MODELS FOR DAMPED WATER WAVES*

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Abstract. In this paper we derive some new weakly nonlinear asymptotic models describing viscous waves in deep water with or without surface tension effects. These asymptotic models take into account several different dissipative effects and are obtained from the free boundary problems formulated in the works of Dias, Dyachenko, and Zakharov [*Phys. Lett. A*, 372 (2008), pp. 1297–1302], Jiang et al. [*J. Fluid Mech.*, 329 (1996), pp. 275–307], and Wu, Liu and Yue [*J. Fluid Mech.*, 556 (2006), pp. 45–54].

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1. Introduction. The motion of a free boundary irrotational and incompressible flow is a classical research topic [40]. In most applications, the flow is also assumed to be inviscid [7, 27].

However, even if in most situations in coastal engineering the assumption of inviscid flow leads to very accurate results, there are other physical scenarios where the viscosity needs to be taken into account. Moreover, there are many situations in which the viscosity is very large and the vorticity is small and its effect negligible. Actually, certain discrepancies between experiments and inviscid theory have been previously reported in the literature. For instance, Wu [44] found that

> *"From this comparison the theory appears quite satisfactory in predicting the wave phases during the inward focusing and the subsequent reflection within a radial distance as far as* $r = 20$*, while the peak amplitudes observed in the experiments are slightly smaller than those predicted by the theory. This discrepancy can be ascribed to the neglect of the viscous effects in the theory and to the approximation that the initial wave generated in the tank was not cylindrical in shape and departed slightly from a perfect solitary wave profile in the experiment."*

In addition to this, Zabusky and Galvin [45] wrote

> *"A laboratory-data/numerical-solution comparison of the number of crests and troughs and their phases (or relative locations within a period) shows only negligible difference. As one expects, the crest-to-trough amplitudes differ somewhat more because they are more sensitive to dissipative forces. To quantify some of the details we recommend a study including dissipation."*

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and, furthermore, Longuet-Higgins [29] stated that

> *"For certain applications, however, viscous damping of the waves is important, and it would be highly convenient to have equations and boundary conditions of comparable simplicity as for undamped waves."*

The purpose of this paper is to derive new weakly nonlinear asymptotic models (in the spirit of [6, 16, 17, 18, 30, 31, 32, 36]) describing damped water waves and, at the same time, keeping the features of potential flows. We observe that, at first sight, the idea of viscous damping of potential flows is somehow paradoxical since the hypothesis of irrotational velocity implies that the viscous term in the Navier–Stokes equations vanishes.

The problem of describing the motion of an irrotational, incompressible, inviscid, and homogeneous fluid with a free surface in two dimensions is known as the two-dimensional water waves problem. The equations for the water waves problem are [46]

$$\Delta\phi = 0 \qquad \text{in } \Omega(t)\times[0,T], \tag{1.1a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) - \gamma\mathcal{K} = 0 \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.1b}$$

$$h_t = \nabla\phi\cdot\left(1+(\partial_1 h)^2\right)^{1/2} n \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.1c}$$

where G stands for the gravity force,

$$\Omega(t) = \left\{(x_1,x_2)\in\mathbb{R}^2 \;\middle|\; -L\pi < x_1 < L\pi\,, -\infty < x_2 < h(x_1,t)\,,\ t\in[0,T]\right\}, \tag{1.2}$$

$$\Gamma(t) = \left\{(x_1,h(x_1,t))\in\mathbb{R}^2 \;\middle|\; x_1\in\mathbb{S}^1\,,\ t\in[0,T]\right\} \tag{1.3}$$

are the the region occupied by the fluid and the surface wave, respectively. We write

$$n = (-\partial_1 h, 1)/\left(1+(\partial_{x_1}h)^2\right)^{1/2}$$

for the unit normal to the surface wave, $2L\pi$ to denote the characteristic wavelength of the surface wave, ϕ for the scalar potential of the flow, i.e., the velocity field u satisfies $u=\nabla\phi$, ρ is the density of the fluid, γ for the surface tension coefficient, and

$$\mathcal{K} = \frac{\partial_1^2 h}{\left(1+(\partial_1 h)^2\right)^{3/2}}$$

is the curvature of the surface wave.

The first attempts to include viscosity effects go back as far as to the works of Boussinesq [5] and Lamb [26]. Later on, Ruvinsky and Freidman [39] formulated a system of equations for weakly damped surfaces waves in deep water and used this system to compute capillary-gravity ripples riding on the forward face of steep gravity waves (see also [38]). Then, these first results were generalized by Ruvinsky, Feldstein and Freidman [37] and the following system is proposed:

$$\Delta\phi = 0 \qquad \text{in } \Omega(t)\times[0,T], \tag{1.4a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) - \gamma\mathcal{K} = -2\mu\partial_2^2\phi \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.4b}$$

$$h_t = \nabla\phi\cdot\left(1+(\partial_1 h)^2\right)^{1/2} n + v \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.4c}$$

$$v_t = \partial_1^2\partial_2\phi \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.4d}$$

where v and μ denote the vertical component of the vortex part of fluid velocity and the dynamic viscosity. Equation (1.4) was also studied by Kharif, Skandrani, and Poitevin [25].

Using a clever change of variables, Longuet-Higgins [29] simplified the previous system and obtained

$$\Delta\phi = 0 \qquad \text{in } \Omega(t)\times[0,T], \tag{1.5a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) - \gamma\mathcal{K} = -4\mu\partial_n^2\phi \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.5b}$$

$$h_t = \nabla\phi\cdot\left(1+(\partial_1 h)^2\right)^{1/2} n \qquad \text{on } \Gamma(t)\times[0,T]. \tag{1.5c}$$

A similar model was also studied by Jiang et al. [21] and Wu, Liu, and Yue [43], namely,

$$\Delta\phi = 0 \qquad \text{in } \Omega(t)\times[0,T], \tag{1.6a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) - \gamma\mathcal{K} = -\delta\mathscr{D}^s\phi \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.6b}$$

$$h_t = \nabla\phi\cdot\left(1+(\partial_1 h)^2\right)^{1/2} n \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.6c}$$

where the dissipative terms are chosen as

$$\mathscr{D}^2\phi = \partial_2^2\phi \text{ or } \mathscr{D}^0\phi = \phi. \tag{1.7}$$

Another similar model where the dissipation acts only on the velocity is the one by Joseph and Wang [22, equations (6.7) and (6.8)] (see also Wang and Joseph [42]).

We would like to remark that, in the models of damped water waves mentioned so far, there are no dissipative effects acting on the free surface.

In a more recent paper, Dias, Dyachenko, and Zakharov [10] proposed a system where the free surface experiences dissipative effects. In particular, based on the linear problem, these authors derived

$$\Delta\phi = 0 \qquad \text{in } \Omega(t)\times[0,T], \tag{1.8a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) = -2\mu\partial_2^2\phi \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.8b}$$

$$h_t = \nabla\phi\cdot\left(1+(\partial_1 h)^2\right)^{1/2} n + 2\frac{\mu}{\rho}\partial_1^2 h \qquad \text{on } \Gamma(t)\times[0,T], \tag{1.8c}$$

as a model of viscous water waves. This model was also considered by several other authors. Dutykh and Dias [15] obtain a new set of viscous potential free-surface flow equations in the spirit of (1.8) taking into account the effects of the bottom topography. These authors also derived a long wave approximation. This approximate model takes the form of a nonlocal (in time) Boussinesq system (see also [13, 14, 15]). Kakleas and Nicholls [23], Kakleas and Nicholls [23], using the analytic dependence of the Dirichlet–Neumann operator, derived a system of two equations modeling (1.8). These equations are the viscous analog of the classical Craig–Sulem WW2 model and were mathematically studied by Ambrose, Bona, and Nicholls [3]. The well-posedness of the full (1.8) was studied very recently by Ngom and Nicholls [35]. In particular these authors proved global existence of solutions starting from small enough initial data for the case of nonvanishing surface tension $\gamma \neq 0$.

Some other related results are those by Kharif et al. [24] and Hunt and Dutykh [20]. Kharif et al. studied a similar situation to (1.8) within the framework of a forced and damped nonlinear Schrödinger equation (see also Touboul and Kharif [41]), while Hunt and Dutykh considered the problem of the interface motion under capillary gravity and an external electric force in the case of an incompressible, viscous, perfectly conducting fluid. Finally, let us mention the recent work by Guyenne and Parau [19] where the authors applied a simplified version of (1.8) to model wave attenuation in sea ice.

1.1. Plan of the paper. First we obtain the dimensionless Eulerian formulation in the moving domain and transform it into a dimensionless arbitrary Lagrangian–Eulerian formulation in a fixed domain in section 2. Then we introduce the asymptotic expansion and obtain the cascade of linear equations for the different scales present in the problem with $s = 0$ corresponding to the models by Jiang et al. [21] and Wu, Liu, and Yue [43] in section 3. After neglecting errors of $\mathcal{O}(\varepsilon^2)$ we find the nonlocal wave equation modeling the case $s = 0$. After that we consider the case $s = 2$ and, following a similar approach, find the nonlocal wave equation for the model of Dias, Dyachenko, and Zakharov [10]. Finally, we conclude with a parabolic system of Craig–Sulem flavor in section 5.

1.2. Notation. Let A be a matrix and b be a column vector. Then, we write A^i_j for the component of A located on row i and column j. We will use the Einstein summation convention for expressions with indexes.

