A note on small data soliton selection for nonlinear Schrödinger equations with potential

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Abstract

In this note, we give an alternative proof of the theorem on soliton selection for small energy solutions of nonlinear Schrödinger equations (NLS) studied in [3, 4]. As in [4] we use the notion of Refined Profile but unlike in [4] we do not modify the modulation coordinates and we do not search for Darboux coordinates.

1 Introduction

In this note we give an alternative and simplified proof of the selection of small energy standing waves for the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\partial_t u = Hu + g(|u|^2)u, \quad (t,x) \in \mathbb{R}^{1+3}, \tag{1.1}$$

where $H := -\Delta + V$ is a Schrödinger operator with $V \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ (Schwartz function) and $g \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies $g(0) = 0$ and the growth condition:

$$\forall n \in \mathbb{N} \cup \{0\},\ \exists C_n > 0,\ |g^{(n)}(s)| \le C_n \langle s \rangle^{2-n} \ \text{ where } \ \langle s \rangle := (1+|s|^2)^{1/2}. \tag{1.2}$$

We consider the Cauchy problem of NLS (1.1) with the initial condition $u(0) = u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$. It is well known that the NLS (1.1) is locally well-posed (LWP) in $H^1 := H^1(\mathbb{R}^3, \mathbb{C})$, see e.g. [2, 7]. It is easy also to conclude, by mass and energy conservation, that for small initial data $u_0 \in H^1$ the corresponding solution is globally defined.

The aim of this paper is to revisit the study of asymptotic behavior of small (in H^1) solutions when the Schrödinger operator H has several simple eigenvalues. In such situation, it has been proved that the solutions decouple into a soliton and a dispersive wave [11, 13, 3]. More recently, in [4], we have introduced the notion of Refined Profile, which simplifies significantly the proof of the result in [3]. In this note we exploit the notion of Refined Profile of [4], but we give an alternative proof of the result in [4] which does not exploit directly the hamiltonian structure of the NLS. In this sense, in this paper we are closer in spirit to Soffer and Weinstein [11] and Tsai and Yau [13], but our proof is at the same time simpler and with stronger results.

To state our main result precisely, we introduce some notation and several assumptions. The following two assumptions for the Schrödinger operator H hold for generic V.

Assumption 1.1. 0 is neither an eigenvalue nor a resonance of H.

Assumption 1.2. There exists $N \ge 2$ s.t.

$$\sigma_d(H) = \{\omega_j \mid j = 1, \cdots, N\}, \text{ with } \omega_1 < \cdots < \omega_N < 0,$$

where $\sigma_d(H)$ is the set of discrete spectrum of H. Moreover, we assume all ω_j are simple and

$$\forall \mathbf{m} \in \mathbb{Z}^N \setminus \{0\},\ \mathbf{m} \cdot \boldsymbol{\omega} \neq 0, \tag{1.3}$$

where $\boldsymbol{\omega} := (\omega_1, \cdots, \omega_N)$. We set ϕ_j to be the eigenfunction of H associated to the eigenvalue ω_j satisfying $\|\phi_j\|_{L^2} = 1$. We also set $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_N)$.

Remark 1.3. The cases $N = 0, 1$ are easier and are not treated it in this paper. Unfortunately, Assumption (1.2) excludes radial potentials $V(r)$, for $r = |x|$, where in general we should expect eigenvalues with multiplicity higher than one.

As it is well known, the ϕ_j's are smooth and decays exponentially. For $s \geq 0, \gamma \geq 0$, we set

$$H^s_\gamma := \{u \in H^s \mid \|u\|_{H^s_\gamma} := \|\cosh(\gamma x) u\|_{H^s} < \infty\}.$$

The following is well known.

Proposition 1.4. *There exists $\gamma_0 > 0$ s.t. for all $1 \leq j \leq N$, we have $\phi_j \in \cap_{s \geq 0} H^s_{\gamma_0}$.*

Using $\gamma_0 > 0$, we set

$$\Sigma^s := H^s_{\gamma_0} \text{ if } s \geq 0,\ \Sigma^s := (H^{-s}_{\gamma_0})^* \text{ if } s < 0,\ \Sigma^{0-} := (\Sigma^0)^* \text{ and } \Sigma^\infty := \cap_{s \geq 0} \Sigma^s.$$

We will not consider any topology in Σ^∞ and we will only consider it as a set.

In order to introduce the notion of refined profile, we need the following combinatorial set up, exactly that of [4].

We start with the following standard basis of $\mathbb{R}^N$, which we view as "non–resonant" indices,

$$\mathbf{NR}_0 := \{\mathbf{e}_j \mid j = 1, \cdots, N\},\ \mathbf{e}_j := (\delta_{1j}, \cdots, \delta_{Nj}) \in \mathbb{Z}^N,\ \delta_{ij} \text{ the Kronecker delta.} \tag{1.4}$$

More generally, the sets of resonant and non–resonant indices $\mathbf{R}$, $\mathbf{NR}$, are

$$\mathbf{R} := \{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1,\ \boldsymbol{\omega} \cdot \mathbf{m} > 0\}, \quad \mathbf{NR} := \{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1,\ \boldsymbol{\omega} \cdot \mathbf{m} < 0\}, \tag{1.5}$$

where $\sum \mathbf{m} := \sum_{j=1}^N m_j$ for $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$.

From Assumption 1.2 it is clear that $\{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1\} = \mathbf{R} \cup \mathbf{NR}$ and $\mathbf{NR}_0 \subset \mathbf{NR}$. For $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, we define

$$|\mathbf{m}| := (|m_1|, \cdots, |m_N|) \in \mathbb{Z}^N,\ \|\mathbf{m}\| := \sum |\mathbf{m}| = \sum_{j=1}^N |m_j|, \tag{1.6}$$

and introduce partial orders $\preceq$ and $\prec$ by

$$\mathbf{m} \preceq \mathbf{n} \ \Leftrightarrow_{\text{def}} \forall j \in \{1, \cdots, N\},\ m_j \leq n_j, \quad \text{and} \quad \mathbf{m} \prec \mathbf{n} \ \Leftrightarrow_{\text{def}} \mathbf{m} \preceq \mathbf{n} \ \text{and } \mathbf{m} \neq \mathbf{n}, \tag{1.7}$$

where $\mathbf{n} = (n_1, \cdots, n_N)$. We define the minimal resonant indices by

$$\mathbf{R}_{\min} := \{\mathbf{m} \in \mathbf{R} \mid \ \nexists \mathbf{n} \in \mathbf{R} \text{ s.t. } |\mathbf{n}| \prec |\mathbf{m}|\}. \tag{1.8}$$

We also consider $\mathbf{NR}_1$, formed by the nonresonant indices not larger than resonant indices:

$$\mathbf{NR}_1 := \{\mathbf{m} \in \mathbf{NR} \mid \forall \mathbf{n} \in \mathbf{R}_{\min},\ |\mathbf{n}| \not\prec |\mathbf{m}|\}. \tag{1.9}$$

Both $\mathbf{R}_{\min}$ and $\mathbf{NR}_1$ are finite sets, see [4] for the elementary proof.

