

HOPF ALGEBROIDS FROM NONCOMMUTATIVE BUNDLES

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ABSTRACT. We present two classes of examples of Hopf algebroids associated with non-commutative principal bundles. The first comes from deforming the principal bundle while leaving unchanged the structure Hopf algebra. The second is related to deforming a quantum homogeneous space; this needs a careful deformation of the structure Hopf algebra in order to preserve the compatibilities between the Hopf algebra operations.

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1. INTRODUCTION

A commutative Hopf algebroid is somehow the dual of a groupoid, in the spirit of Hopf algebras versus groups. One is extending the scalar, similarly to the passage from Hilbert space to Hilbert module: the ground field k gets replaced by an algebra B which could be noncommutative. The result is a bi-algebra over a noncommutative base algebra. In fact, in general not all structures survive: there is a notion of coproduct and counit but in general there is no antipode. The notions of source and target maps are still present.

An important groupoid used in gauge theory, is the gauge groupoid associated with a principal bundle [11]. In [10], as a preliminary step to study the gauge group of a non-commutative principal bundle, we considered the Ehresmann–Schauenburg bialgebroid of the noncommutative bundle which, in a sense, is the quantization of the classical gauge

groupoid. For a monopole bundle over a quantum Podleś sphere and a not faithfully flat Hopf–Galois extension of commutative algebras we gave a suitable invertible antipode so that the corresponding bialgebroids got upgraded to Hopf algebroids.

In the present paper we study two classes of examples of Hopf algebroids associated with noncommutative principal bundles. The first comes from deforming the principal bundle while leaving unchanged the structure Hopf algebra. The prototype for this is the bundle over the noncommutative four-sphere S_θ^4 with classical $SU(2)$ as structure group. The second class is associated to deformations of quantum homogeneous spaces. It is known that one needs a careful deformation of the multiplication in a Hopf algebra in order to preserve the compatibilities between the Hopf algebra structures. And this attention is needed also for deforming homogeneous spaces. Examples of the second class are the principal bundles over the noncommutative spheres S_θ^{2n} with noncommutative orthogonal group $SO_\theta(2n, \mathbb{R})$ as structure group.

This paper is organised as follows. In §2 we give a recap of algebraic preliminaries and notation, and of the relevant concepts for noncommutative principal bundles (Hopf–Galois extensions), bialgebroids and Hopf algebroids. We devote §3 to two well know examples of Hopf–Galois extensions for which in §§ 5.4 and 5.3 we construct the corresponding Hopf algebroids; these are a $SU(2)$ -bundle over the sphere S_θ^4 and $SO_\theta(2n)$ bundles over even spheres S_θ^{2n} . In §4 we review the general scheme of deforming by the action of tori. This is done via \mathbb{Z}^n -graded spaces and deforming relevant structures by means of a bi-character. The discussion is developed along two scenarios to cover the constructions of both §4.1, where the structure Hopf algebra is not changed, and §4.2 where attention is payed to a suitable deformation of the multiplication that is compatible with all Hopf algebra operations, in order to get new Hopf algebras with related comodule algebras. The latter framework accommodates deformed homogeneous spaces. The noncommutative principal bundles that result from both schemes of deformation have natural Ehresmann–Schauenburg bialgebroids. In the context of the present paper the flip map will preserve the bialgebroids and will satisfy all properties for an invertible algebrois antipode. All of these last parts and the examples are worked out in §5.

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2. PRELIMINARY RESULTS

To be definite we work over the field \mathbb{C} of complex numbers. Algebras (coalgebras) are assumed to be unital and associative (counital and coassociative) with morphisms of algebras taken to be unital (of coalgebras taken counital). Tensor product over \mathbb{C} is denoted \otimes while the symbol $\dot{\otimes}$ implies also a matrix sum: for matrices $M = (m_{jk})$ and $N = (n_{kl})$ the product $M \dot{\otimes} N$ have components $M \dot{\otimes} N = (\sum_k m_{jk} \otimes n_{kl})$.

2.1. Rings and corings over an algebra. For an algebra B a *B-ring* is a triple (A, μ, η) . Here A is a B -bimodule with B -bimodule maps $\mu : A \otimes_B A \rightarrow A$ and $\eta : B \rightarrow A$,

satisfying associativity and unit conditions:

$$\mu \circ (\mu \otimes_B \text{id}_A) = \mu \circ (\text{id}_A \otimes_B \mu), \quad \mu \circ (\eta \otimes_B \text{id}_A) = \text{id}_A = \mu \circ (\text{id}_A \otimes_B \eta). \quad (2.1)$$

A morphism of B -rings $f : (A, \mu, \eta) \rightarrow (A', \mu', \eta')$ is an B -bimodule map $f : A \rightarrow A'$, such that $f \circ \mu = \mu' \circ (f \otimes_B f)$ and $f \circ \eta = \eta'$.

From [2, Lemma 2.2] there is a bijective correspondence between B -rings (A, μ, η) and algebra automorphisms $\eta : B \rightarrow A$. Starting with a B -ring (A, μ, η) , one obtains a multiplication map $A \otimes A \rightarrow A$ by composing the canonical surjection $A \otimes A \rightarrow A \otimes_B A$ with the map μ . Conversely, starting with an algebra map $\eta : B \rightarrow A$, a B -bilinear associative multiplication $\mu : A \otimes_B A \rightarrow A$ is obtained from the universality of the coequaliser $A \otimes A \rightarrow A \otimes_B A$ which identifies an element $ar \otimes a'$ with $a \otimes ra'$.

Dually, for an algebra B a B -coring is a triple (C, Δ, ε) . Here C is a B -bimodule with B -bimodule maps $\Delta : C \rightarrow C \otimes_B C$ and $\varepsilon : C \rightarrow B$ that satisfy coassociativity and counit conditions,

$$(\Delta \otimes_B \text{id}_C) \circ \Delta = (\text{id}_C \otimes_B \Delta) \circ \Delta, \quad (\varepsilon \otimes_B \text{id}_C) \circ \Delta = \text{id}_C = (\text{id}_C \otimes_B \varepsilon) \circ \Delta. \quad (2.2)$$

A morphism of B -corings $f : (C, \Delta, \varepsilon) \rightarrow (C', \Delta', \varepsilon')$ is a B -bimodule map $f : C \rightarrow C'$, such that $\Delta' \circ f = (f \otimes_B f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

Let B be an algebra. A *left B -bialgebroid* \mathcal{L} consists of a $(B \otimes B^{op})$ -ring together with a B -coring structures on the same vector space \mathcal{L} , with mutual compatibility conditions [16]. From what said above, a $(B \otimes B^{op})$ -ring \mathcal{L} is the same as an algebra map $\eta : B \otimes B^{op} \rightarrow \mathcal{L}$. Equivalently, one may consider the restrictions

$$s := \eta(\cdot \otimes_B 1_B) : B \rightarrow \mathcal{L} \quad \text{and} \quad t := \eta(1_B \otimes_B \cdot) : B^{op} \rightarrow \mathcal{L}$$

which are algebra maps with commuting ranges in \mathcal{L} , called the *source* and the *target* map of the $(B \otimes B^{op})$ -ring \mathcal{L} . Thus a $(B \otimes B^{op})$ -ring is the same as a triple (\mathcal{L}, s, t) with \mathcal{L} an algebra and $s : B \rightarrow \mathcal{L}$ and $t : B^{op} \rightarrow \mathcal{L}$ both algebra maps with commuting range.

For a left B -bialgebroid \mathcal{L} the compatibility conditions are required to be the following.

- i) The bimodule structures in the B -coring $(\mathcal{L}, \Delta, \varepsilon)$ are related to those of the $B \otimes B^{op}$ -ring (\mathcal{L}, s, t) via

$$b \triangleright a \triangleleft \tilde{b} := s(b)t(\tilde{b})a, \quad \text{for } b, \tilde{b} \in B, a \in \mathcal{L}. \quad (2.3)$$

- ii) Considering \mathcal{L} as a B -bimodule as in (2.3), the coproduct Δ corestricts to an algebra map from \mathcal{L} to

$$\mathcal{L} \times_B \mathcal{L} := \left\{ \sum_j a_j \otimes_B \tilde{a}_j \mid \sum_j a_j t(b) \otimes_B \tilde{a}_j = \sum_j a_j \otimes_B \tilde{a}_j s(b), \quad \forall b \in B \right\}, \quad (2.4)$$

where $\mathcal{L} \times_B \mathcal{L}$ is an algebra via component-wise multiplication.

- iii) The counit $\varepsilon : \mathcal{L} \rightarrow B$ satisfies the properties,

$$\begin{aligned} 1) \quad & \varepsilon(1_{\mathcal{L}}) = 1_B, \\ 2) \quad & \varepsilon(s(b)a) = b\varepsilon(a), \\ 3) \quad & \varepsilon(as(\varepsilon(\tilde{a}))) = \varepsilon(a\tilde{a}) = \varepsilon(at(\varepsilon(\tilde{a}))), \quad \text{for all } b \in B \text{ and } a, \tilde{a} \in \mathcal{L}. \end{aligned}$$

An automorphism of the left bialgebroid $(\mathcal{L}, \Delta, \varepsilon, s, t)$ over the algebra B is a pair (Φ, φ) of algebra automorphisms, $\Phi : \mathcal{L} \rightarrow \mathcal{L}$, $\varphi : B \rightarrow B$ such that:

$$\Phi \circ s = s \circ \varphi, \quad \Phi \circ t = t \circ \varphi, \quad (2.5)$$

$$(\Phi \otimes_B \Phi) \circ \Delta = \Delta \circ \Phi, \quad \varepsilon \circ \Phi = \varphi \circ \varepsilon. \quad (2.6)$$

In fact, the map φ is uniquely determined by Φ via $\varphi = \varepsilon \circ \Phi \circ s$ and one can just say that Φ is a bialgebroid automorphism. Automorphisms of a bialgebroid \mathcal{L} form a group $\text{Aut}(\mathcal{L})$ by map composition. A *vertical* automorphism is one of the type $(\Phi, \varphi = \text{id}_B)$.

From the conditions (2.5), Φ is a B -bimodule map: $\Phi(b \triangleright c \triangleleft \tilde{b}) = b \triangleright_{\varphi} \Phi(c) \triangleleft_{\varphi} \tilde{b}$. The first condition (2.6) is well defined once the conditions (2.5) are satisfied (the balanced tensor product is induced by $s' := s \circ \varphi$ and $t' := t \circ \varphi$). Conditions (2.5) imply Φ is a coring map, therefore (Φ, φ) is an isomorphism between the starting and the new bialgebroid.

Finally, we recall from [3, Def. 4.1] the conditions for a Hopf algebroid with invertible antipode. Given a left bialgebroid $(\mathcal{L}, \Delta, \varepsilon, s, t)$ over the algebra B , an invertible antipode $S : \mathcal{L} \rightarrow \mathcal{L}$ in an algebra anti-homomorphism with inverse $S^{-1} : \mathcal{L} \rightarrow \mathcal{L}$ such that

$$S \circ t = s \tag{2.7}$$

and satisfying compatibility conditions with the coproduct:

$$\begin{aligned} (S^{-1}h_{(2)})_{(1')} \otimes_B (S^{-1}h_{(2)})_{(2')} h_{(1)} &= S^{-1}h \otimes_B 1_{\mathcal{L}} \\ (Sh_{(1)})_{(1')} h_{(2)} \otimes_B S(h_{(1)})_{(2')} &= 1_{\mathcal{L}} \otimes_B Sh, \end{aligned} \tag{2.8}$$

for any $h \in \mathcal{L}$. These then imply $S(h_{(1)}) h_{(2)} = t \circ \varepsilon \circ Sh$.

2.2. Hopf–Galois extensions. We give a brief recall of Hopf–Galois extensions as non-commutative principal bundles. These extensions are H -comodule algebras A with a canonically defined map $\chi : A \otimes_B A \rightarrow A \otimes H$ which is required to be invertible [14].

Definition 2.1. *Let H be a Hopf algebra and let A be a H -comodule algebra with coaction δ^A . Consider the subalgebra $B := A^{\text{co}H} = \{b \in A \mid \delta^A(b) = b \otimes 1_H\} \subseteq A$ of coinvariant elements with balanced tensor product $A \otimes_B A$. The extension $B \subseteq A$ is called a H -Hopf–Galois extension if the canonical Galois map*

$$\chi := (m \otimes \text{id}) \circ (\text{id} \otimes_B \delta^A) : A \otimes_B A \longrightarrow A \otimes H, \quad \tilde{a} \otimes_B a \mapsto \tilde{a} a_{(0)} \otimes a_{(1)}$$

is an isomorphism.

Remark 2.2. For a Hopf–Galois extension $B \subseteq A$, we take the algebra A to be *faithfully flat* as a right B -module. One possible way to state this property is that for any left B -module map $F : M \rightarrow N$, the map F is injective if and only if the map $\text{id}_A \otimes_B F : A \otimes_B M \rightarrow A \otimes_B N$ is injective; injectivity of F implying the injectivity of $\text{id}_A \otimes_B F$ would state that A is flat as a right B -module (see [18, Chap. 13]). \square

Since the canonical Galois map χ is left A -linear, its inverse is determined by the restriction $\tau := \chi_{|_{1_A \otimes H}}^{-1}$, named *translation map*,

$$\tau = \chi_{|_{1_A \otimes H}}^{-1} : H \rightarrow A \otimes_B A, \quad h \mapsto \tau(h) = h^{<1>} \otimes_B h^{<2>}. \tag{2.9}$$

Thus by definition:

$$h^{<1>} h^{<2>}_{(0)} \otimes h^{<2>}_{(1)} = 1_A \otimes h. \tag{2.10}$$

The translation map enjoys a number of properties [15, 3.4] that we listed here for later use. For any $h, k \in H$ and $a \in A, b \in B$:

$$h^{<1>} \otimes_B h^{<2>}_{(0)} \otimes h^{<2>}_{(1)} = h_{(1)}^{<1>} \otimes_B h_{(1)}^{<2>} \otimes h_{(2)}, \tag{2.11}$$

$$h^{<1>}_{(0)} \otimes_B h^{<2>} \otimes h^{<1>}_{(1)} = h_{(2)}^{<1>} \otimes_B h_{(2)}^{<2>} \otimes S(h_{(1)}), \tag{2.12}$$

$$h^{<1>}h^{<2>} = \varepsilon(h)1_A, \quad (2.13)$$

$$a_{(0)}a_{(1)}^{<1>} \otimes_B a_{(1)}^{<2>} = 1_A \otimes_B a, \quad (2.14)$$

$$bh^{<1>} \otimes_B h^{<2>} = h^{<1>} \otimes_B h^{<2>} b, \quad (2.15)$$

$$(hk)^{<1>} \otimes_B (hk)^{<2>} = k^{<1>}h^{<1>} \otimes_B h^{<2>}k^{<2>}, \quad (2.16)$$

$$h_{(1)}^{<1>} \otimes_B h_{(1)}^{<2>} h_{(2)}^{<1>} \otimes_B h_{(2)}^{<2>} = h^{<1>} \otimes_B 1_A \otimes_B h^{<2>}. \quad (2.17)$$

2.3. Ehresmann–Schauenburg bialgebroids. To any Hopf–Galois extension $B = A^{coH} \subseteq A$ one associates a B -coring and a bialgebroid [13] (see [6, §34.13 and §34.14]). These can be viewed as a quantization of the gauge groupoid that is associated to a (classical) principal fibre bundle (see [11]).

