A hybrid level-based learning swarm algorithm with mutation operator for solving large-scale cardinality-constrained portfolio optimization problems

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Annex: Supplementary results

In this complementary file of the article "A hybrid level-based learning swarm algorithm with mutation operator for solving large-scale cardinality-constrained portfolio optimization problems", we first analyse the impact of the Ledoit shrinkage correction with respect to the sample estimation in the calculation of the covariance matrices for the S&P 500, Russell 2000 and Russell 3000 data sets. To this end, we evaluate the Frobenius norm[1] of the difference between the covariance matrices based on these two estimators. Second, for each data set, we adopt different cardinality thresholds in the algorithmic comparison, showing that the results are consistent with the $k_{\%} = 30\%$ case presented in the article.

[1]The Frobenius norm of a matrix A is defined as $\|A\|_F = \sqrt{Trace\,(AA^T)}$.

I. Impact of different estimates of covariance matrices

Table 1: Results for ALLSO-MUT-H with the Ledoit shrinkage correction and the sample estimation for the covariance matrices. The second column displays the Frobenius norm of the difference between the two covariance matrices. The last columns show the p-values for the paired t-test on the best values over 30 runs. In all cases except one, the p-values are under the significance level $\alpha = 0.05$, indicating the rejection of the null hypothesis of equality in terms of best objective function, against the alternative bilateral hypothesis. Note that only for the S&P 500 data set, in the case $k_{\%} = 5\%$ we can not reject the null hypothesis.

Data set	Frobenius norm	p-values		
		$k_{\%} = 30\%$	$k_{\%} = 15\%$	$k_{\%} = 5\%$
S&P 500	0.04226	0.0032	0.0017	0.7529
Russell 2000	10.7240	0.0282	0.0045	0.0158
Russell 3000	11.1220	0.0008	0.0005	0.0003

II. Algorithmic comparisons for different cardinality thresholds

II.I. Results for $k_{\%} = 15\%$

Table 2: Statistics regarding the best values of the objective function over 30 runs.

Data set	Statistics	DLLSO-H	DLLSO-MUT-H	ALLSO-H	ALLSO-MUT-H	RLLPSO-H	RLLPSO-MUT-H
S&P 500	mean	-0.1507	**-0.1712**	-0.1506	-0.1705	-0.1507	-0.1667
	std	0.0003	0.0027	0.0003	0.0026	**0.0002**	0.0035
	min	-0.1514	-0.1791	-0.1514	**-0.1793**	-0.1512	-0.1751
	max	-0.1499	**-0.1674**	-0.1501	-0.1660	-0.1502	-0.1605
Russell 2000	mean	-0.1887	-0.2096	-0.1889	**-0.2132**	-0.1891	-0.2035
	std	**0.0007**	0.0035	0.0008	0.0034	0.0011	0.0039
	min	-0.1898	-0.2163	-0.1905	**-0.2186**	-0.1913	-0.2119
	max	-0.1870	-0.2030	-0.1873	**-0.2059**	-0.1861	-0.1965
Russell 3000	mean	-0.2102	-0.2313	-0.2107	**-0.2341**	-0.2111	-0.2292
	std	**0.0010**	0.0034	0.0013	0.0048	0.0011	0.0055
	min	-0.2123	-0.2381	-0.2137	**-0.2447**	-0.2139	-0.2394
	max	-0.2081	**-0.2246**	-0.2085	**-0.2246**	-0.2094	-0.2171

Table 3: Relative change of the mutated algorithms versus non-mutated counterparts. The p-values for the paired t-tests are displayed in brackets. Note that in all cases the p-values are under the significance level $\alpha = 0.05$, indicating the rejection of the null hypothesis of equality of the means, against the alternative left-sided hypothesis.