We write

$$\partial_j f = \frac{\partial f}{\partial x_j}, \quad f_t = \frac{\partial f}{\partial t}$$

for the space derivative in the jth direction and for a time derivative, respectively. Unless parentheses are involved, every differential operator acts locally. For instance,

$$\partial_1 f \partial_1 \eta = (\partial_1 f)(\partial_1 \eta).$$

Let $f(x_1)$ denote an L^2 function on $\mathbb{S}^1$ (as usual, identified with the interval $[-\pi, \pi]$ with periodic boundary conditions). We define the Hilbert transform $\mathcal{H}$ and the Dirichlet-to-Neumann operator Λ and its powers, respectively, using Fourier series

$$\widehat{\mathcal{H}f}(k) = -i\text{sgn}(k)\hat{f}(k)\,, \quad \widehat{\Lambda f}(k) = |k|\hat{f}(k)\,, \quad \widehat{\Lambda^s f}(k) = |k|^s\hat{f}(k)\,, \tag{1.9}$$

where

$$\hat{f}(k) = \frac{1}{2\pi}\int_{\mathbb{S}^1} f(x_1)\, e^{-ikx_1} dx_1.$$

In particular, for zero-mean functions, we note that

$$\partial_1 \mathcal{H} = \Lambda, \quad \mathcal{H}^2 = -1, \quad \partial_1 \Lambda^{-1} = -\mathcal{H}.$$

These last equalities will be used extensively through the whole text. Finally, we define the commutator as

$$[\![A, B]\!] f = A(Bf) - B(Af).$$

2. Damped water waves.

2.1. The equations in the Eulerian formulation. We consider the system

$$\Delta\phi = 0 \qquad \text{in } \Omega(t)\times[0,T], \tag{2.1a}$$

$$\rho\left(\phi_t + \frac{1}{2}|\nabla\phi|^2 + Gh\right) - \gamma\mathcal{K} = -\delta_1\mathscr{D}^s\phi \qquad \text{on } \Gamma(t)\times[0,T], \tag{2.1b}$$

$$h_t = \nabla\phi\cdot\left(1+(\partial_1 h)^2\right)^{1/2} n + \delta_2\partial_1^2 h \qquad \text{on } \Gamma(t)\times[0,T], \tag{2.1c}$$

where $\delta_i \geq 0$ are constant, the dissipative terms are as in (1.7), ϕ is the scalar potential (units of length^2/time), h denotes the surface wave (units of length), and G (units of length/time^2) is the gravity acceleration. The constant δ_1 has units of mass/(length$^2\cdot$ time) (when $\mathscr{D}^0\phi = \phi$) and of mass/time (when $\mathscr{D}^2\phi = \partial_2^2\phi$) while δ_2 has units of length^2/time. We observe that, for an appropriate choice of δ_i and s we recover (exactly) (1.6) and (1.8). Indeed, if $\delta_2 = 0$ we obtain the same model by Jiang et al. [21] ($\delta_2 = 0$ and $s = 0$) and Wu, Liu, and Yue [43] ($\delta_2 = 0$ and $s = 0$ or $s = 2$), while if $\delta_2 = \delta_1/\rho$ and $s = 2$ we recover the model by Dias, Dyachenko, and Zakharov [10].

Following the pioneer work of Zakharov [46], we use the trace of the velocity potential $\xi(t,x) = \phi(t,x,h(t,x))$ (units of length^2/time). Now we observe that

$$\begin{aligned}\xi_t(t,x) &= \phi_t(t,x,h(t,x)) + \partial_2\phi(t,x,h(t,x))h_t(t,x)\\ &= \phi_t(t,x,h(t,x)) + \partial_2\phi(t,x,h(t,x))\left(\nabla\phi\cdot(-\partial_x h,1) + \delta_2\partial_1^2 h\right).\end{aligned}$$

Thus, (2.1) can be written as

$$\begin{aligned}\Delta\phi &= 0 && \text{in } \Omega(t)\times[0,T],\\ \phi &= \xi && \text{on } \Gamma(t)\times[0,T],\\ \xi_t &= \partial_2\phi\left(\nabla\phi\cdot(-\partial_1 h,1) + \delta_2\partial_1^2 h\right)\\ &\quad - \frac{1}{2}|\nabla\phi|^2 - Gh + \frac{\gamma}{\rho}\mathcal{K} - \frac{\delta_1}{\rho}\mathscr{D}^s\phi && \text{on } \Gamma(t)\times[0,T],\\ h_t &= \nabla\phi\cdot\left(1+(\partial_1 h)^2\right)^{1/2} n + \delta_2\partial_1^2 h && \text{on } \Gamma(t)\times[0,T].\end{aligned} \tag{2.2}$$

The system (2.2) is supplemented with an initial condition for h and ξ:

$$h(x,0) = h_0(x), \tag{2.3}$$

$$\xi(x,0) = \phi(x,h(0,x),0) = \xi_0(x). \tag{2.4}$$

2.2. Nondimensional Eulerian formulation. We denote by H and L the typical amplitude and wavelength of the water wave. We change to dimensionless variables (denoted with $\tilde{\cdot}$)

$$x = L\,\tilde{x}, \qquad t = \sqrt{\frac{L}{G}}\,\tilde{t}, \tag{2.5}$$

and unknowns

$$h(x_1,t) = H\,\tilde{h}(\tilde{x}_1,\tilde{t}), \qquad \phi(x_1,x_2,t) = H\sqrt{GL}\tilde{\phi}(\tilde{x}_1,\tilde{x}_2,\tilde{t}) \tag{2.6}$$

with the nondimensionalized fluid domain

$$\widetilde{\Omega}(t)=\left\{(\tilde{x}_1,\tilde{x}_2)\;\middle|\;-\pi<\tilde{x}_1<\pi\,,-\infty<\tilde{x}_2<\frac{H}{L}\tilde{h}(\tilde{x}_1,t)\,,\;t\in[0,T]\right\},$$
$$\widetilde{\Gamma}(t)=\left\{\left(\tilde{x}_1,\frac{H}{L}\tilde{h}(\tilde{x}_1,t)\right)\,,\;t\in[0,T]\right\}.$$

We find the following dimensionless parameters:

$$\varepsilon=\frac{H}{L},\qquad \alpha_1^s=\frac{\delta_1}{\rho\sqrt{G}L^{s-\frac{1}{2}}},\qquad \alpha_2=\frac{\delta_2}{\sqrt{G}L^{3/2}},\qquad \beta=\frac{\gamma}{\rho G L^2},\tag{2.7}$$

where $s=0$ if $\mathscr{D}\phi=\phi$ and $s=2$ if $\mathscr{D}\phi=\partial_2^2\phi$. The first parameter is known as the *steepness parameter* and measures the ratio between the amplitude and the wavelength of the wave. The $\alpha' s$ consider the ratio between gravity and viscosity forces. Finally, the fourth one is the Bond number that compares gravity forces with capillary forces. Dropping the tildes for the sake of clarity, we have the following dimensionless form of the damped water waves problem:

$$\Delta\phi=0 \qquad \text{in } \Omega(t)\times[0,T],\tag{2.8a}$$
$$\phi=\xi \qquad \text{on } \Gamma(t)\times[0,T],\tag{2.8b}$$
$$\begin{aligned}\xi_t=&-\frac{\varepsilon}{2}|\nabla\phi|^2-h+\frac{\beta\partial_1^2 h}{\left(1+(\varepsilon\partial_1 h)^2\right)^{3/2}}-\alpha_1^s\mathscr{D}^s\phi\\ &+\varepsilon\partial_2\phi\left(\nabla\phi\cdot(-\varepsilon\partial_1 h,1)+\alpha_2\partial_1^2 h\right)\end{aligned} \qquad \text{on } \Gamma(t)\times[0,T],\tag{2.8c}$$
$$h_t=\nabla\phi\cdot(-\varepsilon\partial_1 h,1)+\alpha_2\partial_1^2 h \qquad \text{on } \Gamma(t)\times[0,T],\tag{2.8d}$$

where we have used the nondimensional parameters (2.7).