The coring can be given in a few equivalent ways. Let $B = A^{coH} \subseteq A$ be a Hopf–Galois extension with right coaction $\delta^A : A \rightarrow A \otimes H$. This extends to a diagonal coaction,

$$\delta^{A \otimes A} : A \otimes A \rightarrow A \otimes A \otimes H, \quad a \otimes \tilde{a} \mapsto a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)} \tilde{a}_{(1)}, \quad \text{for } a, \tilde{a} \in A. \quad (2.18)$$

Let τ be the translation map of the Hopf–Galois extension. We have the following:

Lemma 2.3. *The B -bimodule of coinvariant elements for the diagonal coaction,*

$$(A \otimes A)^{coH} = \{a \otimes \tilde{a} \in A \otimes A; a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)} \tilde{a}_{(1)} = a \otimes \tilde{a} \otimes 1_H\} \quad (2.19)$$

is the same as the B -bimodule

$$\mathcal{C}(A, H) := \{a \otimes \tilde{a} \in A \otimes A : a_{(0)} \otimes \tau(a_{(1)}) \tilde{a} = a \otimes \tilde{a} \otimes_B 1_A\}. \quad (2.20)$$

Proof. This is a direct check: using properties of the canonical map χ and of the translation map τ , one shows the two inclusions. \square

We have then the following definition [13] (see [6, §34.13]).

Definition 2.4. *Let $B = A^{coH} \subseteq A$ be a faithfully flat Hopf–Galois extension with translation map τ . Then the B -bimodule $\mathcal{C}(A, H)$ in (2.20) is a B -coring with coproduct,*

$$\Delta(a \otimes \tilde{a}) = a_{(0)} \otimes \tau(a_{(1)}) \otimes \tilde{a} = a_{(0)} \otimes a_{(1)}^{<1>} \otimes_B a_{(1)}^{<2>} \otimes \tilde{a}, \quad (2.21)$$

and counit,

$$\varepsilon(a \otimes \tilde{a}) = a \tilde{a}. \quad (2.22)$$

Applying the map $m_A \otimes \text{id}_H$ to elements of (2.19) one gets $a \tilde{a} \in B$. The above B -coring is called the *Ehresmann* or *gauge coring*; we denote it $\mathcal{C}(A, H)$. Also, using the well know relation between the coinvariants of a tensor product of comodules and their cotensor product [15, Lemma 3.1], the coring $\mathcal{C}(A, H)$ can be given as a cotensor product $A \square^H A$.

The Ehresmann coring of a Hopf–Galois extension is in fact a bialgebroid [13], called the *Ehresmann–Schauenburg bialgebroid* (see [6, 34.14]). One see that $\mathcal{C}(A, H) = (A \otimes A)^{coH}$ is a subalgebra of $A \otimes A^{op}$; indeed, given $x \otimes \tilde{x}, y \otimes \tilde{y} \in (A \otimes A)^{coH}$, one computes

$$\begin{aligned} \delta^{A \otimes A}(xy \otimes \tilde{y} \tilde{x}) &= x_{(0)} y_{(0)} \otimes \tilde{y}_{(0)} \tilde{x}_{(0)} \otimes x_{(1)} y_{(1)} \tilde{y}_{(1)} \tilde{x}_{(1)} \\ &= x_{(0)} y \otimes \tilde{y} \tilde{x}_{(0)} \otimes x_{(1)} \tilde{x}_{(1)} \\ &= xy \otimes \tilde{y} \tilde{x} \otimes 1_H. \end{aligned}$$

Definition 2.5. *Let $\mathcal{C}(A, H)$ be the coring associated with a faithfully flat Hopf–Galois extension $B = A^{coH} \subseteq A$. Then $\mathcal{C}(A, H)$ is a (left) B -bialgebroid with product*

$$(x \otimes \tilde{x}) \bullet_{\mathcal{C}(A, H)} (y \otimes \tilde{y}) = xy \otimes \tilde{y} \tilde{x},$$

for all $x \otimes \tilde{x}, y \otimes \tilde{y} \in \mathcal{C}(A, H)$ (and unit $1_A \otimes 1_A$). The target and the source maps are

$$t(b) = 1_A \otimes b \quad \text{and} \quad s(b) = b \otimes 1_A.$$

We refer to [6, 34.14] for the checking that all defining properties are satisfied. When there is no risk of confusion we drop the decoration $\bullet_{\mathcal{C}(A, H)}$ in the product.

3. TWO EXAMPLES OF HOPF–GALOIS EXTENSION

We review two well know examples of Hopf–Galois extensions for which in §§ 5.4 and 5.3 we shall explicitly construct the corresponding algebroids.

3.1. The $SU(2)$ principal fibration. Consider the sphere S_θ^4 constructed in [8]. With θ a real parameter, the algebra $A(S_\theta^4)$ of polynomial functions on the sphere S_θ^4 is generated by elements $\zeta_0 = \zeta_0^*$ and $\zeta_j, \zeta_j^*, j = 1, 2$, subject to relations

$$\zeta_\mu \zeta_\nu = \lambda_{\mu\nu} \zeta_\nu \zeta_\mu, \quad \zeta_\mu \zeta_\nu^* = \lambda_{\nu\mu} \zeta_\nu^* \zeta_\mu^*, \quad \zeta_\mu^* \zeta_\nu^* = \lambda_{\mu\nu} \zeta_\nu^* \zeta_\mu^*, \quad \mu, \nu = 0, 1, 2, \quad (3.1)$$

with deformation parameters given by

$$\lambda_{12} = \bar{\lambda}_{21} =: \lambda = e^{2\pi i \theta}, \quad \lambda_{j0} = \lambda_{0j} = 1, \quad j = 1, 2, \quad (3.2)$$

and together with the spherical relation $\sum_\mu \zeta_\mu^* \zeta_\mu = 1$. For $\theta = 0$ one recovers the $*$ -algebra of complex polynomial functions on the usual sphere S^4 .

On the sphere S_θ^4 there is an $SU(2)$ noncommutative principal fibration $S_{\theta'}^7 \rightarrow S_\theta^4$ given in [9]. Firstly, with $\lambda'_{ab} = e^{2\pi i \theta'_{ab}}$ and (θ'_{ab}) a real antisymmetric matrix, the algebra $A(S_{\theta'}^7)$ of polynomial functions on the sphere $S_{\theta'}^7$ is generated by elements $\psi_a, \psi_a^*, a = 1, \dots, 4$, subject to relations

$$\psi_a \psi_b = \lambda'_{ab} \psi_b \psi_a, \quad \psi_a \psi_b^* = \lambda'_{ba} \psi_b^* \psi_a, \quad \psi_a^* \psi_b^* = \lambda'_{ab} \psi_b^* \psi_a^*, \quad (3.3)$$

and with the spherical relation $\sum_a \psi_a^* \psi_a = 1$. At $\theta = 0$, it is the $*$ -algebra of complex polynomial functions on the sphere S^7 . For the noncommutative Hopf bundle over the given 4-sphere S_θ^4 , we need to select a particular noncommutative 7 dimensional sphere $S_{\theta'}^7$. We take the one corresponding to the following deformation parameters

$$\lambda'_{ab} = \begin{pmatrix} 1 & 1 & \bar{\mu} & \mu \\ 1 & 1 & \mu & \bar{\mu} \\ \mu & \bar{\mu} & 1 & 1 \\ \bar{\mu} & \mu & 1 & 1 \end{pmatrix}, \quad \mu = \sqrt{\lambda} \quad \text{or} \quad \theta'_{ab} = \frac{\theta}{2} \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix}. \quad (3.4)$$

The previous choice is essentially the only one that allows the algebra $A(S_{\theta'}^7)$ to carry an action of the group $SU(2)$ by automorphisms and such that the invariant subalgebra coincides with $A(S_\theta^4)$. The best way to see this is by means of the matrix-valued function on $A(S_{\theta'}^7)$ (we are changing notations with respect to [9])

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* \\ \psi_2 & \psi_1^* \\ \psi_3 & -\psi_4^* \\ \psi_4 & \psi_3^* \end{pmatrix}. \quad (3.5)$$

Then, the commutation relations of the algebra $A(S_{\theta'}^7)$, with deformation parameter in (3.4), gives that $\Psi^\dagger \Psi = \mathbb{I}_2$. As a consequence, the matrix-valued function $p = \Psi \Psi^\dagger$ is a

projection, $p^2 = p = p^\dagger$, and its entries rather than functions in $A(S_{\theta'}^7)$ are (the generating) elements of $A(S_\theta^4)$. Indeed, the right coaction of $A(\mathrm{SU}(2))$ on $A(S_{\theta'}^7)$ is simply given by

$$\delta(\Psi) = \Psi \dot{\otimes} w, \quad w = \begin{pmatrix} w_1 & -w_2^* \\ w_2 & w_1^* \end{pmatrix} \in A(\mathrm{SU}(2)), \quad ww^\dagger = 1 = w^\dagger w. \quad (3.6)$$

If $\sigma(a \otimes b) = b \otimes a$ is the flip, this gives

$$\delta(\Psi^\dagger) = \sigma(w^\dagger \dot{\otimes} \Psi^\dagger),$$

and the invariance of the entries of p follows at once:

$$p \mapsto \delta(\Psi) \delta(\Psi^\dagger) = p \dot{\otimes} w w^\dagger = p \dot{\otimes} 1. \quad (3.7)$$

The generators of $A(S^4)$, the independent entries of p , are identified as bilinear expressions in the ψ, ψ^* 's. Explicitly,

$$p = \Psi \cdot \Psi^\dagger = \begin{pmatrix} \zeta_0 & 0 & \zeta_1 & -\bar{\mu}\zeta_2^* \\ 0 & \zeta_0 & \zeta_2 & \mu\zeta_1^* \\ \zeta_1^* & \zeta_2^* & 1 - \zeta_0 & 0 \\ -\mu\zeta_2 & \bar{\mu}\zeta_1 & 0 & 1 - \zeta_0 \end{pmatrix}, \quad (3.8)$$

with

$$\begin{aligned} \zeta_1 &= \psi_1 \psi_3^* + \psi_2^* \psi_4, & \zeta_2 &= \psi_2 \psi_3^* - \psi_1^* \psi_4, \\ \zeta_0 &= \psi_1 \psi_1^* + \psi_2^* \psi_2 = 1 - \psi_3 \psi_3^* - \psi_4^* \psi_4. \end{aligned} \quad (3.9)$$

By using the commutation relations of the ψ 's, one computes the commutation rules $\zeta_1 \zeta_2 = \lambda \zeta_2 \zeta_1$, $\zeta_1 \zeta_2^* = \bar{\lambda} \zeta_2^* \zeta_1$, and that ζ_0 is central and hermitian and ζ_1, ζ_2 are normal. The spherical relation for $S_{\theta'}^7$ gives an analogous one, $\zeta_1^* \zeta_1 + \zeta_2^* \zeta_2 = \zeta_0(1 - \zeta_0)$, for S_θ^4 .

There are compatible toric actions on S_θ^4 and $S_{\theta'}^7$ (see e.g. [5], §2.3.) With a slight change of notation, the torus \mathbb{T}^2 acts on $A(S_\theta^4)$ as

$$\sigma_s(\zeta_0, \zeta_1, \zeta_2) = (\zeta_0, e^{2\pi i s_1} \zeta_1, e^{2\pi i s_2} \zeta_2), \quad s \in \mathbb{T}^2. \quad (3.10)$$

This action is lifted to a double cover action on $A(S_{\theta'}^7)$. The double cover map $p : \tilde{\mathbb{T}}^2 \rightarrow \mathbb{T}^2$ is given explicitly by $p : (s_1, s_2) \mapsto (s_1 + s_2, -s_1 + s_2)$. Then $\tilde{\mathbb{T}}^2$ acts on the ψ_a 's as:

$$\tilde{\sigma} : (\psi_1, \psi_2, \psi_3, \psi_4) \mapsto (e^{2\pi i s_1} \psi_1, e^{-2\pi i s_1} \psi_2, e^{-2\pi i s_2} \psi_3, e^{2\pi i s_2} \psi_4) \quad (3.11)$$

The sense in which the algebra inclusion $A(S_\theta^4) \subset A(S_{\theta'}^7)$ is a nontrivial (faithfully flat) noncommutative $\mathrm{SU}(2)$ principal bundle is explained in [9]. Here we mention that there is a canonical Galois maps $\chi : A(S_{\theta'}^7) \otimes_{A(S_\theta^4)} A(S_{\theta'}^7) \rightarrow A(S_\theta^7) \otimes A(\mathrm{SU}(2))$ which is invertible. The corresponding translation map $\tau : A(\mathrm{SU}(2)) \rightarrow A(S_{\theta'}^7) \otimes_{A(S_\theta^4)} A(S_{\theta'}^7)$ on generators is

$$\tau(w) = \Psi^\dagger \dot{\otimes}_{A(S_\theta^4)} \Psi \quad (3.12)$$

Indeed, $\chi \circ \tau(w) = \chi(\Psi^\dagger \dot{\otimes}_{A(S_\theta^4)} \Psi) = \Psi^\dagger \delta(\Psi) = \Psi^\dagger \Psi \dot{\otimes} w = 1 \otimes 1_2 w = 1 \otimes w$.

There is also a copy of the projection p in the opposite algebra:

$$q = \Psi \cdot_{op} \Psi^\dagger = \begin{pmatrix} \zeta_0 & 0 & \bar{\mu}\zeta_1 & -\zeta_2^* \\ 0 & \zeta_0 & \mu\zeta_2 & \zeta_1^* \\ \mu\zeta_1^* & \bar{\mu}\zeta_2^* & 1 - \zeta_0 & 0 \\ -\zeta_2 & \zeta_1 & 0 & 1 - \zeta_0 \end{pmatrix} \quad (3.13)$$

The difference between p and q is due to the multiplication in $A(S_{\theta'}^7)$ versus the one in $A(S_{\theta'}^7)^{op}$. Indeed:

$$p_{mn} = \sum_r \Psi_{mr} \Psi_{rn}^\dagger, \quad q_{mn} = \sum_r \Psi_{mr} \cdot_{op} \Psi_{rn}^\dagger = \sum_r \Psi_{rn}^\dagger \Psi_{mr}. \quad (3.14)$$

With the commutation relations (3.3), the condition $\Psi^\dagger \cdot \Psi = \mathbb{I}_2$ leads also to $\Psi^\dagger \cdot_{op} \Psi = \mathbb{I}_2$.