Data set	DLLSO-MUT-H vs. DLLSO-H (%)	ALLSO-MUT-H vs. ALLSO-H (%)	RLLPSO-MUT-H vs. RLLPSO-H (%)
S&P 500	13.6424	13.2584	10.6324
	$(2.2337 \cdot 10^{-27})$	$(3.0619 \cdot 10^{-27})$	$(1.4020 \cdot 10^{-21})$
Russell 2000	11.0538	12.8299	7.5931
	$(5.8192 \cdot 10^{-24})$	$(1.5370 \cdot 10^{-25})$	$(1.0391 \cdot 10^{-18})$
Russell 3000	10.0522	11.1408	8.5493
	$(8.1912 \cdot 10^{-25})$	$(3.6320 \cdot 10^{-22})$	$(3.2061 \cdot 10^{-17})$

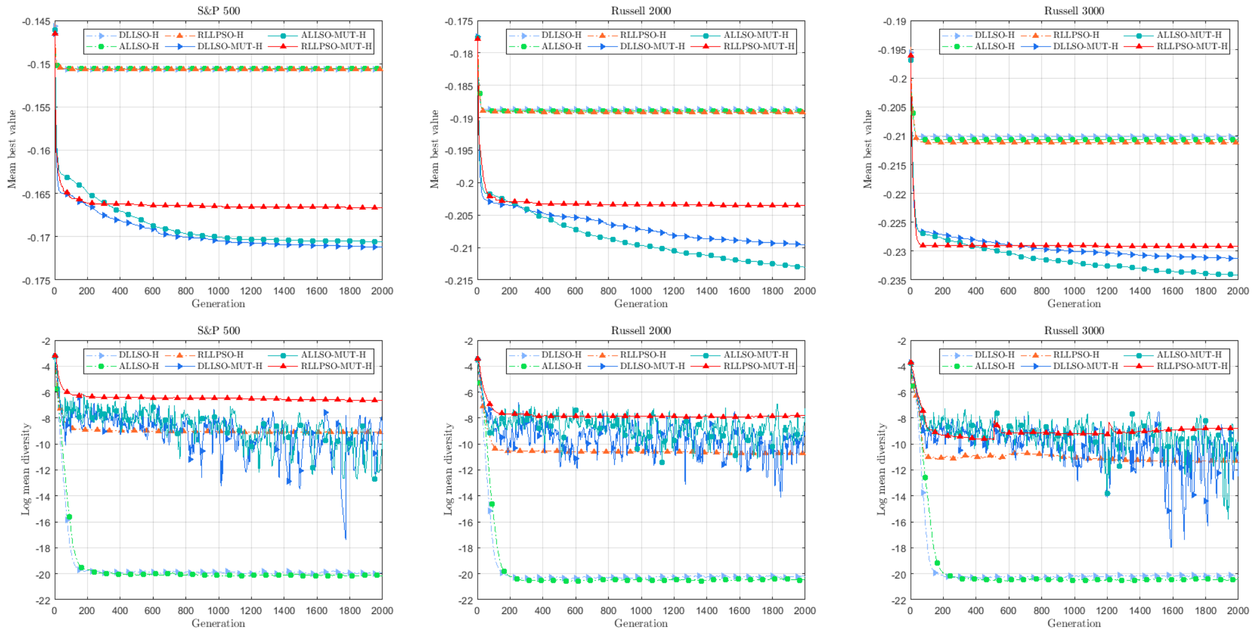

Figure 1: Convergence and diversity analyses on the three data sets for problems with $k_{\%} = 15\%$. Graphs in the first row show the behaviour of algorithms in terms of mean best value of the objective function, while in the row below are displayed the logarithmic scale plots of the diversity scores.

Table 4: Comparison with state-of-the-art swarm optimization algorithms implementing the exact ℓ_1-penalty framework.

Data set	Statistics	PSO-ℓ_1	FA-ℓ_1	ALLSO-MUT-ℓ_1	ALLSO-MUT-H
S&P 500	feasible sol. (%)	70	100	100	100
	mean CV	$3.9937 \cdot 10^{-12}$	0	0	0
	mean F_{ℓ_1}	-0.1221	-0.1221	-0.1221	-0.1738
Russell 2000	feasible sol. (%)	93.3	100	83.3	100
	mean CV	$3.7007 \cdot 10^{-12}$	0	0.0072	0
	mean F_{ℓ_1}	-0.1730	-0.1583	-0.1584	-0.2185
Russell 3000	feasible sol. (%)	96.7	100	83.3	100
	mean CV	$4.1119 \cdot 10^{-12}$	0	0.0043	0
	mean F_{ℓ_1}	-0.2012	-0.1817	-0.1819	-0.2445