2.3. The equations in the arbitrary Lagrangian–Eulerian formulation. In the present section we want to express system (2.8) on the reference domain Ω and reference interface Γ:

$$\Omega=\mathbb{S}^1\times(-\infty,0)\,,\qquad\qquad \Gamma=\mathbb{S}^1\times\{0\}\,.\tag{2.9}$$

The easiest way to do so is, supposing that h is regular, by defining the following family (parametrized in $t\in[0,T]$) of diffeomorphisms

$$\begin{array}{rccl}\psi: & [0,T]\times\Omega & \to & \Omega\,(t)\,,\\ & (x_1,x_2,t) & \mapsto & \psi\,(x_1,x_2,t)=(x_1,x_2+\varepsilon h\,(x_1,t))\,.\end{array}$$

Such a technique has already been used in the past by different authors (see, for instance, [6, 7, 16, 27, 35] and the references therein). We compute

$$\nabla\psi=\begin{pmatrix}1 & 0\\ \varepsilon\partial_1 h\,(x_1,t) & 1\end{pmatrix},\qquad A=(\nabla\psi)^{-1}=\begin{pmatrix}1 & 0\\ -\varepsilon\partial_1 h\,(x_1,t) & 1\end{pmatrix}.\tag{2.10}$$

With such a map we can define the pushback of any application θ defined on $\Omega\,(t)$ simply as $\Theta=\theta\circ\psi$, whence in particular we define

$$\Phi=\phi\circ\psi\,.$$

We let $N = e_2$ denote the outward unit normal to Ω at Γ. We also recall that, if $\Theta = \theta \circ \psi$, the following formula holds:

$$\partial_j \theta \circ \psi = A^k_j \partial_k \Theta,$$

where the Einstein convention is used. Then, we can rewrite (2.8) as the following system of variable coefficient nonlinear PDEs posed on a fixed reference domain:

$$A^\ell_j \partial_\ell \left(A^k_j \partial_k \Phi \right) = 0 \qquad \text{in } \Omega \times [0,T], \tag{2.11a}$$

$$\Phi = \xi \qquad \text{on } \Gamma \times [0,T], \tag{2.11b}$$

$$\begin{aligned} \xi_t = & -\frac{\varepsilon}{2} A^k_j \partial_k \Phi A^\ell_j \partial_\ell \Phi - h + \frac{\beta \partial_1^2 h}{\left(1 + (\varepsilon \partial_1 h)^2\right)^{3/2}} - \alpha_1^s \mathcal{D}^s \Phi \\ & + \varepsilon A^k_2 \partial_k \Phi \left(A^\ell_j \partial_\ell \Phi A^2_j + \alpha_2 \partial_1^2 h \right) \qquad \text{on } \Gamma \times [0,T], \end{aligned} \tag{2.11c}$$

$$h_t = A^k_j \partial_k \Phi A^2_j + \alpha_2 \partial_1^2 h \qquad \text{on } \Gamma \times [0,T], \tag{2.11d}$$

where the operator $\mathcal{D}^s$ is

$$\mathcal{D}^0 \Phi = \xi, \ \mathcal{D}^2 \Phi = A^\ell_2 \partial_\ell \left(A^k_2 \partial_k \Phi \right).$$

Next we make explicit the values of the A^i_j's in the above system (see (2.10)) obtaining hence

$$\Delta \Phi = \varepsilon \left(\partial_1^2 h \ \partial_2 \Phi + 2 \partial_1 h \ \partial_{12} \Phi \right) - \varepsilon^2 (\partial_1 h)^2 \partial_2^2 \Phi \qquad \text{in } \Omega \times [0,T], \tag{2.12a}$$

$$\Phi = \xi \qquad \text{on } \Gamma \times [0,T], \tag{2.12b}$$

$$\begin{aligned} \xi_t = & -\frac{\varepsilon}{2} \left[(\partial_1 \Phi)^2 + (\varepsilon \partial_1 h \partial_2 \Phi)^2 + (\partial_2 \Phi)^2 - 2\varepsilon \partial_1 h \partial_2 \Phi \partial_1 \Phi \right] \\ & - h + \frac{\beta \partial_1^2 h}{\left(1 + (\varepsilon \partial_1 h)^2\right)^{3/2}} - \alpha_1^s \mathcal{D}^s \Phi \\ & + \varepsilon \partial_2 \Phi \left(-\varepsilon \partial_1 h \partial_1 \Phi + \varepsilon^2 (\partial_1 h)^2 \partial_2 \Phi + \partial_2 \Phi + \alpha_2 \partial_1^2 h \right) \qquad \text{on } \Gamma \times [0,T], \end{aligned} \tag{2.12c}$$

$$h_t = -\varepsilon \partial_1 h \partial_1 \Phi + \varepsilon^2 (\partial_1 h)^2 \partial_2 \Phi + \partial_2 \Phi + \alpha_2 \partial_1^2 h \qquad \text{on } \Gamma \times [0,T], \tag{2.12d}$$

where

$$\mathcal{D}^0 \Phi = \Phi, \ \mathcal{D}^2 \Phi = \partial_2^2 \Phi.$$

3. The asymptotic model for damped water waves when $s = 0$. In this section we consider the case $s = 0$ (the model by Jiang et al. [21] and Wu, Liu, and Yue [43]). In this case we have that

$$\mathcal{D}^0 \Phi = \Phi.$$

We introduce the following ansatz:

$$\begin{aligned} \Phi(x_1, x_2, t) &= \sum_n \varepsilon^n \Phi^{(n)}(x_1, x_2, t), \\ \xi(x_1, t) &= \sum_n \varepsilon^n \xi^{(n)}(x_1, t), \\ h(x_1, t) &= \sum_n \varepsilon^n h^{(n)}(x_1, t). \end{aligned} \tag{3.1}$$

With this ansatz we can reprofile the nonlinear system (2.12) in an equivalent sequence of linear systems where the evolution of the nth profile is determined by the evolution of the preceding $n-1$ profiles.

We are interested in a model approximating (2.12) with an error $\mathcal{O}(\varepsilon^2)$. Using that

$$\frac{1}{(1+x^2)^{3/2}} = 1 + \mathcal{O}(x^2),$$

we obtain that

$$\frac{\beta \partial_1^2 h}{\left(1+(\varepsilon\partial_1 h)^2\right)^{3/2}} = \beta\partial_1^2 h + \mathcal{O}(\varepsilon^2).$$

For the case $n=0$, we have that

(3.2a) $$\Delta\Phi^{(0)} = 0 \qquad \text{in } \Omega\times[0,T],$$

(3.2b) $$\Phi^{(0)} = \xi^{(0)} \qquad \text{on } \Gamma\times[0,T],$$

(3.2c) $$\xi_t^{(0)} = -h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^0\Phi^{(0)} \qquad \text{on } \Gamma\times[0,T],$$

(3.2d) $$h_t^{(0)} = \partial_2\Phi^{(0)} + \alpha_2\partial_1^2 h^{(0)} \qquad \text{on } \Gamma\times[0,T].$$

Recalling that

$$\widehat{\Phi^{(0)}}(k,x_2,t) = \xi^{(0)}(k,t)e^{|k|x_2} \qquad \text{in } \Omega\times[0,T]$$

so

$$\partial_2\Phi^{(0)} = \Lambda\xi^{(0)} \qquad \text{on } \Gamma,$$

we find that (3.2d) can be equivalently written as

$$h_{tt}^{(0)} = \Lambda\left(-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^0\xi^{(0)}\right) + \alpha_2\partial_1^2 h_t^{(0)} \qquad \text{on } \Gamma\times[0,T].$$

We note that (3.2d) can be equivalently written as

(3.3) $$\xi^{(0)} = \Lambda^{-1}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right],$$

thus,

(3.4)
$$h_{tt}^{(0)} = \Lambda\left(-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^0\Lambda^{-1}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right) + \alpha_2\partial_1^2 h_t^{(0)} \qquad \text{on } \Gamma\times[0,T].$$

Similarly, in the case $n=1$, we find that

(3.5a) $$\Delta\Phi^{(1)} = \partial_1^2 h^{(0)}\,\partial_2\Phi^{(0)} + 2\partial_1 h^{(0)}\,\partial_{12}\Phi^{(0)} \qquad \text{in } \Omega\times[0,T],$$

(3.5b) $$\Phi^{(1)} = \xi^{(1)} \qquad \text{on } \Gamma\times[0,T],$$

(3.5c) $$\xi_t^{(1)} = \frac{1}{2}\left[(\partial_2\Phi^{(0)})^2 - (\partial_1\Phi^{(0)})^2\right] - h^{(1)} + \beta\partial_1^2 h^{(1)} - \alpha_1^0\Phi^{(1)} + \alpha_2\partial_2\Phi^{(0)}\partial_1^2 h^{(0)} \qquad \text{on } \Gamma\times[0,T],$$

(3.5d) $$h_t^{(1)} = -\partial_1 h^{(0)}\partial_1\Phi^{(0)} + \partial_2\Phi^{(1)} + \alpha_2\partial_1^2 h^{(1)} \qquad \text{on } \Gamma\times[0,T].$$

Let us define

$$b = \partial_1^2 h^{(0)}\partial_2\Phi^{(0)} + 2\partial_1 h^{(0)}\partial_{12}\Phi^{(0)}.$$

We now use Lemma A.1 in order to compute

$$\partial_2 \widehat{\Phi^{(1)}}(k,0,t) = \int_{-\infty}^{0} \hat{b}(k,y_2,t)\, e^{|k| y_2} \mathrm{d}y_2 + |k|\, \widehat{\xi^{(1)}}(k,t)\,.$$

We want to provide an explicit expression for the term $\int_{-\infty}^{0} \hat{b}(k,y_2,t)\, e^{|k|y_2} \mathrm{d}y_2$ considering the form of b. We compute that

$$\begin{aligned}
&\int_{-\infty}^{0} \hat{b}(k,y_2,t)\, e^{|k|y_2} \mathrm{d}y_2 \\
&\quad = -\int_{-\infty}^{0} e^{(|k|+|m|)y_2} (k-m)(k+m)\,|m|\, \widehat{h^{(0)}}(k-m)\, \widehat{\xi^{(0)}}(m)\, \mathrm{d}y_2 \\
&\quad = -\frac{|m| \left[|k|^2 - |m|^2 \right]}{|k|+|m|}\, \widehat{h^{(0)}}(k-m)\, \widehat{\xi^{(0)}}(m) \\
&\quad = -\,|m| \left[|k| - |m| \right]\, \widehat{h^{(0)}}(k-m)\, \widehat{\xi^{(0)}}(m)\,.
\end{aligned}$$

