Existence for elastodynamic Griffith fracture with a weak maximal dissipation condition

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A B S T R A C T

We consider a model of elastodynamics with fracture evolution, based on energy-dissipation balance and a maximal dissipation condition. We prove an existence result in the case of planar elasticity with a free crack path, where the maximal dissipation condition is satisfied among suitably regular competitor cracks.

R É S U M É

Nous considérons un modèle élastodynamique de l'évolution d'une fracture, basé sur un bilan énergie-dissipation et sur une condition de dissipation maximale. Nous obtenons un résultat d'existence dans le cas de l'élasticité plane avec un chemin de fissure libre, lorsque on considère seulement des fissures suffisamment régulières.

1. Introduction

Existence proofs for dynamic fracture models that predict crack paths remain a major challenge. In [1] we proposed a model for dynamic fracture, based on the following ideas:

(a) the displacement solves elastodynamics out of the crack, with traction-free boundary conditions on the crack;
(b) the dynamic energy-dissipation balance is satisfied: the sum of the kinetic energy and of the elastic energy at time t, plus the energy dissipated by the crack between time 0 and time t, is equal to the initial energy plus the total work done by external forces between time 0 and time t;

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(c) a maximal dissipation condition is satisfied, which forces the crack to run as fast as possible, consistent with the energy-dissipation balance.

These general ideas were applied to the case of antiplane displacement with linear elasticity, and a prescribed crack path. We refer to [2,1] for a discussion on the mechanical motivation of conditions (a)–(c) and for the literature on this subject.

The purpose of this paper is to extend these ideas to both predict the crack path and consider linear elasticity (not restricted to antiplane displacements). In particular, we give the first existence proof for a model of dynamic fracture that predicts the crack path.

Our reference configuration is a bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary and the problem is studied in a bounded time interval $[0,T]$. Cracks, as functions of time, will be described as follows. For a prescribed $a_0 < 0$, a sufficiently regular curve parameterized by arc-length $\gamma\colon [a_0, b_\gamma] \to \mathbb{R}^2$, and a function $s\colon [0,T] \to [0, b_\gamma]$, the crack at time t is

$$\Gamma_{s(t)} := \gamma([a_0, s(t)]).$$

Here we also assume $s(0) = 0$, and for every $t \in [0,T]$, $s(t)$ provides the length of the crack produced along the curve γ between time 0 and time T. The goal is then to determine both the curve γ and the length as a function of time, $t \mapsto s(t)$. We assume that

(1) the initial part of the crack is prescribed: $\gamma(s) = \gamma_0(s)$ for every $s \in [a_0, 0]$, where $\gamma_0\colon [a_0, 0] \to \overline{\Omega}$ is a given curve with $\gamma_0(a_0) \in \partial\Omega$ and $\gamma_0(s) \in \Omega$ for $s > a_0$;
(2) the unknown function γ, which describes the geometry of the crack, satisfies some prescribed regularity estimates (see Definition 2.1), in particular a bound on the curvature and an estimate, for every $s \geq 0$, of the distance of $\gamma(s)$ from the complement of Ω;
(3) the unknown function $t \mapsto s(t)$, whose derivative $\dot{s}(t)$ is the speed of the crack tip, satisfies some prescribed regularity estimates (see Definition 2.7), in particular $0 \leq \dot{s}(t) \leq \mu$ for a suitable constant $\mu > 0$.

The results of [3] imply that, for any pair (γ, s) satisfying the properties considered above, there exists one and only one solution $u(t,x)$ of the system of elastodynamics in the time-dependent cracking domains $t \mapsto \Omega \setminus \Gamma_{s(t)}$. The aim of this paper is to prove the following result: among all pairs (γ, s) that, together with the corresponding solution u, satisfy the dynamic energy-dissipation balance, there exists one which satisfies a maximal dissipation condition, whose formulation will be made precise below. For the mechanical interpretation of this result we refer to [1].

We consider the collection $\mathcal{C}^{piec}$ of all pairs (γ, s) satisfying (1)–(3), with s continuous and piecewise regular, such that the triple (γ, s, u) satisfies the dynamic energy-dissipation balance for every time t (see Definition 4.1). It is easy to see that $\mathcal{C}^{piec} \neq \emptyset$. Indeed, if s is constant then the solution of the system of elastodynamics in a time-independent cracked domain satisfies the energy balance (see Remark 4.2). It remains to prove that the collection $\mathcal{C}^{piec}$ contains an element that satisfies the maximal dissipation condition, which we now describe.

Since in our model we neglect the effects of heat production and transfer, the only dissipative mechanism is the process of crack formation and, assuming homogeneity and isotropy, the only dissipated energy is proportional to the crack length $s(t)$. For simplicity we suppose that the proportionality constant is 1. Therefore, a natural formulation of the maximal dissipation condition is as follows: $(\gamma, s) \in \mathcal{C}^{piec}$ satisfies the maximal dissipation condition on $[0,T]$ if there exists no $(\hat{\gamma}, \hat{s}) \in \mathcal{C}^{piec}$ such that, for some $0 \leq \tau_0 < \tau_1 \leq T$,

(MD1) $\operatorname{sing}(\hat{s}) \subset \operatorname{sing}(s)$ (see Definition 2.7),
(MD2) $\hat{s}(t) = s(t)$ and $\hat{\gamma}(\hat{s}(t)) = \gamma(s(t))$ for every $t \in [0, \tau_0]$,
(MD3) $\hat{s}(t) > s(t)$ for every $t \in (\tau_0, \tau_1]$.

Conditions (MD1)–(MD3) say that there is no sufficiently regular crack which satisfies the dynamic energy-dissipation balance, coincides with the crack described by (γ, s) up to time τ_0, and is longer at every time between τ_0 and τ_1. In other words, a crack satisfying this maximal dissipation condition cannot be overcome by longer cracks, still satisfying the dynamic energy-dissipation balance.

As in [1] we can prove the existence of a pair $(\gamma, s) \in \mathcal{C}^{piec}$ satisfying the previous condition only in a quantitative way, depending on a prescribed threshold $\eta > 0$. This leads to the following definition: $(\gamma, s) \in \mathcal{C}^{piec}$ satisfies the η-maximal dissipation condition on $[0, T]$ if there exists no $(\hat{\gamma}, \hat{s}) \in \mathcal{C}^{piec}$ such that (MD1)–(MD3) hold for some $0 \le \tau_0 < \tau_1 \le T$ and, in addition,

(MD4) $\hat{s}(\tau_1) > s(\tau_1) + \eta$.

Our main result (see Theorem 5.2) is that, if the upper bound on crack speed μ that appears in (3) is smaller than a suitable constant related to the speed of elastic waves, then there exists a pair $(\gamma, s) \in \mathcal{C}^{piec}$ which satisfies the η-maximal dissipation condition on $[0, T]$. To be precise, the condition on μ reads $0 < \mu < \sqrt{\lambda}/2$, where $\lambda > 0$ is the ellipticity constant of the elasticity tensor (see Definition 3.1). The same quantitative condition was considered in [4].

Following the scheme introduced in [1], the proof is based on a continuous dependence result: the solutions u of the system of elastodynamics in cracking domains depend continuously on the pair (γ, s) (see [3, Theorem 4.1]). It is easy to see that this theorem can be applied if μ is sufficiently small, but to apply it when $0 < \mu < \sqrt{\lambda}/2$ we must localize the problem, both in space and time, so that all diffeomorphisms used in [3, Theorem 4.1] are very close to the identity. This property is crucial in order to apply this theorem without requiring possibly very small values of μ.

To justify the localization argument we have to use a result on the finite speed of propagation for the system of elastodynamics. We need this result in an irregular domain, due to the presence of the crack. Usually the proof of the finite speed of propagation is given assuming some regularity of the solution u, which is not available here. Therefore, in the Appendix we give a complete proof of this property under minimal assumptions and in arbitrary space dimension (see Theorem A.1).

2. Admissible cracks

In this paper we deal with two dimensional problems whose reference configuration is a fixed bounded open set $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. In this section we describe the admissible cracks of our model. We first introduce the geometric constraints on the curves along which the crack may grow (see Subsection 2.1). Then we consider the admissible time evolutions of the cracks along their paths (see Subsection 2.2).

2.1. Geometry of the admissible cracks

In the following the curves are always parametrized using the arc-length parameter s and for a given curve $\gamma\colon [a_\gamma, b_\gamma] \to \mathbb{R}^2$ we set

$$\Gamma := \gamma([a_\gamma, b_\gamma]) \text{ and } \Gamma_s := \gamma([a_\gamma, s]) \text{ for } s \in [a_\gamma, b_\gamma]\,.$$

We fix an initial curve $\gamma_0\colon [a_0, 0] \to \overline{\Omega}$ of class $C^{3,1}$ such that $\gamma_0(a_0) \in \partial\Omega$, $\gamma_0(s) \in \Omega$ for every $s \in (a_0, 0]$ and we set

$$\Gamma_0 = \gamma_0([a_0, 0]) \,. \tag{2.1}$$

We assume that γ_0 is transversal to $\partial\Omega$ at $\gamma_0(a_0)$, i.e., there exists an isosceles triangle contained in $\overline{\Omega}$ with vertex in $\gamma_0(a_0)$ and axis parallel to $\gamma_0'(a_0)$.

Throughout the paper $r > 0$ and $L > 0$ are fixed constants.

Definition 2.1 *(Geometric constraints).* Let $\mathcal{G}_{r,L}$ be the set of simple curves $\gamma\colon [a_0, b_\gamma] \to \overline{\Omega}$ of class $C^{3,1}$, with $a_0 < 0 \le b_\gamma$, such that

(a) fixed initial crack: $\gamma(s) = \gamma_0(s)$ for every $s \in [a_0, 0]$;
(b) velocity one: $|\gamma'(s)| = 1$ for every $s \in [a_0, b_\gamma]$;
(c) uniform tangent balls condition: the two open disks of radius r tangent to Γ at $\gamma(s)$ do not intersect Γ,
(d) uniform distance: $\mathrm{dist}(\gamma([0, b_\gamma]), \partial\Omega) \ge 2r$,
(e) uniform bounds: $|\gamma^{(3)}(s)| \le L$, $|\gamma^{(3)}(s_2) - \gamma^{(3)}(s_1)| \le L|s_2 - s_1|$, for every $s, s_1, s_2 \in [a_0, b_\gamma]$,

where $\gamma^{(i)}$ denotes the i-th derivative of γ.

We assume that γ_0, r, and L are fixed in such a way that $\mathcal{G}_{r,L} \neq \emptyset$. In particular, by (a) and (d) we must have

$$|a_0| \ge 2r \,. \tag{2.2}$$

Remark 2.2 *(Estimate on the second derivatives).* Condition (c) of Definition 2.1 implies that $|\gamma^{(2)}(s)| \le 1/r$ for every $s \in [a_0, b_\gamma]$.

Definition 2.3 *(Convergence in $\mathcal{G}_{r,L}$).* Let γ_k be a sequence of curves in $\mathcal{G}_{r,L}$ and let $\gamma \in \mathcal{G}_{r,L}$. We say that γ_k converges to γ uniformly if $b_{\gamma_k} \to b_\gamma$ and for every $b \in (0, b_\gamma)$ we have $\gamma_k|_{[a_0,b]} \to \gamma|_{[a_0,b]}$ uniformly in $[a_0, b]$.

Lemma 2.4 (Extension). *There exist two constants $\hat{r}$ and $\widehat{L}$, with $0 < \hat{r} < r$ and $\widehat{L} > L$, depending only on r and L, such that for every $\gamma\colon [a_0, b_\gamma] \to \overline{\Omega}$ with $\gamma \in \mathcal{G}_{r,L}$ there exists an extension $\widehat{\gamma}\colon [a_0, b_\gamma + \hat{r}] \to \overline{\Omega}$ of γ with $\widehat{\gamma} \in \mathcal{G}_{\hat{r},\widehat{L}}$, whose image will be indicated by $\widehat{\Gamma}$. Moreover, the extension can be chosen in such a way that the uniform convergence of γ_k implies the uniform convergence of the corresponding extensions $\widehat{\gamma}_k$.*

Proof. For $s > b_\gamma$ let $\widehat{\gamma}(s)$ be the arc-length parametrization of the curve $\sigma \mapsto \gamma(b_\gamma) + \gamma'(b_\gamma)(\sigma - b_\gamma) + \frac{1}{2}\gamma^{(2)}(b_\gamma)(\sigma - b_\gamma)^2 + \frac{1}{6}\gamma^{(3)}(b_\gamma)(\sigma - b_\gamma)^3$. It is easy to check that the uniform tangent balls condition and the estimate $\mathrm{dist}(\widehat{\gamma}([0, b_\gamma + \hat{r}]), \partial\Omega) \ge 2\hat{r}$ are satisfied if $\hat{r}$ is small enough. Using Arzelà-Ascoli Theorem we see that conditions (b) and (e) of Definition 2.1 and Remark 2.2, together with the uniform convergence of γ_k, imply the convergence of the derivatives up to the third order evaluated at b_{γ_k}. This gives the uniform convergence of the extensions $\widehat{\gamma}_k$. □

Lemma 2.5 (Compactness of $\mathcal{G}_{r,L}$). *Let γ_k be a sequence of curves in $\mathcal{G}_{r,L}$. Then there exist a subsequence, not relabeled, and a curve $\gamma \in \mathcal{G}_{r,L}$ such that γ_k converges to γ uniformly.*

Proof. Since Ω is bounded, it is easy to deduce from the uniform tangent balls condition that the length of the curves γ_k is uniformly bounded. Let b_γ be the limit of a subsequence of b_{γ_k}. The uniform estimates on the derivatives imply that there exists a subsequence, still denoted γ_k, and a curve $\gamma\colon [a_0, b_\gamma] \to \overline{\Omega}$ of class $C^{3,1}$ such that γ_k converges to γ uniformly. The geometric constraints (c) and (d) pass to the limit as shown, e.g., in [5], allowing us to conclude that $\gamma \in \mathcal{G}_{r,L}$. □

To apply [3, Theorems 3.2 and 4.1] we have to construct some time-dependent diffeomorphisms $\Phi(t,\cdot)$ and $\Psi(t,\cdot)$ satisfying conditions (H1)–(H12) of [3]. They will be of the form $\Phi(t,\cdot)=\widehat{\Phi}(s(t),\cdot)$ and $\Psi(t,\cdot)=\widehat{\Psi}(s(t),\cdot)$, where, for every σ, $\widehat{\Phi}(\sigma,\cdot)$ and $\widehat{\Psi}(\sigma,\cdot)$ depend only on Γ, and $t\mapsto s(t)$ is the function describing the length of the crack along Γ (see Subsection 2.2). We want to apply the results of [3] under our hypotheses on $s(t)$ (see Definition 2.7 below) and on the elasticity tensor (see Definition 3.1 below), assuming that the relevant constants satisfy the natural assumption (3.19).

To this aim we have to prove that $\widehat{\Phi}(\sigma,\cdot)$ and $\widehat{\Psi}(\sigma,\cdot)$ are close to the identity and that the norm of the partial derivative $\partial_\sigma\widehat{\Phi}(\sigma,\cdot)$ is bounded by a constant close to 1. This can be obtained only when $\sigma\in[s_0,s_1]$ with s_1-s_0 sufficiently small. Moreover, to apply [3, Theorem 4.1] we also need a continuous dependence of the diffeomorphisms on the curve γ. A technical difficulty is due to the fact that we need uniform estimates depending on the smallness of s_1-s_0, but not on the values of s_0 and s_1, nor on the specific curve γ. The following lemma provides all properties we need.

Lemma 2.6 *(Diffeomorphisms depending on the curves). Let $\varepsilon>0$ and let $0<\rho<\hat{r}/2$ (see Lemma 2.4). Then there exist two constants $\widehat{\delta}\in(0,\rho)$ and $\widehat{C}>0$, depending only on $\hat{r},\widehat{L},\varepsilon$, and ρ, such that for every $\gamma\in\mathcal{G}_{r,L}$ and $0\le s_0<s_1\le b_\gamma$, with $s_1-s_0\le\widehat{\delta}$, we can construct two functions $\widehat{\Phi},\widehat{\Psi}\colon[s_0,s_1]\times\overline{\Omega}\to\overline{\Omega}$ of class $C^{2,1}$ with the following properties:*

(a) *for every $\sigma\in[s_0,s_1]$ we have $\widehat{\Phi}(\sigma,\overline{\Omega})=\overline{\Omega}$, $\widehat{\Phi}(\sigma,\widehat{\Gamma})=\widehat{\Gamma}$ (see Lemma 2.4), $\widehat{\Phi}(\sigma,\Gamma_{s_0})=\Gamma_\sigma$, and $\widehat{\Phi}(\sigma,y)=y$ on $\overline{\Omega}\setminus B(\gamma(s_0),2\rho)$;*
(b) *$\widehat{\Phi}(s_0,y)=y$ for every $y\in\overline{\Omega}$;*
(c) *for every $\sigma\in[s_0,s_1]$, $\widehat{\Psi}(\sigma,\cdot)$ is the inverse of $\widehat{\Phi}(\sigma,\cdot)$ on $\overline{\Omega}$;*
(d) *for every $\sigma\in[s_0,s_1]$ we have $1-\varepsilon\le\det\nabla\widehat{\Phi}(\sigma,y)\le1+\varepsilon$ and $1-\varepsilon\le\det\nabla\widehat{\Psi}(\sigma,x)\le1+\varepsilon$ for every $x,y\in\overline{\Omega}$, where ∇ denotes the spatial gradient;*
(e) *for every $\sigma\in[s_0,s_1]$ we have $|\partial_\sigma\widehat{\Phi}(\sigma,y)|\le1+\varepsilon$ for every $y\in\overline{\Omega}$;*
(f) *the absolute values of all partial derivatives of $\widehat{\Phi}$ and of $\widehat{\Psi}$ of order less than or equal to two, as well as the Lipschitz constants of all second derivatives, are bounded by $\widehat{C}$;*
(g) *if γ_k is a sequence in $\mathcal{G}_{r,L}$ converging to γ uniformly and such that $s_1\le b_{\gamma_k}$ for every k, then the corresponding diffeomorphisms $\widehat{\Phi}_k(\sigma,\cdot)$ satisfy $\widehat{\Phi}_k(\sigma,x)\to\widehat{\Phi}(\sigma,x)$ for every $\sigma\in[s_0,s_1]$ and every $x\in\overline{\Omega}$.*

Proof. Let us fix γ and s_0 as in the statement of the lemma and let $\widehat{\gamma}\colon[a_0,b_\gamma+\hat{r}]\to\overline{\Omega}$ be the extension provided by Lemma 2.4. The construction of $\widehat{\Phi}$ and $\widehat{\Psi}$ requires several steps.

