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RESEARCH ARTICLE

A simple characterization of the existence of upper semicontinuous order-preserving functions

Gianni Bosi[1] · Laura Franzoi[1]

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Abstract
We introduce an upper semicontinuity condition concerning a not necessarily total preorder on a topological space, namely *strong upper semicontinuity*, and in this way we extend to the nontotal case the famous Rader's theorem, which guarantees the existence of an upper semicontinuous order-preserving function for an upper semicontinuous total preorder on a second countable topological space. We show that Rader's theorem is not generalizable if we adopt weaker upper semicontinuity conditions already introduced in the literature. We characterize the existence of an upper semicontinuous order-preserving function for all strongly upper semicontinuous preorders on a metrizable topological space.

Keywords Order-preserving function · Upper semicontinuous function · Strongly upper semicontinuous preorder

JEL classification: C60 · D01

1 Introduction

It is well known that the existence of an *order-preserving* function (or, equivalently, a *strictly isotone function*) for a not necessarily total preorder is important since its maximization leads to the existence of maximal elements. An order-preserving function provides the largest amount of information concerning the preorder, despite for the fact that it cannot fully characterize it in the nontotal case. Indeed, an order-preserving function for a nontotal preorder provides an *extension* of the preorder by means of a total preorder which is representable by a *utility function*.

However, the knowledge of an order-preserving function can be considered "sufficient" for the aforementioned reasons (see, e.g., the introduction in Bosi and Herden

✉ Gianni Bosi
gianni.bosi@deams.units.it

1 DEAMS, University of Trieste, Via Valerio 4/1, 34127 Trieste, Italy

2002). A folk maximum theorem allows to identify the maximal elements of a not necessarily total preorder on a compact topological space, provided that an *upper semicontinuous* order-preserving function exists. Rader (1963) proved that an upper semicontinuous total preorder on a second countable topological space admits an upper semicontinuous order-preserving function. Rader's theorem, which is fundamental and widely referred to in the economic literature, has been discussed, for example, by Mehta (1997).

The generalization of Rader's theorem to the case of nontotal preorders is not immediate and requires some, so to say, ad hoc upper semicontinuity condition. Indeed, for example it is not true that every *weakly upper semicontinuous* preorder on a second countable topological space admits an upper semicontinuous order-preserving function, as we show below in Proposition 2.9 (see Bosi and Herden 2005). Therefore, we introduce the notion of a *strongly upper semicontinuous* preorder, which slightly generalizes that of an *upper semiclosed* preorder, according to which the *weak upper sections* are closed. We show that now Rader's theorem is generalizable by using such concept of upper semicontinuity. We further characterize the existence of an upper semicontinuous order-preserving function for all strongly upper semicontinuous preorders on a metrizable topological space. We underline that the present research is in the spirit of previous papers concerning the existence of upper semicontinuous weak utilities (see, e.g., Andrikopoulos 2011), and in some sense, of resent research about the existence of maximal elements (see, e.g., Andrikopoulos and Zacharias 2012; Quartieri 2022).

2 Notation and preliminary results

We start from the classical definitions concerning binary relations and their real representations, in order to then recall different versions of continuity of a preorder on a topological space.

Definition 2.1 Let $\precsim$ be a binary relation on a nonempty set X (i.e., $\precsim \subset X \times X$). Then $\precsim$ is said to be

(1) *reflexive*, if $x \precsim x$, for every $x \in X$;
(2) *transitive*, if $(x \precsim y)$ and $(y \precsim z)$ imply $(x \precsim z)$, for all $x, y, z \in X$;
(3) *antisymmetric*, if $(x \precsim y)$ and $(y \precsim x)$ imply $x = y$, for all $x, y \in X$;
(4) *total*, if either $(x \precsim y)$ or $(y \precsim x)$, for all $x, y \in X$;
(5) *linear* (or *complete*), if either $(x \precsim y)$ or $(y \precsim x)$, for all $x \neq y$ $(x, y \in X)$;
(6) a *preorder*, if $\precsim$ is reflexive and transitive;
(7) an *order*, if $\precsim$ is an antisymmetric preorder;
(8) a *chain*, if $\precsim$ is a linear order.

The *strict part* (or *asymmetric part*) of a preorder $\precsim$ on X is defined as follows, for all $x, y \in X$: $x \prec y$ if and only if $(x \precsim y)$ *and not*$(y \precsim x)$. Further, the *symmetric part* $\sim$ of a preorder $\precsim$ on X is defined as follows, for all $x, y \in X$: $x \sim y$ if and only if $(x \precsim y)$ *and* $(y \precsim x)$.

Definition 2.2 Let $(X, \precsim)$ be a preordered set. Then a subset A of X is said to be *decreasing* if $x \in A$ and $y \precsim x$ imply that $y \in A$.