We now introduce the functions $\{G_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{R}_{\min}} \subset \Sigma^\infty$ which are crucial in our analysis. For $\mathbf{m} \in \mathbf{NR}_1$, we inductively define $\widetilde{\phi}_{\mathbf{m}}(0)$ and $g_{\mathbf{m}}(0)$ by

$$\widetilde{\phi}_{\mathbf{e}_j}(0) := \phi_j,\ g_{\mathbf{e}_j}(0) = 0,\ j = 1, \cdots, N, \tag{1.10}$$

and, for $\mathbf{m} \in \mathbf{NR}_1 \setminus \mathbf{NR}_0$, by

$$\widetilde{\phi}_{\mathbf{m}}(0) := -(H - \mathbf{m}\cdot\boldsymbol{\omega})^{-1} g_{\mathbf{m}}(0), \tag{1.11}$$

$$g_{\mathbf{m}}(0) := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(\mathbf{m}_1,\cdots,\mathbf{m}_{2m+1})\in A(m,\mathbf{m})} \widetilde{\phi}_{\mathbf{m}_1}(0)\cdots\widetilde{\phi}_{\mathbf{m}_{2m+1}}(0), \tag{1.12}$$

where

$$A(m,\mathbf{m}) := \left\{ \{\mathbf{m}_j\}_{j=1}^{2m+1} \in (\mathbf{NR}_1)^{2m+1} \mid \sum_{j=0}^{m} \mathbf{m}_{2j+1} - \sum_{j=1}^{m} \mathbf{m}_{2j} = \mathbf{m},\ \sum_{j=0}^{2m+1} |\mathbf{m}_j| = |\mathbf{m}| \right\} \tag{1.13}$$

Remark 1.5. For each $m \geq 1$ and $\mathbf{m} \in \mathbf{NR}_1$, $A(m,\mathbf{m})$ is a finite set. Furthermore, for sufficiently large m, we have $A(m,\mathbf{m}) = \emptyset$. Thus, even though we are expressing $g_{\mathbf{m}}(0)$ in (1.12) by a series, the sum is finite.

For $\mathbf{m} \in \mathbf{R}_{\min}$, we define $G_{\mathbf{m}}$ by

$$G_{\mathbf{m}} := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(\mathbf{m}_1,\cdots,\mathbf{m}_{2m+1})\in A(m,\mathbf{m})} \widetilde{\phi}_{\mathbf{m}_1}(0)\cdots\widetilde{\phi}_{\mathbf{m}_{2m+1}}(0). \tag{1.14}$$

Remark 1.6. $g_{\mathbf{m}}(0)$ and $G_{\mathbf{m}}$ are defined similarly. We are using a different notation to emphasize that $g_{\mathbf{m}}(0)$ has $\mathbf{m} \in \mathbf{NR}_1$, while $G_{\mathbf{m}}$ has $\mathbf{m} \in \mathbf{R}_{\min}$.

The following is the nonlinear Fermi Golden Rule (FGR) assumption essential in our analysis.

Assumption 1.7. For all $\mathbf{m} \in \mathbf{R}_{\min}$, we assume

$$\int_{|k|^2=\mathbf{m}\cdot\boldsymbol{\omega}} |\widehat{G}_{\mathbf{m}}(k)|^2\, dS \neq 0, \tag{1.15}$$

where $\widehat{G}_{\mathbf{m}}$ is the distorted Fourier transform associated to H.

Remark 1.8. In the case $N = 2$ and $\omega_1 + 2(\omega_2 - \omega_1) > 0$, we have $G_{\mathbf{m}} = g'(0)\phi_1\phi_2^2$, which corresponds to the condition in Tsai and Yau [14], based on the explicit formulas in Buslaev and Perelman [1] and Soffer and Weinstein [10]. These works are related to Sigal [9]. More general situations are considered in [3], where however the $G_{\mathbf{m}}$ are obtained after a certain number of coordinate changes, so that the relation of the $G_{\mathbf{m}}$ and the ϕ_j's is not discussed in [3] and is not easy to track.

In [4] it is proved that for a generic nonlinear function g the condition (1.15) is a consequence of the following simpler one, which is similar to (11.6) in Sigal [9],

$$\int_{|k|^2=\mathbf{m}\cdot\boldsymbol{\omega}} |\widehat{\phi^{\mathbf{m}}}(k)|^2\, dS \neq 0 \text{ for all } \mathbf{m} \in \mathbf{R}_{\min}\ , \tag{1.16}$$

using again the distorted Fourier transform and where $\phi^{\mathbf{m}} := \prod_{j=1,\ldots,N} \phi_j^{m_j}$. Specifically, in [4] the following is proved.

Proposition 1.9. *Let* $L = \sup\{\frac{\|\mathbf{m}\|-1}{2} : \mathbf{m} \in \mathbf{R}_{\min}\}$ *and suppose that the operator* H *satisfies condition* (1.16)*. Then there exists an open dense subset* Ω *of* $\mathbb{R}^L$ *s.t. if* $(g'(0), ..., g^{(L)}(0)) \in \Omega$ *such that Assumption 1.7 is true for* (1.1).

□

For $\mathbf{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$, $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, we define

$$\mathbf{z}^{\mathbf{m}} := z_1^{(m_1)} \cdots z_N^{(m_N)} \in \mathbb{C}, \text{ where } z^{(m)} := \begin{cases} z^m & m \geq 0 \\ \bar{z}^{-m} & m < 0, \end{cases} \quad \text{and} \tag{1.17}$$

$$|\mathbf{z}|^k := (|z_1|^k, \cdots, |z_N|^k) \in \mathbb{R}^N, \ \|\mathbf{z}\| := \sum |\mathbf{z}| = \sum_{j=1}^N |z_j| \in \mathbb{R}. \tag{1.18}$$

We will use the following notation for a ball in a Banach space B:

$$\mathcal{B}_B(u, r) := \{v \in B \mid \|v - u\|_B < r\}. \tag{1.19}$$

The Refined Profile is of the form $\phi(\mathbf{z}) = \mathbf{z}\cdot\boldsymbol{\phi} + o(\|\mathbf{z}\|)$ and is defined by the following proposition, proved in [4].

Proposition 1.10 (Refined Profile)**.** *For any* $s \geq 0$*, there exist* $\delta_s > 0$ *and* $C_s > 0$ *s.t.* δ_s *is nonincreasing w.r.t.* $s \geq 0$ *and there exist*

$$\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1} \in C^\infty(\mathcal{B}_{\mathbb{R}^N}(0, \delta_s^2), (\Sigma^s)^{\sharp\mathbf{NR}_1}), \ \boldsymbol{\varpi}(\cdot) \in C^\infty(\mathcal{B}_{\mathbb{R}^N}(0, \delta_s^2), \mathbb{R}^N)$$
$$\textit{and } \mathcal{R} \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0, \delta_s), \Sigma^s),$$

s.t. $\boldsymbol{\varpi}(0, \cdots, 0) = \boldsymbol{\omega}$, $\psi_{\mathbf{m}}(0) = 0$ *for all* $\mathbf{m} \in \mathbf{NR}_1$ *and*

$$\|\mathcal{R}(\mathbf{z})\|_{\Sigma^s} \leq C_s \|\mathbf{z}\|^2 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|, \tag{1.20}$$

