\ &\sim \|A\|_2\, e^{\left(r_1 - r_q\right)t}\, t, \quad t \to +\infty. \quad \blacksquare \end{aligned}$$

Remark 4.4. The previous theorem says that

$$K_2(t, A) = O\left(e^{\left(r_1 - r_q\right)t}\, t\right), \quad t \to +\infty$$

$$\frac{1}{K_2(t, A)} = O\left(e^{-\left(r_1 - r_q\right)t}\right), \quad t \to +\infty.$$

We also have

$$\log K_2(t, A) \sim \left(r_1 - r_q\right) t, \quad t \to +\infty.$$

5. Numerical tests

The numerical tests involve the condition number $K_2(t, A, y_0)$. We consider skew symmetric matrices in Example 5.1, with the aim to confirm Theorem 3.4, and symmetric matrices in Example 5.2, with the aim to confirm Theorem 4.2.

Example 5.1. Consider the following two cases of a skew symmetric matrix A in (1):

- the 2×2 matrix

  $$A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix},$$

  which has the pair of pure imaginary eigenvalues $\pm 3\sqrt{-1}$;

- the 4×4 matrix

$$A = \begin{bmatrix} 0 & 2 & -1 & 3 \\ -2 & 0 & -4 & 1 \\ 1 & 4 & 0 & 2 \\ -3 & -1 & -2 & 0 \end{bmatrix}$$

which has the two pairs of pure imaginary eigenvalues $\pm 5.7913\sqrt{-1}$ and $\pm 1.2087\sqrt{-1}$.

In Fig. 2, for both the skew symmetric matrices and for any t in a uniform mesh over the interval [0, 50], we plot the maximum of the values

$$\frac{\frac{\xi(t)}{\epsilon}}{\|A\|_2\, t} = \frac{K_2(t, A, y_0, \widehat{B})}{\|A\|_2\, t} + o(1), \;\; \epsilon \to 0, \tag{26}$$

over 10 000 random selections of the matrix $\widehat{B}$. We consider the initial values $y_0 = (1, 2)$ for the 2×2 matrix and $y_0 = (1, 2, 3, 4)$ for the 4×4 matrix. We take $\epsilon = 10^{-4}$.

For both matrices, as t varies, the maximum of the values (26) is always close to 1, confirming Theorem 3.4.

For the matrix 2×2, we observe a slight deviation from 1 as t increases. This is due to the error $o(1)$, as $\epsilon \to 0$, in (26).

The maximum values for the 2×2 matrix are closer to 1 than the maximum values for the matrix 4×4. This is due to the fact that much more than 10 000 random selections of the matrix $\widehat{B}$ are necessary for having maximum values very close to 1, in case of the matrix 4×4.

Example 5.2. Consider the following two cases of a symmetric matrix A in (1):

- the 2×2 matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix},$$

which has the eigenvalues -1 and -3;

- the 4×4 matrix

$$A = 1/2 \begin{bmatrix} -1 & 2 & 1 & 0 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & -1 & -2 \\ 0 & -1 & -2 & -1 \end{bmatrix},$$

which has the eigenvalues 1, 0, -1 and -2.

In Fig. 3, for both the symmetric matrices and for any t in a uniform mesh over the interval [0, 15], we plot the maximum of the values (26) over 10 000 random selections of matrix $\widehat{B}$. We consider the initial values $y_0 = (1, 2)$ for the 2×2 matrix and $y_0 = (1, 2, 3, 4)$ for the 4×4 matrix. For such initial values we have $j^* = 1$ (the index j^* is defined at the beginning of Section 4). We take $\epsilon = 10^{-4}$.

For both matrices, as t varies, the maximum of the values (26) tends asymptotically to 1, after an initial hump. This confirms Theorem 4.2, case $j^* = 1$. About the initial hump, see Remark 3.2.

In Fig. 4, for the 2×2 matrix and for any t in a uniform mesh over the interval [0, 15], we plot the maximum of the values (26) over 10 000 random selections of matrix $\widehat{B}$, when the initial values are $y_0 = (1, 1)$, which is eigenvector of the rightmost eigenvalue -1, and $y_0 = (1, -1)$, which is eigenvector of the other eigenvalue -3. We take $\epsilon = 10^{-4}$.

For the initial value $y_0 = (1, 1)$, as t varies, the maximum of the values (26) is always close to 1. Since y_0 stays in the rightmost eigenspace, we have the same situation of the case $q = 1$, namely the condition number is equal to $\|A\|_2 t$ (see Theorem 3.5).