Definition 2.3 A real-valued function u on a preordered set $(X, \precsim)$ is said to be

(1) *increasing*, if, for all $x, y \in X$,

$$x \precsim y \Rightarrow u(x) \leq u(y);$$

(2) *order-preserving*, if u is increasing and, for all $x, y \in X$,

$$x \prec y \Rightarrow u(x) < u(y);$$

(3) a *utility function*, if $x \precsim y$ is equivalent to $u(x) \leq u(y)$, for all $x, y \in X$.

It is easily seen that a utility function, which may exist only in case that the preorder is total, is also order-preserving, and that an order-preserving function is a utility function if the preorder is total.

In the sequel, t_{nat} will stand for the *natural topology* (i.e., *interval topology*) on $\mathbb{R}$. If $\precsim$ is a preorder on a topological space (X, t), then we shall denote by $t^{\downarrow}_{\precsim}$ the set of all the open decreasing subsets of X.

Definition 2.4 A preorder $\precsim$ on a topological space (X, t) is said to be

(1) *upper semicontinuous*, if $l_{\precsim}(x) = \{z \in X; z \prec x\}$ is open for every $x \in X$;
(2) *upper semiclosed*, if $i_{\precsim}(x) = \{z \in X; x \precsim z\}$ is closed for every $x \in X$;
(3) *weakly upper semicontinuous*, if for every $(x, y) \in \prec$, there exists a set $O \in t^{\downarrow}_{\precsim}$ such that $(x, y) \in O \times (X \setminus O)$.

We shall limit ourselves to the study of upper semicontinuity of a preorder on a topological space and its variants because upper and lower semicontinuity are dual concepts.

Remark 2.5 Herden and Levin (2012) call *upper semicontinuous of Type 1* an upper semicontinuous preorder and *upper semicontinuous of Type 2* an upper semiclosed preorder.

It is immediate to check that upper semicontinuity and upper semiclosedness are equivalent concepts in case that the preorder is total, while they are not equivalent in general, as the following simple example shows. We recall that $\Delta_X := \{(x, x) \mid x \in X\}$ is the *diagonal* of X.

Example 2.6 Consider the real interval $[0, 1]$ endowed with the natural (induced) topology t_{nat} on $[0, 1]$. Then define the preorder $\precsim$ on X as follows:

$$\precsim = \Delta_X \cup \left\{(z, w) : z = 0,\ \ w \in \left[\frac{1}{2}, 1\right]\right\} \cup \left\{(z, w) : z \in \left]0, \frac{1}{2}\right[,\ \ w \in \left]0, \frac{1}{2}\right[\right\}.$$

We have that $\precsim$ is not upper semicontinuous since, for example, $l_{\precsim}(\frac{1}{2}) = \{0\}$ is not open. On the other hand, $\precsim$ is upper semiclosed, since

$$i(x) := \begin{cases} \{0\} \cup \left[\frac{1}{2}, 1\right], & \text{if } x = 0 \\ \left[x, \frac{1}{2}\right], & \text{if } x \in \left]0, \frac{1}{2}\right] \\ \{x\} & \text{if } x \in \left]\frac{1}{2}, 1\right] \end{cases}.$$

The following function is order-preserving for $\precsim$:

$$u(x) := \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right[\\ 1, & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

The simple proof of the following proposition, characterizing a weakly upper semicontinuous preorder, is left to the reader.

Proposition 2.7 *A preorder $\precsim$ on a topological space (X, t) is weakly upper semicontinuous if and only if one of the following equivalent conditions is verified:*

(1) *For every pair $(x, y) \in \prec$ there exists an upper semicontinuous and increasing real-valued function u_{xy} on X, such that $u_{xy}(x) < u_{xy}(y)$;*
(2) *For every $x \in X$ there exists a set $l^0_{\precsim}(x) \in t^{\downarrow}_{\precsim}$ satisfying the following conditions:*
    (a) *$x \notin l^0_{\precsim}(x)$;*
    (b) *$l_{\prec}(x) \subset l^0_{\precsim}(x)$.*

It is clear that an upper semicontinuous total preorder is weakly upper semicontinuous. Condition (2) of Proposition 2.7 suggests that the sets $l^0_{\precsim}(x)$ $(x \in X)$ surrogate the *strict lower sections* $l_{\prec}(x)$ $(x \in X)$ in the general case when $\precsim$ is a weakly upper semicontinuous preorder on (X, t). Weakly upper semicontinuous (pre)orders were studied by Bosi and Herden (2005).

A real-valued function u on a topological space (X, t) is said to be *upper semicontinuous* if $u^{-1}(]-\infty, \alpha[) = \{x \in X : u(x) < \alpha\}$ is an open set for all $\alpha \in \mathbb{R}$.