where $B_X(a, r) := \{u \in X \mid \|u - a\|_X < r\}$*, and if we set*

$$\phi(\mathbf{z}) := \mathbf{z}\cdot\boldsymbol{\phi} + \sum_{\mathbf{m}\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}} \psi_{\mathbf{m}}(|\mathbf{z}|^2) \ \textit{and} \ z_j(t) = e^{-\mathrm{i}\varpi_j(|\mathbf{z}|^2)t} z_j, \tag{1.21}$$

then, setting $\mathbf{z}(t) = (z_1(t), \cdots, z_n(t))$*, the function* $u(t) := \phi(\mathbf{z}(t))$ *satisfies*

$$\mathrm{i}\partial_t u - Hu - g(|u|^2)u = -\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} - \mathcal{R}(\mathbf{z}), \tag{1.22}$$

where $\{G_{\mathbf{m}}\}_{\mathbf{R}_{\min}} \subset (\Sigma^\infty)^{\sharp\mathbf{R}_{\min}}$ *is given in* (1.14)*. Finally, writing* $\psi_{\mathbf{m}} = \psi_{\mathbf{m}}^{(s)}$*,* $\boldsymbol{\varpi} = \boldsymbol{\varpi}^{(s)}$ *and* $\mathcal{R} = \mathcal{R}^{(s)}$*, for* $s_1 < s_2$ *we have* $\psi_{\mathbf{m}}^{(s_1)}(|\cdot|^2) = \psi_{\mathbf{m}}^{(s_2)}(|\cdot|^2)$*,* $\boldsymbol{\varpi}^{(s_1)}(|\cdot|^2) = \boldsymbol{\varpi}^{(s_2)}(|\cdot|^2)$ *and* $\mathcal{R}^{(s_1)} = \mathcal{R}^{(s_2)}$ *in* $\mathcal{B}_{\mathbb{R}^N}(0, \delta_{s_2})$.

□

Remark 1.11. Notice that solitons, or standing waves, are exact solutions to the NLS generated from the Refined Profile setting

$$\phi_j(z_j) := \phi(z_j \mathbf{e}_j) \text{ for } z_j \in \mathcal{B}_{\mathbb{C}}(0, \delta_s). \tag{1.23}$$

So the Refined Profile fails to be an exact solution precisely when there are at least two nonzero coordinates in $\mathbf{z}$, which, under our hypotheses, make the defect on the right hand side of (1.22) nonzero. Notice in particular that (1.20) states that the error term $\mathcal{R}(\mathbf{z})$ is not just small, but that it has a specific combinatorial structure. A monomial of the form $z_j|z_j|^{2N}$ cannot be a term in $R(\mathbf{z})$, since it does not have the required combinatorial structure. These $z_j|z_j|^{2N}$ terms are in the left hand side of (1.22) and cancel out because the Refined Profile encodes the standing waves, as

$$\phi_j(z_j) = \phi(z_j\mathbf{e}_j) = \left[\mathbf{z}\cdot\boldsymbol{\phi} + \mathbf{z}^{\mathbf{e}_j}\psi_{\mathbf{e}_j}(|\mathbf{z}|^2)\right]\big|_{\mathbf{z}=z_j\mathbf{e}_j}.$$

We give now several formulae related to the refined profile. Let X be a Banach space and $F \in C^1(\mathcal{B}_{\mathbb{C}^N}(0,\delta), X)$ for some $\delta > 0$. For $\mathbf{z} \in \mathcal{B}_{\mathbb{C}^N}(0,\delta)$ and $\mathbf{w} \in \mathbb{C}^N$, we set

$$D_{\mathbf{z}}F(\mathbf{z})\mathbf{w} := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(\mathbf{z}+\epsilon\mathbf{w}).$$

For $\mathbf{z}(t)$ given by the 2nd equation of (1.21), that is $z_j(t) = e^{-\mathrm{i}\varpi_j(|\mathbf{z}|^2)t}z_j$, we have

$$\mathrm{i}\partial_t\mathbf{z} = \boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \text{ where } \boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z} := (\varpi_1(|\mathbf{z}|^2)z_1, \cdots, \varpi_N(|\mathbf{z}|^2)z_N).$$

Thus, $\mathrm{i}\partial_t\phi(\mathbf{z}(t)) = \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z}(t))(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}(t)|^2)\mathbf{z}(t))$ and we have the following formula, identically satisfied by $\phi(\mathbf{z})$,

$$\mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}) = H\phi(\mathbf{z}) + g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}) - \sum_{\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} - \mathcal{R}(\mathbf{z}). \tag{1.24}$$

Furthermore, differentiating (1.24) w.r.t. $\mathbf{z}$ in any given direction $\widetilde{\mathbf{z}} \in \mathbb{C}^N$, we obtain

$$\begin{aligned} H[\mathbf{z}]D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}} =& \mathrm{i}D^2_{\mathbf{z}}\phi(\mathbf{z})(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}) + \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})\left(D_{\mathbf{z}}(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z})\widetilde{\mathbf{z}}\right) \\ &+ \sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})\widetilde{\mathbf{z}}G_{\mathbf{m}} + D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}, \end{aligned} \tag{1.25}$$

where the operator $H[\mathbf{z}]$ is defined by

$$H[\mathbf{z}]f := Hf + g(|\phi(\mathbf{z})|^2)f + 2g'(|\phi(\mathbf{z})|^2)\mathrm{Re}\left(\overline{\phi(\mathbf{z})}\, f\right)\phi(\mathbf{z}) \tag{1.26}$$

and is selfadjoint for the inner product $\langle u, v\rangle = \mathrm{Re}\int_{\mathbb{R}^3} u\bar{v}dx$.

As mentioned above, the refined profile $\phi(\mathbf{z})$ contains as a special case the small standing waves bifurcating from the eigenvalues, when they are simple.

Corollary 1.12. *Let $s > 0$ and $j \in \{1, \cdots, N\}$. Then, $\phi(z(t)\mathbf{e}_j)$ solves* (1.1) *if $z \in \mathcal{B}_{\mathbb{C}}(0, \delta_s)$ and $z(t) = e^{-\mathrm{i}\varpi_j(|z\mathbf{e}_j|^2)t}z$.*

Proof. Since $(z\mathbf{e}_j)^{\mathbf{m}} = 0$ for $\mathbf{m} \in \mathbf{R}_{\min}$, we see that from (1.20) and (1.22) the remainder terms $\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}(t)^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}(t)))$ are 0 in (1.22). Therefore, we have the conclusion. □

Remark 1.13. If the eigenvalues of H are not simple the above does not hold anymore in general. See Gustafson-Phan [6].

The main result, which we have first proved in [3], is the following.

