On Dini derivatives of real functions

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Abstract. For a continuous function f, the set V_f made of those points where the lower left derivative is strictly less than the upper right derivative is totally disconnected. Besides continuity, alternative assumptions are proposed so to preserve this property. On the other hand, for any given totally disconnected closed set A, we construct a function f whose set V_f coincides with the entire domain, and f is continuous on A.

1. Introduction and main result

Dini derivatives take their names after Ulisse Dini, who introduced them in 1878, cf. [5]; let us recall their standard notation:

$$D_+f(x) = \liminf_{h\to 0^+} \frac{f(x+h)-f(x)}{h}, \qquad D^+f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h},$$

$$D_-f(x) = \liminf_{h\to 0^-} \frac{f(x+h)-f(x)}{h}, \qquad D^-f(x) = \limsup_{h\to 0^-} \frac{f(x+h)-f(x)}{h}.$$

Here, and in the rest of the paper, we assume that $f : I \to \mathbb{R}$ is defined on some open interval $I \subseteq \mathbb{R}$. A fundamental step in the study of Dini derivatives was achieved in the first quarter of the twentieth century by Denjoy [4] for continuous functions, Young [15] for measurable functions, and Saks [14] for arbitrary ones.

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The Denjoy–Young–Saks theorem states that at each point x, except for a set of measure zero, one of the following four alternatives holds:

(1) f has a finite derivative at x;

(2) $D_- f(x) = D^+ f(x) \in \mathbb{R}, \quad D^- f(x) = +\infty, \quad D_+ f(x) = -\infty;$

(3) $D^- f(x) = D_+ f(x) \in \mathbb{R}, \quad D^+ f(x) = +\infty, \quad D_- f(x) = -\infty;$

(4) $D^- f(x) = D^+ f(x) = +\infty, \quad D_- f(x) = D_+ f(x) = -\infty.$

Denjoy also explicitly constructed some continuous functions realizing each of the previous four conditions on a perfect set of positive Lebesgue measure. We refer to [2] for a more complete historical account, and to [10] for an extensive study on the possible pathological behaviours of continuous functions.

In this paper, for any function $f : I \to \mathbb{R}$, we are interested in studying the set

$$V_f := \{x \in I : D_- f(x) < D^+ f(x)\}.$$

It should be noticed that, in the above-mentioned example by Denjoy, the set V_f is totally disconnected, i.e., it does not contain any nontrivial interval. The main question is: *How large can this set be?*

We were mainly motivated in studying this problem when dealing with some ordinary differential equations [7]. One of the main tools in solving a given boundary value problem is provided by the lower and upper solutions method. See the book [3] for a comprehensive exposition of the theory for scalar second order equations. In particular, in [3, Definition I-2.1], the notion of lower solution involves explicitly the set V_f without properly analyzing its properties.

It is well known that there exist *non-continuous* functions $f : \mathbb{R} \to \mathbb{R}$ for which $V_f = \mathbb{R}$ (see, for instance [9], where the function $f : \mathbb{R} \to \mathbb{R}$ has a dense graph in $\mathbb{R}^2$). On the contrary, we will prove that there are no *continuous* functions with such a property. To be more precise, let us introduce the following class of functions.

Definition 1. We say that a function $f : I \to \mathbb{R}$ is *upper well behaved* if for every compact interval J contained in I, there is an $x_J \in J$ such that $f(x_J) = \max f(J)$.

Clearly, every continuous function (as well as upper semicontinuous) is upper well behaved. On the other hand, one can easily find examples of upper well behaved functions which are nowhere continuous (e.g., the well-known Dirichlet function).

Here is our first result.

Theorem 2. *If $f : I \to \mathbb{R}$ is upper well behaved, then the set V_f is totally disconnected.*

We will also show that the set V_f can be preassigned, at least in the class of totally disconnected *closed* sets; taking, e.g., $I = \mathbb{R}$, for any given totally disconnected closed set $\mathcal{V} \subseteq \mathbb{R}$, there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $V_f = \mathcal{V}$. This will be a consequence of Lemma 5 below.

Let us emphasize that, as proved in [16], there are functions f (e.g., the Weierstrass function) such that the set V_f is of second Baire category (cf. [13]) and has full measure on any interval $I =]a, b[$. See also [1], [6], [8] and [11] for more recent similar investigations on Takagi's function.

