Research Article

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A view on Liouville theorems in PDEs

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Abstract: Our review of Liouville theorems includes a special focus on nonlinear partial differential equations and inequalities.

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With sincere affection and warm regards, this paper is dedicated to Ermanno Lanconelli on the occasion of his 80th birthday, acknowledging and celebrating his profound contributions to Mathematical Analysis.

1 Augustin Cauchy and Joseph Liouville

In 1844, in a concise communication published in the C. R. Acad. Sci., Paris, Augustin Cauchy unveiled the original version of what we now recognize as the Liouville theorem concerning bounded analytic functions. Within this work, Cauchy proposed that any entire function of a single complex variable, bounded throughout its entire domain, it is necessarily constant. This publication marks the initial introduction of the theorem later attributed to Liouville (Jesper Lutzen, Joseph Liouville 1809–1882: Master of Pure and Applied Mathematics, Studies in the History of Mathematics 15 1990, Springer) as recorded in (C. R. Acad. Sci. Paris, 19 1377–1384 (1844)). It emphasizes the fundamental claim that any bounded entire function of a single complex variable is constant. The narrative surrounding Joseph Liouville's involvement in this theorem is both intriguing and complex, warranting a more indepth examination.

A few weeks before the Cauchy note appeared, Liouville announced to the academy his first results for doubly periodic functions, for which he is justly famous (C. R. Acad. Sci. Paris, 19 1262 (1844)). This announcement includes, without proof, a weak version of the Cauchy theorem, namely the statement that a doubly periodic holomorphic function must be constant. Cauchy was entirely aware of the relation of his result to that of Liouville, as he writes (C. R. Acad. Sci. Paris, 19 1379 (1844)), *If one considers separately the case of doubly periodic functions, one recovers the special theorem regarded with reason, by one of our honorable associates, as particularly applicable to the theory of elliptic functions.* Three years later, Liouville gave a series of informal lectures on his theory for F. Joachimsthal and C. W. Borchardt; these lectures, containing the previously cited weak version of the Cauchy result, but with no reference to Cauchy, were transcribed and edited by Borchardt and (much later) published in J. Reine Angew. Math., 88 277–310 (1880). This is the complete published record of the Liouville work, except for the first announcement and one later note (see below), but it is surprising that it does not contain Liouville own proof, but instead an alternate discussion due to Borchardt. In 1851, Cauchy again wrote explicitly that his work of 1844, "furnished the fundamental principle invoked by Joseph Liouville for doubly- periodic functions" and went on to restate his result of 1844 (see C. R. Acad. Sci. Paris, 32 452–454 (1851); Ouvres completes,tome XI, 373–376).

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At about the same time, Liouville gave a well-prepared course of lectures at College de France on doubly periodic functions, which contained a relatively simple proof of his doubly periodic theorem, but did not cite Cauchy's contribution again. Liouville was clearly much concerned with what he considered his priority to the doubly- periodic result, indeed in J. Math. Pures Appl., 20, 201–208 he republished his 1844 remarks together with a later comment of 1851 containing much the same material; indeed, he even went on to refer explicitly to his lectures at the College de France in the second semester of the year 1850–1851. This degree of concern almost certainly stems from the remarkable fact that near the end of his mathematical notebook for the year 1844 he had written the following "Remarque d'analyse":

Soit f(z) une fonction bien determinee de z. Si le module de f(z) ne depasse jamais M, on a f(z) = Constante. It is clear that he comprehended the function $f(z)$ that was given on the entire complex plane, which is clearly the general result! The proof sketch is limited to one line and is only tentative. From internal evidence, it seems highly likely that these words were written prior to the announcement of 1844. Liouville then devoted his effort to finding a proof of the doubly periodic result, and, upon finding a (difficult) demonstration, he reported this (but only this) result to the academy. He never afterwards referred to the Remarque. Liouville saw the utility and centrality of the doubly periodic theorem for elliptic function theory, but in his preoccupation with this he missed the elegance and beauty of the main result. Cauchy, like all subsequent writers, understood its importance immediately. Liouville was clearly saddened by the outcome and did not mention Cauchy's theorem. Despite the irony, Liouville's name is still associated with the theorem.

2 The classical Liouville theorem

Theorem 2.1. *Let $u \geq 0$ be a harmonic function on $\mathbf{R}^N$, i.e.*

$$\Delta u = 0 \quad \textit{in } \mathbf{R}^N.$$

Then

$$u \equiv \text{const.} \quad \textit{in } \mathbf{R}^N.$$

In 2006, a notably straightforward proof [6], possibly already familiar to many, was devised for this significant result. The proof relies on a lemma, presented herein, which appears to hold intrinsic interest on its own merits.

Lemma 2.2. *Let $u \in L^1_{\text{loc}}(\mathbf{R}^N)$, $u \geq 0$ a.e. on $\mathbf{R}^N$. Define for $x \in \mathbf{R}^N$,*

$$u_R(x) := \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y)\mathrm{d}y. \tag{2.1}$$

Then,

$$l(x) := \liminf_{R\to+\infty} u_R(x) = \ell \in [0, +\infty] \tag{2.2}$$

and

$$L(x) := \limsup_{R\to+\infty} u_R(x) = L \in [0, +\infty] \tag{2.3}$$

are independent of $x \in \mathbf{R}^N$.

Proof. Let $x, y \in \mathbf{R}^N$ be such that $0 < |x - y| = \delta$. Since $u \geq 0$ a.e. on $\mathbf{R}^N$, we have

$$u_R(x) \leq \left(\frac{R+\delta}{R}\right)^N u_{R+\delta}(y)$$

and then

$$l(x) \le l(y).$$

Changing the role of x and y, it follows that $l(y) \le l(x)$. Thus, $l(x) = l(y)$. Similarly $L(x) = L(y)$. □

Theorem 2.3. *Let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ be a nonnegative and superharmonic function on $\mathbf{R}^N$, i.e., $u \ge 0$ a.e. on $\mathbf{R}^N$ and for any $R > 0$,*

$$u(x) \ge u_R(x) \quad a.e.\ x \in \mathbf{R}^N. \tag{2.4}$$

Then,

$$\lim_{R\to+\infty} u_R(x) = \operatorname*{essinf}_{x\in\mathbf{R}^N} u(x). \tag{2.5}$$

If equality holds in (2.4)*, that is, u is harmonic, then u is constant a.e. in $\mathbf{R}^N$.*

Proof. We have

$$u(x) \ge u_R(x) \ge \operatorname*{essinf}_{x\in\mathbf{R}^N} u(x), \tag{2.6}$$

and from the above lemma we deduce

$$u(x) \ge L \ge l \ge \operatorname*{essinf}_{x\in\mathbf{R}^N} u(x). \tag{2.7}$$

This completes the proof. □

Theorem 2.4. *Let u be a harmonic function on $\mathbf{R}^N$, i.e.,*

$$\Delta u = 0 \quad in\ \mathbf{R}^N.$$

If for some $p \ge 1$ we have

$$\frac{1}{|B_R(0)|}\int_{B_R(0)} |\nabla u(y)|^p \mathrm{d}y \to 0$$

as $R \to \infty$, then

$$u \equiv \text{const. } in\ \mathbf{R}^N.$$

Proof. Since u is harmonic it follows that for every $i = 1 \dots n$, $\frac{\partial u}{\partial x_i}$ is harmonic too. Thus, applying the Hölder inequality to the identity

$$\frac{\partial u}{\partial x_i} = \frac{1}{|B_R(x)|}\int_{B_R(x)} \frac{\partial u}{\partial x_i}(y)\mathrm{d}y, \tag{2.8}$$

we obtain

$$|\nabla u(x)|^p \le \frac{1}{|B_R(x)|}\int_{B_R(x)} |\nabla u(y)|^p \mathrm{d}y \to 0,$$

and the claim follows from Lemma (2.2).