Theorem 1.14. *Under the Assumptions 1.1, 1.2 and 1.7, there exist $\delta_0 > 0$ and $C > 0$ s.t. for all $u_0 \in H^1$ with $\epsilon_0 := \|u_0\|_{H^1} < \delta_0$, there exists $j \in \{1, \cdots, N\}$, $z \in C^1(\mathbb{R}, \mathbb{C})$, $\eta_+ \in H^1$ and $\rho_+ \geq 0$ s.t.*

$$\lim_{t\to\infty} \|u(t) - \phi_j(z(t)) - e^{\mathrm{i}t\Delta}\eta_+\|_{H^1} = 0, \tag{1.27}$$

with $C^{-1}\epsilon_0^2 \leq \rho_+^2 + \|\eta_+\|_{H^1}^2 \leq C\epsilon_0^2$ and

$$\lim_{t\to+\infty} |z(t)| = \rho_+ \ . \tag{1.28}$$

When written in the Modulation parameters, the NLS appears like a complicated system where some discrete modes are coupled to radiation. The discrete modes tend to produce complicated patterns, similar to the ones of a linear system with eigenvalues. However, asymptotically in time the nonlinear interaction is responsible of spilling of energy into radiation which disperses at space infinity and to the selection of a unique nonlinear standing wave. Theorem 1.14 is the same of the main theorem in [4] and is very similar to the main theorem in [3]. The proofs here and in [4] are much simpler than in [3] or in earlier papers containing early partial results, like [11, 13]. In [3], in order to detect the nonlinear redistribution of the energy, it was necessary to make full use of the hamiltonian structure of our NLS, by first introducing Darboux coordinates and by then considering a normal forms argument. The discovery of the notion of Refined Profile made in [8] and its further development in [4] allows to forgo the normal forms argument because an almost optimal system of coordinates is provided automatically by the Refined Profile. In [4] we introduced Darboux coordinates in a way much simpler than in [3]. Undoubtedly, Darboux coordinates are quite natural for a Hamiltonian system and in [4] they contribute to simplify the system. In the present note however, we provide a different proof which, except for the information that mass and energy are constant, thus guaranteeing the global existence of our small H^1 solutions, does not make explicit use of the hamiltonian structure of the equations.

2 The proof

We start from constructing the modulation coordinate. First, we have the following.

Lemma 2.1. *There exist $\delta > 0$ and $\mathbf{z} \in C^\infty(\mathcal{B}_{\Sigma^{-1}}(0, \delta), \mathbb{C}^N)$ s.t.*

$$\forall \widetilde{\mathbf{z}} \in \mathbb{C}^N, \ \langle \mathrm{i}\,(u - \phi(\mathbf{z}(u)))\,, D_{\mathbf{z}}\phi(\mathbf{z}(u))\widetilde{\mathbf{z}}\rangle = 0.$$

Proof. Standard. □

We set

$$\eta(u) := u - \phi(\mathbf{z}(u)). \tag{2.1}$$

In the following we write $\eta = \eta(u)$ and $\mathbf{z} = \mathbf{z}(u)$. Substituting $u = \phi(\mathbf{z}) + \eta$ to (1.1) and using (1.24), we have

$$\mathrm{i}\partial_t\eta + \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) = H[\mathbf{z}]\eta + \sum_{\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}) + F(\mathbf{z}, \eta), \tag{2.2}$$

where

$$F(\mathbf{z}, \eta) = g(|\phi(\mathbf{z}) + \eta|^2)(\phi(\mathbf{z}) + \eta) - g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}) - g(|\phi(\mathbf{z})|^2)\eta - 2g'(|\phi(\mathbf{z})|^2)\mathrm{Re}\left(\overline{\phi(\mathbf{z})}\eta\right)\phi(\mathbf{z}).$$

Given an interval $I \subseteq \mathbb{R}$ we set

$$\mathrm{Stz}^j(I) := L_t^\infty H^j(I) \cap L_t^2 W^{j,6}(I), \quad \mathrm{Stz}^{*j}(I) := L_t^1 H^j(I) + L_t^2 W^{j,6/5}(I), \ j = 0, 1, \tag{2.3}$$

where $H^0 = L^2$ and $W^{0,p} = L^p$ and use Yajima's [15] Strichartz inequalities, for $t_0 \in \overline{I}$,

$$\|e^{-\mathrm{i}tH} P_c v\|_{\mathrm{Stz}^j(\mathbb{R})} \lesssim \|v\|_{H^j}, \ \|\int_{t_0}^t e^{-\mathrm{i}(t-s)H} P_c f(s)\, ds\|_{\mathrm{Stz}^j(I)} \lesssim \|f\|_{\mathrm{Stz}^{*j}(I)}, \ j = 0, 1. \tag{2.4}$$

Under the assumptions of Theorem 1.14 we have $\|u\|_{L^\infty H^1(\mathbb{R})} \lesssim \epsilon_0$ from energy and mass conservation. Since $\|u\|_{H^1} \sim \|\mathbf{z}\| + \|\eta\|_{H^1}$, we conclude

$$\|\mathbf{z}\|_{L_t^\infty(\mathbb{R})} + \|\eta\|_{L_t^\infty H^1(\mathbb{R})} \lesssim \epsilon_0.$$

Theorem 2.2 (Main Estimates)**.** *There exist $\delta_0 > 0$ and $C_0 > 0$ s.t. if $\epsilon_0 = \|u_0\|_{H^1} < \delta_0$, we have*

$$\|\eta\|_{\mathrm{Stz}^1(I)} + \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L_t^2(I)} + \|\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\|_{L_t^2(I)} \le C\epsilon_0, \tag{2.5}$$

for $I = [0, \infty)$ and $C = C_0$.

Notice that (2.5), the equation (2.2) satisfied by η, estimate (1.20) for $\mathcal{R}(\mathbf{z})$ and Lemma 2.4 below for $F(\mathbf{z}, \eta)$, allow to prove in a standard and elementary fashion that $\eta(t)$ scatters as $t \to +\infty$, i.e. there exists $\eta_+ \in H^1$ such that $\|\eta(t) - e^{\mathrm{i}t\Delta}\eta_+\|_{H^1} \xrightarrow{t \to +\infty} 0$. From (2.5) we have $\|\eta_+\|_{H^1} \le C\epsilon_0$. Using mass conservation we have

$$\begin{aligned} \|\phi(\mathbf{z}(t))\|_{L^2}^2 &= \|u_0\|_{L^2}^2 - 2\left\langle \phi(\mathbf{z}(t)), e^{\mathrm{i}t\Delta}\eta_+ \right\rangle - 2\left\langle \phi(\mathbf{z}(t)), \eta(t) - e^{\mathrm{i}t\Delta}\eta_+ \right\rangle - \|\eta(t)\|_{L^2}^2 \\ &\xrightarrow{t \to +\infty} \|u_0\|_{L^2}^2 - \|\eta_+\|_{L^2}^2. \end{aligned}$$

So, by $\|\phi(\mathbf{z}(t))\|_{L^2}^2 = \|\mathbf{z}(t)\|^2 + o(\|\mathbf{z}(t)\|^2)$, we get $\lim_{t \to +\infty} \|\mathbf{z}(t)\|^2 = \rho_+^2$ for some $0 \le \rho_+ \le 2C\epsilon_0$.

The fact that $\mathbf{z}^{\mathbf{m}} \in L^2(\mathbb{R}_+)$ and, as it is easy to see, $\partial_t(\mathbf{z}^{\mathbf{m}}) \in L^\infty(\mathbb{R}_+) \cap C^0([0, \infty))$, imply $\mathbf{z}^{\mathbf{m}} \xrightarrow{t \to +\infty} 0$ for any $\mathbf{m} \in \mathbf{R}_{\min}$. This implies $z_k \xrightarrow{t \to +\infty} 0$ for all k except at most for one, yielding the selection of one coordinate j in the statement of Theorem 1.14. The proof that Theorem 2.2 implies Theorem 1.14 is like in [3].