Step 1. Construction of diffeomorphisms from $[s_0-\frac{3}{2}\rho,s_0+\frac{3}{2}\rho]$ into itself.

Let us fix a C^∞ function $\chi\colon\mathbb{R}\to[0,1]$ such that $\chi(s_0)=1$, $\mathrm{supp}\chi\subset(s_0-\frac{3}{2}\rho,s_0+\frac{3}{2}\rho)$, and $|\chi'(s)|\le\frac{3}{4\rho}$ for every $s\in\mathbb{R}$.

Let us fix $\widehat{\delta}\in(0,\rho)$ and let $\zeta\colon[s_0,s_0+\widehat{\delta}]\times[s_0-\frac{3}{2}\rho,s_0+\frac{3}{2}\rho]\to\mathbb{R}$ be the C^∞ function defined by

$$\zeta(\sigma,s)=s+(\sigma-s_0)\chi(s)\,. \tag{2.3}$$

We first observe that since $0\le\sigma-s_0\le\widehat{\delta}$, from the estimate on χ' we obtain

$$\tfrac{1}{4}<1-\tfrac{3}{4}\tfrac{\widehat{\delta}}{\rho}\le\partial_s\zeta\le1+\tfrac{3}{4}\tfrac{\widehat{\delta}}{\rho}<\tfrac{7}{4}\,. \tag{2.4}$$

Moreover, $\zeta(\sigma,s)=s$ for $s=s_0\pm\frac{3}{2}\rho$. Together with (2.4) this shows that $\zeta(\sigma,\cdot)$ is a diffeomorphism from $[s_0-\frac{3}{2}\rho,s_0+\frac{3}{2}\rho]$ into itself for every $\sigma\in[s_0,s_0+\widehat{\delta}]$.

We observe also that

$$\partial_\sigma\zeta(\sigma,s)=\chi(s)\in[0,1]\quad\text{for every }\sigma\in[s_0,s_0+\widehat{\delta}]\text{ and every }s\in[s_0-\tfrac{3}{2}\rho,s_0+\tfrac{3}{2}\rho]\,. \tag{2.5}$$

Step 2. Construction of diffeomorphisms in a neighborhood of $\widehat\gamma((s_0-\frac{3}{2}\rho, s_0+\frac{3}{2}\rho))$.

We begin by observing that $a_0 < s_0 - \frac{3}{2}\rho$, since $\rho < \frac{1}{4}|a_0|$ by (2.2), and that $s_0 + \frac{3}{2}\rho < b_\gamma + \widehat r$. Therefore $\widehat\gamma$ is well-defined in the interval $[s_0-\frac{3}{2}\rho, s_0+\frac{3}{2}\rho]$. For every $s \in [s_0-\frac{3}{2}\rho, s_0+\frac{3}{2}\rho]$ let $\hat\nu(s)$ be the unit normal to $\widehat\gamma$ at $\widehat\gamma(s)$. Let us fix $0 < \ell_0 < \frac{1}{2}\rho$. Since $\ell_0 < \hat r$, the map

$$(s,\ell) \mapsto \widehat\gamma(s) + \ell\hat\nu(s) \tag{2.6}$$

is a diffeomorphism of class $C^{2,1}$ between $[s_0-\frac{3}{2}\rho, s_0+\frac{3}{2}\rho]\times[-\ell_0,\ell_0]$ and its image, indicated by A. Let us note that $A \subset B(\gamma(s_0), 2\rho) \subset\subset \Omega$, where the second inclusion follows from the uniform distance condition (d) of Definition 2.1.

For every $\sigma \in [s_0, s_0+\widehat\delta]$ the diffeomorphism $\zeta(\sigma,\cdot)$ induces a diffeomorphism from $\widehat\gamma([s_0-\frac{3}{2}\rho, s_0+\frac{3}{2}\rho])$ into itself, which coincides with the identity near $\widehat\gamma(s_0\pm\frac{3}{2}\rho)$. We now want to extend it to a diffeomorphism $\widehat\Phi$ between A and itself such that $\widehat\Phi(\sigma,y)=y$ for every y in a neighborhood of ∂A.

To this aim we fix an even C^∞ function $\varphi\colon[-1,1]\to[0,1]$ equal to 1 in a neighborhood of ±1 and equal to 0 in 0. For every $\sigma\in[s_0,s_0+\widehat\delta]$, $\ell\in[-\ell_0,\ell_0]$, and $y\in A$ we set

$$\widehat\sigma(\sigma,\ell) = (1-\varphi(\tfrac{\ell}{\ell_0}))\sigma + \varphi(\tfrac{\ell}{\ell_0})s_0\,, \tag{2.7}$$

$$\widehat\Phi(\sigma,y) = \widehat\gamma(\zeta(\widehat\sigma(\sigma,\ell),s)) + \ell\hat\nu(\zeta(\widehat\sigma(\sigma,\ell),s))\,, \tag{2.8}$$

where $(s,\ell)\in[s_0-\frac{3}{2}\rho, s_0+\frac{3}{2}\rho]\times[-\ell_0,\ell_0]$ is related to y by the equality $y=\widehat\gamma(s)+\ell\hat\nu(s)$.

For every $\sigma\in[s_0,s_0+\widehat\delta]$ and $y\in A$, we have $\widehat\Phi(\sigma,y)\in A$. Moreover, $\widehat\Phi(\sigma,y)=y$ for every y in a neighborhood of ∂A. Using the fact that both $\widehat\Phi(\sigma,y)$ and y are at the same (signed) distance ℓ from $\widehat\gamma([s_0-\frac{3}{2}\rho, s_0+\frac{3}{2}\rho])$, it is easy to see that $\widehat\Phi(\sigma,\cdot)\colon A\to A$ is bijective. As for the regularity of $\widehat\Phi$, the regularity properties of φ, ζ, and $\widehat\gamma$ imply that $\widehat\Phi$ is of class $C^{2,1}$ and that the estimates in (f) hold for $\widehat\Phi$ on A.

Step 3. Extension of the diffeomorphisms and proof of (a)–(f).

To obtain a diffeomorphism from $\overline\Omega$ into $\overline\Omega$ it is enough to set $\widehat\Phi(\sigma,y)=y$ if $y\in\overline\Omega\setminus A$. For every $\sigma\in[s_0,s_0+\widehat\delta]$ let $\widehat\Psi(\sigma,\cdot)$ be the inverse of $\widehat\Phi(\sigma,\cdot)$.

Since $A\subset B(\gamma(s_0),2\rho)$, we have $\widehat\Phi(\sigma,y)=y$ for every $y\in\overline\Omega\setminus B(\gamma(s_0),2\rho)$. It follows from the construction that for every $\sigma\in[s_0,s_0+\widehat\delta]$ we have $\Phi(\sigma,\gamma([s_0-\frac{3}{2}\rho,s_0]))=\widehat\gamma([s_0-\frac{3}{2}\rho,\sigma])$ and $\widehat\Phi(\sigma,\widehat\gamma([s_0-\frac{3}{2}\rho,s_0+\frac{3}{2}\rho]))=\widehat\gamma([s_0-\frac{3}{2}\rho,s_0+\frac{3}{2}\rho])$. Hence $\widehat\Phi(\sigma,\Gamma_{s_0})=\widehat\Gamma_\sigma$ and $\widehat\Phi(\sigma,\widehat\Gamma)=\widehat\Gamma$. As $\widehat\Gamma_\sigma=\Gamma_\sigma$ for $\sigma\in[0,b_\gamma]$, this concludes the proof of (a).

Note that for every $\sigma\in[s_0,s_0+\widehat\delta]$ and $\ell\in[-\ell_0,\ell_0]$ we have

$$|\widehat\sigma(\sigma,\ell)-s_0| \le (1-\varphi(\tfrac{\ell}{\ell_0}))(\sigma-s_0)\le\widehat\delta\,,$$

and that by (2.3), (2.7), and (2.8) we have $\widehat\Phi(s_0,y)=y$ for every $y\in A$, which proves (b).

Since (2.6) is a diffeomorphism, it follows from these remarks that $\widehat\Phi(\sigma,\cdot)$ is C^1-close to the identity for $\widehat\delta$ small enough, so that estimates (d) hold for $x,y\in A$ for a suitable choice of $\widehat\delta\in(0,\rho)$.

Finally, for every $\sigma\in[s_0,s_0+\widehat\delta]$ we have

$$\begin{aligned}\partial_\sigma\widehat\Phi(\sigma,y) &= \widehat\gamma'(\zeta(\widehat\sigma(\sigma,\ell),s))\partial_{\widehat\sigma}\zeta(\widehat\sigma(\sigma,\ell),s)(1-\varphi(\tfrac{\ell}{\ell_0}))\\ &\quad+\ell\widehat\nu'(\zeta(\widehat\sigma(\sigma,\ell),s))\partial_{\widehat\sigma}\zeta(\widehat\sigma(\sigma,\ell),s)(1-\varphi(\tfrac{\ell}{\ell_0}))\,.\end{aligned} \tag{2.9}$$

Recalling that $\widehat\Gamma$ is parametrized by arc-length and satisfies the uniform tangent balls condition, we obtain that $|\widehat\gamma'(\sigma)|=1$ and the curvature is bounded by $1/\widehat r$, hence $|\widehat\nu'(\sigma)|\le1/\widehat r$. Therefore, (2.5) and (2.9) give

$$|\partial_\sigma\widehat\Phi(\sigma,y)|\le 1+\frac{\ell_0}{\widehat r}\,.$$

Taking $0 < \ell_0 \leq \varepsilon \widehat{r}$ we obtain $|\partial_\sigma \widehat{\Phi}(\sigma, y)| \leq 1 + \varepsilon$ for every $\sigma \in [s_0, s_0 + \widehat{\delta}]$ and for every $y \in A$. This proves (e).

Since, by construction, $\widehat{\Phi}(\sigma, \cdot)$ coincides with the identity in a neighborhood of ∂A, it follows from the estimates in A that $\widehat{\Phi}(\sigma, \cdot)$ is of class $C^{2,1}$ in $\overline{\Omega}$ and that the estimates in (f) hold for $\widehat{\Phi}$ as well as for its inverse function $\widehat{\Psi}$. Finally, the last statement concerning the convergence follows easily from the construction. $\square$

2.2. The class of admissible time evolutions of the crack length

In order to use the results of [3], throughout the paper we fix a constant $\mu > 0$, which will bound the speed of the crack tip, and a constant $M > 0$, which will bound some higher order derivatives of the crack length with respect to time. The regularity assumptions and the constraints on the time evolution of the crack length in our model are prescribed by the following definition.

Definition 2.7 *(Time-dependence of the crack length).* Let $T_0 < T_1$. The class $\mathcal{S}^{reg}_{\mu,M}(T_0, T_1)$ is composed of all nonnegative functions satisfying the following conditions:

$$s \in C^{3,1}([T_0, T_1]), \tag{2.10}$$

$$0 \leq \dot{s}(t) \leq \mu, \tag{2.11}$$

$$|\ddot{s}(t)| \leq M, \ |\dddot{s}(t)| \leq M, |\dddot{s}(t_1) - \dddot{s}(t_2)| \leq M|t_1 - t_2|, \tag{2.12}$$

for every $t, t_1, t_2 \in [T_0, T_1]$, where dots denote derivatives with respect to time.

We also consider the class $\mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$ of all functions $s \in C^0([T_0, T_1])$ such that there exists a finite subdivision $T_0 = \tau_0 < \tau_1 < \cdots < \tau_k = T_1$ for which

$$s|_{[\tau_{j-1}, \tau_j]} \in \mathcal{S}^{reg}_{\mu,M}(\tau_{j-1}, \tau_j).$$

The set of these intermediate times, where s may be discontinuous, is denoted by $\operatorname{sing}(s)$.

In our model an admissible crack at time t is given by

$$\Gamma_{s(t)} := \gamma([a_0, s(t)])$$

where $\gamma \in \mathcal{G}_{r,L}$ and $s \in \mathcal{S}^{piec}_{\mu,M}(0, T)$ for some $T > 0$ with $s(T) \leq b_\gamma$. Since γ is an arc-length parametrization, $s(t)$ represents the length of the crack produced along the curve γ between time 0 and time t.

For technical reasons, we assume an upper bound on the speed of the crack tip, related to the speed of the elastic waves. We note that the existence of such a bound might follow from more basic hypotheses, such as energy-dissipation balance, but for now, this is open. Briefly, the reason for the specific bound (3.19) on the constant μ that appears in (2.11) is that it will guarantee condition (3.1) in [3] is satisfied, which is crucial to our results.

On the other hand, the other constraints on s, as well as those on γ (see Definition 2.1), have no mechanical motivation; they are needed in order to apply the existence, uniqueness, and continuous dependence results of [3]. It is possible that at some point this (piecewise) regularity will be established, but this is completely open, and we make no claim about it.

In order to prove our existence result, we construct some time-dependent diffeomorphisms $\Phi(t, \cdot)$ and $\Psi(t, \cdot)$ satisfying conditions (H1)–(H12) and (3.1) of [3]. To obtain (3.1) it is convenient to prove that $\Phi(t, \cdot)$ and $\Psi(t, \cdot)$ are close to the identity and that the norm of the partial derivative $\partial_t \Phi(t, \cdot)$ is bounded by a constant close to μ. This can be done only locally in space and time. Moreover, to apply [3, Theorem 4.1] we also need a continuous dependence of the diffeomorphisms on the curve γ and on the function s. For this application we need uniform estimates depending on the smallness of the time interval, but not on the specific choice of γ and s. The following lemma provides all technical properties we need.

Lemma 2.8 *(Time-dependent diffeomorphisms). Let $\varepsilon > 0$ and let $0 < \rho < \hat{r}/2$ (see Lemma 2.4). Then there exist two constants $\delta \in (0, \rho/\mu)$ and $C > 0$, depending only on $r, L, \mu, M, \varepsilon$, and ρ, with the following property: for every $\gamma \in \mathcal{G}_{r,L}$, for every $t_0 < t_1$, and for every $s \in \mathcal{S}^{reg}_{\mu,M}(t_0, t_1)$, with $t_1 - t_0 \le \delta$, $s(t_1) \le b_\gamma$, we can construct two functions $\Phi, \Psi \colon [t_0, t_1] \times \overline{\Omega} \to \overline{\Omega}$ of class $C^{2,1}$ with the following properties:*

- (a) *for every $t \in [t_0, t_1]$ we have $\Phi(t, \overline{\Omega}) = \overline{\Omega}$, $\Phi(t, \widehat{\Gamma}) = \widehat{\Gamma}$ (see Lemma 2.4), $\Phi(t, \Gamma_{s(t_0)}) = \Gamma_{s(t)}$, and $\Phi(t, y) = y$ on $\overline{\Omega} \setminus B(\gamma(s(t_0)), 2\rho)$;*
- (b) *$\Phi(t_0, y) = y$ for every $y \in \overline{\Omega}$;*
- (c) *for every $t \in [t_0, t_1]$, $\Psi(t, \cdot)$ is the inverse of $\Phi(t, \cdot)$ on $\overline{\Omega}$;*
- (d) *for every $t \in [t_0, t_1]$ we have $1 - \varepsilon \le \det \nabla\Phi(t, y) \le 1 + \varepsilon$ and $1 - \varepsilon \le \det \nabla\Psi(t, x) \le 1 + \varepsilon$ for every $x, y \in \overline{\Omega}$, where ∇ denotes the spatial gradient;*
- (e) *for every $t \in [t_0, t_1]$ we have $|\partial_t \Phi(t, y)| \le \mu(1 + \varepsilon)$ for every $y \in \overline{\Omega}$;*
- (f) *the absolute values of all partial derivatives of Φ and of Ψ of order less than or equal to two, as well as the Lipschitz constants of all second derivatives, are bounded by C;*
- (g) *if $\gamma_k \in \mathcal{G}_{r,L}$ converges to γ uniformly, $s_k \in \mathcal{S}^{reg}_{\mu,M}(t_0, t_1)$ converges to s uniformly, with $s_k(t_1) \le b_{\gamma_k}$ for every k, then the corresponding diffeomorphisms satisfy $\Phi_k(t, x) \to \Phi(t, x)$ for every $t \in [t_0, t_1]$ and every $x \in \overline{\Omega}$.*

Proof. Let $\delta := \widehat{\delta}/\mu$, where $\widehat{\delta}$ is given by Lemma 2.6. Let us fix γ, s, t_0, and t_1 as in the statement, let $s_0 = s(t_0)$ and let $s_1 = s(t_1)$. If $s_0 = s_1$ we take $\Phi(t, y) = y$ for every $t \in [t_0, t_1]$ and every $y \in \overline{\Omega}$. If $s_0 < s_1$ let $\widehat{\Phi}$ and $\widehat{\Psi}$ be the diffeomorphisms provided by Lemma 2.6. For every $t \in [t_0, t_1]$, by (2.11) we have $s(t) \in [s_0, s_1]$, and so we can define $\Phi(t, y) := \widehat{\Phi}(s(t), y)$ and $\Psi(t, x) := \widehat{\Psi}(s(t), x)$, for every $x, y \in \overline{\Omega}$. Properties (a)–(g) of the functions Φ and Ψ follow now from Lemma 2.6. □

3. The wave equation

In our model the displacement satisfies the system of linear elastodynamics out of the crack. In this section we specify the notion of solution to the wave equation in domains with a prescribed time-dependent crack and prove an existence and uniqueness result as well as the continuous dependence of the solutions on the cracks.

Throughout the rest of the paper $T > 0$ is a fixed constant, which determines the time interval $[0, T]$ for the evolution problem, and $\partial_D \Omega$ is a fixed (possibly empty) Borel subset of $\partial\Omega$, where we will prescribe a time-dependent Dirichlet boundary condition. On the complement $\partial_N \Omega := \partial\Omega \setminus \partial_D \Omega$ we will prescribe the traction-free boundary condition. Γ_0 is the initial crack introduced in (2.1) and Ω_0 is defined by $\Omega_0 := \Omega \setminus \Gamma_0$.