3.2. Principal bundles over even quantum spheres. Even noncommutative spheres S_θ^{2n} , introduced in [8], were shown in [17] to be homogeneous spaces of quantum groups $\text{SO}_\theta(2n+1, \mathbb{R})$. The algebra of coordinate functions of the latter $A = \mathcal{O}(\text{SO}_\theta(2n+1, \mathbb{R}))$ is the total space algebra of a principal bundle over the algebra $B = \mathcal{O}(S_\theta^{2n})$ for the Hopf (structure) algebra $H = \mathcal{O}(\text{SO}_\theta(2n, \mathbb{R}))$. These bundles were worked out in details in [1, §4.1.1] that we follows with changes.

Start with the commutative torus \mathbb{T}^n with generators t_j, t_j^* and relations $t_j t_j^* = t_j^* t_j = 1$. Consider the bi-character $\gamma : \mathbb{T}^n \times \mathbb{T}^n \rightarrow \text{U}(1)$ defined on generators by

$$\gamma(t_j, t_k) = e^{i\pi\theta_{jk}}, \quad \theta_{jk} = -\theta_{kj}.$$

We shall denote $\lambda_{jk} = \gamma(t_j, t_k)^2 = e^{2i\pi\theta_{jk}}$. In order for the deformed algebra to still be a Hopf algebra one needs a left and a right action of $\mathbb{T}^n = \text{diag}(t_1, \dots, t_n, t_1^*, \dots, t_n^*)$ (or of $\mathbb{T}^n \times \mathbb{T}^n$). This action then allows one to deform the algebra $\mathcal{O}(\text{SO}(2n))$ into an algebra $\mathcal{O}(\text{SO}_\theta(2n))$ described as follows. It has generators $\mathbf{a} = (a_{jk}), \mathbf{b} = (b_{jk}), \mathbf{a}^* = (a_{jk}^*), \mathbf{b}^* = (b_{jk}^*)$ with commutation relations computed to be

$$\begin{aligned} a_{ij} a_{kl} &= \lambda_{ik} \lambda_{lj} a_{kl} a_{ij}, & a_{ij} b_{kl}^* &= \lambda_{ki} \lambda_{lj} b_{kl}^* a_{ij} \\ a_{ij} b_{kl} &= \lambda_{ik} \lambda_{jl} b_{kl} a_{ij}, & a_{ij} a_{kl}^* &= \lambda_{ki} \lambda_{jl} a_{kl}^* a_{ij} \\ b_{ij} b_{kl} &= \lambda_{ik} \lambda_{lj} b_{kl} b_{ij}, & b_{ij} b_{kl}^* &= \lambda_{ki} \lambda_{jl} b_{kl}^* b_{ij} \end{aligned} \quad (3.15)$$

together with their $*$ -conjugated. In fact, the Hopf algebra structure of $\mathcal{O}(\text{SO}(2n))$ survives the quantization. In matrix notation the deformed $\mathcal{O}(\text{SO}_\theta(2n))$ has coproduct and counit given by

$$M = (M_{JK}) = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b}^* & \mathbf{a}^* \end{pmatrix}, \quad \Delta(M) = M \dot{\otimes} M, \quad \varepsilon(M) = \mathbb{I}. \quad (3.16)$$

To define an antipode there is a suitable determinant $\det_\theta(M)$ and one can pass to the quotient by the $*$ -bialgebra ideal given by

$$I_Q = \langle M^t Q M - Q, M Q M^t - Q, \det_\theta(M) - 1 \rangle, \quad Q = \begin{pmatrix} 0 & \mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix} = Q^{-1}. \quad (3.17)$$

The $*$ -structure is then $*M = Q M Q$ while the antipode is $S(M) = Q M^t Q = M^\dagger$. The previous conditions reads then $M^\dagger M = M M^\dagger = \mathbb{I}_{2n}$.

The odd case of $\mathcal{O}(\text{SO}_\theta(2n+1))$ is defined in a similar fashion by deforming the left and right actions of the torus $\mathbb{T}^n = \text{diag}(t_1, \dots, t_n, t_1^*, \dots, t_n^*, 1)$ on $\mathcal{O}(\text{SO}(2n+1))$. In matrix notation

$$N = (N_{JK}) = \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{u} \\ \mathbf{b}^* & \mathbf{a}^* & \mathbf{u}^* \\ \mathbf{v} & \mathbf{v}^* & x \end{pmatrix},$$

with n -component column vectors $\mathbf{u} = (u_j), \mathbf{u}^* = (u_j^*)$ and row vectors $\mathbf{v} = (v_j), \mathbf{v}^* = (v_j^*)$ and a hermitian scalar x . The commutation relations are found to be given by

$$N_{IJ} N_{KL} = \lambda_{IK} \lambda_{LJ} N_{KL} N_{IJ}. \quad (3.18)$$

Now the coproduct and antipode are as before by $\Delta(N) = N \dot{\otimes} N$ and $\varepsilon(N) = \mathbb{I}$ and one verifies ideal conditions analogue to the ones in (3.17):

$$N^t Q N = Q, \quad N Q N^t = Q, \quad \det_\theta(N) = 1, \quad Q = \begin{pmatrix} 0 & \mathbb{I}_n & 0 \\ \mathbb{I}_n & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q^{-1}. \quad (3.19)$$

The $*$ -structure is $*N = Q N Q$ while the antipode is $S(N) = Q N^t Q = N^\dagger$. Then the previous condition read $N^\dagger N = N N^\dagger = \mathbb{I}_{2n+1}$.

The Hopf algebra $\mathcal{O}(\mathrm{SO}_\theta(2n))$ is a quantum subgroup of $\mathcal{O}(\mathrm{SO}_\theta(2n+1))$ with surjective Hopf algebra morphism

$$\begin{aligned} \pi : \mathcal{O}(\mathrm{SO}_\theta(2n+1)) &\rightarrow \mathcal{O}(\mathrm{SO}_\theta(2n)), \\ \begin{pmatrix} \mathbf{a} & \mathbf{b} & \mathbf{u} \\ \mathbf{b}^* & \mathbf{a}^* & \mathbf{u}^* \\ \mathbf{v} & \mathbf{v}^* & x \end{pmatrix} &\mapsto \begin{pmatrix} \mathbf{a} & \mathbf{b} & 0 \\ \mathbf{b}^* & \mathbf{a}^* & 0 \\ 0 & 0 & 1 \end{pmatrix} =: \begin{pmatrix} \mathbf{h} & \mathbf{k} & 0 \\ \mathbf{k}^* & \mathbf{h}^* & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{w}. \end{aligned} \quad (3.20)$$

This results into a right coaction of $\mathcal{O}(\mathrm{SO}_\theta(2n))$ on $\mathcal{O}(\mathrm{SO}_\theta(2n+1))$:

$$\begin{aligned} \delta^A : \mathcal{O}(\mathrm{SO}_\theta(2n+1)) &\rightarrow \mathcal{O}(\mathrm{SO}_\theta(2n+1)) \otimes \mathcal{O}(\mathrm{SO}_\theta(2n)), \\ \delta^A(N) &= N \dot{\otimes} \pi(N). \end{aligned} \quad (3.21)$$

The subalgebra B of coinvariant elements, generated by the last column of the matrix N : (u_j, u_j^*, x) , is the algebra $\mathcal{O}(S_\theta^{2n})$ of coordinate functions on a quantum $2n$ -sphere S_θ^{2n} . The commutation relations of the generators follows from (3.18):

$$u_i u_j = \lambda_{ij} u_j u_i, \quad u_i^* u_j^* = \lambda_{ij} u_j^* u_i^*, \quad u_i u_j^* = \lambda_{ji} u_j^* u_i, \quad (3.22)$$

and x central. The orthogonality conditions (3.19) imply the sphere relation

$$\sum_{j=1}^n 2u_j^* u_j + x^2 = 1,$$

(each generator is normal $u_j^* u_j = u_j u_j^*$). The algebra extension $\mathcal{O}(S_\theta^{2n}) \subset \mathcal{O}(\mathrm{SO}_\theta(2n+1))$ is a Hopf Galois extension for the Hopf algebra $H = \mathcal{O}(\mathrm{SO}_\theta(2n))$ (cf. [1, §4.1.1]). In particular we record the form of the translation map to be used later on. In components

$$\begin{aligned} \tau(\mathbf{h}) &= \mathbf{a}^\dagger \dot{\otimes}_B \mathbf{a} + (\mathbf{b}^*)^\dagger \dot{\otimes}_B \mathbf{b}^* + \mathbf{v}^\dagger \dot{\otimes}_B \mathbf{v}, \\ \tau(\mathbf{k}) &= \mathbf{a}^\dagger \dot{\otimes}_B \mathbf{b} + (\mathbf{b}^*)^\dagger \dot{\otimes}_B \mathbf{a}^* + \mathbf{v}^\dagger \dot{\otimes}_B \mathbf{v}^*. \end{aligned} \quad (3.23)$$

4. ALGEBRAIC θ -DEFORMATIONS

In this section we review the general scheme of deforming by the action of tori. This will be done in the crudest way via \mathbb{Z}^n -graded spaces and deforming relevant structures by means of a bi-character. The role of \mathbb{Z}^n comes from it being the Pontryagin dual of the torus \mathbb{T}^n and one is effectively deforming objects with a torus action. More details are e.g. in [4] and [5]. In particular we shall deform principal bundles and associated Hopf algebroids. A general scheme of deformations of noncommutative principal bundles via convolution invertible 2-cocycles $\gamma : H \otimes H \rightarrow \mathbb{C}$ on a Hopf algebra H is in [1].

Let \mathcal{T}_n be the category of \mathbb{Z}^n -graded complex vector spaces whose objects are written as (finite) sums of the kind

$$V = \bigoplus_{r \in \mathbb{Z}^n} V_r, \quad p_r : V \rightarrow V_r.$$

Here p_r is the projection onto the r -th component, and most of the time we simply use a subscript to indicate the projection $v_r = p_r(v)$ for $v \in V$. Morphisms $\psi \in \text{Hom}(V, W)$ are linear maps that preserve homogeneity, but not necessarily the degree. More precisely, there always exists a group homomorphism $\rho_\psi : \mathbb{Z}^n \rightarrow \mathbb{Z}^n$ such that

$$\psi(V_r) \subset V_{\rho_\psi(r)}. \quad (4.1)$$

In the θ -deformation literature, one starts with a smooth action of a n -torus on a Fréchet space V , $t \in \mathbb{T}^n \mapsto \alpha_t \in \text{Aut}(V)$. The induced \mathbb{Z}^n -grading, based as mentioned on Pontryagin duality, is given by projections $p_r : V \rightarrow V_r$, $r \in \mathbb{Z}^n$, taking the r -th Fourier coefficients of the vector-valued function $t \rightarrow \alpha_t(v)$,

$$p_r(v) = \int_{\mathbb{T}^n} e^{-2\pi i r \cdot t} \alpha_t(v) dt, \quad v \in V,$$

where dt is the normalized Lebesgue measure on \mathbb{T}^n . Morphisms as in (4.1) corresponds to linear maps $\tilde{\psi} : V \rightarrow W$ which are \mathbb{T}^n -equivariant up-to a group homomorphism $\tilde{\rho}_{\tilde{\psi}} : \mathbb{T}^n \rightarrow \mathbb{T}^n$ so that the diagram commute:

$$\begin{array}{ccc} \mathbb{T}^n \times V & \longrightarrow & V \\ \downarrow \rho_{\tilde{\psi}} \times \tilde{\psi} & & \downarrow \tilde{\psi} \\ \mathbb{T}^n \times W & \longrightarrow & W \end{array} \quad (4.2)$$

The parameter θ in a θ -deformation is a $n \times n$ skew-symmetric matrix and what is actually needed for the deformation is the induced bi-character on \mathbb{Z}^n , that is a map

$$\lambda_\theta : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{T}, \quad (r, l) \mapsto \lambda_\theta(r, l) := e^{\pi i \langle \theta r, l \rangle},$$

which is a 2-cocycle in the sense of

$$\lambda_\theta(r, l) \lambda_\theta(r + l, s) = \lambda_\theta(l, s) \lambda_\theta(r, s + l), \quad r, s, l \in \mathbb{Z}^n. \quad (4.3)$$

The tensor products functor $\otimes : \mathcal{T}_n \times \mathcal{T}_n \rightarrow \mathcal{T}_n$ makes \mathcal{T}_n into a monoidal category, in which the \mathbb{Z}^n -grading is assigned in the usual way,

$$(V \otimes W)_s = \bigoplus_{s=r+l} V_r \otimes W_l, \quad s, r, l \in \mathbb{Z}^n, \quad (4.4)$$

by taking the total degree of the natural bi-grading. One can deform the tensor functor via the following natural transformation c^θ , for any $V, W \in \mathcal{T}_n$,

$$c_{V,W}^\theta : V \otimes W \rightarrow V \otimes_\theta W, \quad v_r \otimes w_l \rightarrow v_r \otimes_\theta w_l := \lambda_\theta(r, l) v_r \otimes w_l \quad (4.5)$$

which is defined firstly on homogeneous elements and then extended by linearity. It is not difficult to see that it has an inverse given by $(c_{V,W}^\theta)^{-1} = c_{V,W}^{-\theta}$.

Given an algebra (A, m) in \mathcal{T}_n , with multiplication $m : A \otimes A \rightarrow A$ preserving the grading,

$$m(A_r \otimes A_l) \subset A_{r+l}, \quad r, l \in \mathbb{Z}^n, \quad (4.6)$$

its deformation $A_\theta = (A, m_\theta)$ maintains the underlying (graded) vector space unchanged, but endowed with a new multiplication:

$$m_\theta = m \circ c_{A,A}^\theta : A \otimes A \xrightarrow{c_{A,A}^\theta} A \otimes A \xrightarrow{m} A. \quad (4.7)$$

As shown in (4.5), m is twisted by a phase factor on homogeneous elements:

$$m_\theta(a_r, \tilde{a}_l) = \lambda_\theta(r, l) m(a_r, \tilde{a}_l), \quad (4.8)$$

which, provided that m is commutative, leads to the commutation relations:

$$m_\theta(a_r, \tilde{a}_l) = \lambda_\theta(r, l)^2 m_\theta(a_l, \tilde{a}_r). \quad (4.9)$$

The required associativity for m_θ follows directly from the 2-cocycle condition in (4.3). For easy of notation in the following we shall denote $m_\theta(a_l, \tilde{a}_r) = a_l \cdot_\theta \tilde{a}_r$.