Thus, we find that

$$\partial_2 \Phi^{(1)} \Big|_{x_2=0} = \Lambda \xi^{(1)} - \left[\!\left[\Lambda, h^{(0)} \right]\!\right] \Lambda \xi^{(0)}. \tag{3.6}$$

The evolution equations for $h^{(1)}$ and $\xi^{(1)}$ hence become

$$h_t^{(1)} = -\partial_1 h^{(0)} \partial_1 \xi^{(0)} + \Lambda \xi^{(1)} - \left[\!\left[\Lambda, h^{(0)} \right]\!\right] \Lambda \xi^{(0)} + \alpha_2 \partial_1^2 h^{(1)}, \tag{3.7}$$

$$\begin{aligned}
\xi_t^{(1)} &= \frac{1}{2} \left[\left(\Lambda \xi^{(0)} \right)^2 - \left(\partial_1 \xi^{(0)} \right)^2 \right] \\
&\quad - h^{(1)} + \beta \partial_1^2 h^{(1)} - \alpha_1^0 \Phi^{(1)} + \alpha_2 \Lambda \xi^{(0)} \partial_1^2 h^{(0)}.
\end{aligned} \tag{3.8}$$

Using the above equation for $h_t^{(1)}$, (3.7), we can express $\xi^{(1)}$ as a function of $h^{(0)}$, $\xi^{(0)}$, and $h^{(1)}$ as follows:

$$\xi^{(1)} = \Lambda^{-1} \left[h_t^{(1)} + \partial_1 h^{(0)} \partial_1 \xi^{(0)} + \left[\!\left[\Lambda, h^{(0)} \right]\!\right] \Lambda \xi^{(0)} - \alpha_2 \partial_1^2 h^{(1)} \right]. \tag{3.9}$$

Time differentiating (3.7) and inserting (3.8), we deduce

$$\begin{aligned}
h_{tt}^{(1)} &= -\partial_1 h_t^{(0)} \partial_1 \xi^{(0)} - \partial_1 h^{(0)} \partial_1 \xi_t^{(0)} + \frac{1}{2} \Lambda \left[\left(\Lambda \xi^{(0)} \right)^2 - \left(\partial_1 \xi^{(0)} \right)^2 \right] \\
&\quad - \Lambda h^{(1)} + \beta \Lambda \partial_1^2 h^{(1)} - \alpha_1^0 \Lambda \Phi^{(1)} + \alpha_2 \Lambda \left(\Lambda \xi^{(0)} \partial_1^2 h^{(0)} \right) \\
&\quad - \left[\!\left[\Lambda, h_t^{(0)} \right]\!\right] \Lambda \xi^{(0)} - \left[\!\left[\Lambda, h^{(0)} \right]\!\right] \Lambda \xi_t^{(0)} + \alpha_2 \partial_1^2 h_t^{(1)}.
\end{aligned}$$

Recalling the definition of the Riesz potential Λ^{-1} and using (3.2c) and (3.3) in order to express $\xi^{(0)}$ and $\xi_t^{(0)}$ in terms of $h^{(0)}$, we find that

$$\begin{aligned}
h_{tt}^{(1)} &= \partial_1 h_t^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] - \partial_1 h^{(0)}\partial_1\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^0\Phi^{(0)}\right] \\
&\quad + \frac{1}{2}\Lambda\left\{\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]^2 - \left(\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right)^2\right\} \\
&\quad - \Lambda h^{(1)} + \beta\Lambda\partial_1^2 h^{(1)} - \alpha_1^0\Lambda\Phi^{(1)} + \alpha_2\Lambda\left[\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right)\partial_1^2 h^{(0)}\right] \\
&\quad - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right]\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right) - \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\Lambda\left(-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^0\Phi^{(0)}\right) \\
&\quad + \alpha_2\partial_1^2 h_t^{(1)}.
\end{aligned}$$

Using the Tricomi identity

$$(\mathcal{H}f)^2 - f^2 = 2\mathcal{H}\left(f\mathcal{H}f\right), \tag{3.10}$$

the previous equation can be further simplified and we find that

$$\begin{aligned}
h_{tt}^{(1)} &= \partial_1 h_t^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] - \partial_1 h^{(0)}\partial_1\left[-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^0\Phi^{(0)}\right] \\
&\quad + \partial_1\left\{\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right\} \\
&\quad - \Lambda h^{(1)} + \beta\Lambda\partial_1^2 h^{(1)} - \alpha_1^0\Lambda\Phi^{(1)} + \alpha_2\Lambda\left[\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right)\partial_1^2 h^{(0)}\right] \\
&\quad - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right]\left(h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right) - \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\Lambda\left(-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^0\Phi^{(0)}\right) \\
&\quad + \alpha_2\partial_1^2 h_t^{(1)}.
\end{aligned}$$

We can express $\alpha_1^0\Phi^{(0)}$ in terms of $h^{(0)}$ as follows

$$\alpha_1^0\Phi^{(0)}\Big|_{x_2=0} = \alpha_1^0\xi^{(0)} = \alpha_1^0\Lambda^{-1}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right],$$

and, inserting the previous formula into (3.9), we find that

$$\begin{aligned}
&\alpha_1^0\Phi^{(1)}\Big|_{x_2=0} \\
&\quad = \alpha_1^0\xi^{(1)}, \\
&\quad = \alpha_1^0\Lambda^{-1}\left[h_t^{(1)} + \partial_1 h^{(0)}\partial_1\xi^{(0)} + \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\Lambda\xi^{(0)} - \alpha_2\partial_1^2 h^{(1)}\right] \\
&\quad = \alpha_1^0\Lambda^{-1}\left\{h_t^{(1)} - \partial_1 h^{(0)}\mathcal{H}\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]\right. \\
&\qquad\qquad \left. + \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right] - \alpha_2\partial_1^2 h^{(1)}\right\}.
\end{aligned}$$

Substituting the previous expressions into the equation for $h_{tt}^{(1)}$, we deduce the following equation:

$$
\begin{aligned}
h_{tt}^{(1)} &= \partial_1 h_t^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \\
&\quad - \partial_1 h^{(0)} \partial_1 \left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^0 \Lambda^{-1}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right] \\
&\quad + \partial_1 \left\{\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right\} - \Lambda h^{(1)} + \beta \Lambda \partial_1^2 h^{(1)} \\
&\quad - \alpha_1^0 \left\{h_t^{(1)} - \partial_1 h^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right. \\
&\qquad\quad \left. + \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \alpha_2 \partial_1^2 h^{(1)}\right\} \\
&\quad + \ \alpha_2 \Lambda \left[\left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) \partial_1^2 h^{(0)}\right] - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right]\left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right) \\
&\quad - \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\left(-\Lambda h^{(0)} + \beta \Lambda \partial_1^2 h^{(0)} - \alpha_1^0\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right) + \alpha_2 \partial_1^2 h_t^{(1)}.
\end{aligned}
$$

We group the nonlinear terms according to the coefficient in front. At $\mathcal{O}(1)$ we find that

$$
\begin{aligned}
\partial_1 h_t^{(0)} \mathcal{H} h_t^{(0)} + \left(\partial_1 h^{(0)}\right)^2 + \frac{\Lambda}{2}\left\{\left[h_t^{(0)}\right]^2 - \left(\mathcal{H} h_t^{(0)}\right)^2\right\} - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right] h_t^{(0)} \\
+ \left[\!\left[\Lambda, h^{(0)}\right]\!\right] \Lambda h^{(0)} = -\Lambda\left(\left(\mathcal{H} h_t^{(0)}\right)^2\right) + \partial_1 \left[\!\left[\mathcal{H}, h^{(0)}\right]\!\right] \Lambda h^{(0)},
\end{aligned}
\tag{3.11}
$$

where we have used the identity (3.10). At $\mathcal{O}(\beta)$ we obtain that

$$
\begin{aligned}
\beta\left(-\partial_1 h^{(0)} \partial_1^3 h^{(0)} - \left[\!\left[\Lambda, h^{(0)}\right]\!\right] \Lambda \partial_1^2 h^{(0)}\right) &= \beta\left(\Lambda\left(h^{(0)} \Lambda^3 h^{(0)}\right) - \partial_1\left(h^{(0)} \partial_1^3 h^{(0)}\right)\right) \\
&= \beta \partial_1 \left[\!\left[\mathcal{H}, h^{(0)}\right]\!\right] \Lambda^3 h^{(0)}.
\end{aligned}
\tag{3.12}
$$

At $\mathcal{O}(\alpha_2)$ we find the following contribution:

$$
\begin{aligned}
& -\alpha_2 \Big[\partial_1 h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + \partial_1\left\{h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)}\right\} + \partial_1\left\{\partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)}\right\} \\
&\qquad\qquad - \Lambda\left[h_t^{(0)} \partial_1^2 h^{(0)}\right] - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right] \partial_1^2 h^{(0)}\Big] \\
&\quad = -\alpha_2 \Big[\partial_1 h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + \partial_1\left\{h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)}\right\} + \partial_1\left\{\partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)}\right\} \\
&\qquad\qquad\qquad - 2\Lambda\left[h_t^{(0)} \partial_1^2 h^{(0)}\right] + h_t^{(0)} \Lambda \partial_1^2 h^{(0)}\Big].
\end{aligned}
\tag{3.13}
$$