Let $\mathbb{M}^{2\times 2}$ be the space of 2×2 real matrices and let $\mathbb{M}^{2\times 2}_{sym}$ be the space of 2×2 real symmetric matrices. For every $F \in \mathbb{M}^{2\times 2}$ the symmetric part F^{sym} of F is defined by $F^{sym} := \frac{1}{2}(F + F^T)$, where F^T is the transpose of F. The space of linear maps from a vector space X into a vector space Y is denoted by $\mathrm{Lin}(X, Y)$.

Throughout the paper λ and Λ are two constants with $0 < \lambda < \Lambda$. The following definition introduces the class of elasticity tensors we are going to consider.

Definition 3.1 *(Elasticity tensors).* $\mathcal{E}(\lambda, \Lambda)$ is the collection of all functions $\mathbb{C} \colon \overline{\Omega} \to \mathrm{Lin}(\mathbb{M}^{2\times 2}, \mathbb{M}^{2\times 2})$ of class C^2 such that for every $x \in \overline{\Omega}$ we have

$$\mathbb{C}(x)F = \mathbb{C}(x)F^{sym} \in \mathbb{M}^{2\times 2}_{sym} \qquad \text{for every } F \in \mathbb{M}^{2\times 2}\,, \tag{3.1}$$

$$\mathbb{C}(x)F\cdot G = \mathbb{C}(x)G\cdot F \qquad \text{for every } F, G \in \mathbb{M}^{2\times 2}\,, \tag{3.2}$$

$$\lambda|F^{sym}|^2 \le \mathbb{C}(x)F\cdot F \le \Lambda|F^{sym}|^2 \qquad \text{for every } F \in \mathbb{M}^{2\times 2}\,. \tag{3.3}$$

Let us fix $\mathbb{C} \in \mathcal{E}(\lambda, \Lambda)$ and $T > 0$. We assume that the body forces f satisfy

$$f \in L^2((0,T); L^2(\Omega;\mathbb{R}^2))\,. \tag{3.4}$$

Given $\gamma \in \mathcal{G}_{r,L}$ and $s \in \mathcal{S}^{piec}_{\mu,M}(0,T)$, with $s(T) \le b_\gamma$, we now consider the wave equation on the time-dependent cracking domains $t \mapsto \Omega \setminus \Gamma_{s(t)}$

$$\ddot{u}(t,x) - \operatorname{div}(\mathbb{C}(x)\nabla u(t,x)) = f(t,x) \qquad \text{for } t \in (0,T) \text{ and } x \in \Omega\setminus\Gamma_{s(t)}\,, \tag{3.5}$$

where $\ddot{u}$ denotes the second partial derivative of u with respect to time, ∇ denotes the spatial gradient, and div denotes the divergence with respect to the space variable, acting here on the rows of the matrix $\mathbb{C}\nabla u$. The equation is complemented with Dirichlet boundary condition on $\partial_D\Omega$

$$u(t,x) = w(t,x) \qquad \text{for } t \in (0,T) \text{ and } x \in \partial_D\Omega\,, \tag{3.6}$$

and homogeneous Neumann boundary condition on $\partial_N\Omega \cup \Gamma_{s(t)}$

$$(\mathbb{C}(x)\nabla u(t,x))\nu(x) = 0 \qquad \text{for } t \in (0,T) \text{ and } x \in \partial_N\Omega\cup\Gamma_{s(t)}\,. \tag{3.7}$$

It is convenient to express the function w used in the Dirichlet boundary condition as the trace on $\partial_D\Omega$ of a function, denoted by the same symbol, satisfying

$$w \in L^2((0,T); H^2(\Omega_0;\mathbb{R}^2)) \cap H^1((0,T); H^1(\Omega_0;\mathbb{R}^2)) \cap H^2((0,T); L^2(\Omega_0;\mathbb{R}^2))\,. \tag{3.8}$$

We also assume that for every $t \in [0,T]$

$$w(t) = 0 \qquad \text{a.e. on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge r)\} \tag{3.9}$$

and that the following integration by parts formula holds

$$-\langle \mathbb{C}\nabla w(t), \nabla\varphi\rangle = \langle \operatorname{div}(\mathbb{C}\nabla w(t)), \varphi\rangle \qquad \text{for every } \varphi \in H^1_D(\Omega,\mathbb{R}^2), \tag{3.10}$$

with $H^1_D(\Omega_0;\mathbb{R}^2) = \{\varphi \in H^1(\Omega_0;\mathbb{R}^2) : \varphi = 0\ \mathcal{H}^1\text{-a.e. on } \partial_D\Omega\}$, where the values of φ on $\partial_D\Omega$ are defined using the trace operator from $H^1(\Omega_0;\mathbb{R}^2)$ to $L^2(\partial\Omega;\mathbb{R}^2)$ and $\mathcal{H}^1$ is the one-dimensional Hausdorff measure (see, e.g., [6, Definition 2.46]). Under suitable regularity assumptions, condition (3.10) holds if $w(t)$ satisfies the homogeneous Neumann boundary condition

$$(\mathbb{C}\nabla w(t))\nu = 0 \qquad \text{on } \partial_N\Omega \cup \Gamma_0\,.$$

To give a precise meaning to the notion of weak solution of the wave equation (3.5) with boundary conditions (3.6) and (3.7) we introduce some additional notation. Given $\gamma \in \mathcal{G}_{r,L}$ and $s \in [0, b_\gamma]$, we set $\Omega^\gamma_s := \Omega\setminus\Gamma_s$ and $H^1_D(\Omega^\gamma_s;\mathbb{R}^2) = \{\varphi \in H^1(\Omega^\gamma_s;\mathbb{R}^2) : \varphi = 0\ \mathcal{H}^1\text{-a.e. on } \partial_D\Omega\}$, where the values of φ on $\partial_D\Omega$ are defined using the trace operator from $H^1(\Omega^\gamma_s;\mathbb{R}^2)$ to $L^2(\partial\Omega;\mathbb{R}^2)$. Note that by property (a) of Definition 2.1 we have $\Omega_0 = \Omega^\gamma_0$ and that (3.9) and (3.10) imply, for every $t \in [0,T]$, the integration by parts formula

$$-\langle \mathbb{C}\nabla w(t), \nabla\varphi\rangle = \langle \mathrm{div}(\mathbb{C}\nabla w(t)), \varphi\rangle \qquad \text{for every } \varphi \in H^1_D(\Omega^\gamma_{s(t)};\mathbb{R}^2). \tag{3.11}$$

Given a function $u \in H^1(\Omega^\gamma_s;\mathbb{R}^2)$ for some $s \in [0, b_\gamma]$, it is convenient to regard its gradient ∇u as an element of $L^2(\Omega;\mathbb{M}^{2\times 2})$, by extending it to 0 on Γ_s. To underline the fact that this extension does not coincide with the distributional gradient of any extension of u, we shall use the notation $\widehat{\nabla} u$.

We now recall the notion of weak solution to this problem.

Definition 3.2 *(Wave equation in cracking domains).* Given $\mathbb{C} \in \mathcal{E}(\lambda, \Lambda)$, $\gamma \in \mathcal{G}_{r,L}$, $0 \le T_0 < T_1 \le T$, and $s \in \mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$ with $s(T_1) \le b_\gamma$, assume that f and w satisfy (3.4), (3.8), (3.9), and (3.10). We say that u is a weak solution of the wave equation (3.5) with boundary conditions (3.6) and (3.7) on the time-dependent cracking domains $t \mapsto \Omega^\gamma_{s(t)}$, $T_0 \le t \le T_1$, if

$$u \in C^1([T_0, T_1]; L^2(\Omega;\mathbb{R}^2)), \tag{3.12}$$

$$u(t) - w(t) \in H^1_D(\Omega^\gamma_{s(t)};\mathbb{R}^2) \quad \text{for every } t \in [T_0, T_1], \tag{3.13}$$

$$\widehat{\nabla} u \in C^0([T_0, T_1]; L^2(\Omega;\mathbb{M}^{2\times 2})), \tag{3.14}$$

$$\dot u \in AC([t, T_1]; H^{-1}_D(\Omega^\gamma_{s(t)};\mathbb{R}^2)) \quad \text{for every } t \in [T_0, T_1), \tag{3.15}$$

$$\frac{1}{h}(\dot u(t+h) - \dot u(t)) \xrightarrow{h\to 0} \ddot u(t) \text{ weakly in } H^{-1}_D(\Omega^\gamma_{s(t)};\mathbb{R}^2) \text{ for a.e. } t \in (T_0, T_1), \tag{3.16}$$

$$t \mapsto \|\ddot u(t)\|_{H^{-1}_D(\Omega^\gamma_{s(t)};\mathbb{R}^2)} \text{ is integrable on } (T_0, T_1), \tag{3.17}$$

and for a.e. $t \in (T_0, T_1)$ satisfies

$$\langle \ddot u(t), \varphi\rangle + \langle \mathbb{C}\nabla u(t), \nabla\varphi\rangle = \langle f(t), \varphi\rangle \quad \text{for every } \varphi \in H^1_D(\Omega^\gamma_{s(t)};\mathbb{R}^2), \tag{3.18}$$

where $\ddot u(t)$ is the element of $H^{-1}_D(\Omega^\gamma_{s(t)};\mathbb{R}^2)$ defined for a.e. $t \in (T_0, T_1)$ by (3.16). Here and in the rest of the paper $\langle\cdot,\cdot\rangle$ denotes the duality product between spaces that are clear from the context. For instance, its first occurrence in (3.18) refers to the duality between $H^{-1}_D(\Omega^\gamma_{s(t)};\mathbb{R}^2)$ and $H^1_D(\Omega^\gamma_{s(t)};\mathbb{R}^2)$, the second one to the duality between $L^2(\Omega;\mathbb{M}^{2\times 2})$ and $L^2(\Omega;\mathbb{M}^{2\times 2})$, while the third one regards the duality between $L^2(\Omega;\mathbb{R}^2)$ and $L^2(\Omega;\mathbb{R}^2)$.

In this paper we consider only the traction-free boundary condition (3.7); the case of a nonhomogeneous Neumann boundary condition on $\partial_N\Omega$ can be obtained under suitable regularity assumptions on the data as in [3].

To obtain an existence and uniqueness result we assume that the constant μ which appears in the Definition 2.7 satisfies

$$0 < \mu < \sqrt{\lambda}/2\,. \tag{3.19}$$

We shall see that the constant $\sqrt{\lambda}$ is related to an estimate on the speed of propagation for the solutions to the wave equation corresponding to $\mathbb{C}$ (see Theorem A.1).

Theorem 3.3 *(Existence and uniqueness). Under the assumptions of Definition 3.2, let $u^0 \in H^1(\Omega^\gamma_{s(T_0)};\mathbb{R}^2)$ and $u^1 \in L^2(\Omega;\mathbb{R}^2)$. Suppose that (3.19) holds and that the compatibility condition*

$$u^0 - w(T_0) \in H^1_D(\Omega^\gamma_{s(T_0)};\mathbb{R}^2)$$

is satisfied. Then there exists a unique weak solution of problem (3.5)–(3.7) on the time-dependent cracking domains $t \mapsto \Omega^\gamma_{s(t)}$, $T_0 \le t \le T_1$, satisfying the initial conditions

$$u(T_0) = u^0 \textit{ and } \dot{u}(T_0) = u^1 \textit{ in } L^2(\Omega;\mathbb{R}^2).$$

The proof is based on an existence and uniqueness result proved in [3, Theorems 3.2 and 3.6]. Unfortunately, these theorems can be applied directly only if μ is very small. In the general case $0 < \mu < \sqrt{\lambda}/2$ we apply them to a localized version of our problem, and show that this is sufficient.

Among the hypotheses of these theorems there is an estimate of the tensor $\mathbb{B}(t,y)$ defined by

$$\mathbb{B}(t,y)F := [\mathbb{C}(x)(F\nabla\Psi(t,x))]\nabla\Psi(t,x)^T - F\dot{\Psi}(t,x)\otimes\dot{\Psi}(t,x) \tag{3.20}$$

with $x = \Phi(t,y)$, where $\Phi(t,\cdot)\colon \Omega_0 \to \Omega^\gamma_{s(t)}$ and $\Psi(t,\cdot)\colon \Omega^\gamma_{s(t)} \to \Omega_0$ are suitable diffeomorphisms (see (3.1) in [3]). To obtain this estimate under the assumption $0 < \mu < \sqrt{\lambda}/2$ we consider a small time interval $[t_0,t_1]$ and use the diffeomorphisms $\Phi(t,\cdot)\colon \Omega^\gamma_{s(t_0)} \to \Omega^\gamma_{s(t)}$ and $\Psi(t,\cdot)\colon \Omega^\gamma_{s(t)} \to \Omega^\gamma_{s(t_0)}$ introduced in Lemma 2.8. The following lemma shows that $\mathbb{B}(t,y)$ satisfies estimate (3.1) in [3] on a suitable ball B_2 for every $t \in [t_0,t_1]$; namely there exist two constants $\alpha > 0$ and $\beta > 0$, independent of the diffeomorphisms, such that

$$\langle \mathbb{B}(t)\nabla v, \nabla v\rangle_{L^2(B_2\setminus\Gamma_{s_0};\mathbb{M}^{2\times 2})} \geq \alpha\|\nabla v\|^2_{L^2(B_2\setminus\Gamma_{s_0};\mathbb{M}^{2\times 2})} - \beta\|v\|^2_{L^2(B_2;\mathbb{R}^2)} \tag{3.21}$$

for every $v \in H^1(B_2 \setminus \Gamma_{s_0};\mathbb{R}^2)$, where $s_0 = s(t_0)$. The proof is based on the results of Lemma 2.8 and on a careful estimate of the constants in the second Korn inequality. In view of the application to the proof of the continuous dependence of the solutions on the cracks, we need an estimate independent of the pair (γ, s) which describes the crack.

Lemma 3.4 *(Estimate for $\mathbb{B}$). Assume that*

$$0 < \mu < \sqrt{\lambda}/2\,. \tag{3.22}$$

Let $\hat{r}$ be the constant introduced in Lemma 2.4, let B_1 and B_2 be two open balls of radii $R_1, R_2 \in (0,\hat{r}/4)$, with $B_1 \subset\subset B_2 \subset\subset \Omega$, and let

$$0 < \rho < R_1/2. \tag{3.23}$$

Then there exist $\alpha > 0$, $\beta > 0$, and $\delta > 0$ with the following property: for every $\gamma \in \mathcal{G}_{r,L}$, for every $t_0 < t_1$, with $t_1 - t_0 \leq \delta$, and for every $s \in \mathcal{S}^{reg}_{\mu,M}(t_0,t_1)$, with $s(t_1) \leq b_\gamma$ and $B(\gamma(s(t_0)),2\rho) \subset\subset B_1$, we can construct two functions $\Phi,\Psi\colon [t_0,t_1]\times\overline{\Omega} \to \overline{\Omega}$ of class $C^{2,1}$ which satisfy properties (a)–(g) of Lemma 2.8 and such that for every $\mathbb{C} \in \mathcal{E}(\lambda,\Lambda)$ the corresponding $\mathbb{B}(t)$, defined by (3.20), satisfies (3.21) in $B_2 \setminus \Gamma_{s_0}$, where $s_0 = s(t_0)$. In addition, we may assume

$$\Phi(t,y) = y \qquad \textit{for every } y \notin B_1 \textit{ and every } t \in [t_0,t_1]\,. \tag{3.24}$$

Proof. Since $\mu^2 < \lambda/4$, we can fix $\varepsilon > 0$ such that

$$\mu^2(1+\varepsilon)^3 < \lambda(1-\varepsilon)\Big(\frac{1}{4} - \varepsilon\Big)\,. \tag{3.25}$$

Let $\delta > 0$ be the constant, depending on $r, L, \mu, M, \varepsilon$, and ρ, provided by Lemma 2.8. Let us fix γ, s, t_0, t_1 as required in the statement of the lemma.

Since $0 \leq \dot{s}(t) \leq \mu$ and $\delta < \rho/\mu$, we have $s(t_0) \leq s(t) \leq s(t_0) + \rho$ for every $t \in [t_0,t_1]$, which implies that $|\gamma(s(t)) - \gamma(s(t_0))| \leq \rho$ by property (b) of Definition 2.1. Since $B(\gamma(s(t_0)),2\rho) \subset\subset B_1$, we conclude that $\gamma(s(t)) \in B_1$ for every $t \in [t_0,t_1]$.

Let us consider the extension $\widehat{\gamma} \in \mathcal{G}_{\hat{r},\hat{L}}$ of γ given in Lemma 2.4. Since $|\widehat{\gamma}'(s_0)| = 1$, from the estimate on the second derivatives (see Remark 2.2), which holds for $\widehat{\gamma}$ with constant $1/\hat{r}$, we obtain $|\widehat{\gamma}(s_0+\hat{r}) - \widehat{\gamma}(s_0)| \geq \frac{1}{2}\hat{r} > 2R_2$. Since $\widehat{\gamma}(s_0) = \gamma(s_0) \in B_2$, we have $\widehat{\gamma}(s_0+\hat{r}) \notin B_2$. On the other hand, we also have $\widehat{\gamma}(a_0) = \gamma(a_0) \notin B_2$. Therefore $\widehat{\Gamma}$ meets ∂B_2 in at least two points. Using the uniform tangent balls condition and the bound on R_2, it follows that $\widehat{\Gamma}$ cannot meet ∂B_2 in more than two points, so that $B_2 \setminus \widehat{\Gamma}$ has two connected components, B_2^+ and B_2^-.

It is then possible to find two connected C^2-domains A^+ and A^- such that $B_2^\pm \cap B_1 \subset A^\pm \subset B_2^\pm$. Therefore, setting $A = A^+ \cup A^-$, for every $t \in [t_0, t_1]$ we have $A^+, A^- \subset B_2 \setminus \hat{\Gamma} \subset B_2 \setminus \Gamma_{s(t)}$ and $B_1 \setminus \Gamma_{s(t)} \subset A \cup \widehat{\Gamma}$. Moreover, the C^2-norms of ∂A^+ and ∂A^- can be bounded uniformly with respect to γ, s, and t_0.