For the initial value $y_0 = (1, -1)$, as t varies, the maximum of the values (26) does not tend asymptotically to 1, but it grows indefinitely, by confirming Theorem 4.2, case $j^* > 1$.

In Fig. 5, for the 4×4 matrix and for t in a uniform mesh over the interval [0, 15], we plot the maximum of the values

$$\frac{\log \frac{\xi(t)}{\varepsilon}}{(r_1 - r_{j^*})t} = \frac{\log K_2(t, A, y_0, \widehat{B})}{(r_1 - r_{j^*})t} + o(1), \;\; \epsilon \to 0, \tag{27}$$

over 10 000 random selections of matrix $\widehat{B}$. We consider the initial values $y_0 = (1, 1, -1, 1)$, which is eigenvector of the eigenvalue 0, $y_0 = (-1, 1, -1, -1)$, which is eigenvector of the eigenvalue -1, and $y_0 = (1, -1, -1, -1)$, which is eigenvector of the eigenvalue -2. For these three initial values, we have $j^* = 2, 3, 4$, respectively.

For all initial values, as t varies, the maximum of the values (27) tends asymptotically to 1, by confirming Remark 4.3. In the lower part, the final red points at the right are numerical artifacts.

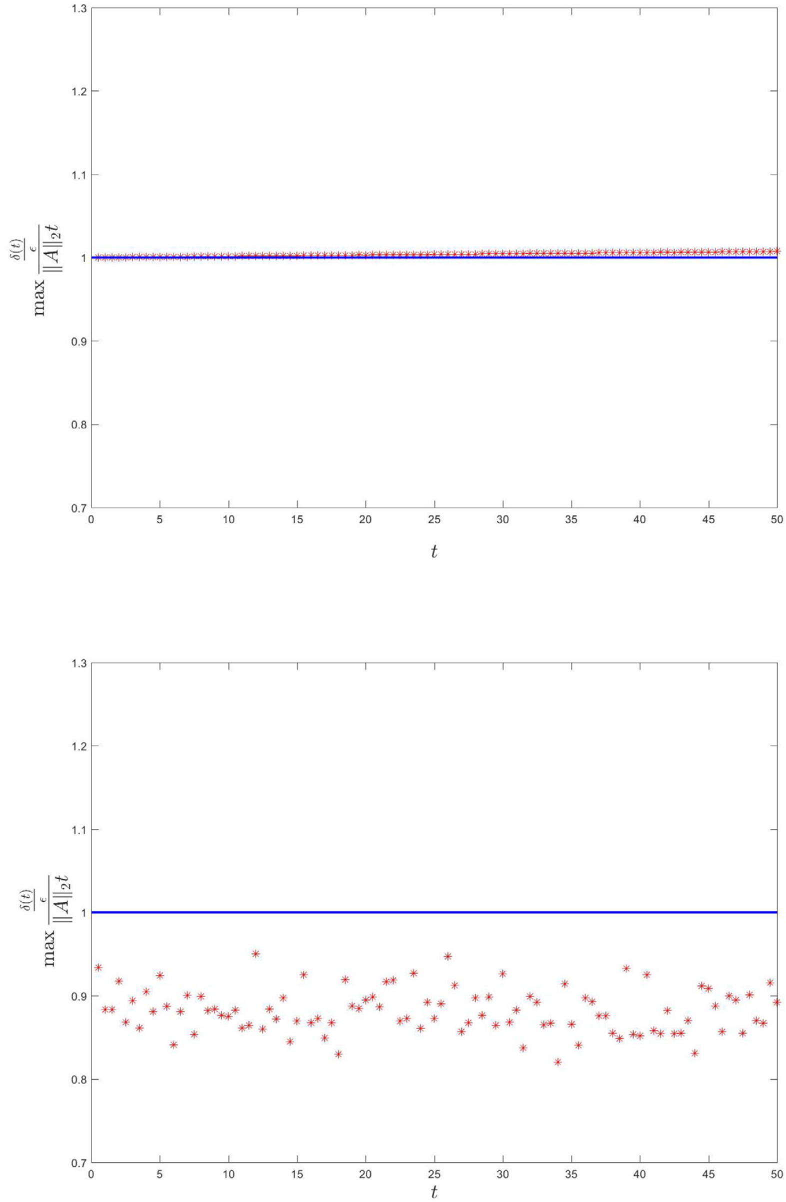


Fig. 2. For the skew symmetric matrices of Example 5.1, maximum value of $\frac{\frac{\xi(t)}{\epsilon}}{\|A\|_2 t}$ over 10 000 random selections the matrix $\widehat{B}$, for any t varying from 0 to 50 with step 0.5. The maximum values are the red points. The blue line is the constant value 1. Upper part: 2×2 matrix. Lower part: 4×4 matrix.