Note that by following the same idea we have that.

Theorem 2.5. *Let u be a harmonic function on $\mathbf{R}^N$, i.e.,*

$$\Delta u = 0 \quad in\ \mathbf{R}^N.$$

If for some $p \ge 1$ we have

$$\frac{1}{|B_R(0)|}\int_{B_R(0)} |u(y)|^p \mathrm{d}y \to 0,$$

as $R \to \infty$, *then*

$$u \equiv 0 \quad \textit{in } \mathbf{R}^N.$$

Once again this follows from

$$u(x) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y) \mathrm{d}y.$$

Indeed by Hölder's inequality, we obtain

$$|u(x)|^p \leq \frac{1}{|B_R(x)|} \int_{B_R(x)} |u(y)|^p \mathrm{d}y,$$

and the claim follows as above. □

Summarizing,

Theorem 2.6. *Let* $1 \leq p < \infty$. *If* u *is harmonic and*

$$u \in L^p(\mathbf{R}^N) \quad \textit{or} \quad |\nabla u| \in L^p(\mathbf{R}^N),$$

then we have, respectively,
(*i*) $u \equiv 0 \quad \textit{in } \mathbf{R}^N$,
(*ii*) $u \equiv \textit{const.} \quad \textit{in } \mathbf{R}^N$.

In the case $p = +\infty$, *we have, respectively,*
(*i*) $u \equiv \textit{const.} \quad \textit{in } \mathbf{R}^N$,
(*ii*) $u(x) = (a, x) + b$ *for some* $a \in \mathbf{R}^N$ *and* $b \in \mathbf{R}$.

3 A generalized form of the classical Liouville theorem

Theorem 3.1. *Let* u *be a harmonic function on* $\mathbf{R}^N$, *i.e.,*

$$\Delta u = 0 \quad \textit{in } \mathbf{R}^N.$$

If

$$\liminf_{|x| \to +\infty} \frac{u(x)}{|x|} \geq 0,$$

then

$$u \equiv \text{const.} \quad \textit{in } \mathbf{R}^N.$$

The proof can be easily obtained from the characterization of the harmonic functions, i.e.,

$$u(x) = \frac{1}{|B_R(x)|} \int_{B_R(x)} u(y) \mathrm{d}y. \tag{3.1}$$

Several generalizations of the above results (in a very general context) are contained in the book [5].

For further results in a more general context than sub-Riemannian, as well as for stationary and evolution problems, refer [24–26].

Definition 3.2. Let $\Omega \subset \mathbf{R}^N$ be a domain. A function $u \in C^2(\Omega)$ is called *superharmonic (subharmonic)* in Ω if,

$$\Delta u \leq (\geq) 0 \quad \text{in } \Omega. \tag{3.2}$$

Theorem 3.3. (Hadamard three circles/spheres theorem) *Let* $A = \{x \in \mathbf{R}^N : \rho < |x| < R\}$. *Let* $u : A \to \mathbf{R}$ *be subharmonic and let*

$$M : (\rho, R) \to \mathbf{R},$$
$$V : (\rho, R) \to \mathbf{R},$$

be defined by

$$M(r) = \max\{u(x) : |x| = R\}$$

and

$$V(r) = \log r \quad \text{if } N = 2,$$
$$V(r) = r^{2-N} \quad \text{if } N > 2.$$

Then, M is a convex function of V, i.e.,

$$M(r) \leq M(a)\frac{V(b) - V(r)}{V(b) - V(a)} + M(b)\frac{V(b) - V(r)}{V(b) - V(a)}$$

for $\rho < a < r < b < R$. *Moreover,*

$$M(r) = \alpha + \beta V(r) \iff u(x) = \alpha + \beta V(|x|)$$

for all $\alpha, \beta \in \mathbf{R}$ *and* $x \in A$.

By using an argument based on Hadamard's three circles theorem [20],[1] it is not difficult to see that.

Theorem 3.4. *Let u be a nonnegative superharmonic function in* $\mathbf{R}^2$. *Then,*

$$u = \text{const.} \quad \text{in } \mathbf{R}^2.$$

Note. The stated result does not hold when $N \geq 3$. To see this, it is enough to consider

$$u(x) \coloneqq (1 + |x|^2)^{\frac{2-N}{2}}.$$

An analysis of the proof of the three circles theorem gives:

Theorem 3.5. (Generalized form) *Let u be a superharmonic function in* $\mathbf{R}^2$. *If*

$$\liminf_{|x| \to +\infty} \frac{u(x)}{\log |x|} \geq 0,$$

then

$$u \equiv \text{const.} \quad \text{in } \mathbf{R}^2.$$

1 In the paper of Hadamard, there is no proof. For a detailed and interesting discussion, see Murray H. Protter, Hans F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, London 1967.

4 The ring condition: recent results for higher order operators

The following results have been obtained jointly with Caristi et al. [7]. See also [14] for earlier results on second-order degenerate elliptic operators in a more general context.

By (5), we know that if u is a superharmonic function which is bounded below, then

$$\lim_{R\to+\infty}\frac{1}{|B_R(x)|}\int_{B_R(x)}(u(y)-l)\mathrm{d}y=0, \tag{4.1}$$

where $l=\operatorname{essinf}_{x\in\mathbf{R}^N}u(x)$.

Of course, this implies that

$$\liminf_{R\to+\infty}\frac{1}{R^N}\int_{R\le|x-y|\le 2R}(u(y)-l)\mathrm{d}y=0. \tag{4.2}$$

This motivates the following,

Definition 4.1. Let $u\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$. We say that u *satisfies the ring condition* if there exists $l\in\mathbf{R}$ such that

$$\liminf_{R\to+\infty}\frac{1}{R^N}\int_{R\le|x-y|\le 2R}|u(y)-l|\mathrm{d}y=0 \tag{4.3}$$

holds for every $x\in\mathbf{R}^N$.

Definition 4.2. Let $m\ge 1$ be an integer and $N>2m$. Let μ be a positive Radon measure on $\mathbf{R}^N$. We say that $u\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ is a *distributional solution* of

$$(-\Delta)^m u=\mu \quad \text{on } \mathbf{R}^N, \tag{4.4}$$

if for any function $\varphi\in C_0^\infty(\mathbf{R}^N)$, we have

$$\int_{\mathbf{R}^N}u(y)(-\Delta)^m\varphi(y)\mathrm{d}y=\int_{\mathbf{R}^N}\varphi(y)\mathrm{d}\mu(y). \tag{4.5}$$

The following property plays a crucial role when studying several questions related to polyharmonic problems.