For every $v \in H^1(A;\mathbb{R}^2)$ let $Ev := (\nabla v + \nabla v^T)/2$ be the symmetric part of ∇v. By the second Korn inequality in C^2-domains with optimal constants (see, e.g., [7, Theorem 5.1]), applied separately to A^+ and A^-, we can find a constant $\beta_1 > 0$, independent of γ, s, t_0, such that

$$\int_A |Ev|^2 dx \geq \Big(\frac{1}{4} - \varepsilon\Big) \int_A |\nabla v|^2 dx - \beta_1 \int_A |v|^2 dx \tag{3.26}$$

for every $v \in H^1(A;\mathbb{R}^2)$.

We fix $t \in [t_0, t_1]$ and define $z(x) = v(\Psi(t,x))$. By (a) of Lemma 2.8 we have $\Phi(t,A) = \Psi(t,A) = A$. Since $1-\varepsilon \leq \det \nabla\Psi(t,x) \leq 1+\varepsilon$, and $\partial_t \Psi(t,x) = -\nabla\Psi(t,x)\partial_t\Phi(t,\Psi(t,x))$, by a change of variables, for every $\mathbb{C} \in \mathcal{E}(\lambda,\Lambda)$, we obtain from (3.20) that

$$\begin{aligned}\int_A \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy \geq (1-\varepsilon)&\int_A \mathbb{C}(x)\nabla z(x)\cdot\nabla z(x)dx \\ -(1+\varepsilon)&\int_A |\nabla z(x)\partial_t\Phi(t,\Psi(t,x)))|^2 dx.\end{aligned}$$

Using the ellipticity of $\mathbb{C}$ in (3.3) and the estimate on $\partial_t\Phi$ given in (e) of Lemma 2.8, from (3.26) we get

$$\int_A \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy \geq \alpha_1 \int_A |\nabla z(x)|^2 dx - \beta_1\lambda \int_A |z(x)|^2 dx\,,$$

where $\alpha_1 = (\frac{1}{4}-\varepsilon)\lambda(1-\varepsilon) - (1+\varepsilon)^3\mu^2 > 0$. By another change of variables we obtain that

$$\int_A \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy \geq \alpha_2 \int_A |\nabla v(y)|^2 dy - \beta_2 \int_A |v(y)|^2 dy\,, \tag{3.27}$$

with $\alpha_2 = (1-\varepsilon)\alpha_1/C^2 > 0$ and $\beta_2 = \beta_1\lambda(1+\varepsilon) > 0$, where C is the constant in (f) of Lemma 2.8.

On the other hand, we have $\Phi(t,y) = y$ on $B_2 \setminus B_1$ by (a) of Lemma 2.8, hence (3.3) and (3.20) give

$$\int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy = \int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} \mathbb{C}(y)\nabla v(y)\cdot\nabla v(y)dy \geq \lambda \int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} |Ev(y)|^2 dy\,.$$

Therefore, by the second Korn inequality in domains with piecewise smooth boundary (see, e.g., [8]), there exist constants $\alpha_3 > 0$ and $\beta_3 > 0$, independent of γ, s, t_0, and t_1, such that

$$\int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy \geq \alpha_3 \int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} |\nabla v(y)|^2 dy - \beta_3 \int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} |v(y)|^2 dy\,. \tag{3.28}$$

Let $\alpha = \frac{1}{2}\min\{\alpha_2, \alpha_3\}$ and $\beta = \max\{\beta_2, \beta_3\}$. Then

$$\begin{aligned}2\alpha \int_{B_2\setminus\Gamma_{s_0}} |\nabla v(y)|^2 dy &\le \alpha_2 \int_A |\nabla v(y)|^2 dy + \alpha_3 \int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} |\nabla v(y)|^2 dy \\ &\le \int_A \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy + \int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy \\ &\quad + \beta_2 \int_A |v(y)|^2 dy + \beta_3 \int_{(B_2\setminus B_1)\setminus\Gamma_{s_0}} |v(y)|^2 dy \\ &\le 2 \int_{B_2\setminus\Gamma_{s_0}} \mathbb{B}(t,y)\nabla v(y)\cdot\nabla v(y)dy + 2\beta \int_{B_2\setminus\Gamma_{s_0}} |v(y)|^2 dy\,.\end{aligned}$$

This proves (3.21). □

We now present the main ideas of the proof of Theorem 3.3. We consider a small constant $\rho > 0$, two small concentric open balls $B_1 \subset\subset B_2 \subset\subset \Omega$, and a small time interval $[t_0, t_1] \subset [T_0, T_1]$ such that $B(\gamma(s(t)), 2\rho) \subset B_1$ for every $t \in [t_0, t_1]$. Supposing that the solution exists and is unique in $[T_0, t_0]$, to extend the solution to $[t_0, t_1]$ we localize the problem to B_2, i.e., we consider the solution of the wave equation in the cracking domains $t \mapsto B_2 \setminus \Gamma_{s(t)}$ and in the time interval $[t_0, t_1]$. Thanks to Lemma 3.4, if $t_1 - t_0$ is sufficiently small we can apply the results of [3] and we find a unique weak solution u^{int} which satisfies the homogeneous Neumann condition on $\partial(B_2 \setminus \Gamma_{s(t)})$ and the initial conditions $u^{int}(t_0) = u(t_0)$ and $\dot{u}^{int}(t_0) = \dot{u}(t_0)$ in $B_2 \setminus \Gamma_{s(t_0)}$.

Similarly, noticing that $\Omega \setminus (\overline{B}_1 \cup \Gamma_{s(t)}) = \Omega \setminus (\overline{B}_1 \cup \Gamma_{s(t_0)})$ for every $t \in [t_0, t_1]$, we consider the wave equation in the time-independent cracked domain $\Omega \setminus (\overline{B}_1 \cup \Gamma_{s(t_0)})$ and in the time interval $[t_0, t_1]$. We find a unique weak solution u^{ext} which satisfies the Dirichlet boundary condition (3.6) on $\partial_D\Omega$, the homogeneous Neumann condition on the rest of the boundary of $\Omega \setminus (\overline{B}_1 \cup \Gamma_{s(t_0)})$, and the initial conditions $u^{ext}(t_0) = u^0$ and $\dot{u}^{ext}(t_0) = u^1$ in $\Omega \setminus (\overline{B}_1 \cup \Gamma_{s(t_0)})$.

Thanks to the finite speed of propagation (see Theorem A.1) we find two balls $\widehat{B}_1$ and $\widehat{B}_2$, with $B_1 \subset\subset \widehat{B}_1 \subset\subset \widehat{B}_2 \subset\subset B_2$, such that $u^{int}(t) = u^{ext}(t)$ in $\widehat{B}_2 \setminus \widehat{B}_1$ for every $t \in [t_0, t_1]$. This shows that the function

$$u(t) = \begin{cases} u^{ext}(t) & \text{in } \Omega \setminus \widehat{B}_1, \\ u^{int}(t) & \text{in } \widehat{B}_2, \end{cases}$$

is well defined and provides a weak solution of the wave equation in the cracking domain $t \mapsto \Omega \setminus \Gamma_{s(t)}$ for $t \in [t_0, t_1]$. Moreover, the uniqueness of u^{int} and u^{ext} leads to the uniqueness of the solution u for $t \in [t_0, t_1]$.

Since Lemma 3.4 ensures that the same argument can be repeated when $t_1 - t_0$ is less than a constant depending only on ρ, B_1, and B_2, existence and uniqueness hold for all times t such that $B(\gamma(s(t)), 2\rho) \subset B_1$. To complete the proof in the global time interval $[T_0, T_1]$ it is enough to consider a finite number of carefully chosen triples (ρ, B_1, B_2).

Proof of Theorem 3.3. Since $s \in \mathcal{S}^{piec}_{\mu,M}(T_0, T_1)$, there exists a finite subdivision $T_0 = \tau_0 < \tau_1 < \cdots < \tau_k = T_1$ for which $s|_{[\tau_{j-1},\tau_j]} \in \mathcal{S}^{reg}_{\mu,M}(\tau_{j-1}, \tau_j)$. It is enough to prove the result in each subinterval $[\tau_{j-1}, \tau_j]$, therefore it is not restrictive to assume that $s \in \mathcal{S}^{reg}_{\mu,M}(T_0, T_1)$.

Let us fix $0 < \rho < \hat{r}/64$ and $\eta \in (4\rho/\mu, 5\rho/\mu)$. Without loss of generality we assume that $T_1 \le T_0 + \eta$. Indeed, the result in the general case can be obtained by repeating the same arguments on $[T_0+\eta, T_0+2\eta]$, $[T_0 + 2\eta, T_0 + 3\eta]$, and so on.

We set $B_1 := B(\gamma(s(T_0)), 8\rho)$ and $B_2 := B(\gamma(s(T_0)), 16\rho)$. We note that $B_2 \subset\subset \Omega$ by property (d) in Definition 2.1. Moreover, since $|\gamma'(s(t))| = 1$, $0 \le \dot{s}(t) \le \mu$, $T_1 \le T_0 + \eta$, and $\mu\eta < 5\rho$, we have also

$$B(\gamma(s(t)), 3\rho) \subset B_1 \quad \text{for every } t \in [T_0, T_1]\,. \tag{3.29}$$

Let $\widehat{\gamma}$ be the extension of γ given by Lemma 2.4. Arguing as in the proof of Lemma 3.4 we obtain that $\widehat{\gamma}(s(T_0) + \hat{r}) \notin \overline{B}_2$. Since $\widehat{\gamma}(a_0) = \gamma(a_0) \notin \overline{B}_2$, the manifold $\widehat{\Gamma} = \widehat{\gamma}([a_0, b_\gamma + \hat{r}])$ meets ∂B_1 and ∂B_2. Since the radii of B_1 and B_2 are sufficiently small, the tangent balls condition implies that $\widehat{\Gamma}$ is transversal to ∂B_1 and ∂B_2. Hence conditions (H3) and (H4) of [3] are satisfied with Ω and Γ replaced by B_2 and $\widehat{\Gamma}$.

Let $\alpha, \beta, \delta > 0$ be the constants given by Lemma 3.4 corresponding to our choice of B_1, B_2, and ρ, and let

$$\delta^* = \min\left\{\delta, 4\rho/\sqrt{\Lambda}\right\}. \tag{3.30}$$

We apply Lemma 3.4 with $t_0 = T_0$ and $t_1 = \min\{t_0 + \delta^*, T_1\}$ and we obtain that there exist functions $\Phi, \Psi \colon [t_0, t_1] \times \overline{\Omega} \to \overline{\Omega}$ of class $C^{2,1}$ which satisfy properties (a)–(f) of Lemma 2.8 and such that the corresponding $\mathbb{B}(t)$ satisfies (3.21) in $B_2 \setminus \Gamma_{s(t_0)}$. In addition, we can suppose that

$$\Phi(t, y) = y \quad \text{for every } y \notin B_1 \text{ and } t \in [t_0, t_1]\,. \tag{3.31}$$

It is easy to check that the diffeomorphisms $\Phi(t, \cdot)$ and $\Psi(t, \cdot)$ satisfy all hypotheses of the existence and uniqueness results [3, Theorems 3.2 and 3.6] in the cracking domains $t \mapsto B_2 \setminus \Gamma_{s(t)}$ and in the time interval $[t_0, t_1]$. Therefore, the boundary value problem for the wave equation (3.5)–(3.7), with Ω replaced by B_2, $\partial_D\Omega$ replaced by $\emptyset$, and $\partial_N\Omega$ replaced by ∂B_2, has a unique weak solution u^{int} which satisfies the initial conditions $u^{int}(t_0) = u^0$ and $\dot{u}^{int}(t_0) = u^1$ in $L^2(B_2; \mathbb{R}^2)$.

Applying the same results of [3] to the set $\Omega \setminus \overline{B}_1$ we find that the same problem, with Ω replaced by $\Omega \setminus \overline{B}_1$ and $\partial_N\Omega$ replaced by $\partial_N\Omega \cup \partial B_1$, has a unique weak solution u^{ext} which satisfies the initial conditions $u^{ext}(t_0) = u^0$ and $\dot{u}^{ext}(t_0) = u^1$ in $L^2(\Omega \setminus \overline{B}_1; \mathbb{R}^2)$ and the Dirichlet boundary condition $u^{ext}(t) = w(t)$ on $\partial_D\Omega$.

Note that by (3.29) we have $(B_2 \setminus \overline{B}_1) \setminus \Gamma_{s(t)} = (B_2 \setminus \overline{B}_1) \setminus \Gamma_{s(t_0)}$ for every $t \in [t_0, t_1]$. We now apply the result on the finite speed of propagation (see Theorem A.1) to the function $u^{ext} - u^{int}$, with $U = (B_2 \setminus \overline{B}_1) \setminus \Gamma_{s(t_0)}$, $S_0 = \emptyset$, and $S_1 = (\partial B_2 \cup \partial B_1) \setminus \Gamma_{s(t_0)}$. We obtain that for every $t \in [t_0, t_1]$ we have $u^{ext}(t) - u^{int}(t) = 0$ a.e. in $(\widehat{B}_2 \setminus \widehat{B}_1) \setminus \Gamma_{s(t)} = (\widehat{B}_2 \setminus \widehat{B}_1) \setminus \Gamma_{s(t_0)}$, where $\widehat{B}_1$ and $\widehat{B}_2$ are the balls concentric to B_1 and B_2 with radius $8\rho + \delta^*\sqrt{\Lambda}$ and $16\rho - \delta^*\sqrt{\Lambda}$ respectively. Since $\widehat{B}_1 \subset\subset \widehat{B}_2$ by (3.30), the function

$$u(t) = \begin{cases} u^{ext}(t) & \text{in } \Omega \setminus \widehat{B}_1, \\ u^{int}(t) & \text{in } \widehat{B}_2, \end{cases} \tag{3.32}$$

is well defined and provides a weak solution of the boundary value problem (3.5)–(3.7) for the wave equation on the cracking domains $t \mapsto \Omega^\gamma_{s(t)}$ for $t \in [t_0, t_1]$, according to Definition 3.2, with initial conditions $u(t_0) = u^0$ and $\dot{u}(t_0) = u^1$ in $L^2(\Omega; \mathbb{R}^2)$.

To prove the uniqueness of this solution on this time interval, by difference we can consider the case when u_0, u_1, and $w(t)$ are identically zero, and we call $v(t)$ a solution of the corresponding problem. We apply the result on the finite speed of propagation (Theorem A.1) with $U = \Omega \setminus \overline{B}_1$, $S_0 = \partial_D\Omega$, and $S_1 = \partial B_1 \cup \partial_D\Omega$ and we obtain that $v(t) = 0$ a.e. in $\Omega \setminus \widehat{B}_1$, for every $t \in [t_0, t_1]$. In particular $v(t)$ vanishes in a neighborhood of ∂B_2.

Now we apply the uniqueness results [3, Theorems 3.2 and 3.6] to the cracking domains $t \mapsto B_2 \setminus \Gamma_{s(t)}$ with the Dirichlet boundary condition $v(t) = 0$ on ∂B_2, and we obtain that $v(t) = 0$ a.e. in $B_2 \setminus \Gamma_{s(t)}$ for every $t \in [t_0, t_1]$. Since $(\Omega \setminus \widehat{B}_1) \cup B_2 = \Omega$ we obtain that $v(t) = 0$ a.e. in Ω for every $t \in [t_0, t_1]$, which proves uniqueness in this time interval.

If $t_1 = T_1$, the proof of existence and uniqueness in $[T_0, T_1]$ is concluded. Otherwise we can repeat the same arguments with the same B_1, B_2, and ρ, with t_0 replaced by t_1, with t_1 replaced by $t_2 := \min\{t_1 + \delta^*, T_1\}$, and with initial data $u(t_1)$ and $\dot{u}(t_1)$. Lemma 3.4 can be applied again because of (3.29). To prove existence and uniqueness in the time interval $[t_1, t_2]$ we have to check that $u(t_1)$ and $\dot{u}(t_1)$ are well defined, which is given by (3.12) in $[t_0, t_1]$, and that $u(t_1)$ satisfies the compatibility condition $u(t_1) - w(t_1) \in H^1_D(\Omega^\gamma_{s(t_1)}; \mathbb{R}^2)$, which is a consequence of (3.13) in $[t_0, t_1]$. Therefore we obtain existence and uniqueness in $[t_1, t_2]$. Since in this argument we always apply Lemma 3.4 with the same B_1, B_2, and ρ, the constant δ^* does not change. Hence, iterating this process, after a finite number of steps we obtain existence and uniqueness in $[T_0, T_1]$. □

We are now ready to prove the continuous dependence of the solutions on the cracks.

Theorem 3.5 *(Continuous dependence). Suppose that $0 < \mu < \sqrt{\lambda}/2$. Let $\mathbb{C} \in \mathcal{E}(\lambda, \Lambda)$, $\gamma_k, \gamma \in \mathcal{G}_{r,L}$, $0 \leq T_0 < T_1 \leq T$, $s_k, s \in \mathcal{S}^{reg}_{\mu,M}(T_0, T_1)$, $u^0_k \in H^1(\Omega^{\gamma_k}_{s_k(T_0)}; \mathbb{R}^2)$, $u^0 \in H^1(\Omega^\gamma_{s(T_0)}; \mathbb{R}^2)$, and $u^1_k, u^1 \in L^2(\Omega; \mathbb{R}^2)$. Assume that*

$$s_k \to s \textit{ uniformly}, \tag{3.33}$$

$$\gamma_k \to \gamma \textit{ uniformly}, \tag{3.34}$$

$$s_k(T_1) \leq b_{\gamma_k} \textit{ for every } k, \tag{3.35}$$

$$u^0_k \to u^0 \textit{ strongly in } L^2(\Omega; \mathbb{R}^2), \tag{3.36}$$

$$\widehat{\nabla} u^0_k \to \widehat{\nabla} u^0 \textit{ strongly in } L^2(\Omega; \mathbb{M}^{2\times 2}), \tag{3.37}$$

$$u^1_k \to u^1 \textit{ strongly in } L^2(\Omega; \mathbb{R}^2). \tag{3.38}$$

For $T_0 \leq t \leq T_1$ let $t \mapsto u_k(t)$ and $t \mapsto u(t)$ be the weak solutions of problems (3.5)-(3.7) on the time-dependent cracking domains $t \mapsto \Omega^{\gamma_k}_{s_k(t)}$ and $t \mapsto \Omega^\gamma_{s(t)}$ respectively, satisfying the initial conditions

$$u_k(T_0) = u^0_k, \; \dot{u}_k(T_0) = u^1_k \textit{ and } u(T_0) = u^0, \; \dot{u}(T_0) = u^1 \textit{ respectively}.$$

Then

$$u_k(t, \cdot) \to u(t, \cdot) \textit{ strongly in } L^2(\Omega; \mathbb{R}^2), \tag{3.39}$$

$$\widehat{\nabla} u_k(t, \cdot) \to \widehat{\nabla} u(t, \cdot) \textit{ strongly in } L^2(\Omega; \mathbb{M}^{2\times 2}), \tag{3.40}$$

$$\dot{u}_k(t, \cdot) \to \dot{u}(t, \cdot) \textit{ strongly in } L^2(\Omega; \mathbb{R}^2), \tag{3.41}$$

for every $t \in [T_0, T_1]$.