Using

$$
\mathcal{H} f \mathcal{H} g - \mathcal{H}\left(f \mathcal{H} g + g \mathcal{H} f\right) = f g,
\tag{3.14}
$$

we find that

$$
\Lambda\left[h_t^{(0)} \partial_1^2 h^{(0)}\right] = \Lambda\left(\mathcal{H} h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)}\right) + \partial_1\left(h_t^{(0)} \mathcal{H} \partial_1^2 h^{(0)} + \partial_1^2 h^{(0)} \mathcal{H} h_t^{(0)}\right).
$$

Thus, we can group terms in (3.13) as follows:

$$
\begin{aligned}
&-\alpha_2\Big[\partial_1\left(h_t^{(0)}\mathcal{H}\partial_1^2h^{(0)}\right)-2\Lambda\left(\mathcal{H}h_t^{(0)}\mathcal{H}\partial_1^2h^{(0)}\right)-\partial_1\left(h_t^{(0)}\mathcal{H}\partial_1^2h^{(0)}+\partial_1^2h^{(0)}\mathcal{H}h_t^{(0)}\right)\Big]\\
&\qquad=\alpha_2\partial_1\left[\!\left[\mathcal{H},\mathcal{H}h_t^{(0)}\right]\!\right]\mathcal{H}\partial_1^2h^{(0)}+\alpha_2\Lambda\left(\mathcal{H}h_t^{(0)}\mathcal{H}\partial_1^2h^{(0)}\right).
\end{aligned}
\tag{3.15}
$$

At $\mathcal{O}(\alpha_2\alpha_2)$, we find that

$$
\alpha_2^2\Big[\partial_1\left\{\partial_1^2h^{(0)}\mathcal{H}\partial_1^2h^{(0)}\right\}-\Lambda\left[\left(\partial_1^2h^{(0)}\right)^2\right]\Big]=-\alpha_2^2\partial_1\left[\!\left[\mathcal{H},\partial_1^2h^{(0)}\right]\!\right]\partial_1^2h^{(0)}.
\tag{3.16}
$$

We group now the $\mathcal{O}(\alpha_1^0)$ terms:

$$
\alpha_1^0\Big[-\partial_1h^{(0)}\mathcal{H}h_t^{(0)}+\partial_1h^{(0)}\mathcal{H}h_t^{(0)}-\left[\!\left[\Lambda,h^{(0)}\right]\!\right]h_t^{(0)}+\left[\!\left[\Lambda,h^{(0)}\right]\!\right]h_t^{(0)}\Big]=0.
\tag{3.17}
$$

Finally, we are left with the $\mathcal{O}(\alpha_1^0\alpha_2)$ terms. These terms are

$$
\alpha_1^0\alpha_2\Big[\partial_1h^{(0)}\mathcal{H}\partial_1^2h^{(0)}-\partial_1h^{(0)}\mathcal{H}\partial_1^2h^{(0)}+\left[\!\left[\Lambda,h^{(0)}\right]\!\right]\partial_1^2h^{(0)}-\left[\!\left[\Lambda,h^{(0)}\right]\!\right]\partial_1^2h^{(0)}\Big]=0.
\tag{3.18}
$$

Thus, using (3.11), (3.12), (3.15), (3.16), (3.17), and (3.18), we conclude that

$$
\begin{aligned}
&h_{tt}^{(1)}+\Lambda h^{(1)}+\beta\Lambda^3h^{(1)}+\alpha_1^0h_t^{(1)}-\alpha_1^0\alpha_2\partial_1^2h^{(1)}-\alpha_2\partial_1^2h_t^{(1)}\\
&\quad=-\Lambda\left(\left(\mathcal{H}h_t^{(0)}\right)^2\right)+\partial_1\left[\!\left[\mathcal{H},h^{(0)}\right]\!\right]\Lambda h^{(0)}+\beta\partial_1\left[\!\left[\mathcal{H},h^{(0)}\right]\!\right]\Lambda^3h^{(0)}\\
&\qquad+\alpha_2\partial_1\left[\!\left[\mathcal{H},\mathcal{H}h_t^{(0)}\right]\!\right]\mathcal{H}\partial_1^2h^{(0)}+\alpha_2\Lambda\left(\mathcal{H}h_t^{(0)}\mathcal{H}\partial_1^2h^{(0)}\right)-\alpha_2^2\partial_1\left[\!\left[\mathcal{H},\partial_1^2h^{(0)}\right]\!\right]\partial_1^2h^{(0)}.
\end{aligned}
$$

We define the renormalized variable

$$
f=h^{(0)}+\varepsilon h^{(1)}.
\tag{3.19}
$$

Using

$$
\varepsilon h^{(0)}=\varepsilon f+\mathcal{O}(\varepsilon^2),
$$

and neglecting errors $\mathcal{O}(\varepsilon^2)$, we conclude the following model:

$$
\begin{aligned}
&f_{tt}+\Lambda f+\beta\Lambda^3f+\alpha_1^0f_t-\alpha_1^0\alpha_2\partial_1^2f-\alpha_2\partial_1^2f_t\\
&\quad=\varepsilon\Big[-\Lambda\left((\mathcal{H}f_t)^2\right)+\partial_1[\![\mathcal{H},f]\!]\Lambda f+\beta\partial_1[\![\mathcal{H},f]\!]\Lambda^3f+\alpha_2\partial_1[\![\mathcal{H},\mathcal{H}f_t]\!]\mathcal{H}\partial_1^2f\\
&\qquad+\alpha_2\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_1^2f\right)-\alpha_2^2\partial_1[\![\mathcal{H},\partial_1^2f]\!]\partial_1^2f\Big].
\end{aligned}
\tag{3.20}
$$

When $\alpha_2=0$, (3.20) is an asymptotic model of the damped water waves system proposed by Jiang et al. [21] and Wu, Liu, and Yue [43]. Also, when $\alpha_2=\alpha_1^0=0$, (3.20) recovers the quadratic h-model in [1, 2, 6, 30, 31, 32].

4. The asymptotic model for damped water waves when $s = 2$. In this section we focus on the case $s = 2$ (the model by Dias, Dyachenko, and Zakharov [10]). In this case we have that

$$\mathcal{D}^2\Phi = \partial_2^2\Phi.$$

We use the ansatz (3.1) and follow the previous steps. The first term in the series solves

$$\Delta\Phi^{(0)} = 0 \quad \text{in } \Omega\times[0,T]\,, \tag{4.1a}$$

$$\Phi^{(0)} = \xi^{(0)} \quad \text{on } \Gamma\times[0,T], \tag{4.1b}$$

$$\xi_t^{(0)} = -h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\partial_2^2\Phi^{(0)} \quad \text{on } \Gamma\times[0,T], \tag{4.1c}$$

$$h_t^{(0)} = \partial_2\Phi^{(0)} + \alpha_2\partial_1^2 h^{(0)} \quad \text{on } \Gamma\times[0,T]. \tag{4.1d}$$

Taking a time derivative of (4.1d), using the fact that

$$\partial_2\Phi^{(0)}\Big|_{x_2=0} = \Lambda\xi^{(0)} = h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}$$

and substituting (4.1c), we find that

$$h_{tt}^{(0)} = \Lambda\left(-h^{(0)} + \beta\partial_1^2 h^{(0)} - \alpha_1^2\partial_2^2\Phi^{(0)}\right) + \alpha_2\partial_1^2 h_t^{(0)} \qquad \text{on } \Gamma\times[0,T].$$

Similarly, due to the fact that

$$\partial_2^2\Phi^{(0)}\Big|_{x_2=0} = \Lambda^2\xi^{(0)} = \Lambda\left[h_t^{(0)} - \alpha_2\partial_1^2 h^{(0)}\right]$$

we find that the previous equation for h_{tt} can be written as

$$h_{tt}^{(0)} = -\Lambda h^{(0)} - \beta\Lambda^3 h^{(0)} + \alpha_1^2\partial_1^2 h_t^{(0)} - \alpha_1^2\alpha_2\partial_1^4 h^{(0)} + \alpha_2\partial_1^2 h_t^{(0)} \quad \text{on } \Gamma\times[0,T]. \tag{4.2}$$

Analogously as in (3.5), for $n = 1$, we find that

$$\Delta\Phi^{(1)} = \partial_1^2 h^{(0)}\ \partial_2\Phi^{(0)} + 2\partial_1 h^{(0)}\ \partial_{12}\Phi^{(0)} \quad \text{in } \Omega\times[0,T]\,, \tag{4.3a}$$

$$\Phi^{(1)} = \xi^{(1)} \quad \text{on } \Gamma\times[0,T], \tag{4.3b}$$

$$\begin{aligned}\xi_t^{(1)} = {}& \frac{1}{2}\left[(\partial_2\Phi^{(0)})^2 - (\partial_1\Phi^{(0)})^2\right] \\ & - h^{(1)} + \beta\partial_1^2 h^{(1)} - \alpha_1^2\partial_2^2\Phi^{(1)} + \alpha_2\partial_2\Phi^{(0)}\partial_1^2 h^{(0)} \quad \text{on } \Gamma\times[0,T],\end{aligned} \tag{4.3c}$$

$$h_t^{(1)} = -\partial_1 h^{(0)}\partial_1\Phi^{(0)} + \partial_2\Phi^{(1)} + \alpha_2\partial_1^2 h^{(1)} \quad \text{on } \Gamma\times[0,T], \tag{4.3d}$$