Definition 4.3. Let $m\ge 1$ be an integer and $N>2m$. A function $u\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ is called *weakly polysuperharmonic* or *polysuperharmonic in the distributional sense*, if for any $i=0,\dots,m$ and for every nonnegative $\varphi\in C_0^\infty(\mathbf{R}^N)$ we have

$$\int_{\mathbf{R}^N}u(-\Delta)^i\varphi\ge 0. \tag{4.6}$$

Remark 4.4. Note that in Definition 4.3, we do not assume that $\Delta^i u\in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ $(i=1,\dots m)$. Indeed, by Theorem 6.22 of [27], the distribution $(-\Delta)^i u$ is a positive Radon measure μ_i, that is,

$$\langle(-\Delta)^i u,\varphi\rangle=\int_{\mathbf{R}^N}u(x)(-\Delta)^i\varphi(x)\mathrm{d}x=\int_{\mathbf{R}^N}u(x)\mathrm{d}\mu_i(x)$$

for any test function $\varphi\in C_0^\infty(\mathbf{R}^N)$.

The following results generalize to polyharmonic inequalities the classical Riesz representation theorem for superharmonic functions.

Theorem 4.5. *Let $m \geq 1$ be an integer and $N > 2m$. Let μ be a positive Radon measure on $\mathbf{R}^N$ and $l \in \mathbf{R}$. The following statements are equivalent*:

(*a*) *u is a distributional solution of* (4.4) *and for a.e. $x \in \mathbf{R}^N$*,

$$\liminf_{R\to+\infty} \frac{1}{R^N} \int\limits_{R\leq|x-y|\leq 2R} |u(y) - l|\mathrm{d}y = 0. \tag{4.7}$$

(*b*) *u is a distributional solution of* (4.4), essinf $u = l$ *and u is weakly polysuperharmonic.*

(*c*) *$u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ and we have*

$$u(x) = l + c(2m)\int\limits_{\mathbf{R}^N} \frac{\mathrm{d}\mu(y)}{|x-y|^{N-2m}} \quad \text{a.e. } x \in \mathbf{R}^N, \tag{4.8}$$

where, for general, $\alpha > 0$ with $0 < \alpha < N$, $c(\alpha) := \Gamma(\frac{N-\alpha}{2})\Big(2^\alpha \pi^{N/2}\Gamma(\frac{\alpha}{2})\Big)^{-1}$.

Theorem 4.6. *Let $1 \leq p < +\infty$ and $N > 2m$. Let $u \in L^p(\mathbf{R}^N)$ be a distributional solution of the equation*

$$(-\Delta)^m u = 0 \quad \text{in } \mathbf{R}^N.$$

Then, $u \equiv 0$ a.e. on $\mathbf{R}^N$.

Corollary 4.7. *Let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ be a distributional solution of the inequality $(-\Delta)^m u \geq 0$ in $\mathbf{R}^N$ satisfying* (4.7). *Then,*

$$u(x) = l + c(2m)\int\limits_{\mathbf{R}^N} \frac{\mathrm{d}\mu(y)}{|x-y|^{N-2m}}, \quad \text{a.e. } x \in \mathbf{R}^N, \tag{4.9}$$

where μ is the unique positive Radon measure such that

$$\int\limits_{\mathbf{R}^N} \phi\mu(\mathrm{d}x) = T(\phi) = \langle(-\Delta)^m u, \phi\rangle = \int\limits_{\mathbf{R}^N} (-\Delta)^m\phi(x)u(x)\mathrm{d}x, \quad \phi \in C_0^\infty(\mathbf{R}^N).$$

Remark 4.8. The existence of the integral

$$\int\limits_{\mathbf{R}^N} \frac{\mathrm{d}\mu(y)}{|x-y|^{N-2m}}$$

for a.e. $x \in \mathbf{R}^N$, is a byproduct of Theorem 4.5.

Remark 4.9. Since $N > 2m$, the constant $c(2m)$ is positive. Hence, if $l \geq 0$, then by (4.9), it follows that $u(x) \geq 0$ a.e. on $\mathbf{R}^N$. In particular, if $l = 0$ a strong maximum principle holds. More precisely, from (4.9) it follows that either $u \equiv 0$ a.e. on $\mathbf{R}^N$ or $u > 0$ a.e. in $\mathbf{R}^N$.

We also note that the following Liouville theorem holds: if $(-\Delta)^m u = 0$ on $\mathbf{R}^N$ and (4.7) holds, then $u(x) = l$ a.e. in $\mathbf{R}^N$.

Theorem 4.10. *Let $m \geq 1$ be an integer and $N > 2m$. Let $p > 1$ and let $u \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ be a distributional solution of the inequality $(-\Delta)^m u \geq 0$ on $\mathbf{R}^N$ satisfying* (4.7) *with $l = 0$.*

Then,

(*i*) *If $u \in L^p(\mathbf{R}^N)$ with $(N-2m)p \leq N$, then $u \equiv 0$ a.e. in $\mathbf{R}^N$.*

(*ii*) *If $u \in L^p_w(\mathbf{R}^N)$ with $(N-2m)p < N$, then $u \equiv 0$ a.e. in $\mathbf{R}^N$.*[2]

2 We recall that the weak L^p space (also known as Marcinkiewicz's space) $L^p_w(\mathbf{R}^N)$ is defined as the space of all measurable functions f such that $\sup_{\alpha>0} |\{x : |f(x)| > \alpha\}|^{\frac{1}{p}} < \infty$.

Remark 4.11. Theorem 4.10 cannot be improved. Indeed, if (i) holds and $q > \frac{N}{N-2m}$, then the function

$$u(x) = \frac{1}{(1 + |x|^{2m})^{\frac{1}{q-1}}},$$

satisfies $(-\Delta)^m u \geq 0 \quad \text{on } \mathbf{R}^N$ and belongs to $L^p(\mathbf{R}^N)$ for any $p > \frac{N(q-1)}{2m}$.

Observe that $\frac{N(q-1)}{2m} > \frac{N}{N-2m}$ is equivalent to $q > \frac{N}{N-2m}$.

To see the sharpness of claim (ii), it is enough to consider the fundamental solution $u(x) = |x|^{2m-N}$. Clearly, we have $u \in L^p_w(\mathbf{R}^N)$ with $(N-2m)p = N$.

5 Nonlinear extensions

In recent years, there has been an increased interest in the question of whether a differential inequality of second (or higher) order in a domain admits only constant solutions. The main reason for this is the diverse implications of this problem in different fields, such as differential geometry, subelliptic theory, Riemannian geometry, and so on.

Certainly, the utilization of negative powers of solutions, combined with special test functions as multipliers, constitutes the foundation of Moser's iteration method. This method is particularly employed in establishing the Harnack inequality for nonlinear elliptic problems in divergence form. Moser's iteration plays a crucial role in obtaining important estimates and insights into the regularity of solutions in such contexts.

One important example is given by the minimal surface operator in nonparametric form and its related minimal surface equation,

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0 \quad x \in \mathbf{R}^N.$$

This is a classical problem as we shall see during the course of this review. We will bound ourself to simple cases showing the need to develop a general method to solve this general question when Harnack's type inequalities are not available.