By completely routine arguments discussed in [3], (2.5) for $I = [0, \infty)$ is a consequence of the following Proposition.

Proposition 2.3. *There exists a constant $c_0 > 0$ s.t. for any $C_0 > c_0$ there is a value $\delta_0 = \delta_0(C_0)$ s.t. if (2.5) holds for $I = [0, T]$ for some $T > 0$, for $C = C_0$ and for $u_0 \in B_{H^1}(0, \delta_0)$, then in fact for $I = [0, T]$ the inequalities (2.5) holds for $C = C_0/2$.*

In the remainder of the paper we prove Proposition 2.3.

2.1 Estimate of the continuous variable η

In the following, we set $\epsilon_0 = \|u_0\|_{H^1}$. Further, when we use $\lesssim$, the implicit constant will not depend on C_0. We start from the estimate of the remainder term F.

Lemma 2.4. *Under the assumption of Proposition 2.3, we have*

$$\|F(\mathbf{z}, \eta)\|_{\mathrm{Stz}^{*1}(I)} \lesssim C_0\epsilon_0^3. \tag{2.6}$$

Proof. By (1.2), we have the pointwise bound

$$|F(\mathbf{z},\eta)|+|\nabla_x F(\mathbf{z},\eta)|\lesssim \left(1+|\eta|^2\right)\left(|\phi(\mathbf{z})|+|\nabla_x\phi(\mathbf{z})|+|\eta|\right)|\eta|\left(|\eta|+|\nabla_x\eta|\right). \tag{2.7}$$

Using this, we obtain the conclusion by Hölder and Sobolev estimates. □

We set

$$\mathcal{H}_c[\mathbf{z}]:=\{v\in L^2 \mid \forall \widetilde{\mathbf{z}}\in\mathbb{C}^N,\ \langle \mathrm{i}v, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle=0\}. \tag{2.8}$$

Notice that for $u\in H^1$, $\eta(u)\in\mathcal{H}_c[\mathbf{z}(u)]\cap H^1$. Following Gustafson, Nakanishi and Tsai [5], we can construct an inverse of P_c on $\mathcal{H}_c[\mathbf{z}]$.

Lemma 2.5. *There exists $\delta>0$ s.t. there exists $\{a_{jA}\}_{j=1,\cdots,N,A=\mathrm{R},\mathrm{I}}\in C^\infty(\mathcal{B}_{\mathbb{C}^N(0,\delta)},\Sigma^1)$ s.t.*

$$\|a_{jA}(\mathbf{z})\|_{\Sigma^1}\lesssim\|\mathbf{z}\|^2,\ j=1,\cdots,N,\ A=\mathrm{R},\mathrm{I} \tag{2.9}$$

and

$$R[\mathbf{z}]:=\mathrm{Id}-\sum_{j=1}^N\left(\langle\cdot,a_{j\mathrm{R}}(\mathbf{z})\rangle\,\phi_j+\langle\cdot,a_{j\mathrm{R}}(\mathbf{z})\rangle\,\mathrm{i}\phi_j\right), \tag{2.10}$$

satisfies $R[\mathbf{z}]P_c|_{\mathcal{H}_c[\mathbf{z}]}=\mathrm{Id}|_{\mathcal{H}_c[\mathbf{z}]},\ P_cR[\mathbf{z}]|_{P_cL^2}=\mathrm{Id}|_{P_cL^2}$.

Proof. A proof is in [3]. □

We set $\widetilde{\eta}=P_c\eta$. By Lemma 2.5, we have $\eta=R[z]\widetilde{\eta}$ and $\|\eta\|_{\mathrm{Stz}^1}\sim\|\widetilde{\eta}\|_{\mathrm{Stz}^1}$. Applying P_c to (2.2), we have

$$\begin{aligned}\mathrm{i}\partial_t\widetilde{\eta}=&H\widetilde{\eta}-\mathrm{i}P_cD_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}P_cG_{\mathbf{m}}\\&+P_c\mathcal{R}(\mathbf{z})+P_cF(\mathbf{z},\eta)+P_c\left(H[\mathbf{z}]-H\right)\eta.\end{aligned} \tag{2.11}$$

Lemma 2.6. *Under the assumption of Proposition 2.3, we have*

$$\|\eta\|_{\mathrm{Stz}^1(I)}\lesssim\epsilon_0+C(C_0)\epsilon_0^3+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}. \tag{2.12}$$

Proof. Obviously, from $\|\eta\|_{\mathrm{Stz}^1}\sim\|\widetilde{\eta}\|_{\mathrm{Stz}^1}$ it is enough to bound the latter. By Strichartz estimates (2.4) and Lemma 2.4 we easily obtain

$$\|\widetilde{\eta}\|_{\mathrm{Stz}^1(I)}\lesssim\epsilon_0+C(C_0)\epsilon_0^3+\|P_cD_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\|_{L^2(I)}+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}.$$

Using the fact that $\|P_cD_{\mathbf{z}}\phi(\mathbf{z})\|_{\Sigma^1}=O\left(\|\mathbf{z}\|^2\right)$, we obtain (2.12). □

We set $Z(\mathbf{z}):=-\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_cG_{\mathbf{m}}$ and $\xi:=\widetilde{\eta}+Z$, where $R_+(\lambda):=(H-\lambda-\mathrm{i}0)^{-1}$. Using the identity

$$\left(D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\right)(\mathrm{i}\boldsymbol{\omega}\mathbf{z})=\mathrm{i}\mathbf{m}\cdot\boldsymbol{\omega}\ \mathbf{z}^{\mathbf{m}} \tag{2.13}$$

with, in the left hand side, $\boldsymbol{\omega}\mathbf{z} := (\omega_1 z_1, \cdots, \omega_N z_N)$, we see that Z satisfies

$$-\mathrm{i}\partial_t Z(\mathbf{z}) + HZ(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} P_c G_{\mathbf{m}} + \mathcal{R}_Z(\mathbf{z}), \tag{2.14}$$

where

$$\mathcal{R}_Z(\mathbf{z}) = \mathrm{i}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\left[\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) + \left(\mathrm{i}\boldsymbol{\omega}\mathbf{z} - \mathrm{i}\boldsymbol{\varpi}(|z|^2)\mathbf{z}\right)\right] R_+(\mathbf{m}\cdot\boldsymbol{\omega}) P_c G_{\mathbf{m}}.$$

Substituting $\widetilde{\eta} = \xi - Z(\mathbf{z})$ into (2.11), we obtain

$$\mathrm{i}\partial_t \xi = H\xi - \mathrm{i}P_c D_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) + P_c\mathcal{R}(\mathbf{z}) + P_c F(\mathbf{z},\eta) + P_c\left(H[\mathbf{z}] - H\right)\eta + \mathcal{R}_Z(\mathbf{z}). \tag{2.15}$$