As in the proof of Theorem 3.3, on a small time interval $[t_0, t_1]$ we consider local problems in the time-dependent cracking domains $t \mapsto B_2 \setminus \Gamma^k_{s_k(t)}$, where B_2 is a suitable small ball. The continuous dependence results of [3] cannot be applied directly, since one of the hypotheses of [3, Theorem 4.1] is that all cracks have a common initial part. This condition is satisfied for the global problem in Ω, but not for the problems localized to B_2. To overcome this difficulty we have to consider a sequence of diffeomorphisms ω_k which map $\overline{\Omega}$ onto $\overline{\Omega}$, B_2 onto B_2, and the image of (an extension of) γ onto the image of (an extension of) γ_k. Then we consider the problem satisfied by $v^{int}_k(t,x) := u^{int}_k(t, \omega_k(x))$ and $v^{ext}_k(t,x) := u^{ext}_k(t, \omega_k(x))$, where u^{int}_k and u^{ext}_k are defined as in the proof of Theorem 3.3. The crucial point in the proof of Theorem 3.5

is the convergence of v_k^{int} to u^{int} and of v_k^{ext} to u^{ext}, which are obtained by using a slight modification of [3, Theorem 4.1].

Proof of Theorem 3.5. Let us fix ρ and η as at the beginning of the proof of Theorem 3.3. Without loss of generality we assume that $T_1 \leq T_0 + \eta$. Let B_1 and B_2 be as in the proof of Theorem 3.3, let $\widehat{\gamma}_k$ and $\widehat{\gamma}$ be the extensions of γ_k and γ provided by Lemma 2.4, and let $\widehat{\Gamma}_k$ and $\widehat{\Gamma}$ be the corresponding images.

Since $\gamma_k(s_k(t)) \to \gamma(s(t))$ uniformly in $[T_0, T_1]$, it is not restrictive to assume that $|\gamma_k(s_k(t)) - \gamma(s(t))| < \rho$, hence (3.29) implies that

$$B(\gamma_k(s_k(t)), 2\rho) \subset\subset B_1 \quad \text{for every } k \text{ and for every } t \in [T_0, T_1]\,. \tag{3.42}$$

As in the proof of Lemma 3.4 we obtain that $\widehat{\gamma}_k(s(T_0)+\hat{r}) \notin \overline{B}_2$. Since we have also $\widehat{\gamma}_k(a_0) = \gamma_k(a_0) \notin \overline{B}_2$, there exist s_k^0, s_k^1, with $a_0 < s_k^0 < s_k(T_0) < s_k^1 < s_k(T_0) + \hat{r}$ such that $\widehat{\gamma}_k(s_k^i) \in \partial B_2$ for $i = 0, 1$. By the uniform tangent balls condition it is easy to see that s_k^0 and s_k^1 are uniquely determined, hence $\widehat{\gamma}_k(s) \in B_2$ for every $s \in (s_k^0, s_k^1)$ and $\widehat{\Gamma}^k \cap \overline{B}_2 = \widehat{\gamma}_k([s_k^0, s_k^1])$. Similarly, there exist s^0 and s^1, with $a_0 < s^0 < s(T_0) < s^1 < s(T_0) + \hat{r}$, such that $\widehat{\gamma}(s^i) \in \partial B_2$ for $i = 0, 1$, $\widehat{\gamma}(s) \in B_2$ for every $s \in (s^0, s^1)$, and $\widehat{\Gamma} \cap \overline{B}_2 = \widehat{\gamma}([s^0, s^1])$.

Since the radii of the balls B_1 and B_2 are sufficiently small with respect to $\hat{r}$, the uniform tangent balls condition implies that $\widehat{\Gamma}^k$ meets ∂B_1 and ∂B_2 transversally. Hence for every $t_0 \in [T_0, T_1]$ and for every k we can construct a diffeomorphism $\omega_k : \overline{\Omega} \to \overline{\Omega}$ of class $C^{3,1}$ such that

$$\omega_k(x) = x \quad \text{for } x \text{ in a neighborhood of } \partial\Omega\,, \tag{3.43}$$

$$\omega_k(\widehat{\Gamma}) = \widehat{\Gamma}^k, \quad \omega_k(B_1) = B_1\,, \quad \omega_k(B_2) = B_2\,, \tag{3.44}$$

$$\omega_k(\Gamma_{s(T_0)}) = \Gamma^k_{s_k(T_0)} \quad \text{and} \quad \omega_k(\gamma(s(T_0))) = \gamma_k(s_k(T_0))\,. \tag{3.45}$$

By (3.42) for every $t \in [T_0, T_1]$ we have $\gamma_k(s_k(t)) \in \widehat{\Gamma}_k \cap B_2$, hence $\omega_k^{-1}(\gamma_k(s_k(t))) \in \widehat{\Gamma} \cap B_2$. This implies that there exists a unique $\tilde{s}_k(t) \in [s^0, s^1]$ such that $\gamma(\tilde{s}_k(t)) = \omega_k^{-1}(\gamma_k(s_k(t)))$. The regularity assumptions on γ, γ_k, s_k, and ω_k imply that $\tilde{s}_k$ is of class $C^{3,1}$. Note that (3.44) implies that $\tilde{s}_k(T_0) = s(T_0)$.

Moreover, since $\gamma_k \to \gamma$ and $s_k \to s$ uniformly, taking into account the bounds on the derivatives contained in Definitions 2.1 and 2.7, we may assume that

$$\omega_k \to id \quad \text{and} \quad \omega_k^{-1} \to id \quad \text{in } C^3(\overline{\Omega}; \mathbb{R}^2)\,, \tag{3.46}$$

$$\tilde{s}_k \to s \text{ in } C^3([s^0, s^1]), \tag{3.47}$$

and that there exists a constant $\widetilde{L}$, independent of T_0 and k, such that the third derivatives of the components of ω_k are Lipschitz continuous with Lipschitz constant less than $\widetilde{L}$.

We now choose $\varepsilon \in (0, \lambda)$ and $\mu_0 > 0$ such that

$$\mu < \mu_0 < \min\left\{\frac{5\rho}{\eta}, \frac{\sqrt{\lambda - \varepsilon}}{2}\right\}. \tag{3.48}$$

Using (3.46), (3.47), and the bounds on the derivatives of γ_k, s_k, and ω_k (see Definitions 2.1 and 2.7, and the remark after (3.47)) we can prove that there exists a constant $M_0 > M$ such that $\tilde{s}_k \in \mathcal{S}^{reg}_{\mu_0, M_0}(T_0, T_1)$ for k large enough.

Let $\alpha, \beta, \delta > 0$ be the constants given by Lemma 3.4 applied with our choice of ρ, B_1, and B_2, and with μ, M, λ, and Λ replaced by μ_0, M_0, $\lambda - \varepsilon$, and $\Lambda + \varepsilon$, respectively. Furthermore, let

$$\delta^* = \min\left\{\delta, \frac{4\rho}{\sqrt{\Lambda + \varepsilon}}\right\}. \tag{3.49}$$

We now choose $t_0 = T_0$ and $t_1 = \min\{t_0 + \delta^*, T_1\}$.

Let u^{int} and u^{ext} be defined as in the proof of Theorem 3.3. We also consider the boundary value problem for the wave equation (3.5)–(3.7), with Ω replaced by B_2, $\partial_D\Omega$ replaced by $\emptyset$, $\partial_N\Omega$ replaced by ∂B_2, and γ replaced by γ_k. Let u_k^{int} be the unique solution of this problem (see Theorem 3.3) with initial conditions $u_k^{int}(t_0) = u_k^0$ and $\dot{u}_k^{int}(t_0) = u_k^1$ in $L^2(B_2;\mathbb{R}^2)$. Moreover, we consider the same problem with Ω replaced by $\Omega \setminus \overline{B}_1$ and $\partial_N\Omega$ replaced by $\partial_N\Omega \cup \partial B_1$, and with γ replaced by γ_k. Let u_k^{ext} be its unique weak solution satisfying the initial conditions $u_k^{ext}(t_0) = u_k^0$ and $\dot{u}_k^{ext}(t_0) = u_k^1$ in $L^2(\Omega \setminus B_1;\mathbb{R}^2)$ and the Dirichlet boundary condition $u_k^{ext}(t) = w(t)$ on $\partial_D\Omega$.

Note that, since $\delta^* \leq \delta \leq \rho/\mu$ and $0 \leq \dot{s}_k(t) \leq \mu$, we have $\gamma_k(s_k(t)) \in B(\gamma_k(s_k(t_0)), 2\rho) \subset B_1$ for every $t \in [t_0, t_1]$, hence $(B_2 \setminus \overline{B}_1) \setminus \Gamma^k_{s_k(t)} = (B_2 \setminus \overline{B}_1) \setminus \Gamma^k_{s_k(t_0)}$.

We now apply the result on the finite speed of propagation (see Theorem A.1) to the function $u_k^{ext} - u_k^{int}$, with $U_k = (B_2 \setminus \overline{B}_1) \setminus \Gamma^k_{s_k(t_0)}$, $S_0 = \emptyset$, and $S_1 = \partial B_2 \cup \partial B_1$. We obtain that for every $t \in [t_0, t_1]$ we have $u_k^{ext}(t) - u_k^{int}(t) = 0$ a.e. in $(\widehat{B}_2 \setminus \widehat{B}_1) \setminus \Gamma^k_{s_k(t)}$ where $\widehat{B}_1$ and $\widehat{B}_2$ are the balls concentric to B_1 and B_2 with radii $8\rho + \delta^*\sqrt{\Lambda}$ and $16\rho - \delta^*\sqrt{\Lambda}$ respectively. Since $\widehat{B}_1 \subset\subset \widehat{B}_2$ by (3.49), the function

$$u_k^*(t) = \begin{cases} u_k^{ext}(t) & \text{in } \Omega \setminus \widehat{B}_1, \\ u_k^{int}(t) & \text{in } \widehat{B}_2, \end{cases}$$

is well defined and provides a weak solution of the boundary value problem (3.5)–(3.7) for the wave equation on the cracking domains $t \mapsto \Omega^{\gamma_k}_{s_k(t)}$ for $t \in [t_0, t_1]$, with initial conditions $u_k^*(t_0) = u_k^0$ and $\dot{u}_k^*(t_0) = u_k^1$ in $L^2(\Omega;\mathbb{R}^2)$. By uniqueness (see Theorem 3.3) we have

$$u_k(t) = \begin{cases} u_k^{ext}(t) & \text{in } \Omega \setminus \widehat{B}_1, \\ u_k^{int}(t) & \text{in } \widehat{B}_2. \end{cases} \tag{3.50}$$

We now want to prove that $u_k^{int}(t) \to u^{int}(t)$ and $u_k^{ext}(t) \to u^{ext}(t)$ for every $t \in [t_0, t_1]$. To this aim we introduce the function $v_k^{int}(t,x) := u_k^{int}(t, \omega_k(x))$. To write the equation satisfied by v_k^{int}, for every $x \in \overline{\Omega}$ we define $\mathbb{C}_k(x) \in \mathrm{Lin}(\mathbb{M}^{2\times2}, \mathbb{M}^{2\times2})$, $a_k(x) \in \mathrm{Lin}(\mathbb{M}^{2\times2}, \mathbb{R}^2)$, and $f_k(x) \in \mathbb{R}^2$ and imposing, for $y = \omega_k(x)$, the equalities

$$\mathbb{C}_k(x)F \cdot G = \mathbb{C}(y)[F\nabla\omega_k^{-1}(y)] \cdot [G\nabla\omega_k^{-1}(y)]\,, \tag{3.51}$$

$$a_k(x)F \cdot \zeta = \mathbb{C}(y)[F\nabla\omega_k^{-1}(y)] \cdot [\zeta \otimes \nabla(\log(\det\nabla\omega_k^{-1}))(y)]\,, \tag{3.52}$$

$$f_k(x) = f(y)\,, \tag{3.53}$$

for every $F, G \in \mathbb{M}^{2\times2}$ and every $\zeta \in \mathbb{R}^2$. By a change of variables we see that

$$\langle \ddot{v}_k^{int}(t), \varphi\rangle + \langle \mathbb{C}_k \nabla v_k^{int}(t), \nabla\varphi\rangle + \langle a_k \nabla v_k^{int}(t), \varphi\rangle = \langle f_k, \varphi\rangle \tag{3.54}$$

for every $\varphi \in H^1(B_2 \setminus \Gamma_{\tilde{s}_k(t)};\mathbb{R}^2)$. By (3.46) we have

$$\mathbb{C}_k \to \mathbb{C} \quad \text{in } C^2(\overline{\Omega}; \mathrm{Lin}(\mathbb{M}^{2\times2}, \mathbb{M}^{2\times2})), \tag{3.55}$$

$$a_k \to 0 \quad \text{in } C^1(\overline{\Omega}; \mathrm{Lin}(\mathbb{M}^{2\times2}, \mathbb{R}^2)), \tag{3.56}$$

$$f_k \to f \quad \text{in } L^2(\overline{\Omega}; \mathbb{R}^2)\,. \tag{3.57}$$

This implies that for every $\varepsilon > 0$ there exists k_ε such that for $k \geq k_\varepsilon$ and every $x \in \overline{\Omega}$

$$(\lambda - \varepsilon)|F^{sym}|^2 \leq \mathbb{C}_k(x)F \cdot F \leq (\Lambda + \varepsilon)|F^{sym}|^2 \qquad \text{for every } F \in \mathbb{M}^{2\times2}\,. \tag{3.58}$$

We now apply Lemma 3.4 with our choice of ρ, B_1, and B_2, and with μ, M, λ, and Λ replaced by μ_0, M_0, $\lambda-\varepsilon$, $\Lambda+\varepsilon$, respectively. Therefore, we can associate to γ and $\tilde{s}_k$ two functions $\Phi_k, \Psi_k\colon [t_0,t_1]\times\overline{B}_2 \to \overline{B}_2$ of class $C^{2,1}$ which satisfy properties (a)–(g) of Lemma 2.8 and such that the tensors $\mathbb{B}_k(t)$ corresponding to these functions and to $\mathbb{C}_k$ satisfy (3.21) in $B_2\backslash\Gamma_{s(t_0)}$, with constants α and β independent of k. Moreover,

$$\Phi_k(t,y) = y \quad \text{for every } y\notin B_1 \text{ and } t\in[t_0,t_1]\,. \tag{3.59}$$

A slight modification of [3, Theorem 4.1], due to the presence of the term a_k, yields

$$\begin{gathered}
v_k^{int}(t,\cdot)\to u^{int}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{R}^2)\,,\\
\widehat{\nabla} v_k^{int}(t,\cdot)\to \widehat{\nabla} u^{int}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{M}^{2\times2})\,,\\
\dot v_k^{int}(t,\cdot)\to \dot u^{int}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{R}^2)\,,
\end{gathered}$$

for every $t\in[t_0,t_1]$. Since $u_k^{int}(t,x) := v_k^{int}(t,\omega_k^{-1}(x))$, by (3.46) we have

$$u_k^{int}(t,\cdot)\to u^{int}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{R}^2)\,, \tag{3.60}$$

$$\widehat{\nabla} u_k^{int}(t,\cdot)\to \widehat{\nabla} u^{int}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{M}^{2\times2})\,, \tag{3.61}$$

$$\dot u_k^{int}(t,\cdot)\to \dot u^{int}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{R}^2)\,, \tag{3.62}$$

for every $t\in[t_0,t_1]$.

We now set $v_k^{ext}(t,x) := u_k^{ext}(t,\omega_k(x))$ for every $t\in[t_0,t_1]$. By (3.44) and (3.45) we have $v_k^{ext}(t)\in H^1((\Omega\setminus\overline{B}_1)\setminus\Gamma_{s(t_0)};\mathbb{R}^2)$. By the same change of variables considered for v_k^{int} we see that

$$\langle \ddot v_k^{ext}(t),\varphi\rangle + \langle \mathbb{C}_k\nabla v_k^{ext}(t),\nabla\varphi\rangle + \langle a_k\nabla v_k^{ext}(t),\varphi\rangle = \langle f_k,\varphi\rangle \tag{3.63}$$

for every $\varphi\in H^1((\Omega\backslash\overline{B}_1)\backslash\Gamma_{s(t_0)};\mathbb{R}^2)$ with $\varphi=0$ on $\partial_D\Omega$, where $\mathbb{C}_k$, a_k, and f_k are given by (3.51)–(3.53).