6 The Bernstein theorem

"Bernstein's theorem is one of the most fascinating results in the theory of nonlinear elliptic differential equations"[3]

A celebrated result obtained by S. N. Bernstein in 1915 is the following:

Theorem 6.1. [1] *Let $u \in C^2(\mathbf{R}^2)$ be a solution of the minimal surface equation in $\mathbf{R}^2$, i.e.,*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0, \quad x \in \mathbf{R}^2.$$

Then, the graph of u is a plane.

3 Ulric Dierkes, Stefan Hildebrant, Albrecht Küster and Ortwin Wohlrab, Minimal Surfaces I, Grundlehren der Mathematischen Wissenschaften, Vol. 295, Springer Verlag (1992).

Theorem 6.2. [33] *Let $u \in C^2(\mathbf{R}^N)$ be a solution of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0,\quad x\in\mathbf{R}^N. \tag{B}$$

Suppose further that u has bounded gradient in $\mathbf{R}^N$. Then, u is an affine function.

Despite the fact that the mean curvature operator does not satisfy the Harnack property for the solutions of the minimal surface equations, if a solution has bounded gradient, then it can be show that an ad hoc argument based on the Harnack inequality can be applied.

We briefly recall that if L is a second-order uniformly elliptic second-order operator in divergence form and bounded coefficients and $u \in C^1(\Omega)$ is a positive solution of the equation

$$Lu = 0,\quad x\in\Omega\subset\mathbf{R}^N \tag{6.1}$$

and $B_{2R}(x)\subset\Omega$, then

$$\operatorname*{Sup}_{x\in B_R} v(x)\le c\operatorname*{Inf}_{x\in B_R} v(x),$$

where c is a universal constant depending only on N.

Let $v(x)=\frac{\partial u}{\partial x_i}(x)$ for $i=1\ldots N$. It is not difficult to check that that if u satisfies (B), then

$$(UE)\qquad \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_i}(x)\right)=0$$

and

$$a_{ij}=\frac{\delta_{ij}(1+|\nabla u|^2)-\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}}{(1+|\nabla u|^{3/2})}\in L^\infty(\mathbf{R}^N).$$

Since $|\nabla u|$ is bounded on $\mathbf{R}^N$, it follows that for every $\zeta\in\mathbf{R}^N$, we have

$$\alpha\,|\zeta|^2\le a_{ij}(x)\zeta_i\zeta j,$$

for some suitable constant $\alpha>0$. This means that (UE) is uniformly elliptic. Since v is bounded, the function

$$w \coloneqq v-\inf_{\mathbf{R}^N} v$$

is a non-negative solution of (UE). By Harnack's inequality, it follows that

$$\operatorname*{Sup}_{x\in B_R} w(x)\le c\operatorname*{Inf}_{x\in B_R} w(x),$$

where c is independent of R.

The claim of the theorem follows by taking the limit as $R\to\infty$ in the above inequality.

Theorem 6.3. [8] *Let $u\in C^\infty(\mathbf{R}^3)$ be a solution of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0.$$

Then, u is an affine function.

Theorem 6.4. [3] *Let $\Omega\subset\mathbf{R}^N$. Let $u\in C^2(\Omega)$ be a solution of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)=0,\quad x\in\Omega.$$

Let $x_0 \in \Omega$ *and* $R < d(x_0, \partial\Omega)$. *Suppose further that*

$$u(x) > 0 \quad in \ |x - x_0| < R.$$

Then

$$|Du(x_0)| \le c_1 \quad \exp\left(c_2 \frac{u(x_0)}{R}\right), \tag{E}$$

where c_1, c_2 *depend only on* N.

Corollary 6.5. (Liouville theorem) *Let* $u \in C^2(\mathbf{R}^N)$ *be a positive solution of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0, \quad x \in \mathbf{R}^N.$$

Then

$$u = const. \quad in \ \mathbf{R}^N.$$

This is indeed a consequence of the exponential estimate (E). This result was announced by Ennio De Giorgi [28].

Another important consequence of the exponential estimate of the gradient of the solution gives the extension of Bernstein's theorem to higher dimensions.

Theorem 6.6. [[2]] *Let* $u \in C^2(\mathbf{R}^N)$ *be a solution of*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = 0, \quad x \in \mathbf{R}^N.$$

Then, either $N \ge 8$ *or the graph of* u *in a hyperplane.*

In the above article, it is proved that if $N \ge 8$, then there exists an entire solution of the minimal surface equation whose graph is not a hyperplane.

7 Recent contributions: the nonlinear capacity method

The general idea of the method was indeed founded by Stanislav I. Pohozaev in 1997 (and developed jointly until 2013). The rough idea of this method is to associate with a pair (L, f), where L is a differential operator and f is a given function, a number (nonlinear capacity). If the capacity is finite, then the problem

$$Lu = f(u)$$

has no non-trivial solutions. The distinctive advantage of this method lies in its applicability to various types of equations and inequalities, including elliptic, parabolic, hyperbolic, nonlocal problems, and systems. What sets this method apart is its independence from comparison principles or any form of maximum principle to derive the results. Notably, within the specific class of problems under consideration, the results achieved through this method are typically sharp, providing precise and accurate information on the solutions. The root of this idea relies on a sophisticated use of test functions. An account of the results and different implications up to 2001 appears in the book:

Ref. [31], and for more recent results in, Ref. [18].

Let us consider a simple contribution in this direction.

Problem: Let u be a $C^1(\mathbf{R}^2)$ solution of the inequality

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \le 0, \quad x \in \mathbf{R}^2.$$

Suppose that u is bounded below. Is it true that

$$u \equiv \text{cost. on } \mathbf{R}^2?$$

In 1996, during a meeting in Perugia, I presented this question to Mario Miranda. Initially, his impression was pessimistic. Nonetheless, I was aware that the result held true under the additional assumption that the function u is radial. After multiple attempts to construct a nonradial counterexample and persistent efforts, in 2001, we successfully resolved the problem affirmatively. Importantly, the solution went beyond the initial scope, addressing a much more general form of the problem. See [32].

Definition 7.1. Let

$$A : \mathbf{R}^+ \to \mathbf{R}^+$$

be a continuous function. Suppose that there exists $C > 0$ satisfying:

$$0 < A(t) \le C \quad \text{for every } t \ge 0.$$

Then, the divergence operator acting (weakly) on C^1 functions:

$$\operatorname{div}(A(|\nabla u|)\nabla u)$$

is said to generate an operator of "mean curvature type."

Theorem 7.2. *Suppose that A generates an operator of mean curvature type. Let $u \in C^1(\mathbf{R}^2)$ be a weak solution of the problem,*

$$\operatorname{div}(A(|\nabla u|)\nabla u) \le 0 \quad x \in \mathbf{R}^2.$$

If u is bounded below, then $u \equiv cost.$ in $\mathbf{R}^2$.

