UNIVERSITÀ DEGLI STUDI DI TRIESTE

UNIVERSITÀ DEGLI STUDI DI TRIESTE

XXXVI CICLO DEL DOTTORATO DI RICERCA IN

SCIENZE DELLA TERRA, FLUIDODINAMICA E MATEMATICA. INTERAZIONI E METODICHE.

STABILITY ESTIMATES

FOR SOME ANISOTROPIC INVERSE PROBLEMS

Settore scientifico-disciplinare: **MAT/05**

DOTTORANDA
SONIA FOSCHIATTI

COORDINATORE
PROF. STEFANO MASET

SUPERVISORE DI TESI
PROF. SSA EVA SINCICH

CO-SUPERVISORE DI TESI
PROF. SSA ROMINA GABURRO

ANNO ACCADEMICO 2022/2023

To my parents

It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another. I imagine the world conqueror must feel thus, who, after one kingdom is scarcely conquered, stretches out his arms for others.

Carl Friedrich Gauss - Letter to Bolyai, 1808.

Most people, if you describe a train of events to them, will tell you what the result would be. They can put those events together in their minds, and argue from them that something will come to pass. There are few people, however, who, if you told them a result, would be able to evolve from their own inner consciousness what the steps were which led up to that result. This power is what I mean when I talk of reasoning backwards, or analytically.

\- Arthur Conan Doyle, *A study in Scarlet*

Acknowledgements

I am very grateful for the support and belief of the many people who have been a part of my life during these years. First of all, I would like to express my deepest gratitude to my supervisors, Prof. Eva Sincich and Prof. Romina Gaburro, for their guidance, encouragement, and patience during these three years of doctoral studies. I owe them the opportunity to work on inverse problems, the knowledge acquired during these years, and the development of my research skills.

I am very grateful to the professors and my colleagues at the University of Trieste who have supported me during these years, as well as to the colleagues I have met during conferences and workshops, for the fruitful discussions and valuable advice, among which I would like to thank Prof. Andrea Aspri for the constructive suggestions.

I would like to thank my family, especially my sister Michela and my nephews Agnese, Rita and Marco, my friends, and the people from the music world for their support, encouragement and understanding.

Finally, I am deeply grateful to my parents, Barbara and Franco, whose constant support and belief have guided me throughout this challenging journey, and to whom this work is dedicated.

Contents

Introduction

An inverse problem deals with the reconstruction of unknown physical parameters from indirect observations. As noted by Sabatier [116], an inverse problem is not a particular type of mathematical problem, but rather a class of problems related to the identification of physical properties or quantities that cannot be directly observed. The term **inverse problem** derives from the fact that one starts from a set of observed effects to derive information about the causal factors, including parameters that are not directly observable. In contrast, a **direct problem** typically starts from a differential equation, the physical model, with a known structure and coefficients and tries to determine the effects, the solution ([86]).

It is difficult to say exactly when research on inverse problems began. While some discoveries may have been made earlier, the roots can be traced back to the 20th century. A 1911 article by Herman Weyl ([137]) is one of the first to consider an inverse problem. Weyl analysed the behaviour of the eigenvalues of the Laplace-Beltrami operator and established a correlation between the asymptotic behaviour of the eigenvalues and the volume of the domain on which the operator acts ("Is it possible to hear the shape of a drum?" Kac [79]). In 1929, the Soviet physicist A. V. A. Ambartsumian [22] formulated the inverse Sturm-Liouville problem, which investigates how the eigenvalues of an ordinary differential operator determine the eigenfunctions and parameters of the operator ([96]). He discovered that in the particular case of a homogeneous string, the eigenvalues determine the operator. Although Ambartsumian's article was initially ignored, it was rediscovered by a team of Swedish mathematicians at the end of World War II. As a result, it became the basis for a whole area of research on inverse problems.

Mathematically, inverse problems are (severely) ill-posed, which is the antonym of well-posed. The notion of well-posedness was first introduced by Jacques Hadamard in his 1902 article [70]. We recall his formulation (see Rivière in [110, Section 3.1] for a recent review). A problem is said to be **well-posed** if it satisfies the following properties:

- The problem has a solution;

- The solution is unique;
- The behaviour of the solution depends continuously on the initial data.

Many direct problems arising from mathematical models of physical phenomena are well-posed. On the other hand, the majority of inverse problems are ill-posed (see Isakov [77, Chapter 1]). We will focus on the analysis of the stability, which quantifies how small perturbations in the measured data affect the solution of the problem at hand. In inverse problems we speak of conditional stability rather than stability. Tikhonov [133] was one of the first mathematicians to notice that the introduction of constraints, the so-called *a-priori* information, on the unknowns could lead to a gain in stability for this problem. In applications, the study of (conditional) stability plays a crucial role as it guarantees the reliability of numerical reconstructions. Indeed, measurement data are typically obtained from a finite number of samples, which may be subject to noise or error.

This dissertation focuses on the study of the (conditional) stability of two main classes of inverse problems: the identification of coefficients and the determination of inclusions. On the one hand, the coefficient identification problem consists in the reconstruction of one or more physical parameters associated with a boundary value problem. On the other hand, the inclusion determination problem refers to the ability to identify the shape, size and location of an inclusion (an object or material) within a given medium based on measurements taken at the surface. This medium can be anything from solid material to biological tissue, and the inclusions might represent defects, anomalies, or substances of interest.

This thesis contributes to the existing literature by providing stability estimates for the anisotropic Calderón problem and two inverse problems for the anisotropic Schrödinger equation. In particular, we have provided a Lipschitz stability estimate for a special class of anisotropic conductivities using a novel quantity, the misfit functional for the classical Calderón problem. We have considered an anisotropic conductivity with a piecewise affine scalar function. There are few stability results in the literature for the anisotropic problem that do not consider the "up to diffeomorphism" approach, as we will discuss later. Moreover, piecewise affine coefficients are well studied in numerical reconstructions. For the inverse boundary value problem described by the generalised Schrödinger equation, we have provided a log-type stability estimate for the inclusion determination problem. We have considered a very general anisotropic inclusion with inhomogeneous coefficients, and the presence of the zeroth order term has required to overcome technical difficulties such as the lack of the Dirichlet to Neumann map, and hence the use of Cauchy data. We have also provided a Lipschitz estimate for the simultaneous determination of coefficients for the generalised Schrödinger equation when the coefficients are known to be piecewise affine. Lipschitz stability is a valuable property as it ensures the robustness and accuracy of the numerical reconstruction of the solution. In the literature, there are many results on the lack of uniqueness for this type of inverse problem, see for example Arridge and Lionheart [24] and Harrach [72], but only a few results on the uniqueness of both the coefficients and the stability. Moreover, our result has been proved in terms of local Cauchy data and without the assumption of monotonicity required in Harrach's work [72]. Another valuable result of this research work has been the extension of existing techniques, such as the singular solutions method, the construction of Green functions and some quantitative estimates of propagation of smallness.

Dissertation Plan

Chapter 1 is devoted to an introduction to the classical Calderón problem, also known as the inverse conductivity problem. We will recall some relevant uniqueness and stability results. In the last part of the chapter, we will focus on the stability issue in the finite-dimensional case. Chapter 2 is devoted to a brief review of the unique continuation tools. We derive a stability estimate for the Cauchy problem ([134]). Then, we focus on the three sphere inequalities in the case of Lipschitz and piecewise Lipschitz coefficients in the leading order term.

In Chapter 3, the inverse conductivity problem is considered. Before discussing the inverse problem, we introduce the direct problem. Let $\Omega \subset \mathbb{R}^n$ be a conductor with a sufficiently smooth boundary. Let $f \in H^{1/2}_{00}(\Sigma)$ be the prescribed voltage on a non-empty portion Σ of the boundary of Ω, $\partial\Omega$. If there are no sources or sinks in the medium, the induced electric potential $u \in H^1(\Omega)$ is a weak solution of the boundary value problem

$$\begin{cases} \mathrm{div}(\sigma \nabla u) = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here, σ represents the anisotropic conductivity, which is a real $n \times n$ symmetric matrix function that is bounded, measurable and positive definite. Moreover, σ satisfies the uniform ellipticity condition, i.e., there exists a constant $\lambda > 1$ such that

$$\lambda^{-1}|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq \lambda|\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for all } \xi \in \mathbb{R}^n.$$

Under these assumptions, the Dirichlet problem (1) is well-posed. As a result, we can introduce a map known as the Dirichlet to Neumann map (DtoN), represented by

$$\Lambda^{\Sigma}_{\sigma} : H^{1/2}_{00}(\Sigma) \to H^{-1/2}_{00}(\Sigma),$$

which assigns to each electric potential f at Σ the corresponding current density $\sigma \nabla u \cdot \nu|_{\Sigma}$. Knowing the DtoN map is equivalent to making an infinite number of boundary measurements.

The inverse problem is to recover the unknown conductivity σ from the knowledge of the given DtoN map.

This problem serves as a mathematical model in several research areas. In geophysics, we have the direct current (DC) resistivity methods, which aim to reconstruct the resistivity of a conductor from measurement data [138]. Conductivity is the inverse of resistivity. In DC resistivity methods, a current is introduced into the ground through two electrodes (C_1, C_2) and the induced voltage is measured through the other two electrodes (P_1, P_2) [119]. The source can be direct current or low frequency (0.1-30 Hz) alternating current. Among the different modes of operation, we mention the profiling methods, the vertical electrical sounding (VES) and the electrical resistivity tomography (ERT). DC resistivity methods are used to investigate lithological underground structures, to estimate the depth, thickness and properties of aquifers and aquicludes, to detect underground cavities, to monitor temporal changes in subsurface electrical properties, to name but a few. The method was introduced by Schlumberger in 1920 and involves the acquisition of voltage and current measurements at the surface of the medium (see [118]). As a matter of example, let us describe how the Schlumberger array works. It consists of four electrodes, the outer ones injecting electric current (sources) and the inner ones measuring the electric potential (receivers). The inner electrodes are situated at a fixed distance and remain stationary. The outer electrodes are moved in opposite directions to take the measurements. The process is repeated until a voltage of zero is reached. Other types of array used in DC resistivity methods are the Wenner array, the dipole-dipole array, the pole-pole electrode configuration, and the (crossed) square array acquisition, which is more sensitive to anisotropy. We also mention the electromagnetic inductive methods, which are used to obtain information about the electrical conductivity of the soil. They can be classified into natural field methods and controlled source methods. The latter are based on the measurement of the electromagnetic fields induced by controlled sources and can be divided into frequency domain electromagnetic methods (FEM) and time domain or transient electromagnetic methods (TEM). We refer to [119] for a survey on geophysical field methods.

The Calderón problem also serves as a mathematical model for electrical impedance tomography (EIT), which is a medical imaging technique developed in the 1980s. EIT produces images that show the distribution of electrical conductivity in human tissue [98, 48, 41, 40, 94]. Electrodes are placed on the surface of the skin and an alternating electric current is injected through a limited number of electrodes, while the remaining electrodes measure the resulting voltage. This process is repeated several times for different electrode configurations. There has been a growing interest in EIT for its potential application in medical diagnostics, such as lung protective ventilation, as it is an ideal tool for monitoring ventilation due to its non-invasive and radiation-free nature, and rapid response. Furthermore, it has potential applications in the early detection of breast cancer and stroke [76]. The ill-posed nature of the inverse problem, however, is evident from the fact that this technique is insensitive to small changes in the conductivity. Moreover, image reconstruction is affected by modelling and measurement errors [75]. However, the ill-posedness can be mitigated by imposing suitable *a-priori* information on the conductivity distribution [76]. For a complete survey on the Calderón problem and its relation to EIT, we refer to Uhlmann [135].

A. Calderón first presented the mathematical formulation of the inverse problem in "On an inverse boundary value problem" [45], which we discuss in Chapter 1. The motivation behind his work was oil exploration. The uniqueness issue was addressed after Calderón's 1980 publication, for measurements that can be made over the entire boundary (see Kohn and Vogelius [82, 83], Sylvester and Uhlmann [128], Alessandrini [8] and Nachmann [101]). For dimension $n \geq 2$, Astala and Päivärinta

[27] proved that L^∞ isotropic conductivities are uniquely determined.

On the issue of stability, Alessandrini proved in [8] that assuming *a-priori* bounds on σ of the form $\|\sigma\|_{H^s(\Omega)} \leq E$ in the isotropic case and for dimension $n \geq 3$, where $s > \frac{n}{2} + 2$, leads to a continuous dependence of σ in Ω on Λ_σ of logarithmic type. It has also been proved the Lipschitz continuous dependence of the restriction of the conductivity at the boundary of the domain and the DtoN map [130, 7]. Mandache [97] proved that in the interior of Ω, the logarithmic stability is sharp for $n \geq 2$, even if Lipschitz stability holds at the boundary (see [12, 13]). To obtain more accurate stability estimates, it is reasonable to replace the *a-priori* assumptions described in terms of regularity bounds by *a-priori* information of a distinct nature, suitable for the physical problem under consideration. Alessandrini and Vessella proved in [21] that when the conductivity σ is isotropic and piecewise constant on a given partition of Ω, there is a Lipschitz continuous dependence between the conductivity and the DtoN map. Additionally, Rondi [111] proved that the Lipschitz constant has an exponential behaviour with respect to the number of subdomains in the partition.

In Chapter 3 of this dissertation, we investigate the issue of stability for anisotropic conductivities belonging to a special class. Anisotropy is a property of solids for which the values of vector attributes are direction dependent. This property is observed in crystalline materials, but not in amorphous materials, which lack a crystalline structure. Although minerals are in general anisotropic, rocks composed of them may appear isotropic. Many tissues in the human body also display anisotropy. In the theory of homogenisation, anisotropy appears as a limit in layered or fibrous structures, such as rock formations or muscles, due to the crystalline structure or the deformation of an isotropic material.

From a mathematical perspective, the inverse problem for anisotropic conductivities is an open problem. Since Tartar's observation [84] that any diffeomorphism of Ω which keeps the boundary points fixed also leaves the DtoN map unchanged, while σ is modified, different lines of research have been pursued. One approach has been to determine the conductivity up to diffeomorphisms that keep the boundary fixed (see [91, 127, 89, 87, 28]). Another approach involves formulating suitable *a-priori* assumptions on the structure of the unknown anisotropic conductivity. For example, one can formulate the hypothesis that the directions of the anisotropy are known, while a scalar space-dependent parameter is not. We refer to the results in [8, 93, 12, 4, 13, 61, 62]. We also refer to [68, 67, 15] for non-uniqueness results in the anisotropic case.

We provide a stability estimate for the anisotropic conductivity in the form of

$$\sigma(x) = \Big[\sum_{m=1}^{N} \gamma_m(x)\, \chi_{D_m}(x)\Big]\, A(x), \qquad \text{for any } x \in \Omega, \tag{2}$$

where $\gamma_m(x)$ is an unknown affine scalar function on D_m, A is a known Lipschitz continuous matrix-valued function on Ω and $\{D_m\}_{m=1}^N$ is a given partition of Ω. Possible partitions include layered media models in the geophysical setting and bodies with multiple inclusions in the medical setting. The inversion problem of EIT is inherently ill-posed, and imaging deeper into the body Ω results in poor image resolution (see [63]).

The novelty of our approach lies in the use of a new method for modelling boundary data. Rather than utilising the DtoN map, which can be proved to be too expensive for numerical simulations, we introduce a misfit functional. This formulation was influenced by the work of [14] in the context of the Full Waveform Inversion (FWI). The FWI method is used in seismic exploration to recover the properties of the Earth's subsurface. The authors have considered data that can be acquired by modern dual sensors that measure pressure and vertical velocity modelled by the Cauchy data. They have conducted their study in the frequency domain and proved that by minimising the misfit functional it is possible to construct the wave speed within the medium. We consider two anisotropic conductivities, $\sigma^{(1)}$ and $\sigma^{(2)}$, of the form (2). We assume that the measurements are taken on an open portion Σ of the boundary of Ω, which is reasonable in view of real-life applications. We find it convenient to enlarge the physical domain Ω to an augmented domain Ω_0 and consider Green functions G_i associated with the elliptic operator $\mathrm{div}(\sigma^{(i)}\nabla\cdot)$ in Ω_0 for $i = 1, 2$. We express the quadratic error in the measurements corresponding to two different conductivities $\sigma^{(1)}$ and $\sigma^{(2)}$ using the misfit functional

$$\mathcal{J}(\sigma^{(1)}, \sigma^{(2)}) = \int_{D_y \times D_z} |S_0(y, z)|^2 \, dydz, \tag{3}$$

where D_y, D_z are suitably chosen sets that are compactly contained in $\Omega_0 \setminus \bar{\Omega}$, and $S_0(y, z)$ is given by the integral

$$S_0(y, z) = \int_\Sigma \left[G_2(\cdot, z)\sigma^{(1)}(\cdot)\nabla G_1(\cdot, y) \cdot \nu - G_1(\cdot, y)\sigma^{(2)}(\cdot)\nabla G_2(\cdot, z) \cdot \nu \right] \, dS. \tag{4}$$

In Theorem 4.5.1, we derive the following stability estimate of Hölder type:

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \leq C \, \left(\mathcal{J}(\sigma^{(1)}, \sigma^{(2)}) \right)^{1/2}, \tag{5}$$

where $C > 0$ is a constant that depends on the *a-priori* information only. The augmented domain Ω_0 is chosen in such a way that $G_1(\cdot, y)|_{\partial\Omega}$, $G_2(\cdot, z)|_{\partial\Omega}$ are supported in Σ in the trace sense, hence belonging to the domain of the local DtoN maps $\Lambda^\Sigma_{\sigma^{(i)}}$, $i = 1, 2$. Therefore, both (5) and the well-known Alessandrini's identity [7] imply the following global Lipschitz stability estimate in terms of the local Dirichlet to Neumann map:

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \leq C \, \|\Lambda^\Sigma_{\sigma^{(1)}} - \Lambda^\Sigma_{\sigma^{(2)}}\|_{\mathcal{L}(H^{1/2}_{00}(\Sigma), H^{-1/2}_{00}(\Sigma))}. \tag{6}$$

Moreover, we notice that the set of measurements $\left\{ G(\cdot, y)|_{\partial\Omega}, \sigma\nabla G(\cdot, y) \cdot \nu|_{\partial\Omega} \right\}$, where $y, \in \Omega_0 \setminus \bar{\Omega}$ is sufficient to determine σ in a stable manner. The use of the misfit functional may result in a simpler numerical implementation when compared to computing the norm of bounded linear operators between a trace space and its dual. Another advantage is the ability to separate computational and observed measurements by introducing sets D_y and D_z respectively, which can be chosen almost arbitrarily outside of the domain Ω. For example, D_y could be chosen to collect numerical data for the simulations, while D_z could represent the set for the actual measurement acquisition.

In Chapter 4, the inclusion determination problem is studied. Let Ω be a bounded domain in $\mathbb{R}^n$, where $n \geq 3$, and let D be an open set contained in Ω. We assume that Ω and D consist of different inhomogeneous and anisotropic materials. Let $u \in H^1(\Omega)$ be a weak solution of

$$\mathrm{div}(\sigma \nabla u) + q\, u = 0 \qquad \text{in } \Omega, \tag{7}$$

with

$$\sigma(x) = (a_b(x) + (a_D(x) - a_b(x))\chi_D(x))\, A(x), \tag{8}$$

and

$$q(x) = q_b(x) + (q_D(x) - q_b(x))\chi_D(x), \tag{9}$$

where a_b, q_b and a_D, q_D are the scalar parameters of the background body Ω and the inclusion D, respectively, χ_D is the characteristic function of D and $A(x)$ denotes a matrix-valued function. The pair of Cauchy data $\left\{u|_\Sigma, \sigma\nabla u \cdot \nu|_\Sigma\right\}$ represents the boundary measurements. The set of all pairs of Cauchy data restricted to the portion Σ is denoted by $\mathcal{C}_D^\Sigma$.

The inverse problem consists in the determination of the shape or the location of the inclusion D given the local boundary measurement $\mathcal{C}_D^\Sigma$.

The prototype for this class of inverse problems is the inclusion determination problem in an isotropic conductor by means of the DtoN map. This problem is known to be ill-posed. Uniqueness has been established by Isakov [78] by combining the Runge approximation theorem with the solutions of the equation (7) with Green function type singularities. Alessandrini and Di Cristo [5] have addressed the stability issue for inclusions with boundary of class $C^{1,\alpha}$ and piecewise constant conductivity. The authors have proved a logarithmic stability estimate using Isakov's arguments in a quantitative form. Their proof relies on the singular solution method, while the Runge approximation argument has been replaced by quantitative unique continuation estimates. The optimality has been proved by Di Cristo and Rondi [53]. This approach has stimulated a line of research, in which certain techniques and results have been extended to more general equations and systems (see [51, 52, 11, 100]). Recently, Lipschitz stability estimates for polygonal or polyhedral inclusions have been provided for the conductivity equation and the Helmholtz equation. The first theoretical result in this direction is due to Bacchelli and Vessella [29]. The authors consider the problem of determining an unknown polyhedral portion of the boundary of a two-dimensional region. They derived a Lipschitz stability estimate by proving that the map associated with the measurements is injective and uniformly continuous on a certain subset of admissible polyhedral profiles, is Frechét differentiable, the Frechét differential is uniformly continuous and the Frechét derivative is locally bounded from below. Lipschitz stability estimates have been provided in Beretta, Francini, Vessella [36] for polygonal inclusions and in Aspri, Beretta, Francini, Vessella [26] for polyhedral inclusions. Both these results are based on a two-step procedure. First, the authors prove a log-type, rough logarithmic estimate for the Hausdorff distance in terms of the boundary measurements. Then, they provide a Lipschitz stability estimate using the distributed representation of the Gateaux derivative of the Dirichlet to Neumann map.

Another class of problems that falls under the umbrella of this problem is optical tomography, a field predominantly studied in medical imaging (see [24]). We would like to underline the fact that the ill-posed nature of the inclusion determination problem in the conductivity equation case is evident in the numerical reconstruction.

In Theorem 4.0.1, we present an optimal stability estimate for the Hausdorff distance between the two inclusions, in terms of the local Cauchy data. Our approach was inspired by the works of Alessandrini and Di Cristo [5] and Alessandrini, De Hoop, Gaburro and Sincich [17].

Let us summarise the new tools and arguments required for the proof, together with the corresponding issues.

The pairs of boundary data $\left\{u|_\Sigma, \sigma\nabla u\cdot\nu|_\Sigma\right\}$ are modelled by the local Cauchy data set

$$\begin{aligned}\mathcal{C}_D^\Sigma(\Sigma) = \Big\{(f,g)\in H_{00}^{1/2}(\Sigma)\times H^{-1/2}(\partial\Omega)|_\Sigma \ :\ \exists\, u\in H^1(\Omega) \text{ weak solution of}\\ \operatorname{div}(\sigma\nabla u) + q\,u = 0 \qquad \text{in } \Omega,\\ u|_{\partial\Omega} = f,\\ \langle \sigma\nabla u\cdot\nu|_{\partial\Omega}, \varphi\rangle = \langle g,\varphi\rangle \qquad \text{for all } \varphi\in H_{00}^{1/2}(\Sigma)\Big\}.\end{aligned} \tag{10}$$

The local Cauchy data set is a closed subspace of the Hilbert space $H_{00}^{1/2}(\Sigma)\times H^{-1/2}(\partial\Omega)|_\Sigma$. Note that when $q=0$, the Dirichlet problem (7) is well-posed, so that the local Dirichlet to Neumann map is well defined and the Cauchy data set is its graph. Given two inclusions D_1 and D_2, we denote the corresponding local Cauchy data sets by $\mathcal{C}_1, \mathcal{C}_2$. Following the ideas of [81, 17], the discrepancy between two local Cauchy data sets is measured by a quantity called distance or aperture, denoted by $d(\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma)$.

Our main result is that if two local Cauchy data $\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma$ are at a distance less than 1, then there is a log-type dependence between the Hausdorff distance of the boundary of the two inclusions and the boundary data. Formally, there exists a positive constant $C>0$ and $0<\eta<1$ such that the following logarithmic estimate holds

$$\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2) \le C\,|\ln d(\mathcal{C}_{D_1}^\Sigma, \mathcal{C}_{D_2}^\Sigma)|^{-\eta}. \tag{11}$$

The proof of this result relies on the method of singular solutions and the study of their blow-up behaviour near the boundary of the inclusions, which has been successful in proving stability since the work of Alessandrini [8]. However, the application of this method is not straightforward.

The first obstacle in this work comes from the fact that the direct problem is not well-posed. Since no sign or spectral condition is assumed on q, the boundary value problem associated with (7) may be in the eigenvalue regime. To overcome this issue, we introduce a slightly different boundary value problem. This idea was first applied in [17] for the Schrödinger equation. Bamberger and Ha Duong in [31] lay the ground of the well-posedness of the mixed boundary value problem of this form. Here, one additional issue to consider is that the principal part has a matrix-valued coefficient which may be discontinuous at the boundary of the inclusion D.

Before presenting the formulation of the boundary value problem, let us introduce some notation. We consider an enlarged domain Ω_0 given by the union of Ω and a set $D_0 \subset \mathbb{R}^n \setminus \Omega$ with the condition that $\partial D_0 \cap \partial \Omega \subset \Sigma$. We choose a non-empty portion Σ_0 from $\partial \Omega_0 \setminus \partial \Omega$. The boundary value problem we consider involves the generalised Schrödinger equation (7) and a prescribed complex Robin boundary condition at Σ_0, with homogeneous Dirichlet boundary condition in the remaining portion. This mixed boundary value problem is well-posed: existence and uniqueness can be derived using the Fredholm alternative. The stability is derived by applying a propagation of smallness estimate for a second order elliptic equation in divergence form with Lipschitz continuous leading coefficients on both sides of a C^2 hyperplane. This result was proved by Carstea and Wang [47].

Now, let $y \in \Omega_0$, and let $G(\cdot, y) \in H^1(\Omega_0)$ be the Green function solution to

$$\begin{cases} \operatorname{div}(\sigma \nabla G(\cdot, y)) + q\,u = 0 & \text{in } \Omega_0, \\ G(\cdot, y) = 0 & \text{on } \partial \Omega_0 \setminus \Sigma_0, \\ \sigma \nabla G(\cdot, y) \cdot \nu + iG(\cdot, y) = 0 & \text{on } \Sigma_0. \end{cases} \tag{12}$$

Let G_1 and G_2 be the Green functions satisfying (12). The singular solution can be expressed as

$$f(y, w) = S_1(y, w) - S_2(y, w), \tag{13}$$

where, for $k = 1, 2$,

$$\begin{aligned} S_k(y, z) = &\int_{D_k} (a_{D_k}(x) - a_b(x)) \nabla G_1(x, y) \cdot \nabla G_2(x, z)\, \mathrm{d}x \\ &+ \int_{D_k} (q_{D_k}(x) - q_b(x)) G_1(x, y)\, G_2(x, z)\, \mathrm{d}x, \end{aligned}$$

for $y, z \in D_0$, by applying the first Green identity, we derive the following inequality

$$\begin{aligned} &|f(y, z)| \leq \\ &\leq d(\mathcal{C}^{\Sigma}_{D_1}, \mathcal{C}^{\Sigma}_{D_2})\, \|(G_1, \sigma_1 \nabla G_1 \cdot \nu)\|_{H^{1/2}_{00}(\Sigma) \times H^{-1/2}_{00}(\Sigma)}\, \|(\bar{G}_2, \sigma_2 \nabla \bar{G}_2 \cdot \nu)\|_{H^{1/2}_{00}(\Sigma) \times H^{-1/2}_{00}(\Sigma)}. \end{aligned} \tag{14}$$

Before proceeding, it is important to mention a geometrical aspect. In a general context, the Hausdorff distance may be achieved at a point that can only be reached from the boundary portion Σ by passing through the boundary of two inclusions. This is an obstruction when applying the propagation of smallness argument. To overcome this issue, Alessandrini and Di Cristo [21] have introduced the so-called modified distance, which is a quantity that allows to bound the Hausdorff distance in terms of a modified one which involves only points that can be reached from the exterior. It is important to note that we can only perform unique continuation estimates on points that are close to the boundary of the two inclusions and that are contained in a subset, denoted as V, that can be reached from the connected component of $\mathbb{R}^n \setminus (D_1 \cup D_2)$ in a quantitative form. This involves using a chain of a finite number of balls whose numbers are suitably bounded and whose radii must be bounded from below (see [20, 11, 113]). In this context, a fundamental step is the proof of the existence of a point $P \in \partial D_1 \setminus \overline{D_2}$ such that the Hausdorff distance between ∂D_1 and ∂D_2 is dominated by the distance $dist(P, D_2)$.

Now, if y and w are located outside Ω, we can control $f(y, w)$ in (13) in terms of the distance $d(\mathcal{C}^{\Sigma}_{D_1}, \mathcal{C}^{\Sigma}_{D_2})$ by applying (14). As y, w are moved inside Ω within a connected set contained in $\mathbb{R}^n \setminus (D_1 \cup D_2)$, we propagate the smallness of f near the boundary of the two inclusions.

We show that as $y = w$ approaches the point P of $\partial D_1 \setminus \overline{D_2}$, $f(y, y)$ blows up. This, combined with the upper bound of $f(y, y)$ in terms of $d(\mathcal{C}^{\Sigma}_{D_1}, \mathcal{C}^{\Sigma}_{D_2})$ discussed above, leads to (11).

We wish to remark that the presence of an anisotropic and inhomogeneous leading coefficient, along with an additional zero-order term, requires a careful analysis of the asymptotic behaviour of the Green functions $G_j(\cdot, y)$ as the pole y approaches the inclusion $D_j,\ j = 1, 2$.

In the concluding part of the chapter, we derive the following stability estimate

$$\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2) \leq C \left|\log \mathcal{J}(D_1, D_2)\right|^{-\eta}, \tag{15}$$

in terms of a misfit functional

$$\begin{aligned} \mathcal{J}(D_1, D_2) = \int_{D_y \times D_z} \Big| \int_{\Sigma} &[\sigma_1(x)\nabla G_1(x, y) \cdot \nu(x)\, G_2(x, z) - \\ &-\sigma_2(x)\nabla G_2(x, z) \cdot \nu(x)\, G_1(x, y)]\ dS(x)\Big|^2 dy\, dz, \end{aligned} \tag{16}$$

where the D_y, D_z are appropriately selected sets that are compactly contained in $\mathbb{R}^n \setminus \bar{\Omega}$. As proved in [14] (see also [59]), the Hölder estimate (15) can be applied in numerical reconstructions. The stability estimate derived in terms of this misfit functional suggests that it is not necessary to know the full local Cauchy data set. It is sufficient to sample it on Green's type functions with sources placed outside the physical domain.

In Chapter 5, we consider the coefficient identification problem associated with the generalised Schrödinger equation. We begin with the formulation of the inverse problem. Let Ω be a bounded domain of $\mathbb{R}^n$ and Σ be a non-empty portion of the boundary $\partial\Omega$. Let $u \in H^1(\Omega)$ denote a weak solution of the equation

$$\mathrm{div}(\sigma \nabla u) + q\, u = 0, \qquad \text{in } \Omega. \tag{17}$$

The inverse problem consists in the simultaneous determination of the pair of coefficients σ and q from the knowledge of all the possible pairs of Cauchy data $\{u|_{\partial\Omega}, \sigma\nabla u \cdot \nu|_{\partial\Omega}\}$.

We address the stability issue, and we prove a Lipschitz stability estimate that holds simultaneously for both the coefficients σ and q under suitable *a-priori* assumptions. Let us introduce the main *a-priori* assumptions. There is a given partition $\{D_j\}_{j=1}^N$, $N \in \mathbb{N}$ consisting of a finite number of bounded domains with boundary of class C^2 such that $\overline{\Omega} = \cup_{j=1}^N \overline{D_j}$. Notice that in previous works the boundary regularity was at most Lipschitz. However, this regularity assumption is necessary in order to have singular solutions with Green's type singularities.

The coefficients σ and q are finite-dimensional and have the form

$$\sigma(x) := \gamma(x)A(x) = \left(\sum_{j=1}^{N} \gamma_j(x)\chi_{D_j}(x)\right) A(x), \qquad q(x) := \sum_{j=1}^{N} q_j(x)\chi_{D_j}(x), \tag{18}$$

where γ_j, q_j are affine functions for $j = 1, \ldots, N$, and $A(x)$ is a known $C^{1,1}(\Omega, Sym_n)$ matrix function, where Sym_n is the space of real $n \times n$ symmetric matrices. Let $\mathcal{C}_i$, for $i = 1, 2$, represent the local Cauchy data sets associated with the pairs of coefficients $\{\sigma^{(i)}, q^{(i)}\}_{i=1,2}$, as defined in (18). We denote the distance between these sets as $d(\mathcal{C}_1, \mathcal{C}_2)$. In Chapter 5 we aim to prove that

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} \leq C\, d(\mathcal{C}_1, \mathcal{C}_2), \tag{19}$$

with $C > 0$ a constant that depends only on the *a-priori* data. To derive the stability estimate, we adopt a constructive approach based on the singular solutions method and the quantitative estimates of unique continuation (see [21, 33, 17] and [136] for a recent survey). It is important to mention that the Green functions considered here are the ones constructed in Chapter 4 for a boundary value problem on an enlarged domain Ω_0. On the boundary of Ω_0, a complex-valued Robin condition on a portion Σ_0 of $\partial\Omega_0 \setminus \partial\Omega$ and a homogeneous Dirichlet condition on the remaining part are prescribed.

The boundary value problem associated with the equation (17) encompasses a wide range of inverse problems that are characterised by their ill-posed nature. The first problem that has motivated our study is the Calderón problem, namely the inverse conductivity problem, for which the boundary data are modelled by the DtoN map. As already discussed earlier on in this thesis, Alessandrini investigated the stability issue in [8] for isotropic conductivities belonging to $H^s(\Omega)$ for $s > \frac{n}{2} + 2$ providing a log-type stability estimate. This result is optimal, as proved by Mandache [97]. The ill-posed character of the inverse conductivity problem is a common denominator in this field and it constitutes an obstruction in numerical reconstructions. To overcome this issue, it is convenient to restrict the space of admissible conductivities by imposing appropriate *a-priori* assumptions on the conductivity. This line of research was pursued by Alessandrini and Vessella [21], who proved a Lipschitz stability estimate for piecewise constant conductivities defined on a finite partition of the domain Ω that satisfy certain *a-priori* bounds. Rondi [112] has proved that the Lipschitz constant appearing in the stability estimate [21, Theorem 2.7] behaves exponentially with respect to the number N of subdomains of the partition. This result was subsequently extended by Di Cristo and Rondi [53] for the inverse scattering problem and the inclusion determination problem, and by Sincich [120] for the corrosion detection problem. Recently, in [3], Alberti, Arroyo and Santacesaria have extended these ideas by proving that for coefficients belonging to finite dimensional manifolds, uniqueness and stability are guaranteed. Lipschitz stability estimates have been proved for real and complex finite dimensional isotropic coefficients ([21, 18, 35]), for a special type of anisotropic conductivities ([62, 59]), for polyhedral inclusions in a conductive medium ([36, 26, 34]), for the nonlocal operator ([114]), and for the elasticity case ([55]). This list is far from being complete! However, it includes results that are all based on the singular solutions method and unique continuation techniques.

When σ is the identity matrix, equation (17) is the Schrödinger equation. Lipschitz stability has been proved both when the DtoN map is defined (hence under suitable spectral conditions) and when only Cauchy data are available, in the case of a finite dimensional potential q (see [39, 17, 115]). When q has a positive sign, (17) is the reduced wave equation or the Helmholtz equation. In [37], the authors succeeded in proving the conditional Lipschitz stability at selected frequencies, using the DtoN map. See also [14] for the related numerical experiments.

When q is a non-positive scalar function, the boundary value problem associated with (17) models the propagation of light in a body and corresponds to the diffusion approximation of the radiative transfer equation in the frequency domain. In this framework, the coefficients σ and q model the diffusion and absorption coefficients, respectively. The corresponding application is the diffuse optical tomography (DOT), a novel, non-invasive technique that allows one to map the optical properties of a tissue (see [23, 25]). In [24], Arridge and Lionheart proved that, under general assumptions, it is not possible to recover the diffusion and the absorption coefficients simultaneously. However, later results showed that if the coefficients belong to a finite dimensional space of bounded functions, it is possible to determine the coefficients simultaneously. In [72], Harrach proved uniqueness under the assumption that the diffusion coefficient is piecewise constant and the absorption coefficient is piecewise analytic. The author used the technique of localised potentials, developed by the same author in [65], and the monotonicity method also used in [73]. Recently, the method of localised potentials has been successfully employed by Harrach and Lin ([74]) to recover piecewise analytic coefficients in a semilinear elliptic equation, under proper hypotheses that ensure the existence of the DtoN map. Results of Lipschitz and Hölder stability at the boundary of the absorption coefficient and its derivatives, respectively, have been proved in [54] and [49] in the anisotropic time-harmonic case. Finally, we would like to mention another application, photoacoustic tomography, an imaging technique that combines the high contrast of optical tomography with the high resolution of acoustic waves ([30]). In a recent paper (see [10]), Alessadrini, Di Cristo, Francini and Vessella simultaneously determine the absorption and diffusion coefficients from the measurements of the energy distribution.

The proof of our main result can be summarised as follows. We define

$$E = \max\{\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)}, \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)}\}.$$

Then, there is an index $K \in \{1, \dots, N\}$ such that E is reached in D_K. Let D_1 be a domain of the given partition such that $\partial D_1 \cap \Sigma \neq \emptyset$. Then we fix a chain of contiguous subdomains $D_0, D_1, \dots, D_K$.

We use an iterative procedure to determine a bound for the coefficients in terms of the boundary data. It is based on the strategy introduced in Alessandrini and Vessella [21] for the determination of a coefficient and generalised in our case. We determine Hölder type boundary estimates on the portion Σ_1 in terms of the Cauchy data, from which we obtain the following estimate:

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_1)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(D_1)} \leq C\,(E+\varepsilon)\left(\frac{\varepsilon}{E+\varepsilon}\right)^{\tilde{\eta}_1}, \qquad (20)$$

where $0 < \tilde{\eta}_1 < 1$ depends only on the *a-priori* data and $\varepsilon = d(\mathcal{C}_1, \mathcal{C}_2)$. The estimate (20) is derived by applying an Alessandrini's type argument, and the study of the blowup rate of the Green function near the discontinuity interface. We then apply the following two-step procedure, which we describe for the domain D_2.

1. We determine an upper bound for $\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(D_2)}$ by exploiting the blow-up rate of the singular solutions near the discontinuity interface for the coefficient and quantitative estimates of unique continuation, which are based on the propagation of smallness estimates proved by Carstea and Wang in [47] that hold for piecewise Lipschitz coefficients.
2. We estimate $\|q_2^{(1)} - q_2^{(2)}\|_{L^\infty(D_2)}$ by taking advantage of the stability estimate in 1, the asymptotic estimates for the Green functions and the quantitative estimates of unique continuation.