Let us now investigate the possibility for a function $f : \mathbb{R} \to \mathbb{R}$ to be such that $V_f = \mathbb{R}$ and, at the same time, to be continuous at some points of its domain. We will prove the following.

Theorem 3. *For any totally disconnected closed set $A \subseteq \mathbb{R}$, there exists a function $f : \mathbb{R} \to \mathbb{R}$, whose set of continuity points coincides with A, such that $V_f = \mathbb{R}$, and more precisely*

$$D_- f(x) = -\infty \quad \text{and} \quad D^+ f(x) = +\infty, \qquad \text{for every } x \in \mathbb{R}.$$

Recall that a Smith–Volterra–Cantor set is a totally disconnected closed set C, contained in $[0, 1]$, having any assigned Lebesgue measure $\mu(C) \in [0, 1[$. Iterating its construction on any interval $[n, n + 1]$, with $n \in \mathbb{Z}$, we could have a totally disconnected closed set A with "almost full" measure.

In the next section, we provide the proofs of Theorems 2 and 3. They are based on the knowledge that every monotone function is differentiable almost everywhere, and on some simple properties of continued fractions.

2. Proofs

We denote by μ be the Lebesgue measure on $\mathbb{R}$.

Proof of Theorem 2. By contradiction, let $[a, b] \subseteq V_f$, with $a < b$. Let $(x_n)_n$ be a sequence in $[a, b]$ such that $f(x_n) \to \inf f([a, b])$. Passing if necessary to a subsequence, we can assume that $x_n \to \check{x}$, for some $\check{x} \in [a, b]$. We have two cases.

Case 1. $\check{x} \in [a, b[$. We will prove that f is increasing in $]\check{x}, b]$, hence almost everywhere differentiable there, a contradiction.

By contradiction, let α, β in $]\check{x}, b]$ be such that $\alpha < \beta$ and $f(\alpha) > f(\beta)$. Being $\check{x} < \alpha$ and $f(\alpha) > \inf f([a,b])$, there exists n such that $x_n < \alpha$ and $f(x_n) < f(\alpha)$. Since f is upper well behaved, there is an $\hat{x} \in [x_n, \beta]$ such that $f(\hat{x}) = \max f([x_n, \beta])$. Being $f(\hat{x}) \geq f(\alpha) > \max\{f(x_n), f(\beta)\}$, it has to be $\hat{x} \in]x_n, \beta[$, whence $D_- f(\hat{x}) \geq 0 \geq D^+ f(\hat{x})$, a contradiction, since $\hat{x} \in V_f$.

Case 2. $\check{x} = b$. One proves in an analogous way that f is decreasing in $[a, b[$, hence almost everywhere differentiable there, a contradiction.

The proof is thus completed. □

Remark 4. If we define a function $f : I \to \mathbb{R}$ to be *lower well behaved* when $(-f)$ is upper well behaved, then it can be proved that the set

$$\Lambda_f := \{x \in I : D^- f(x) > D_+ f(x)\}$$

is totally disconnected.

Let us now go for the proof of Theorem 3. In the following, we allow an *interval* to be reduced to a single point. It will be useful to consider the function $F : \mathbb{R} \to [0,1]$ defined as

$$F(x) = \begin{cases} 2\sqrt{x(1-x)}, & \text{if } x \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

We first need to prove the following two lemmas.

Lemma 5. *Let A be a totally disconnected closed set. Then, there exists a nonnegative continuous function $\sigma_A : \mathbb{R} \to \mathbb{R}$ such that:*

- *σ_A is differentiable on $\mathbb{R} \setminus A$;*
- *for all $x \in A$, one has $D_- \sigma_A(x) = -\infty$ and $D^+ \sigma_A(x) = +\infty$;*
- *$\sigma_A(x) = 0$ if and only if $x \in A$.*

Proof. We first prove the result in the case when A is bounded. Without loss of generality, we can assume that $A \subseteq]0,1[$. Since A is closed, its complement in $]0,1[$ can be written as an at most countable union of pairwise disjoint open intervals $U_n =]a_n, b_n[$, with $n \geq 1$. We will treat in detail only the case when there are infinitely many of them (in the other case, A has only finitely many points, and the proof is much easier). We can then write

$$A =]0,1[\setminus \bigcup_{n \geq 1} U_n.$$

We define $R_1 = [0,1]$ and, for every $n \geq 2$,

$$R_n = [0,1] \setminus \bigcup_{j=1}^{n-1} U_j.$$

The following properties hold true:

- $U_n \subseteq R_n$, for every $n \geq 1$;
- $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_n \supseteq \cdots$;
- $\bigcap_{n\geq 1} R_n = A \cup \{0,1\}$.