Clearly, $\lambda_\theta(r, \pm r) = 1$ since θ is skew-symmetric. We record this simple observation as a lemma which will be used often later on.

Lemma 4.1. *For homogeneous element $a, \tilde{a} \in A$ of the same degree or of the opposite degree, that is $\deg a \pm \deg \tilde{a} = 0$, the deformed multiplication agrees with the original one:*

$$m_\theta(a, \tilde{a}) = m(a, \tilde{a}). \quad (4.10)$$

In particular, (4.10) holds whenever the product $m(a, \tilde{a}) \in A_0$ belongs to the degree zero component, in this case, a, \tilde{a} are not required to be homogeneous.

In a similar manner, for an A -module V in \mathcal{T}_n such that the action $\triangleright : A \otimes V \rightarrow V$ preserves the grading as in (4.6), the deformation $\triangleright_\theta := \triangleright \circ c_{A,V}^\theta$ makes V_θ into an A_θ -module. The ‘associativity’ (the action properties) for \triangleright_θ again follows directly from the 2-cocycle condition in (4.3). There is clearly a right-module version of this.

And finally, if (C, Δ) is a coalgebra in \mathcal{T}_n with $\Delta : C \rightarrow C \otimes C$ that preserves the degree in the sense of (4.4): $\Delta(c_s) = \sum_{r+l=s} c_{r(1)} \otimes c_{l(2)}$ the deformation $\Delta_\theta := c_{C,C}^{-\theta} \circ \Delta$ makes C_θ into a coalgebra with co-associativity again following from the 2-cocycle condition.

The next step in deforming a bialgebra (or even a Hopf algebra) structures needs some extra care. Also, for deforming a Hopf–Galois extension with structure Hopf algebra H , and aiming at including both examples in §§ 3.1 and 3.2, it turns out that the construction of gradings on the algebra involved and the related assumptions are quite different depending on whether the Hopf algebra is deformed or not.

We will break the discussion into two scenarios to cover the constructions of both §§ 4.1 and 4.2 in which our aim is to get a (possible new) structure Hopf algebra with related comodule algebras out of the θ -deformation scheme. After that, the deformation of the Ehresmann–Schauenburg bialgebroids can be handled in a uniform way, and will be carried out in §§ 5.1 and 5.2.

4.1. Scenario I: No Hopf algebra is deformed. We start with a setting in which the Hopf algebra H is not touched. Thus, we assume that H has trivial \mathbb{Z}^n -grading and a H -comodule algebra A is \mathbb{Z}^n -graded so that the multiplication preserves the grading as in (4.6), and the coaction $\delta^A : A \rightarrow A \otimes H$ also behaves the same way:

$$\delta^A(A_r) \subset A_r \otimes H. \quad (4.11)$$

Thus when writing $\delta^A(a) = a_{(0)} \otimes a_{(1)}$ one has $\deg a = \deg a_{(0)}$.

The following is an almost free version of [1, Cor. 3.16].

Proposition 4.2. *Let $A_\theta = (A, m_\theta)$ be the deformation of A as in (4.7). It is still a H -comodule algebra with the same coaction treated as $\delta^A : A_\theta \rightarrow A_\theta \otimes H$. Then, the coinvariant subspace $B = A^{\text{co}H}$ remains the same and the θ -multiplication can be restricted onto B to form $B_\theta = (B, m_\theta)$. Moreover, if the starting pair (A, H) is a Hopf–Galois extension with algebra of coinvariant elements B , such is its deformation (H, A_θ) , with algebra of coinvariant elements B_θ .*

Proof. The first part is evident. As for the final (almost evident) statement, consider the starting canonical map $\chi : A \otimes_B A \longrightarrow A \otimes H$, $\chi(\tilde{a} \otimes_B a) = \tilde{a} a_{(0)} \otimes a_{(1)}$ and define

$$\chi_\theta : A_\theta \otimes_{B_\theta} A_\theta \longrightarrow A_\theta \otimes H, \quad \chi_\theta(\tilde{a} \otimes_B a) = \tilde{a} \cdot_\theta a_{(0)} \otimes a_{(1)}.$$

Then, for $h \in H$ consider the starting canonical map $\tau(h) = h^{<1>} \otimes_B h^{<2>}$ with components (sum of terms) of opposite degree $\deg h^{<1>} = -\deg h^{<2>}$ since H has zero degree which is preserved by τ . Then, from Lemma 4.1,

$$\begin{aligned} \chi_\theta(h^{<1>} \otimes_B h^{<2>}) &= h^{<1>} \cdot_\theta h^{<2>}_{(0)} \otimes h^{<2>}_{(1)} = h^{<1>} h^{<2>}_{(0)} \otimes_B h^{<2>}_{(1)} \\ &= \chi(h^{<1>} \otimes_B h^{<2>}) \end{aligned}$$

(the latter being just $1 \otimes h$ from (2.10)) and χ_θ is invertible if and only if χ is.

Thus the translation map of χ_θ is the same as the starting undeformed one that can be considered as a map $\tau : H \rightarrow A_\theta \otimes_{B_\theta} A_\theta$. \square

Remark 4.3 (On the degree of the translation map). The fact that, in writing for the translation map $\tau(h) = h^{<1>} \otimes_B h^{<2>}$, one can take $\deg h^{<1>} = -\deg h^{<2>}$ does not depend on the representatives: suppose $h^{<1>} \otimes_B h^{<2>} = \tilde{h}^{<1>} b \otimes_B h^{<2>} = \tilde{h}^{<1>} \otimes_B b h^{<2>} = \tilde{h}^{<1>} \otimes_B \tilde{h}^{<2>}$. Then it follows that $\deg h^{<1>} = -\deg h^{<2>}$ if and only if $\deg \tilde{h}^{<1>} = -\deg \tilde{h}^{<2>}$.

Example 4.4 (Noncommutative Hopf-fibration). As mentioned, the \mathbb{Z}^n -grading we consider is derived from a torus action. To construct the $SU(2)$ -fibration $S_\theta^7 \rightarrow S_\theta^4$ in §3.1, one begins with a two torus action defined in (3.10) and (3.11), in which all generators in 3.1 and 3.3 are \mathbb{T}^2 -eigenfunctions. Then,

$$\begin{aligned} \deg \zeta_0 &= \deg \zeta_0^* = 0 \\ \deg \zeta_1 &= (1, 0), \quad \deg \zeta_2 = (0, 1) \\ \deg \psi_1 &= -\deg \psi_2 = (1, 0), \quad \deg \psi_4 = -\deg \psi_3 = (0, 1). \end{aligned}$$

The deformation matrix just reads $\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}$, with $\theta \in \mathbb{R}$. We have, according to (4.8),

$$\zeta_\mu \cdot_\theta \zeta_\nu = \sqrt{\lambda_{\mu\nu}} \zeta_\mu \zeta_\nu, \quad \psi_a \cdot_\theta \psi_b = \sqrt{\lambda'_{ab}} \psi_a \psi_b,$$

so that the commutation relations 3.1 and 3.3 follow immediately from (4.9). It is also worth noting that the double covering between the two actions (3.10) and (3.11) is exactly dual to the following map $\mathbb{Z}^2 \rightarrow \mathbb{Z}^2$:

$$\begin{aligned} (1, 0) &= \deg \zeta_1 \mapsto (1, 1) = \deg \psi_1 \psi_3^* = \deg \psi_2^* \psi_4, \\ (0, 1) &= \deg \zeta_2 \mapsto (-1, 1) = \deg \psi_2 \psi_3^* = \deg \psi_1^* \psi_4. \end{aligned}$$

revealed in the embedding $A(S_\theta^4) \rightarrow A(S_\theta^7)$ given by 3.9. \square

The details of the construction of the Ehresmann–Schauenburg bialgebroid related to the Hopf–Galois extension (H, A_θ) , are postponed to §5.1.

4.2. Scenario II: Deforming Hopf algebras and homogeneous spaces. Unlike the previous section, in order to deform a Hopf algebra H (or in a more accurate context, to only deform the algebra structure of H), in a way that the all compatibilities axioms for Hopf algebras remains, one needs a more delicate setup for the grading and the θ -matrix. To motivate the long list of requisites below, the reader is referred to App. A, where we recall the original formulation in terms of torus actions due to Rieffel [12].

Let $H = \bigoplus_{r,s \in \mathbb{Z}^n} H_{(r,s)}$ be a Hopf algebra with a bi-grading of \mathbb{Z}^n (in particular, a grading of \mathbb{Z}^{2n}), such that the group homomorphism ρ_ψ on gradings in (4.1) induced via the structure maps of H are given as follows:

i) the multiplication preserves the grading as in (4.6)

$$m(H_{(r,s)} \otimes H_{(p,q)}) \subset H_{(r+p,s+q)}, \quad (4.12)$$

ii) the coproduct $\Delta : H \rightarrow H \otimes H$, is required to be such that

$$\Delta(H_{(r,l)}) \subset \bigoplus_{s \in \mathbb{Z}^n} H_{(r,s)} \otimes H_{(s,l)}, \quad (4.13)$$

iii) the counit factors through the projection:

$$\varepsilon : H \rightarrow \bigoplus_{s \in \mathbb{Z}^n} H_{(s,s)} \rightarrow \mathbb{C}, \quad (4.14)$$

that is $\varepsilon(h_{(r,l)}) = 0$ for all homogeneous elements $h_{(r,l)}$ with $r \neq l$,

iv) for the antipode and the $*$ -operator (if H has one), one assumes

$$S(H_{(r,l)}) \subset H_{(-l,-r)}, \quad *(H_{(r,l)}) \subset H_{(-r,-l)}. \quad (4.15)$$

Remark 4.5 (On condition (4.13)). From the general assigning of the total degree in (4.4), on the right hand side of (4.12) one would have $\bigoplus_{a+c=r, b+d=l} H_{(a,b)} \otimes H_{(c,d)}$. The subspaces $\bigoplus_{s \in \mathbb{Z}^n} H_{(r,s)} \otimes H_{(s,l)}$ of $H \otimes H$ when summed on the indices r, l corresponds to the subspace \mathcal{D} in (A.4) in which the coproduct lands. \square

Next, let θ be a $n \times n$ skew-symmetric matrix and put

$$\Theta = \begin{bmatrix} \theta & 0 \\ 0 & -\theta \end{bmatrix}, \quad (4.16)$$

so that their 2-cocycles are related as follows: for $r = (r_1, r_2)$ and $l = (l_1, l_2)$,

$$\lambda_\Theta(r, l) = \lambda_\theta(r_1, l_1) \lambda_{-\theta}(r_2, l_2).$$

Denote by $H_\Theta = (H, \cdot_\Theta)$ the deformed algebra, with the new multiplication given, on homogeneous elements h_r, g_l of degree $r, l \in \mathbb{Z}^n$ respectively, by

$$h_r \cdot_\Theta g_l = \lambda_\Theta(r, l) h_r g_l = \lambda_\theta(r_1, l_1) \lambda_{-\theta}(r_2, l_2) h_r g_l. \quad (4.17)$$

Lemma 4.6. *With the condition in (4.13), the (undeformed) coproduct Δ is still an algebra homomorphism for the product \cdot_Θ :*

$$\Delta(h \cdot_\Theta g) = h_{(1)} \cdot_\Theta g_{(1)} \otimes h_{(2)} \cdot_\Theta g_{(2)}.$$

Proof. It suffices to work with homogeneous elements. Take $h, g \in H$, with $\deg h = (r, l)$ and $\deg g = (p, q)$ with their components in the coproduct, $\Delta x = x_{(1)} \otimes x_{(2)}$ in Sweedler notation, that can be assumed to be homogeneous as well:

$$\deg h_{(1)} = (r, s), \quad \deg h_{(2)} = (s, l), \quad \deg g_{(1)} = (p, k), \quad \deg g_{(2)} = (k, q).$$

where only s, k vary within to the components. Then,

$$\begin{aligned} (h_{(1)} \otimes h_{(2)}) \cdot_\Theta (g_{(1)} \otimes g_{(2)}) &= h_{(1)} \cdot_\Theta g_{(1)} \otimes h_{(2)} \cdot_\Theta g_{(2)} \\ &= \lambda_\theta(r, p) \lambda_{-\theta}(s, k) \lambda_\theta(s, k) \lambda_{-\theta}(l, q) h_{(1)} g_{(1)} h_{(2)} g_{(2)} \\ &= \lambda_{-\theta}(l, q) \lambda_\theta(r, p) h_{(1)} g_{(1)} h_{(2)} g_{(2)} \\ &= \lambda_{-\theta}(l, q) \lambda_\theta(r, p) \Delta(hg) \\ &= \Delta(h \cdot_\Theta g), \end{aligned}$$

as stated. \square

Proposition 4.7. *By θ -deforming the multiplication of H as in (4.17), we obtain a new Hopf algebra $H_\Theta = (H, \cdot_\Theta, \Delta, \varepsilon, S)$ with the same coproduct, counit and antipode.*

Proof. The compatibility between the algebra \cdot_Θ and the coalgebra Δ structures has been dealt with in Lemma 4.6. The coproduct Δ and counit ε are not deformed at all, thus property $(\varepsilon \otimes 1)\Delta = 1 = (1 \otimes \varepsilon)\Delta$ remains. We are left to verify

$$S(h_{(1)}) \cdot_\Theta h_{(2)} = \varepsilon(h) = h_{(1)} \cdot_\Theta S(h_{(2)}), \quad \forall h \in H. \quad (4.18)$$

Suppose h , $h_{(1)}$ and $h_{(2)}$ are homogeneous of degree (r, l) , (r, s) and (s, l) respectively. By the assumptions in (4.15), $S(h_{(1)})$ is of degree $(-s, -r)$, thus

$$\begin{aligned} S(h_{(1)}) \cdot_\Theta h_{(2)} &= \lambda_\theta(-s, s)\lambda_{-\theta}(r, l)S(h_{(1)}) \cdot h_{(2)} = \lambda_{-\theta}(r, l)S(h_{(1)})h_{(2)} \\ &= \lambda_{-\theta}(r, l)\varepsilon(h) = \varepsilon(h). \end{aligned}$$

For the last step, we need to invoke (4.14), so that $\varepsilon(h) = 0$ whenever $l \neq r$, while for $r = l$, we have $\lambda_{-\theta}(r, l) = 1$. \square

Next, let \mathcal{M}^H be the category of H -comodule with a bi-grading of \mathbb{Z}^n and such that the coaction $\delta^V : V \rightarrow V \otimes H$ with $V \in \mathcal{M}^H$, behaves in a similar way to the coproduct in (4.13) as regarding the grading:

$$\delta^V(V_{(r,l)}) \subset \bigoplus_{s \in \mathbb{Z}^n} V_{(r,s)} \otimes H_{(s,l)}. \quad (4.19)$$

The co-representations \mathcal{M}^{H_Θ} of H_Θ , keep the same objects and morphisms as \mathcal{M}^H . Modification only occurs on the coaction on the monoidal structure. Namely, in the coaction on $V \otimes W$, where V, W are in \mathcal{M}^H , we must use of the multiplication of H_Θ :

$$\delta^{V \otimes_\Theta W} : V \otimes W \rightarrow V \otimes W \otimes H_\Theta, \quad v \otimes w \mapsto v_{(0)} \otimes w_{(0)} \otimes v_{(1)} \cdot_\Theta w_{(1)}. \quad (4.20)$$

When deforming a comodule algebra A in \mathcal{M}^H , which play the role of function algebra on the noncommutative principal bundle, we have to impose similar conditions. That is, we have that A also admits a bi-grading of \mathbb{Z}^n such that

- (1) the product of A preserves the bi-grading as in (4.12);
- (2) the coaction $\delta^A : A \rightarrow A \otimes H$ satisfies (4.19) on the bi-grading.