We use Lemma A.1 and (3.6) to find that

$$\partial_2\Phi^{(1)}\Big|_{x_2=0} = \Lambda\xi^{(1)} - \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\Lambda\xi^{(0)},$$

$$\partial_2^2\Phi^{(1)}\Big|_{x_2=0} = \Lambda^2\xi^{(1)} + \partial_1^2 h^{(0)}\Lambda\xi^{(0)} + 2\partial_1 h^{(0)}\partial_1\Lambda\xi^{(0)}.$$

Then we find the following system of equations:

$$h_t^{(1)} = -\partial_1 h^{(0)} \partial_1 \xi^{(0)} + \Lambda \xi^{(1)} - \left[\!\left[\Lambda, h^{(0)}\right]\!\right] \Lambda \xi^{(0)} + \alpha_2 \partial_1^2 h^{(1)}, \tag{4.4}$$

$$\begin{aligned} \xi_t^{(1)} = {} & \frac{1}{2}\left[\left(\Lambda \xi^{(0)}\right)^2 - \left(\partial_1 \xi^{(0)}\right)^2\right] \\ & - h^{(1)} + \beta \partial_1^2 h^{(1)} - \alpha_1^2 \left[\Lambda^2 \xi^{(1)} + \partial_1^2 h^{(0)} \Lambda \xi^{(0)} + 2\partial_1 h^{(0)} \partial_1 \Lambda \xi^{(0)}\right] \\ & + \alpha_2 \Lambda \xi^{(0)} \partial_1^2 h^{(0)}. \end{aligned} \tag{4.5}$$

These equations are the analog (when $s = 2$) of (3.7) and (3.8).

As before, we want to reduce everything to a single equation for $h^{(1)}$ and $h^{(0)}$. Using (4.1d), we find that

$$\Lambda \xi^{(1)} = h_t^{(1)} - \partial_1 h^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \alpha_2 \partial_1^2 h^{(1)} + \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right].$$

As a consequence, we have that

$$\begin{aligned} & \alpha_1^2 \partial_2^2 \Phi^{(1)}\Big|_{x_2=0} \\ & = \alpha_1^2 \Lambda \Big\{ h_t^{(1)} - \partial_1 h^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \\ & \qquad + \left[\!\left[\Lambda, h^{(0)}\right]\!\right]\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \alpha_2 \partial_1^2 h^{(1)} \Big\} \\ & \quad + \alpha_1^2 \left(\partial_1^2 h^{(0)} \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] + 2\partial_1 h^{(0)} \partial_1 \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \right). \end{aligned}$$

Time differentiating (4.4) and inserting (4.5), we deduce

$$\begin{aligned} h_{tt}^{(1)} = {} & -\partial_1 h_t^{(0)} \partial_1 \xi^{(0)} + \Lambda \xi_t^{(1)} - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right] \Lambda \xi^{(0)} \\ & + \alpha_2 \partial_1^2 h_t^{(1)} - \partial_1 h^{(0)} \partial_1 \xi_t^{(0)} - \left[\!\left[\Lambda, h^{(0)}\right]\!\right] \Lambda \xi_t^{(0)} \\ = {} & \partial_1 h_t^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right]\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \\ & + \alpha_2 \partial_1^2 h_t^{(1)} - \partial_1 h^{(0)} \partial_1 \left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^2 \Lambda \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right] \\ & - \left[\!\left[\Lambda, h^{(0)}\right]\!\right] \Lambda \left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^2 \Lambda \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right] \\ & + \frac{1}{2}\Lambda \left[\left(\Lambda \xi^{(0)}\right)^2 - \left(\partial_1 \xi^{(0)}\right)^2\right] \\ & + \Lambda \Big[-h^{(1)} + \beta \partial_1^2 h^{(1)} - \alpha_1^2 \left[\Lambda^2 \xi^{(1)} + \partial_1^2 h^{(0)} \Lambda \xi^{(0)} + 2\partial_1 h^{(0)} \partial_1 \Lambda \xi^{(0)}\right] \\ & \qquad + \alpha_2 \Lambda \xi^{(0)} \partial_1^2 h^{(0)}\Big] \\ = {} & \partial_1 h_t^{(0)} \mathcal{H}\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] - \left[\!\left[\Lambda, h_t^{(0)}\right]\!\right]\left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right] \\ & + \alpha_2 \partial_1^2 h_t^{(1)} - \partial_1 h^{(0)} \partial_1 \left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^2 \Lambda \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right] \\ & - \left[\!\left[\Lambda, h^{(0)}\right]\!\right] \Lambda \left[-h^{(0)} + \beta \partial_1^2 h^{(0)} - \alpha_1^2 \Lambda \left[h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right]\right] \\ & + \frac{1}{2}\Lambda \left[\left(h_t^{(0)} - \alpha_2 \partial_1^2 h^{(0)}\right)^2 - \left(\mathcal{H} h_t^{(0)} - \alpha_2 \mathcal{H} \partial_1^2 h^{(0)}\right)^2\right] \end{aligned}$$

$$
\begin{aligned}
&+\Lambda\Bigg\{-h^{(1)}+\beta\partial_1^2 h^{(1)}-\alpha_1^2\left[\partial_1^2 h^{(0)}\left[h_t^{(0)}-\alpha_2\partial_1^2 h^{(0)}\right]\right.\\
&\qquad\qquad\qquad\qquad\left.+2\partial_1 h^{(0)}\partial_1\left[h_t^{(0)}-\alpha_2\partial_1^2 h^{(0)}\right]\right]\\
&\qquad\qquad\qquad\qquad\qquad\qquad+\ \alpha_2\left[h_t^{(0)}-\alpha_2\partial_1^2 h^{(0)}\right]\partial_1^2 h^{(0)}\Bigg\}\\
&-\alpha_1^2\Lambda^2\Big\{h_t^{(1)}-\partial_1 h^{(0)}\mathcal{H}\left[h_t^{(0)}-\alpha_2\partial_1^2 h^{(0)}\right]\\
&\qquad-\alpha_2\partial_1^2 h^{(1)}+\left[\!\left[\Lambda,h^{(0)}\right]\!\right]\left[h_t^{(0)}-\alpha_2\partial_1^2 h^{(0)}\right]\Big\},
\end{aligned}
$$

where we have used the previous expression for $\Lambda\xi^{(1)}$. We group the different nonlinear contributions according to the coefficient in front: at $\mathcal{O}(1)$ we find (3.11), while at $\mathcal{O}(\beta)$ we have (3.12). Using the Tricomi identity (3.10) to obtain

$$
\begin{aligned}
&\frac{1}{2}\Lambda\left[\left(h_t^{(0)}-\alpha_2\partial_1^2 h^{(0)}\right)^2-\left(\mathcal{H}h_t^{(0)}-\alpha_2\mathcal{H}\partial_1^2 h^{(0)}\right)^2\right]\\
&\qquad\qquad=\partial_1\left[\left(h_t^{(0)}-\alpha_2\partial_1^2 h^{(0)}\right)\left(\mathcal{H}h_t^{(0)}-\alpha_2\mathcal{H}\partial_1^2 h^{(0)}\right)\right],
\end{aligned}
$$

we find that the $\mathcal{O}(\alpha_2)$ contribution is given by (3.13) and, as a consequence, it can be further simplified to conclude (3.15). At $O(\alpha_2\alpha_2)$ we have the terms (3.16). We collect now the $\mathcal{O}(\alpha_1^2)$ terms:

$$
\begin{aligned}
&\alpha_1^2\Big[\partial_1 h^{(0)}\partial_1\Lambda h_t^{(0)}-\partial_1^2\left(\partial_1 h^{(0)}\mathcal{H}h_t^{(0)}\right)-\left[\!\left[\Lambda,h^{(0)}\right]\!\right]\partial_1^2 h_t^{(0)}+\partial_1^2\left[\!\left[\Lambda,h^{(0)}\right]\!\right]h_t^{(0)}\\
&\qquad-\Lambda\left\{\partial_1^2 h^{(0)}h_t^{(0)}+2\partial_1 h^{(0)}\partial_1 h_t^{(0)}\right\}\Big]\\
&=\alpha_1^2\Big[-\partial_1^3 h^{(0)}\mathcal{H}h_t^{(0)}-2\partial_1^2 h^{(0)}\Lambda h_t^{(0)}+2\left[\!\left[\Lambda,\partial_1 h^{(0)}\right]\!\right]\partial_1 h_t^{(0)}\\
&\qquad+\left[\!\left[\Lambda,\partial_1^2 h^{(0)}\right]\!\right]h_t^{(0)}-\Lambda\left\{\partial_1^2 h^{(0)}h_t^{(0)}+2\partial_1 h^{(0)}\partial_1 h_t^{(0)}\right\}\Big]\\
&=-\alpha_1^2\partial_1\left[\!\left[\partial_1^2,h^{(0)}\right]\!\right]\mathcal{H}h_t^{(0)}.
\end{aligned}\tag{4.6}
$$