We can now apply [3, Theorem 4.1] with a sequence of elasticity tensors and with a time-independent crack, so that all diffeomorphisms considered there are the identity maps. A slight modification of this theorem, due to the presence of the term a_k, implies that

$$\begin{gathered}
v_k^{ext}(t,\cdot)\to u^{ext}(t,\cdot) \text{ strongly in } H^1((\Omega\setminus B_1)\setminus\Gamma_{s(t_0)};\mathbb{R}^2)\,,\\
\dot v_k^{ext}(t,\cdot)\to \dot u^{ext}(t,\cdot) \text{ strongly in } L^2(\Omega\setminus\overline{B}_1;\mathbb{R}^2)\,,
\end{gathered}$$

for every $t\in[t_0,t_1]$. Since $u_k^{ext}(t,x) := v_k^{ext}(t,\omega_k^{-1}(x))$, by (3.46) we have

$$u_k^{ext}(t,\cdot)\to u^{ext}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{R}^2)\,, \tag{3.64}$$

$$\widehat{\nabla} u_k^{ext}(t,\cdot)\to \widehat{\nabla} u^{ext}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{M}^{2\times2})\,, \tag{3.65}$$

$$\dot u_k^{ext}(t,\cdot)\to \dot u^{ext}(t,\cdot) \text{ strongly in } L^2(B_2;\mathbb{R}^2)\,, \tag{3.66}$$

for every $t\in[t_0,t_1]$. By (3.32), (3.50), (3.60)–(3.62), and (3.64)–(3.66) we conclude that (3.39)–(3.41) hold for every $t\in[t_0,t_1]$. To obtain the result for every $t\in[T_0,T_1]$ we argue as in the final part of the proof of Theorem 3.3. □

4. Energy balance

In this section we consider the issue of the dynamic energy-dissipation balance on $[T_0, T_1]$, which plays an important role in our model: the sum of the kinetic energy and of the elastic energy at time T_1, plus the energy dissipated by the crack between time T_0 and time T_1, is equal to the initial energy at time T_0 plus the total work done between time T_0 and time T_1. We are here in a situation similar to that considered in [1, Section 3].

The sum of the elastic and kinetic energies of a solution u at time t is given by

$$\mathcal{E}(\widehat{\nabla}u(t), \dot{u}(t)) := \frac{1}{2}\langle \mathbb{C}\widehat{\nabla}u(t), \widehat{\nabla}u(t)\rangle + \frac{1}{2}\|\dot{u}(t)\|^2 . \tag{4.1}$$

The work of the external forces on the solution u over a time interval $[t_1, t_2] \subset [T_0, T_1]$ is given by

$$\mathcal{W}_{load}(u; t_1, t_2) := \int_{t_1}^{t_2} \langle f(t), \dot{u}(t)\rangle dt , \tag{4.2}$$

which is well defined by (3.4) and (3.12).

As explained in [1, Proposition 3.1] it is convenient to express the work $\mathcal{W}_{bdry}(u; t_1, t_2)$ due to the time-dependent boundary conditions w in the form

$$\begin{gathered}\mathcal{W}_{bdry}(u; t_1, t_2) := \langle \dot{u}(t_2), \dot{w}(t_2)\rangle - \langle \dot{u}(t_1), \dot{w}(t_1)\rangle \\ - \int_{t_1}^{t_2} \langle \ddot{w}(t), \dot{u}(t)\rangle dt - \int_{t_1}^{t_2} \langle f(t), \dot{w}(t)\rangle dt + \int_{t_1}^{t_2} \langle \mathbb{C}\widehat{\nabla}u(t), \nabla\dot{w}(t)\rangle dt ,\end{gathered} \tag{4.3}$$

which has good continuity properties with respect to u.

The total work on the solution u over a time interval $[t_1, t_2] \subset [T_0, T_1]$ is defined by

$$\mathcal{W}(u; t_1, t_2) := \mathcal{W}_{load}(u; t_1, t_2) + \mathcal{W}_{bdry}(u; t_1, t_2) .$$

According to Griffith's theory (see [9]), the energy dissipated by the crack in the interval $[t_1, t_2]$ is proportional to the length of the crack produced in the same interval, since we are assuming that the toughness of the material is homogeneous and isotropic. For simplicity it is assumed that the proportionality constant is one, hence the energy dissipated is given by $s(t_2) - s(t_1)$.

Definition 4.1 *(Cracks satisfying the energy-dissipation balance).* Assume that $0 < \mu < \sqrt{\lambda}/2$. Let $\mathbb{C} \in \mathcal{E}(\lambda, \Lambda)$, $0 \le T_0 < T_1 \le T$, $s_0 \ge 0$, and $\bar{\gamma} \in \mathcal{G}_{r,L}$, with $b_{\bar{\gamma}} = s_0$. Assume that f and w satisfy (3.4), (3.8), (3.9), and (3.11). Let $u^0 \in H^1(\Omega^{\bar{\gamma}}_{s_0}; \mathbb{R}^2)$, with $u^0 - w(T_0) \in H^1_D(\Omega_0; \mathbb{R}^2)$, and let $u^1 \in L^2(\Omega; \mathbb{R}^2)$.

The class $\mathcal{C}^{reg}(T_0, T_1) = \mathcal{C}^{reg}(T_0, T_1, s_0, \bar{\gamma}, \mathbb{C}, f, w, u^0, u^1)$ is composed of all pairs (γ, s), with $\gamma \in \mathcal{G}_{r,L}$, $\gamma|_{[a_0,s_0]} = \bar{\gamma}|_{[a_0,s_0]}$, $s \in \mathcal{S}^{reg}_{\mu,M}([T_0, T_1])$, $s(T_0) = s^0$, and $s(T_1) \le b_\gamma$, such that the unique weak solution u of (3.5)–(3.7) on the time-dependent cracking domains $t \mapsto \Omega^{\gamma}_{s(t)}$ for $T_0 \le t \le T_1$, with the initial conditions $u(T_0) = u^0$, $\dot{u}(T_0) = u^1$, satisfies the dynamic energy-dissipation balance

$$\mathcal{E}(\widehat{\nabla}u(t_2), \dot{u}(t_2)) - \mathcal{E}(\widehat{\nabla}u(t_1)), \dot{u}(t_1)) + s(t_2) - s(t_1) = \mathcal{W}(u; t_1, t_2) \tag{4.4}$$

for every interval $[t_1, t_2] \subset [T_0, T_1]$.

Similarly, the class $\mathcal{C}^{piec}(T_0,T_1) = \mathcal{C}^{piec}(T_0,T_1,s_0,\bar{\gamma},\mathbb{C},f,w,u^0,u^1)$ is defined in the same way replacing $s \in \mathcal{S}^{reg}_{\mu,M}([T_0,T_1])$ by $s \in \mathcal{S}^{piec}_{\mu,M}([T_0,T_1])$.

As remarked in [1], equality (4.4) expresses conservation of energy: The work $\mathcal{W}$ done on the system is balanced by the change in mechanical energy $\mathcal{E}(\widehat{\nabla}u(t_2),\dot{u}(t_2)) - \mathcal{E}(\widehat{\nabla}u(t_1)),\dot{u}(t_1))$ and by the energy dissipated in the process of crack growth in the same time interval $[t_1,t_2]$.

Remark 4.2 *(Nonempty class).* The class $\mathcal{C}^{reg}(T_0,T_1)$ is nonempty. Indeed, it is well known that the wave equation in a time-independent domain satisfies the energy balance. In the case $w=0$, we refer to [10, Chapter 3, Lemma 8.3]. The general case can be obtained by considering the equation satisfied by $u-w$, taking into account the integration by parts formula (3.11) and using the identity

$$\int_0^t \langle \mathbb{C}\nabla u(\tau), \nabla \dot{w}(\tau)\rangle d\tau = \langle \mathbb{C}\nabla u(t), \nabla w(t)\rangle - \langle \mathbb{C}\nabla u(0), \nabla w(0)\rangle - \frac{1}{2}\langle \mathbb{C}\nabla w(t), \nabla w(t)\rangle$$

$$+ \frac{1}{2}\langle \mathbb{C}\nabla w(0), E\nabla w(0)\rangle + \int_0^t \langle \mathrm{div}(\mathbb{C}\nabla w(\tau)), \dot{u}(\tau) - \dot{w}(\tau)\rangle d\tau\,,$$

which can be easily proved by regularizing u with respect to time and using (3.11) again. The energy balance for the wave equation in a time-independent domain implies that the pair $(\bar{\gamma},s)$, with $s(t)=s_0$ for every $t\in[T_0,T_1]$, belongs to $\mathcal{C}^{reg}(T_0,T_1)$.

Remark 4.3 *(Concatenation).* Under the assumptions of Definition 4.1, let

$$(\gamma_1,s_1) \in \mathcal{C}^{piec}(T_0,T_1,s_0,\bar{\gamma},\mathbb{C},f,w,u^0,u^1)\,.$$

Let $T_1 < T_2 \le T$ and let

$$(\gamma_2,s_2) \in \mathcal{C}^{piec}(T_1,T_2,s_1(T_1),\gamma_1,\mathbb{C},f,w,u(T_1),\dot{u}(T_1))\,,$$

where u is as in Definition 4.1. Let $s\colon [T_0,T_2] \to \mathbb{R}$ be defined by

$$s(s) := \begin{cases} s_1(t) & \text{if } t\in[T_0,T_1]\,,\\ s_2(t) & \text{if } t\in[T_1,T_2]\,.\end{cases}$$

Then $(\gamma_2,s) \in \mathcal{C}^{piec}(T_0,T_2,s_0,\bar{\gamma},\mathbb{C},f,w,u^0,u^1)$.

Theorem 4.4 *(Compactness). Under the assumptions of Definition 4.1, let $(\gamma_k,s_k)\in\mathcal{C}^{reg}(T_0,T_1)$. Then there exist $(\gamma,s)\in\mathcal{C}^{reg}(T_0,T_1)$ and a subsequence (not relabeled) such that $\gamma_k\to\gamma$ uniformly (in the sense of Definition 2.3) and $s_k\to s$ in $C^3([T_0,T_1])$.*

Proof. By the compactness of $\mathcal{G}_{r,L}$ (see Lemma 2.5) there exist $\gamma\in\mathcal{G}_{r,L}$ and a subsequence γ_k such that $\gamma_k\to\gamma$ uniformly. By the Arzelà-Ascoli Theorem there exist $s\in C^3([T_0,T_1])$ and a further subsequence such $s_k\to s$ in $C^3([T_0,T_1])$. It is easy to see that the estimates on the third derivatives also hold for s, so that $s\in\mathcal{S}^{reg}_{\mu,M}([T_0,T_1])$. It remains to prove (4.4) for the solution corresponding to (γ,s). For every k let u_k be the unique weak solution of (3.5)–(3.7) on the time-dependent cracking domains $t\mapsto\Omega^{\gamma_k}_{s_k(t)}$ for $T_0\le t\le T_1$, with the initial conditions $u_k(T_0)=u^0$, $\dot{u}_k(T_0)=u^1$. Since $(\gamma_k,s_k)\in\mathcal{C}^{reg}(T_0,T_1)$ we have

$$\mathcal{E}(\widehat{\nabla}u_k(t_2), \dot{u}_k(t_2)) - \mathcal{E}(\widehat{\nabla}u_k(t_1)), \dot{u}_k(t_1)) + s_k(t_2) - s_k(t_1) = \mathcal{W}(u_k; t_1, t_2) \tag{4.5}$$

for every interval $[t_1, t_2] \subset [T_0, T_1]$. By (3.40) and (3.41) proved in Theorem 3.5 we can pass to the limit in (4.5) and obtain (4.4). $\square$

5. Existence of an η-maximal dissipation evolution

In our model the crack satisfies a maximality condition, which forces the crack tip to move, when possible, and to choose a path which allows for a maximal speed. In this section we introduce this maximality condition (see Definition 5.1), which depends on a threshold parameter $\eta > 0$, as explained in the Introduction. Then we prove the main result of the paper: the existence of a crack satisfying this η-maximality condition (see Theorem 5.2).

Given $s \in \mathcal{S}^{piec}_{\mu,M}(0,T)$, we consider its singular set $\mathrm{sing}(s)$ introduced in Definition 2.7.

Definition 5.1 *(η-maximal dissipation).* Assume that $0 < \mu < \sqrt{\lambda}/2$ and that f and w satisfy (3.4), (3.8), (3.9), and (3.10). Let $\mathbb{C} \in \mathcal{E}(\lambda, \Lambda)$, $u^0 \in H^1(\Omega_0; \mathbb{R}^2)$, with $u^0 - w(0) \in H^1_D(\Omega_0; \mathbb{R}^2)$, and $u^1 \in L^2(\Omega; \mathbb{R}^2)$.

Given $\eta > 0$ we say that $(\gamma, s) \in \mathcal{C}^{piec}(0,T)$ satisfies the η-maximal dissipation condition on $[0,T]$ if there exists no $(\hat{\gamma}, \hat{s}) \in \mathcal{C}^{piec}(0, \tau_1)$, for some $0 < \tau_1 \le T$, such that

(a) $\mathrm{sing}(\hat{s}) \subset \mathrm{sing}(s)$,
(b) $\hat{s}(t) = s(t)$ and $\hat{\gamma}(\hat{s}(t)) = \gamma(s(t))$ for every $t \in [0, \tau_0]$, for some $0 \le \tau_0 < \tau_1$,
(c) $\hat{s}(t) > s(t)$ for every $t \in (\tau_0, \tau_1]$ and $\hat{s}(\tau_1) > s(\tau_1) + \eta$.

Theorem 5.2 *(Existence of an η-maximally dissipative crack). Under the assumptions of Definition 5.1, for every $\eta > 0$ there exists a pair $(\gamma, s) \in \mathcal{C}^{piec}(0,T)$ satisfying the η-maximal dissipation condition on $[0,T]$.*

Proof. We proceed as in [1]. Let us fix $\eta > 0$ and a finite subdivision $0 = T_0 < T_1 < \cdots < T_k = T$ of the time interval $[0,T]$ such that $T_j - T_{j-1} < \frac{\eta}{\mu}$ for every j.

The solution will be constructed recursively in the intervals $[T_{j-1}, T_j]$. Fix $j \in \{1, \dots, k\}$ and assume the pair $(\gamma_{j-1}, s_{j-1}) \in \mathcal{C}^{piec}(0, T_{j-1}) = \mathcal{C}^{piec}(0, T_{j-1}, 0, \gamma_0, \mathbb{C}, f, w, u^0, u^1)$ has already been defined, where γ_0 is the function that appears in condition (a) of Definition 2.1.

To define the next pair (γ_j, s_j) we consider the class $\mathcal{A}_j$ of pairs $(\gamma, s) \in \mathcal{C}^{piec}(0, T_j) = \mathcal{C}^{piec}(0, T_j, 0, \gamma_0, \mathbb{C}, f, w, u^0, u^1)$ such that $s|_{[T_{j-1}, T_j]} \in \mathcal{S}^{reg}_{\mu,M}(T_{j-1}, T_j)$, $s(t) = s_{j-1}(t)$, and $\gamma(s(t)) = \gamma_{j-1}(s_{j-1}(t))$ for every $t \in [0, T_{j-1}]$. For $j = 1$ we define $\mathcal{A}_1$ as the set of all pairs $(\gamma, s) \in \mathcal{C}^{piec}(0, T_1) = \mathcal{C}^{piec}(0, T_1, 0, \gamma_0, \mathbb{C}, f, w, u^0, u^1)$ such that $s \in \mathcal{S}^{reg}_{\mu,M}(0, T_1)$ and $s(0) = 0$.

Note that $\mathcal{A}_j \neq \emptyset$. Indeed, $(\gamma_{j-1}, \bar{s}_{j-1}) \in \mathcal{A}_j$ if $\bar{s}_{j-1}$ is defined by $\bar{s}_{j-1}(t) = s_{j-1}(t)$ for $0 \le t \le T_{j-1}$ and $\bar{s}_{j-1}(t) = s_{j-1}(T_{j-1})$ for $T_{j-1} \le t \le T_j$ (see Remarks 4.2 and 4.3). In the case of $\mathcal{A}_1$ we consider the pair $(\gamma_0, 0)$.

We choose $(\gamma_j, s_j) \in \mathcal{A}_j$ such that

$$\int_{T_{j-1}}^{T_j} s_j(t)\, dt = \max_{(\gamma,s) \in \mathcal{A}_j} \int_{T_{j-1}}^{T_j} s(t)\, dt \,. \tag{5.1}$$

The existence of (γ_j, s_j) is guaranteed by Lemma 5.3 below.

We now define $(\gamma, s) := (\gamma_k, s_k)$, where (γ_k, s_k) is the pair obtained in the final step $j = k$ of our construction. Let us prove that (γ, s) satisfies the η-maximal dissipation condition on $[0,T]$. Assume, by contradiction, that there exist $0 \le \tau_0 < \tau_1 \le T$, and $(\hat{\gamma}, \hat{s}) \in \mathcal{C}^{piec}(0, \tau_1)$ such that:

(a) $\mathrm{sing}(\hat{s}) \subset \mathrm{sing}(s) \subset \{T_1, \dots, T_{k-1}\}$,
(b) $s(t) = \hat{s}(t)$ and $\gamma(s(t)) = \hat{\gamma}(\hat{s}(t))$ for every $t \in [0, \tau_0]$,
(c) $s(t) < \hat{s}(t)$ for every $t \in (\tau_0, \tau_1]$ and $\hat{s}(\tau_1) - s(\tau_1) > \eta$.

Let $j \in \{1, \dots, k\}$ be the index such that $T_{j-1} \leq \tau_0 < T_j$. Let us prove that $\tau_1 > T_j$. The monotonicity of s, together with (b) and (c), gives $\hat{s}(\tau_1) > s(\tau_1) + \eta \geq s(\tau_0) + \eta = \hat{s}(\tau_0) + \eta$, which implies that $\hat{s}(\tau_1) - \hat{s}(\tau_0) > \eta$. On the other hand, by the definition of the class $\mathcal{S}^{piec}_{\mu,M}(0, \tau_1)$ we have $\hat{s}(\tau_1) - \hat{s}(\tau_0) \leq \mu(\tau_1 - \tau_0)$, hence $\tau_1 - \tau_0 > \eta/\mu > T_j - T_{j-1}$. This implies $\tau_1 > T_j$.