We conclude this section by illustrating the procedure of the random selection of the matrix $\widehat{B}$, namely the random selection of the direction of perturbation.

Fixed the order n of the matrix, we construct the Singular Value Decomposition

$$\widehat{B} = UTV$$

of the matrix $\widehat{B}$, where U and V are $n \times n$ randomly selected orthonormal matrices and T is a $n \times n$ diagonal matrix with diagonal $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, where $\sigma_1 = 1$ and $\sigma_2, \ldots, \sigma_n \in [0, 1]$ are randomly selected. Our computations are implemented in MATLAB and, for the random selections of U, V and T, we use:

$$U = orth(rand(n))$$
$$V = orth(rand(n))$$
$$T = diag([1, rand(1, n-1)]),$$

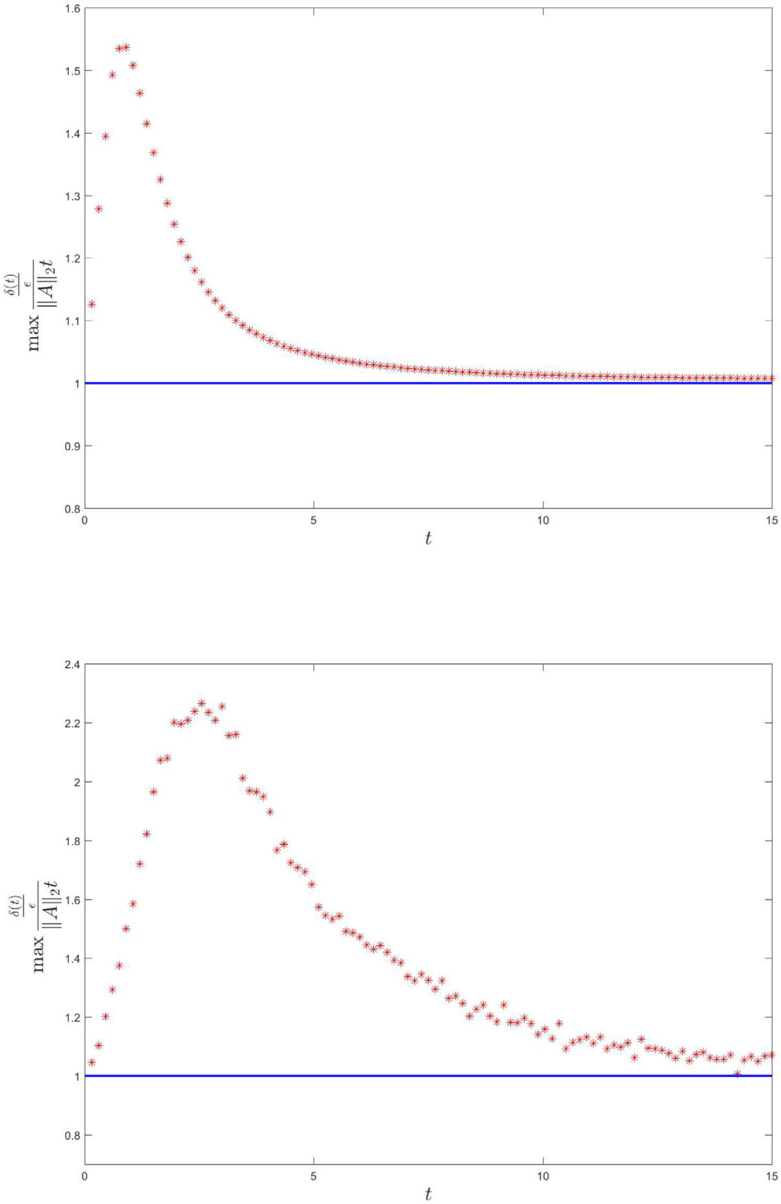


Fig. 3. For the symmetric matrices of Example 5.2, maximum value of $\frac{\frac{\xi(t)}{\epsilon}}{\|A\|_2 t}$ over 10 000 random selections of the matrix $\widehat{B}$, for any t varying from 0 to 15 with step 0.15. The maximum values are the red points. The blue line is the constant value 1. Upper part: 2×2 matrix. Lower part: 4×4 matrix.

where the MATLAB function orth(C) computes a matrix whose columns are an orthonormal basis of the range of C, and the MATLAB function rand(p, q) computes a $p \times q$ matrix of uniformly distributed elements in [0, 1].