Let us recall the famous Rader's utility representation theorem (see Rader 1963).

Theorem 2.8 (Rader's theorem) *Let $\precsim$ be an upper semicontinuous total preorder on a second countable topological space (X, t). Then there exists an upper semicontinuous utility function $u : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$.*

Unfortunately, Rader's theorem is not generalizable to weakly upper semicontinuous preorders. Indeed, the following very restrictive result holds, as a corollary of Bosi and Herden (2005, Theorem 3.1).

Proposition 2.9 *If every weakly upper semicontinuous order $\lesssim$ has an upper semicontinuous order-preserving function on a Hausdorff topological space (X, t), then either t is the discrete topology on X, or there exists a point $x \in X$ such that $t \cap (X \setminus \{x\})$ is the discrete topology on $X \setminus \{x\}$.*

Let us introduce the new definition of a *strongly upper semicontinuous preorder* on a topological space.

Definition 2.10 We say that a preorder $\precsim$ on a topological space (X, t) is *strongly upper semicontinuous* if there exists a mapping $l^0_{\precsim} : X \to t^{\downarrow}_{\precsim}$ satisfying the following conditions:

(1) For every $x \in X$, $x \notin l^0_{\precsim}(x) \supset l_{\precsim}(x)$;

(2) $x \prec y \Rightarrow l^0_{\precsim}(x) \subsetneq l^0_{\precsim}(y)$.

The following simple proposition motivates the above Definition 2.10.

Proposition 2.11 *If there exists an upper semicontinuous order-preserving function* $u : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$*, then the preorder* $\precsim$ *is strongly upper semicontinuous on the topological space* (X, t).

Proof If u is an upper semicontinuous order-preserving function for $\precsim$ on (X, t), then just define, for every $x \in X$,

$$l^0_{\precsim}(x) = u^{-1}(]-\infty, u(x)[),$$

in order to immediately verify that $\precsim$ is strongly upper semicontinuous. □

Remark 2.12 An upper semiclosed preorder on a topological space (X, t) is strongly upper semicontinuous. Indeed, if $i_{\precsim}(x) = \{z \in X : x \precsim z\}$ is closed for every $x \in X$, then just define, for every $x \in X$, $l^0_{\precsim}(x) = X \setminus i_{\precsim}(x)$.

The concept of a strongly upper semicontinuous preorder strengthens that of a weakly upper semicontinuous preorder, as the following easy proposition shows.

Proposition 2.13 *If* $\precsim$ *is a strongly upper semicontinuous preorder on a topological space* (X, t)*, then* $\precsim$ *is weakly upper semicontinuous on* (X, t).

Proof Just consider that, if $x \prec y$, then $(x, y) \in (l^0_{\precsim}(y), X \setminus l^0_{\precsim}(y))$. □

We are now ready to present a generalization of Rader's theorem to the case of a nontotal preorder.

Theorem 2.14 (Generalized Rader Theorem) *Let* $\precsim$ *be a strongly upper semicontinuous preorder on a second countable topological space* (X, t)*. Then there exists an upper semicontinuous order-preserving function* $u : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$.

Proof Let $\mathcal{B} = \{B_n\}_{n \in \mathbb{N}^+}$ be a countable base for t. Define, for every $x \in X$,

$$u(x) = \begin{cases} \sum\limits_{k \in \mathbb{N}^+, B_k \subset l^0_{\precsim}(x)} \dfrac{1}{2^k} & \text{if } l_{\precsim}(x) \neq \emptyset \\ 0, & \text{if } l_{\precsim}(x) = \emptyset \end{cases}.$$

in order to immediately verify that u is an upper semicontinuous order-preserving function for $\precsim$ on (X, t). □

Another useful property of strongly upper semicontinuous preorders is in order now. We first recall the concept of an upper semicontinuous multi-utility representation (see e.g. Bosi and Herden 2016).

Definition 2.15 A preorder $\precsim$ on a topological space (X, t) is said to have an *upper semicontinuous multi-utility representation* if there exists a family $\mathcal{U}$ of upper semicontinuous increasing real-valued functions u on $(X, \precsim, t)$ such that, for all $x \in X$ and all $y \in Y$, the following equivalence holds:

$$x \precsim y \Leftrightarrow \forall u \in \mathcal{U}\ (u(x) \leq u(y)).$$

It is easily seen that a preorder admitting an upper semicontinuous multi-utility representation is upper semiclosed, and therefore also strongly upper semicontinuous. Therefore, the following proposition holds.