Finally, we consider the $\mathcal{O}(\alpha_2\alpha_1^2)$ terms and obtain

$$
\begin{aligned}
&\alpha_1^2\alpha_2\Big[-\partial_1 h^{(0)}\Lambda\partial_1^3 h^{(0)}+\left[\!\left[\Lambda,h^{(0)}\right]\!\right]\partial_1^4 h^{(0)}-\partial_1^2\left[\!\left[\Lambda,h^{(0)}\right]\!\right]\partial_1^2 h^{(0)}\\
&\qquad+\Lambda\left[\left(\partial_1^2 h^{(0)}\right)^2+2\partial_1 h^{(0)}\partial_1^3 h^{(0)}\right]+\partial_1^2\left(\partial_1 h^{(0)}\partial_1\Lambda h^{(0)}\right)\Big]\\
&=\alpha_1^2\alpha_2\partial_1\left[\!\left[\partial_1^2,h^{(0)}\right]\!\right]\Lambda\partial_1 h^{(0)}.
\end{aligned}\tag{4.7}
$$

Collecting (3.11), (3.12), (3.15), (3.16), (4.6), and (4.7), we conclude the following equation for $h^{(1)}$:

$$
\begin{aligned}
&h_{tt}^{(1)}-(\alpha_1^2+\alpha_2)\partial_1^2 h_t^{(1)}+\Lambda h^{(1)}+\beta\Lambda^3 h^{(1)}+\alpha_1^2\alpha_2\partial_1^4 h^{(1)}\\
&\quad=-\Lambda\left(\left(\mathcal{H}h_t^{(0)}\right)^2\right)+\partial_1\left[\!\left[\mathcal{H},h^{(0)}\right]\!\right]\Lambda h^{(0)}+\beta\partial_1\left[\!\left[\mathcal{H},h^{(0)}\right]\!\right]\Lambda^3 h^{(0)}
\end{aligned}
$$

$$+ \alpha_2\partial_1 \left[\!\left[\mathcal{H}, \mathcal{H}h_t^{(0)}\right]\!\right]\mathcal{H}\partial_1^2 h^{(0)} + \alpha_2\Lambda\left(\mathcal{H}h_t^{(0)}\mathcal{H}\partial_1^2 h^{(0)}\right) + \alpha_1^2\alpha_2\partial_1\left[\!\left[\partial_1^2, h^{(0)}\right]\!\right]\Lambda\partial_1 h^{(0)}$$
$$- \alpha_1^2\partial_1\left[\!\left[\partial_1^2, h^{(0)}\right]\!\right]\mathcal{H}h_t^{(0)} - \alpha_2\alpha_2\partial_1\left[\!\left[\mathcal{H}, \partial_1^2 h^{(0)}\right]\!\right]\partial_1^2 h^{(0)}.$$

Thus, neglecting errors of order $\mathcal{O}(\varepsilon^2)$, we conclude the following model for the renormalized variable (3.19):

$$\begin{aligned} &f_{tt} - (\alpha_1^2+\alpha_2)\partial_1^2 f_t + \Lambda f + \beta\Lambda^3 f + \alpha_1^2\alpha_2\partial_1^4 f \\ &\quad = \varepsilon\Big\{ - \Lambda\left((\mathcal{H}f_t)^2\right) + \partial_1[\![\mathcal{H}, f]\!]\Lambda f + \beta\partial_1[\![\mathcal{H}, f]\!]\Lambda^3 f \\ &\qquad + \alpha_2\partial_1[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_1^2 f + \alpha_2\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_1^2 f\right) + \alpha_1^2\alpha_2\partial_1[\![\partial_1^2, f]\!]\Lambda\partial_1 f \\ &\qquad - \alpha_1^2\partial_1[\![\partial_1^2, f]\!]\mathcal{H}f_t - \alpha_2\alpha_2\partial_1[\![\mathcal{H}, \partial_1^2 f]\!]\partial_1^2 f\Big\}. \end{aligned} \tag{4.8}$$

When $\alpha_2 = \alpha_1^2$, (4.8) is an asymptotic model of the damped water waves system proposed by Dias, Dyachenko, and Zakharov [10]. Also, when $\alpha_2 = \alpha_1^2 = 0$, (4.8) again recovers the quadratic h-model in [1, 2, 6, 30, 31, 32].

5. Craig–Sulem models for damped water waves. The pioneer work of Craig and Sulem [9] (see also [8, 33, 34]) leads, among other things, to several asymptotic models obtained by truncating a Taylor series for the Dirichlet-to-Neumann operator present in the Zakharov formulation of the water waves problem [46]. Probably the most famous model of this type is the Craig–Sulem WW2 (see [4, 6, 28]):

$$f_t = -\varepsilon\partial_1 f\partial_1\zeta + \Lambda\zeta - \varepsilon[\![\Lambda, f]\!]\Lambda\zeta, \tag{5.1}$$
$$\zeta_t = \frac{\varepsilon}{2}\left[(\Lambda\zeta)^2 - (\partial_1\zeta)^2\right] - f + \beta\partial_1^2 f. \tag{5.2}$$

Using the Tricomi identity (3.10), the previous system can be equivalently written as

$$f_t = -\varepsilon\partial_1 f\partial_1\zeta + \Lambda\zeta - \varepsilon[\![\Lambda, f]\!]\Lambda\zeta, \tag{5.3}$$
$$\zeta_t = \varepsilon\mathcal{H}\left(\partial_1 f\Lambda f\right) - f + \beta\partial_1^2 f. \tag{5.4}$$

5.1. Case $s = 0$. Using (3.7) and (3.8) we find that, up to an error $\mathcal{O}(\varepsilon^2)$, the variables

$$f = h^{(0)} + \varepsilon h^{(1)},\ \zeta = \xi^{(0)} + \varepsilon\xi^{(1)}, \tag{5.5}$$

solve the system

$$f_t = -\varepsilon\partial_1 f\partial_1\zeta + \Lambda\zeta - \varepsilon[\![\Lambda, f]\!]\Lambda\zeta + \alpha_2\partial_1^2 f, \tag{5.6}$$
$$\zeta_t = \varepsilon\mathcal{H}\left(\partial_1 f\Lambda f\right) - f + \beta\partial_1^2 f - \alpha_1^0\zeta + \alpha_2\varepsilon\Lambda\zeta\partial_1^2 f. \tag{5.7}$$

5.2. Case $s = 2$. Using (4.4) and (4.5), we also find the viscous analog (called Craig–Sulem WWV2 [3, 23]) of the Craig–Sulem WW2 model corresponding to the model of Dias, Dyachenko, and Zakharov [10] of water waves with viscosity

(5.8) $f_t = -\varepsilon\partial_1 f\partial_1\zeta + \Lambda\zeta - \varepsilon[\![\Lambda, f]\!]\Lambda\zeta + \alpha_2\partial_1^2 f,$

(5.9) $\zeta_t = \varepsilon\mathcal{H}\left(\partial_1 f\Lambda f\right) - f + \beta\partial_1^2 f - \alpha_1^2\left(\Lambda^2\zeta + \varepsilon\partial_1^2 f\Lambda\zeta + 2\varepsilon\partial_1 f\partial_1\Lambda\zeta\right) + \alpha_2\varepsilon\Lambda\zeta\partial_1^2 f.$

6. Study of the models and discussion. In this paper we have obtained a number of new models for damped water waves. Of course, one may ask why viscosity effects are required when studying water waves. Besides the fact that every liquid is viscous, there are a number of scenarios where the viscous damping needs to be taken into account. For instance, damping has been used to study standing surface waves generated in a vertically oscillating container (these waves are called Faraday waves) or the question of stabilization of the Benjamin–Feir stability [43].

In particular, we derived two nonlocal wave equations, namely,

$$\begin{aligned}
&f_{tt} + \Lambda f + \beta\Lambda^3 f + \alpha_1^0 f_t - \alpha_1^0\alpha_2\partial_1^2 f - \alpha_2\partial_1^2 f_t \\
&\quad = \varepsilon\Big[-\Lambda\left((\mathcal{H}f_t)^2\right) + \partial_1[\![\mathcal{H}, f]\!]\Lambda f + \beta\partial_1[\![\mathcal{H}, f]\!]\Lambda^3 f + \alpha_2\partial_1[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_1^2 f \\
&\qquad + \alpha_2\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_1^2 f\right) - \alpha_2^2\partial_1\left[\!\left[\mathcal{H}, \partial_1^2 f\right]\!\right]\partial_1^2 f\Big],
\end{aligned} \tag{6.1}$$

and

$$\begin{aligned}
&f_{tt} - (\alpha_1^2 + \alpha_2)\partial_1^2 f_t + \Lambda f + \beta\Lambda^3 f + \alpha_1^2\alpha_2\partial_1^4 f \\
&\quad = \varepsilon\Big\{ -\Lambda\left((\mathcal{H}f_t)^2\right) + \partial_1[\![\mathcal{H}, f]\!]\Lambda f + \beta\partial_1[\![\mathcal{H}, f]\!]\Lambda^3 f \\
&\qquad + \alpha_2\partial_1[\![\mathcal{H}, \mathcal{H}f_t]\!]\mathcal{H}\partial_1^2 f + \alpha_2\Lambda\left(\mathcal{H}f_t\mathcal{H}\partial_1^2 f\right) + \alpha_1^2\alpha_2\partial_1\left[\!\left[\partial_1^2, f\right]\!\right]\Lambda\partial_1 f \\
&\qquad - \alpha_1^2\partial_1\left[\!\left[\partial_1^2, f\right]\!\right]\mathcal{H}f_t - \alpha_2\alpha_2\partial_1\left[\!\left[\mathcal{H}, \partial_1^2 f\right]\!\right]\partial_1^2 f\Big\}.
\end{aligned} \tag{6.2}$$

Equation (6.1) is an asymptotic model of the damped water waves system proposed by Jiang et al. [21] and Wu, Liu, and Yue [43], while (6.2) is an asymptotic model of the water waves with viscosity system proposed by Dias, Dyachenko, and Zakharov [10].