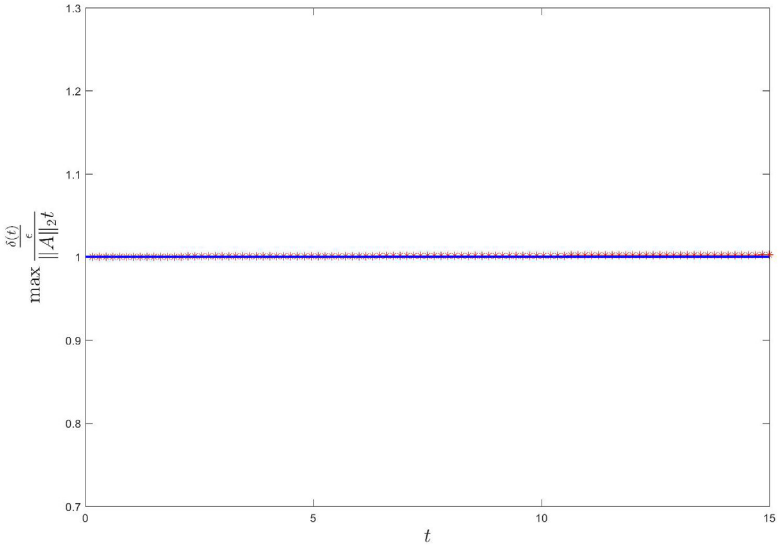
By constructing the matrix $\widehat{B}$ as

$$\widehat{B} = \frac{B}{\|B\|_2},$$

where B is obtained in MATLAB by

$$B = rand(n, n),$$

does not give good results, since this procedure misses some directions of perturbation.



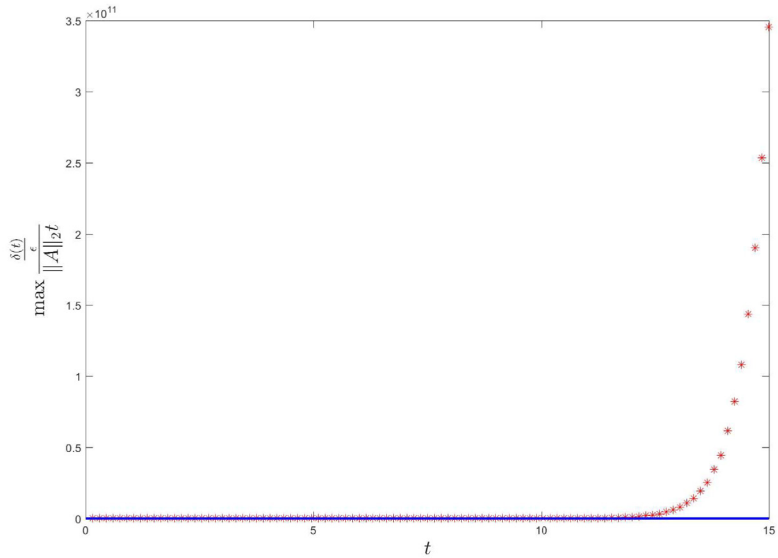


Fig. 4. For the 2×2 symmetric matrix of Example 5.2, maximum value of $\frac{\frac{\xi(t)}{\epsilon}}{\|A\|_2 t}$ over 10 000 random selections the matrix $\widehat{B}$, for any t varying from 0 to 15 with step 0.15. The maximum values are the red points. The blue line is the constant value 1. Upper part: $y_0 = (1, 1)$. Lower part: $y_0 = (1, -1)$.

6. Conclusion

In this paper, we have studied the conditioning of the problem

$$A \mapsto e^{tA} y_0,$$

namely how a perturbation in the matrix $A \in \mathbb{R}^{n \times n}$ propagates to $e^{tA} y_0$. We have considered the case of a normal matrix A, perturbed to a possibly non-normal matrix, and three condition numbers have been analyzed:

- the condition number $K_2(t, A, y_0, \widehat{B})$ with direction of perturbation defined in (4);
- the condition number $K_2(t, A, y_0)$ defined in (6);
- the condition number $K_2(t, A)$ independent of y_0 defined in (8).