Proposition 2.16 *If a preorder $\precsim$ on a topological space (X, t) has an upper semicontinuous multi-utility representation $\mathcal{U}$, then $\precsim$ is strongly upper semicontinuous.*

Definition 2.17 We say that a topology t on a set X is

(1) *useful*, if every continuous total preorder $\precsim$ on (X, t) admits a continuous utility function $u : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$;
(2) *upper useful*, if every upper semicontinuous total preorder $\precsim$ on (X, t) has an upper semicontinuous utility function $u : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$;
(3) *strongly upper useful*, if every strongly upper semicontinuous preorder $\precsim$ on (X, t) has an upper semicontinuous order-preserving function $u : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$.

Remark 2.18 We recall that the concept of a useful topology was introduced by Herden (1991). The notion of an upper useful topology was inaugurated by Bosi and Herden (2002), who referred to a *completely useful topology* (see also Campión et al. 2009).

It is known that an upper useful topology is useful. The following proposition holds.

Proposition 2.19 *A strongly upper useful topology t on a set X is upper useful.*

Proof By contraposition, assume that a topology t on a set X is not upper useful. Then there exists an upper semicontinuous total preorder $\precsim$ on (X, t) that doesn't admit an upper semicontinuous utility function u. Then, $\precsim$ is in particular a strongly upper semicontinuous preorder, which can't admit an upper semicontinuous order-preserving function (otherwise, an order-preserving function would be a utility function for $\precsim$ since $\precsim$ is total). This consideration completes the proof. □

Let us present an example of a useful topology, which is not strongly upper useful.

Example 2.20 Let $\aleph_1$ be the first uncountable ordinal number (cardinal number). Then we consider the set X of all ordinal numbers $\alpha \leq \aleph_1$ and assume X to be assigned with the topology t that consists of the empty set and all subsets Y of X the complement of which is finite. (X, t) is an irreducible topological space, in the sense that the intersection of any two disjoint non-empty open subsets of X is non-empty. Therefore, the constant functions are the only continuous real-valued functions on X, and the only continuous total preorder on X is the preorder $\precsim := \sim$. Now, we choose some fixed point $z \in X$, and consider for every $x \in X \setminus \{z\}$ the upper semicontinuous real-valued function $u_x : (X, t) \to (\mathbb{R}, t_{nat})$ that is defined by

$$u_x(y) := \begin{cases} 0, & \text{if } y \in X \setminus \{x\} \\ 1, & \text{if } y = x \end{cases}.$$

Let $\mathcal{U}$ be the set of these functions. Then we define a weakly upper semicontinuous preorder $\precsim$ on X as follows:

$$x \precsim y \Leftrightarrow \forall v \in \mathcal{U}\ (v(x) \leq v(y)).$$

Since $\mathcal{U}$ provides an upper semicontinuous multi-utility representation of $\precsim$, we have that actually $\precsim$ is a strongly upper semicontinuous preorder by Proposition 2.16. We still must verify that there exist no upper semicontinuous order-preserving function u for $\precsim$. Let us assume, in contrast, that such function u exists. The definition of $\precsim$ implies that z is the point of minimum of u. In addition, since $u^{-1}([u(x), +\infty[)$ is closed for every $x \in X$, and $X \setminus \{z\}$ is an uncountable set, there must exist at least one $x \in X \setminus \{z\}$ such that $u^{-1}([u(x), +\infty[)$ is an infinite set. Indeed, otherwise $X \setminus \{z\} = \bigcup_{x \in X \setminus \{z\}} u^{-1}([u(x), +\infty[)$ is a countable set. This observation contradicts the fact that X is the only uncountable closed subset of X. Hence, t is not strongly upper useful.

Remark 2.21 Candeal et al. (1998, Theorem 1) proved that, for metrizable topologies, separability (or, equivalently, second countability) and usefulness are equivalent concepts. Bosi and Herden (2002, Corollary 4.5) proved that a metrizable topology is upper useful if and only if t is second countable. Therefore, for metrizable topologies, separability, usefulness and upper usefulness are equivalent concepts.

Remark 2.22 Bosi and Herden (2002, Corollary 4.5) proved that a metrizable topology is upper useful if and only if it is separable. Therefore, this latter result, Theorem 2.14 and Proposition 2.19 guarantee that, for metrizable topologies, strongly upper usefulness and separability are equivalent concepts. Therefore, for metrizable topologies, separability, usefulness, upper usefulness and strongly upper usefulness are equivalent concepts.

3 Conclusions

In this paper the concept of a *strongly upper semicontinuous* preorder has been introduced in order to generalize the classical Rader's utility representation theorem to the case of nontotal preorders on second countable topological spaces. We have also shown that, when we consider metrizable topological spaces, second countability is equivalent to the requirement according to which every strongly upper semicontinuous preorder admits an upper semicontinuous order-preserving function. The case of topological spaces which are not metrizable will be considered in a future paper in case that interesting results can be achieved.

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