We just mention another result that involves as special case the p-Laplacian inequality:

$$\Delta_p u \coloneqq \operatorname{div}(|\nabla u|^{p-2}\nabla u) \le 0, \quad x \in \mathbf{R}^N.$$

Definition 7.3. Let $\mathscr{A} : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}^N$ be a continuous function. We say that A generates an (**SpC**) operator if there exist $a, b > 0$ such that

$$(\mathscr{A}(x,t,\xi),\xi) \ge a\,|\xi|^m \ge b|\mathscr{A}(x,t,\xi)|^{m'},$$

for every $(x,t,\xi) \in \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^N$, where $p > 1$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

The divergence operator defined by

$$\operatorname{div}(\mathscr{A}\,(x,u,\nabla u)),$$

and acting (weakly) on C^1-functions is called the differential operator generated by $\mathscr{A}$.

Theorem 7.4. *Let $N \ge 1$ and suppose that A generates an* (**SpC**) *operator. Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of the problem,*

$$\operatorname{div}(A(x,u,\nabla u)) \le 0 \quad x \in \mathbf{R}^N.$$

If u is bounded below, and $p \ge N$, then $u \equiv$ cost. in $\mathbf{R}^N$.

Corollary 7.5. *Let $N \geq 1$. Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of the problem,*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) \leq 0, \quad x \in \mathbf{R}^N.$$

If u is bounded below, and $p \geq N$, then $u \equiv cost.$ in $\mathbf{R}^N$.

In the proof of the above results, *we do not use any argument related to the Harnack inequality*.

The question of classifying the solutions of the equation,

$$\operatorname{div}(A(|\nabla u|)\nabla u) = 0, \quad x \in \mathbf{R}^N$$

for general functions A that generate an operator of mean curvature type and for solutions that are *a priori*, bounded above or below, remains completely untouched in higher dimensions ($N > 2$). Very likely “Bernstein’s type” theorems hold for these equations with dimensional obstructions ($N = 7$, for the mean curvature operator), depending on the structure assumptions on the function A. Similar problem for operators of p-Laplacian type is also widely open. An example in this direction is given by the p-harmonic equation,

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad x \in \mathbf{R}^N.$$

An interesting question (*yet unknown the answer*) is the following: Consider the problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad x \in \mathbf{R}^N. \tag{7.1}$$

Suppose that $u \in C^1(\mathbf{R}^N)$ is a solution of (7.1) such that

$$\liminf_{|x|\to+\infty} \frac{u(x)}{|x|^{p-1}} \geq 0$$

holds. Is it true that $u \equiv$ const. in $\mathbf{R}^N$?

However, the following immediate consequence of the quasilinear version of Harnack’s inequality for the p-Laplacian operator (see Moser-Trudinger-Serrin, see for instance [36]) or [21] is known.

Theorem 7.6. (Classical Liouville theorem) *Let $N > p$. Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of the problem,*

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad x \in \mathbf{R}^N.$$

If u is bounded from below, then $u \equiv const.$ in $\mathbf{R}^N$.

Proof. Let $l = \inf_{x\in\mathbf{R}^N} u(x)$. Then, $v(x) := u(x) - l$ is nonnegative and satisfies

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0, \quad x \in \mathbf{R}^N.$$

From Harnack’s inequality, it follows that

$$\operatorname*{Sup}_{x\in B_R} v(x) \leq c \operatorname*{Inf}_{x\in B_R} v(x),$$

where c is a universal constant depending only on N and p. Clearly,

$$\operatorname*{Inf}_{x\in\mathbf{R}^N} v(x) = 0,$$

and the claim follows from (H) by passing to the limit as $R \to \infty$.

Another interesting contribution to this problem in “low-dimensions” has been obtained by James Serrin in 2011. See Ref. [37]. □

Suppose that the function A satisfies the following properties:

(i) $p > 1$,

(ii) $A \in C[0, \infty)$ with $A(0) > 0$,

(iii) $t^{p-1}A(t)$ is strictly increasing for $t > 0$.

Theorem 7.7. *Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of the problem,*

$$\operatorname{div}(A(|\nabla u|)|\nabla u|^{p-2}\nabla u) \le 0, \quad x \in \mathbf{R}^N.$$

If u is bounded below and $p \ge N$, then $u \equiv$ const. in $\mathbf{R}^N$.

When the function A is continuously differentiable, condition (iii) can be dropped. Indeed, the following result holds:

Theorem 7.8. *Suppose that the function A satisfies:*

$$A \in C^1[0,\infty) \quad \text{with} \quad A(t) > 0 \quad \text{for} \quad t \ge 0.$$

Let $u \in C^1(\mathbf{R}^N)$ be a weak solution of the problem,

$$\operatorname{div}(A(|\nabla u|)|\nabla u|^{p-2}\nabla u) \le 0, \quad x \in \mathbf{R}^N.$$

If u is bounded below and $p \ge N$, then $u \equiv$ const. in $\mathbf{R}^N$.

The clever idea used in the article is a sophisticated quasilinear variation of the three sphere Hadamard's theorem.[4]

8 Problems with a source: positivity results and related Liouville theorems

These are some samples of the results proved in [10]. Throughout this section, we will consider solutions of class $C^1(\mathbf{R}^N)$.

Theorem 8.1. *Let $p > 1$ and $N > 1$. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that*

$$f(t) > 0 \quad \text{if} \quad t < 0, \quad f \quad \text{is non increasing in }]-\infty, 0[\tag{8.1}$$

and

$$\int_{-\infty}^{-1}\left(\int_t^{-1} f(s)\,\mathrm{d}s\right)^{-\frac{1}{p}} \mathrm{d}t < +\infty. \tag{8.2}$$

If u is a solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) \ge f(u) \quad \text{in } \mathbf{R}^N, \tag{8.3}$$

then $u \ge 0$ on $\mathbf{R}^N$. Moreover, if $f(t) \ge 0$ for $t \ge 0$ then, either $u \equiv 0$ or $u > 0$ in $\mathbf{R}^N$.

Corollary 8.2. *Let $p > 1$. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $f(t) \ge C\,|t|^q$ for $t < 0$. Let u be a solution of*

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) \ge f(u) \quad \text{in } \mathbf{R}^N. \tag{8.4}$$

If $q > p - 1$, then $u \ge 0$ in $\mathbf{R}^N$. Moreover, if $f(t) \ge 0$ for $t \ge 0$, then, either $u \equiv 0$ or $u > 0$ in $\mathbf{R}^N$.

4 For the standard versions of the three sphere theorem, see Murray H. Protter, Hans F. Weinberger, Maximum Principles in Differential Equations, Prentice-Hall, London 1967.

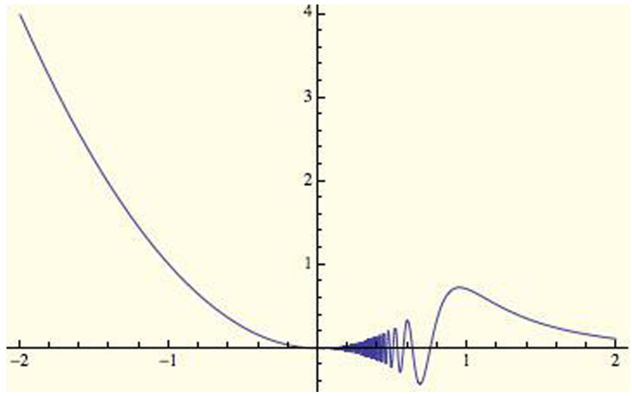

Figure 1: Piecewise $[x^2, x < 0, x\sin[1/x^4]\sin[x], x > 0], x, 2, 2]$.