We continue this iterative process until we reach the domain D_K, where we derive the following inequality:

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_K)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(D_K)} \leq C\,(E+\varepsilon)\,\omega_{\tilde{\eta}_K}^{(3(K-1))}\left(\frac{\varepsilon}{E+\varepsilon}\right), \qquad (21)$$

where $0 < \tilde{\eta}_K < 1$ is a constant that depends only on the *a-priori* data, and $\omega_{\tilde{\eta}_K}$ is a modulus of continuity of logarithmic type of the form

$$\omega_{\tilde{\eta}_K}(t) \leq C\,|\ln t^{-1}|^{-\tilde{\eta}_K} \quad \text{for } t \in (0,1).$$

Inequality (21), along with the invertibility of the modulus of continuity, leads to the desired Lipschitz stability estimate (19).

As a corollary, we derive a stability estimate on the portion Σ of Hölder type for both the coefficients σ and q of the form

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Sigma)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Sigma)} \leq C\,(d(\mathcal{C}_1, \mathcal{C}_2) + E)^{1-\eta}\, d(\mathcal{C}_1, \mathcal{C}_2)^{\eta}, \qquad (22)$$

for $0 < \eta < 1$, and $C > 0$ is a positive constant depending on the *a-priori* data only. Notice that this estimate is consistent with previous results. Indeed, for the Calderón problem, the estimate at the boundary is Lipschitz [7, 130], whereas for the Schrödinger equation with Cauchy data it is Hölder [17].

Abbreviations and Notation

- a.e.: almost everywhere.
- e.g.: exempli gratia.
- i.e.: id est.
- $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- $x = (x', x_n)$: $x \in \mathbb{R}^n$, $x' = (x_1, \dots, x_{n-1})$.
- $\mathbb{R}^n_{\pm} = \{x \in \mathbb{R}^n : x_n \gtrless 0\}$.
- $B(x, r)$: the n-dimensional open ball centred at x with radius $r > 0$.
- $B'(x', r)$: the $(n-1)$-dimensional open ball centred at x' with radius $r > 0$
- $Q(x, r) = B'(x', r) \times (x_n - r, x_n + r)$: the cylinder centred at x with height $2r$.
- $B_r = B(0, r)$, $B'_r = B'(0, r)$.
- $Q_r = Q(0, r)$, $Q'_r = Q'(0, r)$.
- $B^{\pm}_r = B_r \cap \mathbb{R}^n_{\pm}$.
- $\omega_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$: the volume of the unit ball in $\mathbb{R}^n$.
- $A \subset\subset B$ or $A \Subset B$: A has compact closure in B.
- $\partial\,\Omega$: the boundary of a bounded set Ω.
- $|\Omega|$: the n-dimensional Lebesgue measure of a measurable set Ω.
- $supp f$: the support of the function f.
- $\nabla u = \left(\dfrac{\partial\, u}{\partial\, x_1}, \dots, \dfrac{\partial\, u}{\partial\, x_n}\right)$: the gradient of u.

- $\operatorname{div}(u) = \sum_{i=1}^{n} \frac{\partial\, u}{\partial\, x_i}$: the divergence operator.
- $\Delta u = \sum_{i=1}^{n} \frac{\partial^2\, u}{\partial\, x_i^2}$: the Laplacian operator.
- $C^k(\Omega)$: the space of continuous scalar functions with continuous derivatives up to order $k \in \mathbb{N}_0$ in an open set Ω of $\mathbb{R}^n$.
- $C_c^k(\Omega)$: the space of continuous scalar functions with continuous derivatives up to order $k \in \mathbb{N}_0$ with compact support in an open set Ω of $\mathbb{R}^n$.
- $C^{0,\alpha}(\Omega)$: the space of scalar Hölder continuous functions with exponent $\alpha \in (0,1]$ (Lipschitz continuous functions if $\alpha = 1$).
- $C^{k,\alpha}(\Omega)$: the space of C^k continuous functions whose derivatives of order $k \in \mathbb{N}$ belong to $C^{0,\alpha}(\Omega)$.
- $L^p(\Omega)$: the space of measurable functions that are p-integrable for $p \geq 1$.
- $W^{m,p}(\Omega)$: the Sobolev space of functions in $L^p(\Omega)$ whose derivatives of order $\leq m$ belong to $L^p(\Omega)$ for $m \in \mathbb{N}, p \geq 1$.

CHAPTER **1**

The inverse conductivity problem

Contents

In this chapter, we discuss the inverse conductivity problem, a widely studied reference model in the field of inverse problems known as the Calderón problem. This problem is the basis for many other inverse problems, including travel-time tomography ([124]) and boundary rigidity problems ([123]).

In Section 1.1, we analyse Alberto Calderón's formulation of the inverse conductivity problem, as presented in his 1980 paper. Calderón aimed to determine the conductivity of a medium by measuring the voltage and current at its boundary. We examine the well-posedness of the direct problem.

Section 1.2 gives an overview of certain uniqueness results. In Section 1.3, we will outline the most recent stability results. Subsection 1.3.1 will cover the known stability results for the Calderón problem, including the stability estimates of Alessandrini [7] and the Lipschitz stability estimates of Alessandrini and Vessella [21] for piecewise constant conductivities, and the results of Gaburro and Sincich [62] for special anisotropic conductivities and Beretta and Francini [35] for complex conductivities.

Finally, Subsection 1.3.2 is devoted to the stable determination of an inclusion in the case of a piecewise constant conductivity established by Alessandrini and Di Cristo [21], and to a Lipschitz stability estimate due to Aspri, Beretta, Francini and Vessella [26] for polyhedral inclusions.

First, we establish the notation. We denote a multi-index as any n-uple of elements of $\mathbb{N}_0^n$ of the form

$$\alpha = (\alpha_1, \ldots, \alpha_n), \quad \alpha_j \in \mathbb{N}_0, \quad j = 1, 2 \ldots, n.$$

For any $\alpha \in \mathbb{N}_0^n$, let $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$ and $\alpha! = \alpha_1!\,\alpha_2! \ldots \alpha_n!$ be the length (modulus) and the factorial, respectively. For any $x \in \mathbb{R}^n$, set

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

Next, we introduce quantitative notions of smoothness for the boundary of a domain, which will be referred to throughout this monograph.

Definition 1.0.1 ($C^{k,\alpha}$ regularity)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Let $k \geq 0$ be a positive integer and $\alpha \in (0, 1]$. The boundary $\partial\Omega$ is considered to be of* class $C^{k,\alpha}$ *with constants r_0 and $M_0 > 0$ if, for each point $P \in \partial\,\Omega$, there exists a rigid transformation under which P coincides with the origin O and the following condition is satisfied:*

$$\Omega \cap Q_{r_0} = \{x \in Q_{r_0} \, : \, x_n > \varphi(x')\} \, .$$

The function φ is a $C^{k,\alpha}$ function on the ball B'_{r_0} which satisfies the following conditions:

$$\varphi(0') = |D^\beta \varphi(0')| = 0 \quad \textit{and} \quad \|\varphi\|_{C^{k,\alpha}(B'_{r_0})} \leq M_0\, r_0,$$

where β is a multi-index with $0 \leq |\beta| \leq k$.

When $k = 0$ and $\alpha = 1$, the boundary is said to be of Lipschitz class *with constants r_0, M_0.*

Remark 1.0.1. *If $\partial\,\Omega$ is of class $C^{0,1}$, then Radamacher's theorem proves that the unit exterior normal $\nu(x)$ to $\partial\,\Omega$ exists for $\mathcal{H}^{n-1}$ for almost every $x \in \partial\,\Omega$ (see [57, Section 3.1.2]).*

Definition 1.0.2 (C^k regularity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $k \geq 1$ be a positive integer. We say that the boundary $\partial\Omega$ is of* class C^k *with constants $r_0, M_0 > 0$ if for every point $P \in \partial\,\Omega$ there is a rigid transformation under which P coincides with the origin O, and*

$$\Omega \cap Q_{r_0} = \{x \in Q_{r_0} \, : \, x_n > \varphi(x')\} \, ,$$

where φ is a C^k function on the ball B'_{r_0} that satisfies the following condition:

$$\varphi(0') = |D^\beta \varphi(0')| = 0 \qquad \textit{and} \qquad \|\varphi\|_{C^k(B'_{r_0})} \leq M_0\, r_0,$$

where β is a multi-index with $0 \leq |\beta| \leq k$.

Definition 1.0.3. *Let Ω be a bounded domain of $\mathbb{R}^n$. We define a portion Σ of the boundary $\partial\,\Omega$ to be a* flat portion *of size r_0 if for every point $P \in \Sigma$ there exists a rigid* *transformation under which P coincides with the origin 0 and*

$$\begin{aligned}\Sigma \cap Q_{r_0} &= \{x \in Q_{r_0} \,:\, x_n = 0\}\,,\\ \Omega \cap Q_{r_0} &= \{x \in Q_{r_0} \,:\, x_n > 0\}\,,\\ (\mathbb{R}^n \setminus \Omega) \cap Q_{r_0} &= \{x \in Q_{r_0} \,:\, x_n < 0\}\,.\end{aligned}$$

Remark 1.0.2. *In this monograph, we will follow the convention of normalising norms to be dimensionally equivalent to their argument. Therefore, the norm $C^{k,\alpha}$ in the definition 1.0.1 is normalised as follows:*

$$\|\varphi\|_{C^{k,\alpha}(B'_{r_0})} = \sum_{j=0}^{k} r_0^j \sum_{|\beta|=j} \|D^\beta \varphi\|_{L^\infty(B'_{r_0})} + r_0^{k+\alpha} \sum_{|\beta|=k} \sup_{\substack{x,y\in B'_{r_0}\\ x\neq y}} \frac{|D^\beta\varphi(x) - D^\beta\varphi(y)|}{|x-y|^\alpha}.$$

In particular, for any $\varphi \in C^{0,1}(B'_{r_0})$, the norm can be expressed as

$$\|\varphi\|_{C^{0,1}(B'_{r_0})} = \|\varphi\|_{L^\infty(B'_{r_0})} + r_0 \sup_{\substack{x,y\in B'_{r_0}\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|}.$$

1.1 Background and Literature Review

The inverse conductivity problem was introduced by Alberto P. Calderón, an Argentinian mathematician known for his work on singular integral operators. Calderón first worked as an engineer at Yacimientos Petroliferos Fiscales (YPF), a state-owned Argentine energy company involved in the exploration, production, and transportation of oil and gas, before becoming one of the most important analysts of the 20th century. While working in the company's geophysics department in the late 1940s, Calderón developed an interest in oil exploration and worked on the problem of determining the electrical conductivity of a body using boundary measurements. These measurements take the form of voltage and current measurements on the conductor's surface.

Calderón initially ignored these results, but later published them in a note to the Brazilian Mathematical Society (SBM) in ATAS pp. 65-73 in the 1980's. This was subsequently republished in [45]. In this paper, Calderón proposed a mathematical formulation of the inverse conductivity problem, which we will briefly consider here. Assume that D is a bounded domain in $\mathbb{R}^n$, with a Lipschitz boundary as in Definition 1.0.1, where $n \geq 2$. Let γ be a measurable function, bounded on D, and with a positive lower bound of the form $\gamma(x) \geq \theta > 0$. Consider the second-order elliptic operator

$$\mathrm{L}_\gamma(u) = \mathrm{div}(\gamma \nabla u) \qquad \text{for } u \in H^1(\Omega). \tag{1.1}$$

Let Q_γ be the quadratic form associated with the operator L_γ, given by

$$\mathrm{Q}_\gamma(\phi) = \int_D \gamma |\nabla u|^2 \,\mathrm{d}x,$$

where $u \in H^1(\Omega)$ is a weak solution of

$$\begin{cases} \mathrm{L}_\gamma(u) = 0 & \text{in } D, \\ u|_{\partial D} = \phi \in H^{1/2}(\partial D). \end{cases}$$

Calderón introduced the map

$$\mathrm{Q} : \gamma \to \mathrm{Q}_\gamma$$

which associates each conductivity on Ω with the corresponding induced electric power. The problem is to determine whether it is possible to invert the map Q.

This problem arises in electrical prospecting, where the aim is to find the unknown conductivity of a conductive medium by making steady-state DC electrical measurements at the surface of the body. In this framework, D represents a possibly inhomogeneous conductor, the trace function ϕ represents the voltage at the conductor's surface, and the quadratic form $\mathrm{Q}_\gamma(\phi)$ represents the energy necessary to maintain the conductor's electric potential ϕ at the surface. Calderón proved that the map Q is bounded and analytic on a subset of conductivities which has a positive lower bound; it is thus Fréchet differentiable. As noted by Uhlmann [135], the Fréchet derivative of Q at $\gamma = \gamma_0$, where γ_0 represents a constant conductivity, takes the following form

$$d\,\mathrm{Q}\Big|_{\gamma=\gamma_0}(h)(f,g) = \int_\Omega h \nabla u \cdot \nabla v,$$

where $u, v \in H^1(\Omega)$ are weak solutions of the following Dirichlet problems:

$$\begin{cases} \operatorname{div}(\gamma_0 \nabla u) = \operatorname{div}(\gamma_0 \nabla v) = 0, & \text{in } \Omega, \\ u\Big|_{\partial\Omega} = f \in H^{1/2}(\partial\Omega), \quad v\Big|_{\partial\Omega} = g \in H^{1/2}(\partial\Omega). \end{cases} \tag{1.2}$$

Calderón proved the unique solvability of the linearized problem by showing that the Fréchet derivative $d\,\mathrm{Q}\,|_{\gamma=\gamma_0}$ evaluated at a constant conductivity γ_0 is injective. When $\gamma_0 = 1$, this is equivalent to proving that the L^2 inner product between the gradients of two harmonic functions is dense in L^2.

Since Calderón's initial contribution, there has been a growing interest in the international community of mathematicians to study the well-posedness of this problem and to develop efficient algorithms for reconstructing the unknown parameters. Rather than considering the induced electric power and attempting to invert the map Q, mathematicians have found it more convenient to formulate the inverse problem in terms of the measurements. We present the current formulation of the inverse conductivity problem. Let Ω be a bounded domain with Lipschitz boundary in $\mathbb{R}^n$. Consider the elliptic operator defined in (1.1) with coefficient σ. We assume that σ is a real $n \times n$ symmetric matrix function in $L^\infty(\Omega, Sym_n)$ which satisfies the condition of *uniform ellipticity*, i.e., there exists a constant $\bar{\lambda} > 1$ such that for any $\xi \in \mathbb{R}^n$ and almost any $x \in \Omega$,

$$\bar{\lambda}^{-1}|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq \bar{\lambda}|\xi|^2. \tag{1.3}$$

The lower index for the operator defined in (1.1) will be dropped from now on. The dual space of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$. The duality between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$

is denoted by the standard bracket notation $\langle\cdot,\cdot\rangle$. Let $H^{1/2}(\partial\Omega)$ represent the space of traces of H^1 functions with traces on $\partial\Omega$.

For $f \in H^{1/2}(\partial\Omega)$ and $F \in H^{-1}(\Omega)$, we say that a scalar function $u \in H^1(\Omega)$ is a weak solution of:

$$\begin{cases} \mathrm{L}(u) = F & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

if u satisfies the following identity

$$\int_\Omega \sigma(x)\nabla u(x)\cdot\nabla\varphi(x)\,\mathrm{d}x = \int_\Omega F(x)\varphi(x)\,\mathrm{d}x, \qquad \text{for any } \varphi \in H_0^1(\Omega), \tag{1.5}$$

with $u|_{\partial\Omega} = f$ in the trace sense. The hypothesis on σ allows us to define a bilinear form B corresponding to the elliptic operator L, as follows

$$\mathrm{B}[u,v] := \int_\Omega \sigma\,\nabla u\cdot\nabla v\,\mathrm{d}x, \qquad \text{for any } u,v \in H^1(\Omega). \tag{1.6}$$

Therefore, $u \in H^1(\Omega)$ is a weak solution of (1.4) if and only if

$$\mathrm{B}[u,\varphi] = \langle F,\varphi\rangle, \qquad \text{for any } \varphi \in H_0^1(\Omega),$$

and $u|_{\partial\Omega} = f$ in the trace sense (see [56, Chapter 6]). For a bounded domain $\Omega \subset \mathbb{R}^n$, the trace space $H^{-1/2}(\partial\Omega)$ is defined as the dual space of $H^{1/2}(\partial\Omega)$. The duality between these trace spaces is expressed by the brackets $\langle\cdot,\cdot\rangle$, as usual. If the boundary $\partial\Omega$ is of class C^k with $k \geq 1$, the duality reduces to the L^2 inner product on $\partial\Omega$. We denote the Banach space of bounded linear operators over the trace space $H^{1/2}(\partial\Omega)$ and its dual with $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. In the following theorem we introduce a classical result of well-posedness for boundary value problem (1.4) ([58, Theorem 2.52]).

Theorem 1.1.1 (Well-posedness of the Direct Problem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let L be the second-order elliptic operator as in* (1.1) *with the coefficient $\sigma \in L^\infty(\Omega, Sym_n)$ which satisfies* (1.3)*. Then, for any $f \in H^{1/2}(\partial\Omega)$ and $F \in H^{-1}(\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of the boundary value problem* (1.4).

Moreover, there exists a constant $C > 0$ independent of f and F such that

$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{H^{1/2}(\partial\Omega)} + \|F\|_{H^{-1}(\Omega)}). \tag{1.7}$$

Proof. We divide the proof into two parts.

Homogeneous Dirichlet problem. Let $F \in H^{-1}(\Omega)$ and consider the boundary value problem

$$\begin{cases} \mathrm{L}(u) = F & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

Notice that, thanks to (1.3) and the Poincarè inequality A.2.1, the bilinear form B defined in (1.6) is an inner product on $H_0^1(\Omega)$ and induces a norm that is equivalent to the original one. It follows that the inner product space $(H_0^1(\Omega), \mathrm{B}[\cdot,\cdot])$ has the same Cauchy sequences as the original $H_0^1(\Omega)$, so it is a Hilbert space. Moreover, since B is coercive, it turns out that

$$|\langle F, v\rangle| \leq \|F\|_{H^{-1}(\Omega)}\|v\|_{H^1(\Omega)} \leq C\|F\|_{H^{-1}(\Omega)}\,\mathrm{B}[v,v]^{1/2}$$

for any $v \in H^1_0(\Omega)$. By the Riesz representation theorem it follows that there exists a unique $u \in H^1_0(\Omega)$ such that

$$\mathrm{B}[u,v] = \langle F, v\rangle \qquad \text{for any } v \in H^1_0(\Omega).$$

Since $u|_{\partial\Omega} = 0$, it follows that u is the unique solution of (1.8). Moreover, $\|u\|_{H^1(\Omega)} \leq C\|F\|_{H^{-1}(\Omega)}$.

Inhomogeneous Dirichlet problem. Now, let $f \in H^{1/2}(\partial\Omega)$ and assume that $u|_{\partial\Omega} = f$ in (1.4). Consider a function $u_f \in H^1(\Omega)$ such that the following inequality is satisfied:

$$\|u_f\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}$$

(see for instance the right-inverse of the trace operator [58, Theorem 2.44]). Let $\tilde{u} = u - u_f$, where u is the unique solution of (1.8). Then the Dirichlet problem (1.4) is equivalent to the following weak formulation:

$$\mathrm{B}[\tilde{u}, v] = \langle F, v\rangle - \mathrm{B}[u_f, v] \qquad \text{for any } v \in H^1_0(\Omega),\ \tilde{u}|_{\partial\Omega} = 0.$$

Consider the map $\tilde{F} : w \to \langle F, w\rangle - \mathrm{B}[u_f, w]$. For every $w \in H^1_0(\Omega)$, it follows from the triangle inequality and the Schwarz inequality, it follows that

$$|\,\mathrm{B}[u_f, w]| \leq C\int_\Omega (|\nabla u|\cdot|\nabla w| + |u|\cdot|w|) \leq C\|u_f\|_{H^1(\Omega)}\,\|w\|_{H^1(\Omega)},$$

for $C > 0$ a suitable constant independent of u_f and w. Thus, $\tilde{F}$ is a bounded linear operator and its norm is bounded as

$$\|\tilde{F}\|_{H^{-1}(\Omega)} \leq C(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}).$$

Hence, thanks to the results of the previous step, $\tilde{u} \in H^1_0(\Omega)$ is the unique solution of

$$\begin{cases} \mathrm{L}(\tilde{u}) = \tilde{F} & \text{in } \Omega, \\ \tilde{u} = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\|\tilde{u}\|_{H^1(\Omega)} \leq C(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{1/2}(\partial\Omega)}).$$

Due to the choice of u_f, (1.7) follows straightforwardly. □

Thanks to Theorem 1.1.1, there exists a unique solution u to the boundary value problem (1.4) for every $f \in H^{1/2}(\partial\Omega)$. In the case $F = 0$, i.e. there are no sinks or sources in the conductor, it is possible to define an operator of the form

$$f \to \sigma\nabla u\cdot\nu|_{\partial\Omega},$$

which assigns to each trace function defined on $\partial\Omega$ the corresponding trace of the conormal derivative of u on $\partial\Omega$. Let us check this result for $F = 0$ (see [58, Theorem 2.63] and [117, Lemma 3.4])

Theorem 1.1.2. *Suppose that the hypotheses of Theorem 1.1.1 are satisfied. Then there exists a unique bounded linear map*

$$\Lambda_\sigma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

which satisfies

$$\langle \Lambda_\sigma f, g\rangle = \int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x, \tag{1.9}$$

where $u \in H^1(\Omega)$ is the unique solution of (1.4) and v is any function in $H^1(\Omega)$ such that $v|_{\partial\Omega} = g$. Moreover, the following identity holds:

$$\langle \Lambda_\sigma f, g\rangle = \langle f, \Lambda_\sigma g\rangle, \qquad \text{for any } f, g \in H^{1/2}(\partial\Omega). \tag{1.10}$$

Proof. Existence. Using the Divergence Theorem, (1.6) can be expressed as follows: for any $v \in H^1(\Omega)$,

$$\mathrm{B}[u, v] = \int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\partial\Omega} (\sigma \nabla u \cdot \nu) v \, \mathrm{d}S. \tag{1.11}$$

By Theorem A.1.2, we derive that the right-hand side of the equation (1.11) depends only on the trace of v on $\partial\Omega$. As a result, if v_1 and v_2 are two functions in $H^1(\Omega)$ with the same trace on $\partial\Omega$, then their difference, $v_1 - v_2$, belongs to $H^1_0(\Omega)$. Therefore, we derive the following identity:

$$\mathrm{B}[u, v_1] = \mathrm{B}[u, v_2].$$

Then, for any $f \in H^{1/2}(\partial\Omega)$, the map $\mathrm{T}_f : H^{1/2}(\partial\Omega) \to \mathbb{R}$ defined by

$$\mathrm{T}_f(g) = \int_\Omega \sigma \nabla u \cdot \nabla v_g \qquad \text{for any } g \in H^{1/2}(\partial\Omega),$$

where $v_g \in H^1(\Omega)$ is such that $v_g|_{\partial\Omega} = g$ and $\|v_g\|_{H^1(\Omega)} \le C\|g\|_{H^{1/2}(\partial\Omega)}$ (see Theorem A.1.3), is well-defined.

The linearity of T_f is trivial to prove, so we only need to prove its boundedness. By applying the Cauchy-Schwarz inequality, we derive

$$|\,\mathrm{T}_f(g)| \le C\|u\|_{H^1(\Omega)}\,\|v_g\|_{H^1(\Omega)}.$$

By Theorem 1.1.1, it follows that:

$$\|u\|_{H^1(\Omega)} \le C\|f\|_{H^{1/2}(\partial\Omega)},$$

which implies that:

$$|\,\mathrm{T}_f(g)| \le C\|f\|_{H^{1/2}(\partial\Omega)}\,\|g\|_{H^{1/2}(\partial\Omega)}.$$

Hence, the mapping T_f is bounded, and its norm satisfies the inequality:

$$\| \mathrm{T}_f \|_{H^{-1/2}(\partial\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}, \qquad \text{for any } f \in H^{1/2}(\partial\Omega). \tag{1.12}$$

The inequality (1.12) implies that the map

$$H^{1/2}(\partial\Omega) \ni f \to \mathrm{T}_f \in H^{-1/2}(\partial\Omega)$$

is bounded. Let $\mathrm{T}_f := \sigma\nabla u \cdot \nu$. We define the map

$$\begin{aligned} \Lambda_\sigma : H^{1/2}(\partial\Omega) &\to H^{-1/2}(\partial\Omega) \\ f \quad &\mapsto \mathrm{T}_f := \sigma \nabla u \cdot \nu|_{\partial\Omega}. \end{aligned}$$

The map Λ_σ satisfies (1.9) and it is unique.
Simmetry. Let $f, g \in H^{1/2}(\partial\Omega)$ and let u_f, u_g be the weak solutions to (1.4) with $F = 0$ and Dirichlet data $u_f|_{\partial\Omega} = f, u_g|_{\partial\Omega} = g$. Since $B[\cdot,\cdot]$ is a real symmetric bilinear form, it follows that

$$\langle\Lambda_\sigma f, g\rangle = B[u_f, u_g] = B[u_g, u_f] = \langle f, \Lambda_\sigma g\rangle.$$

□

Definition 1.1.1. *The operator Λ_σ defined in* (1.9) *is called the* Dirichlet to Neumann (DtoN) map.

The DtoN map Λ_σ maps the trace of the solution $u|_{\partial\Omega}$ at the boundary $\partial\Omega$ (the *Dirichlet condition*) to the trace of the corresponding conormal derivative $\sigma\nabla u \cdot \nu|_{\partial\Omega}$ (the *Neumann condition*). The DtoN map is a bounded linear operator belonging to the Banach space $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. Its norm is defined as follows:

$$\|\Lambda_\sigma\|_{\mathcal{L}(H^{1/2}(\partial\Omega),H^{-1/2}(\partial\Omega))} = \sup_{\substack{f,g\in H^{1/2}(\partial\Omega) \\ \|g\|_{H^{1/2}(\partial\Omega)}=\|f\|_{H^{1/2}(\partial\Omega)}=1}} |\langle\Lambda_\sigma g, f\rangle|. \tag{1.13}$$

In this monograph, the norm on the Banach space of bounded linear operators $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$ is denoted with $\|\cdot\|_*$.

The inverse conductivity problem can be expressed as follows:

Given the DtoN map Λ_σ, determine the conductivity σ.

From Theorem 1.1.2, the following well-known identity, known as the Alessandrini's identity, can be derived:

Theorem 1.1.3 (The Alessandrini's identity). *Let $\sigma_1, \sigma_2 \in L^\infty(\Omega, Sym_n)$ be two matrix-valued functions that satisfy the uniform ellipticity condition* (1.3). *Let $f_1, f_2 \in H^{1/2}(\partial\Omega)$ and let $u_1, u_2 \in H^1(\Omega)$ be the weak solutions to*

$$\begin{cases} \mathrm{div}(\sigma_1\nabla u_1) = 0, & \text{in } \Omega, \\ u_1 = f_1, & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} \mathrm{div}(\sigma_2\nabla u_2) = 0, & \text{in } \Omega, \\ u_2 = f_2, & \text{on } \partial\Omega. \end{cases}$$

Then, the following identity holds:

$$\langle (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) f_1, f_2 \rangle = \int_\Omega (\sigma_1 - \sigma_2) \nabla u_1 \cdot \nabla u_2 \, \mathrm{d}x. \tag{1.14}$$

Proof.

$$\langle (\Lambda_{\sigma_1} - \Lambda_{\sigma_2}) f_1, f_2 \rangle = \langle \Lambda_{\sigma_1} f_1, f_2 \rangle - \langle \Lambda_{\sigma_2} f_1, f_2 \rangle.$$

Let $w \in H^1(\Omega)$ be a weak solution of

$$\begin{cases} \operatorname{div}(\sigma_2 \nabla w) = 0 & \text{in } \Omega, \\ w = f_1 & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Then, because of the symmetry of σ_2,

$$\langle \Lambda_{\sigma_2} f_1, f_2 \rangle = \int_\Omega \sigma_2 \nabla w \cdot \nabla u_2 = \int_\Omega \sigma_2 \nabla u_2 \cdot \nabla w = \langle \Lambda_{\sigma_2} f_2, f_1 \rangle,$$

which gives the thesis. □

1.2 Uniqueness for the Calderón problem

After Calderón's work, a first uniqueness result was proved by Robert V. Kohn and Michael Vogelius in their 1984 paper ([83]). They considered a bounded domain with a smooth boundary and studied isotropic conductivities modelled by real, bounded, integrable functions, with a positive lower bound. They assumed the existence of a point $x_0 \in \partial\Omega$ and a neighbourhood B of x_0 where the conductivities are smooth. They showed that if the values of the electric power corresponding to the two conductivities coincide on the trace functions supported in $B \cap \partial\Omega$, then the two isotropic conductivities and their partial derivatives evaluated at x_0 coincide.

Theorem 1.2.1 (Kohn-Vogelius 1984). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^∞. Let $\sigma_1, \sigma_2 \in L^\infty(\Omega)$ be two real isotropic conductivities with positive lower bound. Let $x_0 \in \partial\Omega$ and let B be a neighbourhood of x_0 in $\bar{\Omega}$. Suppose that $\sigma_i \in C^\infty(B)$ for $i = 1, 2$ and*

$$\mathrm{Q}_{\sigma_1}(f) = \mathrm{Q}_{\sigma_2}(f) \qquad \textit{for any } f \in H^{1/2}(\partial\Omega) \textit{ with supp}(f) \subset B \cap \partial\Omega.$$

Then

$$D^\beta \sigma_1(x_0) = D^\beta \sigma_2(x_0) \qquad \textit{for any } \beta = (\beta_1, \dots, \beta_n),\ \beta_i \geq 0.$$

In their 1985 paper [82, Theorem 1], Kohn and Vogelius extended the uniqueness result to piecewise analytic conductivities. They considered two piecewise analytic functions σ_1 and σ_2 on $\bar{\Omega}$ with a positive lower bound. If the electric powers corresponding to the two conductivities are equal for any trace function $f \in H^{1/2}(\partial\Omega)$, then σ_1 and σ_2 coincide. The proofs of these results rely on the application of the Runge approximation theorem that we state here (see [82, Lemma 2] for a proof).

Lemma 1.2.2 (The Runge Approximation Property). *Let ω be a bounded domain with boundary of class C^∞ contained in the analytic curvilinear polygon Ω, and such that every connected component of $\Omega \setminus \omega$ has a boundary curve in common with $\partial\Omega$. Let γ be a piecewise analytic function on $\bar{\Omega}$ with a positive lower bound. Suppose $u \in H^1(\omega)$ satisfies*

$$\mathrm{L}_\gamma(u) = 0 \quad \textit{in } \omega,$$

where L_γ is defined in (1.1). Given any compact subset $K \subset \omega$ and any $\varepsilon > 0$ there exists $U \in H^1(\Omega)$ such that

$$\mathrm{L}_\gamma(U) = 0 \quad \textit{in } \Omega,$$

and

$$\int_K |\nabla(U-u)|^2 \,\mathrm{d}x < \varepsilon.$$

Sylvester and Uhlmann ([128, 129]) established a uniqueness result based on the knowledge of the Dirichlet to Neumann map in dimension $n \geq 2$. Motivated by Calderón's use of exponential solutions in the study of the linearised problem, they constructed the complex geometric optics (CGO) solutions of the conductivity equation for isotropic conductivities of class C^2. This can be reduced to constructing solutions in $\mathbb{R}^n$ by extending $\gamma = 1$ outside Ω for the Schrödinger equation. For an isotropic conductivity $\sigma \in C^2(\mathbb{R}^n)$ with positive lower bound, they transformed the conductivity operator $\mathrm{L}(u)$ into the Schrödinger operator by a Liouville transformation

$$\sigma^{-1/2}\,\mathrm{L}(\sigma^{1/2}u) = (\Delta - q)u, \quad \text{with } q = \frac{\Delta(\sqrt{\sigma})}{\sqrt{\sigma}}.$$

Hence, the construction of solutions to the conductivity equation boils down to constructing solutions to the Schrödinger equation.

Theorem 1.2.3 (Silvester-Uhlmann 1987). *For $\sigma_j \in C^2(\overline{\Omega})$ with $j = 1, 2$ and σ_j strictly positive, if $\Lambda_{\sigma_1} = \Lambda_{\sigma_2}$ then $\sigma_1 = \sigma_2$.*

These results from the 1980s greatly influenced the development of the field of inverse problems, with many uniqueness and stability results being obtained for isotropic conductivities (uniqueness at the boundary [82], nonlinear conductivities [126], conductivities with derivatives of higher order [102, 43], uniqueness for conormal conductivities in $C^{1+\epsilon}$ [88], Lipschitz conductivities [46]).

In the case of anisotropic conductivity, the problem of identifiability becomes more challenging. A natural obstruction to uniqueness arises from the fact that anisotropic conductivities can only be determined up to diffeomorphisms that leave the boundary of the domain fixed. This was first observed by Tartar and later confirmed by Kohn and Vogelius ([84]). They introduced a diffeomorphism $\phi : \bar{\Omega} \to \bar{\Omega}$ which satisfies $\phi(x) = x$ and $D\phi(x) = Id_n$ for any $x \in \partial\Omega$. For an anisotropic conductivity $\sigma(x) = \{\sigma_{ij}(x)\}_{i,j=1}^n$ satisfying the uniform ellipticity condition (1.3), a transformed conductivity can be defined as:

$$\sigma^\phi(x) := \frac{[(D\phi)^T\sigma(D\phi)](\phi^{-1}(x))}{|\det(D\phi)(\phi^{-1}(x))|},$$

where $D\phi$ is the Jacobian matrix of ϕ. It can be shown that the Dirichlet to Neumann maps associated with σ and σ^ϕ are the same, i.e., $\Lambda_\sigma = \Lambda_{\sigma^\phi}$.

Kohn and Vogelius were able to establish a uniqueness result at the boundary by mitigating the ill-posedness of the problem. They proved that if $n-1$ eigenvalues of the anisotropic conductivity σ are known, then the remaining eigenvalue and its partial derivatives can be determined by boundary measurements.

Theorem 1.2.4 (Kohn-Vogelius 1983)**.** *Let $\sigma, \tilde{\sigma}$ be two symmetric, positive definite matrices with entries in $L^\infty(\Omega)$, and let $\{\lambda_i\}, \{\tilde{\lambda}_i\}$ and $\{e_i\}, \{\tilde{e}_i\}$ be the corresponding eigenvalues and eigenvectors. For $x_0 \in \partial\Omega$, let B be a neighbourhood of x_0 in $\bar{\Omega}$. Moreover, assume that $\sigma, \tilde{\sigma} \in C^\infty(B)$ and $\partial\Omega \cap B$ is of class C^∞,*

$$\begin{aligned}
&e_j = \tilde{e}_j, \quad \lambda_j = \tilde{\lambda}_j \quad \text{in } B, \text{ for } 1 \le j \le n-1, \\
&e_n(x_0)\cdot \nu(x_0) \neq 0, \\
&\mathrm{Q}_\sigma(\phi) = \mathrm{Q}_{\tilde{\sigma}}(\phi) \qquad \text{for any } \phi \in H^{1/2}(\partial\Omega) \text{ with } supp(\phi) \subset B \cap \partial\Omega.
\end{aligned}$$

Then

$$D^\beta \lambda_n(x_0) = D^\beta \tilde{\lambda}_n(x_0)$$

for all $\beta = (\beta_1, \dots, \beta_n)$, $\beta_i \ge 0$.

Since the discovery of the obstruction identified by Luc Tartar, which states that conductivities can only be determined up to diffeomorphisms that leave the boundary points fixed, researchers have pursued different approaches. On the one hand, one approach is to prove uniqueness up to diffeomorphisms that leave the boundary points fixed. Lassas and Uhlmann, in their work [89], were able to reconstruct the conformal class of a smooth, compact Riemannian surface (M, g) with boundary by utilising the Cauchy data of harmonic functions given on an open subset of the boundary of M. They employed the Laplace-Beltrami operator Δ_g associated with the metric g, defined as

$$\Delta_g u = \frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^n \frac{\partial}{\partial x_i}\Big(\sqrt{\det(g)} g^{ij} \frac{\partial}{\partial x_j} u\Big),$$

where (g^{ij}) is the inverse matrix of (g_{ij}). This formulation is equivalent to the conductivity equation through the transformation

$$(\sigma^{ij}) = \sqrt{\det(g)}(g_{ij})^{-1} \quad \text{or} \quad (g_{ij}) = (\det(\sigma))^{\frac{1}{n-2}}(\sigma^{ij})^{-1}. \tag{1.16}$$

The operator analogous to the DtoN map is denoted as

$$\Lambda_g(f) = (\sqrt{\det(g)}) \sum_{i,j=1}^n \nu_i g^{ij} \frac{\partial u}{\partial x_j}\Big|_{\partial M}.$$

For examples of non-uniqueness in the anisotropic case, see [84, 68, 13]. From the point of view of applications, the lack of uniqueness in EIT corresponds to the existence of anisotropic structures that can act as barriers, making them undetectable by measurements taken at the boundary and leading to the appearance of a homogeneous conductor.

The other approach consists in assuming the *a-priori* structure of the anisotropic conductivity on a finite number of spatially independent parameters (see for example [8, 12]). We would like to highlight two results of uniqueness. In [15], G. Alessandrini, R. Gaburro and M. de Hoop have considered the problem of recovering an anisotropic conductivity σ known to be given by a piecewise constant matrix function on a given partition of Ω. The authors considered the Neumann to Dirichlet map, which is the inverse of the Dirichlet to Neumann map, and is defined as:

$$\mathcal{N}_\sigma : {}_0H^{-1/2}(\partial\Omega) \to {}_0H^{1/2}(\partial\Omega), \quad \mathcal{N}_\sigma := \left(\Lambda_\sigma\Big|_{{}_0H^{1/2}(\partial\Omega)}\right)^{-1},$$

where

$$\begin{aligned} {}_0H^{1/2}(\partial\Omega) &= \Big\{f \in H^{1/2}(\partial\Omega) : \int_{\partial\Omega} f = 0\Big\}, \\ {}_0H^{-1/2}(\partial\Omega) &= \Big\{\psi \in H^{-1/2}(\partial\Omega) : \langle\psi, 1\rangle = 0\Big\}. \end{aligned}$$

This is due to the fact that the Dirichlet to Neumann map Λ_σ restricted to ${}_0H^{1/2}(\partial\Omega)$ is injective with bounded inverse. From the self-adjointness of $\mathcal{N}_\sigma$, it follows that the Neumann to Dirichlet map satisfies the following Alessandrini's identity:

$$\langle(\mathcal{N}_{\sigma_1} - \mathcal{N}_{\sigma_2})g_1, g_2\rangle = \int_\Omega (\sigma_1(x) - \sigma_2(x))\nabla u_1 \cdot \nabla u_2 \,\mathrm{d}x,$$

where $g_i \in {}_0H^{-1/2}(\partial\Omega)$ and $u_i \in H^1(\Omega)$ are the weak solution of the Neumann problem

$$\begin{cases} \mathrm{div}(\sigma_i\nabla u_i) = 0 & \text{in } \Omega, \\ \sigma_i\nabla u_i \cdot \nu = g_i & \text{on } \partial\Omega, \\ \int_{\partial\Omega} u_i = 0. \end{cases}$$

The authors have considered a finite partition $\{D_j\}_{j=1}^N$ of a bounded domain Ω consisting of connected, non-overlapping domains of Lipschitz class. Each of these domains contains a non-empty portion of class $C^{1,\alpha}$ as in Definition 1.0.1. The anisotropic conductivities are of the form

$$\sigma(x) = \sum_{j=1}^N \sigma_j \chi_{D_j}(x) \qquad x \in \Omega,$$

where σ_j for $j = 1, \ldots, N$ are constant real symmetric, positive definite matrices satisfying the uniform ellipticity condition. For a non-empty flat portion $\Sigma \subset \partial\Omega$ as in Definition 1.0.3, the authors considered the local Neumann-to-Dirichlet map (NtoD)

$$\mathcal{N}_\sigma^\Sigma : {}_0H^{-1/2}(\Sigma) \to ({}_0H^{-1/2}(\Sigma))^* \subset {}_0H^{1/2}(\Sigma), \tag{1.17}$$

where $({}_0H^{-1/2}(\Sigma))^*$ is the dual space to ${}_0H^{-1/2}(\Sigma)$. The local NtoD map also satisfies Alessandrini's identity.