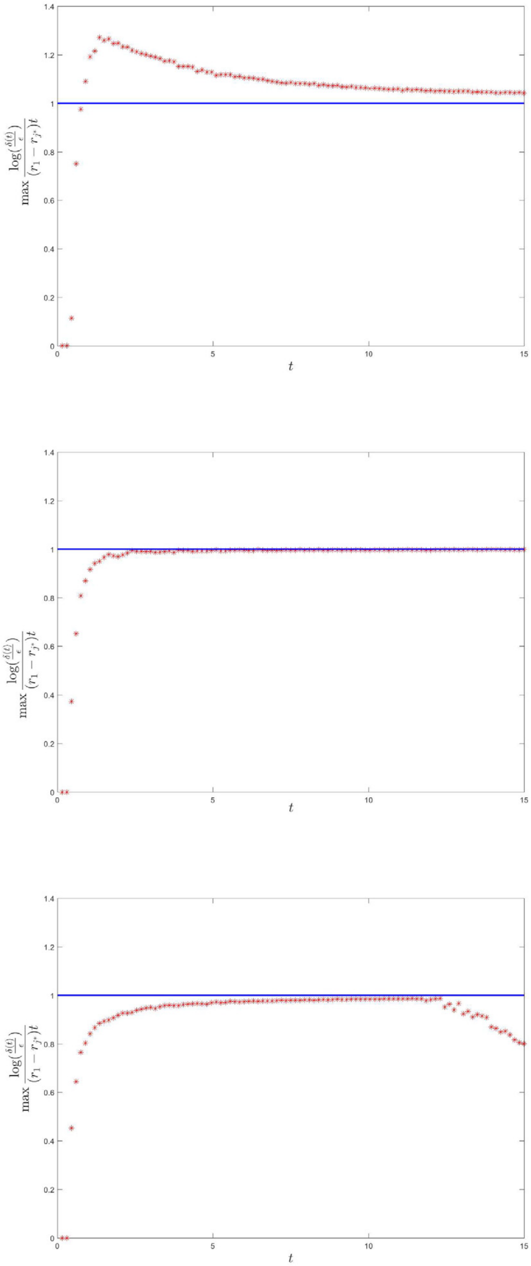


Fig. 5. For the 4×4 matrix of Example 5.2, maximum value of $\frac{\log \frac{\xi(t)}{\epsilon}}{(r_j - r_{j^*})t}$ over 10 000 random selections the matrix $\widehat{B}$, for any t varying from 0 to 15 with step 0.15. The maximum values are the red points. The blue line is the constant value 1. Upper part: $y_0 = (1, 1, -1, 1)$ and $j^* = 2$. Middle part: $y_0 = (-1, 1, -1, -1)$ and $j^* = 3$. Lower part: $y_0 = (1, -1, -1, -1)$ and $j^* = 4$.

The spectrum of the normal matrix A has been partitioned by decreasing real parts in the subsets $\Lambda_1, \ldots, \Lambda_q$, where the eigenvalues in Λ_j, $j = 1, \ldots, q$, have real part r_j, and $r_1 > \cdots > r_q$ holds. We have denoted by j^* the minimum index in $\{1, \ldots, q\}$ such that y_0 has a non-zero component on the sum of the eigenspaces relevant to the eigenvalues in Λ_j. The generic situation for y_0 is $j^* = 1$.

Regarding the condition number $K_2(t, A, y_0)$, we have obtained the following results:

- if A is shifted skew-symmetric, then $K_2(t, A, y_0)$ equals $\|A\|_2 t$.
- If A is not shifted skew-symmetric and $j^* = 1$, then $K_2(t, A, y_0)$ asymptotically, as $t \to +\infty$, equals $\|A\|_2 t$.
- If A is not shifted skew symmetric and $j^* > 1$, then $K_2(t, A, y_0)$ grows exponentially in t and $\log K_2(t, A, y_0)$ asymptotically, as $t \to +\infty$, equals $(r_1 - r_{j^*})t$.

Regarding the condition number $K_2(t, A)$ independent of y_0, we have obtained the following result:

- $K_2(t, A)$ grows exponentially in t and $\log K_2(t, A)$ asymptotically, as $t \to +\infty$, equals $(r_1 - r_q)t$.

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