Lemma 2.7. *Under the assumption of Proposition 2.3, we have*

$$\|\xi\|_{L^2\Sigma^{0-}(I)} \lesssim \epsilon_0 + C_0\epsilon_0^3.$$

Proof. By $\|\cdot\|_{L^2\Sigma^{0-}} \lesssim \|\cdot\|_{\mathrm{Stz}^0}$ and Strichartz estimates (2.4), we have

$$\begin{aligned} \|\xi\|_{L^2\Sigma^{0-}} \leq \|\widetilde{\eta}(0)\|_{L^2} + \|e^{-\mathrm{i}tH} Z(\mathbf{z}(0))\|_{L^2\Sigma^{0-}} + \|\int_0^t e^{-\mathrm{i}(t-s)H}\mathcal{R}_Z(\mathbf{z}(u(s)))\,ds\|_{L^2\Sigma^{0-}} \\ + \|\mathrm{i}P_c D_{\mathbf{z}}\phi(\mathbf{z})\left(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) - \mathcal{R}(\mathbf{z}) - F(\mathbf{z},\eta) - \left(H[\mathbf{z}] - H\right)\eta\|_{\mathrm{Stz}^{*0}} \end{aligned} \tag{2.16}$$

where $\mathbf{z}(t) = \mathbf{z}(u(t))$. One can bound the contribution of the 2nd line of (2.16) by $\lesssim C(C_0)\epsilon_0^3$ using, as in Lemma 2.6, $\|P_c D_{\mathbf{z}}\phi(\mathbf{z})\|_{\Sigma^1} = O\left(\|\mathbf{z}\|^2\right)$ and

$$D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\mathrm{i}\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}|^2)\right)\mathbf{z} = \mathrm{i}\mathbf{m}\cdot\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}|^2)\right)\mathbf{z}^{\mathbf{m}} = O\left(\|\mathbf{z}\|^2\right)\mathbf{z}^{\mathbf{m}} \tag{2.17}$$

by (2.13) and $\boldsymbol{\varpi}(|\mathbf{z}|^2)|_{\mathbf{z}=0} = \boldsymbol{\omega}$. Similarly, the first term in the r.h.s. of (2.16) can be bounded by $\lesssim \epsilon_0$. For the 2nd and 3rd term in the r.h.s. of (2.16), we will now use the estimate

$$\|e^{-\mathrm{i}tH} R_+(\mathbf{m}\cdot\boldsymbol{\omega}) P_c f\|_{\Sigma^{0-}} \lesssim \langle t\rangle^{-3/2}\|f\|_{\Sigma^0}. \tag{2.18}$$

By (2.18), we have

$$\|e^{-\mathrm{i}tH} Z(\mathbf{z}(0))\|_{L^2\Sigma^{0-}(I)} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}(0)|\,\|\langle t\rangle^{-3/2}\|_{L^2}\|G_{\mathbf{m}}\|_{\Sigma^0} \lesssim \epsilon_0,$$

and

$$\begin{aligned} &\|\int_0^t e^{-\mathrm{i}(t-s)H}\mathcal{R}_Z(\mathbf{z}(u(s)))\,ds\|_{L^2\Sigma^{0-}(I)} \leq \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \Big\|\int_0^t \|e^{-\mathrm{i}(t-s)H} R_+(\mathbf{m}\cdot\boldsymbol{\omega}) P_c G_{\mathbf{m}}\|_{\Sigma^{0-}} \\ &\left(\left|D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)(s)\left(\partial_t \mathbf{z}(s) + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}(s)|^2)\mathbf{z}(s)\right)\right| + \left|D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)(s)\mathrm{i}\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}(s)|^2)\right)\mathbf{z}(s)\right|\right) ds\Big\|_{L^2(I)} \\ &\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\epsilon_0^2\int_0^t \left(|\partial_t \mathbf{z}(s) + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2(s))\mathbf{z}(s)| + |\mathbf{z}^{\mathbf{m}}(s)|\right)\langle t-s\rangle^{-3/2}\|_{L^2(I)} \lesssim C(C_0)\epsilon_0^3, \end{aligned}$$

where we have used (2.17) in the 2nd inequality and Young's convolution inequality in the 3rd inequality. Therefore, we have the conclusion. ∎

2.2 Estimate of discrete variables

We next estimate the quantities $\|\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\|_{L^2}$ and $\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2}$. To do so, we first compute the inner product $\langle (2.2), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle$ for any given $\widetilde{\mathbf{z}} \in \mathbb{C}^N$. First, notice that by $\eta \in \mathcal{H}_c[\mathbf{z}]$ we obtain the orthogonality relation

$$\langle \mathrm{i}\partial_t \eta, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle = -\left\langle \mathrm{i}\eta, D^2_{\mathbf{z}}\phi(\mathbf{z})(\partial_t \mathbf{z}, \widetilde{\mathbf{z}})\right\rangle.$$

Second, applying the inner product $\langle \eta, \cdot\rangle$ to equation (1.25), we have

$$\langle H[\mathbf{z}]\eta, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle = \left\langle \mathrm{i}\eta, D^2_{\mathbf{z}}\phi(\mathbf{z})(\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}})\right\rangle + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right) G_{\mathbf{m}}\rangle + \langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}\rangle,$$

where we exploited the selfadjointness of $H[\mathbf{z}]$ and the orthogonality in Lemma 2.1. Thus, applying $\langle \cdot, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle$ to equation (2.2) for η and using the last two equalities, we obtain

$$\begin{aligned}\left\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle &= \left\langle \mathrm{i}\eta, D^2_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}\right)\right\rangle + \langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}\rangle \\ &\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right) G_{\mathbf{m}}\rangle \\ &\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle + \langle \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle \\ &\quad + \langle F(\mathbf{z},\eta), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle.\end{aligned} \tag{2.19}$$

Using $\widetilde{\mathbf{z}} = \mathbf{e}_j, \mathrm{i}\mathbf{e}_j$ we have the following.

Lemma 2.8. *Under the assumption of Proposition 2.3, we have*

$$\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j = -\mathrm{i}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\langle G_{\mathbf{m}}, \phi_j\rangle + r_j(\mathbf{z},\eta), \tag{2.20}$$

where $r_j(\mathbf{z},\eta)$ satisfies

$$\|r_j(\mathbf{z},\eta)\|_{L^2(I)} \lesssim C(C_0)\epsilon_0^3.$$

In particular, we have

$$\|\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\|_{L^2(I)} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + C(C_0)\epsilon_0^3. \tag{2.21}$$

Proof. First since $D_{\mathbf{z}}\phi(0)\widetilde{\mathbf{z}} = \widetilde{\mathbf{z}}\cdot\boldsymbol{\phi}$, we have

$$\left\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle = \sum_{j=1}^N \mathrm{Re}(\mathrm{i}(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j)\overline{\widetilde{z}_j}) + r(\mathbf{z},\widetilde{\mathbf{z}}), \tag{2.22}$$

where

$$\begin{aligned} r(\mathbf{z},\widetilde{\mathbf{z}}) = &\left\langle \mathrm{i}\left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\right\rangle \\ &+ \left\langle \mathrm{i}D_{\mathbf{z}}\phi(0)(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), \left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)\widetilde{\mathbf{z}}\right\rangle.\end{aligned} \tag{2.23}$$