By (a) we have $\hat{s}|_{[T_{j-1}, T_j]} \in \mathcal{S}^{reg}_{\mu,M}(T_{j-1}, T_j)$. Taking (b) into account it follows that $(\hat{\gamma}, \hat{s}) \in \mathcal{A}_j$. By construction $s = s_j$ on $[T_{j-1}, T_j]$ and, by (c), $\hat{s}(t) > s(t) = s_j(t)$ for every $t \in (\tau_0, T_j]$. This contradicts (5.1) and concludes the proof. $\square$

Lemma 5.3 *(Solution of a maximum problem). For every $j = 1, \dots, k$ there exists $(\gamma_j, s_j) \in \mathcal{A}_j$ such that*

$$\int_{T_{j-1}}^{T_j} s_j(t)\, dt = \max_{(\gamma,s) \in \mathcal{A}_j} \int_{T_{j-1}}^{T_j} s(t)\, dt\,. \tag{5.2}$$

Proof. Fix $j = 1, \dots, k$ and set $I_{max} := \sup\limits_{(\gamma,s) \in \mathcal{A}_j} \int_{T_{j-1}}^{T_j} s(t) dt$ and, for every $n \in \mathbb{N}$, let $(\gamma^n, s^n) \in \mathcal{A}_j$ be such that

$$\int_{T_{j-1}}^{T_j} s^n(t) dt \geq I_{max} - \frac{1}{n}\,. \tag{5.3}$$

Let u_{j-1} be the unique weak solution of (3.5)–(3.7) on the time-dependent cracking domains $t \mapsto \Omega^{\gamma_{j-1}}_{s_{j-1}(t)}$ for $0 \leq t \leq T_{j-1}$, with initial conditions $u_{j-1}(0) = u^0$ and $\dot{u}_{j-1}(0) = u^1$. We now define the new initial conditions at time T_{j-1}, by setting $s^0_{j-1} := s_{j-1}(T_{j-1})$, $u^0_{j-1} = u_{j-1}(T_{j-1})$, and $u^1_{j-1} = \dot{u}_{j-1}(T_{j-1})$. By the compactness of $\mathcal{C}^{reg}(T_{j-1}, T_j) = \mathcal{C}^{reg}(T_{j-1}, T_j, s^0_{j-1}, \gamma_{j-1}, \mathbb{C}, f, w, u^0_{j-1}, u^1_{j-1})$ (see Theorem 4.4) there exists a subsequence of $(\gamma^n, s^n|_{[T_{j-1}, T_j]})$, not relabeled, and a pair $(\gamma_j, \hat{s}) \in \mathcal{C}^{reg}(T_{j-1}, T_j)$ such that $\gamma^n \to \gamma_j$ and $s^n \to \hat{s}$ uniformly. Let us define $s_j(t) = s_{j-1}(t)$ for $t \in [0, T_{j-1}]$ and $s_j(t) = \hat{s}(t)$ for $t \in [T_{j-1}, T_j]$. Since $(\gamma^n, s^n) \in \mathcal{A}_j$ we have $\gamma^n(s_{j-1}(t)) = \gamma^n(s^n(t)) = \gamma_{j-1}(s_{j-1}(t))$ for all $t \in [0, T_{j-1}]$. Passing to the limit as $n \to \infty$ and using the definition of s_j on $[0, T_{j-1}]$ we obtain that $\gamma_j(s_j(t)) = \gamma_{j-1}(s_{j-1}(t))$ for all $t \in [0, T_{j-1}]$. From Lemma 4.3 we obtain that $(\gamma_j, s_j) \in \mathcal{C}^{piec}(0, T_j)$. Hence $(\gamma_j, s_j) \in \mathcal{A}_j$. Passing to the limit in n, from (5.3) we get $\int_{T_{j-1}}^{T_j} s_j(t) dt = I_{max}$, which immediately gives (5.2). $\square$

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Appendix

In this section we prove the finite speed of propagation for the system of elastodynamics under very weak assumptions. Since the proof does not depend on the dimension, we will state the result in any dimension $n \geq 1$.

Let U be a bounded open subset of $\mathbb{R}^n$ and let $\partial_L U$ be the Lipschitz part of the boundary ∂U, defined as the set of points $x \in \partial U$ with the following property: there exist an orthogonal coordinate system $y_1, \dots, y_n$, a neighborhood V of x of the form $A \times I$, with A open in $\mathbb{R}^{n-1}$ and I open interval in $\mathbb{R}$, and a Lipschitz function $g \colon A \to I$, such that $V \cap U = \{(y_1, \dots, y_n) \in V : y_n < g(y_1, \dots, y_{n-1})\}$.

Let $\mathbb{M}^{n\times n}$ be the space of $n\times n$ real matrices and let $\mathbb{M}^{n\times n}_{sym}$ be the space of $n\times n$ real symmetric matrices. The elasticity tensor $\mathbb{A} \colon U \to \mathrm{Lin}(\mathbb{M}^{n\times n}, \mathbb{M}^{n\times n})$ is a measurable function with the following properties: for a.e. $x \in U$ we have

$$\mathbb{A}(x)F = \mathbb{A}(x)F^{sym} \in \mathbb{M}^{n\times n}_{sym} \qquad \text{for every } F \in \mathbb{M}^{n\times n}, \tag{A.1}$$

$$\mathbb{A}(x)F \cdot G = \mathbb{A}(x)G \cdot F, \qquad \text{for every } F, G \in \mathbb{M}^{n\times n}, \tag{A.2}$$

$$\lambda |F^{sym}|^2 \leq \mathbb{A}(x)F \cdot F \leq \Lambda |F^{sym}|^2, \qquad \text{for every } F \in \mathbb{M}^{n\times n}. \tag{A.3}$$

Let us fix $T > 0$, $f \in L^2(0,T; L^2(U;\mathbb{R}^n))$, $u^0 \in H^1(U;\mathbb{R}^n)$, $u^1 \in L^2(U;\mathbb{R}^n)$, and two Borel sets S_0 and S_1, with $S_0 \subset S_1 \subset \partial_L U$. We consider a weak solution u of the system of elastodynamics

$$\ddot{u} - \mathrm{div}(\mathbb{A}\nabla u) = f \quad \text{in } (0,T) \times U \tag{A.4}$$

with boundary conditions

$$u = 0 \quad \text{on } (0,T) \times S_0, \tag{A.5}$$

$$(\mathbb{A}\nabla u)\nu = 0 \quad \text{on } (0,T) \times (\partial U \setminus S_1), \tag{A.6}$$

and initial conditions

$$u(0) = u^0 \quad \text{and} \quad \dot{u}(0) = u^1 \quad \text{in } U. \tag{A.7}$$

To give a precise meaning to (A.4)–(A.6) for every Borel set $S \subset \partial_L U$ we introduce the space

$$H^1_S(U;\mathbb{R}^n) := \{u \in H^1(U;\mathbb{R}^n) : u = 0 \ \mathcal{H}^{n-1}\text{-a.e. on } S\},$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (see, e.g., [6, Definition 2.46]) and the equality on S refers to the trace of u. It is clear that $H^1_S(U;\mathbb{R}^n)$, endowed with the norm of $H^1(U;\mathbb{R}^n)$, is a Hilbert space. Its dual is denoted by $H^{-1}_S(U,\mathbb{R}^n)$.

By a weak solution of (A.4)–(A.6) we mean a function u such that

$$u \in L^2(0,T; H^1_{S_0}(U;\mathbb{R}^n)), \tag{A.8}$$

$$\dot{u} \in L^2(0,T; L^2(U;\mathbb{R}^n)), \tag{A.9}$$

$$\ddot{u} \in L^2(0,T; H^{-1}_{S_1}(U;\mathbb{R}^n)), \tag{A.10}$$

and for a.e. $t \in (0,T)$ satisfies

$$\langle \ddot{u}(t), \varphi \rangle + \langle \mathbb{A}\nabla u(t), \nabla\varphi \rangle = 0 \text{ for every } \varphi \in H^1_{S_1}(U;\mathbb{R}^n). \tag{A.11}$$

By (A.8)–(A.10), a weak solution u satisfies

$$u \in C^0([0,T]; L^2(U;\mathbb{R}^n))\,, \tag{A.12}$$

$$\dot u \in C^0([0,T]; H^{-1}_{S_1}(U;\mathbb{R}^n))\,, \tag{A.13}$$

therefore the initial conditions (A.7) have to be interpreted as equalities in $L^2(U;\mathbb{R}^n)$ and $H^{-1}_{S_1}(U;\mathbb{R}^n)$, respectively.

We are now in a position to state the main result of this section.

Theorem A.1 *(Finite speed of propagation). Let $T>0$, let $U\subset\mathbb{R}^n$ be a bounded open set, let $\mathbb{A}: U\to \mathrm{Lin}(\mathbb{M}^{n\times n},\mathbb{M}^{n\times n})$ be a measurable function satisfying (A.1)–(A.3), let S_0 and S_1 be Borel sets with $S_0\subset S_1\subset\partial_L U$, and for every $t\in[0,T]$ let*

$$U_t := \{x\in U : \mathrm{dist}(x, S_1\setminus S_0) > t\sqrt{\Lambda}\,\}\,. \tag{A.14}$$

If u is a weak solution of (A.4)–(A.7) in the sense of (A.8)–(A.11), with $f=0$, $u^0=0$, and $u^1=0$, then

$$u(t) = 0 \qquad \textit{a.e. in } U_t \tag{A.15}$$

for every $t\in[0,T]$.

To prove the theorem we need the following lemma.

Lemma A.2 *(Auxiliary estimates). Let E be a bounded set in $\mathbb{R}^n$ and let $a\ge 0$, $b>0$, and $T>0$. For every $t\in[-a/b,T]$ let*

$$E_t := \{x\in\mathbb{R}^n : \mathrm{dist}(x,E)\le a+bt\,\} \quad \textit{and} \quad \psi(t) := 1_{E_t} * \rho\,,$$

where 1_{E_t} is the characteristic function of E_t, $\rho\in C^\infty_c(B_1(0))$ is a nonnegative function with $\int_{\mathbb{R}^n}\rho\,dx = 1$, and $*$ denotes the convolution with respect to the spatial variable. Let B be an open ball in $\mathbb{R}^n$ containing $E_T + B_1(0)$. Then $\psi\colon[-a/b,T]\to L^\infty(B)$ is absolutely continuous and for a.e. $t\in[-a/b,T]$ there exists $\dot\psi(t)\in L^\infty(B)$ such that

$$(\psi(t+h)-\psi(t))/h \to \dot\psi(t) \tag{A.16}$$

weakly* in $L^\infty(B)$ and strongly in $L^p(B)$ for every $1\le p<+\infty$. Moreover, for a.e. $t\in[-a/b,T]$ we have

$$|\nabla\psi(t)| \le |\dot\psi(t)|/b \quad \textit{a.e. in } B\,. \tag{A.17}$$

Proof. We begin by proving that $t\mapsto\mathcal{L}^n(E_t)$ is absolutely continuous on $[-a/b,T]$. For every $x\in B$ let $g(x) := \mathrm{dist}(x,E)$ and let P denote the perimeter of a set in $\mathbb{R}^n$ (see [6, Definition 3.35]). Since $|\nabla g(x)| = 1$ for a.e. $x\in B\backslash\overline{E}$, by the co-area formula [6, Theorem 3.40], the function $s\mapsto P(\{g\le s\})$ is integrable and for every $t\in[-a/b,T]$

$$\mathcal{L}^n(E_t) - \mathcal{L}^n(\overline{E}) = \int_B |\nabla(g\wedge(a+bt))|dx = \int_0^{a+bt} P(\{g\le s\})ds\,,$$

where $\alpha\wedge\beta := \min\{\alpha,\beta\}$. This shows that $t\mapsto\mathcal{L}^n(E_t)$ is absolutely continuous on $[-a/b,T]$.

Since for $s < t$ we have

$$\|1_{E_t} - 1_{E_s}\|_{L^1(B)} \le \mathcal{L}^n(E_t) - \mathcal{L}^n(E_s)\,,$$

the function $t \mapsto 1_{E_t}$ is absolutely continuous from $[-a/b, T]$ into $L^1(B)$.

Let us prove that

$$\frac{1}{h}(1_{E_{t+h}} - 1_{E_t}) \rightharpoonup b\,\mathcal{H}^{n-1} \llcorner \partial^* E_t \quad \text{weakly}^* \text{ in } \mathcal{M}_b(B)\,, \tag{A.18}$$

where the space $\mathcal{M}_b(B)$ of bounded Radon measures on B is regarded as the dual of the Banach space $C_0^0(B)$ of continuous functions on $\overline{B}$ vanishing on ∂B. Here and in the rest of the paper, ∂^* denotes the reduced boundary (see [6, Definition 3.54]) and, for every Borel set F, $\mathcal{H}^{n-1} \llcorner F$ denotes the measure defined by $(\mathcal{H}^{n-1} \llcorner F)(A) = \mathcal{H}^{n-1}(F \cap A)$ for every Borel set A.

Let $\varphi \in C_0^0(B)$. Using again the co-area formula, together with De Giorgi's characterization of the derivative of a characteristic function (see [6, Theorem 3.59]), we obtain that the function $s \mapsto \int_{\partial^*\{g \le s\}} \varphi\, d\mathcal{H}^{n-1}$ is integrable and that for every $t \in [-a/b, T]$ we have

$$\int_{E_t} \varphi dx = \int_B |\nabla(g \wedge (a + bt))| \varphi dx = \int_0^{a+bt} \int_{\partial^*\{g \le s\}} \varphi\, d\mathcal{H}^{n-1} ds\,,$$

therefore

$$\lim_{h\to 0} \frac{1}{h}\Big(\int_{E_{t+h}} \varphi dx - \int_{E_t} \varphi dx\Big) = b \int_{\partial^* E_t} \varphi\, d\mathcal{H}^{n-1}$$

for a.e. $t \in [-a/b, T]$. This proves (A.18).

Since the convolution by ρ is a continuous linear operator mapping $L^1(B)$ into $L^\infty(B)$, the absolute continuity of $t \mapsto 1_{E_t}$ implies that ψ is absolutely continuous from $[-a/b, T]$ into $L^\infty(B)$. Moreover, since the convolution by ρ maps weakly* convergent sequences in $\mathcal{M}_b(B)$, supported by E_T, into weakly* convergent sequences in $L^\infty(B)$, from (A.18) we obtain (A.16) weakly* in $L^\infty(B)$, with

$$\dot\psi(t) = b(\mathcal{H}^{n-1} \llcorner \partial^* E_t) * \rho. \tag{A.19}$$

As for the strong convergence in $L^p(B)$ for $1 < p < +\infty$, we observe that the absolute continuity of $\psi\colon [-a/b, T] \to L^\infty(B)$ implies the absolute continuity of $\psi\colon [-a/b, T] \to L^p(B)$. Since $L^p(B)$ is reflexive we can apply [11, Corollaire A.2] and we obtain (A.16) strongly in $L^p(B)$ for $1 < p < +\infty$. The result for $p = 1$ is now obvious.

To prove (A.17) we observe that for every $t \in [-a/b, T]$ we have

$$\nabla\psi(t) = D1_{E_t} * \rho\,,$$

where D denotes the distributional gradient. By the co-area formula for a.e. $t \in [-a/b, T]$ the set E_t has finite perimeter and therefore

$$D1_{E_t} = \nu_t \mathcal{H}^{n-1} \llcorner \partial^* E_t\,,$$

where ν_t is the inner unit normal of E_t. It follows that

$$|\nabla\psi(t)| \leq (\mathcal{H}^{n-1} \llcorner \partial^* E_t) * \rho$$

which, together with (A.19), gives (A.17). $\square$

Proof of Theorem A.1. Let u be as in the statement of the theorem. We extend u by setting

$$u(t) = 0 \quad \text{for every } t \in (-T, 0]\,. \tag{A.20}$$

Since $u(0+) = 0$ in $L^2(U;\mathbb{R}^n)$ and $\dot u(0+) = 0$ in $H^{-1}_{S_1}(U;\mathbb{R}^n)$, we have that

$$u \in L^2(-T,T; H^1_{S_0}(U;\mathbb{R}^n))\,, \tag{A.21}$$

$$\dot u \in L^2(-T,T; L^2(U;\mathbb{R}^n))\,, \tag{A.22}$$

$$\ddot u \in L^2(-T,T; H^{-1}_{S_1}(U;\mathbb{R}^n))\,, \tag{A.23}$$

$$\langle \ddot u(t), \varphi\rangle + \langle \mathbb{A}\nabla u(t), \nabla\varphi\rangle = 0 \text{ for a.e. } t \in (-T,T) \text{ and for every } \varphi \in H^1_{S_1}(U;\mathbb{R}^n)\,. \tag{A.24}$$

For a.e. $t \in (0,T)$ we define

$$e(t) := \frac{1}{2}\int_{U_t} |\dot u(t)|^2 dx + \frac{1}{2}\int_{U_t} \mathbb{A}\nabla u(t)\cdot\nabla u(t)\, dx\,. \tag{A.25}$$

We want to prove that

$$e(t) = 0 \quad \text{for a.e. } t \in (0,T). \tag{A.26}$$

Before doing this, let us show that (A.26) implies that for every $t \in [0,T]$ we have $u(t) = 0$ a.e. in U_t. Let us fix $t \in (0,T]$. Since $U_t \subset U_s$ for $0 < s < t$, (A.25) and (A.26) give $\dot u(s) = 0$ a.e. in U_t for a.e. $s \in (0,t)$. Since $u \in H^1(0,t; L^2(U_t,\mathbb{R}^n))$ and $u(0) = 0$, we conclude that $u(t) = 0$ a.e. in U_t. Therefore, to prove the theorem it is enough to show that (A.26) holds.

To obtain an estimate for (A.25) we consider the set

$$V_t := \{x \in \mathbb{R}^n : \operatorname{dist}(x, S_1 \setminus S_0) > t\sqrt{\Lambda}\,\}, \tag{A.27}$$

so that $U_t = V_t \cap U$. To regularize the characteristic function 1_{V_t} of V_t we fix a nonnegative $\rho \in C_c^\infty(\mathbb{R}^n)$ with $\rho(x) = 0$ for $|x| \geq 1$ and $\int_{\mathbb{R}^n} \rho\, dx = 1$. For every $\varepsilon \in (0, T\sqrt{\Lambda})$ let $\rho_\varepsilon(x) = 1/\varepsilon^n \rho(x/\varepsilon)$ and, for every $t \in (-\varepsilon/\sqrt{\Lambda}, T)$ let $\psi_\varepsilon(t) = 1_{V_{t+\alpha\varepsilon}} * \rho_\varepsilon$, where $\alpha = 2/\sqrt{\Lambda}$. We remark that by (A.27) we have

$$\psi_\varepsilon(t) = 0 \quad \text{in a neighborhood of } S_1 \setminus S_0 \tag{A.28}$$

for every $\varepsilon \in (0, T\sqrt{\Lambda})$ and for every $t \in (-\varepsilon/\sqrt{\Lambda}, T)$.