Moreover, for $n \geq 2$, the set R_n is the union of n pairwise disjoint closed intervals

$$R_n = S_{n,1} \cup S_{n,2} \cup \cdots \cup S_{n,n}.$$

We set $S_{1,1} = R_1 = [0,1]$. For every $n \geq 1$, there exists an integer $H(n) \in \{1,\ldots,n\}$ such that $U_n \subseteq S_{n,H(n)}$. For simplicity, let us introduce the notation

$$\rho_n = \mu\big(S_{n,H(n)}\big).$$

Note that, since A is totally disconnected, we have

$$\lim_n \rho_n = 0. \tag{1}$$

We define the function $\tilde\sigma_A : \mathbb{R} \to \mathbb{R}$ as

$$\tilde\sigma_A(x) = \sum_{n=1}^{\infty} \sqrt{\rho_n}\, F\left(\frac{x-a_n}{b_n-a_n}\right).$$

Notice that, for each $x \in \mathbb{R}$, the above sum has at most one non-zero addend. It is clear that $\tilde\sigma_A(x) \geq 0$ for all $x \in \mathbb{R}$, and that

$$A = \{x \in]0,1[\,:\, \tilde\sigma_A(x) = 0\}.$$

If $x \in]0,1[\setminus A$, then $x \in U_n$ for some n, hence $\tilde\sigma_A$ is differentiable there. However, $\tilde\sigma_A(x) = 0$ for every $x \in \mathbb{R} \setminus]0,1[$. We thus need to modify $\tilde\sigma_A$ outside some interval $[\delta, 1-\delta]$, with $\delta \in]0,1[$, containing A in its interior. It is indeed possible to find a function $\sigma_A : \mathbb{R} \to \mathbb{R}$, which coincides with $\tilde\sigma_A$ on $[\delta, 1-\delta]$, and is continuously differentiable on $]-\infty, \delta] \cup [1-\delta, +\infty[$, being strictly positive there, and

$$\sigma_A(x) = 1, \qquad \text{for every } x \in]-\infty, 0] \cup [1, +\infty[.$$

This function $\sigma_A : \mathbb{R} \to \mathbb{R}$ is differentiable on $\mathbb{R} \setminus A$, and it is such that

$$A = \{x \in \mathbb{R} : \sigma_A(x) = 0\}.$$

We would like to prove that, for any $x \in A$, the function σ_A is continuous at x, with $D_-\sigma_A(x) = -\infty$ and $D^+\sigma_A(x) = +\infty$.

Suppose then $x \in A$, and so $\sigma_A(x) = 0$. For every $n \geq 1$, we can find an index $N(x, n) \in \{1, \dots, n\}$ such that $x \in S_{n,N(x,n)}$. Let us first focus our attention on a right neighborhood of x. We consider two cases.

Case 1. $\inf\{y \in A : y > x\} > x$. Then $x = a_n$, for a certain index n. In particular, $U_n \cup \{x\} = [a_n, b_n[$ is a right neighborhood of x, and it is easily seen that $\lim_{y \to x^+} \sigma_A(y) = 0$ and $D^+\sigma_A(x) = +\infty$.