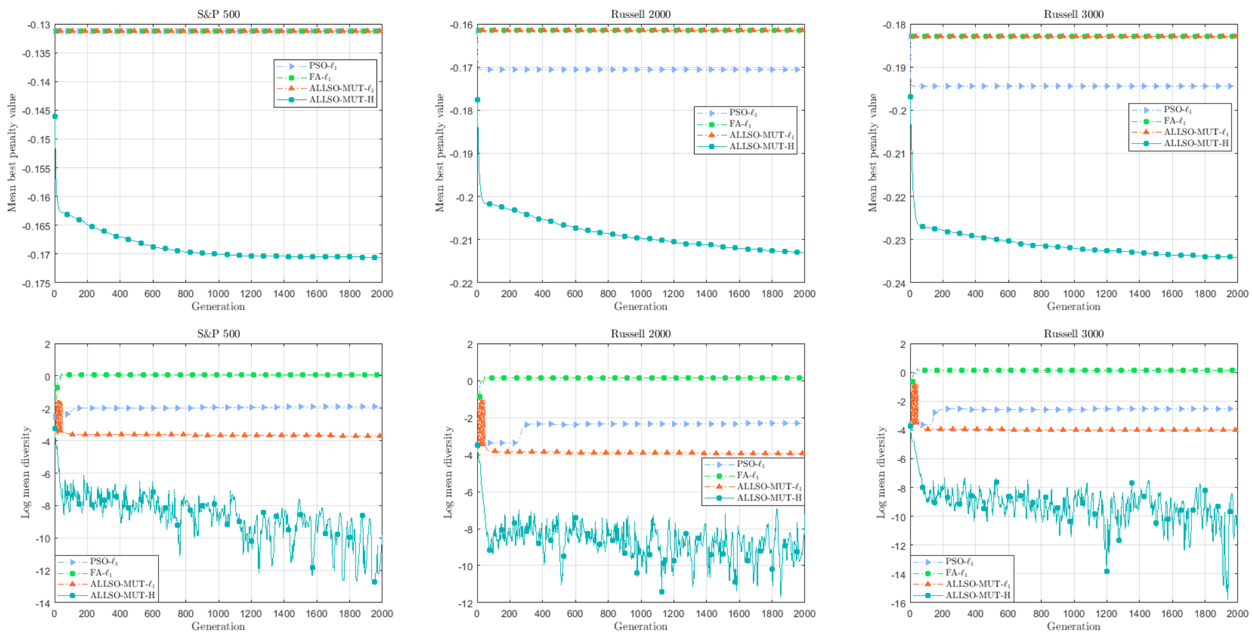

Figure 2: Plots in the first row show the behaviour of the algorithms in terms of mean best value of the penalty function, while in the second row are presented the logarithmic scale graphs of the diversity.

II.II. Results for $k_{\%} = 5\%$

Table 5: Statistics regarding the best values of the objective function over 30 runs.

Data set	Statistics	DLLSO-H	DLLSO-MUT-H	ALLSO-H	ALLSO-MUT-H	RLLPSO-H	RLLPSO-MUT-H
S&P 500	mean	-0.1516	-0.1704	-0.1523	**-0.1738**	-0.1523	-0.1718
	std	**0.0027**	0.0045	0.0028	0.0062	0.0034	0.0047
	min	-0.1565	-0.1785	-0.1588	**-0.1809**	-0.1584	-0.1794
	max	-0.1473	-0.1585	-0.1466	-0.1588	-0.1445	**-0.1628**
Russell 2000	mean	-0.1796	-0.2147	-0.1795	**-0.2185**	-0.1796	-0.2023
	std	0.0007	0.0040	0.0005	0.0039	**0.0004**	0.0056
	min	-0.1824	-0.2216	-0.1805	**-0.2272**	-0.1806	-0.2115
	max	-0.1789	-0.2065	-0.1783	**-0.2121**	-0.1789	-0.1929
Russell 3000	mean	-0.2067	-0.2406	-0.2065	**-0.2445**	-0.2069	-0.2334
	std	**0.0008**	0.0045	0.0010	0.0045	0.0010	0.0060
	min	-0.2084	-0.2487	-0.2085	**-0.2526**	-0.2087	-0.2461
	max	-0.2048	-0.2319	-0.2047	**-0.2364**	-0.2052	-0.2243