Since $\|D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\|_{L^2} \lesssim |\mathbf{z}|^2 \lesssim \epsilon_0^2$, by the assumptions of Proposition 2.3 we have

$$\|r(\mathbf{z},\widetilde{\mathbf{z}})\|_{L^2(I)} \lesssim C(C_0)\epsilon_0^3 \text{ for all } \widetilde{\mathbf{z}} = \mathbf{e}_1, \mathrm{i}\mathbf{e}_1, \cdots, \mathbf{e}_N, \mathrm{i}\mathbf{e}_N. \tag{2.24}$$

Setting

$$\widetilde{r}(\mathbf{z},\widetilde{\mathbf{z}},\eta) := \left\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})\left(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z},\widetilde{\mathbf{z}}\right)\right\rangle + \langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}\rangle + \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right)G_{\mathbf{m}}\rangle \quad (2.25)$$

$$+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \left(D_{\mathbf{z}}\phi(\mathbf{z})-D_{\mathbf{z}}\phi(0)\right)\widetilde{\mathbf{z}}\rangle + \langle \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle + \langle F(\mathbf{z},\eta), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle,$$

by the assumptions of Proposition 2.3 we have

$$\|\widetilde{r}(\mathbf{z},\widetilde{\mathbf{z}},\eta)\|_{L^2(I)} \lesssim C(C_0)\epsilon_0^3 \text{ for all } \widetilde{\mathbf{z}} = \mathbf{e}_1, \mathrm{i}\mathbf{e}_1, \cdots, \mathbf{e}_N, \mathrm{i}\mathbf{e}_N. \quad (2.26)$$

Therefore, since $D\phi(0)\mathrm{i}^k\mathbf{e}_j = \mathrm{i}^k\phi_j$ $(k=0,1)$, we have

$$-\mathrm{Im}\left(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j\right) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \phi_j\rangle - r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta),$$

$$\mathrm{Re}\left(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j\right) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \mathrm{i}\phi_j\rangle - r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta).$$

Since $G_{\mathbf{m}}$ (as can be seen from the proof in [4]) and ϕ_j are $\mathbb{R}$-valued, we have

$$\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j = -\mathrm{i}\sum_{\mathbf{m}}\langle G_{\mathbf{m}}, \phi_j\rangle\,\mathbf{z}^{\mathbf{m}} - r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \mathrm{i}r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta) - \mathrm{i}\widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta).$$

Therefore, from (2.24) and (2.26), we have the conclusion with $r_j(\mathbf{z},\eta) = -r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \mathrm{i}r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta) - \mathrm{i}\widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta)$. □

Having estimated η and $\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}$ in terms of $\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}$, we need to estimate the latter quantity. Here we use the Fermi Golden Rule.

Lemma 2.9. *Under the assumption of Proposition 2.3, we have*

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2} \lesssim \epsilon_0 + (C_0\epsilon_0)\epsilon_0. \quad (2.27)$$

Proof. We substitute $\widetilde{\mathbf{z}} = \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}$ in (2.19) and we make various simplifications. The first, by $\langle f, \mathrm{i}f\rangle = 0$ the right hand side of (2.19) can be rewritten as

$$\left\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right\rangle = \left\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right\rangle. \quad (2.28)$$

Next, we consider the 3rd line of (2.19), which we rewrite as

$$\left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right\rangle = \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle$$

$$-\left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\partial_t\mathbf{z}\right\rangle. \quad (2.29)$$

The term in the 1st line of the r.h.s. of (2.29) can be written as

$$\left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, D_{\mathbf{z}}\phi(0)\left(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle + R_1(\mathbf{z}), \quad (2.30)$$

where

$$R_1(\mathbf{z}) = \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) \right\rangle$$
$$+ \left\langle \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle,$$

satisfies

$$\int_0^T |R_1(\mathbf{z}(t))|\,dt \lesssim C_0^2\epsilon_0^4. \tag{2.31}$$

Using the stationary Refined Profile equation (1.24), the last line of (2.29) can be written as

$$-\left\langle H\phi(\mathbf{z}) + g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\partial_t\mathbf{z}\right\rangle + \left\langle D_{\mathbf{z}}\phi(\mathbf{z})(\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2))\mathbf{z}, \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})\partial_t\mathbf{z}\right\rangle. \tag{2.32}$$

Notice that the 2nd term of (2.32) coincides with the right hand side of (2.28), which lies in the left hand side of (2.19), so that the two cancel each other. On the other hand, we have

$$\left\langle H\phi(\mathbf{z}) + g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\partial_t\mathbf{z}\right\rangle = \frac{d}{dt}E(\phi(\mathbf{z})). \tag{2.33}$$

Therefore, from (2.19) with $\widetilde{\mathbf{z}} = \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}$, (2.28), (2.29), (2.30), (2.32) and (2.33), we have

$$\frac{d}{dt}E(\phi(\mathbf{z})) - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\left\langle \eta, \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\right\rangle = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, D_{\mathbf{z}}\phi(0)\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle + R_2(\mathbf{z},\eta), \tag{2.34}$$

where

$$R_2(\mathbf{z},\eta) = R_1(\mathbf{z}) + \left\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle + \left\langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right\rangle \tag{2.35}$$
$$+ \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left(\boldsymbol{\varpi}(|\mathbf{z}|^2) - \boldsymbol{\omega}\right)\left\langle \eta, \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\right\rangle + \left\langle F(\mathbf{z},\eta), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right\rangle,$$

satisfies

$$\int_0^T |R_2(\mathbf{z}(t),\eta(t))|\,dt \lesssim \left(C_0^2\epsilon_0^2 + C_0^5\epsilon_0^5\right)\epsilon_0^2. \tag{2.36}$$

By Lemma 2.8 and $D_{\mathbf{z}}\phi(0)\widetilde{\mathbf{z}} = \widetilde{\mathbf{z}}\cdot\boldsymbol{\phi}$, the 1st term of right hand side of (2.34) can be written as

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \boldsymbol{\phi}\cdot\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle = \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}}\sum_{j=1}^N \left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \phi_j\left(-\mathrm{i}\mathbf{z}^{\mathbf{n}}g_{\mathbf{n},j} + r_j(\mathbf{z},\eta)\right)\right\rangle$$
$$= \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^N \operatorname{Re}\left(\mathrm{i}\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right) g_{\mathbf{m},j}g_{\mathbf{n},j} + \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}}\sum_{j=1}^N \left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, r_j(\mathbf{z},\eta)\phi_j\right\rangle,$$

where we have set $g_{\mathbf{m},j} := \langle G_{\mathbf{m}}, \phi_j\rangle$ and used the fact that $\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, -\mathrm{i}\mathbf{z}^{\mathbf{m}}\phi_j\rangle = 0$ due to $G_{\mathbf{m}}$ and ϕ_j being $\mathbb{R}$ valued. Now, for $\mathbf{m}\neq\mathbf{n}$, we have

$$\partial_t(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}) = \mathrm{i}(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}} + \mathrm{i}(\mathbf{m}-\mathbf{n})\cdot\left(\boldsymbol{\varpi}(|\mathbf{z}|^2) - \boldsymbol{\omega}\right)\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}$$
$$+ D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}})(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z}))\overline{\mathbf{z}^{\mathbf{m}}} + \mathbf{z}^{\mathbf{n}}\overline{D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})((\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z})))}$$