The first condition allows one to form the deformed algebra A_Θ and the second one makes sure we still have a comodule algebra after deformation.

Proposition 4.8. *Consider a Hopf–Galois extension (A, H) with both H and A endowed with a bi-grading of \mathbb{Z}^n , and algebra of coinvariants $B = A^{\text{co}H}$ (with a heredity bi-grading from A). Then, their bi-grading leads to the deformed algebras $H_\Theta = (H, \cdot_\Theta)$ and $A_\Theta = (A, \cdot_\Theta)$ according to (4.17). Moreover, A_Θ is a H_Θ -comodule algebra with the same coaction viewed as a map $\delta_\Theta^A : A_\Theta \rightarrow A_\Theta \otimes H_\Theta$. Also, the coinvariant subspace $B_\Theta = A_\Theta^{\text{co}H_\Theta} = (A^{\text{co}H}, \cdot_\Theta) = (B, \cdot_\Theta)$ maintains its starting vector space sitting inside A_Θ as a subalgebra.*

Proof. Observe that, from (4.13) and (4.19), the coaction δ^A and coproduct Δ of H change the bi-grading in a similar manner, hence compatibility between the coaction and multiplication of A can be proved along the lines of Lemma 4.6.

Since the coaction is taken directly from (A, H) , the coinvariant subspace remains the same as a vectors space. Moreover, the θ -multiplication differs from the original one by a phase factor on homogeneous elements, thus it maps $B \otimes B$ into B . In other words, \cdot_θ can be restricted onto the coinvariant subspace B to form B_θ . \square

Example 4.9. Let us specialize the discussion in Appendix [A](#) to the case $G = \text{SO}(2n)$ and $\tilde{G} = \text{SO}(2n + 1)$ and discuss the bi-grading behind the quantum spheres in [\(5.4\)](#) in great detail. The torus action α of $\mathbb{T}^n \times \mathbb{T}^n$ is now given by matrix multiplications from two sides so that all the generators in [\(3.15\)](#) and [\(3.18\)](#) are eigenfunctions:

$$\begin{aligned} \alpha_{t,\tilde{t}}(a_{jk}) &= t_j \tilde{t}_k a_{jk}, & \alpha_{t,\tilde{t}}(b_{jk}) &= t_j \tilde{t}_k^* b_{jk}, & \alpha_{t,\tilde{t}}(u_j) &= t_j u_j, & \alpha_{t,\tilde{t}}(v_k) &= \tilde{t}_k v_k, \\ \alpha_{t,\tilde{t}}(a_{jk}^*) &= t_j^* \tilde{t}_k^* a_{jk}^*, & \alpha_{t,\tilde{t}}(b_{jk}^*) &= t_j^* \tilde{t}_k^* b_{jk}^*, & \alpha_{t,\tilde{t}}(u_j^*) &= t_j^* u_j^*, & \alpha_{t,\tilde{t}}(v_k^*) &= \tilde{t}_k^* v_k^*. \end{aligned} \quad (4.21)$$

Therefore, we can reconstruct the algebras $H_\theta = \mathcal{O}(\text{SO}_\theta(2n))$ and $\tilde{H}_\theta = \mathcal{O}(\text{SO}_\theta(2n + 1))$ by assigning the following degrees to generators:

$$\begin{aligned} \deg a_{ij} &= (e_i, e_j) = -\deg a_{ij}^*, & \deg b_{ij} &= (e_i, -e_j) = -\deg b_{ij}^*, \\ \deg u_i &= -\deg u_i^* = (e_i, 0), & \deg v_i &= -\deg v_i^* = (0, e_i), \end{aligned} \quad (4.22)$$

where $\{e_j, j = 1, \dots, n\}$ is the standard basis of \mathbb{Z}^n , and extends to the whole algebra according to [\(4.12\)](#). For homogeneous elements, the new multiplication differs from the commutative one by the phase factors as in [\(4.17\)](#) instance,

$$a_{ij} \cdot_\theta a_{kl} = \lambda_\theta(e_i, e_k) \lambda_{-\theta}(e_j, e_l) a_{ij} a_{kl} = \sqrt{\lambda_{ik} \lambda_{lj}} a_{ij} a_{kl},$$

and similarly, $a_{ij} \cdot_\theta b_{kl} = \sqrt{\lambda_{ik} \lambda_{jl}} b_{ij} b_{kl}$, while for generators \mathbf{u} and \mathbf{v} in [\(3.18\)](#),

$$\begin{aligned} u_i \cdot_\theta u_j &= \lambda_\theta(e_i, e_j) u_i u_j = \sqrt{\lambda_{ij}} u_i u_j, \\ v_i \cdot_\theta v_j &= \lambda_{-\theta}(e_i, e_j) v_i v_j = \sqrt{\lambda_{ji}} v_i v_j. \end{aligned}$$

One recovers the commutation relations in [\(3.15\)](#) and [\(3.18\)](#) by taking [\(4.9\)](#) into account.

Let us sample the assumptions [\(4.13\)](#) - [4.15](#) on some of generators. For the coproduct:

$$\Delta(a_{jl}) = \sum_s a_{js} \otimes a_{sl} + b_{js} \otimes b_{sl}^*,$$

the right hand side indeed fulfils $a_{js} \otimes a_{sl} \in H_{(j,s)} \otimes H_{(s,l)}$ and $b_{js} \otimes b_{sl}^* \in H_{(j,-s)} \otimes H_{(-s,l)}$. For the counit ε defined by $\varepsilon(N) = \mathbb{I}$, only the diagonal entries of N will survive after applying ε and they indeed belong to $\bigoplus_{s \in \mathbb{Z}^n} H_{(s,s)}$ as required in [\(4.14\)](#). For the $*$ -operator $*N = QNQ$, we see, for instance, that $\deg(b_{ij})^* = \deg b_{ij}^* = -\deg b_{ij}$. For the antipode $S(N) = N^\dagger$, we would like to check on, say u_j : $\deg u_j = (e_j, 0)$ compared with $\deg S(u_j) = \deg v_j^* = (0, -e_j)$. Lastly, the analysis of [\(4.19\)](#) for the coaction $\delta^{\tilde{H}}$ is just the same as that for the coproduct Δ . \square

Example 4.10. The matrix representation of $\text{SO}(2n)$ in [\(3.16\)](#) and [\(3.17\)](#) require an extra structure on \mathbb{R}^{2n} , that is a choice of polarization. Concretely, one identifies $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and choose a basis formed by complex coordinates $\{z_j, \bar{z}_j, j = 1 \dots n\}$, with respect to which the coefficient matrix of the Euclidean inner product is of the form Q in [\(3.17\)](#). We recall a remark made in [\[7, §8\]](#) which further motivates the bigrading setting of our scenario II in connection with the grading in scenario I.

At the level of the endomorphisms $M(2n, \mathbb{R})$, the identification is achieved by realising $M(2n, \mathbb{R}) \subset \text{End}(\mathbb{C}^n) \cong (\mathbb{C}^n)^* \otimes \mathbb{C}^n$. One first applies the θ -deformation to $(\mathbb{C}^n)^* \otimes \mathbb{C}^n$ following the setting in scenario I, which gives rise to two deformed algebra $A(\mathbb{R}_\theta^{2n}) =$

$A(\mathbb{C}_\theta^n)$ and $A(\mathbb{R}_{-\theta}^{2n}) = A(\mathbb{C}_{-\theta}^n)$, with generators $\{z^j, \bar{z}^j := (z^j)^*, j = 1, \dots, n\}$ for $A(\mathbb{R}_\theta^{2n})$ and $\{z_j, \bar{z}_j := (z_j)^*, j = 1, \dots, n\}$ for $A(\mathbb{R}_{-\theta}^{2n})$. The n -torus action α is the standard one:

$$\alpha_t(z_j) = t_j z_j, \quad \alpha_t(\bar{z}_j) = \bar{t}_j \bar{z}_j, \quad \alpha_t(z^j) = t_j z^j, \quad \alpha_t(\bar{z}^j) = \bar{t}_j \bar{z}^j, \quad (4.23)$$

which leads to the \mathbb{Z}^n -grading:

$$e_j = \deg z^j = \deg z_j = -\deg \bar{z}^j = -\deg \bar{z}_j, \quad (4.24)$$

where $\{e_j, j = 1, \dots, n\}$ is the standard basis of \mathbb{Z}^n . The algebra structure of $A(\mathbb{R}_\theta^{2n})$ is determined by the commutation relations:

$$z_j z_k = \lambda_{jk} z_k z_j, \quad \bar{z}_j z_k = \lambda_{kj} z_k \bar{z}_j.$$

Those for $A(\mathbb{R}_{-\theta}^{2n})$ are obtained by replacing λ_{jk} with $\bar{\lambda}_{jk}$. Now, the deformed $*$ -algebra $\mathcal{O}(M_\theta(2n, \mathbb{R}))$ (One can forget the coalgebra structure for the time being.) has already been defined in (3.16) and (3.15) in terms of generators and relations. A key point is that there is a $*$ -algebra homomorphism $\varphi : \mathcal{O}(M_\theta(2n, \mathbb{R})) \rightarrow A(\mathbb{R}_\theta^{2n}) \otimes A(\mathbb{R}_{-\theta}^{2n})$ induced by

$$\varphi(a_{ij}) = z^i \otimes z_j, \quad \varphi(b_{ij}) = z^i \otimes \bar{z}_j.$$

Furthermore, the map φ is injective and transfers the torus action $\alpha \otimes \alpha$ (cf. (4.23)), or equivalently, the bi-grading of $A(\mathbb{R}_\theta^{2n}) \otimes A(\mathbb{R}_{-\theta}^{2n})$ (cf. (4.24)), to those described in the Example 4.9: see (4.21) and (4.22). \square

Let us now take a closer look at the algebra of coinvariants and at the balanced product.

Lemma 4.11. *With the assumptions on H and A as before, the coinvariant subalgebra $B = A^{\text{co}H}$ is contained in*

$$B \subset \bigoplus_{r \in \mathbb{Z}^n} A_{(r,0)}. \quad (4.25)$$

Proof. The (algebra of functions on) the torus \mathbb{T}^n acting on the right is contained in H and gets washed away when passing to the coinvariant elements for the coaction of H . Explicitly, consider a homogeneous element $b \in B$ with $\deg b = (r, l)$. From (4.19) we have $\deg b_{(0)} = (r, s)$ and $\deg b_{(1)} = (s, l)$ where $s \in \mathbb{Z}^n$ depends on the components. The condition of being coinvariant $b_{(0)} \otimes b_{(1)} = b \otimes 1$ forces that $(r, s) = (r, l)$ and $(s, l) = (0, 0)$, hence $\deg b$ is always of the form $(r, 0)$ for some $r \in \mathbb{Z}^n$. \square

The $2n$ -sphere $B = \mathcal{O}(S^{2n})$ and its deformation in (4.9) indeed satisfy (4.25): the generators of B (or B_Θ) are $\{u_j, u_j^*, j = 1, \dots, n\}$ which are of degree $(\pm e_j, 0)$.

When forming the balanced tensor product $a \otimes_B \tilde{a}$, where $a, \tilde{a} \in A$, the degrees of a and \tilde{a} (assumed to be homogeneous) depend on the choice of the representative. However, from the previous lemma, the action of B only varies the left degree. As we shall see in next lemma, by slightly abusing the notation, for the translation map we can write

$$\tau(H_{(r,l)}) \subset \bigoplus_{p \in \mathbb{Z}^n} A_{(-p,-r)} \otimes_B A_{(p,l)}. \quad (4.26)$$

Let us check this on $\tau(\mathbf{h})$ and $\tau(\mathbf{k})$ in (3.23). For the (r, l) -entry of \mathbf{h} , we have

$$\begin{aligned} \tau(h_{rl}) &= \sum_s (\mathbf{a})_{rs}^\dagger \otimes_B \mathbf{a}_{sl} + (\mathbf{b}^*)_{rs}^\dagger \otimes_B (\mathbf{b}^*)_{sl} + (\mathbf{v})_r^\dagger \otimes_B \mathbf{v}_l \\ &= \sum_s a_{sr}^* \otimes_B a_{sl} + b_{sr} \otimes_B b_{sl}^* + v_r^* \otimes_B v_l. \end{aligned} \quad (4.27)$$

We see that $a_{sr}^* \otimes a_{sl} \in \tilde{H}_{(-e_s, -e_r)} \otimes \tilde{H}_{(e_s, e_l)}$, $b_{sr} \otimes b_{sl}^* \in \tilde{H}_{(e_s, -e_r)} \otimes \tilde{H}_{(-e_s, e_l)}$ as well as $v_r^* \otimes v_l \in \tilde{H}_{(0, -e_r)} \otimes \tilde{H}_{(0, e_l)}$ all satisfy (4.29). Similarly, for the (r, l) -entry of \mathbf{k} , we have

$$\begin{aligned} \tau(k_{rl}) &= \sum_s (\mathbf{a})_{rs}^\dagger \otimes_B \mathbf{b}_{sl} + (\mathbf{b}^*)_{rs}^\dagger \otimes_B (\mathbf{a}^*)_{sl} + (\mathbf{v})_r^\dagger \otimes_B \mathbf{v}_l^* \\ &= \sum_s a_{sr}^* \otimes_B b_{sl} + b_{sr} \otimes_B a_{sl}^* + v_r^* \otimes_B v_l^*. \end{aligned} \quad (4.28)$$

We see that $a_{sr}^* \otimes b_{sl} \in \tilde{H}_{(-e_s, -e_r)} \otimes \tilde{H}_{(e_s, -e_l)}$, $b_{sr} \otimes a_{sl}^* \in \tilde{H}_{(e_s, -e_r)} \otimes \tilde{H}_{(-e_s, -e_l)}$ as well as $v_r^* \otimes v_l^* \in \tilde{H}_{(0, -e_r)} \otimes \tilde{H}_{(0, -e_l)}$ and again they all satisfy (4.29). We also have

$$\begin{aligned} m_\Theta(\tau(\mathbf{h})) &= \mathbf{a}^\dagger \cdot_\Theta \mathbf{a} + \mathbf{b}^t \cdot_\Theta \mathbf{b}^* + \mathbf{v}^\dagger \cdot_\Theta \mathbf{v} = \varepsilon(\mathbf{h})\mathbb{I} = \mathbb{I}, \\ m_\Theta(\tau(\mathbf{k})) &= \mathbf{a}^\dagger \cdot_\Theta \mathbf{b} + \mathbf{b}^t \cdot_\Theta \mathbf{a}^* + \mathbf{v}^\dagger \cdot_\Theta \mathbf{v}^* = \varepsilon(\mathbf{k})\mathbb{I} = 0 \end{aligned}$$

and both agree with (2.13) and Corollary 4.13 below.