Theorem 1.2.5 (Alessandrini-De Hoop-Gaburro, 2017)**.** *Let Ω be a bounded domain of $\mathbb{R}^n$, let $\{D_j\}_{j=1}^N$ be a fixed partition of Ω as introduced above, and let Σ be a non-empty portion of the boundary $\partial\Omega$. Let σ_k, $k = 1, 2$ be two anisotropic conductivities of the* *form*

$$\sigma_k(x) = \sum_{j=1}^{N} (\sigma_k)_j \chi_{D_j}(x), \qquad x \in \Omega, \quad (\sigma_k)_j \in Sym_n.$$

Let $\mathcal{N}^{\Sigma}_{\sigma_k}$*,* $k = 1, 2$ *be the corresponding local NtoD map. If* $\mathcal{N}^{\Sigma}_{\sigma_1} = \mathcal{N}^{\Sigma}_{\sigma_2}$ *then* $\sigma_1 = \sigma_2$.

The proof of Theorem 1.2.5 relies on several key ideas. First, it is important to note that the solutions of the conductivity equation are harmonic functions on the Riemannian manifold $\{\Omega, g\}$, where g is the metric associated with σ by the equation (1.16). In [15, Lemma 3.5], it is shown that the tangential part of the metric $g(P)$ can be uniquely determined from the knowledge of the local Neumann to Dirichlet map near a point $P \in \partial\Omega$. The tangential part of $g(P)$ refers to the $(n-1) \times (n-1)$ minor of $g(P)$ with respect to the tangential (hyper)plane to $\partial\Omega$ at P. Furthermore, if the local NtoD map is known on a non-flat portion of $\partial\Omega$, and σ is constant nearby, then we will have enough tangent planes to fully recover g and hence σ. The proof relies on the unique continuation property.

The authors introduced an example of non-uniqueness to show that knowing the Neumann to Dirichlet map on the half space is not sufficient to uniquely determine a constant anisotropic conductivity. This highlights that flat boundaries and interfaces can be an obstacle to uniquely recovering the coefficients, thereby justifying the assumption of curved interfaces and boundaries.

A second uniqueness result can be found in [16], where Alessandrini, Gaburro, De Hoop and Sincich provide a global uniqueness result for finite-dimensional anisotropic conductivities defined on nested domains. The authors consider a family of nested domains $\{\Omega_k\}_{k=1}^{K}$ such that $\Omega_{k+1} \subset\subset \Omega_k \subset\subset \Omega$. Let $\Omega_0 = \Omega$ and $\Omega_{K+1} = \emptyset$, the anisotropic conductivities have the form

$$\sigma(x) = \sum_{j=1}^{K+1} \sigma_j \chi_{\Omega_{j-1}\setminus\Omega_j}(x),$$

where $\sigma_j \in Sym_n$ are real symmetric, positive definite matrices and σ satisfies the uniform ellipticity condition.

The authors proved the following uniqueness theorem.

Theorem 1.2.6 (Alessandrini-De Hoop-Gaburro-Sincich, 2018). *Let* Ω, $\{\Omega_j\}_{j=1}^{N}$, Σ *be as stated in [16, Theorem 2.6], and let* σ_k*,* $k = 1, 2$ *be two anisotropic conductivities of the form*

$$\sigma_k(x) = \sum_{j=1}^{K_k+1} (\sigma_k)_j \chi_{\Omega_{j-1}\setminus\Omega_j}(x), \qquad x \in \Omega,$$

satisfying

$$(\sigma_k)_j \neq (\sigma_k)_{j+1} \quad \textit{for } j = 1, \ldots, K_k.$$

If $\mathcal{N}^{\Sigma}_{\sigma_1} = \mathcal{N}^{\Sigma}_{\sigma_2}$*, then* $K_1 = K_2 = K$*, and*

$$\Omega_j^{(1)} = \Omega_j^{(2)} \quad \textit{and} \quad (\sigma_1)_{j+1} = (\sigma_2)_{j+1},$$

for $j = 0, \ldots, K$.

1.3 Stability for the inverse conductivity problem

In this section, we discuss the stability issue for the inverse conductivity problem and describe some relevant results.

Before we consider the stability issue, let us define the modulus of continuity. Let Ω be a subset of $\mathbb{R}^n$.

Definition 1.3.1. *Let $f \in C^0(\Omega)$. The* modulus of continuity *of f in Ω is defined as*

$$\omega(\delta) = \sup\{|f(x) - f(y)| \,:\, x, y \in \Omega,\, |x - y| \leq \delta\} \qquad \textit{for } \delta > 0.$$

Here, ω is an increasing function with $\omega(0) = 0$. Furthermore, if f is uniformly continuous, we have

$$\lim_{t \to 0^+} \omega(t) = 0.$$

For bounded functions, it may be useful to consider the *concave modulus of continuity*, which is defined as follows:

$$\tilde{\omega}(\delta) = \inf\{f(\delta) \,:\, f \text{ concave},\, f \geq \omega \text{ in } [0, +\infty)\}, \quad \text{for } \delta > 0. \tag{1.18}$$

It can be shown that $\lim_{\delta \to 0^+} \tilde{\omega}(\delta) = 0$.

Now, let us consider two isotropic conductivities γ_j, $j = 1, 2$ and their corresponding Dirichlet to Neumann maps Λ_{γ_j}. The aim is to obtain an estimate of the form

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq \omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*).$$

This estimate quantifies the distance between two conductivities based on the distance between the corresponding Dirichlet to Neumann maps.

A first stability result at the boundary for the Calderón problem was established by Sylvester and Uhlmann in [130] by constructing CGO solutions.

Theorem 1.3.1 (Sylvester-Uhlmann, 1988)**.** *Let γ_1, γ_2 be two smooth conductivities defined on $\overline{\Omega} \subset \mathbb{R}^n$ which satisfy the following conditions:*

i) $0 < \dfrac{1}{E} \leq \gamma_i \leq E$,

ii) $\|\gamma_i\|_{C^2(\overline{\Omega})} \leq E$.

For some positive number σ, $\sigma < \frac{1}{n+1}$, there exists $C = C(\Omega, E, n, \sigma)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*,$$

and

$$\left\|\frac{\partial \gamma_1}{\partial \nu} - \frac{\partial \gamma_2}{\partial \nu}\right\|_{L^\infty(\partial\Omega)} \leq C\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*^\sigma.$$

A stronger result, a consequence of Theorem 1.3.1, is the stability estimate proved by Alessandrini [8]. In his 1988 paper, Alessandrini showed that in dimension $n \geq 3$, under minimal assumptions, the isotropic conductivity γ depends continuously on Λ_γ with a modulus of continuity of logarithmic type ([8, Theorem 1]).

Theorem 1.3.2 (Alessandrini, 1988). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, $n \geq 3$. Let m, E be two positive numbers such that $m > n/2 + 2$, and let γ_j, $j = 1, 2$ be two positive functions in $H^m(\Omega)$ which satisfy the following conditions:*

$$E^{-1} \leq \gamma_j(x), \qquad j = 1, 2; \tag{1.19}$$

$$\|\gamma_j\|_{H^m(\Omega)} \leq E. \tag{1.20}$$

For any $\phi \in H^{1/2}(\partial\Omega)$, let $u_j \in H^1(\Omega)$ be a weak solution of

$$\begin{cases} \operatorname{div}(\gamma_j \nabla u_j) = 0 & \text{in } \Omega, \\ u_j = \phi & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Define $\Lambda_j : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ with $\Lambda_j \phi = \gamma_j \frac{\partial u_j}{\partial \nu}$, where ν is the exterior unit normal of $\partial\Omega$ as in Theorem 1.1.2. Then there exists a positive constant $C = C(\Omega, E, n, m)$ such that

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\omega(\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_*), \tag{1.22}$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq |\ln t|^{-\delta} \qquad \text{for any } t \text{ such that } 0 < t < e^{-1},$$

with $0 < \delta < 1$ depending on m, E and n.

The proof of Theorem 1.3.2 relies on the construction of special solutions to the conductivity equation, inspired by the work of Sylvester and Uhlmann [128] and of Kohn and Vogelius [83], and the Alessandrini's identity. In two subsequent papers by Alessandrini [7, 6], the author derived that the logarithmic estimate (1.22) holds also in the case where the *a-priori* bound (1.20) is replaced by the following bound:

$$\|\gamma\|_{W^{2,\infty}(\Omega)} \leq E.$$

In the two-dimensional case, a logarithmic stability estimate was proved by Barceló, Faraco, and Ruiz in [32], assuming that γ satisfies a bound of the form

$$\|\gamma\|_{C^\alpha(\bar{\Omega})} \leq E \qquad \text{for some } \alpha,\ 0 < \alpha \leq 1.$$

When little is known about the problem, the best stability rate is logarithmic, as was pointed out by Mandache in [97].

1.3.1 Stability for finite-dimensional conductivities

Alessandrini's work has provided important insights into the stability of the Calderón problem. However, Mandache proved that the optimality for the stability is logarithmic under mild regularity assumptions on the conductivity. In order to obtain better stability estimates, such as Hölder or Lipschitz type, Alessandrini and Vessella introduced an innovative approach in their 2005 paper [21]. They incorporated suitable *a-priori* information about the unknown conductivity and the geometry of the problem.

The authors considered piecewise constant conductivities of the form:

$$\gamma(x) := \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x). \tag{1.23}$$

Here, $D_1, \ldots, D_N$ is a given partition of Ω, and $\gamma_1, \ldots, \gamma_N$ are *unknown* real numbers. This choice is motivated by applications where the disjoint domains D_j represent different tissue regions of known geometry.

We give the result established by Alessandrini and Vessella [21]. Let γ_k, $k = 1, 2$ be two isotropic conductivities satisfying (1.23) and (1.3). Define the local Dirichlet to Neumann map $\Lambda^{\Sigma}_{\gamma_k}$ as the linear operator:

$$H^{1/2}_{co}(\Sigma) \ni \varphi \to \Lambda^{\Sigma}_{\gamma_k}\varphi = \gamma \frac{\partial u}{\nu}\Big|_{\Sigma} \in H^{-1/2}_{co}(\Sigma),$$

where $u \in H^1(\Omega)$ is the weak solution of the Dirichlet problem

$$\begin{cases} \mathrm{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.24}$$

Let N, r_0, L, M, α, $\bar{\gamma}$, be given positive numbers with $N \in \mathbb{N}$ and $\alpha \in (0, 1]$. We refer to this set of numbers, along with the space dimension n, as to the *a-priori* data.

Theorem 1.3.3 (Alessandrini - Vessella, 2005)**.** *Let Ω, Σ be a domain and an open portion of $\partial\Omega$ respectively, satisfying the a-priori assumptions listed above. Let γ_k, $k = 1, 2$ be two scalar piecewise constant functions of the form* (1.23)*, satisfying* (1.3)*. Then we have*

$$\|\gamma_1 - \gamma_2\|_{L^\infty(\Omega)} \leq C\|\Lambda^{\Sigma}_{\gamma_1} - \Lambda^{\Sigma}_{\gamma_2}\|_*, \tag{1.25}$$

where C is a positive constant depending on the a-priori data only.

In their papers [21, 4], Alessandrini and Vessella, Alessandrini pointed out that the Lipschitz constant C in Theorem 1.3.3 depends on the number of domains N. As was proved by Rondi [111], the following lower bound holds:

$$C \geq A \exp(BN^{1/(2n-1)}), \tag{1.26}$$

where A and B are positive constants that depend on the *a-priori* data. The inequality (1.26) provides evidence for the exponential relationship between C and the number of domains N. This observation highlights the inherently ill-posed nature of the inverse conductivity problem. Further Lipschitz stability estimates have been provided in [18] for piecewise linear conductivities for the Calderón problem and [17] for piecewise linear potentials for the Schrödinger equation.

The stability issue in the case of anisotropic conductivities is particularly challenging. Indeed, the obstacle highlighted by Tartar provides evidence that the knowledge of the Dirichlet to Neumann map alone is not sufficient to recover anisotropic conductivities. Alessandrini [12] established uniqueness and stability results for anisotropic conductivities with the following structure:

$$A(x) = A(a(x)),$$

where $t \to A(t)$ is a given matrix function and $a = a(x)$ is an unknown scalar function. The matrix function A satisfies a monotonicity condition of the form:

$$D_t A(t) \geq \text{Const.} I > 0.$$

Alessandrini's identity and the method of singular solutions played a crucial role in proving the stability estimate. Alessandrini and Gaburro [12], and [13] for local data, extended this result to anisotropic conductivities with the following structure:

$$A(x) = A(x, a(x)),$$

where $a(x)$ is an unknown scalar function and $A(x,t)$ is a given matrix function that satisfies the following monotonicity assumption:

$$D_t A(x,t) \geq \text{Const.} I > 0.$$

The authors proved Lipschitz stability estimates at the boundary [12, Theorem 2.1] and [13] and Hölder estimates for higher order derivatives of $A(x, a(x))$ [12, Theorem 2.2] and [13].

In a related study, Lionheart [93] proved a uniqueness result for anisotropic conductivities with the structure:

$$\sigma(x) = a_0(x) A(x),$$

where $A = A(x)$ is a given matrix function and $a_0 = a_0(x)$ is an unknown scalar function. In this case, the anisotropic nature of the body is known, but it may be subject to scalar perturbations.

Inspired by the 2005 paper [21], Gaburro and Sincich [62] were able to prove a Lipschitz stability estimate for a certain class of anisotropic conductivities, which we describe in the following definition.

Definition 1.3.2. *We shall say that $\sigma_A \in \mathcal{C}$ if σ_A is of the type*

$$\sigma_A(x) = \sum_{j=1}^{N} \gamma_j\, A(x) \chi_{D_j}(x), \qquad x \in \Omega, \tag{1.27}$$

where γ_j are unknown real numbers, D_j, $j = 1, \dots, N$ are the given subdomains of the partition and

$$\bar{\gamma} \leq \gamma_j \leq \bar{\gamma}^{-1}, \qquad \textit{for any } j = 1, \dots n. \tag{1.28}$$

$A(x)$ is a known Lipschitz matrix function satisfying

$$\|A\|_{C^{0,1}(\Omega)} \leq \bar{A}, \tag{1.29}$$

where $\bar{A} > 0$ is a constant and, for some $\lambda > 1$,

$$\lambda^{-1}|\xi|^2 \leq A(x)\xi \cdot \xi \leq \lambda|\xi|^2, \qquad \textit{for almost every } x \in \Omega, \textit{ for every } \xi \in \mathbb{R}^n. \tag{1.30}$$

We refer to the set $\{N, r_0, L, M, \alpha, \lambda, \bar{\gamma}, \bar{A}\}$, with $N \in \mathbb{N}$ and $\alpha \in (0,1]$, along with the space dimension n, as the *a-priori* data.

Gaburro and Sincich proved the following Lipschitz stability estimate.

Theorem 1.3.4 (Gaburro - Sincich, 2015)**.** *Let Ω, D_j, $j = 1, \dots, N$ and Σ be a domain, N subdomains and a portion of $\partial\Omega$ as in the hypothesis of Theorem 1.3.3. If $\sigma_A^{(k)} \in \mathcal{C}$, $k = 1, 2$ are two conductivities of type*

$$\sigma_A^{(k)}(x) = \sum_{j=1}^{N} \gamma_j^{(k)} A(x) \chi_{D_j}(x) \qquad x \in \Omega,\ k = 1, 2, \tag{1.31}$$

then we have

$$\|\sigma_A^{(1)} - \sigma_A^{(2)}\|_{L^\infty(\Omega)} \leq C \|\Lambda^{\Sigma}_{\sigma_A^{(1)}} - \Lambda^{\Sigma}_{\sigma_A^{(2)}}\|_*, \tag{1.32}$$

where C is a positive constant that depends on the a-priori data only.

A Lipschitz stability estimate for the Calderón problem for a special type of complex conductivity, also known as admittivity, has been established by Beretta and Francini [35]. The authors consider a bounded domain $\Omega \subset \mathbb{R}^n$ for $n \geq 2$ with a given partition of disjoint Lipschitz domains $\{D_j\}_{j=1}^N$ such that the boundaries of contiguous subdomains share a non-empty flat portion. They consider conductivities of the form

$$\gamma^{(k)}(x) = \sum_{j=1}^{N} \gamma_j^{(k)} \chi_{D_j}(x), \quad k = 1, 2, \tag{1.33}$$

with $\Re(\gamma^{(k)}) \geq \lambda^{-1} > 0$. Here, $\gamma_j^{(k)} \in \mathbb{C}$ are constants.

Theorem 1.3.5 (Beretta - Francini, 2011)**.** *Let Ω, D_j, $j = 1, \dots, N$ and Σ be a domain, N subdomains and a portion of $\partial\Omega$ as in the hypothesis of Theorem 1.3.3. If $\gamma^{(k)} \in L^\infty(\Omega)$, $k = 1, 2$ are two admittivities of the form (1.33), then we have*

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \leq C \|\Lambda^{(\Sigma)}_{\gamma^{(1)}} - \Lambda^{(\Sigma)}_{\gamma^{(2)}}\|_*, \tag{1.34}$$

where C is a positive constant that depends on $|\Omega|, \lambda, N$.

1.3.2 Stable determination of an inclusion

Another class of inverse problems related to the inverse conductivity problem is the determination of an inclusion within a body. This problem involves the determination of the shape, size, and location of an object or material within a given body based on measurements taken from the surface of the body. We consider an optimal stability estimate established by Alessandrini and Di Cristo [21]. The authors consider a conductor modelled by a bounded and measurable domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. Within this domain, there is a subdomain D compactly contained in Ω with boundary of class $C^{1,\alpha}$ as in Definition 1.0.1. The domain D is made of a different material than the background body, and we call it an *inclusion*. The isotropic conductivity σ_D has a jump at the interface ∂D. For a fixed electric potential $f \in H^{1/2}(\partial\Omega)$, let $u \in H^1(\Omega)$

be the weak solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma_D \nabla u) = 0, & \text{in } \Omega, \\ u|_{\partial\Omega} = f. \end{cases} \tag{1.35}$$

We define the DtoN map Λ_D as follows:

$$H^{1/2}(\partial\Omega) \ni f \longmapsto \sigma_D \nabla u \cdot \nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

where ν is the exterior unit normal of $\partial\Omega$.

The inverse problem aims to determine the inclusion D from the knowledge of the complete map Λ_D.

In [78, Theorem 1.1], Viktor Isakov derived a uniqueness result for open sets $D_1, D_2 \subset \Omega$ with Lipschitz boundary, showing that if $\Lambda_{D_1} = \Lambda_{D_2}$, then $D_1 = D_2$ and $b_1 = b_2$, for isotropic conductivities of the form:

$$\sigma_i(x) = a(x) + b_i(x)\chi_{D_i}(x), \qquad b_i \in C^2(D_i), \quad i = 1, 2,$$

where $a(x)$ is the background conductivity of Ω. The proof of uniqueness is based on the Runge approximation theorem and the unique continuation property for elliptic equations.

Subsequently, Alessandrini and Di Cristo in [21] introduced a first stability estimate for a piecewise constant conductivity of the form:

$$\gamma(x) = 1 + (k-1)\chi_D(x), \quad \text{with } x \in \Omega,\ k \neq 1,\ k > 0.$$

They used the Hausdorff distance between the boundaries of the two inclusions.

Definition 1.3.3. *Let X, Y be two non-empty subsets of a metric space M. Their* Hausdorff distance *is defined by*

$$\mathrm{d}_{\mathcal{H}}(X, Y) = \max\left\{ \sup_{x \in X} \operatorname{dist}(x, Y), \sup_{y \in Y} \operatorname{dist}(X, y) \right\},$$

with $\operatorname{dist}(x, Y) = \inf_{y \in Y} \operatorname{dist}(x, y)$.

The authors proved the following optimal stability estimate.

Theorem 1.3.6 (Alessandrini - Di Cristo, 2005)**.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ and let $D_1, D_2 \subset \Omega$ be two inclusions with boundary of class $C^{1,\alpha}$. Let $k \neq 1, k > 0$ be given. If, given $\varepsilon > 0$, we have that $\|\Lambda_{D_1} - \Lambda_{D_2}\|_* \leq \varepsilon$, then*

$$d_H(\partial D_1, \partial D_2) \leq \omega(\varepsilon), \tag{1.36}$$

where ω is a modulus of continuity satisfying

$$\omega(t) \leq C|\ln t|^{-\eta}, \quad \textit{for any } 0 < t < 1,$$

with $C > 0$, $0 < \eta < 1$ depending on the a-priori data only.

The proof of Theorem 1.3.6 is based on the method of singular solutions and the application of propagation of smallness. As was first pointed out in [20] and [11], propagation of smallness is a technique that allows one to propagate the estimate of a weak solution of an elliptic equation in an open set along a chain of spheres centred on points lying on a Jordan curve contained in a connected portion of the complementary set of the union of the two inclusions. To apply these unique continuation tools, the authors introduced a quantity called *modified distance*, which we will discuss and use in Chapter 4. This result was extended by Di Cristo and Ren in [52] for special anisotropic conductivities of the form

$$\sigma_D(x) = (1 + (k-1)\chi_D(x))A(x), \quad \text{with } x \in \Omega,\ k \neq 1,\ k > 0,$$

where $A(x)$ is a known $n \times n$ real symmetric positive definite Lipschitz continuous matrix function. The authors obtained an optimal stability estimate similar to (1.36).

These results are optimal because of the mild *a-priori* assumptions made about the unknown inclusion D. As pointed out by Bacchelli and Vessella [29], the stability rate can be improved if the unknowns depend on a finite number of parameters. This observation has led to the development of a line of research that has been able to prove Lipschitz stability estimates for polygonal (in two dimensions [38, 36]) or polyhedral (in three dimensions [26]) inclusions. We recall here the result regarding the polyhedral inclusions.

Fix the parameters $R_0 > r_0 > 0$, $\theta_0 \in (0, \pi/2)$ and $M_0 > 0$.

The electrical conductor is modelled by a bounded domain Ω of $\mathbb{R}^3$ with Lipschitz boundary of constants M_0 and r_0. We assume that $\mathrm{diam}(\Omega) \leq R_0$. We consider Σ as the flat non-empty portion of $\partial\,\Omega$ of size r_0. We assume that there exists a point $P_\Sigma \in \Sigma$ such that $\mathrm{dist}(P_\Sigma, \partial\,\Omega \setminus \Sigma) \geq r_0$.

We recall the formal definition of a polyhedron and the notation for faces and vertices as introduced in [26, Definition 2.1].

Definition 1.3.4. *A* polyhedron *is a closed subset D of $\mathbb{R}^3$ that is homeomorphic to a ball.*

The boundary of the polyhedron, denoted as $\partial\, D$, is composed of multiple faces, which are closed simply connected plane polygons. The boundary $\partial\, D$ can be expressed as the union of these faces:

$$\partial\, D = \bigcup_{j=1}^{H} F_j^D,$$

where F_j^D represents the j-th face of D. Two distinct faces, F_i^D and F_j^D, do not intersect in their interior regions:

$$\mathit{Int}_{\mathbb{R}^2}(F_i^D) \cap \mathit{Int}_{\mathbb{R}^2}(F_j^D) = \emptyset.$$

The edges of the polyhedron are the non-empty intersections of two contiguous faces F_i^D and F_j^D, denoted as σ_{ij}^D:

$$\sigma_{ij}^D := F_i^D \cap F_j^D.$$

The non-empty intersection of two edges is called the vertex V^D of D.

Now, we introduce the class of non-degenerate polyhedra with parameters

r_0, R_0, θ_0 and M_0.

We define a polyhedron $D \subset \Omega$ to be in the class of non-degenerate polyhedra $\mathfrak{D} = \mathfrak{D}(r_0, R_0, \theta_0, M_0)$, if it satisfies the following conditions:

1. *Strict Inclusion*: The distance between the polyhedral inclusion D and the boundary $\partial\Omega$ is greater than or equal to r_0.

2. *Dihedral Angle Non-degeneracy*: At each edge σ_{ij}^D of D, the angle between the intersecting faces F_i^D and F_j^D has width α, where $\alpha \in (\theta_0, \pi-\theta_0)\cup(\pi+\theta_0, 2\pi-\theta_0)$.

3. *Face Non-degeneracy*: For any polygonal face F^D, there exists a point $x_0 \in F^D$ such that a ball $B'_{r_0}(x_0)$ is contained in F^D.

4. *Edge Non-degeneracy*: For each edge σ^D of D, the length of the edge is greater than or equal to r_0.

5. *Face Angle Non-degeneracy*: Each interior angle β of each face F^D satisfies $\beta \in (\theta_0, \pi - \theta_0) \cup (\pi + \theta_0, 2\pi - \theta_0)$.

6. *Lipschitz Regularity*: The complement $\Omega \setminus D$ is connected and has a Lipschitz boundary with constants r_0 and M_0.

The conductivity is modelled by a piecewise constant function of the form

$$\gamma_D(x) = 1 + (k-1)\chi_D(x), \tag{1.37}$$

where $D \in \mathfrak{D}$ and k is a positive constant such that

$$\min(k, |k-1|) \geq \kappa_0, \quad \text{for some } \kappa_0 > 0. \tag{1.38}$$

We refer to constants belonging to the set $\{R_0, M_0, r_0, \theta_0, \eta_0, \kappa_0\}$ as the *a-priori* data.

Let $\Lambda_{\gamma_D}^{\Sigma} : H_{co}^{1/2}(\Sigma) \to H_{co}^{-1/2}(\Sigma)$ be the Dirichlet to Neumann map associated with γ_D.

Theorem 1.3.7 (Aspri - Beretta - Francini - Vessella, 2022). *Let Ω be a bounded domain in $\mathbb{R}^3$ with Lipschitz boundary satisfying the previous assumptions. Let $D_0, D_1 \in \mathfrak{D}$ be two admissible polyhedral inclusions, let k satisfy (1.38), and let Σ be an open portion of $\partial\Omega$. Then, there exists a positive constant C depending only on the a-priori data such that*

$$\mathrm{d}_{\mathcal{H}}(\partial D_0, \partial D_1) \leq C\|\Lambda_{\gamma_{D_0}}^{\Sigma} - \Lambda_{\gamma_{D_1}}^{\Sigma}\|_*, \tag{1.39}$$

where

$$\gamma_{D_i} = 1 + (k-1)\chi_{D_i}, \quad \textit{for } i = 0, 1.$$

CHAPTER 2

Quantitative estimates of unique continuation

Contents

This chapter is devoted to a topical review of the unique continuation property. We begin by recalling the main definitions and ideas connected with unique continuation. For a deeper understanding, we refer to the notes of Daniel Tataru [132] and Sergio Vessella [136]. In Section 2.1, we introduce a stability estimate for the Cauchy problem related to the conductivity equation (Lemma 2.1.1). In Section 2.2, we examine the three sphere inequality in two cases: when the leading term coefficient is Lipschitz continuous and when it is piecewise constant continuous. This section is based on the results proved in [19] and [60].

To begin, we state the unique continuation property for linear partial differential operators (PDOs), which have the following form:

$$P(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha, \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \qquad x \in \mathbb{R}^n.$$

The number m represents the order of the PDOs.

We define the principal symbol $P(x,\xi)$ of $P(x,D)$ as

$$P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha.$$

Definition 2.0.1. *Let Ω be an open, connected set of $\mathbb{R}^n$. We say that the linear differential operator $P(x,D)$ satisfies the* weak unique continuation property *if, for any open subset ω of Ω,*

$$P(x,D)u = 0 \qquad \text{in } \Omega, \tag{2.1}$$

and $u = 0$ in ω, then $u \equiv 0$.

Definition 2.0.2. *We say that the differential operator $P(x,D)$ satisfies the* strong unique continuation property *if, for any $x_0 \in \Omega$ and for any solution u that satisfies*

$$\lim_{r\to 0^+} r^{-k} \int_{B_r(x_0)} u^2 = 0 \qquad \text{for any } k \in \mathbb{N},$$

it follows that $u \equiv 0$ in Ω.

Notice that the strong unique continuation property implies the weak unique continuation. It can be proved that the unique continuation property is equivalent to uniqueness in the Cauchy problem. If the Cauchy problem is well-posed, then the unique continuation property holds. Conversely, if the problem is ill-posed, one needs to verify if the unique continuation property holds.

These definitions can be reformulated in the case of elliptic operators as follows. An elliptic operator $\mathcal{L}$ is said to have the *weak unique continuation* property if every solution u to $\mathcal{L}[u] = 0$ in Ω that vanishes on an open domain strictly contained in Ω must vanish in Ω. The weak unique continuation property guarantees the uniqueness of solutions to the Cauchy problem. In particular, Pliš proved that the optimal threshold is the Lipschitz continuity of the leading term coefficient (see [107]), since the unique continuation property fails if the coefficients are Hölder continuous.

On the other hand, an elliptic operator $\mathcal{L}$ is said to have the *strong unique continuation property* if every solution u to $\mathcal{L}[u] = 0$ in Ω that vanishes of infinite order at a point $x_0 \in \Omega$ must be identically zero in Ω.

As pointed out in [19], we mention the connection between the unique continuation property and the stability of the Cauchy problem. We define a *stability estimate* as an inequality of the form

$$\|u\|_{L^2(\Omega)} \leq \omega(\|u\|_{H^1(\Omega)}), \qquad \text{if } \|u\|_{H^1(\Omega)} \leq 1. \tag{2.2}$$

We will see that ω has different behaviours such as logarithmic, Hölder continuous, or Lipschitz continuous.

The proofs of the unique continuation properties are based on inequalities which are also used to derive stability estimates. These inequalities can be divided into two classes: the Carleman estimates and the three sphere inequality. We will discuss the three sphere inequalities in the last section of this chapter.

2.1 A stability estimate for the Cauchy problem

In this section, we present a stability estimate for a function that satisfies the Cauchy problem for an elliptic equation in divergence form with Lipschitz coefficients. This estimate can be traced back to Trytten [134] and will be used in Chapter 3 to prove a quantitative estimate of propagation of smallness (Proposition 3.2.8).

It is known that the Cauchy problem for the n-dimensional Laplace equation is not well-posed according to Hadamard definition of well-posedness (see [71, 106]). However, Lavrent'ev [90] showed that if an additional condition is imposed on the solution along with the Cauchy data, namely that the harmonic solution can be uniformly bounded by a positive constant on its domain Ω, then the Cauchy problem for the Laplacian has a stable solution, and the problem is well-posed. Payne [103] considered the Cauchy problem taking into account the error in the Cauchy data, since they represents physical measurements. Payne showed that by knowing the error in the measurement data and an upper bound on the absolute value of the harmonic function in Ω, one can obtain *a-priori* bounds on the value of the harmonic function at any point in Ω. He used a Rayleigh-Ritz technique to obtain upper and lower bounds for the harmonic function evaluated at a point in Ω. Payne and Weinberg [105] extended this result to the second-order uniformly elliptic operator

$$\mathrm{L}(u) = \mathrm{div}(A(x)\nabla u).$$

Trytten [134] extended these results further by considering an approximating function φ which is piecewise C^2 in Ω and allows for non-linear problems, in particular operators of the form

$$\mathrm{L}(w) = h(x, w, \nabla w),$$

where h is Lipschitz continuous with respect to the last two arguments.

Before giving the Trytten estimate, we introduce some notation. We consider a bounded domain $\Omega \subset \mathbb{R}^n$ with the Lipschitz boundary as per Definition 1.0.1. Let $\Sigma \subset \partial\Omega$ be a portion of the boundary of class $C^{1,\alpha}$ as per Definition 1.0.1 with constants r_0, M_0 for $\alpha \in (0, 1]$. Let $r_1 = r_0/(\sqrt{1 + M_0^2})$. For $\rho \in (0, r_1)$, let:

$$\begin{aligned} \Omega_\rho^\Sigma &:= \{x \in \overline{\Omega} \,:\, \mathrm{dist}(x, \partial\Omega \setminus \Sigma) > \rho\}, \\ \Sigma^\rho &:= \Omega_\rho^\Sigma \cap \Sigma. \end{aligned} \tag{2.3}$$

Hence, for any $P \in \Sigma^\rho$ there exists a rigid transformation under which $P \equiv 0$ and

$$\Omega_\rho^\Sigma \cap B_{r_0} = \{x \in B_{r_0} \,:\, x_n > \varphi(x')\},$$

where φ is a $C^{1,\alpha}$ function on $B'_{r_0} \subset \mathbb{R}^{n-1}$. We define

$$\begin{aligned} \Sigma_0^\rho &= \{(x', x_n) \,:\, |x'| < \rho,\, x_n = \varphi(x')\}, \\ \Sigma_0 &= \{(x', x_n) \,:\, |x'| < r_0,\, x_n = \varphi(x')\}. \end{aligned}$$

Consider a bounded, measurable, symmetric matrix function $A(x) = \{a_{ij}(x)\}_{i,j=1}^n$

that satisfies the following uniform ellipticity condition: for $\lambda > 1$,

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2, \qquad \text{for a.e. } x\in\Omega, \text{ for all } \xi\in\mathbb{R}^n, \tag{2.4}$$

and the following Lipschitz condition for some $\Lambda > 0$:

$$|a_{ij}(x) - a_{ij}(y)| \le \frac{\Lambda}{r_0}|x-y| \quad \text{for every } i,j = 1,\dots,n,\ x,y\in\Omega. \tag{2.5}$$

Let $u \in H^1(\Omega)$ be a weak solution of the Cauchy problem:

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } \Omega,\\ u = \psi & \text{on } \Sigma,\\ A\nabla u\cdot\nu = g & \text{on } \Sigma, \end{cases} \tag{2.6}$$

with $\psi \in H^{1/2}_{co}(\Sigma)$ and $g \in L^2(\Sigma)$.

Given $E > 0$, we assume that u satisfies the following energy bound:

$$\int_\Omega u^2 \,\mathrm{d}x + r_0^2\int_\Omega |\nabla u|^2 \,\mathrm{d}x \le E. \tag{2.7}$$

Lemma 2.1.1 (Trytten, 1963). *Let $u \in H^1(\Omega)$ be a weak solution of (2.6) that satisfies (2.7). Let $P_1 \in \Sigma_0^\rho$ and set $P_0 = P_1 + (r_1/4)\nu(P_1)$, where $\nu(P_1)$ is the exterior unit normal of Σ_0^ρ at P_1. Then there exist $C > 0$, $\eta\in(0,1)$ constant depending on $\lambda, \Lambda, M_0, r_0$ only such that*

$$\begin{aligned}\int_{B_{\frac{3}{8}r_1}(P_0)\cap\Omega^\Sigma_{r_0}} |\nabla u|^2 \le C\Big(\int_{\Sigma_0}|\psi|^2 + r_0^2\int_{\Sigma_0}|\nabla u|^2 + r_0\int_{\Omega^\Sigma_{r_0}} A\nabla u\cdot\nabla u\Big)^{1-\eta}\cdot\\ \cdot\Big(\int_{\Sigma_0}|\psi|^2 + r_0^2\int_{\Sigma_0}|\nabla u|^2\Big)^{\eta}.\end{aligned} \tag{2.8}$$

Proof. The proof is based on the stability estimate for the Cauchy problem in Trytten [134] (see also a later work of Payne [104], [9, p.778] and Sincich [121]). By Trytten [134], for suitably chosen $p > 1$ and $K > 0$ that depend on ρ and on λ, Λ, M_0, only, the following estimate holds:

$$\begin{aligned}\mathcal{F}\left(\frac{r_1}{2}\right) \le \frac{C}{r_0^p}\left(\int_{\Sigma_0} u^2 + r_0^2\int_{\Sigma_0}|\nabla u|^2 + r_0\int_{\Omega^\Sigma_{r_0}} A\nabla u\cdot\nabla u\right)^{1-\eta}\cdot\\ \cdot\left(\int_{\Sigma_0^\rho} u^2 + r_0^2\int_{\Sigma_0^\rho}|\nabla u|^2\right)^{\eta},\end{aligned} \tag{2.9}$$

where

$$\mathcal{F}(r) = \int_{\frac{1}{4}r_1}^{r} s^{-p}\int_{B_s(P_0)\cap\Omega^\Sigma_{r_0}} A\nabla u\cdot\nabla u + \frac{K}{r_0^p}\left(\int_{\Sigma_0^{r_1/4}} u^2 + r_0^2\int_{\Sigma_0^{r_1/4}}|\nabla u|^2\right),$$

with $0 < \eta < 1$ and $C > 0$ constants that depend on ρ and on λ, Λ, M_0, only. On the other hand, by (2.4), we can determine a lower bound for the function $\mathcal{F}$ in terms of the L^2 norm of ∇u as follows:

$$\mathcal{F}\left(\frac{r_1}{2}\right) \geq \int_{\frac{3}{8}r_1}^{\frac{1}{2}r_1} s^{-p} \int_{B_s(P_0)\cap\Omega_{r_0}^{\Sigma}} A\nabla u\cdot\nabla u \geq c_1 r_0^{1-p} \int_{B_{\frac{3r_1}{8}}(P_0)\cap\Omega_{r_0}^{\Sigma}} |\nabla u|^2. \quad (2.10)$$

By combining (2.9) and (2.10), we derive (2.8). □

2.2 The three sphere inequality

In this subsection, we introduce two versions of the three sphere inequality. The first one holds for second-order elliptic equations in divergence form when the leading term coefficient is Lipschitz continuous on the domain. The second version holds for the same kind of equations with a piecewise Lipschitz leading coefficient.

2.2.1 Three sphere inequality for Lipschitz continuous coefficients

Consider the ball B_R centred at the origin with radius $R > 0$. Consider the elliptic operator in pure divergence form:

$$\mathrm{L}(u) = \mathrm{div}(A\,\nabla u),$$

where $A = A(x) = \{A_{ij}(x)\}_{i,j=1}^n$ is a real symmetric $n\times n$ matrix function such that its entries are bounded, measurable functions and it satisfies the uniform ellipticity condition (2.4) with constant $\lambda > 1$. Moreover, we assume that σ is Lipschitz continuous as in (2.5) with constants $\Lambda, r_0 > 0$.