Case 2. $\inf\{y \in A : y > x\} = x$. In this case, $S_{n,N(x,n)}$ contains a right neighborhood of x, for every $n \geq 1$, and

$$S_{n,N(x,n)} \cap \{y \in]x, 1[: y \notin A\} = \bigcup_{j \in J_n} U_j,$$

where J_n is an infinite set of integers, such that

$$\lim_n (\min J_n) = +\infty. \tag{2}$$

We first prove that σ_A is continuous from the right at x. Fix $\varepsilon > 0$. By (1) and (2), there exists $\bar{n} \geq 1$ such that

$$n \geq \bar{n} \Rightarrow \rho_j < \varepsilon^2, \qquad \text{for every } j \in J_n. \tag{3}$$

For any $y \in S_{\bar{n},N(x,\bar{n})} \cap]x, 1[$, we have that, either $y \in A$, hence $\sigma_A(y) = 0$, or $y \in U_j$ for a certain $j \in J_{\bar{n}}$; in this case, by (3),

$$\sigma_A(y) = \sqrt{\rho_j}\, F\left(\frac{y - a_j}{b_j - a_j}\right) \leq \sqrt{\rho_j} < \varepsilon.$$

We have thus proved that $0 \leq \sigma_A(y) < \varepsilon$ for every y in a right neighborhood of x, and so $\lim_{y \to x^+} \sigma_A(y) = 0$. We now prove that $D^+\sigma_A(x) = +\infty$. We claim that there exists a strictly increasing sequence $(n_k)_k$ of positive integers such that

$$S_{n_k,H(n_k)} = S_{n_k,N(x,n_k)}. \tag{4}$$

Indeed, set $n_1 = 1$. Then, for some $m \geq 2$, we know that it will be

$$S_{2,N(x,2)} = S_{3,N(x,3)} = \cdots = S_{m,N(x,m)} \neq S_{m+1,N(x,m+1)}$$

if and only if the sets $U_1, U_2, \ldots, U_{m-1}$ have an empty intersection with $S_{2,N(x,2)}$, while $U_m \subseteq S_{2,N(x,2)}$. We see that in this case, $S_{m,H(m)} = S_{m,N(x,m)}$; such an m is denoted by n_2. Then, one proceeds inductively: assume that n_k has been defined, for a certain $k \geq 2$; for some $m \geq n_k + 1$, it will be

$$S_{n_k+1,N(x,n_k+1)} = S_{n_k+2,N(x,n_k+2)} = \cdots = S_{m,N(x,m)} \neq S_{m+1,N(x,m+1)}$$

if and only if the sets $U_{n_k}, U_{n_k+1}, \ldots, U_{m-1}$ have an empty intersection with $S_{n_k+1,N(x,n_k+1)}$, while $U_m \subseteq S_{n_k+1,N(x,n_k+1)}$. We see that $S_{m,H(m)} = S_{m,N(x,m)}$; such an m is denoted by n_{k+1}. We have thus defined the sequence $(n_k)_k$ for which (4) holds. Denote by $\hat{x}_{n_k}$ the midpoints of the intervals U_{n_k}. Since, by (4),

$$x \in S_{n_k,N(x,n_k)} = S_{n_k,H(n_k)} \quad \text{and} \quad \hat{x}_{n_k} \in U_{n_k} \subseteq S_{n_k,H(n_k)},$$

it has to be $[x, \hat{x}_{n_k}] \subseteq S_{n_k,H(n_k)}$, hence $\hat{x}_{n_k} - x \leq \rho_{n_k}$. Then, by (1),

$$D^+\sigma_A(x) \geq \lim_k \frac{\sigma_A(\hat{x}_{n_k}) - \sigma_A(x)}{\hat{x}_{n_k} - x} \geq \lim_k \frac{\sqrt{\rho_{n_k}}}{\rho_{n_k}} = \lim_k \frac{1}{\sqrt{\rho_{n_k}}} = +\infty.$$

A similar argument shows that $\lim_{y\to x^-} \sigma_A(y) = 0$ and $D_-\sigma_A(x) = -\infty$, thus the proof is completed, in the case when A is bounded.

Let us now consider the case when A is unbounded both from below and from above. We can define a bilateral sequence $(x_n)_{n\in\mathbb{Z}}$ of points, not belonging to A, such that $x_{n+1} - x_n \geq 1$ for every $n \in \mathbb{Z}$. Define $A_n = A \cap [x_n, x_{n+1}]$, for every $n \in \mathbb{Z}$. Notice that A_n is closed, totally disconnected and bounded, for every $n \in \mathbb{Z}$. Applying the above procedure with A_n instead of A, we obtain the corresponding functions σ_{A_n}, which we denote by σ_n. Notice that, by construction, for every n, we have that

$$\sigma_n(x_n) = 1, \quad \sigma_n(x_{n+1}) = 1, \quad \text{and} \quad \sigma_n'(x_n) = \sigma_n'(x_{n+1}) = 0.$$

We define the function $\sigma_A : \mathbb{R} \to \mathbb{R}$ as

$$\sigma_A(x) = \sigma_n(x), \qquad \text{for every } n \in \mathbb{Z} \quad \text{and} \quad x \in [x_n, x_{n+1}].$$

It is readily verified that σ_A is well-defined, continuous on all $\mathbb{R}$, and differentiable on $\mathbb{R} \setminus A$.