In the case of the mean curvature operator, the above results can be improved. Indeed, the claim follows without the assumption (8.2) on f, see the Figure 1 below.

Theorem 8.3. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function satisfying* (8.1). *Let u be a solution of*

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) \geq f(u) \quad \text{in } \mathbf{R}^N. \tag{8.5}$$

Then, $u \geq 0$ in $\mathbf{R}^N$.

A first important consequence of the above results is the following *a priori* estimate.

Theorem 8.4. *Let $p > 1$ and $N > 1$. Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that there exists $\alpha, \beta \in \mathbf{R}$, $\alpha \leq \beta$ such that*

$$f_{]-\infty,\alpha[} > 0 \text{ and nonincreasing}, \tag{8.6}$$

$$f_{]\beta,+\infty[} < 0 \text{ and nonincreasing}, \tag{8.7}$$

and

$$\int_{-\infty}^{\alpha}\left(\int_t^{\alpha} f(s)\,\mathrm{d}s\right)^{-\frac{1}{p}} \mathrm{d}t < +\infty, \tag{8.8}$$

$$\int_{\beta}^{\infty}\left(\int_{\beta}^{t} -f(s)\,\mathrm{d}s\right)^{-\frac{1}{p}} \mathrm{d}t < +\infty. \tag{8.9}$$

If u is a solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{in } \mathbf{R}^N, \tag{8.10}$$

then u is bounded and $\alpha \leq u(x) \leq \beta$ for any $x \in \mathbf{R}^N$.

Again, for the mean curvature operator, we can require more general assumptions on f.

Theorem 8.5. *Let $f : \mathbf{R} \to \mathbf{R}$ be a continuous function such that*

$$\liminf_{t\to-\infty} f(t) > 0.$$

If u is a solution of (8.5), *then f has at least a zero, and set* $\alpha :=$ *the first zero of f* (*i.e.,* $\alpha := \min S$ *where* $S := f^{-1}(0)$) *we have* $u \geq \alpha$. *In particular if* $f > 0$, *then* (8.5) *has no solution.*

Moreover, if

$$\limsup_{t\to+\infty} f(t) < 0$$

and u solves

$$-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = f(u) \quad \text{on } \mathbf{R}^N, \tag{8.11}$$

then u is bounded and $\alpha \leq u(x) \leq \beta$ *for any* $x \in \mathbf{R}^N$, *where* $\beta :=$ *last zero of f* (*i.e.,* $\beta := \max S$).

A direct consequence of Theorems 8.4 and 8.3, 8.5 is the following Liouville theorem.

Corollary 8.6. *Let* $p > 1$ *and* $N > 1$. *Let* $f : \mathbf{R} \to \mathbf{R}$ *be a nonincreasing continuous function such that*

$$f(t) > 0 \quad \text{if } t < 0, \quad \text{and} \quad f(t) < 0 \quad \text{if } t > 0, \tag{8.12}$$

and

$$\int_{-\infty}^{-1}\left(\int_t^{-1} f(s)\,ds\right)^{-\frac{1}{p}} dt < +\infty, \tag{8.13}$$

$$\int_1^{\infty}\left(\int_1^t -f(s)\,ds\right)^{-\frac{1}{p}} dt < +\infty. \tag{8.14}$$

If u is a solution of

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{on } \mathbf{R}^N, \tag{8.15}$$

then $u \equiv 0$ *in* $\mathbf{R}^N$.

In particular, if $q > p - 1$ *and u is a solution of*

$$\Delta_p u = |u|^{q-1}u \quad \text{in } \mathbf{R}^N, \tag{8.16}$$

then $u \equiv 0$ *in* $\mathbf{R}^N$.

9 Coercive problems: weak solutions, possible sign-changing solutions

It is well known that when looking for Liouville theorems of *noncoercive* nonlinear equations or inequalities, the fact that the nonlinearity has definite sign is of fundamental importance. This is because, in general, examples of this type show that when the nonlinearity changes sign, the problem may possess infinitely many solutions with no *a priori* bound. A canonical example in this direction is the following:

$$-\Delta u = |u|^{q-1}u \quad \text{in } \mathbf{R}^N. \tag{9.1}$$

Indeed, it is well known that if $1 < q < \frac{N+2}{N-2}$, $N > 2$, (9.1) admits infinitely many radial solutions with increasing number of zeroes.

Conversely, when the problem is *coercive*, the situation may be completely different as the following striking result due to Brezis [4] shows.

Theorem [4] *Let $q > 1$. If $u \in L^q_{loc}(\mathbf{R}^N)$ is a distributional solution of*

$$\Delta u \geq |u|^{q-1}u \quad \text{in } \mathbf{R}^N, \tag{9.2}$$

then $u \leq 0$ a.e. on $\mathbf{R}^N$. In particular, if equality holds in (9.2), *then $u \equiv 0$ a.e. in $\mathbf{R}^N$.*

It is worth pointing out that, besides the quite general functional framework (distributional solutions), there are no assumptions on the behavior of the possible solutions of (9.2) at infinity.

Brezis' technique is based on a form of Kato's inequality [4,22] and on a construction of a suitable Loewner-Nirenberg barrier function [23,35].

Some generalizations of Brezis's result for quasilinear elliptic inequalities of second order have been obtained in [9,10,17] and more in a series of articles by Farina and Serrin [15,16].

One common aspect in these recent contributions is that from the technical point of view, none of them use a form of Kato's inequality. This is why they required strong condition on the regularity of the solutions.

Thus, one natural question is the extent to which Kato's inequality might be satisfied in the quasilinear case. A positive answer to this problem will allow us to develop a general strategy for proving positivity-type results, Liouville theorems, and uniqueness results for wide classes of quasilinear equations and inequalities. This will bring together some aspects of qualitatively different problems, namely, coercive and noncoercive quasilinear elliptic inequalities of second order.

In what follows, we shall assume that $\mathscr{A} : \mathbf{R}^N \times \mathbf{R} \times \mathbf{R}^l \to \mathbf{R}^l$ is a Caratheodory function, that is, for each $t \in \mathbb{R}$ and $\xi \in \mathbf{R}^l$ the function $\mathscr{A}(\cdot, t, \xi)$ is measurable; and for a.e. $x \in \mathbf{R}^N$, $\mathscr{A}(x, \cdot, \cdot)$ is continuous.

We consider operators L "generated" by $\mathscr{A}$, that is

$$L(u)(x) = \mathrm{div}_L(\mathscr{A}(x, u(x)), \nabla u(x)).$$

Our model cases are the p-Laplacian operator, the mean curvature operator and some related generalizations. See the **Examples** below.