Theorem 2.2.1 (Alessandrini-Rondi-Rosset-Vessella, 2009). *Let $u \in H^1(B_R)$ be a weak solution of $L(u) = 0$ in B_R with the coefficient A as described above. Then, for any r_1, r_2, r_3 such that $0 < r_1 < r_2 < \frac{r_3}{\lambda} \leq r_3 \leq R$, the following inequality holds:*

$$\|u\|_{L^2(B_{r_2})} \leq Q\|u\|_{L^2(B_{r_1})}^{\alpha}\,\|u\|_{L^2(B_{r_3})}^{1-\alpha}, \quad (2.11)$$

where $Q \geq 1$, $C = C(\lambda, \Lambda, \max\{R/r_0, 1\})$ and

$$\alpha = \frac{\ln\frac{r_3}{\lambda r_2}}{\ln\frac{r_3}{\lambda r_2} + C\ln\frac{\lambda r_2}{r_1}}.$$

We introduce a preliminary result needed for the proof of Theorem 2.2.1. For $r \in (0, R)$, set

$$\mu = \frac{A(x)x\cdot x}{|x|^2} \quad \text{and} \quad H(r) = \int_{\partial B_r} \mu(x)u^2(x)\,\mathrm{d}x.$$

Theorem 2.2.2 (Alessandrini-Rondi-Rosset-Vessella, 2009). *Let $u \in H^1(B_R)$ be a weak solution of $L(u) = 0$ in B_R with the coefficient A as described above, and assume that $A(0) = Id$. Then, for any r_1, r_2, r_3 with $0 < r_1 < r_2 < r_3 \leq R$, the following inequality holds:*

$$H(r_2) \leq QH(r_1)^{\alpha}H(r_3)^{1-\alpha}, \quad (2.12)$$

where $Q \geq 1$ *and* $Q = Q(\lambda, \Lambda, \max\{R/r_0, 1\})$,

$$\alpha = \frac{\ln \frac{r_3}{r_2}}{\ln \frac{r_3}{r_2} + C \ln \frac{r_2}{r_1}}, \qquad C = C(\lambda, \Lambda, \max\{R/r_0, 1\}).$$

Proof. The proof is based on estimates for the frequency function introduced by Garofalo and Lin in [64] and a well-known Rellich identity [109]. See [19, Theorem 2.3] for more details. □

Proof of Theorem 2.2.1. The proof is based on [19, Theorem 2.1]. Assume that $A(0) \neq Id$. The idea is to introduce a change of variable that allows one to reduce the operator $\mathrm{L} = \mathrm{div}(A(x)\,\nabla\cdot)$ to an elliptic operator $\tilde{\mathrm{L}} = \mathrm{div}(\tilde{A}(y)\,\nabla\cdot)$ for which $\tilde{A}(0) = Id$.

Set

$$y = \phi(x) = Jx, \qquad \text{with } J = \sqrt{A^{-1}(0)}.$$

For $r > 0$, since $A(0)$ is symmetric and positive definite, we can consider the open ellipsoid

$$\mathcal{E}_r = \{x \in \mathbb{R}^n : \|\phi(x)\| < r\} = \phi^{-1}(B_r).$$

By (2.4), the following inclusions hold:

$$B_{\frac{r}{\sqrt{\lambda}}} \subset \mathcal{E}_r \subset B_{\sqrt{\lambda} r}. \tag{2.13}$$

Set $\psi = \phi^{-1}$ and define

$$v(y) := u(\psi(y)), \quad \text{and} \quad \tilde{A}(y) := (D\psi(y))^{-1} A(\psi(y))((D\psi(y))^{-1})^T,$$

where $D\psi$ is the Jacobian matrix of ψ. It turns out that $v \in H^1(B_R)$ is a weak solution of

$$\mathrm{div}_y(\tilde{A}\nabla_y v) = 0 \qquad \text{in } B_{\frac{R}{\sqrt{\lambda}}}. \tag{2.14}$$

The matrix function $\tilde{A}$ satisfies the following properties:

1. The uniform ellipticity condition:

$$\lambda^{-2}|\xi|^2 \leq \tilde{A}(y)\xi \cdot \xi \leq \lambda^2 |\xi|^2 \qquad \text{for a.e. } y \in B_r, \text{ for every } \xi \in \mathbb{R}^n. \tag{2.15}$$

2. The Lipschitz continuity condition:

$$|\tilde{A}(y_1) - \tilde{A}(y_2)| \leq \frac{\lambda^{\frac{3}{2}}\Lambda}{r_0}|y_1 - y_2| \qquad \text{for every } y_1 \neq y_2, y_1, y_2 \in \mathbb{R}^n. \tag{2.16}$$

3. $\tilde{A}(0) = Id$.

Hence the operator $\mathrm{div}(\tilde{A}\,\nabla\cdot)$ and the solution v verify the conditions of Theorem 2.2.2. Fix r_1, r_2, r_3 as in the assumptions of the Theorem. Set

$$\rho_1 = \frac{r_1}{\sqrt{\lambda}}, \quad \rho_2 = \sqrt{\lambda} r_2, \quad \rho_3 = \frac{r_3}{\sqrt{\lambda}}.$$

By Theorem 2.2.2, it follows that

$$H(\rho_2) \leq QH(\rho_1)^\alpha H(\rho_3)^{1-\alpha}, \tag{2.17}$$

with

$$H(\rho) = \int_{\partial B_\rho} \mu v^2, \qquad \mu(y) = \frac{\tilde{A}(y)y\cdot y}{|y|^2},$$

and

$$Q \geq 1, \qquad \alpha = \frac{\ln\frac{\rho_3}{\rho_2}}{\ln\frac{\rho_3}{\rho_2} + C\ln\frac{\rho_2}{\rho_1}}.$$

Since $\lambda^{-1} \leq \mu \leq \lambda$,

$$\int_{\partial B_{\rho_2}} v^2 \leq Q\Big(\int_{\partial B_{\rho_1}} v^2\Big)^\alpha \cdot \Big(\int_{\partial B_{\rho_3}} v^2\Big)^{1-\alpha}. \tag{2.18}$$

Integrating the first term on the left-hand side of (2.18), by (2.17), we have

$$\int_{B_{\rho_2}} v^2 \leq \rho_2 \int_0^1 H(\rho_2 t)\ \mathrm{d}t \leq \rho_2 \int_0^1 H(\rho_1 t)^\alpha\, H(\rho_3 t)^{1-\alpha}\ \mathrm{d}t.$$

Then, by the Hölder inequality, we derive

$$\int_{B_{\rho_2}} v^2 \leq \tilde{Q}\Big(\int_0^{\rho_1} H(t)\ \mathrm{d}t\Big)^\alpha \cdot \Big(\int_0^{\rho_3} H(t)\ \mathrm{d}t\Big)^{1-\alpha} \leq \tilde{Q}\Big(\int_{B_{\rho_1}} v^2\Big)^\alpha \cdot \Big(\int_{B_{\rho_3}} v^2\Big)^{1-\alpha}.$$

Back to the old variables,

$$\int_{B_{\rho_2}} v^2\ \mathrm{d}y = \int_{B_{\rho_2}} u(\psi(y))^2\ \mathrm{d}y = \int_{\mathcal{E}_{\rho_2}} u^2 |\det J|.$$

Hence, (2.19) can be written as

$$\int_{\mathcal{E}_{\rho_2}} u^2 \leq \tilde{Q}\Big(\int_{\mathcal{E}_{\rho_1}} u^2\Big)^\alpha \cdot \Big(\int_{\mathcal{E}_{\rho_3}} u^2\Big)^{1-\alpha}. \tag{2.19}$$

Due to (2.13), it turns out that

$$\int_{B_{\rho_2}} u^2 \leq \tilde{Q}\Big(\int_{B_{\rho_1}} u^2\Big)^\alpha \cdot \Big(\int_{B_{\rho_3}} u^2\Big)^{1-\alpha}.$$

□

Consider the elliptic operator

$$\mathcal{L}(u) = \mathrm{div}(A\nabla u) + cu, \tag{2.20}$$

with A that satisfies (2.4) and (2.5). We assume that $c \in L^\infty(\Omega)$ and

$$\|c\|_{L^\infty(\Omega)} \leq \frac{\kappa}{r_0^2}. \tag{2.21}$$

We prove a three sphere inequality for the elliptic operator (2.20) following the lines of [19, Theorem 4.1].

Theorem 2.2.3 (Alessandrini-Rondi-Rosset-Vessella, 2009)**.** *Let $u \in H^1(B_R)$ be a weak solution of*

$$\mathcal{L}(u) = 0 \qquad \text{in } B_R.$$

Then, there exists a constant C_0 with $0 < C_0 \leq 1$ only depending on λ, Λ, κ such that if we set $R_0 = \min\{R, C_0 r_0\}$, for every r_1, r_2, r_3 such that $0 < r_1 < r_2 < \frac{r_3}{4\lambda} \leq r_3 \leq R_0$,

$$\int_{B_{r_2}} u^2 \leq Q\Big(\int_{B_{r_1}} u^2\Big)^{\alpha} \cdot \Big(\int_{B_{r_3}} u^2\Big)^{1-\alpha}, \tag{2.22}$$

with $Q \geq 1$ and α, $0 < \alpha < 1$ depending on $\lambda, \Lambda, r_0, \kappa, \max\left\{\frac{R}{r_0}, 1\right\}, \frac{r_2}{r_1}$ and $\frac{r_3}{r_2}$.

The proof of Theorem 2.2.3 is based on the following two Lemmas.

Lemma 2.2.4 (Alessandrini-Rondi-Rosset-Vessella, 2009)**.** *For every $\delta > 0$ there exists a constant $C_0 \in (0,1]$ that only depends on λ, Λ, κ, and δ such that, for $R_0 := \min\{R, C_0 r_0\}$, there exists a positive solution $w \in C^1(B_{R_0})$ to*

$$\operatorname{div}(A\nabla w) + cw = 0 \qquad \text{in } B_{R_0}, \tag{2.23}$$

such that

$$\frac{1}{1+\delta^2} \leq w \leq 1 + \delta^2, \tag{2.24}$$

and

$$|\nabla w| \leq \frac{\delta}{r_0}. \tag{2.25}$$

Proof of Lemma 2.2.4. The proof is based on [19, Lemma 4.2]. Notice that by (2.4), for $r \leq R$,

$$\mathcal{L}(w,w) = \int_{B_r} (A\nabla w \cdot \nabla w - cw\, w)\, \mathrm{d}x \geq \lambda^{-1} \int_{B_r} |\nabla w|^2 - \frac{\kappa}{r_0^2} \int_{B_r} w^2,$$

where $\mathcal{L}(\cdot,\cdot)$ is the bilinear form associated with (2.23). Hence, for a suitable $C_1 > 0$ depending on λ and κ, there is a radius $R_1 := \min\{R, C_1 r_0\}$ such that for any $r \leq R_1$, the bilinear form $\mathcal{L}$ is coercive on $H^1_0(B_r)$. Therefore, let w be the unique solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(A\nabla w) + c\, w = 0 & \text{in } B_r, \\ w = 1 & \text{on } \partial B_r. \end{cases} \tag{2.26}$$

Set $z = w - 1$, then z is the weak solution of

$$\begin{cases} \operatorname{div}(A\nabla z) = f & \text{in } B_r, \\ z = 0 & \text{on } \partial B_r, \end{cases} \tag{2.27}$$

where $f = -c(1+z)$. By *a-priori* estimates in L^∞ (see [66, Theorem 8.16]), it follows that

$$\|z\|_{L^\infty(B_r)} \leq C\kappa \frac{r^2}{r_0^2}(1 + \|z\|_{L^\infty(B_r)}),$$

where C is a positive constant that depends only on Λ. Hence,

$$(1 - C\kappa\frac{r^2}{r_0^2})\|z\|_{L^\infty(B_r)} \leq C\kappa\frac{r^2}{r_0^2}.$$

There is a constant $C_2 \leq C_1$ only depending on Λ, κ such that for any $r \leq R_2$, with $R_2 := \min\{R, C_2 r_0\}$, the following inequality holds:

$$\|z\|_{L^\infty(B_r)} \leq C\kappa\frac{r^2}{r_0^2}. \tag{2.28}$$

By a global estimate of Schauder type (see [66, Theorem 8.33]), for any $r \leq R_2$ we derive

$$\|\nabla z\|_{L^\infty(B_r)} \leq C\kappa\frac{r}{r_0^2}(1 + \|z\|_{L^\infty(B_r)}), \tag{2.29}$$

where C depends on λ, M_0 only. Finally, there is a constant $C_0 \leq C_2$ for which, for $R_0 := \min\{R, C_0 r_0\}$, by (2.28), (2.29),

$$\begin{aligned} \|z\|_{L^\infty(B_r)} &\leq \frac{\delta^2}{1+\delta^2}, \\ \|\nabla z\|_{L^\infty(B_r)} &\leq \frac{\delta}{r_0}. \end{aligned} \tag{2.30}$$

Hence, by the definition of z, the thesis follows. □

Lemma 2.2.5 (Alessandrini-Rondi-Rosset-Vessella, 2009)**.** *Let R_0 be the quantity defined in Lemma 2.2.4 and choose $\delta = \min\left\{\frac{\Lambda}{\lambda}, 1\right\}$. Then u can be factored on B_{R_0} as $u = wv$, where w is the positive function of Lemma 2.2.4 and v solves*

$$\mathrm{div}(\tilde{A}\nabla v) = 0 \qquad \textit{in } B_{R_0},$$

with $\tilde{A} = w^2 A$.

proof of Lemma 2.2.5. By direct computation, the thesis follows. □

proof of Theorem 2.2.3. Since $w \in C^1(B_{R_0})$, we have that $\tilde{A} \in C^{0,1}(B_{R_0}, Sym_n)$. Due to (2.4) and (2.24), we can show that

$$\frac{1}{4\lambda}|\xi|^2 \leq \tilde{A}(x)\xi\cdot\xi \leq 4\lambda|\xi|^2 \qquad \text{for a.e. } x \in B_{R_0}, \text{ for every } \xi \in \mathbb{R}^n. \tag{2.31}$$

Due to (2.5) and (2.24), it turns out that for every $x \neq y$,

$$|\tilde{A}(x) - \tilde{A}(y)| \leq \frac{8\Lambda}{r_0}|x-y|.$$

Hence, for the solution $v \in H^1(B_{R_0})$ to

$$\mathrm{div}(\tilde{A}\nabla v) = 0 \qquad \text{in } B_{R_0},$$

the three sphere inequality (2.11) holds. Let r_1, r_2, r_3, R_0 be as in the Theorem

(2.2.3). Then, by (2.24) and (2.11),

$$\int_{B_{r_2}} u^2 \leq 4\int_{B_{r_2}} v^2 \leq Q\Big(\int_{B_{r_1}} v^2\Big)^{\alpha}\cdot\Big(\int_{B_{r_3}} v^2\Big)^{1-\alpha} \leq \tilde{Q}\Big(\int_{B_{r_1}} u^2\Big)^{\alpha}\cdot\Big(\int_{B_{r_3}} u^2\Big)^{1-\alpha}.$$

□

We can extend this result to the L^∞ norms by applying the $L^\infty - L^2$ estimate due to Moser and Stampacchia (see [122, Corollaire 5.2] and [44, Theorem 6.1]).

Theorem 2.2.6. *Assume that u is a weak solution of*

$$\operatorname{div}(A\nabla u) + c\,u = 0 \qquad \textit{in } B_{R_0},$$

with A, c satisfying (2.4), (2.5) and (2.21). Then there exists a constant C only depending on n, λ, Λ, and κ such that if $B_r \subset B_{R_0}$, we have

$$\sup_{x\in B_{r/2}} |u(x)| \leq \frac{C}{r^{n/2}}\|u\|_{L^2(B_r)}. \tag{2.32}$$

The proof of this result can be found in [44, Theorem 6.1] and Stampacchia [122, Corollaire 5.2]. One can derive an upper bound for the L^∞ norm over a ball in terms of the L^2 norm of the solution u over a ball slightly bigger than the first one.

Corollary 2.2.7. *Let u be as in Theorem 2.2.6. Let $0 < r < \rho$, such that $B_\rho \subset \Omega$. Then*

$$\sup_{x\in B_r} |u(x)| \leq \frac{C}{(\rho - r)^{n/2}}\,\|u\|_{L^2(B_\rho)}. \tag{2.33}$$

Proof. Choose $x \in B_r$, then $B_{\rho-r}(x) \subset B_\rho$. Hence, by Theorem 2.2.6, it follows that

$$|u(x)| \leq \frac{C}{(\rho-r)^{n/2}}\|u\|_{L^2(B_{\rho-r}(x))} \leq \frac{C}{(\rho-r)^{n/2}}\|u\|_{L^2(B_\rho)}.$$

□

By Corollary 2.2.7, we derive the following three sphere inequality for L^∞ norms.

Theorem 2.2.8. *Assume that the hypotheses of Theorem 2.2.3 hold. For r_1, r_2, r_3 such that $0 < r_1 < r_2 < \frac{r_3}{4\lambda} \leq r_3 \leq R_0$ and $\rho \in (r_2, r_3)$, there exists a constant $C > 1$ that depends only on Λ and n such that*

$$\|u\|_{L^\infty(B_{r_2})} \leq \frac{C}{(\rho - r_2)^{n/2}}\,\|u\|^{\alpha}_{L^\infty(B_{r_1})}\,\|u\|^{1-\alpha}_{L^\infty(B_{r_3})}. \tag{2.34}$$

Proof. By Corollary 2.2.7, there exists a constant $C > 1$ such that

$$\|u\|_{L^\infty(B_{r_2})} \leq \frac{C}{(\rho - r_2)^{n/2}}\|u\|_{L^2(B_\rho)}. \tag{2.35}$$

By Theorem 2.2.3, we have

$$\frac{C}{(\rho-r_2)^{n/2}}\|u\|_{L^2(B_\rho)} \le \frac{C}{(\rho-r_2)^{n/2}}\|u\|^{\alpha}_{L^2(B_{r_1})}\|u\|^{1-\alpha}_{L^2(B_{r_3})}$$
$$\le \frac{C}{(\rho-r_2)^{n/2}}|B_{r_1}|^{\frac{\alpha}{2}}|B_{r_3}|^{\frac{1-\alpha}{2}}\|u\|^{\alpha}_{L^\infty(B_{r_1})}\|u\|^{1-\alpha}_{L^\infty(B_{r_3})}.$$

This completes the proof. □

2.2.2 Three sphere inequality for piecewise Lipschitz coefficients

Before stating the three sphere inequality, which applies to the case of piecewise Lipschitz coefficients, we introduce some notation. Let $\Omega \subset \mathbb{R}^n$ be an open Lipschitz domain. Suppose that Σ is a hypersurface contained in Ω of class C^2 with constants r_0, K_0. Assume that $\Omega \setminus \Sigma$ has two connected components $\Omega_\pm$. Let $A \in L^\infty(\Omega, Sym_n)$ be a real symmetric $n \times n$ matrix function of the form

$$A(x) = A^+(x)\,\chi_{\Omega^+}(x) + A^-(x)\,\chi_{\Omega^-}(x), \quad A^\pm \in C^{0,1}(\Omega^\pm),\, x \in \Omega.$$

Assume that A satisfies the uniform ellipticity condition, i.e. there exists a constant $\lambda > 1$ such that

$$\lambda^{-1}|\xi|^2 \le A(x)\xi\cdot\xi \le \lambda|\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for every } \xi \in \mathbb{R}^n.$$

Let $q \in L^\infty(\Omega)$. Furthermore, we assume that there are positive constants $\bar{q}, \bar{A}$, such that

$$\|A^\pm\|_{C^{0,1}(\Omega)} \le \bar{A} \quad \text{and} \quad \|q\|_{L^\infty(\Omega)} \le \bar{q}.$$

Proposition 2.2.9 (Carstea-Wang, 2020). *Let $u \in H^1(\Omega)$ be a weak solution of*

$$\operatorname{div}(A\nabla u) + q\,u = f + \nabla\cdot F \quad \text{in } \Omega,$$

where $f \in L^2(\Omega)$ and $F \in (L^2(\Omega))^N$ satisfy

$$\|f\|_{L^2(\Omega)} + \|F\|_{(L^2(\Omega))^N} \le \varepsilon \qquad \text{for } \varepsilon \ge 0.$$

There exists a constant $\bar{r}$, which depends on K_0 and r_0, such that if $0 < r_1 < r_2 < r_3 \le \bar{r}$, with $Q \in \Omega$ such that dist$(Q, \partial\Omega) > r_3$, then

$$\|u\|_{L^2(B_{r_2}(Q))} \le C(\|u\|_{L^2(B_{r_1}(Q))} + \varepsilon)^{\delta}(\|u\|_{L^2(B_{r_3}(Q))} + \varepsilon)^{1-\delta}, \tag{2.36}$$

where $C > 1$ and $0 < \delta < 1$ are constants that depend on $\lambda, r_0, K_0, \bar{A}, \bar{q}, \frac{r_1}{r_2}, \frac{r_2}{r_3}$, and diam$(\Omega)$.

From this result, we can deduce a three-sphere inequality in terms of L^∞ norms.

Corollary 2.2.10. *Let $u \in H^1(B_{\bar{r}})$ be a weak solution of*

$$\operatorname{div}(\sigma\nabla u) + q\,u = 0 \qquad \text{in } B_{\bar{r}}.$$

Then, for any $0 < r_1 < r_2 < r_3 \leq \bar{r}$, the following inequality holds:

$$\|u\|_{L^\infty(B_{r_2})} \leq C_\infty \|u\|^{\beta}_{L^\infty(B_{r_1})} \|u\|^{1-\beta}_{L^\infty(B_{r_3})}, \tag{2.37}$$

where $\beta = \dfrac{\ln \frac{2r_3}{r_2+r_3}}{\ln \frac{r_3}{r_1}} \in (0,1)$ *and* $C_\infty > 1$ *depends on* $\bar{q}, \bar{A}, \frac{r_1}{r_2}, \frac{r_2}{r_3}, r_0, K_0, \lambda$ *and* n.

Proof. The proof of Corollary 2.2.10 is based on [39, Corollary 3.8]. By [66, Theorem 8.17], there exists a constant $C > 1$ that depends only on $\lambda, \bar{q}, \bar{A}$, and n such that

$$\|u\|_{L^\infty(B_{r_2})} \leq \frac{C}{(r_3 - r_2)^{n/2}} \|u\|_{L^2(B_{r_3})}. \tag{2.38}$$

By Proposition 2.2.9 and (2.38), we have

$$\begin{aligned}
\|u\|_{L^\infty(B_{r_2})} &\leq \frac{C}{\left(\frac{r_2+r_3}{2} - r_2\right)^{n/2}} \|u\|_{L^2(B_{\frac{r_2+r_3}{2}})} \\
&\leq \frac{C}{\left(\frac{r_2+r_3}{2} - r_2\right)^{n/2}} \|u\|^{\delta}_{L^2(B_{r_1})} \|u\|^{1-\delta}_{L^2(B_{r_3})} \\
&\leq \frac{C}{\left(\frac{r_2+r_3}{2} - r_2\right)^{n/2}} |B_{r_1}|^{\frac{\delta}{2}} |B_{r_3}|^{\frac{1-\delta}{2}} \|u\|^{\delta}_{L^\infty(B_{r_1})} \|u\|^{1-\delta}_{L^\infty(B_{r_3})},
\end{aligned}$$

from which (2.37) follows. □

CHAPTER 3

Stability estimates for the Calderón problem

Contents

This chapter is devoted to the study of the stability issue in the inverse conductivity problem for a specific class of anisotropic conductivities. Recall that stability estimates help to quantify the relation between the unknown coefficient and the known boundary data. In this chapter, we prove a stability estimate in terms of an ad hoc misfit functional that measures the discrepancy between the boundary data produced by two different conductivities.

This chapter is divided into three sections. First, we introduce the a priori assumptions about the conductor Ω and the anisotropic conductivity σ. In Section 3.1, we define the misfit functional and state the stability estimate (Theorem 3.1.1) and the Lipschitz stability estimate in terms of the local Dirichlet to Neumann map (Corollary 3.1.2). In Section 3.2, we introduce the main tools needed to prove Theorem 3.1.1: the asymptotic estimates for the Green function (Proposition 3.2.2) and the quantitative estimates of unique continuation (Proposition 3.2.7) along with the corresponding proofs. Finally, Section 3.3 is devoted to the proof of Theorem 3.1.1 and Corollary 3.1.2.

Assumptions on the domain

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary $\partial\,\Omega$ of Lipschitz class with constants r_0 and L. Let $D > 0$ be a constant such that

$$|\Omega| \leq D\, r_0^n,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Moreover, let Σ be an open non-empty flat portion of $\partial\,\Omega$ of size r_0.

For $N \in \mathbb{N}$, we assume that there exists a finite collection of bounded domains $\{D_m\}_{m=1}^N$ such that the following conditions hold:

1. The domains $D_m \subset \Omega$ for $m = 1, \ldots, N$ are connected and pairwise non-overlapping. The boundaries $\partial\, D_m$ are of Lipschitz class with constants r_0 and L.

2. The closure of Ω is equal to the union of the closure of the domains of the collection:
$$\overline{\Omega} = \bigcup_{m=1}^{N} \overline{D}_m.$$

3. There exists one region that we call D_1 for simplicity, such that the intersection $\partial D_1 \cap \Sigma$ contains a flat portion $\sigma^{(1)}$ of size $r_0/3$.

4. For every index $i \in \{2, \ldots, N\}$, there exists a collection of indices $m_1, \ldots, m_K \in \{1, \ldots, N\}$ such that
$$D_{m_1} = D_1 \quad \text{and} \quad D_{m_K} = D_i.$$
The domains $D_{m_1}, \ldots, D_{m_K}$ are contiguous and pairwise disjoint, and for every index $k = 1, \ldots, K-1$, the intersection $\partial D_{m_k} \cap \partial D_{m_{k+1}}$ contains a flat portion $\Sigma_{m_{k+1}}$ of size $r_0/3$ so that $\Sigma_{m_{k+1}} \subset \Omega$.

   Moreover, we assume that for each of these flat subportions $\Sigma_{m_{k+1}}$, $k = 1, \ldots, K-1$, there exist a point $P_{k+1} \in \Sigma_{m_{k+1}}$ and a rigid transformation under which P_{k+1} coincides with the origin O, and
$$\begin{aligned}
\Sigma_{m_{k+1}} \cap B_{r_0/3} &= \Big\{x \in B_{r_0/3} \,:\, x_n = 0\Big\},\\
D_{m_k} \cap B_{r_0/3} &= \Big\{x \in B_{r_0/3} \,:\, x_n < 0\Big\},\\
D_{m_{k+1}} \cap B_{r_0/3} &= \Big\{x \in B_{r_0/3} \,:\, x_n > 0\Big\}.
\end{aligned}$$

   Later, we will add a domain $D_0 \subset \mathbb{R}^n \setminus \overline{\Omega}$ so that, when indexing the chain of subdomains, we agree that $D_{m_0} = D_0$.

Assumptions on the conductivity

The anisotropic conductivity $\sigma(x) = \{\sigma_{ij}(x)\}_{i,j=1}^n$ is a real symmetric $n \times n$ matrix function, $\sigma \in L^\infty(\Omega, Sym_n)$, of the form

$$\sigma(x) = \gamma(x)A(x), \tag{3.1}$$

$$\gamma(x) = \sum_{m=1}^{N} \gamma_m(x)\chi_{D_m}(x), \quad \gamma_m(x) = s_m + S_m \cdot x, \quad \text{for every } x \in \Omega, \tag{3.2}$$

with $s_m \in \mathbb{R}$, $S_m \in \mathbb{R}^n$ for $m = 1, \dots, N$. $A(x)$ is a $n \times n$ matrix function and $\{D_m\}_{m=1}^N$ is the collection of subsets of Ω described above.

The scalar functions γ_m are bounded, namely there is a constant $\bar{\gamma} > 1$ such that

$$\bar{\gamma}^{-1} \leq \gamma_m(x) \leq \bar{\gamma}, \qquad \text{for every } m = 1, \dots N, \ \text{ for all } x \in \Omega.$$

The matrix function $A(x)$ belongs to the space $W^{1,\infty}(\Omega)$. We recall that a function belonging to the Sobolev space $W^{1,\infty}(\Omega)$ for Ω bounded domain is a Lipschitz continuous function (see [56, Theorem 4, Section 5]). We assume that there exists a constant $\bar{A} > 0$ such that

$$\|A\|_{W^{1,\infty}(\Omega)} \leq \bar{A}. \tag{3.3}$$

Furthermore, we assume that σ satisfies the uniform ellipticity condition, namely there exists a constant $\lambda > 1$ such that

$$\lambda^{-1}|\xi|^2 \leq \sigma(x)\,\xi \cdot \xi \leq \lambda|\xi|^2, \qquad \text{for a.e. } x \in \Omega, \text{ for every } \xi \in \mathbb{R}^n. \tag{3.4}$$

In the sequel, we shall refer to the set of positive constants $\{D, N, r_0, L, \lambda, \bar{\gamma}, \bar{A}, n\}$ with $N \in \mathbb{N}$ and the space dimension $n \geq 3$ as the *a priori data*.

Remark 3.0.1. *The class of functions $\gamma(x)$ of the form (3.2) is a finite dimensional linear subspace. The L^∞ norm $\|\gamma\|_{L^\infty(\Omega)}$ is equivalent to the following norm:*

$$|||\gamma||| = \max_{m=1,\dots,N}\{|s_m| + |S_m|\} \tag{3.5}$$

modulo constants that depend on the a priori data only.

The local Dirichlet to Neumann map

We recall the definition of the local Dirichlet to Neumann map.

Definition 3.0.1. *Let $\Omega \subset \mathbb{R}^n$ and Σ be a bounded domain with Lipschitz boundary and a non-empty flat portion, respectively. Let $\sigma \in L^\infty(\Omega, Sym_n)$ be a matrix function satisfying (3.4). The* local Dirichlet to Neumann map *associated to σ and Σ is the operator*

$$\Lambda_\sigma^\Sigma \, : \, H_{00}^{1/2}(\Sigma) \, \to \, H_{00}^{-1/2}(\Sigma)$$

defined by

$$\langle \Lambda_\sigma^\Sigma \, g, \, \eta \rangle \; = \; \int_\Omega \sigma(x)\, \nabla u(x) \cdot \nabla \varphi(x)\, dx, \tag{3.6}$$

for any $g, \eta \in H^{1/2}_{00}(\Sigma)$*. Here,* $u \in H^1(\Omega)$ *is the weak solution of the boundary value problem*

$$\begin{cases} \operatorname{div}(\sigma \nabla u) = 0 & \textit{in } \Omega, \\ u = g & \textit{on } \partial\Omega, \end{cases} \tag{3.7}$$

and $\varphi \in H^1(\Omega)$ *is any function such that* $\varphi|_\Sigma = \eta$ *in the trace sense. In* (3.6)*, the bracket* $\langle \cdot, \cdot \rangle$ *denotes the* $L^2(\partial\Omega)$*-pairing between* $H^{1/2}_{00}(\Sigma)$ *and its dual* $H^{-1/2}_{00}(\Sigma)$*.*

Let σ_j for $j = 1, 2$ be two conductivities that satisfy the a priori assumptions and let Λ_j^Σ be the corresponding local Dirichlet to Neumann maps. Let $u_j \in H^1(\Omega)$ be the weak solutions to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma_j \nabla u_j) = 0 & \text{in } \Omega, \\ u_j|_{\partial\Omega} \in H^{1/2}_{00}(\Sigma). \end{cases} \tag{3.8}$$

The *Alessandrini's identity* is given by

$$\langle (\Lambda_1^\Sigma - \Lambda_2^\Sigma) u_1, u_2 \rangle = \int_\Omega (\sigma^{(1)} - \sigma^{(2)})(x) \nabla u_1 \cdot \nabla u_2 \, \mathrm{d}x. \tag{3.9}$$

3.1 The misfit functional and the stability estimate

In this section, we introduce the Green function for an enlarged domain Ω_0 and we define the misfit functional. We assume that the point $P_1 \in \Sigma$ of the a priori assumptions coincides with the origin up to a rigid transformation. For simplicity, assume that the flat portion Σ_1 coincides with the entire portion Σ.

Next, we define

$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \; : \; |x_i| < \frac{2}{3} r_0, \text{ for } i = 1, \ldots, n-1, \; \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6} r_0 \right\}.$$

We assume that the intersection $\partial D_0 \cap \partial\Omega$ is compactly contained in Σ. We define the augmented domain Ω_0 as

$$\Omega_0 = \operatorname{Int}_{\mathbb{R}^n}(\overline{\Omega \cup D_0}). \tag{3.10}$$

Notice that $\partial\Omega_0$ is of Lipschitz class with constants $r_0/3$ and $\tilde{L}$, where $\tilde{L}$ depends on L only. For $r \in (0, r_0/6)$, we further define the set $(D_0)_r$ as

$$(D_0)_r = \{ x \in D_0 \; : \; \operatorname{dist}(x, \partial D_0) > r \}.$$

Let σ_j for $j = 1, 2$ be two anisotropic conductivities of the form (3.1). We extend them on the augmented domain Ω_0 as $\sigma_j|_{D_0} = Id_n$, and $\gamma^{(j)}|_{D_0} = 1$.

Next, we introduce the Green function G_j associated with the elliptic operator $\operatorname{div}(\sigma_j \nabla \cdot)$ in Ω_0. For every $y \in \Omega_0$, let $G_j(\cdot, y)$ be the weak solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma_j \nabla G_j(\cdot, y)) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ G_j(\cdot, y) = 0 & \text{on } \partial\Omega_0, \end{cases} \tag{3.11}$$

where $\delta(\cdot - y)$ represents the Dirac distribution centred at y.

We would like to make a remark on the notation. When we consider a function $G(x, y)$ with $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$, we use either the symbols $\nabla_x G(x, y)$, $\nabla_x G$ or $\nabla G(\cdot, y)$ to denote the gradient of G with respect to the variable x.

Before introducing the misfit functional, we consider two bounded subsets of D_0 called D_y and D_z. It is important to notice that these subsets are compactly contained within D_0. The misfit functional is defined as an integral over the product $D_y \times D_z$ of a quadratic expression. This expression evaluates the difference between the mixed product of the trace of the Green function G_j on Σ and the trace of the conormal derivative of the Green function G_i for $i \neq j$, and vice versa, integrated over the boundary portion Σ. The subscripts y and z emphasize that the variable y is integrated over D_y and the variable z is integrated over D_z. This notation is intended for future numerical implementation, where the variable y represents the sources and the variable z represents the receivers.

Definition 3.1.1. *For $(y, z) \in D_y \times D_z$, the* misfit functional *is given by*

$$\begin{aligned}\mathcal{J}(\sigma^{(1)}, \sigma^{(2)}) = \int_{D_y \times D_z} \Big| \int_\Sigma \Big[G_1(x, y)\, \sigma^{(2)}(x) \nabla_x G_2(x, z) \cdot \nu - \\ - G_2(x, z)\, \sigma^{(1)}(x) \nabla_x G_1(x, y) \cdot \nu \Big] \, \mathrm{d}S(x) \Big|^2 \, \mathrm{d}y \, \mathrm{d}z,\end{aligned} \tag{3.12}$$

where $\mathrm{d}S$ *is the* $(n-1)-$*surface element.*

Notice that there is a connection between the formula of the misfit functional (3.12) and the local Dirichlet to Neumann map (Definition 3.0.1). Indeed, if we define the following integral

$$S_0(y, z) := \int_\Omega (\sigma^{(1)} - \sigma^{(2)})(x) \nabla_x G_1(x, y) \cdot \nabla_x G_2(x, z) \, \mathrm{d}x, \tag{3.13}$$

then, by the Alessandrini's identity (3.9), for $u_1 = G_1(\cdot, y)$ and $u_2 = G_2(\cdot, z)$, it follows that

$$\langle (\Lambda_1^\Sigma - \Lambda_2^\Sigma) G_1(\cdot, y)|_{\partial\Omega}, G_2(\cdot, z)|_{\partial\Omega} \rangle = S_0(y, z) \quad \text{for every } (y, z) \in D_y \times D_z. \tag{3.14}$$

By Green's identity (A.4), it follows that

$$S_0(y, z) = \int_\Sigma \Big[G_1(x, y)\, \sigma^{(2)}(x) \nabla_x G_2(x, z) \cdot \nu - G_2(x, z)\, \sigma^{(1)}(x) \nabla_x G_1(x, y) \cdot \nu \Big] \, \mathrm{d}S(x).$$

Hence, we can write

$$\mathcal{J}(\sigma^{(1)}, \sigma^{(2)}) = \int_{D_y \times D_z} |S_0(y, z)|^2 \, \mathrm{d}y \, \mathrm{d}z. \tag{3.15}$$

Now, we state the stability estimate in terms of the misfit functional.

Theorem 3.1.1. *Let Ω be a bounded domain, Σ be a non-empty portion of $\partial\Omega$ and $\{D_m\}_{m=1}^N$ be N subdomains that satisfy the a priori assumptions. Let σ_j for $j = 1, 2$ be two anisotropic conductivities of the form (3.1) satisfying (3.4). Then there exists a* *positive constant C that depends only on the a priori data such that*

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \leq C \left(\mathcal{J}(\sigma^{(1)}, \sigma^{(2)})\right)^{1/2}. \tag{3.16}$$

From this result, we derive the following Lipschitz stability estimate in terms of the local Dirichlet to Neumann map.

Corollary 3.1.2. *Assume that the hypotheses of Theorem 3.1.1 hold. Let Λ_j^Σ be the local Dirichlet to Neumann map associated with the conductivity σ_j for $j = 1, 2$. Then there exists a constant $\tilde{C} > 0$ that depends only on the a priori data such that*

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \leq \tilde{C}\, \|\Lambda_1^\Sigma - \Lambda_2^\Sigma\|_*.$$

The constant C appearing in Theorem 3.1.1 depends on the number of domains of the partition of Ω (see [111]).

3.2 Auxiliary Propositions

Fix an index $K \in \{1, \dots, N\}$. We consider a chain $\{D_m\}_{m=0}^K$ of contiguous subdomains that joins D_0 to D_K (up to a reordering of indices). We also consider the corresponding sequence of flat portions $\Sigma_1, \dots, \Sigma_K$ with points $P_1, \dots, P_K$ as in the a priori assumptions. We denote with $\nu(P_{m+1})$ the exterior unit normal to ∂D_m at the point P_{m+1}. Set $\gamma^- = \gamma_m(P_{m+1})$, $\gamma^+ = \gamma_{m+1}(P_{m+1})$, $J = \sqrt{A(P_{m+1})^{-1}}$, and $|J| = \det\, J$. We define $\chi_+ = \chi_{\mathbb{R}^n_+}$ and let H be the fundamental solution defined in (A.15) associated with the elliptic operator $\mathrm{div}((\gamma^- + (\gamma^+ - \gamma^-)\chi_+)J^{-2}\nabla \cdot)$.

Let G be the Green function associated to the elliptic operator $\mathrm{div}(\sigma\nabla\cdot)$ in Ω_0. For every $y \in \Omega_0$, let $G(\cdot, y)$ be the weak solution of the Dirichlet problem (3.11). It is well-known (see [95, 69]) that the Green function G satisfies the following properties: for every $x, y \in \Omega_0$, $x \neq y$,

$$G(x, y) = G(y, x),$$

and

$$0 < G(x, y) < C\,|x - y|^{2-n}, \qquad \forall x \neq y,\ x, y \in \Omega_0, \tag{3.17}$$

where $C > 0$ is a constant that depends on λ and n.

Proposition 3.2.1. *For every $y \in \Omega_0$ and every $r > 0$, the following inequalities hold:*

$$\int_{\Omega_0 \setminus B_r(y)} |\nabla G(\cdot, y)|^2 \leq C r^{2-n}, \tag{3.18}$$

where C is a positive constant depending on λ and n.