Lemma 4.12. *Let (A, H) be a Hopf–Galois extension fulfilling all assumptions of earlier. For any homogeneous elements $h \in H_{(r,l)}$, there are suitable representatives for the translation map $\tau(h) = h^{(1)} \otimes_B h^{(2)}$ such that*

$$\deg h^{(1)} = (-p, -r), \quad \deg h^{(2)} = (p, l), \quad (4.29)$$

where, by taking (4.25) into account, the left degree p depends on the components $h^{(1)}, h^{(2)}$ and the choice of the representatives (cf. also Remark 4.4).

Proof. The constraint on degrees in (4.29) follows from (2.10): $h^{(1)}(h^{(2)})_{(0)} \otimes (h^{(2)})_{(1)} = 1 \otimes h$. Suppose $\deg h = (r, l)$, $\deg h^{(2)} = (p, q)$ and $\deg h^{(1)} = (p', q')$, so that $\deg(h^{(2)})_{(0)} = (p, s)$ and $\deg(h^{(2)})_{(1)} = (s, q)$ for some $s \in \mathbb{Z}^n$. By comparing the two sides of (2.10), we have $(s, q) = (r, l)$ and $(p', q') = -(p, s)$. Thus (4.29) follows: $q = l$, $p' = -p$ and $q' = -s = -r$. \square

Corollary 4.13. *Let $h \in H_{(r,l)}$ be a homogeneous element with $\tau(h) = h^{(1)} \otimes_B h^{(2)}$. Then,*

- i) for $r = l$ one has $\deg h^{(1)} + \deg h^{(2)} = 0$;
- ii) for $r \neq l$, one has $\tau(h) = 0$.

Proof. With $r = l$, the first statement follows from (4.29). The latter also says that $h^{(1)}h^{(2)} \in A_{(0, l-r)}$, which is non-zero unless $h^{(1)} \otimes h^{(2)} = 0$. \square

This result is in accordance with (4.27) and (4.28) by recalling that h_{jl} has bi-degree $((e_j, 0), (e_l, 0))$ while k_{jl} has bi-degree $((e_j, 0), (-e_l, 0))$. It allows one to repeat the second part of Proposition 4.2 and deform the starting Hopf–Galois extension into a new one.

Proposition 4.14. *Consider the deformed pair (H_Θ, A_Θ) obtained in Proposition 4.8 and define a deformed canonical Galois map χ_Θ by*

$$\chi_\Theta : A_\Theta \otimes_{B_\Theta} A_\Theta \rightarrow A_\Theta \otimes H_\Theta, \quad a' \otimes_{B_\Theta} a \mapsto a' \cdot_\Theta a_{(0)} \otimes a_{(1)}. \quad (4.30)$$

This is invertible if and only if the starting canonical Galois map is with the same translation map, but viewed as a map $\tau : H_\Theta \rightarrow A_\Theta \otimes_{B_\Theta} A_\Theta$.

Proof. Let $h \in H_{(r,l)}$ be a homogeneous element with $\tau(h) = h^{(1)} \otimes_B h^{(2)}$, the starting translation map. From Corollary 4.13, $\deg h^{(1)} = -\deg h^{(2)}$. Then, with a slight abuse of notation ($B = B_\theta$ as a vector space), from Lemma 4.1,

$$\begin{aligned} \chi_\Theta(h^{<1>} \otimes_B h^{<2>}) &= h^{<1>} \cdot_\Theta h^{<2>}_{(0)} \otimes h^{<2>}_{(1)} = h^{<1>} h^{<2>}_{(0)} \otimes_B h^{<2>}_{(1)} \\ &= \chi(h^{<1>} \otimes_B h^{<2>}) \end{aligned}$$

and χ_Θ is invertible if and only if χ is. Or, the pair (H_Θ, A_Θ) is a Hopf–Galois extension if and only if the pair (H, A) is such. \square

5. HOPF ALGEBROIDS

We are ready for the Hopf algebroid structure. We start with a bialgebroid $\mathcal{C}(H_\Theta, A_\Theta)$ associated to the Hopf–Galois extension (H_Θ, A_Θ) of the previous section. We next show that the flip can serve as an antipode. In §§ 5.4 and 5.3, we present two examples. Firstly an algebroid for the principal $SU(2)$ -principal bundle over the four-sphere S_θ^4 described in §3.2 followed by the one for the bundles over the even spheres of §3.1.

5.1. The bialgebroid $\mathcal{C}(H_\Theta, A_\Theta)$. We know from Lemma 4.11 that the coinvariant elements for the action of H_Θ have trivial right grading. This will clearly be the case also for the coinvariant elements for the diagonal action that is needed for the Ehresmann–Schauenburg bialgebroid. Thus the construction of the bialgebroid will be the same for the Hopf–Galois extension (H_Θ, A_Θ) in Proposition 4.14 of our scenario II and for the pair (H, A_θ) discussed for Scenario I in Proposition 4.2. We describe the former here.

Consider then the Hopf–Galois extension (H_Θ, A_Θ) . The diagonal coaction is in (4.20):

$$\delta^{A \otimes_\Theta A} : A_\Theta \otimes A_\Theta \rightarrow A_\Theta \otimes A_\Theta \otimes H_\Theta, \quad \delta^{A \otimes_\Theta A}(a \otimes \tilde{a}) = a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)} \cdot_\Theta \tilde{a}_{(1)}. \quad (5.1)$$

with $V = W = A$. From the analysis before, and in particular from the fact that the canonical map and translation maps are the same as maps between vector spaces, the conclusion is that all the structure equations listed in §2.3 hold true after deformation (which means replacing every occurrence of multiplication by the deformed one).

Lemma 5.1. *Let (H, A) be a Hopf–Galois extension that fulfil the assumptions on the bigradings of earlier and let (H_Θ, A_Θ) be the deformed (H_Θ, A_Θ) Hopf–Galois extension obtained in Proposition 4.30. Then the deformed coaction $\delta^{A \otimes_\Theta A}$ in (5.1), gives rise to the same coinvariant subspace as that of $\delta^{A \otimes A}$:*

$$(A_\Theta \otimes A_\Theta)^{\text{co}H_\Theta} = (A \otimes A)^{\text{co}H}.$$

Also, the deformed Ehresmann–Schauenburg bialgebroid

$$\mathcal{C}(H_\Theta, A_\Theta) = \left((A_\Theta \otimes A_\Theta)^{\text{co}H_\Theta}, \bullet_\Theta \right)$$

with respect to $\mathcal{C}(A, H)$ in Def. 2.5, has only the algebra structure changed, given by:

$$(x \otimes y) \bullet_\Theta (\tilde{x} \otimes \tilde{y}) := x \cdot_\theta \tilde{x} \otimes \tilde{y} \cdot_\theta y. \quad (5.2)$$

Proof. The results follows from the identification in Lemma 2.3, which uses only the translation map that is unchanged (as a map between vector spaces) when deforming. \square

5.2. The flip map as the antipode. The bialgebroids of the previous section gets in fact a structure of Hopf algebroid with a suitable antipode. Now, when the structure Hopf algebra H is commutative, the flip map preserves the coinvariant elements of the diagonal coaction. Indeed, given

$$S : A \otimes A \rightarrow A \otimes A, \quad a \otimes \tilde{a} \mapsto \tilde{a} \otimes a, \quad (5.3)$$

for any coinvariant $a \otimes \tilde{a} \in A \otimes A$, by swapping a and \tilde{a} in $a \otimes \tilde{a} \otimes 1 = a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)} \tilde{a}_{(1)}$, we see that $\tilde{a} \otimes a$ is coinvariant as well:

$$\tilde{a} \otimes a \otimes 1 = \tilde{a}_{(0)} \otimes a_{(0)} \otimes a_{(1)} \tilde{a}_{(1)} = \tilde{a}_{(0)} \otimes a_{(0)} \otimes \tilde{a}_{(1)} a_{(1)},$$

where the last equal sign invokes the commutativity of H . Therefore, when restricted to the coinvariant subspaces the flip is a candidate for the antipode of $\mathcal{C}(H, A)$ and $\mathcal{C}(H, A_\theta)$.

In the more general situation, despite H_θ needs no longer stay commutative after the θ -deformation, the flip S still maps $\mathcal{C}(H_\theta, A_\theta)$ into itself since we have shown in Lemma 5.1 that $\mathcal{C}(H_\theta, A_\theta)$ and $\mathcal{C}(H, A)$ are identical as vector spaces (This fact will be explicitly seen for the example in §5.4 below.)

The main result of this section is that the flip S makes $\mathcal{C}(H_\theta, A_\theta)$ into a Hopf algebroid.

Theorem 5.2. *By only deforming multiplication related structures of the Hopf algebroid $\mathcal{C}(H, A)$ over B , the resulting $\mathcal{C}(H_\theta, A_\theta)$ is a Hopf algebroid, but with base algebra B_θ .*

Proof. One needs to verify the compatibility conditions in (2.7) and (2.8). The latter one is the less nontrivial one and is handled in Lemma 5.3 below. We point out that the computations below work for both $\mathcal{C}(H, A)$ and $\mathcal{C}(H_\theta, A_\theta)$ since they do not rely on the commutativity of the underlying algebra structures in the Hopf–Galois extension. \square

Lemma 5.3. *The flip $S : \mathcal{C}(H_\theta, A_\theta) \rightarrow \mathcal{C}(H_\theta, A_\theta)$ with $S^{-1} = S$ fulfils the compatibility conditions in (2.8), that is, for all $h \in \mathcal{C}(H_\theta, A_\theta)$:*

$$\begin{aligned} (S^{-1}h_{(2)})_{(1)} \otimes_{B_\theta} (S^{-1}h_{(2)})_{(2)} \bullet_\theta h_{(1)} &= S^{-1}h \otimes_{B_\theta} 1, \\ (Sh_{(1)})_{(1)} \bullet_\theta h_{(2)} \otimes_{B_\theta} (Sh_{(1)})_{(2)} &= 1 \otimes_{B_\theta} S(h). \end{aligned}$$

Proof. We shall prove the first one as an example and leave the second one to avid readers.

Write $h = a \otimes \tilde{a} \in \mathcal{C}(H_\theta, A_\theta)$, where $a, \tilde{a} \in A_\theta$, then the coproduct in (2.21) reads

$$\Delta(h) = h_{(1)} \otimes_{B_\theta} h_{(2)} = (a_{(0)} \otimes (a_{(1)})^{(1)}) \otimes_{B_\theta} ((a_{(1)})^{(2)} \otimes \tilde{a}).$$

We compute:

$$\begin{aligned} (S^{-1}h_{(2)})_{(1)} \otimes_{B_\theta} (S^{-1}h_{(2)})_{(2)} \bullet_\theta h_{(1)} &= (\tilde{a} \otimes (a_{(1)})^{(2)})_{(1)} \otimes_{B_\theta} (\tilde{a} \otimes (a_{(1)})^{(2)})_{(2)} \bullet_\theta (a_{(0)} \otimes (a_{(1)})^{(1)}) \\ &= \tilde{a}_{(0)} \otimes (\tilde{a}_{(1)})^{(1)} \otimes_{B_\theta} ((\tilde{a}_{(1)})^{(2)} \otimes (a_{(1)})^{(2)}) \bullet_\theta (a_{(0)} \otimes (a_{(1)})^{(1)}) \\ &= \tilde{a}_{(0)} \otimes (\tilde{a}_{(1)})^{(1)} \otimes_{B_\theta} (\tilde{a}_{(1)})^{(2)} \cdot_\theta a_{(0)} \otimes (a_{(1)})^{(1)} \cdot_\theta (a_{(1)})^{(2)} \\ &= \tilde{a}_{(0)} \otimes (\tilde{a}_{(1)})^{(1)} \otimes_{B_\theta} (\tilde{a}_{(1)})^{(2)} \cdot_\theta a_{(0)} \otimes \varepsilon(a_{(1)}) 1_{A_\theta} \\ &= \tilde{a}_{(0)} \otimes (\tilde{a}_{(1)})^{(1)} \otimes_{B_\theta} (\tilde{a}_{(1)})^{(2)} \cdot_\theta a \otimes 1, \end{aligned}$$

where, in the last two steps, we have used (2.13) and the compatibility between the counit $\varepsilon : H_\Theta \rightarrow \mathbb{C}$ and the coaction δ^A . To continue:

$$\begin{aligned} (S^{-1}h_{(2)})_{(1)} \otimes_{B_\Theta} (S^{-1}h_{(2)})_{(2)} \bullet_\Theta h_{(1)} &= (\tilde{a}_{(0)} \otimes (\tilde{a}_{(1)})^{(1)}) \otimes_{B_\Theta} ((\tilde{a}_{(1)})^{(2)} \bullet_\Theta a \otimes 1) \\ &= \tilde{a}_{(0)} \otimes \tau(\tilde{a}_{(1)}) \bullet_\Theta a \\ &= \tilde{a} \otimes a \otimes_{B_\Theta} 1 = S^{-1}h \otimes_{B_\Theta} 1, \end{aligned}$$

where we need (2.20), which is an equivalent description for $a \otimes \tilde{a} \in \mathcal{C}(H_\Theta, A_\Theta)$, to complete the second line. \square

5.3. The algebroid with SU(2)-symmetry. With respect to the example in §3.1, denote $A = A(S_{\theta'}^7)$, $H = A(\text{SU}(2))$ and $B = A(S_\theta^4) = A^{coH}$ the subalgebra of invariants and, as usual $\delta^A(a) = a_{(0)} \dot{\otimes} a_{(1)}$ and $\tau(h) = h^{<1>} \otimes_B h^{<2>}$.