The cases when A is unbounded only from below or only from above can be obtained adapting the procedure adopted in the previous two cases. $\square$

Lemma 6. *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function, and define $f(x) = \psi(x) \cdot \mathscr{R}(x)$, where*

$$\mathscr{R}(x) = \begin{cases} 1, & \text{if } x = 0 \text{ or } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 2 - \dfrac{1}{p}, & \text{if } x \in \mathbb{Q} \setminus \{0\} \text{ and } |x| = \dfrac{p}{q} \text{ with } \gcd(p,q) = 1. \end{cases}$$

Then, the set of continuity points of f coincides with the set of zeros of ψ; moreover,

- *if $\psi(x) \neq 0$, then $D_- f(x) = -\infty$ and $D^+ f(x) = +\infty$;*
- *if $\psi(x) = 0$, then $D_- f(x) = 2D_-\psi(x)$ and $D^+ f(x) = 2D^+\psi(x)$.*

Proof. The result is proved by means of the theory of continued fractions, for which we refer to [12]. We fix $x \in \mathbb{R}$ and consider two cases.

Case 1. $\psi(x) \neq 0$. It is easy to prove that f is not continuous at these points.

If $x \in]0, +\infty[\setminus \mathbb{Q}$, let $(c_n(x))_{n \in \mathbb{N}}$ be the sequence of convergents of the continued fraction representing x. Define

$$x_n^+ = \frac{a_{2n}}{b_{2n}} = c_{2n}(x), \qquad x_n^- = \frac{a_{2n+1}}{b_{2n+1}} = c_{2n+1}(x).$$

The sequence $(x_n^+)_n$ converges to the right, while $(x_n^-)_n$ converges to the left to x. Since the fractions $c_n(x)$ are in lowest terms, we have

$$\frac{f(x_n^+) - f(x)}{x_n^+ - x} = \frac{\left(2 - \dfrac{1}{a_{2n}}\right) \psi(c_{2n}(x)) - \psi(x)}{c_{2n}(x) - x} \to +\infty,$$

because the numerator tends to $\psi(x) > 0$ as $n \to +\infty$. Analogously,

$$\frac{f(x_n^-) - f(x)}{x_n^- - x} = \frac{\left(2 - \dfrac{1}{a_{2n+1}}\right) \psi(c_{2n+1}(x)) - \psi(x)}{c_{2n+1}(x) - x} \to -\infty.$$

Hence, $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$.

If $x \in]0, +\infty[\cap \mathbb{Q}$, let $x = \frac{p}{q}$ with $\gcd(p,q) = 1$, and define, for every $n \in \mathbb{N}$,

$$y_n^+ = \frac{p}{q} + \frac{1}{(2q)^n} = \frac{2^n p q^{n-1} + 1}{2^n q^n}, \qquad y_n^- = \frac{p}{q} - \frac{1}{(2q)^n} = \frac{2^n p q^{n-1} - 1}{2^n q^n}.$$

For every $n \geq 2$, the fractions are reduced to lowest terms, while their numerators tend to infinity as $n \to +\infty$. So,

$$\frac{f(y_n^+) - f(x)}{y_n^+ - x} = \frac{\left(2 - \dfrac{1}{2^n p q^{n-1} + 1}\right) \psi\left(y_n^+\right) - \left(2 - \dfrac{1}{p}\right) \psi\left(x\right)}{(2q)^{-n}} \to +\infty,$$

because the numerator tends to $\frac{1}{p}\psi(x) > 0$ as $n \to +\infty$. Analogously,

$$\frac{f(y_n^-) - f(x)}{y_n^- - x} = -\frac{\left(2 - \dfrac{1}{2^n p q^{n-1} - 1}\right) \psi\left(y_n^-\right) - \left(2 - \dfrac{1}{p}\right) \psi\left(x\right)}{(2q)^{-n}} \to -\infty.$$

Hence, $D^+ f(x) = +\infty$ and $D_- f(x) = -\infty$. We have thus proved the conclusion, in this case, for every $x > 0$.