Proof of Proposition 3.2.1. Fix $y \in \Omega_0$, let $G(\cdot, y)$ be a weak solution of the Dirichlet problem (3.11). By the Caccioppoli inequality (Theorem A.2.2),

$$\int_{[B_{2r}(y)\setminus B_r(y)]\cap\Omega_0} |\nabla G(x, y)|^2 \,\mathrm{d}x \leq \frac{c}{r^2} \int_{[B_{3r}(y)\setminus B_{r/2}(y)]\cap\Omega_0} |G(x, y)|^2 \,\mathrm{d}x,$$

where C is a positive constant depending only on λ. Then, by (3.17),

$$\int_{[B_{3r}(y)\backslash B_{r/2}(y)]\cap\Omega_0} |G(x,y)|^2 \,\mathrm{d}x \le c \int_{[B_{3r}(y)\backslash B_{r/2}(y)]\cap\Omega_0} |x-y|^{2(2-n)} \,\mathrm{d}x.$$

By a change of variables, it follows that

$$\int_{[B_{3r}(y)\backslash B_{r/2}(y)]\cap\Omega_0} |x-y|^{2(2-n)} \,\mathrm{d}x \le c \int_{r/2}^{3r} \rho^{3-n} \,\mathrm{d}\rho \le c r^{4-n},$$

which leads to

$$\int_{[B_{2r}(y)\backslash B_{r}(y)]\cap\Omega_0} |\nabla G(x,y)|^2 \,\mathrm{d}x \le c r^{2-n}. \tag{3.19}$$

Consider the sequence of annuli $\{B_{2^{k+1}r}(y)\backslash B_{2^k r}(y)\}_{k\in\mathbb{N}}$, then

$$\Omega_0\backslash B_r(y) = \bigcup_{k=0}^{\infty} [B_{2^{k+1}r}(y)\backslash B_{2^k r}(y)] \cap \Omega_0. \tag{3.20}$$

By (3.19) and (3.20), we can conclude that

$$\begin{aligned}\int_{\Omega_0\backslash B_r(y)} |\nabla G(\cdot,y)|^2 &\le \sum_{k=0}^{\infty} \int_{[B_{2^{k+1}r}(y)\backslash B_{2^k r}(y)]\cap\Omega_0} |\nabla G(\cdot,y)|^2 \\ &\le C\sum_{k=0}^{\infty} (2^k r)^{2-n} \le c r^{2-n}.\end{aligned}$$

□

The following Proposition describes the asymptotic behaviour of the Green function near the discontinuity interface.

Proposition 3.2.2. *Fix an index $m \in \{0,\dots,K-1\}$. There exist constants α,θ_1,θ_2 with $0<\alpha,\theta_1,\theta_2<1$ and $C_1,C_2,C_3>0$ depending on the a priori data only such that for any $x \in B_{r_0/4}(P_{m+1})\cap D_{m+1}$ and $y = P_{m+1} - r\nu(P_{m+1})$, where $r\in(0,r_0/4)$ and $\nu(P_{m+1})$ is the exterior unit normal of ∂D_m at P_{m+1}, the following inequalities hold true:*

$$|G(x,y) - H(x,y)| \le C_1|x-y|^{3-n-\alpha}, \tag{3.21}$$

$$|\nabla_x G(x,y) - \nabla_x H(x,y)| \le C_2|x-y|^{1-n+\theta_1}, \tag{3.22}$$

$$|\nabla_y\nabla_x G(x,y) - \nabla_y\nabla_x H(x,y)| \le C_3|x-y|^{-n+\theta_2}, \tag{3.23}$$

with the Green function G solution of (3.11).

To prove Proposition 3.2.2, we need some preliminary results. We introduce first some related notation. Let $0<\mu<1$ and $A^+ \in C^\mu(Q_r^+)$, $A^- \in C^\mu(Q_r^-)$ be real symmetric $n\times n$ positive definite matrix functions and define

$$A(x) = A^+(x)\chi_{Q_r^+}(x) + A^-(x)\chi_{Q_r^-}(x). \tag{3.24}$$

We assume that A satisfies the uniform ellipticity condition: for some constant $\lambda_0 > 1$,

$$\lambda_0^{-1}|\xi|^2 \;\le\; A(x)\xi\cdot\xi \;\le\; \lambda_0|\xi|^2 \quad \text{for a.e. } x \in Q_r, \text{ for every } \xi \in \mathbb{R}^n.$$

Let $U \in H^1(Q_r)$ be a solution of

$$\operatorname{div}(A\nabla U) = 0 \qquad \text{in } Q_r. \tag{3.25}$$

Proposition 3.2.3. *Let $r > 0$ be a fixed number. Let A be of the form (3.24), and suppose that α' is such that $\alpha' \in (0,1)$, and let $\epsilon > 0$. There exists a constant C such that for any $\rho \le r/2$, and for any $x \in Q_{r-2\rho}$, the following estimate holds:*

$$\|\nabla U\|_{L^\infty(Q_\rho(x))} + \rho^{\alpha'}|\nabla U|_{\alpha',Q_\rho(x)\cap Q_r^+} + \rho^{\alpha'}|\nabla U|_{\alpha',Q_\rho(x)\cap Q_r^-} \le \frac{C}{\rho^{1+n/2}}\|U\|_{L^2(Q_r(x))}. \tag{3.26}$$

Proof. For the proof, we refer to [85, Chapter 3, Theorem 16.2], where the authors obtained piecewise $C^{1,\alpha'}$ estimates for solutions to linear second-order elliptic equations with piecewise Hölder coefficients and $C^{1,1}$ discontinuity interfaces (see also [92, Theorem 1.1] for more recent regularity results). □

Proof of Proposition 3.2.2. Fix an index $m \in \{0,\dots,K-1\}$. Up to a rigid transformation, we assume that P_{m+1} coincides with the origin 0 and $\nu(P_{m+1})$ is the n-th standard unit vector e_n. Let $y = y_n e_n$, with $y_n \in (-r_0/4, 0)$. For any $x = (x', x_n)$, we define the reflected point $x^* := (x', -x_n)$ with respect to the hyperplane $\{x_n = 0\}$. Set $\sigma_0(x) = (\gamma^- + (\gamma^+ - \gamma^-)\chi_+(x))A(0)$. Let H be the fundamental solution associated with the elliptic operator $\operatorname{div}(\sigma_0\nabla\cdot)$ in $\mathbb{R}^n$ (see equation (A.15) for the explicit formula). For $y \in \Omega_0$, let $G(\cdot,y) \in H^1(\Omega_0)$ be the weak solution of (3.11). Define the distribution

$$R(x,y) = G(x,y) - H(x,y).$$

For any $y \in \Omega_0$, $R(\cdot,y)$ is a weak solution of the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla R(\cdot,y)) = -\operatorname{div}((\sigma-\sigma_0)\nabla H(\cdot,y)) & \text{in } \Omega_0,\\ R(\cdot,y) = -H(\cdot,y) & \text{on } \partial\Omega_0. \end{cases} \tag{3.27}$$

The above system can be derived as follows. Consider $\varphi \in C_c^\infty(\Omega_0)$, by the weak formulation applied to (3.11), it follows that

$$\int_{\Omega_0} \sigma\nabla G(\cdot,y)\cdot\nabla\varphi = \varphi(y). \tag{3.28}$$

Moreover,

$$\int_{\mathbb{R}^n} \sigma\nabla H(\cdot,y)\cdot\nabla\varphi = \int_{\Omega_0} \sigma\nabla H(\cdot,y)\cdot\nabla\varphi = \varphi(y). \tag{3.29}$$

By (3.28) and (3.29), it follows that

$$\int_{\Omega_0} \sigma\nabla R(\cdot,y)\cdot\nabla\varphi = \varphi(y) - \int_{\Omega_0} \sigma\nabla H(\cdot,y)\cdot\nabla\varphi\, \mathrm{d}x = \int_{\Omega_0} (\sigma_0-\sigma)\nabla H(\cdot,y)\cdot\nabla\varphi.$$

Regarding the trace at the boundary $\partial\Omega$,

$$R(\cdot,y)|_{\partial\Omega} = (G(\cdot,y) - H(\cdot,y))|_{\partial\Omega} = -H(\cdot,y)|_{\partial\Omega},$$

from which (3.27) follows. By the Green's identity, one can derive the following representation formula:

$$\begin{aligned} R(x,y) = &\int_{\Omega_0} (\sigma_0(\xi) - \sigma(\xi))\nabla H(\xi,y)\cdot\nabla G(\xi,x)\ \mathrm{d}\xi + \\ &+ \int_{\partial\Omega_0} \sigma(\xi)\nabla G(\xi,x)\cdot\nu\, H(\xi,y)\ \mathrm{d}\xi. \end{aligned} \tag{3.30}$$

Indeed, using $G(\cdot,y)$ as test function in (3.27), we derive

$$\int_{\Omega_0} \sigma\nabla R(\cdot,y)\cdot\nabla G(\cdot,x) + \int_{\Omega_0} (\sigma-\sigma_0)\nabla H(\cdot,y)\cdot\nabla G(\cdot,x) = 0. \tag{3.31}$$

Using $R(\cdot,y)$ as test function in (3.11), we have

$$\int_{\Omega_0} \sigma\nabla G(\cdot,x)\cdot\nabla R(\cdot,y) + \int_{\partial\Omega_0} \sigma\nabla G(\cdot,x)\cdot\nu H(\cdot,y) = R(x,y). \tag{3.32}$$

By summing (3.31) and (3.32), equation (3.30) follows.

To estimate the right-hand side of the representation formula (3.30), we write the integral as the sum of the following terms:

$$\int_{\Omega_0} (\sigma_0(\xi) - \sigma(\xi))\nabla H(\xi,y)\cdot\nabla G(\xi,x)\ \mathrm{d}\xi = R_1(x,y) + R_2(x,y),$$

where

$$R_1(x,y) = \int_{\Omega_0\setminus Q_{r_0}} (\sigma_0(\xi) - \sigma(\xi))\nabla H(\xi,y)\cdot\nabla G(\xi,x)\ \mathrm{d}\xi, \tag{3.33}$$

$$R_2(x,y) = \int_{Q_{r_0}} (\sigma_0(\xi) - \sigma(\xi))\nabla H(\xi,y)\cdot\nabla G(\xi,x)\ \mathrm{d}\xi. \tag{3.34}$$

To estimate (3.33), we can apply the Hölder inequality and Proposition 3.2.1 to obtain

$$\begin{aligned} |R_1(x,y)| &\le \|\sigma_0-\sigma\|_{L^\infty(\Omega_0)} \int_{\Omega_0\setminus Q_{r_0}} |\nabla H(\xi,y)|\,|\nabla G(\xi,x)|\ \mathrm{d}\xi \\ &\le C\|\nabla H(\cdot,y)\|_{L^2(\Omega_0\setminus Q_{r_0})}\,\|\nabla G(\cdot,y)\|_{L^2(\Omega_0\setminus Q_{r_0})} \le C. \end{aligned}$$

To estimate (3.34), notice that for every $y,\xi\in Q_{r_0}$, by (A.15), we have

$$|\nabla H(\xi,y)| \le C\,|\xi-y|^{1-n},$$

where C is a positive constant that depends on γ^+, γ^-, and the a priori data.

Let $\xi, x\in Q_{r_0}$ be such that $|\xi-x|\le r_0/2$, and set $\rho = |\xi-x|$. By Proposition 3.2.3,

$$|\nabla G(\xi,x)| \le \frac{C}{|\xi-x|^{1+n/2}}\|G(\cdot,x)\|_{L^2(Q_{2\rho}(x)\setminus\{x\})} \le C\,|\xi-x|^{1-n}.$$

Set $\gamma_0(\xi) = \gamma^+\chi_+(\xi) + \gamma^-\chi_-(\xi)$. In a neighbourhood of the origin, we have the following estimate:

$$|\sigma(\xi) - \sigma_0(\xi)| \leq |\gamma(\xi)A(\xi) - \gamma_0(\xi)A(0)| \leq C\,|\xi|, \tag{3.35}$$

where $C > 0$ depends on $\bar{\gamma}, \bar{A}$ only. Therefore,

$$|R_2(x,y)| \leq C \int_{Q_{r_0}} |\xi|\,|\xi - x|^{1-n}\,|\xi - y|^{1-n}\,\mathrm{d}\xi.$$

Let $h = |x - y|$ and define

$$I_1 = \int_{B_{4h}} |\xi|\,|\xi - x|^{1-n}\,|\xi - y|^{1-n}\,\mathrm{d}\xi,$$
$$I_2 = \int_{\mathbb{R}^n\setminus B_{4h}} |\xi|\,|\xi - x|^{1-n}\,|\xi - y|^{1-n}\,\mathrm{d}\xi.$$

This allows to write

$$|R_2(x,y)| \leq C\,(I_1 + I_2).$$

To begin with, we estimate I_1. We introduce the following change of variables:

$$\xi = hw \qquad t = \frac{x}{h} \qquad s = \frac{y}{h},$$

which leads to (see Miranda [99])

$$I_1 = h^{3-n} \int_{B_4} |w|\,|w - t|^{1-n}\,|w - s|^{1-n}\,\mathrm{d}w \leq C\,h^{3-n}.$$

Next, we estimate the integral I_2. Since $y = y_n e_n$ for $y_n \in (-r_0/4, 0)$ and $x \in B^+_{r_0/4}$, we have

$$\frac{3}{4}|\xi| \leq |\xi - y| \qquad \text{and} \qquad \frac{1}{2}|\xi| \leq |\xi - x|.$$

Hence,

$$I_2 \leq \left(\frac{8}{3}\right)^{1-n} \int_{\mathbb{R}^n\setminus B_{4h}} |\xi|^{3-2n}\,\mathrm{d}\xi \leq \begin{cases} C\,h^{3-n} & n > 3, \\ C\,h^{-\alpha} & n = 3, \end{cases}$$

for some positive value of α, since $\ln(1/h) \leq Ch^{-\alpha}$. In conclusion,

$$|R(x,y)| \leq C\,|x - y|^{3-n-\alpha}. \tag{3.36}$$

Now, let us estimate $|\nabla R(x,y)|$. Let $x \in {B^+}_{r_0/4}$ and $y = y_n e_n$ with $y_n \in (-r_0/4, 0)$. We define the cylinder $Q := B'_{h/4}(x') \times (x_n, x_n + h/4)$, where $h = |x - y|$. Notice that

$$Q \subset {Q^+}_{r_0/2}, \quad Q \subset Q_{h/2}(x), \quad x \in \partial Q, \quad y \notin Q.$$

By [7, Lemma 3.2], for $\alpha' \in (0,1]$, we have

$$\|\nabla_x R(\cdot,y)\|_{L^\infty(Q)} \leq C\left[\|R(\cdot,y)\|^{\frac{\alpha'}{\alpha'+1}}{}_{L^\infty(Q)}\,|\nabla_x R(\cdot,y)|^{\frac{1}{\alpha'+1}}{}_{\alpha',Q} + \frac{1}{h}\|R(\cdot,y)\|_{L^\infty(Q)}\right]. \tag{3.37}$$

By (3.36), we find

$$\|R(\cdot,y)\|_{L^\infty(Q)} \le C\,|x-y|^{3-n-\alpha}.$$

To estimate $|\nabla_x R(\cdot,y)|_{\alpha',Q}$, notice that

$$|\nabla_x R(\cdot,y)|_{\alpha',Q} \le |\nabla_x G(\cdot,y)|_{\alpha',Q} + |\nabla_x H(\cdot,y)|_{\alpha',Q}.$$

By Theorem 3.2.3, (3.17) and using cylindrical coordinates, we obtain

$$|\nabla_x G(\cdot,y)|_{\alpha',Q} \le \frac{C}{h^{1+n/2}} h^{-\alpha'} \|G(\cdot,y)\|_{L^2(Q_h(x))} \le C\,h^{1-n-\alpha'}. \tag{3.38}$$

Moreover, we have

$$|\nabla_x H(\cdot,y)|_{\alpha',Q} \le C\,|\nabla_x \Gamma(\cdot,y)|_{\alpha',Q} \le C\,h^{1-n-\alpha'}.$$

Therefore, we have

$$|\nabla_x R(\cdot,y)|_{\alpha',Q} \le C\,h^{1-n-\alpha'}. \tag{3.39}$$

By (3.37), (3.36), (3.38), and (3.39), we conclude that

$$\|\nabla_x R(\cdot,y)\|_{L^\infty(Q)} \le C\,|x-y|^{1-n+\theta_1} \quad \text{with } \theta_1 = \frac{\alpha'(1-\alpha)}{1+\alpha'}. \tag{3.40}$$

Next, we estimate $|\nabla_x \nabla_y R(x,y)|$. Define the cylinder $\hat{Q} = B'_{h/8} \times (y_n - h/8, y_n)$, then

$$\hat{Q} \subset Q^-{}_{r_0/4}, \quad \hat{Q} \subset Q_{h/4}(y) \quad \text{and} \quad x \notin Q_{h/4}(y).$$

Let $k \in \{1,\dots,n\}$, $\partial_{x_k}\Gamma(x,\cdot)$ is a weak solution of the Laplace equation

$$\Delta_y(\partial_{x_k}\Gamma(x,\cdot)) = 0 \quad \text{in } \; Q_{h/4}(y),$$

and $\partial_{x_k} G(x,\cdot)$ is a weak solution of the equation

$$\mathrm{div}_y(\sigma \nabla_y\, \partial_{x_k} G(x,\cdot)) = -\delta(x-\cdot) \qquad \text{in } Q_{h/4}(y).$$

By Theorem 3.2.3, it follows that

$$|\nabla_y \partial_{x_k} G(x,\cdot)|_{\alpha',\hat{Q}} \le C\,h^{-\alpha'-1-n/2} \|\partial_{x_k} G(x,\cdot)\|_{L^2(Q_{h/4}(y))}. \tag{3.41}$$

Fix $\bar{y} \in Q_{h/4}(y)$, then $\bar{y} \notin Q_{h/16}(x)$. By Theorem 3.2.3, it follows that

$$\|\nabla_x G(\cdot,\bar{y})\|_{L^\infty(Q_{h/32}(x))} \le C\,h^{-1-n/2} \|G(\cdot,\bar{y})\|_{L^\infty(Q_{h/16}(x))} \le C\,h^{1-n}. \tag{3.42}$$

From (3.41) and (3.42) it follows that

$$|\nabla_y \partial_{x_k} G(x,\cdot)|_{\alpha',\hat{Q}} \le C\,h^{-\alpha'-n}. \tag{3.43}$$

Moreover,

$$|\nabla_y \partial_{x_k} \Gamma(x,\cdot)|_{\alpha',\hat{Q}} \le C\,h^{-\alpha'-n}, \tag{3.44}$$

and by (3.43) and (3.44),

$$|\nabla_y \partial_{x_k} R(x,\cdot)|_{\alpha',\hat{Q}} \le C\, h^{-\alpha'-n}. \tag{3.45}$$

By (3.40), it turns out that

$$\|\partial_{x_k} R(x,\cdot)\|_{L^\infty(\hat{Q})} \le C\, h^{1-n+\theta_1}. \tag{3.46}$$

By the following interpolation inequality

$$\|\nabla_y \partial_{x_k} R(x,\cdot)\|_{L^\infty(\hat{Q})} \le C\, \|\partial_{x_k} R(x,\cdot)\|^{\frac{\alpha'}{\alpha'+1}}{}_{L^\infty(\hat{Q})} |\nabla_y \partial_{x_k} R(x,\cdot)|^{\frac{1}{\alpha'+1}}{}_{\alpha',\hat{Q}},$$

and by (3.46) and (3.45), we conclude that

$$|\nabla_y \partial_{x_k} R(x,y)| \le C\, |x-y|^{\theta_2 - n} \qquad \text{with } \theta_2 = \frac{\theta_1 \alpha'}{1+\alpha'}.$$

□

3.2.1 Quantitative estimates of unique continuation

For a given positive number b, let ω_b denote a non-decreasing, concave function defined on the interval $(0,+\infty)$ that has the following form:

$$\omega_b(t) = \begin{cases} 2^b e^{-2} |\ln t|^{-b} & t \in (0, e^{-2}), \\ e^{-2} & t \in [e^{-2}, +\infty). \end{cases}$$

The function ω_b satisfies the following properties:

$$(0,+\infty) \ni t \to \ t\omega_b\left(\frac{1}{t}\right) \qquad \text{is an increasing function,} \tag{3.47}$$

and for every $\beta \in (0,1)$ we have that

$$\omega_b\left(\frac{t}{\beta}\right) \le |\ln e\beta^{-1/2}|^b \omega_b(t), \qquad \omega_b(t^\beta) \le \left(\frac{1}{\beta}\right)^b \omega_b(t). \tag{3.48}$$

Furthermore, we shall denote the iterative compositions of ω with itself as

$$\omega_b^{(1)} = \omega_b, \qquad \omega_b^{(j)} = \omega_b \circ \omega_b^{(j-1)} \qquad \text{for } j = 2, 3, \ldots$$

and we set $\omega_b^{(0)}(t) = t^b$ for $0 < b < 1$.

We introduce the following parameters:

$$\begin{aligned} &\beta = \arctan\left(\frac{1}{L}\right), \qquad \beta_1 = \arcsin\left(\frac{\sin\beta}{4}\right), \\ &\lambda_1 = \frac{r_0}{1+\sin\beta_1}, \qquad \rho_1 = \lambda_1 \sin\beta_1, \qquad a = \frac{1-\sin\beta_1}{1+\sin\beta_1}, \\ &\lambda_m = a\lambda_{m-1}, \qquad \rho_m = a\rho_{m-1}, \qquad \text{for every } m \ge 2, \\ &d_m = \lambda_m - \rho_m, \qquad \text{for every } m \ge 1. \end{aligned} \tag{3.49}$$

Fix a point $\bar{y} \in \Sigma_{m+1}$, let $l \in \mathbb{N}$ and define

$$w_l(\bar{y}) = \bar{y} - \lambda_l \nu(\bar{y}) \qquad \text{for every } l \geq 1,$$

so that $w_l(\bar{y})$ is a point into the domain D_m near the interface Σ_{m+1}. For a given $r \in (0, d_1]$, denote

$$\bar{h} := \min\{l \in \mathbb{N} \,:\, d_l \leq r\}. \tag{3.50}$$

The following inequality holds:

$$\frac{|\ln(r/d_1)|}{|\ln a|} \leq \bar{h} - 1 \leq \frac{|\ln(r/d_1)|}{|\ln a|} + 1. \tag{3.51}$$

We further define the sets

$$\mathcal{W}_k = \bigcup_{m=0}^{k} D_m, \qquad \mathcal{U}_k = \Omega_0 \setminus \overline{\mathcal{W}_k}, \quad \text{for } k = 0, \dots, K. \tag{3.52}$$

Definition 3.2.1. *For every $y, z \in \mathcal{W}_k$, we define the singular solution $S_k(y, z)$ for $k = 0, \dots, K$ as*

$$S_k(y, z) = \int_{\mathcal{U}_k} (\sigma^{(1)} - \sigma^{(2)})(x) \nabla G_1(x, y) \cdot \nabla G_2(x, z) \, \mathrm{d}x.$$

The set $\{S_k(y, z)\}_{k=0}^{K}$ represents a family of real-valued singular integrals, and Proposition 3.2.1 guarantees that the following inequality is satisfied:

$$|S_k(y, z)| \leq C \|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \, (d(y)d(z))^{1-n/2} \quad \text{for any } y, z \in \mathcal{W}_k, \tag{3.53}$$

where $d(y) = \mathrm{dist}(y, \mathcal{U}_k)$ and C is a positive constant depending on λ and n only.

Proposition 3.2.4. *Let $k \in \{0, \dots, K\}$. For every $y, z \in \mathcal{W}_k$, the functions $S_k(\cdot, z), S_k(y, \cdot)$ belong to $H^1_{loc}(\mathcal{W}_k)$ and are weak solutions, respectively, to*

$$\mathrm{div}_y(\sigma^{(1)} \nabla_y S_k(\cdot, z)) = 0, \qquad \mathrm{div}_z(\sigma^{(2)} \nabla_z S_k(y, \cdot)) = 0 \quad \textit{in } \mathcal{W}_k.$$

To prove Proposition 3.2.4, we need a preliminary result. Let $\{\rho_h\}_{h \in \mathbb{N}}$ be a sequence of mollifiers (see [42, p. 108]). We define $\gamma_h(x) := (\rho_h * \gamma)(x)$ for any $x \in \Omega_0$. Set $\sigma_h(x) = \gamma_h(x) A(x)$, we obtain a sequence of measurable functions $\{\sigma_h\}_{h \in \mathbb{N}}$ that satisfies the uniform ellipticity condition: $\lambda^{-1}|\xi|^2 \leq \sigma_h(x)\xi \cdot \xi \leq \lambda|\xi|^2$ for a.e. $x \in \Omega_0$ and every $\xi \in \mathbb{R}^n$, for every $h \in \mathbb{N}$. Let G_h be the Green function associated to the elliptic operator $\mathrm{div}(\sigma_h \nabla \cdot)$ in Ω_0. For $y \in \Omega_0$, let $G_h(\cdot, y)$ be a weak solution of

$$\begin{cases} \mathrm{div}(\sigma_h \nabla_x G_h(\cdot, y)) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ G_h(\cdot, y) = 0 & \text{on } \partial\Omega_0. \end{cases} \tag{3.54}$$

Proposition 3.2.5. *Assume that $\{\sigma_h\}_{h \in \mathbb{N}}$ converges to the function σ in $L^s(\Omega_0)$ for every $s \in [1, \infty)$. Let $\mathcal{U}, K$ be two open subsets of Ω_0 such that $\Omega_0 \setminus \overline{\mathcal{U}} \neq \emptyset$ and*

$K \Subset \Omega_0 \setminus \overline{\mathcal{U}}$. *Then,*

$$\lim_{h\to+\infty}\sup_{y\in K}\|G_h(\cdot,y)-G(\cdot,y)\|_{H^1(\mathcal{U})}=0. \tag{3.55}$$

To prove Proposition 3.2.5, we introduce the following Proposition (see [21, Proposition 5.1]).

Proposition 3.2.6. *Let $\{\gamma_h\}_{h\in\mathbb{N}}$ be a sequence of Lebesgue measurable functions that converges almost everywhere to $\gamma\in L^\infty(\Omega_0)$ in Ω_0. Set $\sigma_h=\gamma_h\,A$ and $\sigma=\gamma\,A$, where A is a Lipschitz symmetric positive definite real-valued matrix function and assume that σ_h and σ satisfies the uniform ellipticity condition (3.4) for a constant $\lambda>1$. Let $f\in L^{q/2}(\Omega_0)$ for $q>n$, and let $u_h,u\in H^1_0(\Omega_0)$ be weak solutions to*

$$\operatorname{div}(\sigma_h\nabla u_h)=-f,\qquad \textit{in } \Omega_0, \tag{3.56}$$

and

$$\operatorname{div}(\sigma\nabla u)=-f,\qquad \textit{in } \Omega_0. \tag{3.57}$$

Then

$$u_h\;\to\;u\quad \textit{strongly in } H^1_0(\Omega_0), \tag{3.58}$$

and

$$u_h\;\to\;u\quad \textit{strongly in } L^\infty(\Omega_0). \tag{3.59}$$

Proof of Proposition 3.2.6. Let u_h,u be weak solutions of (3.56) and (3.57), respectively. Then u_h,u satisfy the following equation:

$$\operatorname{div}(\sigma_h\nabla(u_h-u))=\operatorname{div}((\sigma-\sigma_h)\nabla u)\qquad \text{in } \Omega_0. \tag{3.60}$$

By the weak formulation associated to (3.60), if we choose (u_h-u) as test function, the following integral equation holds:

$$\int_{\Omega_0}\sigma_h\nabla(u_h-u)\cdot\nabla(u_h-u)\;\mathrm{d}x=\int_{\Omega_0}(\sigma-\sigma_h)\nabla u\cdot\nabla(u_h-u)\;\mathrm{d}x. \tag{3.61}$$

The left-hand side of (3.61) can be bounded from below by applying (3.4):

$$\int_{\Omega_0}\sigma_h\nabla(u_h-u)\cdot\nabla(u_h-u)\;\mathrm{d}x\geq\lambda^{-1}\int_{\Omega_0}|\nabla(u_h-u)|^2\,\mathrm{d}x. \tag{3.62}$$

The right-hand side can be bounded from above by the Hölder inequality:

$$\int_{\Omega_0}(\sigma-\sigma_h)\nabla u\cdot\nabla(u_h-u)\;\mathrm{d}x\leq\|\sigma-\sigma_h\|_{L^\infty(\Omega_0)}\|\nabla u\|_{L^2(\Omega_0)}\|\nabla(u_h-u)\|_{L^2(\Omega_0)}. \tag{3.63}$$

Hence, we obtain

$$\|\nabla(u_h-u)\|_{L^2(\Omega_0)}\leq C\,\|\nabla u\|_{L^2(\Omega_0)}, \tag{3.64}$$

where C is a positive constant that depends only on λ, $\bar{A}$. By the dominated convergence theorem, it follows that $\nabla u_h\to\nabla u$ strongly in $L^2(\Omega_0)$, hence by the Poincarè inequality, (3.58) is proved.

Consider the weak formulation of (3.56), using u_h as test function, we have

$$\int_{\Omega_0} \sigma_h \nabla u_h \cdot \nabla u_h = \int_{\Omega_0} f\, u_h.$$

Hence, by (3.4), the Hölder inequality, and the Poincarè inequality, we derive

$$\|\nabla u_h\|_{L^2(\Omega_0)} \le C\|f\|_{L^{q/2}(\Omega_0)}.$$

An analogous bound holds for ∇u. Hence,

$$\|\nabla u_h\|_{L^\infty(\Omega_0)}, \|\nabla u\|_{L^\infty(\Omega_0)} \le C\|f\|_{L^{q/2}(\Omega_0)}. \tag{3.65}$$

By [66, Theorem 8.24], u and u_h satisfy the following interior Hölder estimate: for $\alpha \in (0,1)$,

$$|u_h|_{\alpha,\Omega},\ |u|_{\alpha,\Omega} \le C\|f\|_{L^{q/2}(\Omega)} \quad \text{for every } h \in \mathbb{N}. \tag{3.66}$$

Recall the following interpolation inequality (see [9, Equation (5.30) pag. 777]),

$$\|v\|_{L^\infty(\Omega)} \le C\|v\|^{\eta}_{L^2(\Omega)}\|v\|^{1-\eta}_{C^{1,\alpha}(\Omega)} \quad \text{for } \eta = \eta(n,\alpha) \in (0,1), \tag{3.67}$$

By (3.65), (3.66) and (3.67), (3.59) holds. □

Proof of Proposition 3.2.5. Let $\mathcal{U}, K$ be two bounded subsets of Ω_0 such that $K \Subset \Omega_0 \setminus \overline{\mathcal{U}}$. Notice that, since $\gamma_h \to \gamma$ in $L^s(\Omega_0)$ for $s \in [1,+\infty)$, then by [42, Theorem 4.9], γ_h converges a.e. to γ in Ω_0 up to subsequences. Let G and G_h be the Green functions associated with the elliptic operator $\operatorname{div}(\sigma\nabla\cdot)$ and $\operatorname{div}(\sigma_h\nabla\cdot)$ in Ω_0, respectively. For $q > n$, let $f \in L^{q/2}(\Omega_0)$, and for $y \in K$ define

$$u_h(y) = \int_{\Omega_0} G_h(z,y)\, f(z)\ \mathrm{d}z, \qquad u(y) = \int_{\Omega_0} G(z,y)\, f(z)\ \mathrm{d}z.$$

Then u_h and u belong to the Sobolev space $H^1_0(\Omega_0)$ and are weak solutions, respectively, of

$$\operatorname{div}(\sigma_h \nabla u_h) = -f \quad \text{and} \quad \operatorname{div}(\sigma\nabla u) = -f \qquad \text{in } \Omega_0.$$

By Proposition 3.2.6, we have that $u_h \to u$ strongly in $L^\infty(\Omega_0)$, then, for a.e. $y \in \Omega_0$,

$$\int_{\Omega_0} G_h(z,y)\, f(z)\ \mathrm{d}z \to \int_{\Omega_0} G(z,y)\, f(z)\ \mathrm{d}z, \qquad \text{for any } f \in L^{q/2}(\Omega_0).$$

Hence,

$$G_h(\cdot,y) \rightharpoonup G(\cdot,y) \qquad \text{in } L^{(q/2)'}(\Omega_0) \quad \text{for any } y \in K. \tag{3.68}$$

Here, $(q/2)'$ denotes the conjugate exponent of $q/2$.

Let Q be a smooth domain such that $\mathcal{U} \Subset Q \subset \Omega_0$ such that $dist(Q,K) > 0$. For any $x \in Q$, by the symmetry of G_h and G,

$$\operatorname{div}(\sigma_h \nabla G_h(x,\cdot)) = 0, \qquad \operatorname{div}(\sigma\nabla G(x,\cdot)) = 0, \quad \text{in } K.$$

By [92, Theorem 1.1], $G(x,\cdot)$ and $G_h(x,\cdot)$ satisfy a $C^{1,\theta}(\bar{K})$ uniform bound for $\theta \in (0,1)$, for any $x \in Q$ and $h \in \mathbb{N}$. Hence, the functions $\Theta_h(y) = \|G_h(\cdot,y) -$

$G(\cdot,y)\|_{L^\infty(Q)}$ satisfy a $C^{1,\theta}(\bar{K})$ uniform bound with respect to $h\in\mathbb{N}$, so that $\{\Theta_h\}_{h\in\mathbb{N}}$ is a sequence of uniformly bounded functions on K. Thus, there exists a sequence $\{y_h\}_{h\in\mathbb{N}}$ in $\bar{K}$ such that

$$\sup_{y\in K}\|G_h(\cdot,y)-G(\cdot,y)\|_{L^\infty(Q)}=\|G_h(\cdot,y_h)-G(\cdot,y_h)\|_{L^\infty(Q)}\qquad\forall h\in\mathbb{N},$$

and $y_h\to\bar{y}$ in $\bar{K}$ (up to subsequences). Therefore,

$$\lim_{h\to+\infty}\|G_h(\cdot,y_h)-G_h(\cdot,\bar{y})\|_{L^\infty(Q)}=0. \tag{3.69}$$

By (3.68), (3.69) and the triangle inequality,

$$G_h(\cdot,y_h)\rightharpoonup G(\cdot,\bar{y}),\quad\text{weakly in } L^{(q/2)'}(Q).$$

As a solution of $\mathrm{div}(\sigma_h\nabla G_h(\cdot,y_h))=0$ in $\Omega_0\setminus\bar{K}$, the sequence $\{G_h(\cdot,y_h)\}$ is equicontinuous in Q. Hence, up to subsequences,

$$G_h(\cdot,y_h)\to G(\cdot,\bar{y}),\quad\text{strongly in } L^\infty(Q). \tag{3.70}$$

Up to subsequences, taking into account the definition of y_h, (3.70) and the triangle inequality, we can conclude that

$$\sup_{y\in K}\|G_h(\cdot,y)-G(\cdot,y)\|_{L^\infty(Q)}\to 0. \tag{3.71}$$

By Caccioppoli inequality (Theorem A.2.2),

$$\sup_{y\in K}\|G_h(\cdot,y)-G(\cdot,y)\|_{H^1(\mathcal{U})}\le C\sup_{y\in K}\|G_h(\cdot,y)-G(\cdot,y)\|_{L^\infty(Q)}, \tag{3.72}$$

hence, $\sup_{y\in K}\|G_h(\cdot,y)-G(\cdot,y)\|_{H^1(\mathcal{U})}$ is uniformly bounded and (3.55) follows straightforwardly. □

Proof of Proposition 3.2.4. For this proof, we follow the lines of [21, Proposition 3.3]. Let $\{(\sigma^{(1)})_h\}_{h\in\mathbb{N}}$ be a sequence of anisotropic conductivities of the form $(\sigma^{(1)})_h(x)=(\gamma^{(1)})_h(x)A(x)$, and $(\gamma^{(1)})_h=\gamma^{(1)}*\rho_h$. Let $G_{1,h}(\cdot,y)$ be a weak solution of the problem (3.54) with conductivity $(\sigma^{(1)})_h$. Let $K\Subset\mathcal{W}_k$ be an open set, by Proposition 3.2.5,

$$\lim_{h\to\infty}\sup_{y\in K}\|G_{1,h}(\cdot,y)-G_1(\cdot,y)\|_{H^1(\mathcal{U}_k)}=0. \tag{3.73}$$

Set

$$S_k^h(y,z):=\int_{\mathcal{U}_k}((\sigma^{(1)})_h(x)-\sigma^{(2)}(x))\nabla_x G_{1,h}(x,y)\cdot\nabla_x G_2(x,z)\,\mathrm{d}x\quad\text{for } y,z\in\mathcal{W}_k.$$

Fix $z\in\mathcal{W}_k$, by differentiating under the integral sign, we obtain that $S_k^h(\cdot,z)$ is a weak solution of

$$\mathrm{div}_y((\sigma^{(1)})_h\nabla_y S_k^h(\cdot,z))=0\quad\text{in } \mathcal{W}_k\quad\text{for every } h\in\mathbb{N}. \tag{3.74}$$

Notice that

$$|S_k^h(y,z) - S_k(y,z)| = \Big| \int_{\mathcal{U}_k} [((\sigma^{(1)})_h - \sigma^{(2)})\nabla G_{1,h}(x,y) - (\sigma^{(1)} - \sigma^{(2)})\nabla G_1(x,y)] \cdot \nabla G_2(x,z)\,\mathrm{d}x \Big|$$
$$\leq C\|G_{1,h}(\cdot,y) - G_1(\cdot,y)\|_{H^1(\mathcal{U}_k)}\|G_2(\cdot,z)\|_{H^1(\mathcal{U}_k)}.$$

By (3.73) and the Theorem of dominated convergence, it follows that

$$S_k^h(\cdot,z) \to S_k(\cdot,z), \qquad \text{in } L^\infty(K). \tag{3.75}$$

Consider $\tilde{K}$ such that $K \Subset \tilde{K} \Subset \mathcal{W}$. By Caccioppoli inequality (Theorem A.2.2), it follows that

$$\|S_k^h(\cdot,z)\|_{H^1(K)} \leq C\|S_k^h(\cdot,z)\|_{L^2(\tilde{K})}, \qquad \text{for every } h \in \mathbb{N},$$

where C depends on λ and $\mathrm{dist}(K, \mathbb{R}^n \setminus \tilde{K})$ only. By (3.53),

$$\|S_k^h(\cdot,z)\|_{L^\infty(\tilde{K})} \leq C(\mathrm{dist}(\tilde{K},\mathcal{U})d(z))^{1-n/2}, \qquad \text{for every } h \in \mathbb{N},$$

where C depends on λ and n only. Therefore, the norm $\|S_k^h(\cdot,z)\|_{H^1(K)}$ is uniformly bounded. Hence, there exists a subsequence of $\{S_k^h(\cdot,z)\}_{h\in\mathbb{N}}$ that weakly converges in $H^1(K)$ to a function f. By (3.75), the function f is equal to $S_k(\cdot,z)$ and by (3.74), it follows that $S_k(\cdot,z)$ satisfies $\mathrm{div}(\sigma^{(1)}\nabla S_k(\cdot,z))$. A similar proof holds for $S_k(y,\cdot)$. □

At this point, since we have proved that S_k for $k = 1,\dots,K$ are locally weak solutions to an elliptic equation, we can provide a quantitative result of unique continuation for the singular solutions. Let $E = \|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)}$.