Consider then the diagonal coaction of H on the tensor product algebra $A \otimes A$:

$$\delta^{A \otimes A} : A \otimes A \rightarrow A \otimes A \otimes H, \quad a \otimes \tilde{a} \mapsto a_{(0)} \otimes \tilde{a}_{(0)} \otimes a_{(1)} \tilde{a}_{(1)}.$$

Lemma 5.4. *The B -bimodule $\mathcal{C}(A, H)$ of coinvariant elements for the diagonal coaction is generated by elements of the tensor products $p \otimes 1$ and $1 \otimes q$ together with*

$$V = \Psi \dot{\otimes} \Psi^\dagger.$$

Proof. It is clear that elements of $p \otimes 1$ and $1 \otimes q$ are coinvariants. For $V = \Psi \dot{\otimes} \Psi^\dagger$:

$$\delta^{A \otimes A}(V) = \Psi_{(0)} \dot{\otimes} \Psi_{(0)}^\dagger \dot{\otimes} \Psi_{(1)} \Psi_{(1)}^\dagger = \Psi \otimes \Psi^\dagger \dot{\otimes} (w w^\dagger) = V \dot{\otimes} (w w^\dagger) = V \dot{\otimes} \mathbb{I}_2,$$

in parallel with the coinvariance (3.7). \square

With the flip $\sigma(a \otimes b) = b \otimes a$ we define

$$S_{\mathcal{C}}(V) := \sigma(\Psi \dot{\otimes} \Psi^\dagger) = V^\dagger, \quad \text{or} \quad S_{\mathcal{C}}(V_{mn}) = V_{mn}^\dagger = \sum_r \Psi_{rn}^\dagger \dot{\otimes} \Psi_{mr}. \quad (5.4)$$

Then, a direct computation shows that

$$\begin{aligned} S_{\mathcal{C}}(V)V &= V^\dagger V = 1 \otimes \Psi \cdot_{op} \Psi^\dagger = 1 \otimes q \\ VS_{\mathcal{C}}(V) &= VV^\dagger = \Psi \cdot \Psi^\dagger \otimes 1 = p \otimes 1. \end{aligned} \quad (5.5)$$

These then are relations among the elements of V as generators of the B -bimodule $\mathcal{C}(A, H)$. The latter has the structure of a Hopf algebroid.

Firstly, the projections $VV^\dagger = p \otimes 1$ and $V^\dagger V = 1 \otimes q$ are the two embedded copies of the 4-sphere $A(S_\theta^4)$ in $\mathcal{C}(A, H)$: $A(S_\theta^4) \otimes 1$ and $1 \otimes A(S_\theta^4)$, via source and target map respectively, as explicitly described in Lemma 5.6 below.

Next, according to the definition (2.21) a coproduct $\Delta : \mathcal{C}(A, H) \rightarrow \mathcal{C}(A, H) \otimes_B \mathcal{C}(A, H)$, is given on the matrix V of generators by

$$\Delta(V) = \Psi_{(0)} \dot{\otimes} \Psi_{(1)}^{<1>} \dot{\otimes}_B \Psi_{(1)}^{<2>} \dot{\otimes} \Psi^\dagger = \Psi \otimes \Psi^\dagger \dot{\otimes}_B \Psi \otimes \Psi^\dagger = V \dot{\otimes}_B V. \quad (5.6)$$

In components this reads:

$$\Delta(V_{mn}) = \sum_r V_{mr} \otimes_B V_{rn}. \quad (5.7)$$

From (5.6) one also gets

$$\Delta(S_{\mathcal{C}}(V)) = \Delta(V^\dagger) = \sigma(V^\dagger \dot{\otimes}_B V^\dagger) \quad (5.8)$$

or, in components:

$$\Delta(V_{mr}^\dagger) = \sum_r V_{rn}^\dagger \otimes_B V_{mr}^\dagger. \quad (5.9)$$

Finally, the map S_C in (5.4) is indeed an antipode for $\mathcal{C}(A, H)$. Since S_C is the flip, condition (2.7) is obvious. We are left to show condition (2.8). For this, take $h = V$. With expressions (5.6) and (5.8) for the coproducts, and using (5.5):

$$\begin{aligned} (S_C h_{(1)})_{(1')} h_{(2)} \otimes_B S_C(h_{(1)})_{(2')} &= (S(V_{(1)}))_{(1')} V_{(2)} \dot{\otimes}_B (S(V_{(1)}))_{(2')} \\ &= (S_C(V))_{(1')} V \dot{\otimes}_B (S_C(V))_{(2')} \\ &= (V^\dagger)_{(1')} V \dot{\otimes}_B (V^\dagger)_{(2')} = V^\dagger V \dot{\otimes}_B V^\dagger \\ &= 1 \otimes q \dot{\otimes}_B V^\dagger = 1 \otimes 1 \otimes_B q V^\dagger \\ &= 1 \otimes 1 \otimes_B V^\dagger = 1 \otimes 1 \otimes_B S_C(V). \end{aligned} \quad (5.10)$$

Since elements of q are in B they can be crossed over B -tensor products, and we used the relation $q V^\dagger = V^\dagger$. The other condition in (2.8) is similar since $S_C^{-1} = S_C$.

In components of V this works as follows. Take $h = V_{mn}$, with $S_C(h) = V_{mn}^\dagger$. Using the expressions for the coproduct, we compute:

$$\begin{aligned} (S_C h_{(1)})_{(1')} h_{(2)} \otimes_B S_C(h_{(1)})_{(2')} &= \sum_r (S_C(V)_{mr})_{(1')} V_{rn} \otimes_B (S_C(V)_{mr})_{(2')} \\ &= \sum_r (V_{mr}^\dagger)_{(1')} V_{rn} \otimes_B (V_{mr}^\dagger)_{(2')} \\ &= \sum_{rs} V_{sr}^\dagger V_{rn} \otimes_B V_{ms}^\dagger = \sum_s (V^\dagger V)_{sn} \otimes_B V_{ms}^\dagger \\ &= \sum_s 1 \otimes (\Psi \cdot_{op} \Psi^\dagger)_{sn} \otimes_B V_{ms}^\dagger = \sum_s 1 \otimes 1 \otimes_B (\Psi \cdot_{op} \Psi^\dagger)_{sn} V_{ms}^\dagger \\ &= \sum_{s,j,k} 1 \otimes 1 \otimes_B \Psi_{jn}^\dagger \Psi_{sj} \Psi_{ks}^\dagger \otimes \Psi_{mk} \\ &= \sum_{s,j,k} 1 \otimes 1 \otimes_B \Psi_{jn}^\dagger (\Psi_{ks}^\dagger \cdot_{op} \Psi_{sj}) \otimes \Psi_{mk} \\ &= \sum_{j,k} 1 \otimes 1 \otimes_B \Psi_{jn}^\dagger (\delta_{kj} 1) \otimes \Psi_{mk} = \sum_k 1 \otimes 1 \otimes_B \Psi_{kn}^\dagger \otimes \Psi_{mk} \\ &= 1 \otimes 1 \otimes_B V_{mn}^\dagger = 1 \otimes 1 \otimes_B S_C(V)_{mn} = 1 \otimes 1 \otimes_B S_C(h). \end{aligned} \quad (5.11)$$

Here we crossed elements of $(\Psi \cdot_{op} \Psi^\dagger)$ over the B -tensor product, since they are in B , and the relation $\sum_s \Psi_{ks}^\dagger \cdot_{op} \Psi_{sj} = \delta_{kj} 1$.

5.3.1. *Generators and relations.* In term of generators and relations let us write

$$V = \begin{pmatrix} P_1 & Q_2 \\ Q_1 & P_2 \end{pmatrix} \quad \text{with} \quad P_1 = \begin{pmatrix} Z_0 & -\tilde{X}_0 \\ X_0 & \tilde{Z}_0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} Z_2 & -\tilde{W}_2 \\ W_2 & \tilde{Z}_2 \end{pmatrix}, \\ Q_1 = \begin{pmatrix} Z_1 & -\tilde{W}_1 \\ W_1 & \tilde{Z}_1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} W_0 & -\tilde{Y}_0 \\ Y_0 & \tilde{W}_0 \end{pmatrix}. \quad (5.12)$$

An explicit computation leads to

$$\begin{aligned}
Z_0 &= \psi_1 \otimes \psi_1^* + \psi_2^* \otimes \psi_2, & \tilde{Z}_0 &= \psi_1^* \otimes \psi_1 + \psi_2 \otimes \psi_2^* = Z_0^*, \\
X_0 &= \psi_2 \otimes \psi_1^* - \psi_1^* \otimes \psi_2, & \tilde{X}_0 &= \psi_2^* \otimes \psi_1 - \psi_1 \otimes \psi_2^* = X_0^*, \\
W_0 &= \psi_3 \otimes \psi_3^* + \psi_4^* \otimes \psi_4, & \widetilde{W}_0 &= \psi_3^* \otimes \psi_3 + \psi_4 \otimes \psi_4^* = W_0^*, \\
Y_0 &= \psi_4 \otimes \psi_3^* - \psi_3^* \otimes \psi_4, & \widetilde{Y}_0 &= \psi_4^* \otimes \psi_3 - \psi_3 \otimes \psi_4^* = Y_0^*, \\
Z_1 &= \psi_3 \otimes \psi_1^* + \psi_4^* \otimes \psi_2, & \tilde{Z}_1 &= \psi_3^* \otimes \psi_1 + \psi_4 \otimes \psi_2^* = Z_1^*, \\
W_1 &= \psi_4 \otimes \psi_1^* - \psi_3^* \otimes \psi_2, & \widetilde{W}_1 &= \psi_4^* \otimes \psi_1 - \psi_3 \otimes \psi_2^* = W_1^*, \\
Z_2 &= \psi_1 \otimes \psi_3^* + \psi_2^* \otimes \psi_4, & \tilde{Z}_2 &= \psi_2 \otimes \psi_4^* + \psi_1^* \otimes \psi_3 = Z_2^*, \\
W_2 &= \psi_2 \otimes \psi_3^* - \psi_1^* \otimes \psi_4, & \widetilde{W}_2 &= \psi_2^* \otimes \psi_3 - \psi_1 \otimes \psi_4^* = W_2^*.
\end{aligned} \tag{5.13}$$

It is then immediate to check that

$$S_C(V) = V^\dagger.$$

We know that the generators are not independent. Indeed:

Lemma 5.5. *There are four sphere relations:*

$$\begin{aligned}
\tilde{Z}_0 Z_0 + \tilde{X}_0 X_0 &= Z_0 \tilde{Z}_0 + X_0 \tilde{X}_0 = \zeta_0 \otimes \zeta_0, \\
\widetilde{W}_0 W_0 + \widetilde{Y}_0 Y_0 &= W_0 \widetilde{W}_0 + Y_0 \widetilde{Y}_0 = (1 - \zeta_0) \otimes (1 - \zeta_0), \\
\tilde{Z}_1 Z_1 + \widetilde{W}_1 W_1 &= Z_1 \tilde{Z}_1 + W_1 \widetilde{W}_1 = (1 - \zeta_0) \otimes \zeta_0, \\
\tilde{Z}_2 Z_2 + \widetilde{W}_2 W_2 &= Z_1 \tilde{Z}_1 + W_1 \widetilde{W}_1 = \zeta_0 \otimes (1 - \zeta_0).
\end{aligned} \tag{5.14}$$

Proof. One computes these from the relations (5.5). \square

In a sense this says that the four matrices in (5.12) are all equivalent and for the generators of the B -bimodule $\mathcal{C}(A, H)$ of coinvariant elements one can take any one of those together with $A(S_\theta^4) \otimes 1$ and $1 \otimes A(S_\theta^4)$. Alternatively, one could express the generators of the latter spheres in terms of the generators in (5.13).

Lemma 5.6. *The source map:*

$$\begin{aligned}
\tilde{Z}_0 Z_0 + \tilde{X}_0 X_0 + \tilde{Z}_2 Z_2 + \widetilde{W}_2 W_2 &= \zeta_0 \otimes 1, \\
Z_0 \tilde{Z}_1 + \tilde{X}_0 W_2 + Z_2 \widetilde{W}_0 + \widetilde{W}_2 Y_0 &= \zeta_1 \otimes 1, \\
X_0 \tilde{Z}_1 + W_2 \widetilde{W}_0 - \tilde{Z}_0 W_1 - \tilde{Z}_2 Y_0 &= \zeta_2 \otimes 1.
\end{aligned} \tag{5.15}$$

and the target map:

$$\begin{aligned}
\tilde{Z}_0 Z_0 + \tilde{X}_0 X_0 + \tilde{Z}_1 Z_1 + \widetilde{W}_1 W_1 &= 1 \otimes \zeta_0, \\
W_2 \tilde{X}_0 + Z_2 \tilde{Z}_0 + Y_0 \widetilde{W}_1 + W_0 \tilde{Z}_1 &= 1 \otimes \zeta_1, \\
W_2 Z_0 - Z_2 X_0 + Y_0 Z_1 - W_0 W_1 &= 1 \otimes \zeta_2.
\end{aligned} \tag{5.16}$$

Proof. The direct way for these is just to use again the relations in (5.5). \square

5.4. A Hopf algebra with quantum orthogonal symmetry. Let us denote $A = \mathcal{O}(\mathrm{SO}_\theta(2n+1))$, $H = \mathcal{O}(\mathrm{SO}_\theta(2n))$ and $B = \mathcal{O}(S_\theta^{2n})$. With the notations of §3.2 consider the matrix valued function

$$\Phi = (\Phi_{JK}) = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b}^* & \mathbf{a}^* \\ \mathbf{v} & \mathbf{v}^* \end{pmatrix}. \quad (5.17)$$

Then, the orthogonality conditions $N^\dagger N = \mathbb{I}$ gives that $\Phi^\dagger \cdot \Phi = \mathbb{I}_{2n}$. Moreover, the entries of the matrix $\Phi \cdot \Phi^\dagger$ (a projection from the condition $\Phi^\dagger \cdot \Phi = \mathbb{I}_{2n}$) are coinvariants for the coaction (3.20). In fact, one computes explicitly that

$$\Phi \cdot \Phi^\dagger = \begin{pmatrix} 1 - \mathbf{u}\mathbf{u}^\dagger & -\mathbf{u}\mathbf{u}^t & -\mathbf{u}x \\ -\mathbf{u}^*\mathbf{u}^\dagger & 1 - \mathbf{u}^*\mathbf{u}^t & -\mathbf{u}^*x \\ -x\mathbf{u}^\dagger & -x\mathbf{u}^t & 1 - x^2 \end{pmatrix} \quad (5.18)$$