A similar argument leads to the conclusion when $x < 0$. Finally, if $x = 0$, we define, for every $n \geq 1$,

$$z_n^+ = \frac{n+1}{n^2}, \qquad z_n^- = -\frac{n+1}{n^2},$$

therefore,

$$\frac{f(z_n^\pm) - f(0)}{z_n^\pm - 0} = \frac{\left(2 - \dfrac{1}{n+1}\right) \psi\left(z_n^\pm\right) - \psi\left(0\right)}{z_n^\pm} \to \pm\infty,$$

since $\psi(0) > 0$, hence proving again that $D^+ f(0) = +\infty$ and $D_- f(0) = -\infty$.

Case 2. $\psi(x) = 0$. The continuity of f at x is trivial, since

$$\psi(y) \leq f(y) \leq 2\psi(y), \qquad \text{for every } y \in \mathbb{R}. \tag{5}$$

The function

$$r_x(y) = \frac{\psi(y) - \psi(x)}{y - x} = \frac{\psi(y)}{y - x}$$

is continuous in its domain $\mathbb{R} \setminus \{x\}$, and

$$r_x(y)(y - x) \geq 0, \qquad \text{for every } y \in \mathbb{R} \setminus \{x\}. \tag{6}$$

Moreover,

$$D^+\psi(x) = \limsup_{y \to x^+} r_x(y), \qquad D_-\psi(x) = \liminf_{y \to x^-} r_x(y).$$

Correspondingly, we can find two sequences of irrational numbers $(\xi_n^-)_n$ in $]-\infty, x[$ and $(\xi_n^+)_n$ in $]x, +\infty[$ such that $\lim_n \xi_n^\pm = x$ and

$$\lim_n r_x(\xi_n^+) = D^+\psi(x), \qquad \lim_n r_x(\xi_n^-) = D_-\psi(x).$$

We now assume $x > 0$. Recalling the notation $(c_n(\zeta))_n$ for the sequence of the convergents of the continued fraction representing $\zeta \notin \mathbb{Q}$, we can find two sequences of positive rational numbers $(\zeta_n^\pm)_n$ such that

$$\zeta_n^- = c_{2\kappa(n)+1}(\xi_n^-) = \frac{\gamma_n^-}{\delta_n^-} \quad \text{and} \quad \zeta_n^+ = c_{2\kappa(n)}(\xi_n^+) = \frac{\gamma_n^+}{\delta_n^+},$$

where the choice $\kappa(n) > n$ is such that $|\xi_n^\pm - \zeta_n^\pm| < n^{-1}$, $|r_x(\xi_n^\pm) - r_x(\zeta_n^\pm)| < n^{-1}$, and $\gamma_n^\pm > n$. In particular, we can ensure that $\lim_n \zeta_n^\pm = x$ and

$$\lim_n r_x(\zeta_n^+) = D^+\psi(x), \qquad \lim_n r_x(\zeta_n^-) = D_-\psi(x).$$

Finally,

$$\frac{f(\zeta_n^+) - f(x)}{\zeta_n^+ - x} = \frac{f(\zeta_n^+)}{\zeta_n^+ - x} = \left(2 - \frac{1}{\gamma_n^+}\right)\frac{\psi(\zeta_n^+)}{\zeta_n^+ - x} \to 2D^+\psi(x),$$

$$\frac{f(\zeta_n^-) - f(x)}{\zeta_n^- - x} = \frac{f(\zeta_n^-)}{\zeta_n^- - x} = \left(2 - \frac{1}{\gamma_n^-}\right)\frac{\psi(\zeta_n^-)}{\zeta_n^- - x} \to 2D_-\psi(x).$$

Hence, $D^+f(x) = 2D^+\psi(x)$ and $D_-f(x) = 2D_-\psi(x)$, taking into account (5) and (6).

The cases when $x < 0$ or $x = 0$ can be carried out similarly. The proof is thus completed. $\square$

The proof of Theorem 3 is now an immediate consequence of Lemma 6, taking as ψ the function σ_A provided by Lemma 5.

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