Proposition 3.2.7. *(Quantitative estimates of unique continuation) For $k \in \{1,\dots,K\}$, assume that for positive constants ε and $\bar{r}$ we have*

$$|S_k(y,z)| \leq r_0^{2-n}\varepsilon_0, \quad \textit{for every } (y,z) \in (D_0)_{\bar{r}} \times (D_0)_{\bar{r}}, \tag{3.76}$$

then the following inequalities hold true for every $r \in (0, d_1]$:

$$|S_k\left(w_{\bar{h}}(Q_{k+1}), w_{\bar{h}}(Q_{k+1})\right)| \leq C_1^{\bar{h}}(E+\varepsilon_0)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{(1/C)^{\bar{h}}}, \tag{3.77}$$

$$\left|\partial_{y_j}\partial_{z_i} S_k\left(w_{\bar{h}}(Q_{k+1}), w_{\bar{h}}(Q_{k+1})\right)\right| \leq C_2^{\bar{h}}(E+\varepsilon_0)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{(1/C)^{\bar{h}}}, \tag{3.78}$$

for any $i,j = 1,\dots,n$, where $Q_{k+1} \in \Sigma_{k+1} \cap B_{r_0/8}(P_{k+1})$, $w_{\bar{h}(r)}(Q_{k+1}) = Q_{k+1} - \lambda_{\bar{h}(r)}\nu(Q_{k+1})$, with $\lambda_{\bar{h}(r)}$ as in (3.49), $\nu(Q_{k+1})$ is the exterior unit normal to ∂D_k at the point Q_{k+1} and $C_1, C_2 > 0$ depend only on the a priori data.

To prove Proposition 3.2.7, we state and prove the preliminary Proposition 3.2.8 that provides a pointwise bound for the weak solution of the conductivity equation near one of the discontinuity interfaces for the conductivity contained in Ω_0.

Proposition 3.2.8. *Let $v \in H^1(\mathcal{W}_k)$ be a weak solution of*

$$\operatorname{div}(\sigma\,\nabla v) = 0 \qquad \text{in } \mathcal{W}_k. \tag{3.79}$$

We assume that, for given positive numbers E_0 and ε_0, the function v satisfies the following inequalities:

$$|v(x)| \leq r_0^{2-n}\varepsilon_0, \qquad \text{for any } x \in (D_0)_r, \tag{3.80}$$

and

$$|v(x)| \leq r_0^{1-n/2}\, E_0, \qquad \text{for any } x \in (\mathcal{W}_k)_r, \tag{3.81}$$

for certain $r > 0$, where $d(x) = dist(x, \mathcal{U}_k)$. Then, for every $r \in (0, d_1]$,

$$|v(w_{\bar{h}}(P_{k+1}))| \leq r_0^{2-n} C^{\bar{h}} (E_0 + \varepsilon_0) \Big(\omega^{(k)}_{1/C}\Big(\frac{\varepsilon_0}{\varepsilon_0 + E_0}\Big)\Big)^{(1/C)^{\bar{h}}}, \tag{3.82}$$

where $C > 1$ is a constant that depends only on the a priori data, $P_{k+1} \in \Sigma_{k+1}$, $w_{\bar{h}(r)}(P_{k+1}) = P_{k+1} - \lambda_{\bar{h}(r)}\nu(P_{k+1})$, with $\lambda_{\bar{h}(r)}$ as in (3.49), $\nu(P_{k+1})$ is the exterior unit normal to ∂D_k at the point P_{k+1}.

Proof of Proposition 3.2.8. Let us first introduce the following parameters. For $m \in \{1, \dots, N\}$,

$$\begin{aligned} \bar{r} &= \frac{r_0}{4}, \quad \bar{\rho} = \frac{\bar{r}}{128\sqrt{1+L^2}}, \qquad v_m = v|_{D_m}, \\ y_m &= P_m - \frac{\bar{r}}{32}\nu(P_m), \qquad \tilde{y}_m = P_m + \frac{\bar{r}}{32}\nu(P_m), \end{aligned} \tag{3.83}$$

where $P_m \in \Sigma_m$, $\nu(P_m)$ is the exterior unit normal of $\partial\, D_{m-1}$. Fix $K \in \{1, \dots, N\}$. We claim that for every $m \in \{0, \dots, K\}$,

$$\|v\|_{L^\infty(B_{\bar{\rho}}(y_m))} \leq r_0^{2-n} C^{m+1} (E_0 + \varepsilon_0)\, \omega^{(m+1)}_{1/C}\Big(\frac{\varepsilon_0}{\varepsilon_0 + E_0}\Big), \tag{3.84}$$

and prove (3.84) by induction. The case $m = 0$ is straightforward, so we focus our attention on the inductive step. We assume that (3.84) holds for $m \in \{0, \dots, K-1\}$ and prove it for $m+1$.

Up to a rigid transformation, we can assume that the point $P_{m+1} \in \Sigma_{m+1}$ coincides with the origin and $\tilde{y}_m = \frac{\bar{r}}{32} e_n$. Set

$$\varepsilon_m := C^{m+1}(E_0 + \varepsilon_0)\, \omega^{(m+1)}_{1/C}\Big(\frac{\varepsilon_0}{\varepsilon_0 + E_0}\Big).$$

By the inductive hypothesis, we know that v satisfies

$$\|v\|_{L^\infty(B_{\bar{\rho}}(\tilde{y}_m))} \leq r_0^{2-n}\varepsilon_m. \tag{3.85}$$

By (3.81), it follows that

$$\|v\|_{L^\infty(D_0)} \leq E_0 (r_0 \sup_{x \in D_m} d(x))^{1-n/2}. \tag{3.86}$$

Choose an arbitrary point $\bar{y} \in \Sigma_{m+1}$. Let $\phi : [0,1] \to \mathbb{R}^n$ be a Jordan curve joining $\tilde{y}_m$ to $w_1(\bar{y})$ such that $\phi([0,1]) \subset (D_m)_{\bar{d}}$, where $\bar{d} = 4\bar{r}$, and $(D_m)_{\bar{d}}$ is connected. Notice that $w_1(\bar{y}) \in (D_m)_{\bar{d}}$. Define a set of points $\{\phi_i\}$, $i = 1, \dots, s$ via the following process:

- $\phi_1 = \phi(0) = \tilde{y}_m$;
- for $i > 1$, set

$$\phi_{i+1} = \begin{cases} \phi(t_i) & \text{if } |\phi_i - w_1(\bar{y})| > 2r_1 \text{ where } t_i = \max\{t_i : |\phi(t) - \phi_i| = 2r_1\}, \\ w_1(\bar{y}) & \text{if } |\phi_i - w_1(\bar{y})| < 2r_1 \text{ and set } s = i+1. \end{cases}$$

Apply Theorem 2.2.8 to spheres centred at $\phi_1 = \tilde{y}_m$ with suitable radii $r, 3r, 4r$ with estimates (3.86) and (3.80),

$$\|v\|_{L^\infty(B_{3r}(y_1))} \le Q\, \|v\|^{\delta}_{L^\infty(B_r(y_1))} \|v\|^{1-\delta}_{L^\infty(B_{4r}(y_1))} \le Q\, r_0^{2-n} \varepsilon_m^{\delta}\, E_0^{1-\delta}, \tag{3.87}$$

where $\delta = \frac{\ln(4/3\lambda)}{\ln(4/3\lambda) + C\ln(3\lambda)}$ and $Q > 1$ is a constant which depends on λ, L, $\max\left\{\frac{4r}{r_0}, 1\right\}$.

Fix $r \in (0, d_1]$, recall (3.49), the following inclusions hold:

$$B_{\rho_{k+1}}(w_{k+1}(\bar{y})) \subset B_{3\rho_k}(w_k(\bar{y})) \subset B_{4\rho_k}(w_k(\bar{y})) \subset C\Big(\bar{y}, \nu(\bar{y}), \beta_1, r_0/3\Big),$$

for any $k = 1, 2, \dots$. Notice that $\rho_1 < \bar{r}/l$ for a suitable $l > 1$, then $B_{\rho_1}(w_1(\bar{y})) \subset B_{\bar{r}}(w_1(\bar{y}))$. We proceed by moving from one centre to the successive one along the axis of the cone $C\Big(\bar{y}, \nu(\bar{y}), \beta_1, r_0/3\Big)$ allowing us to approach the vertex $\bar{y}$. We stop this process when we reach the sphere with a radius $\rho_{\bar{h}}$. Then, from (3.87), we have

$$\|v\|_{L^\infty(B_{\rho_{\bar{h}}}(w_{\bar{h}}(\bar{y})))} \le C \varepsilon_m^{\delta^{s+\bar{h}-1}} E_0^{1-\delta^{s+\bar{h}-1}}. \tag{3.88}$$

By the triangle inequality, we have

$$|v(\bar{y})| \le |v(\bar{y}) - v(\bar{y} - r\nu(\bar{y}))| + |v(\bar{y} - r\nu(\bar{y}))|. \tag{3.89}$$

We proceed to estimate the second term on the right-hand side of (3.89). Since $\bar{y} - r\nu(\bar{y}) \in B_{\rho_{\bar{h}}}(w_{\bar{h}}(\bar{y}))$,

$$|v(\bar{y} - r\nu(\bar{y}))| \le C r_0^{2-n} \varepsilon_m^{\delta^{s+\bar{h}-1}} E_0^{1-\delta^{s+\bar{h}-1}} \le C r_0^{2-n} (\varepsilon_m + E_0) \Big(\frac{\varepsilon_m}{E_0 + \varepsilon_m}\Big)^{1-\delta^{s+\bar{h}-1}}.$$

Secondly, we estimate the first term on the right-hand side of (3.89). Since $\bar{y} \in \mathcal{W}_k$, by (3.81),

$$|v(\bar{y})| \le C\, r_0^{2-n}\, E_0.$$

Hence, by Theorem 3.2.3, we have

$$|v(\bar{y}) - v(\bar{y} - r\nu(\bar{y}))| \le \|\nabla v\|_{L^\infty(Q_{r_0/3})} r \le \frac{c}{r_0^{1+n/2}} \|v\|_{L^2(Q_{2r_0/3})} r \le C r_0^{2-n} (E_0 + \varepsilon_m) \Big(\frac{r}{r_0}\Big).$$

Therefore,

$$|v(\bar{y})| \leq C\, r_0^{2-n}\, (E_0+\varepsilon_m)\Big(\frac{r}{r_0} + \Big(\frac{\varepsilon_m}{E_0+\varepsilon_m}\Big)^{\delta^{s+\bar{h}-1}}\Big).$$

We define the following quantities:

$$B = \frac{|\log a|}{2|\log \tau|}, \quad \mu = \exp\Big(-\frac{1}{\tau^S}\Big) \quad \text{and} \quad \tilde{r} = d_1\Big|\log\Big(\frac{\varepsilon_m}{E_0+\varepsilon_m}\Big)^{\tau^S}\Big|^{-B}.$$

If $\varepsilon_m/(\varepsilon_m+E_0) \leq \mu$, then $\tilde{r} \in (0, d_1]$. We can minimise the right-hand side with respect to r by choosing $r = \tilde{r}$ and obtain

$$|v(\bar{y})| \leq C\, r_0^{2-n}\, (E_0+\varepsilon_m)\Big|\log\Big(\frac{\varepsilon_m}{E_0+\varepsilon_m}\Big)^{\delta^s}\Big|^{-B},$$

for a suitable constant $C>0$. On the other hand, if $\varepsilon_m/(\varepsilon_m+E_0) > \mu$, then

$$|v(\bar{y})| \leq C\, r_0^{2-n}\, E_0\Big(\frac{\varepsilon_m}{\mu(E_0+\varepsilon_m)}\Big).$$

Given a differentiable function f on the domain Ω, let $\nabla_T f(x)$ denote the $(n-1)$ dimensional tangential derivative of the function f on Σ_{m+1} and let $\partial_\nu f(x)$ denote the normal derivative of f on Σ_{m+1} for $m \in \{0,\dots,K-1\}$.

Set $\widetilde{\Sigma}_{m+1} = \Sigma_{m+1} \cap Q_{r_0/4}(P_{m+1})$. By the arbitrariness of $\bar{y}$, we obtain

$$\|v_m\|_{L^\infty(\widetilde{\Sigma}_{m+1})} \leq C r_0^{2-n}(E_0+\varepsilon_m)\omega_{1/C}\Big(\frac{\varepsilon_m}{\varepsilon_m+E_0}\Big). \tag{3.90}$$

By standard interior estimates [2] and the estimate (3.88), we derive

$$|\nabla v_m(w_{\bar{h}}(\bar{y}))| \leq \frac{1}{\rho_{\bar{h}}} C r_0^{2-n} \varepsilon_m^{\delta^{S+\bar{h}-1}} E_0^{1-\delta^{S+\bar{h}-1}}.$$

Hence,

$$\|\nabla v_m\|_{L^\infty(\widetilde{\Sigma}_{m+1})} \leq C r_0^{2-n}(E_0+\varepsilon_m)\omega_{1/C}\Big(\frac{\varepsilon_m}{\varepsilon_m+E_0}\Big). \tag{3.91}$$

From [92], as v is continuous across the interface Σ_{m+1}, it follows that $v_m = v_{m+1}$ on Σ_{m+1}. Therefore, $\nabla_T v_m = \nabla_T v_{m+1}$, and by (3.91), it follows that

$$\begin{aligned}\|\nabla_T v_{m+1}\|_{L^\infty(\widetilde{\Sigma}_{m+1})} = \|\nabla_T v_m\|_{L^\infty(\widetilde{\Sigma}_{m+1})} &\leq \|\nabla v_m\|_{L^\infty(\widetilde{\Sigma}_{m+1})} \\ &\leq C r_0^{2-n}(E_0+\varepsilon_m)\omega_{1/C}\Big(\frac{\varepsilon_m}{\varepsilon_m+E_0}\Big).\end{aligned} \tag{3.92}$$

Now, apply Lemma 2.1.1:

$$\begin{aligned}\int_{D_{m+1}\cap B_{3\bar{r}/8}(P_{m+1})} |\nabla v_{m+1}|^2 \leq \frac{c}{r_0}\Big(\int_{\widetilde{\Sigma}_{m+1}} v_{m+1}^2 + r_0^2\int_{\widetilde{\Sigma}_{m+1}} |\nabla v_{m+1}|^2\Big)^{\delta_1} \times \\ \times\Big(\int_{\widetilde{\Sigma}_{m+1}} v_{m+1}^2 + r_0^2\int_{\widetilde{\Sigma}_{m+1}} |\nabla v_{m+1}|^2 + r_0\int_{D_{m+1}\cap B_{\bar{r}/4}(P_{m+1})} \gamma_{m+1} A\nabla v_{m+1}\cdot\nabla v_{m+1}\Big)^{1-\delta_1}.\end{aligned} \tag{3.93}$$

In order to bound the left-hand side of (3.93), we have to estimate the following quantities:

i) $\int_{\widetilde{\Sigma}_{m+1}} v_{m+1}^2$;

ii) $\int_{\widetilde{\Sigma}_{m+1}} |\nabla v_{m+1}|^2$;

iii) $\int_{D_{m+1}\cap B_{\bar{r}/4}(P_{m+1})} \gamma_{m+1} A\nabla v_{m+1}\cdot\nabla v_{m+1}$.

For i), we can just use (3.90). For ii), since $\nabla v_{m+1} = \nabla_T v_{m+1} + (\nabla v_{m+1}\cdot\nu)\nu$,

$$\int_{\widetilde{\Sigma}_{m+1}} |\nabla v_{m+1}|^2 \le \int_{\widetilde{\Sigma}_{m+1}} |\nabla_T v_{m+1}|^2 + \int_{\widetilde{\Sigma}_{m+1}} |(\nabla v_{m+1}\cdot\nu)\nu|^2. \tag{3.94}$$

The first integral on the right-hand side of (3.94) can be estimated using (3.92). For the other term, we take into account the transmission condition

$$\gamma_m(x)A(x)\nabla v_m\cdot\nu = \gamma_{m+1}(x)A(x)\nabla v_{m+1}\cdot\nu, \qquad \text{on } \Sigma_{m+1}. \tag{3.95}$$

Then,

$$\|\nabla v_{m+1}\|_{L^\infty(\widetilde{\Sigma}_{m+1})} \le C r_0^{1-n}(E_0+\varepsilon_m)\omega_{1/C}\Big(\frac{\varepsilon_m}{\varepsilon_m+E_0}\Big). \tag{3.96}$$

Finally, iii) follows from standard energy estimates.

From the following trace estimate

$$\int_{D_1\cap B_{3\bar{r}/16}(P_1)} v_1^2 \le C\bigg(r_0\int_{\widetilde{\Sigma}_1} v_1^2 + r_0^2\int_{D_1\cap B_{3\bar{r}/8}(P_1)} |\nabla v_1|^2\bigg), \tag{3.97}$$

the inequalities (3.90), (3.93), (3.96) and (3.97) it follows that

$$\|v_{m+1}\|_{L^\infty(B_{\overline{\rho}}(\tilde{y}_{m+1}))} \le C r_0^{1-n}(E_0+\epsilon)\omega_{1/C}\Big(\frac{\varepsilon_m}{\varepsilon_m+E_0}\Big). \tag{3.98}$$

If $m\le K-1$, (3.84) follows by applying the inequality (3.88) with $\bar{y}=P_{m+1}$.

If $m=K$, by condition (3.81), arguing similarly to the inequality (3.88), and applying the claim, it follows that

$$\begin{aligned}
|v(w_{\bar{h}}(P_{K+1}))| &\le C\big(r_0^{2-n}\,\varepsilon_{K+1}\big)^{\delta^{s+\bar{h}-1}}(r_0\,d_1\,a^{\bar{h}-1}\,E_0)^{1-\delta^{s+\bar{h}-1}}\\
&\le C r_0^{2-n}(\varepsilon_{K+1}+E_0)\omega_{1/C}\Big(\frac{\varepsilon_{K+1}}{\varepsilon_{K+1}+E_0}\Big)\\
&\le C^{\bar{h}} r_0^{2-n}\,(\varepsilon_0+E_0)\,\Big(\omega_{1/C}^{(K)}\Big(\frac{\varepsilon_0}{\varepsilon_0+E_0}\Big)\Big)^{(1/C)^{\bar{h}}}.
\end{aligned}$$

□

Proof of Proposition 3.2.7. To begin with, by (3.53), for any $(y,z)\in(D_0)_{\bar{r}}\times(D_0)_{\bar{r}}$, where $\bar{r}>0$ so that $(D_0)_{\bar{r}}$ is connected, the following bound holds:

$$|S_k(y,z)| \le C\,E.$$

By (3.76), for any $y,z\in B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1}))$, we first apply Proposition 3.2.8 to

$v = S_k(\cdot, z)$ and then to $v = S_k(y, \cdot)$ to obtain

$$|S_k(y,z)| \leq r_0^{2-n} C^{\bar{h}(r)}(E+\varepsilon_0)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon_0}{E+\varepsilon_0}\right)\right)^{(1/C)^{\bar{h}(r)}}. \quad (3.99)$$

Hence, (3.77) can be derived from (3.99).

Since $S_k(y_1, \ldots, y_n, z_1, \ldots, z_n)$ is (locally) a weak solution in $\mathcal{W}_k \times \mathcal{W}_k$ of the elliptic equation

$$\mathrm{div}_y(\sigma^{(1)}(y)\nabla_y S_k(y,z)) + \mathrm{div}_z(\sigma^{(2)}(z)\nabla_z S_k(y,z)) = 0. \quad (3.100)$$

For any $i, j = 1, \ldots, n$, classical Schauder interior estimates [2] allows us to conclude that

$$\begin{aligned}&\|\partial_{x_i}\partial_{x_j} S_k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})\|_{L^\infty(B_{\frac{\rho_{\bar{h}(r)}}{2}}(w_{\bar{h}(r)}(Q_{k+1}))\times B_{\frac{\rho_{\bar{h}(r)}}{2}}(w_{\bar{h}(r)}(Q_{k+1})))}\\ &\leq \frac{C}{\rho^2_{\bar{h}(r)-1}}\|S_k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n})\|_{L^\infty(B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1}))\times B_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1})))}, \end{aligned} \quad (3.101)$$

where $x_i = y_i, x_{i+n} = z_i$ for $i = 1, \ldots, n$.

Moreover, since $d_{\bar{h}(r)-1} > r$, it follows that $r < \frac{d_0}{a\rho_0}\rho_{\bar{h}(r)}$, which in turn leads to

$$\begin{aligned}&\|\partial_{x_i}\partial_{x_j} S_k(x_1, \ldots, x_{2n})\|_{L^\infty(\tilde{Q}_{\frac{\rho_{\bar{h}(r)}}{2}}(w_{\bar{h}(r)}(Q_{k+1})))}\\ &\leq \frac{C}{r^2}\|S_k(x_1, \ldots, x_{2n})\|_{L^\infty(\tilde{Q}_{\rho_{\bar{h}(r)}}(w_{\bar{h}(r)}(Q_{k+1})))}. \end{aligned} \quad (3.102)$$

By (3.51), it follows that $r^{-2} \leq \left(\frac{a}{r_0}\right)^2\left(\frac{1}{a^2}\right)^{\bar{h}(r)}$, and by combining (3.102) and the above inequality we get the desired estimate. □

3.3 Proof of the stability estimate

Let $K \in \{1, \ldots, N\}$ be the index such that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} = \|\gamma_K^{(1)} - \gamma_K^{(2)}\|_{L^\infty(D_K)}. \quad (3.103)$$

By (3.3) and (3.103), it is is enough to prove that

$$\|\gamma_K^{(1)} - \gamma_K^{(2)}\|_{L^\infty(D_K)} \leq C\Big(\mathcal{J}(\sigma^{(1)}, \sigma^{(2)})\Big)^{1/2}, \quad (3.104)$$

where $C > 1$ is a constant depending on the a priori data.

We begin by choosing a chain of contiguous domains $D_0, D_1, \ldots, D_K$. Let $\Sigma_1, \ldots, \Sigma_K$ be the corresponding flat portions. For the sake of simplicity, we introduce the following notation:

$$\varepsilon = \Big(\mathcal{J}(\sigma^{(1)}, \sigma^{(2)})\Big)^{1/2}, \qquad E = \|\gamma_K^{(1)} - \gamma_K^{(2)}\|_{L^\infty(D_K)},$$

$$\delta_k = \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\mathcal{W}_k)}, \qquad \text{for } k = 1, \ldots, K.$$

Note that the norm $\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(D_k)}$ can be evaluated in terms of the following quantities

$$\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))} \quad \text{and} \quad \left|\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)\right|,$$

for any $k \in \{1, \ldots, K\}$. Indeed, let $\{e_j^k\}_{j=1,\ldots,n-1}$ be the orthonormal basis which generates the hyperplane that contains the flat region Σ_k. Let ν be the exterior unit normal of Σ_k. Set

$$\alpha_k + \beta_k \cdot x = (\gamma_k^{(1)} - \gamma_k^{(2)})(x), \qquad x \in D_k,\ \alpha_k \in \mathbb{R},\ \beta_k \in \mathbb{R}^n.$$

If we evaluate $(\gamma_k^{(1)} - \gamma_k^{(2)}$ at the points P_k and $P_k + \frac{r_0}{6} e_j^k$, for $j = 1, \ldots, n-1$, it follows that

$$\left|\alpha_k + \beta_k \cdot (P_k + \frac{r_0}{6} e_j^k)\right| \le |\alpha_k + \beta_k \cdot P_k| + \frac{r_0}{6} \sum_{j=1}^{n-1} |\beta_k \cdot e_j^k| \le C \|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))}.$$

Moreover,

$$|\beta_k \cdot \nu| = \left|\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)\right|.$$

Hence, for $k = 1, \ldots, K$, it turns out that for $C > 0$ depending on r_0,

$$|\alpha_k| + |\beta_k| \le C\left(\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))} + \left|\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)\right|\right)$$

Boundary estimates

Assume that $\Sigma = \sigma^{(1)}$. For $k = 1$, we show that the following estimate holds:

$$\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma \cap B_{r_0/4}(P_1))} + \left|\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)\right| \le C(\varepsilon + E)\omega_{1/C}^{(0)}\left(\frac{\varepsilon}{\varepsilon + E}\right). \tag{3.105}$$

Proof of (3.105).

For every $y, z \in D_0$, from Green's identity and (3.13), it follows that

$$\begin{aligned} &\int_\Sigma \left[G_1(x,y)\,\sigma^{(2)}(x)\nabla G_2(x,z)\cdot\nu - G_2(x,z)\,\sigma^{(1)}(x)\nabla G_1(x,y)\cdot\nu\right] dS(x) = \\ &\quad = \int_\Omega (\sigma^{(1)} - \sigma^{(2)})(x)\nabla G_1(x,y)\cdot\nabla G_2(x,z)\,\mathrm{d}x = S_0(y,z), \end{aligned} \tag{3.106}$$

and

$$\begin{aligned} &\int_\Sigma \left[\partial_{y_n} G_1(x,y)\ \sigma^{(2)}(x)\nabla\,\partial_{z_n} G_2(x,z)\cdot\nu - \partial_{z_n} G_2(x,z)\ \sigma^{(1)}(x)\nabla\,\partial_{y_n} G_1(x,y)\cdot\nu\right] dS(x) = \\ &\quad = \int_\Omega (\sigma^{(1)} - \sigma^{(2)})(x)\,\nabla\partial_{y_n} G_1(x,y)\cdot\nabla\partial_{z_n} G_2(x,z)\,\mathrm{d}x = \partial_{y_n}\partial_{z_n} S_0(y,z). \end{aligned} \tag{3.107}$$

By Proposition 3.2.4, it is known that $S_0(y,z)$ is locally a weak solution of an elliptic equation in $(D_0)_{\bar r} \times (D_0)_{\bar r}$, for some $\bar r \in (0, r_0/4)$ such that $(D_0)_{\bar r}$ is connected.

From [66, Theorem 8.17], we derive that the weak solutions are locally bounded. As a result, the supremum of $S_0(y,z)$ can be bounded by its L^2-norm as follows:

$$\sup_{(y,z)\in(D_0)_{\bar{r}/2}\times(D_0)_{\bar{r}/2}} |S_0(y,z)| \leq \frac{C}{\bar{r}^n}\Big(\int_{D_y\times D_z} |S_0(y,z)|^2 dy\, dz\Big)^{1/2} = \frac{C}{\bar{r}^n}\Big(\mathcal{J}(\sigma^{(1)},\sigma^{(2)})\Big)^{1/2}, \tag{3.108}$$

where C depends on n, λ, $|\Omega|$ and $\bar{r}\in(0,r_0/4)$.

Let $\rho_0 = r_0/C_4$, where C_4 is the constant introduced in Proposition 3.2.2. Let $r\in(0,\overline{\rho_0}]$, where $\overline{\rho_0} = \frac{\rho_0}{2}(1-\sin\beta_1)$. We define the parameter $\tau = \lambda_{\bar{h}(r)} = a^{\bar{h}-1}\lambda_1$, where $\bar{h} = \bar{h}(r)$ is defined in (3.50). We define the point $w = w(P_1) = P_1 + \tau\nu(P_1)$ where $\nu(P_1)$ is the exterior unit normal of ∂D_1 at the point P_1.

Set $y = z = w$, then split the right hand side of (3.106) into the sum of two integrals $I_1(w)$ and $I_2(w)$:

$$S_0(w,w) = I_1(w) + I_2(w),$$

where

$$\begin{aligned} I_1(w) &= \int_{B_{\rho_0}(P_1)\cap D_1} (\gamma_1^{(1)}-\gamma_1^{(2)})(x)A(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\,\mathrm{d}x,\\ I_2(w) &= \int_{\Omega\setminus(B_{\rho_0}(P_1)\cap D_1)} (\sigma^{(1)}-\sigma^{(2)})(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\,\mathrm{d}x. \end{aligned}$$

The integral $I_2(w)$ can be estimated by applying Proposition 3.2.1:

$$|I_2(w)| \leq E\,\|\nabla G_1(\cdot,w)\|_{L^2(\Omega)}\,\|\nabla G_2(\cdot,w)\|_{L^2(\Omega)} \leq CE\rho_0^{2-n}. \tag{3.109}$$

Let us estimate $I_1(w)$ from below in terms of $\|\gamma_1^{(1)}-\gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))}$. Let $\overline{x}\in\overline{\Sigma_1\cap B_{r_0/4}(P_1)}$ be such that

$$(\gamma_1^{(1)}-\gamma_1^{(2)})(\overline{x}) = \|\gamma_1^{(1)}-\gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))}.$$

Hence, we can split $I_1(w)$ as the sum of the following integrals:

$$\begin{aligned} I_1(w) = &\int_{B_{\rho_0}(P_1)\cap D_1} (\gamma_1^{(1)}-\gamma_1^{(2)})(\overline{x})A(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\ \mathrm{d}x\\ &+\int_{B_{\rho_0}(P_1)\cap D_1} \beta_1\cdot(x-\overline{x})A(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\ \mathrm{d}x, \end{aligned} \tag{3.110}$$

which leads to

$$\begin{aligned} |I_1(w)| \geq &\left|\int_{B_{\rho_0}(P_1)\cap D_1} (\gamma_1^{(1)}-\gamma_1^{(2)})(\bar{x})\ A(x)\ \nabla G_1(x,w)\cdot\nabla G_2(x,w)\ \mathrm{d}x\right|\\ &-\bar{A}\int_{B_{\rho_0}(P_1)\cap D_1} |\beta_1\cdot(x-\bar{x})|\,|\nabla G_1(x,w)|\,|\nabla G_2(x,w)|\ \mathrm{d}x. \end{aligned} \tag{3.111}$$

By applying the asymptotic estimate (3.23) at the right-hand side of (3.111), one derives

$$
\begin{aligned}
|I_1(w)| \geq (\gamma_1^{(1)} - \gamma_1^{(2)})(\bar{x}) C \Big\{ \int_{B_{\rho_0}(P_1)\cap D_1} & A(x)\,\nabla_x\Gamma(Jx,Jw)\cdot\nabla_x\Gamma(Jx,Jw)\;\mathrm{d}x -\\
&- \int_{B_{\rho_0}(P_1)\cap D_1} |x-w|^{1-n+\theta_1}|\nabla_x\Gamma(Jx,Jw)|\;\mathrm{d}x -\\
&- \int_{B_{\rho_0}(P_1)\cap D_1} |\nabla_x\Gamma(Jx,Jw))|\,|x-w|^{1-n+\theta_1}\;\mathrm{d}x -\\
&- \int_{B_{\rho_0}(P_1)\cap D_1} |x-w|^{2(1-n+\theta_1)}\;\mathrm{d}x \Big\} -\\
&- CE\int_{B_{\rho_0}(P_1)\cap D_1} |(x-\bar{x})|\,|x-w|^{2(1-n)}\;\mathrm{d}x,
\end{aligned}
$$

for $J = \sqrt{A^{-1}(P_1)}$. Notice that, up to a rigid transformation, it can be assumed that P_1 coincides with the origin 0 of the coordinate system. From equation (3.4), it follows that

$$
\begin{aligned}
|I_1(w)| \geq \|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4})} C \int_{B_{\rho_0}\cap D_1} & \frac{|J^2(x-w)|}{|J(x-w)|^n}^2\;\mathrm{d}x\\
- C\,E \int_{B_{\rho_0}\cap D_1} & \frac{|J^2(x-w)|}{|J(x-w)|^n}\,|x-w|^{\theta_1+1-n}\;\mathrm{d}x\\
- C\,E \int_{B_{\rho_0}\cap D_1} & |x-w|^{2\theta_1+2-n}\;\mathrm{d}x\\
- C\,E \int_{B_{\rho_0}\cap D_1} & |x-\overline{x}|\,|x-w|^{2-2n}\;\mathrm{d}x.
\end{aligned}
\tag{3.112}
$$

By evaluating the integrals of the right-hand side of (3.112) in terms of the parameter τ, one derives:

$$
\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma\cap B_{r_0/4})} C\tau^{2-n} \leq |I_1(w)| + C\,E\,\tau^{2-n+\theta_1} + C\tau^{2-n+2\theta_1} + C\,E\,\tau^{3-n}.
\tag{3.113}
$$

By (3.108) and (3.109), it follows that

$$
|I_1(w)| \leq |S_0(w,w)| + |I_2(w)| \leq C\,\varepsilon\tau^{-n} + C\,E\,\rho_0^{2-n}.
\tag{3.114}
$$

By (3.113) and (3.114), it turns out that

$$
\begin{aligned}
\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma\cap B_{r_0/4}(P_1))}\tau^{(2-n)} \leq C\,\varepsilon\tau^{2-n} +\\
+ C\,E\,\rho_0^{2-n} + C\,E\,\tau^{2-n+\theta_1} + C\tau^{2-n+2\theta_1} + C\,E\,\tau^{3-n}.
\end{aligned}
\tag{3.115}
$$

If we multiply (3.115) by τ^{n-2} and optimise with respect to τ, we can deduce that

$$
\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma\cap B_{r_0/4}(P_1))} \leq C(\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\frac{\theta_1}{\theta_1+2}},
\tag{3.116}
$$

with $C > 0$ is a constant that depends on the a priori data only. Let us evaluate

$|\,\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1)|$. Let the parameters ρ_0, r and the point w be defined as above. For $y=z=w$, we can write

$$\partial_{y_n}\,\partial_{z_n}\,S_0(w,w)=\partial_{y_n}\,\partial_{z_n}\,I_1(w)+\partial_{y_n}\,\partial_{z_n}\,I_2(w). \tag{3.117}$$

From (3.109), we derive the following bound

$$|\,\partial_{y_n}\,\partial_{z_n}\,I_2(w)|\le C\,E\,\rho_0^{-n}, \tag{3.118}$$

where C depends on the a priori data. For any point $x\in B_{\rho_0}(P_1)\cap D_1$, we can write

$$\begin{aligned}(\gamma_1^{(1)}-\gamma_1^{(2)})(x)=(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1)+(\nabla_T(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1))\cdot(x-P_1)'+\\+(\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1))(x-P_1)_n.\end{aligned}$$

It turns out that

$$\begin{aligned}|\,\partial_{y_n}\,\partial_{z_n}\,I_1(w)|\ge&\left|\int_{B_{\rho_0}(P_1)\cap D_1}(\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1))(x-P_1)_nA(\cdot)\nabla\partial_{y_n}G_1(\cdot,w)\cdot\nabla\partial_{z_n}G_2(\cdot,w)\right|\\&-\int_{B_{\rho_0}(P_1)\cap D_1}|(\nabla_T(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1))\cdot(x-P_1)'|\,|A(\cdot)\nabla\partial_{y_n}G_1(\cdot,w)\cdot\nabla\partial_{z_n}G_2(\cdot,w)|\\&-\int_{B_{\rho_0}(P_1)\cap D_1}|(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1)|\,|A(\cdot)\nabla\partial_{y_n}G_1(\cdot,w)\cdot\nabla\partial_{z_n}G_2(\cdot,w)|.\end{aligned}$$

Up to a rigid transformation, we can assume that P_1 coincides with the origin 0 of the coordinate system. By (3.23) and (3.116), one derives

$$\begin{aligned}|\,\partial_{y_n}\,\partial_{z_n}\,I_1(w)|\ge|\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(0)|\,C&\int_{B_{\rho_0}\cap D_1}|\nabla_x\partial_{y_n}\Gamma(Jx,Jw)|^2|x_n|\,\mathrm{d}x-\\-\,C\,E&\int_{B_{\rho_0}\cap D_1}|\nabla_x\partial_{y_n}\Gamma(Jx,Jw)|\,|x-w|^{\theta_2-n}|x_n|\,\mathrm{d}x-\\-\,C\,E&\int_{B_{\rho_0}\cap D_1}|x-w|^{\theta_2-2n}|x_n|\,\mathrm{d}x-\\-\,C\,(\varepsilon+E)&\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\theta_1}{\theta_1+2}}\int_{B_{\rho_0}\cap D_1}|x'|\,|\nabla\partial_{y_n}G_1(x,w)|\,|\nabla\partial_{z_n}G_2(x,w)|\,\mathrm{d}x-\\-\,C\,(\varepsilon+E)&\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\theta_1}{\theta_1+2}}\int_{B_{\rho_0}\cap D_1}|\nabla\partial_{y_n}G_1(x,w)|\,|\nabla\partial_{z_n}G_2(x,w)|\,\mathrm{d}x.\end{aligned}$$

It turns out that

$$\begin{aligned}|\,\partial_{y_n}\,\partial_{z_n}\,I_1(w)|\ge C\,|\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(0)|\,\tau^{1-n}-C\,E\,\tau^{1-n+\theta_2}-\\-\,C\,E\,\theta^{2\theta_2-2n+1}-C\,(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\theta_1}{\theta_1+2}}\tau^{-n}.\end{aligned}$$

Hence, we have

$$|\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(0)|\tau^{1-n} \leq |\,\partial_{y_n}\partial_{z_n} I_1(w)|+C\Big[(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\theta_1}{\theta_1+2}}\tau^{-n}+E\tau^{1-n+\theta_2}\Big]. \tag{3.119}$$

By Schauder interior estimates (see [2]), we derive

$$\|\nabla_y\nabla_z S_0(y,z)\|_{L^\infty(Q_{\tau/2}(w)\times Q_{\tau/2}(w))} \leq \frac{C}{\tau^2}\|S_0(y,z)\|_{L^\infty(Q_\tau(w)\times Q_\tau(w))},$$

hence, by the triangle inequality, we have

$$|\,\partial_{y_n}\partial_{z_n} I_1(w)| \leq C\,\varepsilon\tau^{-n-2}+C\,E\,\rho_0^{-n}. \tag{3.120}$$

Thus, by combining together (3.119) and (3.120), it turns out that

$$|\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(0)|\tau^{1-n} \leq C\Big[\varepsilon\tau^{-n-2}+E\rho_0^{-n}+(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\theta_1}{\theta_1+2}}\tau^{-n}+E\tau^{1-n+\theta_2}\Big]. \tag{3.121}$$

If we multiply both sides of (3.121) by τ^{n-1}, we derive

$$|\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(0)| \leq C(\varepsilon\,\tau^{-3}+E\tau^{\theta_2}).$$

Finally, optimizing the right-hand side with respect to τ, it turns out that

$$|\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(0)| \leq C(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\theta_2}{\theta_2+3}},$$

so that (3.105) is proved.