Thus, since $(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}\neq 0$ from Assumption 1.2, we have

$$\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}=\frac{1}{\mathrm{i}((\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega})}\partial_t(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}})+r_{\mathbf{n},\mathbf{m}}(\mathbf{z}), \tag{2.37}$$

where

$$\begin{aligned} r_{\mathbf{n},\mathbf{m}}(\mathbf{z})=&-\frac{(\mathbf{m}-\mathbf{n})\cdot\left(\boldsymbol{\varpi}(|\mathbf{z}|^2)-\boldsymbol{\omega}\right)}{(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}}\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}\\ &+\frac{\mathrm{i}}{(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}}\left(D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}})(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z}))\overline{\mathbf{z}^{\mathbf{m}}}+\mathbf{z}^{\mathbf{n}}\overline{D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})((\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z})))}\right). \end{aligned}$$

Then, by the hypotheses of Proposition 2.3 we have

$$\int_0^T |r_{\mathbf{m},\mathbf{n}}(\mathbf{z})|\,dt\lesssim C_0^2\epsilon_0^4. \tag{2.38}$$

Thus, we have

$$\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^N \mathrm{Re}\left(\mathrm{i}\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right)g_{\mathbf{m},j}g_{\mathbf{n},j}=\partial_t A_1(\mathbf{z})+R_3(\mathbf{z}),$$

where

$$A_1(\mathbf{z})=\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^N\frac{1}{(\mathbf{n}-\mathbf{m})\cdot\boldsymbol{\omega}}\mathrm{Re}(\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}})g_{\mathbf{m},j}g_{\mathbf{n},j},\ \text{and}$$

$$R_3(\mathbf{z})=\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^N \mathrm{Re}\left(\mathrm{i}r_{\mathbf{n},\mathbf{m}}(\mathbf{z})\right)g_{\mathbf{m},j}g_{\mathbf{n},j}.$$

Thus,

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},\boldsymbol{\phi}\cdot\left(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\right\rangle=\partial_t A_1(\mathbf{z})+R_4(\mathbf{z},\eta),$$

where

$$R_4(\mathbf{z},\eta)=R_3(\mathbf{z})+\sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}}\sum_{j=1}^N\left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},r_j(\mathbf{z},\eta)\phi_j\right\rangle.$$

By (2.38) and Lemma 2.8, we have

$$\int_0^T |R_4(\mathbf{z}(t),\eta(t))|\,dt\lesssim C_0^2\epsilon_0^4.$$

Substituting $\eta=R[\mathbf{z}]\xi-(R[\mathbf{z}]-1)Z(\mathbf{z})-Z(\mathbf{z})$ into the 2nd term of the l.h.s. of (2.34), we have

$$\begin{aligned} \sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{m}\cdot\boldsymbol{\omega}\left\langle\eta,\mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\right\rangle=&-\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{m}\cdot\boldsymbol{\omega}|\mathbf{z}^{\mathbf{m}}|^2\left\langle R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_cG_{\mathbf{m}},\mathrm{i}G_{\mathbf{m}}\right\rangle \\ -\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\mathbf{m}\cdot\boldsymbol{\omega}\left\langle\mathbf{z}^{\mathbf{n}}R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_cG_{\mathbf{n}},\mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\right\rangle&+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{m}\cdot\boldsymbol{\omega}\left\langle R[\mathbf{z}]\xi-(R[\mathbf{z}]-1)Z(\mathbf{z}),\mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\right\rangle. \end{aligned} \tag{2.39}$$

By (2.37), the 2nd term of the r.h.s. of (2.39) can be written as

$$-\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \mathbf{m}\cdot\boldsymbol{\omega}\,\langle \mathbf{z}^{\mathbf{n}} R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_c G_{\mathbf{n}}, \mathrm{i}\mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}\rangle = \partial_t A_2(\mathbf{z}) + R_5(\mathbf{z}),$$

where

$$A_2(\mathbf{z}) = -\mathrm{Re}\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \frac{\mathbf{m}\cdot\boldsymbol{\omega}}{\mathrm{i}\,(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}} \mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}\left\langle R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_c G_{\mathbf{n}}, \mathrm{i}G_{\mathbf{m}}\right\rangle,$$

$$R_5(\mathbf{z}) = -\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \mathbf{m}\cdot\boldsymbol{\omega}\left\langle r_{\mathbf{n},\mathbf{m}}(\mathbf{z}) R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_c G_{\mathbf{n}}, \mathrm{i}G_{\mathbf{m}}\right\rangle,$$

with

$$\int_0^T |R_5(\mathbf{z}(t))|\,dt \lesssim C_0^2\epsilon_0^4.$$

The last term of r.h.s. of (2.39) can be written as

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\,\langle R[z]\xi, \mathrm{i}\mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}\rangle + R_6(\mathbf{z}),$$

with $R_6(\mathbf{z})$ satisfying

$$\int_0^T |R_6(\mathbf{z}(t))|\,dt \lesssim C_0^2\epsilon_0^4.$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}\left(E(\phi(\mathbf{z})) - A_1(\mathbf{z}) - A_2(\mathbf{z})\right) = &-\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}|\mathbf{z}^{\mathbf{m}}|^2\left\langle R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_c G_{\mathbf{m}}, \mathrm{i}P_c G_{\mathbf{m}}\right\rangle \\ &+\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\,\langle R[\mathbf{z}]\xi, \mathrm{i}\mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}\rangle + R_7(\mathbf{z},\eta)\end{aligned} \tag{2.40}$$

where $R_7(\mathbf{z},\eta) = R_2(\mathbf{z}) + R_4(\mathbf{z}) + R_5 + R_6$.

Now, by $R_+(\boldsymbol{\omega}\cdot\mathbf{m}) = \mathrm{P.V.}\frac{1}{H-\boldsymbol{\omega}\cdot\mathbf{m}} + \mathrm{i}\pi\delta(H-\boldsymbol{\omega}\cdot\mathbf{m})$ and formula (2.5) p. 156 [12] and Assumption 1.7, we have

$$\left\langle \mathrm{i}G_{\mathbf{m}}, (H-\boldsymbol{\omega}\cdot\mathbf{m}-\mathrm{i}0)^{-1}G_{\mathbf{m}}\right\rangle = \frac{1}{16\pi\sqrt{\boldsymbol{\omega}\cdot\mathbf{m}}}\int_{|k|^2=\boldsymbol{\omega}\cdot\mathbf{m}} |\widehat{G_{\mathbf{m}}}(k)|\,dS(k) \gtrsim 1,$$

with $\widehat{G_{\mathbf{m}}}(k)$ like in Assumption 1.7. Thus, we have

$$\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2 \lesssim \epsilon_0^2 + \delta^{-1}\|\xi\|_{L^2\Sigma^{0-}(I)}^2 + \delta\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2 + C_0^2\epsilon_0^4,$$

where we have used Schwartz inequality. Taking δ so that the $\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}^2 \lesssim \epsilon_0^2 + \delta^{-1}\|\xi\|_{L^2\Sigma^{0-}(I)}^2 + C_0^2\epsilon_0^4$ and using $\|\xi\|_{L^2\Sigma^{0-}(I)} \lesssim \epsilon_0$ by Lemma 2.7, we obtain (2.27).

□

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