It is a natural question to ask whether these ideas can be extended to three-dimensional waves. Although the extension would not be trivial, these ideas can be applied to three dimensions. This should be addressed in a future work.

Another reasonable question is which model is better for which application. In general, it is assumed in the literature that (2.1) is more realistic when $s = 2$, regardless of whether $\delta_2 = 0$ or not (see [43] for instance). That would mean that (6.2) corresponds to a more realistic description of viscous damping of water waves.

One of the advantages of having an asymptotic model akin to (6.1) or (6.2) is that, as there is no Dirichlet–Neumann operator nor elliptic problem involved, it is easier and cheaper to simulate than the full problem (2.1). However, when $\alpha_1^2, \alpha_2 \neq 0$, the presence of higher order operators as the bi-Laplacian may cause numerical difficulties. Thus, although (6.2) is linked to a more realistic description, its implementation may not be straightforward. A careful numerical study of these models should be addressed elsewhere. Also, this numerical study could help make the decision of which models are better for which applications.

6.1. Typical values of the dimensionless parameters. Let's consider a numerical example. The values of the physical parameters are (see [27])

$$G = 9.8 \text{ m/s}^2,\ \gamma = 72\cdot 10^{-3} \text{ kg/s}^2,\ \rho = 1029 \text{ kg/m}^3.$$

We consider a wave of size

$$H = 0.02 \text{ m}, \quad L = 0.6 \text{ m}.$$

This wave follows the scenario in [21]. Recalling (2.7) (where $\delta_2 = \nu$), we have that

$$\varepsilon \approx 0.03, \qquad \beta = \frac{72 \cdot 10^{-3}}{1029 \cdot 9.8 \cdot (0.6)^2} \approx 2 \cdot 10^{-5}. \tag{6.3}$$

According to [21, section 4], the experimental decay rate in the scenario modeled by (6.1) is estimated as 0.05 s^{-1}. Also, following [10] we have that the right viscosity to be used in these applications is the eddy viscosity value

$$\nu = 10^{-3}.$$

That means that

$$\alpha_1^0 = \frac{0.05}{\sqrt{9.8} \cdot (0.6)^{-\frac{1}{2}}} \approx 0.01, \qquad \alpha_1^2 = \alpha_2 = \frac{10^{-3}}{\sqrt{9.8}(0.6)^{3/2}} \approx 6.8 \cdot 10^{-4}. \tag{6.4}$$

Then, we see that viscous damping effects are at the same level as ε^2 and are somehow more relevant than surface tension effects.

6.2. Linear analysis and dispersion relations. In this section we are going to study the dispersion relation of the models (see also [12]). In the case where viscous effects are neglected ($\alpha_1^s = \alpha_2 = 0$) the model was studied in [6]. In this case, the dispersion relation is

$$\omega_I(k) = \sqrt{|k| \left(1 + \beta |k|^2\right)}, \qquad \beta \geqslant 0, \tag{6.5}$$

which is, of course, the same dispersion relation as for the full water waves problem with infinite depth.

We want to understand now how this dispersion relation is affected by the viscous effects. Keeping only the linear terms in (6.1) and (6.2) and inserting the standard plane wave ansatz

$$f(x,t) = e^{ikx - i\omega t},$$

we obtain the following dispersion relations:

$$\begin{aligned} \omega_\pm^{[0]}(k) &= \pm \frac{\sqrt{-(|k|^2\alpha_2 + \alpha_1^0)^2 + 4(|k|(1+\beta|k|^2) + |k|^2\alpha_1^0\alpha_2)}}{2} \\ &\quad - \frac{i\left(\alpha_1^0 + \alpha_2 |k|^2\right)}{2}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} \omega_\pm^{[2]}(k) &= \pm \frac{\sqrt{-\left(\alpha_1^2 + \alpha_2\right)^2 |k|^4 + 4(|k|(1+\beta|k|^2) + \alpha_1^2\alpha_2|k|^4)}}{2} \\ &\quad - \frac{i\left(\alpha_1^2 + \alpha_2\right)|k|^2}{2}, \end{aligned} \tag{6.7}$$

where $\omega_{\pm}^{[0]}$ and $\omega_{\pm}^{[2]}$ correspond to (6.1) and (6.2), respectively. These dispersion relations are valid for the whole range of values of the dimensionless parameters. We emphasize that the imaginary parts present in the previous expressions for the dispersion relations imply parabolic behavior or, if $\alpha_2 = 0$ in (6.6), at least absorption.

Using the previous numerical values (6.3) and (6.4), we find that, neglecting terms of order $O(10^{-5})$ for a large range of k's, the dispersion relation (6.6) can be approximated by

$$\omega_{\pm}^{[0]}(k) \approx \pm\frac{\sqrt{-(\alpha_1^0)^2 + 4|k|}}{2} - \frac{i\left(\alpha_1^0 + \alpha_2 |k|^2\right)}{2}.$$

Similarly,

$$\omega_{\pm}^{[2]}(k) \approx \pm\sqrt{|k|} - \frac{i\left(\alpha_1^2 + \alpha_2\right)|k|^2}{2},$$

From the previous dispersion relations, $\omega_{\pm}^{[s]}$, and the dispersion relation for the inviscid model ω_I, we see that both models (6.1) and (6.2) have a parabolic behavior. In fact, the dissipation rate in (6.1) when $\alpha_2 = 0$ is independent of the Fourier mode k, while, for model (6.2) the dissipation is purely of parabolic type $O(|k|^2)$ (see [26]).

Appendix A. The explicit solution of an elliptic problem.

LEMMA A.1. *Let us consider the Poisson equation*

$$\text{(A.1)}\qquad \begin{cases} \Delta u(x_1, x_2) = b(x_1, x_2), & (x_1, x_2) \in \mathbb{S}^1 \times (-\infty, 0), \\ u(x_1, 0) = g(x_1), & x_1 \in \mathbb{S}^1, \\ \lim\limits_{x_2 \to -\infty} \partial_2 u(x_1, x_2) = 0, & x_1 \in \mathbb{S}^1, \end{cases}$$

where we assume that the forcing b and the boundary data g are smooth and decay sufficiently fast at infinity. Then, the unique solution u of (A.1) *is given by*

(A.2)

$$\begin{aligned} u(x_1, x_2) = -\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \Bigg\{ & \frac{1}{|k|}\left[\frac{1}{2}\int_{-\infty}^{0} \hat{b}(k, y_2)\, e^{|k|y_2} \mathrm{d}y_2 - |k|\, \hat{g}(k)\right] e^{|k|x_2} \\ & - \frac{1}{2|k|}\int_{-\infty}^{0} \hat{b}(k, y_2)\, e^{|k|y_2} \mathrm{d}y_2 \; e^{-|k|x_2} \\ & + \int_{0}^{x_2} \frac{\hat{b}(k, y_2)}{2|k|}\left[e^{|k|(y_2 - x_2)} - e^{|k|(x_2 - y_2)}\right] \mathrm{d}y_2 \Bigg\} e^{ikx_1}, \end{aligned}$$

where the operator $\hat{\cdot}$ *denotes the Fourier transform in the variable x_1. In particular*

$$\text{(A.3)}\qquad \partial_2 u(x_1, 0) = \int_{-\infty}^{0} e^{y_2 \Lambda} b(x_1, y_2)\, \mathrm{d}y_2 + \Lambda g(x_1),$$

$$\text{(A.4)}\qquad \partial_2^2 u(x_1, 0) = -\partial_1^2 g(x_1) + b(x_1, 0).$$

Proof. Let us apply the Fourier transform to (A.1); this transforms the PDE (A.1) into the following series of second order inhomogeneous costant coefficient ODEs:

$$\text{(A.5)}\qquad \begin{cases} -k^2 \hat{u}(k, x_2) + \partial_2^2 \hat{u}(k, x_2) = \hat{b}(k, x_2), & (k, x_2) \in \mathbb{Z} \times (-\infty, 0), \\ \hat{u}(k, 0) = \hat{g}(k), & k \in \mathbb{Z}, \\ \lim\limits_{x_2 \to -\infty} \partial_2 \hat{u}(k, x_2) = 0, & k \in \mathbb{Z}. \end{cases}$$

The generic solution of (A.5) can be deduced using the variation of parameters method, whence
(A.6)

$$\hat{u}\left(k, x_2\right) = C_1\left(k\right) e^{|k| x_2} + C_2\left(k\right) e^{-|k| x_2} - \int_0^{x_2} \frac{\hat{b}\left(k, y_2\right)}{2\left|k\right|}\left[e^{|k|(y_2 - x_2)} - e^{|k|(x_2 - y_2)}\right] \mathrm{d}y_2.$$

The boundary conditions determine the values of the C_i's:

$$C_2\left(k\right) = -\frac{1}{2\left|k\right|}\int_{-\infty}^{0} \hat{b}\left(k, y_2\right) e^{|k| y_2} \mathrm{d}y_2, \qquad C_1\left(k\right) = -C_2\left(k\right) + \hat{g}\left(k\right). \tag{A.7}$$ □

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