Interior estimates

We show that for the case $k=2$ one can derive the following estimates:

$$\|\gamma_2^{(1)}-\gamma_2^{(2)}\|_{L^\infty(\sigma^{(2)}\cap B_{r_1}(P_2))} \leq C(\varepsilon+E)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}, \tag{3.122}$$

$$\Big|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(P_2)\Big| \leq C(\varepsilon+E)\left(\omega^{(4)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}. \tag{3.123}$$

Proof of (3.122)

For $y,z\in D_0$, we have

$$\begin{aligned}\int_\Sigma\Big[G_1(x,y)\,\sigma^{(2)}(x)\nabla G_2(x,z)\cdot\nu-G_2(x,z)\,\sigma^{(1)}(x)\nabla G_1(x,y)\cdot\nu\Big]\,dS(x)=\\ =S_1(y,z)+\int_{\mathcal{W}_1}(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x G_1(x,y)\cdot\nabla_x G_2(x,z)\,\mathrm{d}x.\end{aligned} \tag{3.124}$$

Let ρ_0, r be the same quantities appearing in the boundary estimates. Set $w=w(P_2)=P_2+\tau\nu(P_2)$. By Schwarz inequality, by (3.124) and trace estimates, it follows that for any $y, z \in (D_0)_{r_0/3}$,

$$|S_1(y,z)| \leq C(\varepsilon_0 + \delta_1), \tag{3.125}$$

Set $y = z = w$, we can write

$$S_1(w,w) = I_1(w) + I_2(w),$$

where

$$\begin{aligned} I_1(w) &= \int_{B_{\rho_0}(P_1)\cap D_2} (\gamma_2^{(1)} - \gamma_2^{(2)})(x)A(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\ \mathrm{d}x, \\ I_2(w) &= \int_{\mathcal{U}_1\setminus(B_{\rho_0}(P_1)\cap D_2)} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\ \mathrm{d}x. \end{aligned}$$

Using Proposition 3.2.1, we derive the following bound:

$$|I_2(w)| \leq CE\rho_0^{2-n}. \tag{3.126}$$

Let us estimate $I_1(w)$ from below in terms of $\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))}$. Let $\overline{x} \in \overline{\Sigma_2 \cap B_{r_0/4}(P_2)}$ be the point such that

$$(\gamma_2^{(1)} - \gamma_2^{(2)})(\overline{x}) = \|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))}.$$

Hence, we can split $I_1(w)$ as the sum of the following integrals:

$$\begin{aligned} I_1(w) = &\int_{B_{\rho_0}(P_2)\cap D_2} (\gamma_2^{(1)} - \gamma_2^{(2)})(\overline{x})A(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\ \ \mathrm{d}x+ \\ &+\int_{B_{\rho_0}(P_2)\cap D_2} \beta_2\cdot(x-\overline{x})A(x)\nabla G_1(x,w)\cdot\nabla G_2(x,w)\ \ \mathrm{d}x. \end{aligned}$$

By (3.23), one derives

$$\begin{aligned} |I_1(w)| \geq (\gamma_2^{(1)} - \gamma_2^{(2)})(\bar{x})C\Big\{ &\int_{B_{\rho_0}(P_2)\cap D_2} A(x)\,\nabla_x\Gamma(Jx,Jw)\cdot\nabla_x\Gamma(Jx,Jw)\ \mathrm{d}x- \\ &-\int_{B_{\rho_0}(P_2)\cap D_2} |x-w|^{1-n+\theta_1}|\nabla_x\Gamma(Jx,Jw)|\ \mathrm{d}x- \\ &-\int_{B_{\rho_0}(P_2)\cap D_2} |\nabla_x\Gamma(Jx,Jw))|\,|x-w|^{1-n+\theta_1}\ \mathrm{d}x- \\ &-\int_{B_{\rho_0}(P_2)\cap D_2} |x-w|^{2(1-n+\theta_1)}\ \ \mathrm{d}x\Big\}- \\ &-CE\int_{B_{\rho_0}(P_2)\cap D_2} |(x-\bar{x})|\,|x-w|^{2(1-n)}\ \ \mathrm{d}x, \end{aligned}$$

for $J = \sqrt{A^{-1}(P_2)}$. Notice that up to a transformation of coordinates, we can assume that P_2 coincides with the origin 0 of the coordinate system. By (3.4) it follows that

$$\begin{aligned}|I_1(w)| \geq \|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}})} C \int_{B_{\rho_0}\cap D_2} \frac{|J^2(x-w)|}{|J(x-w)|^n}^2 \, dx \\ - C\, E \int_{B_{\rho_0}\cap D_2} \frac{|J^2(x-w)|}{|J(x-w)|^n} |x-w|^{\theta_1+1-n} \, dx \\ - C\, E \int_{B_{\rho_0}\cap D_2} |x-w|^{2\theta_1+2-n} \, dx \\ - C\, E \int_{B_{\rho_0}\cap D_2} |x-\overline{x}|\, |x-w|^{2-2n} \, dx.\end{aligned} \tag{3.127}$$

By evaluating the integrals of the right-hand side of (3.127) in terms of the parameter τ, one derives:

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4})} C\tau^{2-n} \leq |I_1(w)| + C\, E\, \tau^{2-n+\theta_1} + C\tau^{2-n+2\theta_1} + C\, E\, \tau^{3-n}. \tag{3.128}$$

By the quantitative estimate (3.77) for the singular solution S_1 and (3.126), it follows that

$$|I_1(w)| \leq |S_1(w,w)| + |I_2(w)| \leq C_1^{\bar{h}}(E+\varepsilon)\left(\omega_{1/C}^{(2)}\left(\frac{\varepsilon}{E+\varepsilon}\right)\right)^{(1/C)^{\bar{h}}} + C\, E\, \rho_0^{2-n},$$

so that (3.128) reads:

$$\begin{aligned}\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_1))}\tau^{(2-n)} \leq C_1^{\bar{h}}(E+\delta_1+\varepsilon)\left(\omega_{1/C}^{(2)}\left(\frac{\varepsilon+\delta_1}{E+\varepsilon+\delta_1}\right)\right)^{(1/C)^{\bar{h}}} + \\ + C\, E\, \rho_0^{2-n} + C\, E\, \tau^{2-n+\theta_1} + CE\tau^{2-n+2\theta_1} + C\, E\, \tau^{3-n}.\end{aligned} \tag{3.129}$$

Since $\bar{h}$ is a function of r, we can estimate $C^{\bar{h}}$ and $(1/C)^{\bar{h}}$ in terms of r. By (3.51), it turns out that

$$C^{\bar{h}} \leq C^2\left(\frac{d_1}{r}\right)^C \quad \text{and} \quad \left(\frac{1}{C}\right)^{\bar{h}} \leq \left(\frac{1}{C}\right)^2\left(\frac{r}{d_1}\right)^C.$$

Hence, we obtain

$$\begin{aligned}\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_1))}\tau^{(2-n)} \leq \left(\frac{r}{d_1}\right)^{-C}(E+\delta_1+\varepsilon)\left(\omega_{1/C}^{(2)}\left(\frac{\varepsilon+\delta_1}{E+\varepsilon+\delta_1}\right)\right)^{\left(\frac{r}{d_1}\right)^C} + \\ + C\, E\, \rho_0^{2-n} + C\, E\, \tau^{2-n+\theta_1} + CE\tau^{2-n+2\theta_1} + C\, E\, \tau^{3-n}.\end{aligned} \tag{3.130}$$

Moreover, notice that $\tau \leq \lambda_1 \cdot \frac{r}{d_1}$. Hence, if we multiply (3.129) by τ^{n-2} and optimising with respect to τ, it follows that

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_1))} \leq C(E+\delta_1+\varepsilon)\left(\omega_{1/C}^{(3)}\left(\frac{\varepsilon+\delta_1}{E+\varepsilon+\delta_1}\right)\right)^{(1/C)}, \tag{3.131}$$

with $C>0$ is a constant that depends on the a priori data only. By the properties of

$\omega^{(3)}_{1/C}$, we have

$$(E+\delta_1+\varepsilon)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon+\delta_1}{E+\varepsilon+\delta_1}\right)\right)^{(1/C)}\leq C\,(E+\varepsilon)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{E+\varepsilon}\right)\right)^{1/C},$$

and this will prove (3.122). Indeed, notice that

$$\omega^{(3)}_{1/C}\left(\frac{\varepsilon+\delta_1}{E+\varepsilon+\delta_1}\right)\leq\omega^{(3)}_{1/C}\left(\frac{2(\varepsilon+E)\omega^{(0)}\left(\frac{\varepsilon}{\varepsilon+E}\right)}{E+\varepsilon+\delta_1}\right).$$

By applying (3.47), (3.48) and (3.105), we have

$$\begin{aligned}(E+&\delta_1+\varepsilon)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon+\delta_1}{E+\varepsilon+\delta_1}\right)\right)\leq\\
&\leq\left(\varepsilon+E+(\varepsilon+E)\omega^{(0)}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)\omega^{(3)}_{1/C}\left(\frac{2(\varepsilon+E)\omega^{(0)}\left(\frac{\varepsilon}{\varepsilon+E}\right)}{E+\varepsilon+\varepsilon+E+(\varepsilon+E)\omega^{(0)}\left(\frac{\varepsilon}{\varepsilon+E}\right)}\right)\leq\\
&\leq 2(\varepsilon+E)\omega^{(3)}\Big(\frac{2(\varepsilon+E)\omega^{(0)}\left(\frac{\varepsilon}{\varepsilon+E}\right)}{2(\varepsilon+E)}\Big)\\
&\leq(E+\varepsilon)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{E+\varepsilon}\right)\right)\end{aligned}$$

Proof of (3.123)

For $y,z\in D_0$, one derives

$$\begin{aligned}\int_{\Sigma}\Big[\partial_{y_n}G_1(x,y)\;\sigma^{(2)}(x)\nabla\,\partial_{z_n}G_2(x,z)\cdot\nu-\partial_{z_n}G_2(x,z)\;\sigma^{(1)}(x)\nabla\,\partial_{y_n}G_1(x,y)\cdot\nu\Big]\;dS(x)=\\
=\partial_{y_n}\partial_{z_n}S_1(y,z)+\int_{\mathcal{W}_1}(\sigma^{(1)}-\sigma^{(2)})(x)\partial_{y_n}\nabla G_1(x,y)\cdot\partial_{z_n}\nabla G_2(x,z)\;\mathrm{d}x.\end{aligned}\tag{3.132}$$

We split the integral solution into two parts:

$$\partial_{y_n}\partial_{z_n}S_1(w,w)=\partial_{y_n}\partial_{z_n}I_1(w)+\partial_{y_n}\partial_{z_n}I_2(w).\tag{3.133}$$

As in the boundary estimates, we can bound $\partial_{y_n}\partial_{z_n}I_2(w)$ as follows:

$$|\partial_{y_n}\partial_{z_n}I_2(w)|\leq CE\rho_0^{-n}.\tag{3.134}$$

Now, let us estimate from below the integral $I_1(w)$. First, notice that for any $x\in B_{\rho_0}(P_2)\cap D_2$ we can rewrite $\gamma_2^{(i)}$ for $i=1,2$ as

$$\gamma_2^{(i)}(x)=\gamma_2^{(i)}(P_2)+\nabla_T\gamma_2^{(i)}(P_2)\cdot(x-P_2)'+\partial_\nu(\gamma_2^{(i)}(P_2))(x-P_2)_n.$$

Therefore, we obtain

$$|\partial_{y_n}\partial_{z_n} I_1(w)| \geq \left| \int_{B_{\rho_0}(P_1)\cap D_2} (\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(P_2))(x-P_2)_n\, A(x)\, \partial_{y_n}\nabla G_1(\cdot,w)\cdot\partial_{z_n}\nabla G_2(\cdot,w)\right|$$
$$-\int_{B_{\rho_0}(P_2)\cap D_2} |(\nabla_T(\gamma_2^{(1)}-\gamma_2^{(2)})(P_2))\cdot(x-P_2)'|\,|A(x)\,\partial_{y_n}\nabla G_1(\cdot,w)\cdot\partial_{z_n}\nabla G_2(\cdot,w)|$$
$$-\int_{B_{\rho_0}(P_2)\cap D_2} |(\gamma_2^{(1)}-\gamma_2^{(2)})(P_2)|\,|A(x)\,\partial_{y_n}\nabla G_1(\cdot,w)\cdot\partial_{z_n}\nabla G_2(\cdot,w)|.$$

Up to a rigid transformation of coordinates, we can assume that P_2 coincides with the origin 0 of the coordinate system. By Proposition 3.2.2, (A.15) and (3.4),

$$\begin{aligned}|\partial_{y_n}\partial_{z_n} I_1(w)| &\geq |\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(0)|C\int_{B_{\rho_0}\cap D_2} |\partial_{y_n}\nabla_x\Gamma(Jx,Jw)|^2\,|x_n| \\ &-CE\int_{B_{\rho_0}\cap D_2} |\partial_{y_n}\nabla_x\Gamma(Jx,Jw)|\,|x-w|^{\theta_2-n}|x_n| \\ &-CE\int_{B_{\rho_0}\cap D_2} |x-w|^{2\theta_2-2n}|x_n| \\ &-\int_{B_{\rho_0}\cap D_2} |\nabla_T(\gamma_2^{(1)}-\gamma_2^{(2)})(0)|\,|x'|\,|A(x)\,\partial_{y_n}\nabla G_1(\cdot,w)\cdot\partial_{z_n}\nabla G_2(\cdot,w)| \\ &-\int_{B_{\rho_0}\cap D_2} |(\gamma_2^{(1)}-\gamma_2^{(2)})(0)|\,|A(x)\,\partial_{y_n}\nabla G_1(\cdot,w)\cdot\partial_{z_n}\nabla G_2(\cdot,w)|.\end{aligned} \tag{3.135}$$

Applying (3.122), we obtain

$$\begin{aligned}|\partial_{y_n}\partial_{z_n} I_1(w)| &\geq |\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(0)|C\int_{B_{\rho_0}\cap D_2} |x-w|^{1-2n} \\ &-CE\int_{B_{\rho_0}\cap D_2} |x-w|^{\theta_2+1-2n} \\ &-CE\int_{B_{\rho_0}\cap D_2} |x-w|^{2\theta_2+1-2n} \\ &-C(\varepsilon+E)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}\int_{B_{\rho_0}\cap D_2} |x-w|^{1-2n} \\ &-C(\varepsilon+E)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}\int_{B_{\rho_0}\cap D_2} |x-w|^{-2n},\end{aligned}$$

where the constant $C>0$ depends on the a priori data and on J. This leads to

$$\left|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(0)\right|\tau^{1-n} \leq |I_1(w)|+C\left\{(\varepsilon+E)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}\tau^{-n}+E\frac{\tau^{1-n+\theta_2}}{\rho_0^{\theta_2}}\right\}. \tag{3.136}$$

Then, by (3.133) and (3.134),

$$|\partial_{y_n}\partial_{z_n} I_1(w)| \leq |\partial_{y_n}\partial_{z_n} S_1(w,w)| + CE\rho_0^{-n}. \tag{3.137}$$

Combining (3.136) with (3.137), it follows that

$$\left|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})\right|\tau^{1-n}\le|\partial_{y_n}\partial_{z_n}S_1(w,w)|+C\Big\{E\rho_0^{-n}+ \\ +(\varepsilon+E)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}\tau^{-n}+E\frac{\tau^{1-n+\theta_2}}{\rho_0^{\theta_2}}\Big\}.$$

By Proposition 3.2.7, it follows that

$$|\partial_{y_n}\partial_{z_n}S_1(w,w)|\le r_0^{-n}C^{\bar h}(\varepsilon+\delta_1+E)\left(\omega^{(2)}_{1/C}\left(\frac{\varepsilon+\delta_1}{E+\delta_1+\varepsilon}\right)\right)^{(1/C)^{\bar h}},$$

so that

$$\left|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(0)\right|\le C^{\bar h}(\varepsilon+\delta_1+E)\Big(\omega^{(2)}_{1/C}\left(\frac{\varepsilon+\delta_1}{E+\delta_1+\varepsilon}\right)\Big)^{(1/C)^{\bar h}}\tau^{n-1}+ \\ +C\tau^{-1}(\varepsilon+E)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}+CE\frac{\tau^{\theta_2}}{\rho_0^{\theta_2}}. \tag{3.138}$$

By the definition of $\bar h$, it turns out that

$$|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(0)|\le C\Big\{\left(\frac{r}{d_1}\right)^{n-1-C}(\varepsilon+\delta_1+E)\Big(\omega^{(2)}_{1/C}\left(\frac{\varepsilon+\delta_1}{E+\delta_1+\varepsilon}\right)\Big)^{\left(\frac{r}{d_1}\right)^C}+ \\ +\left(\frac{r}{d_1}\right)^{-1}(\varepsilon+E)\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{\varepsilon+E}\right)\right)^{1/C}+\left(\frac{r}{d_1}\right)^{\theta_2}\Big\}. \tag{3.139}$$

Then, it turns out that

$$\left|\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(0)\right|\le C(\varepsilon+E)\Big\{\left(\frac{r}{d_1}\right)^{n-1-C}\left(\omega^{(3)}_{1/C}\left(\frac{\varepsilon}{E+\varepsilon}\right)\right)^{\left(\frac{r}{d_1}\right)^C}+\left(\frac{r}{d_1}\right)^{\theta_2}\Big\}.$$

Finally, optimizing with respect to r, (3.123) follows.

Conclusion.

For $y,z\in D_0$,

$$\int_\Sigma\Big[G_2(x,z)\,\sigma^{(1)}(x)\nabla G_1(x,y)\cdot\nu-G_1(x,y)\sigma^{(2)}(x)\nabla G_2(x,z)\cdot\nu\Big]\,dS(x)= \\ =S_{k-1}(y,z)+\int_{\mathcal{W}_{k-1}}(\sigma^{(1)}-\sigma^{(2)})(x)\nabla G_1(x,y)\cdot\nabla G_2(x,z)\,\mathrm{d}x, \tag{3.140}$$

and

$$\int_\Sigma\Big[\partial_{z_n}G_2(x,z)\,\sigma^{(1)}(x)\nabla\,\partial_{y_n}G_1(x,y)\cdot\nu-\partial_{y_n}G_1(x,y)\sigma^{(2)}(x)\nabla\,\partial_{z_n}G_2(x,z)\cdot\nu\Big]\,dS(x) \\ =\partial_{y_n}\partial_{z_n}S_{k-1}(y,z)+\int_{\mathcal{W}_{k-1}}(\sigma^{(1)}-\sigma^{(2)})(x)\nabla\partial_{y_n}G_1(x,y)\cdot\nabla\partial_{z_n}G_2(x,z)\,\mathrm{d}x. \tag{3.141}$$

For $(y,z) \in \mathcal{W}_k \times \mathcal{W}_k$,

$$|S_{k-1}(y,z)| \leq C r_0^{2-n}(\varepsilon + \delta_{k-1}).$$

Proceeding as above, for $k = 3, \dots, K$, one can show that the following inequalities hold:

$$\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))} \leq C(\varepsilon + E)\left(\omega_{1/C}^{(2k-1)}\left(\frac{\varepsilon}{\varepsilon + E}\right)\right)^{1/C}, \tag{3.142}$$

$$\left|\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)\right| \leq C(\varepsilon + E)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon}{\varepsilon + E}\right)\right)^{1/C}, \tag{3.143}$$

Notice that

$$\delta_k \leq \delta_{k-1} + \|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(D_k)},$$

hence

$$\delta_k \leq \delta_{k-1} + C(\varepsilon + E)\left(\omega_{1/C}^{(2k)}\left(\frac{\varepsilon}{\varepsilon + E}\right)\right)^{1/C}.$$

By (3.142) and (3.143) it follows that

$$\delta_k + \varepsilon \leq C(\varepsilon + E)\Big(\omega_{1/C}^{(2k)}\Big(\frac{\varepsilon}{\varepsilon + E}\Big)\Big)^{1/C}.$$

Now, consider $k = K$, for $E = \delta_K$, it turns out that

$$\varepsilon + E \leq C(\varepsilon + E)\left(\omega_{1/C}^{(2K)}\left(\frac{\varepsilon}{\varepsilon + E}\right)\right)^{1/C}.$$

If $\varepsilon < e^{-2}E$ (otherwise the statement is proven), then it follows that

$$E \leq C\left(\frac{E}{e^2} + E\right)\left(\omega_{1/C}^{(2K)}\left(\frac{\varepsilon}{E}\right)\right)^{1/C},$$

therefore

$$E \leq \frac{1}{\omega_{1/C}^{(-(2K))}\left(\frac{1}{C}\right)}\varepsilon,$$

where $\omega_{1/C}^{(-(2K))}$ denotes the inverse of $\omega_{1/C}^{(2K)}$. This completes the proof.

□

Proof of Corollary 3.1.2. By (3.14),

$$S_0(y,z) = \langle (\Lambda_1^\Sigma - \Lambda_2^\Sigma) G_1(\cdot, y)|_{\partial\Omega}, G_2(\cdot, z)|_{\partial\Omega}\rangle, \qquad \text{for } y, z \in D_0. \tag{3.144}$$

By (3.144), it follows that

$$|S_0(y,z)| \leq C\|\Lambda_1^\Sigma - \Lambda_2^\Sigma\|_{\mathcal{L}(H_{00}^{1/2}(\Sigma), H_{00}^{-1/2}(\Sigma))}. \tag{3.145}$$

Since

$$\mathcal{J}(\sigma^{(1)}, \sigma^{(2)}) = \int_{D_y \times D_z} |S_0(y,z)|^2 \, \mathrm{d}y \, \mathrm{d}z,$$

by (3.145), it follows that

$$\left(\mathcal{J}(\sigma^{(1)},\sigma^{(2)})\right)^{1/2} \leq C\|\Lambda_1^{\Sigma}-\Lambda_2^{\Sigma}\|_{\mathcal{L}(H_{00}^{1/2}(\Sigma),H_{00}^{-1/2}(\Sigma))}, \tag{3.146}$$

with $\tilde{C}>0$ depends on the *a priori* data only. Then the thesis follows by (3.16) and (3.146). □

CHAPTER 4

Stability for the determination of an anisotropic inclusion

Contents

In this chapter, we address the problem of determining an inclusion D of a body Ω for the generalised Schrödinger equation from the knowledge of the Cauchy data set. Our main result is a log-type stability estimate, which bounds the Hausdorff distance between the boundaries of two inclusions in terms of the distance between their corresponding Cauchy data sets (Theorem 4.0.1). This result is in line with the instability results of Mandache [97] and Di Cristo and Rondi [53]. This is one of the first results proved in the case of anisotropy in the leading order term of the equation.

This chapter is structured into five sections. First, we introduce the a priori assumptions, the formal definition of the local Cauchy data set and the main result, Theorem 4.0.1. In Section 4.1, we introduce relevant geometric observations. Section 4.2 is dedicated to the construction of the Green function for a mixed boundary value problem defined on an augmented domain with complex Robin data on a portion of its boundary and homogeneous Dirichlet data on the remaining part. This fictitious construction ensures the well-posedness of the direct problem, which will be explained in Lemma 4.2.1.

In Section 4.3, we derive upper and lower bounds for the singular solution, provided in Propositions 4.3.1 and 4.3.4.

In Section 4.4, we prove the log-type stability estimate, Theorem 4.0.1.

Finally, in Section 4.5, we provide a stability estimate based on an ad hoc misfit functional. This provides an alternative perspective on the stability of the problem.

We begin by setting out the initial assumptions for the domain, the coefficients, and the boundary conditions.

Assumptions about the domain

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with $\partial\Omega$ of the Lipschitz class with constants r_0 and L. For a real number $N > 0$, we assume that the Lebesgue measure of Ω is bounded by $N \cdot r_0^n$.

Let $D \subset \Omega$ be an open set. We say that D is an *inclusion* of Ω if the following conditions hold:

$$D \Subset \Omega \quad \text{and} \quad \text{dist}(D, \partial\Omega) \geq \delta_0 > 0, \tag{4.1}$$

$$\partial D \text{ is of class } C^2 \text{ with constants } r_0,\ L, \tag{4.2}$$

$$\Omega \setminus \overline{D} \text{ is connected.} \tag{4.3}$$

Assumptions about the coefficients

Consider the generalised Schrödinger equation

$$\text{div}(\sigma\, \nabla u) + q\, u = 0 \quad \text{in } \Omega. \tag{4.4}$$

The coefficient $\sigma \in L^\infty(\Omega, Sym_n)$ is a real $n \times n$ symmetric matrix function with the following structure:

$$\sigma(x) := (a_b(x) + (a_D(x) - a_b(x))\chi_D(x))\, A(x). \tag{4.5}$$

Here, a_b and a_D are scalar functions in $C^{0,1}(\overline{\Omega})$. Additionally, there exist constants $\bar{\gamma} > 1$ and $\eta_0 > 0$ such that

$$\bar{\gamma}^{-1} \leq a_b(x), a_D(x) \leq \bar{\gamma}, \quad \text{for } x \in \Omega, \tag{4.6}$$

$$(a_D(x) - a_b(x))^2 \geq \eta_0^2 > 0, \quad \text{for } x \in \Omega. \tag{4.7}$$

The real $n \times n$ matrix-valued function $A(x)$ is a symmetric Lipschitz continuous function satisfying $\|A\|_{C^{0,1}(\Omega)} \leq \bar{A}$ for some $\bar{A} > 0$. The matrix-valued function σ also satisfies the uniform ellipticity condition, i.e., there exists a constant $\bar{\lambda} > 1$ such that

$$\bar{\lambda}^{-1}|\xi|^2 \leq \sigma(x)\xi \cdot \xi \leq \bar{\lambda}|\xi|^2, \quad \text{for a.e. } x \in \Omega, \text{ for every } \xi \in \mathbb{R}^n. \tag{4.8}$$

The scalar function q is defined by the formula

$$q(x) := q_b(x) + (q_D(x) - q_b(x))\chi_D(x), \tag{4.9}$$

where q_b and q_D are functions in $L^\infty(\overline{\Omega})$. Furthermore, there exists a constant $\bar{q} > 0$ such that $\|q\|_{L^\infty(\Omega)} \leq \bar{q}$.

The set $\{n, N, r_0, L, \bar{A}, \bar{\gamma}, \bar{\lambda}, \delta_0, \bar{q}\}$ is called the *a-priori* data.

The local Cauchy data

To formally define the local Cauchy data set on the boundary Σ, we first introduce the definition of Cauchy data. Cauchy data refers to a set of boundary conditions associated with an inclusion, namely a collection of pairs $\{u|_{\partial\Omega}, \sigma\nabla u \cdot \nu|_{\partial\Omega}\}$.

Let $D \subset \Omega$ be an inclusion. First, we give the general definition of Cauchy data.

Definition 4.0.1. *The* Cauchy data *on Σ associated with the inclusion D whose first component vanishes on $\partial\Omega \setminus \overline{\Sigma}$ is defined as the set*

$$\begin{aligned}\mathcal{C}_D^\Sigma(\Omega) = \Big\{(f,g) \in H_{00}^{1/2}(\Sigma) \times H^{-1/2}(\partial\Omega) \ :\ \exists\, u \in H^1(\Omega) \text{ weak solution of} \\ \operatorname{div}(\sigma\nabla u) + qu = 0 \qquad \text{in } \Omega, \\ u|_{\partial\Omega} = f, \qquad \sigma\nabla u \cdot \nu|_{\partial\Omega} = g\Big\}.\end{aligned} \tag{4.10}$$

For a characterisation of the trace spaces $H_{00}^{1/2}(\Sigma)$ and $H^{-1/2}(\partial\Omega)$, see the Appendix.

We will now introduce some trace spaces that will be useful in defining local Cauchy data. Consider the trace space of functions with compact support in $\partial\Omega \setminus \overline{\Sigma}$:

$$H_{00}^{-1/2}(\partial\Omega \setminus \overline{\Sigma}) := \left\{\psi \in H^{-1/2}(\partial\Omega) \ :\ \langle\psi, \varphi\rangle = 0, \quad \forall\, \varphi \in H_{00}^{1/2}(\Sigma)\right\},$$

where $\langle\cdot,\cdot\rangle$ represents the dual pairing between the complex-valued trace spaces $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ based on the $L^2(\partial\Omega)$ inner product

$$\langle\psi, \varphi\rangle = \int_{\partial\Omega} \psi\overline{\varphi}.$$

We define $H^{1/2}(\partial\Omega)|_\Sigma$ and $H^{-1/2}(\partial\Omega)|_\Sigma$ as the restrictions to Σ of the trace spaces $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$, respectively. These trace spaces can be defined equivalently as quotient spaces. Indeed, consider the equivalence relation:

$$\varphi_1 \sim \varphi_2 \qquad \Longleftrightarrow \qquad \varphi_1 - \varphi_2 \in H_{00}^{1/2}(\partial\Omega \setminus \overline{\Sigma}).$$

Then

$$H^{1/2}(\partial\Omega)|_\Sigma = H^{1/2}(\partial\Omega)/\sim\, = H^{1/2}(\partial\Omega)/H_{00}^{1/2}(\partial\Omega \setminus \overline{\Sigma}).$$

and, similarly,

$$H^{-1/2}(\partial\Omega)|_\Sigma = H^{-1/2}(\partial\Omega)/H_{00}^{-1/2}(\partial\Omega \setminus \overline{\Sigma}).$$

Definition 4.0.2. *The* local Cauchy data *on Σ associated with the inclusion D, with* *the first component vanishing at $\partial\Omega \setminus \overline{\Sigma}$, is defined as*

$$\mathcal{C}_D^{\Sigma}(\Sigma) = \Big\{(f,g) \in H_{00}^{1/2}(\Sigma) \times H^{-1/2}(\partial\Omega)|_{\Sigma} \;:\; \exists\, u \in H^1(\Omega) \text{ weak solution of}$$
$$\operatorname{div}(\sigma\,\nabla u) + q\,u = 0 \qquad \text{in } \Omega,$$
$$u|_{\partial\Omega} = f,$$
$$\langle \sigma\nabla u\cdot\nu|_{\partial\Omega}, \varphi\rangle = \langle g, \varphi\rangle \qquad \text{for all } \varphi \in H_{00}^{1/2}(\Sigma)\Big\}.$$

It is important to note that $\mathcal{C}_D^{\Sigma}(\Sigma)$ is a subspace of the product space

$$\mathcal{H} := H_{00}^{1/2}(\Sigma) \times H^{-1/2}(\partial\Omega)|_{\Sigma}, \tag{4.11}$$

which is a Hilbert space with norm

$$\|(f,g)\|_{\mathcal{H}} = \Big(\|f\|^2_{H_{00}^{1/2}(\Sigma)} + \|g\|^2_{H^{-1/2}(\partial\Omega)|_{\Sigma}}\Big)^{1/2} \qquad \text{for each } (f,g) \in \mathcal{H}. \tag{4.12}$$

We denote by $\mathcal{C}_i$ the local Cauchy data associated with the inclusion D_i, for $i = 1, 2$. To compare two local Cauchy data, we use the definition of the distance between closed subspaces of a Hilbert space. Given $\mathcal{F}$ and $\mathcal{G}$, two subspaces of a Hilbert space, the *distance* or *aperture* between them is given by the following formula:

$$d(\mathcal{F},\mathcal{G}) = \max\Big\{\sup_{h\in\mathcal{G},h\neq 0}\inf_{k\in\mathcal{F}}\frac{\|h-k\|}{\|h\|}, \sup_{k\in\mathcal{F},k\neq 0}\inf_{h\in\mathcal{G}}\frac{\|h-k\|}{\|k\|}\Big\}. \tag{4.13}$$

For more properties and applications, see Kato's book [80]. To simplify the calculation of the aperture, in [81, Corollary 2.13] we can find the following property: if $d(\mathcal{F},\mathcal{G}) < 1$, then the two quantities in (4.13) coincide. In our context, the distance between two local Cauchy data $\mathcal{C}_1$ and $\mathcal{C}_2$ is considered smaller than 1, then we can assume that it has the form

$$d(\mathcal{C}_1,\mathcal{C}_2) = \sup_{(f_2,g_2)\in\mathcal{C}_2\setminus\{(0,0)\}}\inf_{(f_1,g_1)\in\mathcal{C}_1}\frac{\|(f_1,g_1)-(f_2,g_2)\|_{\mathcal{H}}}{\|(f_2,g_2)\|_{\mathcal{H}}}. \tag{4.14}$$

Now we will state the main theorem of this chapter.

Theorem 4.0.1. *Let $\Omega \subset \mathbb{R}^n$, Σ and D_1, D_2 be a bounded domain, a non-empty portion of $\partial\Omega$ and two inclusions of Ω satisfying the a-priori assumptions. Let σ_1, σ_2 be the anisotropic coefficients of the form* (4.5) *and let q_1, q_2 be the coefficients of the zero-order term of the form* (4.9)*. Let $\mathcal{C}_1$, $\mathcal{C}_2$ be the local Cauchy data corresponding to the inclusions D_1, D_2, respectively. If $d(\mathcal{C}_1,\mathcal{C}_2)$ is less than a given positive constant $\varepsilon \in (0,1)$, then*

$$\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2) \leq C\omega(\varepsilon), \tag{4.15}$$

where C is a positive constant that depends only on the a-priori data, and where $\omega : [0,+\infty) \to [0,+\infty)$ is a non-decreasing, positive function such that

$$\omega(t) \leq |\ln t|^{-\eta} \quad \text{for } t \in (0,1), \quad \text{and} \quad \omega(t) \to 0 \quad \text{as} \quad t \to 0^+, \tag{4.16}$$

where η is a positive constant.

4.1 Geometrical lemmas

We begin with some geometrical remarks. The proof of Theorem 4.0.1 relies on a propagation of smallness argument, based on the three sphere inequality applied along a chain of spheres contained in a connected domain. A potential problem arises when the point at which the Hausdorff distance is reached is contained in the portion of the boundary of the two inclusions that cannot be reached from the portion Σ without crossing those boundaries. Let us be more precise.

We denote by $\mathcal{G}$ the connected component of $\Omega \setminus \overline{(D_1 \cup D_2)}$. We set

$$\Omega_D := \Omega \setminus \overline{\mathcal{G}}.$$

The obstruction highlighted above reads as follows: the value $\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2)$ could be reached at a point that does not belong to $\overline{\mathcal{G}}$ and is therefore inaccessible from the outside. To deal with this problem, Alessandrini and Di Cristo [21] have introduced a quantity called the *modified distance*.

Definition 4.1.1. *The* modified distance *between two subsets D_1 and D_2 of $\mathbb{R}^n$ is defined as*

$$\mathrm{d}_{\mu}(D_1, D_2) = \max\left\{ \max_{x \in \partial D_1 \cap \partial \Omega_D} \mathrm{dist}(x, D_2), \max_{x \in \partial D_2 \cap \partial \Omega_D} \mathrm{dist}(x, D_1) \right\}. \tag{4.17}$$

In general, the modified distance is not a true metric, and it does not bound from above the Hausdorff distance. However, Alessandrini and Di Cristo provide a lemma that guarantees that under our *a-priori* information d_μ dominates $\mathrm{d}_{\mathcal{H}}$.

Lemma 4.1.1 (Alessandrini and Di Cristo 2005)**.** *Let Ω, D_1, D_2 be, respectively, a bounded domain and two inclusions satisfying the a-priori assumptions. There is a positive constant c_0, which depends only on the a-priori data, such that*

$$\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2) \leq c_0 \, \mathrm{d}_{\mu}(D_1, D_2). \tag{4.18}$$

Proof. For a proof, we refer to [21, Proposition 3.3]. □

On the other hand, it is easy to prove that $\mathrm{d}_{\mu}(D_1, D_2) \leq \mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2)$ (see [11]), so that these two quantities are comparable.

Now, let us assume that there is a point on $\partial D_1 \cap \partial \Omega_D$ that realises the modified distance. In order to apply the quantitative estimates for unique continuation, based on an iterated application of the three-sphere inequality, we need to control the radii of the spheres involved. To avoid the cases where points of $\partial \Omega_D$ are not reachable by such a chain of balls, we find convenient to apply the ideas first presented in the paper by Alessandrini and Sincich [20] in the context of crack detection, and then applied by Alessandrini, Di Cristo, Morassi and Rosset [11] in the elasticity case. Before presenting the procedure, we introduce some notation.

Let O denote the origin in $\mathbb{R}^n$, ν be a unit vector, h a positive constant and $\theta \in (0, \frac{\pi}{2})$. The closed truncated cone with vertex at O, axis along the direction ν, height h and aperture 2θ is denoted by $C(O, \nu, h, \theta)$ and is given by

$$C(O, \nu, h, \theta) = \{x \in \mathbb{R}^n \, : \, |x - (x \cdot \nu)\nu| \leq |x| \, \sin\theta, \, 0 \leq x \cdot \nu \leq h\}. \tag{4.19}$$

Given d, R such that $0 < R < d$ and $e_n = (0, \dots, 0, 1)$ the n-th vector of the standard basis on $\mathbb{R}^n$, consider the closed truncated cone

$$C\left(O, -e_n, \frac{d^2 - R^2}{d}, \arcsin \frac{R}{d}\right), \tag{4.20}$$

whose oblique sides are tangent to the sphere $\partial B_R(O)$.

Let $P \in \partial D_1 \cap \partial \Omega_D$ and let ν be the exterior unit normal to ∂D_1 at P. For a suitable $d > 0$, let $[P + d\,\nu, P]$ be the segment contained in $\mathbb{R}^n \setminus \overline{\Omega_D}$ for some $d > 0$. For $P_0 \in \mathbb{R}^n \setminus \overline{\Omega}$, let $\gamma : [0, 1] \to \mathbb{R}^n \setminus \overline{\Omega_D}$ be the path such that $\gamma(0) = P_0$ and $\gamma(1) = P + d\,\nu$. We define the tubular neighbourhood of γ attached to the truncated cone (4.20) with vertex at P and axis along ν as

$$V(\gamma) = \left[\bigcup_{S \in \gamma} B_R(S)\right] \cup C\left(P, \nu, \frac{d^2 - R^2}{d}, \arcsin \frac{R}{d}\right). \tag{4.21}$$

Note that the tubular neighbourhood $V(\gamma)$ depends on the parameters d and R, which will be specified later, as well as on the curve γ. The following lemma guarantees the application of the three-sphere inequality along the tubular neighbourhood contained in $\mathbb{R}^n \setminus \overline{\Omega_D}$. As in [11], we would like to remark that we will use a reference frame where $P = O$ and $\nu(P) = \nu(O) = \nu = -e_n$. The following geometrical Lemma corresponds to [50, Lemma 2.7]

Lemma 4.1.2 (Alessandrini-Di Cristo-Morassi-Rosset, 2014)**.** *There exist positive constants $\bar{d}$ and $\tilde{c}_1$ depending on L and r_0, and there exists a point $P \in \partial D_1$ satisfying*

$$\tilde{c}_1 \mathrm{d}_\mu \leq \mathrm{dist}(P, D_2), \tag{4.22}$$

and such that, for any $\bar{P} \in B_{r_0/16}(P_0)$, where $B_{r_0/16}(P_0) \Subset \mathbb{R}^n \setminus \overline{\Omega_D}$, there exists a path $\gamma \subset \mathbb{R}^n \setminus \overline{\Omega_D}$ joining $\bar{P}$ to $P + \bar{d}\,\nu$ where ν is the exterior unit normal to ∂D_1 at P such that, if we are in a coordinate system where $P \equiv O$ and $\nu = -e_n$, it holds that

$$V(\gamma) \subset \mathbb{R}^n \setminus \Omega_D,$$

where $V(\gamma)$ is the tubular neighbourhood introduced in (4.21) with parameters $R = \frac{\bar{d}}{\sqrt{1+L_0^2}}$ and $L_0 > 0$, which depends only on L.

The proof of Lemma 4.1.2 can be found in [50, Lemma 2.7] and relies on the applications of two related results. The first result, stated as Lemma 5.5 in [19], establishes the connectedness of a set obtained by shrinking a bounded domain $U \subset \mathbb{R}^n$ with the Lipschitz boundary. The second result, stated as Theorem 3.6 in [9], establishes the existence of positive constants d_0, ρ_0, L_0, with $L_0 \leq L$, with $\frac{d_0}{r_0}, \frac{\rho_0}{r_0}$ only depending on L and L_0 only depending on L, such that if

$$\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2) \leq d_0,$$

then $\partial \Omega_D$ is Lipschitz with constants r_0 and L_0.