Definition 9.1. Let $\mathscr{A} : \mathbf{R}^N \times \mathbb{R} \times \mathbf{R}^l \to \mathbf{R}^l$ be a Caratheodory function. The function $\mathscr{A}$ is called *weakly elliptic* if it generates a weakly elliptic operator L, i.e.,

$$\begin{gathered} \mathscr{A}(x, t, \xi)\cdot \xi \geq 0 \quad \text{for each } x \in \mathbf{R}^N,\ t \in \mathbf{R},\ \xi \in \mathbf{R}^l, \\ \mathscr{A}(x, 0, \xi) = 0 \quad \text{or} \quad \mathscr{A}(x, t, 0) = 0. \end{gathered} \tag{WE}$$

Let $p \geq 1$, the function $\mathscr{A}$ is called (**WpC**) (weakly-p-coercive), if $\mathscr{A}$ is (**WE**) and it generates a weakly-p-coercive operator L, i.e., if there exists a constant $k_2 > 0$ such that

$$(\mathscr{A}(x, t, \xi)\cdot \xi)^{p-1} \geq k_2 \, |\mathscr{A}(x, t, \xi)|^p \tag{(WpC)}$$

for each $x \in \mathbf{R}^N$, $t \in \mathbf{R}$, $\xi \in \mathbf{R}^l$.

Definition 9.2. Let $\Omega \subset \mathbf{R}^N$ be an open set and let $f : \Omega \times \mathbf{R} \times \mathbf{R}^l \to \mathbf{R}$ be a Caratheodory function. Let $p \geq 1$. We say that $u \in W^{1,p}_{loc}(\Omega)$ is a *weak solution* of

$$\mathrm{div}_L(\mathscr{A}(x, u, \nabla u)) \geq f(x, u, \nabla u) \quad \text{in } \Omega,$$

if $\mathscr{A}(\cdot, u, \nabla u) \in L^{p'}_{loc}(\Omega)$, $f(\cdot, u, \nabla u) \in L^1_{loc}(\Omega)$, and for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have

$$-\int_\Omega \mathscr{A}(x, u, \nabla u)\cdot\nabla_L \phi \geq \int_\Omega f(x, u, \nabla u)\phi.$$

Theorem 9.3. (Kato's inequality: The quasilinear case) *Let $\mathscr{A}$ be such that*

$$\mathscr{A}(x, t, \xi)\cdot \xi \geq 0 \quad \text{for any } x \in \Omega, \quad t \in \mathbf{R}, \quad \xi \in \mathbf{R}^l. \tag{9.3}$$

Let $f \in L^1_{loc}(\Omega)$ and let $u \in W^{1,p}_{loc}(\Omega)$ be a weak solution of

$$\mathrm{div}_L(\mathscr{A}(x, u, \nabla u)) \geq f \quad \text{in } \Omega. \tag{9.4}$$

Then

$$\operatorname{div}(\operatorname{sign} u\mathscr{A}(x,u,\nabla u)) \geq \operatorname{sign} u^+ f \quad on\ \Omega. \tag{9.5}$$

Moreover, if

$$\operatorname{div}(\mathscr{A}(x,u,\nabla u)) = f \quad in\ \Omega, \tag{9.6}$$

then

$$\operatorname{div}(\operatorname{sign} u \quad \mathscr{A}(x,u,\nabla u)) \geq \operatorname{sign} u\, f \quad in \quad \Omega. \tag{9.7}$$

In particular, if $\mathscr{A}$ *is (WE) and u is a weak solution of* (9.4), *then* u^+ *is a weak solution of*

$$\operatorname{div}_L(\mathscr{A}(x,u^+,\nabla u^+)) \geq \operatorname{sign} u^+ f \quad in\ \Omega. \tag{9.8}$$

If in addition $\mathscr{A}$ *is odd, i.e.,*

$$\mathscr{A}(x,-t,-\xi) = -\mathscr{A}(x,t,\xi), \tag{9.9}$$

and u is a solution of (9.6), *then* $|u|$ *satisfies*

$$\operatorname{div}(\mathscr{A}(x,|u|,\nabla|u|)) \geq \operatorname{sign} u\, f \quad in\ \Omega. \tag{9.10}$$

See Ref. [12].

Examples.

(1) Let $p > 1$. The p-Laplacian operator acting on suitable functions u by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

is an operator generated by $\mathscr{A}(x,t,\xi) \coloneqq |\xi|^{p-2}\xi$, which is **WpC** (indeed it is **S**pC).

(2) If $\mathscr{A}$ is of mean curvature type, that is, $\mathscr{A}$ can be written as $\mathscr{A}(x,t,\xi) \coloneqq A(|\xi|)\xi$ with $A:\mathbb{R}\to\mathbb{R}$ a positive bounded continuous function, then $\mathscr{A}$ is **(W2C)**.

(3) The mean curvature operator in nonparametric form

$$Tu \coloneqq \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)$$

is generated by $\mathscr{A}(x,t,\xi) \coloneqq \frac{\xi}{\sqrt{1+|\xi|^2}}$. In this case, $\mathscr{A}$ is **(WpC)** with $1 \leq p \leq 2$.

(4) Let $m > 1$. The operator

$$T_m u \coloneqq \operatorname{div}\left(\frac{|\nabla u|^{m-2}\nabla u}{\sqrt{1+|\nabla u|^m}}\right)$$

is **(WpC)** for $m \geq p \geq m/2$.

(5) Let T_M be the operator defined as

$$T_M u \coloneqq \operatorname{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right).$$

The operator T_M is the mean curvature operator; hence **(W2C)** in the Lorentz-Minkowski space

$$L^{N+1} \coloneqq \{(x,t) : x \in \mathbf{R}^N, t \in \mathbf{R}\}$$

endowed with the metric $-\mathrm{d}t^2 + \sum_{j=1}^N \mathrm{d}x_j^2$.

The simplest Liouville theorem in this framework is the following:

Theorem 9.4. *Let* $q > p-1 > 0$ *and let* $\mathscr{A}$ *be* **(WpC)**. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N) \cap L^q_{\mathrm{loc}}(\mathbf{R}^N)$ *be a weak solution of*

$$\operatorname{div}(\mathscr{A}(x,u,\nabla u)) = |u|^{q-1}u, \quad in\ \mathbf{R}^N. \tag{9.11}$$

Then, $u \equiv 0$ *a.e. on* $\mathbf{R}^N$.

By passing we mention a related application of Kato's inequality and the capacity method to the question of existence of comparison principles and uniquenees theorems see [13].

The main results (in their simplest form) are the following.

Theorem 9.5. *Let* $1 < p < 2$, $q \geq 1$, $h \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$, *then the problem*

$$-\Delta_p u + |u|^{q-1}u = h \quad \text{in } \mathbf{R}^N$$

has at most one distributional solution $u \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N) \cap L^q_{\mathrm{loc}}(\mathbf{R}^N)$. *Moreover,*

$$\inf_{\mathbf{R}^N} h \leq |u|^{q-1}u \leq \sup_{\mathbf{R}^N} h.$$

Theorem 9.6. *Let* $q \geq 1$, $h \in L^1_{\mathrm{loc}}(\mathbf{R}^N)$ *then the problem*

$$-\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) + |u|^{q-1}u = h \quad \text{in} \quad \mathbf{R}^N$$

has at most one distributional solution $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N) \cap L^q_{\mathrm{loc}}(\mathbf{R}^N)$. *Moreover,*

$$\inf_{\mathbf{R}^N} h \leq |u|^{q-1}u \leq \sup_{\mathbf{R}^N} h.$$

The above results are based on the following comparison principle.