Table 6: Relative change of the mutated algorithms versus non-mutated counterparts. The p-values for the paired t-tests are displayed in brackets. Note that in all cases the p-values are under the significance level $\alpha = 0.05$, indicating the rejection of the null hypothesis of equality of the means, against the alternative left-sided hypothesis.

Data set	DLLSO-MUT-H vs. DLLSO-H (%)	ALLSO-MUT-H vs. ALLSO-H (%)	RLLPSO-MUT-H vs. RLLPSO-H (%)
S&P 500	12.4789	14.1845	12.8293
	$(4.4665 \cdot 10^{-18})$	$(1.7859 \cdot 10^{-16})$	$(1.8329 \cdot 10^{-16})$
Russell 2000	19.5046	21.6973	12.6373
	$(6.4908 \cdot 10^{-29})$	$(4.7644 \cdot 10^{-31})$	$(1.1010 \cdot 10^{-19})$
Russell 3000	16.3895	18.4128	12.8034
	$(1.3297 \cdot 10^{-27})$	$(5.9493 \cdot 10^{-28})$	$(1.2127 \cdot 10^{-20})$

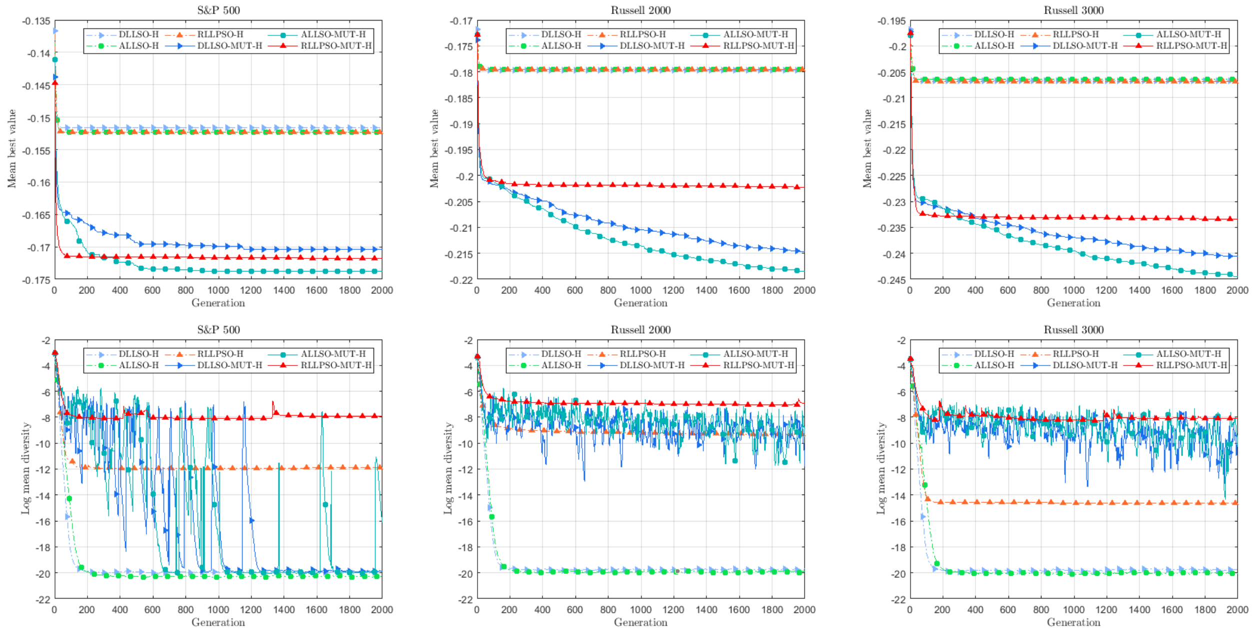

Figure 3: Convergence and diversity analyses on the three data sets for problems with $k_{\%} = 5\%$. Graphs in the first row show the behaviour of algorithms in terms of mean best value of the objective function, while in the row below are displayed the logarithmic scale plots of the diversity scores.