Let us denote

$$\mathbf{w} = \begin{pmatrix} \mathbf{h} & \mathbf{k} \\ \mathbf{k}^* & \mathbf{h}^* \end{pmatrix}, \quad (5.19)$$

the defining matrix of $\mathcal{O}(\mathrm{SO}_\theta(2n))$, with $\mathbf{w}^\dagger \mathbf{w} = \mathbf{w}\mathbf{w}^\dagger = \mathbb{I}_2$. Then the coaction (3.20) reduces to a coaction

$$\delta^A(\Phi) = \Phi \dot{\otimes} \mathbf{w} \quad (5.20)$$

or

$$\begin{aligned} \delta^A(\mathbf{a}) &= \mathbf{a} \dot{\otimes} \mathbf{h} + \mathbf{b} \dot{\otimes} \mathbf{k}^* & \delta^A(\mathbf{a}^*) &= \mathbf{a}^* \dot{\otimes} \mathbf{h}^* + \mathbf{b}^* \dot{\otimes} \mathbf{k} \\ \delta^A(\mathbf{b}) &= \mathbf{a} \dot{\otimes} \mathbf{k} + \mathbf{b} \dot{\otimes} \mathbf{h}^* & \delta^A(\mathbf{b}^*) &= \mathbf{a}^* \dot{\otimes} \mathbf{k}^* + \mathbf{b}^* \dot{\otimes} \mathbf{h} \\ \delta^A(\mathbf{v}) &= \mathbf{v} \dot{\otimes} \mathbf{h} + \mathbf{v}^* \dot{\otimes} \mathbf{k}^* & \delta^A(\mathbf{v}^*) &= \mathbf{v}^* \dot{\otimes} \mathbf{h}^* + \mathbf{v} \dot{\otimes} \mathbf{k}. \end{aligned} \quad (5.21)$$

In turns this gives

$$\delta^A(\Phi^\dagger) = \sigma(\mathbf{w}^\dagger \dot{\otimes} \Phi^\dagger). \quad (5.22)$$

The translation map is easily seen to be given by

$$\tau(\mathbf{w}) = \Phi^\dagger \dot{\otimes}_B \Phi. \quad (5.23)$$

One could show the coinvariance of the entries of $\Phi \cdot \Phi^\dagger$ by using the explicit form (5.21) of the coaction. Then in exactly the same way, one shows the following:

Lemma 5.7. *The B -bimodule $\mathcal{C}(A, H)$ of coinvariant elements for the diagonal coaction of H on $A \otimes A$ is generated by elements $1 \otimes \mathbf{u}, 1 \otimes \mathbf{u}^*, 1 \otimes x$, and $\mathbf{u} \otimes 1, \mathbf{u}^* \otimes 1, x \otimes 1$, together with the entries of the matrix*

$$\mathbf{V} = \Phi \dot{\otimes} \Phi^\dagger = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b}^* & \mathbf{a}^* \\ \mathbf{v} & \mathbf{v}^* \end{pmatrix} \dot{\otimes} \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b}^* & \mathbf{a}^* \\ \mathbf{v} & \mathbf{v}^* \end{pmatrix}^\dagger.$$

Moreover, the flip $S_{\mathcal{C}}(x \otimes y) = y \otimes x$ leaves unchanged the space of coinvariants, and in particular

$$S_{\mathcal{C}}(\mathbf{V}) = \sigma(\Phi \dot{\otimes} \Phi^\dagger) = \mathbf{V}^\dagger.$$

Proof. For $\mathbf{V} = \Phi \dot{\otimes} \Phi^\dagger$:

$$\delta^{A \otimes A}(\mathbf{V}) = \Phi_{(0)} \dot{\otimes} \Phi_{(0)}^\dagger \dot{\otimes} \Phi_{(1)} \dot{\otimes} \Phi_{(1)}^\dagger = \Phi \dot{\otimes} \Phi^\dagger \dot{\otimes} (\mathbf{w}\mathbf{w}^\dagger) = \mathbf{V} \dot{\otimes} (\mathbf{w}\mathbf{w}^\dagger) = \mathbf{V} \dot{\otimes} \mathbb{I}_{2n}$$

and directly: $\delta^{A \otimes A}(\mathbf{V}^\dagger) = \sigma(\Phi_{(0)} \dot{\otimes} \Phi_{(0)}^\dagger) \dot{\otimes} \Phi_{(1)}^\dagger \dot{\otimes} \Phi_{(1)} = \sigma(\Phi \dot{\otimes} \Phi^\dagger) \dot{\otimes} (\mathbf{w}^\dagger \mathbf{w}) = \mathbf{V}^\dagger \dot{\otimes} \mathbb{I}_{2n}$. \square

Lemma 5.8. *Using the conditions $N^\dagger N = \mathbb{I} = NN^\dagger$ one finds the following relations*

$$\Phi^\dagger \cdot_{op} \Phi = \begin{pmatrix} (\Phi^\dagger \cdot \Phi)_{22} & (\Phi^\dagger \cdot \Phi)_{12} \\ (\Phi^\dagger \cdot \Phi)_{21} & (\Phi^\dagger \cdot \Phi)_{11} \end{pmatrix} = Q(\Phi^\dagger \cdot \Phi)^t Q = \begin{pmatrix} \mathbb{I}_n & 0 \\ 0 & \mathbb{I}_n \end{pmatrix} = \mathbb{I}_{2n} \quad (5.24)$$

$$\Phi \cdot_{op} \Phi^\dagger = \begin{pmatrix} 1 - \mathbf{u}^* \mathbf{u}^t & -\mathbf{u} \mathbf{u}^t & -\mathbf{u}^* x \\ -\mathbf{u}^* \mathbf{u}^\dagger & 1 - \mathbf{u} \mathbf{u}^\dagger & -\mathbf{u} x \\ -x \mathbf{u}^t & -x \mathbf{u}^\dagger & 1 - x^2 \end{pmatrix} = Q(\Phi \cdot \Phi^\dagger)^t Q \quad (5.25)$$

It is evident that $(\Phi \cdot \Phi^\dagger)_{JK} \in B$; as well as $(\Phi \cdot_{op} \Phi^\dagger)_{JK} \in B$. Also,

$$\mathbf{V} \mathbf{V}^\dagger = (\Phi \cdot \Phi^\dagger) \otimes 1, \quad \mathbf{V}^\dagger \mathbf{V} = 1 \otimes (\Phi \cdot_{op} \Phi^\dagger), \quad (5.26)$$

which express relations among the generators of $\mathcal{C}(A, H)$.

In parallel with the projection $\Phi \cdot \Phi^\dagger$, the matrix $\Phi \cdot_{op} \Phi^\dagger$ is a projection due to $\Phi^\dagger \cdot_{op} \Phi = \mathbb{I}_{2n}$.

We are ready for the Hopf algebroid structure.

Proposition 5.9. *On $\mathcal{C}(A, H)$, the coproduct $\Delta : \mathcal{C}(A, H) \rightarrow \mathcal{C}(A, H) \otimes_B \mathcal{C}(A, H)$, according to the definition (2.21) and using the translation map (5.23), is given by*

$$\Delta(\mathbf{V}) = \Phi_{(0)} \dot{\otimes} \Phi_{(1)} \langle 1 \rangle \dot{\otimes}_B \Phi_{(1)} \langle 2 \rangle \dot{\otimes} \Phi^\dagger = \Phi \dot{\otimes} \Phi^\dagger \dot{\otimes}_B \Phi \dot{\otimes} \Phi^\dagger = \mathbf{V} \dot{\otimes}_B \mathbf{V}. \quad (5.27)$$

In components this reads:

$$\Delta(V_{JK}) = \sum_L V_{JL} \otimes_B V_{LK}. \quad (5.28)$$

Also,

$$\Delta(S_C(\mathbf{V})) = \sigma(S_C(\mathbf{V}) \dot{\otimes}_B S_C(\mathbf{V})), \quad (5.29)$$

or

$$\Delta(S_C(V)_{JK}) = \sum_L S_C(V)_{LK} \otimes_B S_C(V)_{JL}, \quad \Delta(V_{JK}^\dagger) = \sum_L V_{LK}^\dagger \otimes_B V_{JL}^\dagger. \quad (5.30)$$

Very much in the lines of the proof (5.11), we have the following:

Proposition 5.10. *The flip S_C is the antipode of $\mathcal{C}(A, H)$.*

Proof. Since S_C is just the flip, condition (2.7) is obvious. For conditions (2.8) take $h = V_{JK}$, with $S_C(h) = V_{JK}^\dagger$ and use the explicit form of the coproduct (5.28). Then the proof proceeds verbatim as in the proof of (5.11). \square

APPENDIX A. DEFORMING COMPACT LIE GROUPS ALONG A TORAL SUBGROUP

We recall Rieffel's construction in [12] that deforms function algebras of compact Lie groups and make comparison with the algebraic setup in §4.2. Let G be a compact Lie group and $K \subset G$ be a toral subgroup of rank n (for example, but not necessarily, a maximal torus). Denote by $H = \mathcal{O}(G)$ the Hopf algebra of representative functions on G . One would like to only vary the multiplication of H to get a new Hopf algebra H_Θ . In order to retain the compatibility between the algebra and coalgebra structures, one must cautiously pick the torus action and keep track of the equivariant properties of the structure maps of H .

A suitable deformation begins with an action α of $T = K \times K$ on G from two sides:

$$\alpha_{k_1, k_2}(g) = k_1^{-1} g k_2, \quad k_1, k_2 \in K, \quad g \in G \quad (\text{A.1})$$

which gives rise to a \mathbb{Z}^n -bigrading on H via Pontryagin duality, and a matrix of deformation like in (4.16):

$$\Theta = \begin{bmatrix} \theta & 0 \\ 0 & -\theta \end{bmatrix}, \quad (\text{A.2})$$

with θ a $n \times n$ antisymmetric matrix.

Of course, the pointwise multiplication between functions is indeed equivariant and thus respects the bigrading as in 4.12. Potential problems appear with the observation that after deformation, on the one hand the algebra structure of $\mathcal{O}(G)_\Theta \otimes \mathcal{O}(G)_\Theta$ is inherited from $\mathcal{O}(G \times G)_{\Theta \oplus \Theta}^{\alpha \otimes \alpha}$ whose underlying torus action is $\alpha \otimes \alpha$ of $T \times T$ on $G \times G$: in more detail, $\alpha \otimes \alpha : \mathcal{O}(G \times G) \rightarrow \mathcal{O}(G \times G)$ is given by

$$\alpha_{k_1, k_2} \otimes \alpha_{k'_1, k'_2}(f)(g_1, g_2) = f(k_1^{-1}g_1k_2, (k'_1)^{-1}g_2k'_2), \quad (\text{A.3})$$

with $f \in \mathcal{O}(G \times G)$ and $k_1, k_2, k'_1, k'_2 \in K$ and $g_1, g_2 \in G$. On the other hand, for this action the coproduct

$$\Delta : \mathcal{O}(G) \rightarrow \mathcal{O}(G \times G), \quad \Delta(f)(g_1, g_2) = f(g_1g_2),$$

where $f \in \mathcal{O}(G)$ and $g_1, g_2 \in G$, is not equivariant. Nevertheless, the image of Δ is contained in the subalgebra

$$\mathcal{D} = \{f \in \mathcal{O}(G \times G) : f(g_1k, g_2) = f(g_1, kg_2), \quad g_1, g_2 \in G, \quad k \in K\}, \quad (\text{A.4})$$

and there is a $T = K \times K$ action β on \mathcal{D} , given by:

$$\beta_{k_1, k_2}(f)(g_1, g_2) = f(k_1^{-1}g_1, g_2k_2), \quad (\text{A.5})$$

such that:

- i) the coproduct $\Delta : \mathcal{O}(G) \rightarrow \mathcal{D}$ is equivariant ,
- ii) the θ -deformation \mathcal{D}_Θ^β is a subalgebra of $\mathcal{O}(G \times G)_{\Theta \oplus \Theta}^{\alpha \otimes \alpha}$.

From the bigrading point of view, the subspace \mathcal{D} corresponds to

$$\bigoplus_{r, s, l \in \mathbb{Z}^n} H_{(r, s)} \otimes H_{(s, l)} \subset H \otimes H,$$

and Δ being equivariant is exactly the dual condition of (4.13).

The counit $\varepsilon : \mathcal{O}(G) \rightarrow \mathbb{C}$ is not equivalent in any way, but it factors through

$$\varepsilon : \mathcal{O}(G) \xrightarrow{\pi} \mathcal{O}(K) \xrightarrow{\varepsilon_{\mathcal{O}(K)}} \mathbb{C},$$

where the projection $\pi(f) = f|_K$ for $f \in \mathcal{O}(G)$ is the restriction map of functions on G onto the subgroup K , and π is equivariant when K is equipped with the action α . Hence, it is still an equivariant algebra homomorphism viewed as $\pi : \mathcal{O}(G)_\Theta \rightarrow \mathcal{O}(K)_\Theta$. Also, since K is abelian with the choice of Θ in (4.16), it is not difficult to see that such θ -deformation alters nothing: $\mathcal{O}(K)_\Theta = \mathcal{O}(K)$. All these properties are reflected in term of the bigrading in the condition (4.14). For the antipode and the $*$ -operator

$$S(f)(g) = \overline{f(g^{-1})}, \quad f^*(g) = \overline{f(g)}, \quad \forall f \in \mathcal{O}(G)$$

it is a routine verification to check that for the bigradings, (4.15) is indeed satisfied.

To sum up, let K be a torus subgroup of G of rank n and G is a subgroup of \tilde{G} . Denote by $H = \mathcal{O}(G)$ and $\tilde{H} = \mathcal{O}(\tilde{G})$ the associated Hopf algebras of representative functions. They can be deformed to two new Hopf algebras H_Θ and \tilde{H}_Θ , thanks to Proposition 4.7

If one forgets the coalgebra structure and view \tilde{H} as a H -comodule algebra, one has to repeat some of the arguments of earlier to check that the coaction

$$\delta^{\tilde{H}} : \tilde{H} \rightarrow \tilde{H} \otimes H, \quad \delta^{\tilde{H}}(f)(\tilde{g}, g) = f(\tilde{g}, g), \quad \tilde{g} \in \tilde{G}, g \in G$$

indeed satisfies all the requirements of Proposition [4.8](#). As a result, one obtains a quantum homogeneous space given by $(H_{\Theta}, \tilde{H}_{\Theta})$, with coinvariant subalgebra $B_{\Theta} = \tilde{H}_{\theta}^{\text{co}H_{\Theta}}$ which plays the role of coordinate functions on the base. Thus, in this way, one θ -deforms the Hopf algebra structures (on H and \tilde{H}) and the H -comodule algebra structure of \tilde{H} in such a way that compatibilities among the three survive.

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