Theorem 9.7.

(1) *Let* $1 < p < 2$ *and* $q \geq 1$. *Let* $u, v \in W^{1,p}_{\mathrm{loc}}(\mathbf{R}^N) \cap L^q_{\mathrm{loc}}(\mathbf{R}^N)$ *such that*

$$\Delta_p v - |v|^{q-1}v \geq \Delta_p u - |u|^{q-1}u \quad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{9.12}$$

Then, $v \leq u$ *a.e. in* $\mathbf{R}^N$.

(2) *Let* $q \geq 1$. *Let* $u, v \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^N) \cap L^q_{\mathrm{loc}}(\mathbf{R}^N)$ *such that*

$$\mathrm{div}\left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}}\right) - |v|^{q-1}v \geq \mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) - |u|^{q-1}u \quad \text{in } \mathcal{D}'(\mathbf{R}^N). \tag{9.13}$$

Then, $v \leq u$ *a.e. in* $\mathbf{R}^N$.

10 Noncoercive problems

For noncoercive problems, as recalled above the following problem possesses infinitely many solution if $1 < q < \frac{N+2}{N-2}$, $N > 2$.

$$-\Delta u = |u|^{q-1}u \quad \text{in } \mathbf{R}^N. \tag{10.1}$$

Clearly, when looking for Liouville's theorems, it is *natural* to restrict our analysis to positive solutions.

Theorem 10.1. [19] *If*

$$1 < q < \frac{N+2}{N-2},$$

then the problem,

$$-\Delta u = u^q \quad \textit{in } \mathbf{R}^N, \tag{10.2}$$

has no nontrivial positive solutions.

After the publication of the Gidas-Spruck result, the number of papers concerning Liouville's type results for semilinear and quasilinear second-order equations increased exponentially. The original proof of this fundamental result is very long and technically involved. The main idea is to use a kind of Pohozaev's identity for a suitable vector field and a clever combination of *a priori* estimates on the solutions and Harnack's inequality. The original idea to use a vector field comes from Morio Obata [34].

The analogue of Gidas-Spruck result for the p-Laplacian equation has been proved by James Serrin and Henghui Zou in 2001.

Theorem 10.2. [36] *Let*

$$p-1 < q < \frac{N(p-1)+p}{N-p},$$

then the problem

$$-\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^q \quad \textit{in} \quad \mathbf{R}^N, \tag{10.3}$$

has no nontrivial positive C^1- solutions.

The proof of this extraordinary result is based again on the construction of a vector field acting on the possible solutions, *a priori* estimates of the type proved in [29,30], and Harnack's inequality. The highly nontrivial construction of the vector field uses in a fundamental way the fact that the operator Δ_p is homogeneous.

Several results are know to hold in the semilinear and quasilinear context for nonlinearities which are not pure power functions. We end this brief discussion with a contribution in this direction [11].

Consider the following problem,

$$\begin{cases} -\operatorname{div}\mathscr{A}(x,u,\nabla u) \geq f(u), & x \in \mathbf{R}^N, \\ u \geq 0, & x \in \mathbf{R}^N. \end{cases} \tag{P}$$

Here, $\mathscr{A} : \mathbf{R}^N \times \mathbf{R}^+ \times \mathbf{R}^N \to \mathbf{R}^N$ is a Caratheodory function satisfying the following structure condition: there exist $a,\ b > 0$ and $p > 1$ such that for every $(x,u,w) \in \mathbf{R}^N \times \mathbf{R}^+ \times \mathbf{R}^N$ we have,

$$(\mathscr{A}(x,u,w), w) \geq b\ |w|^p \geq a\ |\mathscr{A}(x,u,w)|^{p'}. \tag{SpC}$$

Here, p' denotes the conjugate exponent of p.

It appears that one of the crucial assumptions concerning the function f for establishing *a priori* estimates of the solutions for inequality (P) is the following.

f is nonnegative and continuous and satisfies the following local condition at zero:

$$\text{there exists } q > 0 \text{ such that } \liminf_{t\to 0^+} \frac{f(t)}{t^q} = l, \quad \text{with } l \in (0,\infty]. \tag{f$_0$}$$

The main result on "*a priori* bounds" on the solutions of (**P**) is as follows.

Theorem 10.3. *Let $N > p$. Assume that $\mathscr{A}$ satisfies* (**SpC**). *Let $f : [0,+\infty[\to[0,+\infty[$ be a continuous function satisfying (f_0) with $q > p-1$. Let u be a weak solution of* (P) *such that* $\operatorname{ess\,inf}_{\mathbb{R}^N} u = 0$. *Then, there exists a constant $c > 0$ such that for R sufficiently large, the following estimates hold*:

(*i*) $\operatorname{ess\,inf}_{B_R} u \leq cR^{-\frac{p}{q-p+1}}$,

(*ii*) $\int_{B_R} f(u)\mathrm{d}x \leq cR^{N-p-\frac{N(p-1)}{q}}\left(\int_{B_R\setminus B_{R/2}} f(u)\mathrm{d}x\right)^{\frac{p-1}{q}}$, (*iii*) *if* $0 < s < \frac{\sigma}{\sigma+1}$ *where* $p - 1 < \sigma < \frac{N(p-1)}{N-p}$, *then we have*

$$\left(\frac{1}{|B_R|}\int_{B_R}(\mathscr{A}(x,u,\nabla u)\cdot\nabla u)^s\right)^{\frac{1}{sp}} \leq cR^{-\frac{q+1}{q-p+1}}. \tag{10.4}$$

As a very special case, we mention.

Theorem 10.4. (Liouville theorem) *Let* $N > p$. *Assume that* $\mathscr{A}$ *satisfies* (**SpC**). *Let* $f : [0,+\infty[\to[0,+\infty[$ *be a continuous function satisfying* (f_0) *with* $0 < q \leq \frac{N(p-1)}{N-p}$. *Let u be a weak solution of* (**P**). *Then, the following statements hold.*
(1) *If* $f(t) > 0$ *for* $t > 0$, *then* $f(0) = 0$ *and* $u \equiv 0$ *a.e. in* $\mathbf{R}^N$.
(2) *If* $\operatorname{ess\,inf}_{\mathbf{R}^N} u = 0$, *then* $f(0) = 0$ *and* $u \equiv 0$ *a.e. in* $\mathbf{R}^N$.

Remark 10.5. We emphasize that Theorem 10.4 is sharp even for equations. To see this, consider the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u), & x \in \mathbf{R}^N, \\ u \geq 0, & x \in \mathbf{R}^N. \end{cases} \tag{E}$$

If $N > p > 1$, $\lambda, c > 0$ and $q > \frac{N(p-1)}{(N-p)}$, then the function u defined by

$$u(x) \coloneqq c\ (\lambda + |x|^{\frac{p}{p-1}})^{-\frac{p-1}{q-p+1}}$$

is a positive solution of (E) with $f(u) \coloneqq u^q + \mu\ \ u^{\frac{pq-p+1}{p-1}}$, for some suitable $\mu > 0$.

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