Let $e_\varepsilon(t)$ be the approximation of $e(t)$ defined by

$$\begin{aligned} e_\varepsilon(t) &:= \frac{1}{2}\int_U |\dot u(t)|^2 \psi_\varepsilon(t) dx + \frac{1}{2}\int_U \mathbb{A}\nabla u(t)\cdot\nabla u(t)\psi_\varepsilon(t) dx \\ &= \frac{1}{2}\langle \dot u(t), \dot u(t)\psi_\varepsilon(t)\rangle + \frac{1}{2}\langle \mathbb{A}\nabla u(t), \nabla u(t)\psi_\varepsilon(t)\rangle \end{aligned} \tag{A.29}$$

for every $\varepsilon \in (0, T\sqrt{\Lambda})$ and for a.e. $t \in (-\varepsilon/\sqrt{\Lambda}, T)$. By (A.21) and (A.22) we have that $e_\varepsilon \in L^1(-\varepsilon/\sqrt{\Lambda}, T)$. Moreover, by standard properties of convolutions and by the integrability properties of $|\dot u(t)|^2$ and $\mathbb{A}\nabla u(t)\cdot\nabla u(t)$, we obtain

$$e_\varepsilon(t) \to e(t) \quad \text{for a.e. } t \in (0,T)\,. \tag{A.30}$$

To obtain an estimate for $e_\varepsilon(t)$ we first consider the differences $e_\varepsilon(t+h)-e_\varepsilon(t)$ for a given $h \in (0,\varepsilon/\sqrt{\Lambda})$. We have

$$\begin{aligned}2(e_\varepsilon(t+h)-e_\varepsilon(t)) &= \langle \dot u(t+h)+\dot u(t), (\dot u(t+h)-\dot u(t))\psi_\varepsilon(t+h)\rangle \\ &+ \langle \dot u(t), \dot u(t)(\psi_\varepsilon(t+h)-\psi_\varepsilon(t))\rangle \\ &+ \langle \mathbb{A}\nabla u(t+h)+\mathbb{A}\nabla u(t), (\nabla u(t+h)-\nabla u(t))\psi_\varepsilon(t+h)\rangle \\ &+ \langle \mathbb{A}\nabla u(t), \nabla u(t)(\psi_\varepsilon(t+h)-\psi_\varepsilon(t))\rangle = I_1+I_2+I_3+I_4\,.\end{aligned} \tag{A.31}$$

It is convenient to write I_1 and I_3 as

$$\begin{aligned}I_1 &= \langle \dot u(t+h)+\dot u(t)), \tfrac{d}{dt}(u(t+h)-u(t))\psi_\varepsilon(t+h))\rangle \\ &\quad - \langle \dot u(t+h)+\dot u(t), (u(t+h)-u(t))\dot\psi_\varepsilon(t+h)\rangle\,, \end{aligned}\tag{A.32}$$

$$\begin{aligned}I_3 &= \langle \mathbb{A}\nabla u(t+h)+\mathbb{A}\nabla u(t), \nabla((u(t+h)-u(t))\psi_\varepsilon(t+h))\rangle \\ &\quad - \langle \mathbb{A}\nabla u(t+h)+\mathbb{A}\nabla u(t), (u(t+h)-u(t))\otimes\nabla\psi_\varepsilon(t+h)\rangle\,. \end{aligned}\tag{A.33}$$

We now integrate by parts we respect to t.

Since u satisfies (A.22) and (A.23), if

$$\zeta \in L^2(-\varepsilon/\sqrt{\Lambda},T;H^1_{S_1}(U;\mathbb{R}^n)) \quad \text{and} \quad \dot\zeta \in L^2(-\varepsilon/\sqrt{\Lambda},T;L^2(U;\mathbb{R}^n)),$$

it is easy to prove by approximation that the function $t\mapsto\langle \dot u(t),\zeta(t)\rangle$ is absolutely continuous in $[-\varepsilon/\sqrt{\Lambda},T]$ and

$$\frac{d}{dt}\langle \dot u(t),\zeta(t)\rangle = \langle \ddot u(t),\zeta(t)\rangle + \langle \dot u(t),\dot\zeta(t)\rangle\,. \tag{A.34}$$

By Lemma A.2 and by (A.21), (A.22), and (A.28), we can apply this formula with

$$\zeta(t) := (u(t+h)-u(t))\psi_\varepsilon(t+h) \tag{A.35}$$

and we obtain that

$$\begin{aligned}I_1 &= \frac{d}{dt}\langle \dot u(t+h)+\dot u(t), (u(t+h)-u(t))\psi_\varepsilon(t+h)\rangle \\ &\quad - \langle \ddot u(t+h)+\ddot u(t), (u(t+h)-u(t))\psi_\varepsilon(t+h)\rangle \\ &\quad - \langle \dot u(t+h)+\dot u(t), (u(t+h)-u(t))\dot\psi_\varepsilon(t+h)\rangle\end{aligned} \tag{A.36}$$

for a.e. $t\in(-\varepsilon/\sqrt{\Lambda},T)$.

Let us now fix $t\in[0,T]$. By integrating (A.31) between $-h$ and $t-h$, and using (A.20), (A.33), and (A.36) we obtain

$$\begin{aligned}2\int_{-h}^{t-h}(e_\varepsilon(s+h)-e_\varepsilon(s))ds &= \langle \dot u(t)+\dot u(t-h)), (u(t)-u(t-h))\psi_\varepsilon(t))\rangle \\ &- \int_{-h}^{t-h}\langle \ddot u(s+h)+\ddot u(s), (u(s+h)-u(s))\psi_\varepsilon(s+h)\rangle ds\end{aligned}$$

$$
\begin{aligned}
&- \int_{-h}^{t-h} \langle \dot{u}(s+h) + \dot{u}(s), (u(s+h) - u(s))\dot{\psi}_\varepsilon(s+h)\rangle ds \\
&+ \int_{-h}^{t-h} \langle \dot{u}(s), \dot{u}(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds \\
&+ \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s+h) + \mathbb{A}\nabla u(s), \nabla((u(s+h) - u(s))\psi_\varepsilon(s+h))\rangle ds \\
&- \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s+h) + \mathbb{A}\nabla u(s), (u(s+h) - u(s)) \otimes \nabla\psi_\varepsilon(s+h)\rangle ds \\
&+ \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s), \nabla u(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds\,.
\end{aligned} \tag{A.37}
$$

Note that by (A.24) for a.e. $s \in (-h, t-h)$ we have

$$
\langle \ddot{u}(s) + \ddot{u}(s+h), \varphi\rangle + \langle \mathbb{A}(\nabla u(s) + \nabla u(s+h)), \nabla\varphi\rangle = 0 \quad \text{for every } \varphi \in H^1_{S_1}(U; \mathbb{R}^n)\,. \tag{A.38}
$$

By (A.28), for a.e. $s \in (-h, t-h)$ we may take $\varphi_\varepsilon(s) = (u(s+h) - u(s))\psi_\varepsilon(s+h)$ as a test function in (A.38) obtaining

$$
\begin{aligned}
&\langle \ddot{u}(s+h) + \ddot{u}(s), u(s+h) - u(s))\psi_\varepsilon(s+h)\rangle \\
&\quad + \langle \mathbb{A}(\nabla u(s+h) + \nabla u(s)), \nabla((u(s+h) - u(s))\psi_\varepsilon(s+h))\rangle = 0\,.
\end{aligned} \tag{A.39}
$$

Substituting in (A.37) we get

$$
\begin{aligned}
2\int_{-h}^{t-h} (e_\varepsilon(s+h) - e_\varepsilon(s))ds &= \langle \dot{u}(t) + \dot{u}(t-h)), (u(t) - u(t-h))\psi_\varepsilon(t))\rangle \\
&- 2\int_{-h}^{t-h} \langle \ddot{u}(s+h) + \ddot{u}(s), (u(s+h) - u(s))\psi_\varepsilon(s+h))\rangle ds \\
&- \int_{-h}^{t-h} \langle \dot{u}(s+h) + \dot{u}(s), (u(s+h) - u(s))\dot{\psi}_\varepsilon(s+h)\rangle ds \\
&+ \int_{-h}^{t-h} \langle \dot{u}(s), \dot{u}(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds \\
&- \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s+h) + \mathbb{A}\nabla u(s), (u(s+h) - u(s)) \otimes \nabla\psi_\varepsilon(s+h)\rangle ds \\
&+ \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s), \nabla u(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds\,.
\end{aligned} \tag{A.40}
$$

Integrating by parts, thanks to (A.20) and (A.34) we obtain

$$-2\int_{-h}^{t-h} \langle \ddot u(s+h)+\ddot u(s), (u(s+h)-u(s))\psi_\varepsilon(s+h))\rangle ds$$

$$= -2\langle \dot u(t)+\dot u(t-h), (u(t)-u(t-h))\psi_\varepsilon(t))\rangle$$

$$+ 2\int_{-h}^{t-h} \langle \dot u(s+h)+\dot u(s), (\dot u(s+h)-\dot u(s))\psi_\varepsilon(s+h))\rangle ds$$

$$+ 2\int_{-h}^{t-h} \langle \dot u(s+h)+\dot u(s), (u(s+h)-u(s))\dot\psi_\varepsilon(s+h))\rangle ds\,. \tag{A.41}$$

Hence, substituting in (A.40) and using again (A.20) we obtain

$$2\int_{t-h}^{t} e_\varepsilon(s)\,ds = 2\int_{t-h}^{t} e_\varepsilon(s)\,ds - 2\int_{-h}^{0} e_\varepsilon(s)\,ds = -\langle \dot u(t)+\dot u(t-h), (u(t)-u(t-h))\psi_\varepsilon(t)\rangle$$

$$+ 2\int_{-h}^{t-h} \langle \dot u(s+h)+\dot u(s), (\dot u(s+h)-\dot u(s))\psi_\varepsilon(s+h)\rangle ds$$

$$+ \int_{-h}^{t-h} \langle \dot u(s+h)+\dot u(s), (u(s+h)-u(s))\dot\psi_\varepsilon(s+h)\rangle ds$$

$$+ \int_{-h}^{t-h} \langle \dot u(s), \dot u(s)(\psi_\varepsilon(s+h)-\psi_\varepsilon(s))\rangle ds$$

$$- \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s+h)+\mathbb{A}\nabla u(s), (u(s+h)-u(s))\otimes\nabla\psi_\varepsilon(s+h)\rangle ds$$

$$+ \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s), \nabla u(s)(\psi_\varepsilon(s+h)-\psi_\varepsilon(s))\rangle ds\,. \tag{A.42}$$

Note that

$$2\int_{-h}^{t-h} \langle \dot u(s+h)+\dot u(s), (\dot u(s+h)-\dot u(s))\psi_\varepsilon(s+h)\rangle ds$$

$$+ \int_{-h}^{t-h} \langle \dot u(s), \dot u(s)(\psi_\varepsilon(s+h)-\psi_\varepsilon(s))\rangle ds$$

$$= 2\int_{-h}^{t-h} \langle \dot u(s+h), \dot u(s+h)\psi_\varepsilon(s+h)\rangle ds - 2\int_{-h}^{t-h} \langle \dot u(s), \dot u(s)\psi_\varepsilon(s)\rangle ds$$

$$- \int_{-h}^{t-h} \langle \dot u(s), \dot u(s)\psi_\varepsilon(s+h)\rangle ds + \int_{-h}^{t-h} \langle \dot u(s), \dot u(s)\psi_\varepsilon(s)\rangle ds$$

$$= 2\int_{t-h}^{t} \langle \dot u(s), \dot u(s)\psi_\varepsilon(s)\rangle ds - \int_{-h}^{t-h} \langle \dot u(s), \dot u(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds\,, \tag{A.43}$$

where in the last equality we used again (A.20). Therefore substituting in (A.42) we obtain

$$\begin{aligned}
2\int_{t-h}^{t} e_\varepsilon(s)\,ds &= -\langle \dot u(t) + \dot u(t-h), (u(t) - u(t-h))\psi_\varepsilon(t)\rangle \\
&\quad + \int_{-h}^{t-h} \langle \dot u(s+h) + \dot u(s), (u(s+h) - u(s))\dot\psi_\varepsilon(s+h)\rangle ds \\
&\quad + 2\int_{t-h}^{t} \langle \dot u(s), \dot u(s)\psi_\varepsilon(s)\rangle ds - \int_{-h}^{t-h} \langle \dot u(s), \dot u(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds \\
&\quad - \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s+h) + \mathbb{A}\nabla u(s), (u(s+h) - u(s)) \otimes \nabla\psi_\varepsilon(s+h)\rangle ds \\
&\quad + \int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s), \nabla u(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds\,.
\end{aligned} \tag{A.44}$$

We divide by h the terms in the right-hand side of (A.44). Thanks to (A.21) and (A.22) we can pass to the limit in $L^1(0,T)$ as $h \to 0+$ and we obtain

$$-\frac{1}{h}\langle \dot u(t) + \dot u(t-h), (u(t) - u(t-h))\psi_\varepsilon(t)\rangle \to -2\langle \dot u(t), \dot u(t)\psi_\varepsilon(t)\rangle\,, \tag{A.45}$$

$$\frac{1}{h}\int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s), \nabla u(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds \to \int_0^t \langle \mathbb{A}\nabla u(s), \nabla u(s)\dot\psi_\varepsilon(s)\rangle ds\,, \tag{A.46}$$

$$\begin{aligned}
&\frac{1}{h}\int_{-h}^{t-h} \langle \mathbb{A}\nabla u(s+h) + \mathbb{A}\nabla u(s), (u(s+h) - u(s)) \otimes \nabla\psi_\varepsilon(s+h)\rangle ds \\
&\qquad \to 2\int_0^t \langle \mathbb{A}\nabla u(s), \dot u(s) \otimes \nabla\psi_\varepsilon(s)\rangle ds\,,
\end{aligned} \tag{A.47}$$

$$\begin{aligned}
&\frac{1}{h}\int_{-h}^{t-h} \langle \dot u(s+h) + \dot u(s), (u(s+h) - u(s))\dot\psi_\varepsilon(s+h))\rangle ds \\
&\qquad -\frac{1}{h}\int_{-h}^{t-h} \langle \dot u(s), \dot u(s)(\psi_\varepsilon(s+h) - \psi_\varepsilon(s))\rangle ds \to \int_0^t \langle \dot u(s), \dot u(s)\dot\psi_\varepsilon(s)\rangle ds\,.
\end{aligned} \tag{A.48}$$

Since

$$\frac{2}{h}\int_{t-h}^{t} e_\varepsilon(s)\,ds \to 2e_\varepsilon(t)$$

in $L^1(0,T)$ as $h\to 0+$, from (A.44)–(A.48) we get

$$2e_\varepsilon(t) = \int_0^t \langle \dot u(s), \dot u(s)\dot\psi_\varepsilon(s)\rangle ds + \int_0^t \langle \mathbb{A}\nabla u(s), \nabla u(s)\dot\psi_\varepsilon(s)\rangle ds$$
$$+ 2\int_0^t \langle \mathbb{A}\nabla u(s), \dot u(s)\otimes\nabla\psi_\varepsilon(s)\rangle ds \tag{A.49}$$

for a.e. $t\in(0,T)$.

Let $\xi_\varepsilon(s)$ be the function on U defined by $\xi_\varepsilon(s) = \nabla\psi_\varepsilon(s)/|\nabla\psi_\varepsilon(s)|$ on $\{\nabla\psi_\varepsilon(s)\neq 0\}\cap U$ and $\xi_\varepsilon(s)=0$ on $\{\nabla\psi_\varepsilon(s)=0\}\cap U$. By the Cauchy inequality for the quadratic form on $L^2(U;\mathbb{R}^{n\times n})$ determined by $\mathbb{A}$, for every $\alpha>0$ we have

$$\begin{aligned}
&2\langle \mathbb{A}\nabla u(s), \dot u(s)\otimes\nabla\psi_\varepsilon(s)\rangle\\
&\leq 2\langle \mathbb{A}\nabla u(s), \nabla u(s)|\nabla\psi_\varepsilon(s)|\rangle^{1/2}\,\langle \mathbb{A}\dot u(s)\otimes\xi_\varepsilon(s), \dot u(s)\otimes\xi_\varepsilon(s)|\nabla\psi_\varepsilon(s)|\rangle^{1/2}\\
&\leq \alpha\langle \mathbb{A}\nabla u(s), \nabla u(s)|\nabla\psi_\varepsilon(s)|\rangle + \frac{1}{\alpha}\langle \mathbb{A}\dot u(s)\otimes\xi_\varepsilon(s), \dot u(s)\otimes\xi_\varepsilon(s)|\nabla\psi_\varepsilon(s)|\rangle
\end{aligned}$$

for a.e. $s\in(0,T)$. Therefore, by (A.3) and (A.17) we obtain

$$\begin{aligned}
&2\langle \mathbb{A}\nabla u(s), \dot u(s)\otimes\nabla\psi_\varepsilon(s)\rangle\\
&\leq \frac{\alpha}{\sqrt{\Lambda}}\langle \mathbb{A}\nabla u(s), \nabla u(s)|\dot\psi_\varepsilon(s)|\rangle + \frac{\sqrt{\Lambda}}{\alpha}\langle \dot u(s), \dot u(s)|\dot\psi_\varepsilon(s)|\rangle\,.
\end{aligned}$$

Taking $\alpha=\sqrt{\Lambda}$ and recalling that $\dot\psi_\varepsilon(s)\leq 0$ we obtain

$$2\langle \mathbb{A}\nabla u(s), \dot u(s)\otimes\nabla\psi_\varepsilon(s)\rangle + \langle \mathbb{A}\nabla u(s), \nabla u(s)\dot\psi_\varepsilon(s)\rangle + \langle \dot u(s), \dot u(s)\dot\psi_\varepsilon(s)\rangle \leq 0$$

for a.e. $s\in(0,T)$. This inequality together with (A.49) gives $e_\varepsilon(t)\leq 0$ for a.e. $t\in(0,T)$, hence $e(t)\leq 0$ for a.e. $t\in(0,T)$, by (A.30). Since $e(t)\geq 0$, this concludes the proof. □

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