Table 7: Comparison with state-of-the-art swarm optimization algorithms implementing the exact ℓ_1-penalty framework.

Data set	Statistics	PSO-ℓ_1	FA-ℓ_1	ALLSO-MUT-ℓ_1	ALLSO-MUT-H
S&P 500	feasible sol. (%)	0	0	100	100
	mean CV	$7.3552 \cdot 10^{-12}$	$7.3552 \cdot 10^{-12}$	0	0
	mean F_{ℓ_1}	-0.1313	-0.1313	-0.1313	-0.1705
Russell 2000	feasible sol. (%)	73.3	100	100	100
	mean CV	$6.8731 \cdot 10^{-5}$	0	0	0
	mean F_{ℓ_1}	-0.1707	-0.1615	-0.1615	-0.2132
Russell 3000	feasible sol. (%)	80	100	96.7	100
	mean CV	$4.7698 \cdot 10^{-12}$	0	0.0108	0
	mean F_{ℓ_1}	-0.1945	-0.1829	-0.1829	-0.2341

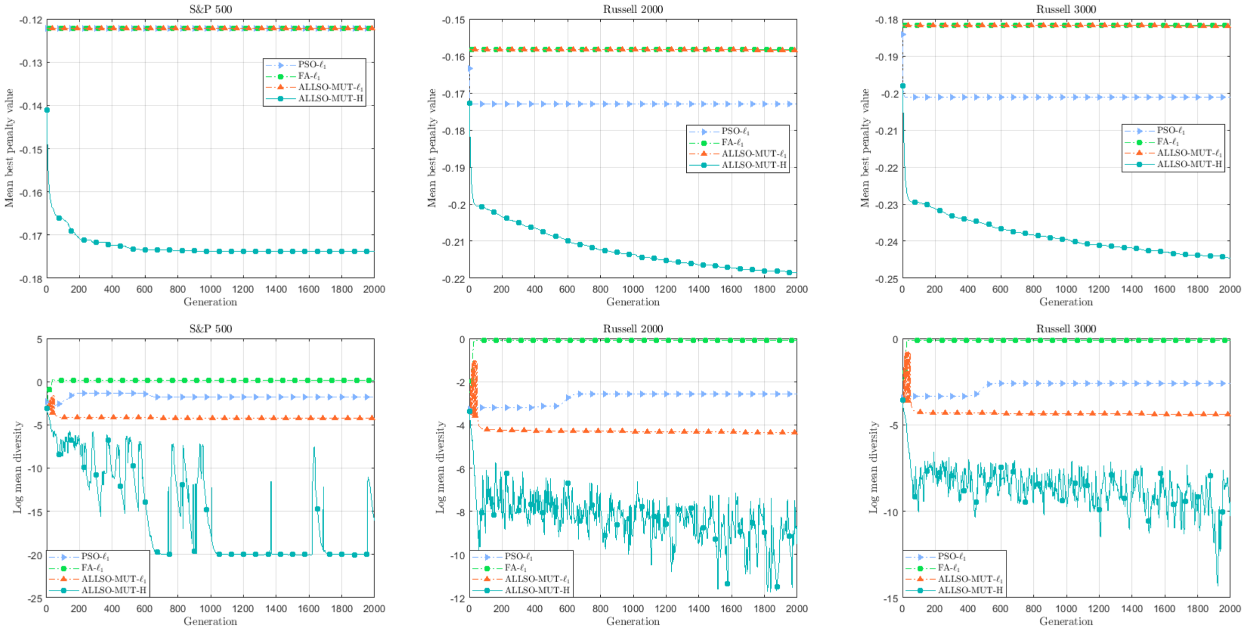

Figure 4: Plots in the first row show the behaviour of the algorithms in terms of mean best value of the penalty function, while in the second row are presented the logarithmic scale graphs of the diversity.