4.2 Singular solutions

In this section, we address the construction of the Green function associated with equation (4.4) under mixed boundary conditions and we define the notion of a singular solution. It is important to note that without specifying any spectral condition or imposing additional assumptions, equation (4.4) with homogeneous boundary conditions does not have a unique solution. As a result, the Green functions associated with this boundary value problem, which are essential for defining singular solutions, are not defined. To overcome this issue, we introduce suitable mixed boundary conditions, building on the ideas presented in [17].

Fix a point $P_\Sigma \in \Sigma$. Up to a rigid transformation, we can assume that P_Σ coincides with the origin. Let D_0 denote the bounded domain

$$D_0 = \left\{x \in (\mathbb{R}^n \setminus \overline{\Omega}) \cap B_{r_0} \,:\, |x_i| < r_0,\, i = 1, \dots, n-1,\, -r_0 < x_n < 0\right\}$$

such that $\partial D_0 \cap \partial \Omega \Subset \Sigma$. We define Ω_0 as the augmented domain given by

$$\Omega_0 := \text{Int}_{\mathbb{R}^n}(\overline{\Omega \cup D_0}).$$

It can be shown that the boundary $\partial \Omega_0$ is of Lipschitz class with constants r_0 and $\tilde{L}$, where $\tilde{L}$ depends only on L. Let $\Sigma_0 \subset \partial D_0$ be the non-empty flat portion of the form

$$\Sigma_0 = \left\{x \in \Omega_0 \,:\, |x_i| \leq r_0,\, i = 1, \dots, n-1,\, x_n = -r_0\right\}$$

that is contained in $\partial \Omega_0 \setminus \partial \Omega$.

We consider two inclusions contained in the domain Ω, denoted by D_1 and D_2, which satisfy the given assumptions. The coefficients σ_1 and σ_2, q_1 and q_2 associated with D_1 and D_2 are extended to the augmented domain Ω_0 by setting their value equal to the identity matrix and the identity function on D_0, respectively. We use the same symbols to denote the extended coefficients.

In the following lemma, we provide the existence of the Green function associated with a Cauchy problem with mixed boundary data.

Lemma 4.2.1. *For any $\sigma \in L^\infty(\Omega_0, Sym_n)$ that satisfies the uniform ellipticity condition and any $q \in L^\infty(\Omega_0)$, there exists a unique distributional solution $G(\cdot, y)$ of the boundary value problem*

$$\begin{cases} \text{div}(\sigma\, \nabla G(\cdot, y)) + q\, G(\cdot, y) = -\delta(\cdot - y) & \text{in} \quad \Omega_0, \\ G(\cdot, y) = 0 & \text{on} \quad \partial \Omega_0 \setminus \Sigma_0, \\ \sigma\, \nabla G(\cdot, y) \cdot \nu + i\, G(\cdot, y) = 0 & \text{on} \quad \Sigma_0. \end{cases} \tag{4.23}$$

Here, $\delta(\cdot - y)$ is the Dirac distribution centred at y and ν is the exterior unit normal at Σ_0. Moreover, for any $x, y \in \Omega_0$ with $x \neq y$,

$$G(x, y) = G(y, x), \tag{4.24}$$

and there exists a positive constant C depending on the a-priori data such that

$$0 < |G(x,y)| \leq C|x-y|^{2-n}. \tag{4.25}$$

Proof. Our proof is based on the reasoning introduced in [17, Proposition 3.1]. We find it more convenient to divide the proof into three steps: in the first step, we prove the well-posedness of the problem (4.23); in the second step, we construct the Green function; and in the final step, we prove the symmetry of the Green function.

First step (Well-posedness). We consider the mixed boundary value problem for $f \in L^2(\Omega_0)$ given by

$$\begin{cases} \operatorname{div}(\sigma\nabla v) + q\, v = f & \text{in} \quad \Omega_0, \\ v = 0 & \text{on} \quad \partial\Omega_0 \setminus \Sigma_0, \\ \sigma\, \nabla v \cdot \nu + i\, v = 0 & \text{on} \quad \Sigma_0. \end{cases} \tag{4.26}$$

Our goal is to prove the existence and uniqueness for (4.26). We consider the adjoint mixed boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla w) + q\, w = f & \text{in} \quad \Omega_0, \\ w = 0 & \text{on} \quad \partial\Omega_0 \setminus \Sigma_0, \\ \sigma\, \nabla w \cdot \nu - i\, w = 0 & \text{on} \quad \Sigma_0. \end{cases} \tag{4.27}$$

By applying the Fredholm alternative, it can be concluded that there exists a solution of (4.26) if and only if uniqueness holds for (4.27), and vice versa (see Evans [56, Theorem 4, §6.2]). Therefore, we find it convenient to prove uniqueness for both boundary value problems. Consider the homogeneous problems

$$\begin{cases} \operatorname{div}(\sigma\nabla u) + q\, u = 0 & \text{in} \quad \Omega_0, \\ u = 0 & \text{on} \quad \partial\Omega_0 \setminus \Sigma_0, \\ \sigma\, \nabla u \cdot \nu \pm i\, u = 0 & \text{on} \quad \Sigma_0. \end{cases} \tag{4.28}$$

By the weak formulation of (4.28), using $\overline{u}$ as test function, it follows that

$$\int_{\Omega_0} \sigma(x)\nabla u(x) \cdot \overline{\nabla u(x)}\, \mathrm{d}x - \int_{\Omega_0} q(x)|u(x)|^2\, \mathrm{d}x \pm i \int_{\Sigma_0} |u(x)|^2\, \mathrm{d}x = 0. \tag{4.29}$$

From (4.29), we can conclude that $u = 0$ on Σ_0, therefore $\partial_\nu u = 0$ on Σ_0. From the uniqueness of the Cauchy problem, it follows that that $u \equiv 0$ in Ω_0. Therefore, there exists a unique solution of (4.26). The next step is to prove stability. Let $v \in H^1(\Omega_0)$ be the weak solution of (4.26). By the weak formulation of (4.26), the following identities hold:

$$\int_{\Sigma_0} |v|^2 = -\Im\Big(\int_{\Omega_0} f\overline{v}\Big), \tag{4.30}$$

$$\int_{\Omega_0} \sigma(x)\, \nabla v(x) \cdot \overline{\nabla v(x)}\, \mathrm{d}x = -\Re\Big(\int_{\Omega_0} f\, \overline{v}\Big) + \int_{\Omega_0} q(x)|v(x)|^2\, \mathrm{d}x. \tag{4.31}$$

Define the following quantities:

$$\varepsilon^2 = \int_{\Sigma_0} |v|^2 + i \int_{\Sigma_0} \sigma(x)\,\nabla v(x)\cdot\nu(x)\,\overline{v(x)}\,\mathrm{d}x,$$

and

$$\eta = \|f\|_{L^2(\Omega_0)}, \qquad \delta = \|v\|_{L^2(\Omega_0)}, \qquad E = \|\nabla v\|_{L^2(\Omega_0)}.$$

From the Schwarz inequality and (4.30), it follows that

$$\int_{\Sigma_0} |v|^2 \le \eta\,\delta,$$

and combined with the impedance condition, we derive

$$\varepsilon^2 \le 2\eta\,\delta. \tag{4.32}$$

From (4.31) and the uniform ellipticity condition, we derive

$$E^2 \le \eta\,\delta + \|q\|^2_{L^\infty(\Omega_0)}\delta^2. \tag{4.33}$$

Now, we prove that there exists a positive constant, denoted by C, which depends on the *a-priori* data, such that the following inequality holds:

$$E^2 \le C\eta^2. \tag{4.34}$$

We can divide the proof of this claim into two cases:

- If $\delta^2 \le \eta^2$, then from (4.33) we can see that $E^2 \le (1+\|q\|^2_{L^\infty(\Omega_0)})\,\eta^2$, which implies the claim.

- If $\delta^2 \ge \eta^2$, we can rely on a quantitative estimate of unique continuation provided by Carstea and Wang [47, Theorem 5.3], which gives us the following inequality:

$$\delta^2 \le \Big(E^2+\varepsilon^2+\eta^2\Big)\;\omega\Big(\frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2}\Big).$$

  Here, $\omega(t)$ is a non-decreasing function satisfying $\omega(t) \le C|\ln t|^{-\mu}$ for $t\in(0,1)$, $\lim_{t\to 0^+}\omega(t)=0$, and C and $\mu\in(0,1)$ are positive constants. By using (4.32) and (4.33), we can obtain the following inequalities:

$$\begin{aligned}\delta^2 &\le \Big(\eta\delta + \|q\|_{L^\infty(\Omega_0)}\delta^2 + 2\eta\delta + \eta^2\Big)\;\omega\Big(\frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2}\Big)\\ &\le \Big(4\delta^2 + \|q\|_{L^\infty(\Omega_0)}\delta^2\Big)\;\omega\Big(\frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2}\Big).\end{aligned}$$

  By multiplying both sides by δ^2, we obtain the inequality

$$1 \le (4+\|q\|_{L^\infty(\Omega_0)})\;\omega\Big(\frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2}\Big),$$

and inverting with respect to ω leads to

$$\omega^{-1}\Big(\frac{1}{4+\|q\|_{L^\infty(\Omega_0)}}\Big) \leq \frac{\varepsilon^2+\eta^2}{E^2+\varepsilon^2+\eta^2},$$

which implies

$$\omega^{-1}\Big(\frac{1}{4+\|q\|_{L^\infty(\Omega_0)}}\Big)(E^2+\varepsilon^2+\eta^2) \leq \varepsilon^2+\eta^2.$$

Setting $C=\omega^{-1}\Big(\frac{1}{4+\|q\|_{L^\infty(\Omega_0)}}\Big)$, we have

$$CE^2 \leq C(E^2+\varepsilon^2+\eta^2) \leq \varepsilon^2+\eta^2 \leq 2\eta\delta+\eta^2 \leq 3\eta^2,$$

which proves the claim.

To summarise, by using (4.32) and (4.34), we can conclude that

$$\|v\|_{H^1(\Omega_0)} \leq C\|f\|_{L^2(\Omega_0)}.$$

Second step (Construction of the Green function). Consider $y \in \Omega_0$. We define $\tilde{G}(\cdot,y)$ as a Green function solution for the boundary value problem

$$\begin{cases} \mathrm{div}(\sigma\nabla\tilde{G}(\cdot,y)) = -\delta(\cdot-y) & \text{in} \quad \Omega_0, \\ \tilde{G}(\cdot,y)=0 & \text{on} \quad \partial\,\Omega_0\setminus\Sigma_0, \\ \sigma\,\nabla\tilde{G}(\cdot,y)\cdot\nu + i\,\tilde{G}(\cdot,y)=0 & \text{on} \quad \Sigma_0. \end{cases} \tag{4.35}$$

We note that $\tilde{G}(\cdot,y)$ satisfies the symmetry property $\tilde{G}(x,y)=\tilde{G}(y,x)$ and the boundedness property $|\tilde{G}(x,y)| \leq C|x-y|^{2-n}$ for any $x\neq y, x,y\in\Omega_0$, as shown in [95].

We then define the distribution $R_0(x,y)$ as $R_0(x,y)=\tilde{G}(x,y)$. For $j=1,\dots,J$, with $J=\left\lfloor \frac{n-1}{2}\right\rfloor$, we define

$$R_j(x,y)=\int_{\Omega_0} q(z)\tilde{G}(x,z)R_{j-1}(z,y)\,\mathrm{d}z.$$

The distribution $R_j(\cdot,y)$ is a weak solution of the boundary value problem

$$\begin{cases} \mathrm{div}(\sigma\,\nabla R_j(\cdot,y)) = -q\,R_{j-1}(\cdot,y) & \text{in } \Omega_0, \\ R_j(\cdot,y)=0 & \text{on } \partial\,\Omega_0\setminus\Sigma_0, \\ \sigma\,\nabla R_j(\cdot,y)\cdot\nu + i\,R_j(\cdot,y)=0 & \text{on } \Sigma_0, \end{cases}$$

and it satisfies the estimate $|R_j(x,y)| \leq C\,|x-y|^{2j+2-n}$ for every $j=0,\dots,J-1$ (see [99, Chapter 2]).

When $j=J$, the estimate for $|R_J(x,y)|$ depends on the parity of n: for n even, $|R_J(x,y)| \leq C\,(|\ln|x-y||+1)$, for n odd, $|R_J(x,y)|\leq C$. In both cases, C is a positive constant that depends only on the *a-priori* data. Furthermore, in either case, $\|R_J(\cdot,y)\|_{L^p(\Omega_0)} \leq C$, for $1\leq p<\infty$.

Finally, we define $R_{J+1}(\cdot, y)$ as the weak solution of the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\,\nabla R_{J+1}(\cdot,y)) + q\,R_{J+1}(\cdot,y) = -q\,R_J(\cdot,y) & \text{in } \Omega_0,\\ R_{J+1}(\cdot,y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0,\\ \sigma\,\nabla R_{J+1}(\cdot,y)\cdot\nu + i\,R_{J+1}(\cdot,y) = 0 & \text{on } \Sigma_0. \end{cases}$$

It can be shown that $\|R_{J+1}(\cdot, y)\|_{H^1(\Omega_0)} \le C$ for some positive constant C, and $|R_{J+1}(x, y)| \le C$ for $x \neq y,\ x, y \in \Omega_0$, based on interior regularity estimates.

Finally, we define the Green function $G(x, y)$ as the sum of the modified Green function $\tilde{G}(x, y)$ and the sum of the terms $R_j(x, y)$ from $j = 1, \dots, J + 1$:

$$G(x,y) = \tilde{G}(x,y) + \sum_{j=1}^{J+1} R_j(x,y). \tag{4.36}$$

It can be easily verified that for $y \in \Omega_0$, $G(\cdot, y)$ is a distributional solution of the boundary value problem (4.23), thus confirming that G is the desired Green function.

Third step (Symmetry of the Green function). Let $f, g \in C_c^\infty(\Omega_0)$ and define the functions

$$u(x) := \int_{\Omega_0} G(x,y) f(y)\,\mathrm{d}y, \qquad v(x) := \int_{\Omega_0} G(x,y) g(y)\,\mathrm{d}y. \tag{4.37}$$

Then $u, v \in H_0^1(\Omega_0)$ and they satisfy the following equations:

$$\begin{aligned} \operatorname{div}(\sigma\nabla u) + q\,u = f \qquad & \text{in } \Omega_0,\\ \operatorname{div}(\sigma\nabla v) + q\,v = g \qquad & \text{in } \Omega_0. \end{aligned} \tag{4.38}$$

Here, $G(\cdot, y)$ is the Green function solution of (4.23). By the Green's identity, it follows that

$$\int_{\Omega_0} u(x)\,g(x)\,\mathrm{d}x = \int_{\Omega_0} f(x)\,v(x)\,\mathrm{d}x.$$

Hence,

$$\int_{\Omega_0} \left[\int_{\Omega_0} G(x,y) f(y)\,\mathrm{d}y\right] g(x)\,\mathrm{d}x = \int_{\Omega_0} \left[\int_{\Omega_0} G(y,x) g(x)\,\mathrm{d}x\right] f(y)\,\mathrm{d}y, \tag{4.39}$$

Now, by applying Fubini's theorem, we can interchange the order of integration of f and g, and we conclude that $G(x, y) = G(y, x)$ for any $x, y \in \Omega_0$. □

Consider two Green functions G_1 and G_2 associated with the inclusions D_1 and D_2, respectively. Let $G_k(\cdot, y)$ be the solution of the boundary value problem (4.23) with $k = 1$ or 2. By multiplying the equation in (4.23) by $G_j(\cdot, y)$ for $j \neq k$ and integrating by parts on Ω, for $y, z \in D_0$, we obtain the following identity:

$$\begin{aligned} &\int_\Sigma \Big[\sigma_1(x)\nabla G_1(x,y)\cdot\nu(x)\,G_2(x,z) - \sigma_2(x)\nabla G_2(x,z)\cdot\nu(x)\,G_1(x,y)\Big]\,\mathrm{d}S(x)\\ &= \int_\Omega (\sigma_1(x) - \sigma_2(x))\nabla G_1(x,y)\cdot\nabla G_2(x,z)\,\mathrm{d}x + \int_\Omega (q_2(x) - q_1(x))\,G_1(x,y)\,G_2(x,z)\,\mathrm{d}x. \end{aligned} \tag{4.40}$$

For $y, z \in \mathcal{G}$, define

$$S_1(y,z) := \int_{D_1} (a_{D_1}(x) - a_b(x))A(x)\nabla G_1(x,y) \cdot \nabla G_2(x,z)\,\mathrm{d}x - \int_{D_1} (q_{D_1}(x) - q_b(x))\,G_1(x,y)\,G_2(x,z)\,\mathrm{d}x,$$

$$S_2(y,z) := \int_{D_2} (a_{D_2}(x) - a_b(x))A(x)\nabla G_1(x,y) \cdot \nabla G_2(x,z)\,\mathrm{d}x - \int_{D_2} (q_{D_2}(x) - q_b(x))G_1(x,y)\,G_2(x,z)\,\mathrm{d}x,$$

and

$$f(y,z) := S_1(y,z) - S_2(y,z). \tag{4.41}$$

For $y, z \in \Omega_0 \setminus \overline{\Omega}$, by (4.40) and (4.41), it turns out that

$$f(y,z) = \int_{\Sigma} a_b(x)\Big[A(x)\nabla G_1(x,y)\cdot\nu(x)G_2(x,z) - A(x)\nabla G_2(x,z)\cdot\nu(x)G_1(x,y)\Big]\,\mathrm{d}S(x).$$

Moreover, for $y, z \in \Omega_0 \setminus \overline{\Omega_D}$, $f(y,\cdot), f(\cdot,z) \in H^1_{loc}(\Omega_0 \setminus \overline{\Omega_D})$ and are weak solutions, respectively, to

$$\mathrm{div}_z(a_b\,A(\cdot)\nabla_z f(y,\cdot)) + q_b\,f(y,\cdot) = 0 \quad \text{in } \Omega_0 \setminus \overline{\Omega_D},$$
$$\mathrm{div}_y(a_b\,A(\cdot)\nabla_y f(\cdot,z)) + q_b\,f(\cdot,z) = 0 \quad \text{in } \Omega_0 \setminus \overline{\Omega_D}.$$

4.3 Upper and lower bounds of the Singular solutions

In this Section, we prove an upper bound (Proposition 4.3.1) and a lower bound (Proposition 4.3.4) of the singular solution f in terms of the local Cauchy data $d(\mathcal{C}_1, \mathcal{C}_2)$ and the geometrical quantities related with the problem.

Upper bound

In the following Proposition, we provide a quantitative estimate of propagation of smallness for the singular integral f. We begin by considering the upper bound of f in D_0 in terms of the local Cauchy data, and then proceed to propagate the error estimate along a curve within the connected domain $\mathcal{G}$ up to a point close to the boundary $\partial\Omega_D$. This allows us to give an estimate of the blowing up behaviour of the function at that location.

Proposition 4.3.1. *Let D_1 and D_2 be two inclusions of Ω satisfying the a-priori assumptions. Let $\mathcal{C}_1, \mathcal{C}_2$ be the local Cauchy data associated with the inclusions D_1, D_2, respectively. Under the notation of Lemma 4.1.2, define*

$$y_h := P + h\,\nu(P),$$

where

$$0 < h \le h_1 := \bar{d}\left(1 - \frac{\sin\theta_0}{4}\right) \quad \text{and} \quad \theta_0 = \arctan\left(\frac{1}{L_0}\right), \tag{4.42}$$

and $\nu(P)$ *is the exterior unit normal of* ∂D_1 *at* P*. For* $\varepsilon \in (0,1)$*, if* $d(\mathcal{C}_1, \mathcal{C}_2) < \varepsilon$*, it follows that*

$$|f(y_h, y_h)| \le c_1 \frac{\varepsilon^{B} h^{F}}{h^{A}}, \tag{4.43}$$

where $A, B, F, c_1 > 0$ *are constants that depend on the a-priori data only.*

In the first step of the proof of Proposition 4.3.1, we make use of Alessandrini's type of identity, which we derive in the following Lemma.

Lemma 4.3.2. *Let* $u_j \in H^1(\Omega)$ *with* $j = 1, 2$ *be weak solutions to the Dirichlet problem*

$$\begin{cases} \mathrm{div}(\sigma_j \nabla u_j) + q_j\, u_j = 0 & \textit{in } \Omega, \\ u_j|_{\partial\Omega} \in H^{1/2}_{00}(\Sigma). \end{cases} \tag{4.44}$$

The following inequality holds:

$$\begin{aligned} &\int_\Omega (\sigma_2 - \sigma_1)\nabla u_1 \cdot \nabla u_2 + \int_\Omega (q_1 - q_2)\, u_1\, u_2 \\ &\quad \le d(\mathcal{C}_1, \mathcal{C}_2)\, \|(u_1, \sigma_1 \nabla u_1 \cdot \nu)\|_{\mathcal{H}}\, \|(u_2, \sigma_2 \overline{\nabla u_2} \cdot \nu)\|_{\mathcal{H}}, \end{aligned} \tag{4.45}$$

where $\mathcal{H}$ *is the trace space defined in* (4.11).

Proof. By the weak formulation of (4.44), it follows that:

$$\int_\Omega (\sigma_2 - \sigma_1)\nabla u_1 \cdot \nabla u_2 + \int_\Omega (q_1 - q_2) u_1 u_2 = \langle \sigma_2 \nabla \bar{u}_2 \cdot \nu|_{\partial\Omega}, u_1\rangle - \langle \sigma_1 \nabla u_1 \cdot \nu|_{\partial\Omega}, \overline{u}_2\rangle, \tag{4.46}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between $H^{1/2}_{00}(\Sigma)$ and $H^{1/2}(\partial\Omega)|_\Sigma$ based on the $L^2(\partial\Omega)$ inner product.

Let v_j for $j = 1, 2$ with $v_j \in H^1(\Omega)$ be weak solutions to $\mathrm{div}(\sigma_j \nabla\, v_j) + q_j\, v_j = 0$ in Ω.

The following identity holds:

$$\langle \sigma_j \nabla v_j \cdot \nu|_{\partial\Omega}, \overline{u_j}\rangle - \langle \sigma_j \overline{\nabla u_j} \cdot \nu|_{\partial\Omega}, v_j\rangle = 0, \qquad \text{for } j = 1, 2. \tag{4.47}$$

By adding (4.46) to (4.47) in the case $j = 2$, the following identity holds:

$$\begin{aligned} &\int_\Omega (\sigma_2 - \sigma_1)\nabla u_1 \cdot \nabla u_2 + \int_\Omega (q_1 - q_2) u_1 u_2 \\ &\quad = \langle \sigma_2 \overline{\nabla u_2} \cdot \nu|_{\partial\Omega}, (u_1 - v_2)\rangle - \langle \sigma_1 \nabla u_1 \cdot \nu|_{\partial\Omega} - \sigma_2 \nabla v_2 \cdot \nu|_{\partial\Omega}, \overline{u_2}\rangle. \end{aligned} \tag{4.48}$$

By using (4.48) and the Schwarz inequality, we obtain

$$\begin{aligned}
&\Big|\int_\Omega (\sigma_2-\sigma_1)\nabla u_1\cdot\nabla u_2+\int_\Omega (q_1-q_2)\,u_1\,u_2\Big|\\
&\quad\le \|\sigma_2\overline{\nabla u_2}\cdot\nu\|_{H^{-1/2}(\partial\Omega)|_\Sigma}\cdot\|u_1-v_2\|_{H^{1/2}_{00}(\Sigma)}\\
&\quad+\|\sigma_1\nabla u_1\cdot\nu-\sigma_2\nabla v_2\cdot\nu\|_{H^{-1/2}(\partial\Omega)|_\Sigma}\cdot\|\overline{u_2}\|_{H^{1/2}_{00}(\Sigma)}.
\end{aligned}$$

By simple calculations, we obtain

$$\begin{aligned}
&\Big|\int_\Omega (\sigma_2-\sigma_1)\nabla u_1\cdot\nabla u_2+\int_\Omega (q_1-q_2)\,u_1\,u_2\Big|\\
&\le \Big(\|\overline{u_2}\|_{H^{1/2}_{00}(\Sigma)}+\|\sigma_2\overline{\nabla u_2}\cdot\nu\|_{H^{-1/2}(\partial\Omega)|_\Sigma}\Big)\cdot\\
&\cdot\Big(\|u_1-v_2\|^2_{H^{1/2}_{00}(\Sigma)}+\|\sigma_1\nabla u_1\cdot\nu-\sigma_2\nabla v_2\cdot\nu\|^2_{H^{-1/2}(\partial\Omega)|_\Sigma}\Big)^{1/2}\\
&=\|(u_1-v_2,\sigma_1\nabla u_1\cdot\nu-\sigma_2\nabla v_2\cdot\nu)\|_{\mathcal{H}}\cdot\|(\overline{u_2},\sigma_2\overline{\nabla u_2}\cdot\nu)\|_{\mathcal{H}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big|\int_\Omega (\sigma_2-\sigma_1)\nabla u_1\cdot\nabla u_2+\int_\Omega (q_1-q_2)\,u_1\,u_2\Big|\\
&\quad\le\Big(\|u_1-v_2\|^2_{H^{1/2}_{00}(\Sigma)}+\|\sigma_1\nabla u_1\cdot\nu-\sigma_2\nabla v_2\cdot\nu\|^2_{H^{-1/2}(\partial\Omega)|_\Sigma}\Big)^{1/2}\|(u_2,\sigma_2\overline{\nabla u_2}\cdot\nu)\|_{\mathcal{H}}.
\end{aligned}\tag{4.49}$$

By multiplying and dividing the right-hand side of (4.49) by $\|(u_1,\sigma_1\nabla u_1\cdot\nu)\|_{\mathcal{H}}$ and taking the infimum over the pairs $(v_2,\sigma_2\nabla v_2\cdot\nu)\in\mathcal{C}_2$ and the supremum over the pairs $(u_1,\sigma_1\nabla u_1\cdot\nu)\in\mathcal{C}_1\backslash\{(0,0)\}$, we obtain the desired inequality. □

We will now introduce the asymptotic estimates for the gradient of the Green function, denoted as G, which will play a crucial role in proving Theorem 4.0.1. To begin, we state the following upper bound for the gradient of the Green function.

Proposition 4.3.3. *Let Ω and D be a bounded domain and an inclusion, respectively, satisfying the a priori assumptions. There exists a positive constant C_1 that depends only on the a-priori data such that the gradient of the Green function satisfies the following inequality:*

$$|\nabla_x G(x,y)|\le C_1|x-y|^{1-n}\qquad \text{for every } x,y\in\mathbb{R}^n. \tag{4.50}$$

Proof of Proposition 4.3.3. For a proof of (4.50), we refer to [21, Proposition 3.4]. □

Proof of Proposition 4.3.1. The proof of this result is based on the arguments used in [21, Proposition 3.3] and [11, Theorem 6.4]. To begin, we establish an upper bound for f in the fictitious domain D_0 in terms of the local Cauchy data by means of (4.45). We then propagate this estimate within the domain $\mathcal{G}$ near the point where the Hausdorff distance is attained. This process is carried out first for the second argument of f and then for the first argument of f.

Let $\bar{y} \in D_0$ be such that $\mathrm{dist}(\bar{y}, \partial\,\Omega) \geq \tilde{c}\, r_0$, where $0 < \tilde{c} < 1$ is a suitable constant. It turns out that $f(\bar{y}, \cdot)$ is a weak solution of

$$\mathrm{div}_w\, (a_b(\bar{w})\, A(\bar{w})\, \nabla_w f(\bar{y}, \bar{w})) + q_b(\bar{w})\, f(\bar{y}, \bar{w}) = 0 \qquad \text{for } \bar{w} \in \mathbb{R}^n \setminus \overline{\Omega_D} \tag{4.51}$$

Let $\bar{w} \in D_0$ be such that $\mathrm{dist}(\bar{w}, \partial\,\Omega) \geq \tilde{c}\, r_0$, for $0 < \tilde{c} < 1$. By equation (4.41), we have that

$$f(\bar{y}, \bar{w}) = \int_\Omega (\sigma_1 - \sigma_2) \nabla G_1(\cdot, \bar{y}) \cdot \nabla G_2(\cdot, \bar{w}) + \int_\Omega (q_2 - q_1)\, G_1(\cdot, \bar{y})\, G_2(\cdot, \bar{w}).$$

We can apply (4.45) with $u_1(x) = G_1(x, \bar{y})$ and $u_2(x) = G_2(x, \bar{w})$, and we have

$$|f(\bar{y}, \bar{w})| \leq d(\mathcal{C}_1, \mathcal{C}_2)\ \|(G_1(\cdot, \bar{y}), \sigma_1 \nabla G_1(\cdot, \bar{y}) \cdot \nu)\|_{\mathcal{H}}\ \|(\overline{G_2}(\cdot, \bar{w}), \sigma_2 \nabla \overline{G_2}(\cdot, \bar{w}) \cdot \nu)\|_{\mathcal{H}},$$

where $\mathcal{H}$ is defined in (4.11). By definition of $\mathcal{H}$ norm, by the Schwarz inequality, and by Proposition 4.3.3, it follows that

$$\|(G_1(\cdot, \bar{w}), \sigma_1 \nabla G_1(\cdot, \bar{w}) \cdot \nu)\|_{\mathcal{H}} = \left(\|G_1(\cdot, \bar{w})\|^2_{H^{1/2}_{00}(\Sigma)} + \|\sigma_1 \nabla G_1(\cdot, \bar{w}) \cdot \nu)\|^2_{H^{-1/2}(\partial\,\Omega)|_\Sigma} \right)^{1/2} \leq C,$$

where C is a positive constant depending on $|\Omega|$, n, $\bar{\gamma}$, $\bar{A}$, M_0, and r_0. A similar bound holds for $\|(\overline{G_2}(\cdot, \bar{y}), \sigma_2 \overline{\nabla G_2}(\cdot, \bar{y}) \cdot \nu)\|_{\mathcal{H}}$. Hence, we conclude that

$$|f(\bar{y}, \bar{w})| \leq C\, \varepsilon. \tag{4.52}$$

Now, let $\bar{w} \in \overline{[(\Omega)^{r_0} \cup \Omega_0]} \setminus \overline{\Omega_D}$, where $(\Omega)^{r_0} = \{x \in \mathbb{R}^n \,:\, \mathrm{dist}(x, \Omega) < r_0\}$, and let $\bar{y} \in D_0$ be as above. By Proposition 4.3.3 and since $|x - \bar{y}| \geq r_0$,

$$|f(\bar{y}, \bar{w})| \leq C \sum_{j=1}^{2} \int_{D_j} |x - \bar{y}|^{1-n}\, |x - \bar{w}|^{1-n}\, \mathrm{d}x \leq C \sum_{j=1}^{2} \int_{D_j} |x - \bar{w}|^{1-n}\, \mathrm{d}x.$$

Set $\tilde{R} = diam(\Omega) + r_0 \leq C\, r_0$, where C is a constant depending only on L and N. Then, $\Omega \subset B_{\tilde{R}}(\bar{w})$ and for $j = 1, 2$, there is some positive constant C such that

$$\int_{D_j} |x - \bar{w}|^{1-n}\, \mathrm{d}x \leq \int_{B_{\tilde{R}}(\bar{w})} |x - \bar{w}|^{1-n}\, \mathrm{d}x \leq C\tilde{R}.$$

Hence,

$$|f(\bar{y}, \bar{w})| \leq C,$$

where C depends on the *a-priori* data only.

Now, we consider the case when $w \in \mathcal{G}$. For $h > 0$, define

$$(\mathcal{G})^h = \left\{ x \in \mathbb{R}^n \setminus \overline{\Omega_D} \,:\, \mathrm{dist}(x, \partial\,\Omega_D) \geq h \right\}.$$

Later, we will specify the value of h. For $w \in (\mathcal{G})^h$, by Proposition 4.3.3,

$$|S_1(\bar{y}, w)| \leq C \int_{D_1} |x - \bar{y}|^{1-n} |x - w|^{1-n}\, \mathrm{d}x \leq C\, h^{1-n}.$$

Similarly, $|S_2(\bar{y}, w)| \le C\, h^{1-n}$, so that there is a positive constant C depending on the *a-priori* data only such that

$$|f(\bar{y}, w)| \le C\, h^{1-n}. \tag{4.53}$$

The next step is to propagate the smallness for the function f within $(\mathcal{G})^h$ with respect to the second variable near the boundary $\partial\Omega_D$. Let P be the point of Lemma 4.1.2 and assume that we are in a coordinate system where P coincides with the origin O, and set $y_h = O + h\,\nu(O)$ and $\nu(O) = -e_n$. Our goal is to propagate (4.52) inside $\mathcal{G}$ up to y_h.

To accomplish this, we consider $\bar{y}$ and $\bar{w} \in D_0$ such that $\mathrm{dist}(\bar{y}, \partial\Omega) \ge r_0$ and $\mathrm{dist}(\bar{w}, \partial\Omega) = \frac{3r_0}{4}$. By Lemma 4.1.2, we know that there exists a curve $\gamma \subset \overline{[(\Omega)^{r_0} \cup \Omega_0]} \setminus \overline{\Omega_D}$ that connects $\bar{w}$ to the point $Q = O + \bar{d}\nu(O)$. Moreover, we have $V(\gamma) \subset \mathbb{R}^n \setminus \Omega_D$ with parameters $R = \frac{\bar{d}}{\sqrt{1+L_0^2}}$ and $\theta_0 = \arcsin\frac{R}{d}$.

Observe that since $f(\bar{y}, \cdot)$ is a weak solution of (4.51), we can apply the three sphere inequality in the ball $B_{r_0}(\bar{w})$, which we can assume, without loss of generality, contained in D_0. Choose $r = \frac{r_0}{4}$, and by applying Corollary 2.2.10 for radii $r, 3r, 4r$, we have the following estimate:

$$\|f(\bar{y},\cdot)\|_{L^\infty(B_{3r}(\bar{w}))} \le C\, \|f(\bar{y},\cdot)\|^{\tau}_{L^\infty(B_r(\bar{w}))}\, \|f(\bar{y},\cdot)\|^{1-\tau}_{L^\infty(B_{4r}(\bar{w}))},$$

where $0 < \tau < 1$ and $C > 0$ depends on $\bar{\lambda}, L, r_0$. We construct a sequence of points on γ that will represent the centres of the spheres. First, let $\phi_1 = \bar{w}$. Then, we iterate the following steps:

1. If $|\phi_{j-1} - Q| > r$, then set $\phi_j = \gamma(t_j)$ where $t_j = \max\{t : |\gamma(t) - \phi_{j-1}| = r\}$.

2. Otherwise, set $s = j$, $\phi_s = Q$, and stop the process.

By iterating the three sphere inequality along the chain of balls centred at ϕ_j for $j = 1, \dots, s$, and assuming that $s \le S$ where S only depends on n, we derive the following inequality for every r_1 with $0 < r_1 < r$:

$$\|f(\bar{y},\cdot)\|_{L^\infty(B_{r_1/2}(Q))} \le C\, \|f(\bar{y},\cdot)\|^{\tau^s}_{L^\infty(B_{r_1/2}(\bar{w}))}\, \|f(\bar{y},\cdot)\|^{1-\tau^s}_{L^\infty(\mathcal{G})}.$$

Using this inequality and (4.52) and (4.53), we can conclude that

$$\|f(\bar{y},\cdot)\|_{L^\infty(B_{r_1/2}(Q))} \le C\varepsilon^{\tau^S}(h^{1-n})^{1-\tau^S}. \tag{4.54}$$

The goal is to propagate the smallness from Q to y_h. We consider a truncated cone $C(O, \nu(O), d, \theta_0)$, where $d = \frac{\bar{d}^2 - R^2}{\bar{d}}$. We define the following parameters:

$$\lambda_1 = \min\left\{\frac{d}{1+\sin\theta_0}, \frac{d}{3\sin\theta_0}\right\},$$
$$\theta_1 = \arcsin\left(\frac{\sin\theta_0}{4}\right),$$
$$w_1 = O + \lambda_1\nu(O), \qquad \rho_1 = \lambda_1 \sin\theta_1.$$

It follows that $B_{\rho_1}(w_1) \subset C(O, \nu(O), d, \theta_1)$ and $B_{4\rho_1}(w_1) \subset C(O, \nu(O), d, \theta_0)$. Set

$$a = \frac{1 - \sin\theta_1}{1 + \sin\theta_1}, \qquad \rho_k = a\rho_{k-1}, \qquad \lambda_k = a\lambda_{k-1}.$$

Since $\rho_1 < r_0/2$, we can apply the inequality (4.54) in the cone $C(O, \nu(O), d, \theta_1)$ over a chain of balls of shrinking radius centred at points $w_k = O + \lambda_k \nu(O)$. Denote $d_k := |w_k - O| - \rho_k$. We consider $h \le d_1$ and define a natural number $k_h = \min\{k \in \mathbb{N} : d_k \le h\}$. Hence, we have the following inequalities:

$$\frac{|\ln(h/d_1)|}{|\ln a|} \le k_h - 1 \le \frac{|\ln(h/d_1)|}{|\ln a|} + 1.$$

By iterating the three-sphere inequality over the chain of balls $B_{\rho_1}(w_1), \ldots, B_{\rho_{k_h}}(w_{k_h})$, we obtain

$$\|f(\bar{y}, \cdot)\|_{L^\infty(B_{\rho_{k_h}}(w_{k_h}))} \le c(h^{1-n})^{A''} \varepsilon^{\beta\, \tau^{k_h - 1}}, \tag{4.55}$$

where $\beta = \tau^S$ and $A'' = 1 - \beta$.

Let us now proceed to propagate the smallness with respect to the first argument of f. It turns out that $f(\cdot, w)$ is a weak solution of

$$\mathrm{div}_y\,(a_b(y)\, A(y) \nabla_y f(y, w)) + q_b(y)\, f(y, w) = 0 \qquad \text{for } y \in \mathbb{R}^n \setminus \overline{\Omega_D}.$$

For every $y, w \in (\mathcal{G})^h$, by Proposition 4.3.3, we have:

$$|S_1(y, w)| \le c \int_{D_1} |x - y|^{1-n}\, |x - z|^{1-n}\ \mathrm{d}x \le c\, h^{2(1-n)}.$$

Similarly, $|S_2(y, w)| \le c\, h^{2(1-n)}$, so we can conclude that

$$|f(y, w)| \le c\, h^{2(1-n)} \qquad \text{for any } y, w \in (\mathcal{G})^h.$$

Now, for $y \in D_0$ such that $\mathrm{dist}(y, \partial\Omega) \ge \tilde{c}\, r_0$, for $w \in (\mathcal{G})^h$ and by (4.55), we have

$$|f(y, w)| \le c(h^{1-n})^{A''} \varepsilon^{\beta\, \tau^{k_h - 1}},$$

where A'', β are defined as above.

Choosing $w \in \mathcal{G}$ such that $\mathrm{dist}(w, \partial\Omega_D) = h$, and $\bar{y} \in D_0$ such that $\mathrm{dist}(\bar{y}, \partial\Omega) \ge 3r_0/2$, then for $\bar{r} = r_0/2$, $3\bar{r}$, $4\bar{r}$ and $y_1 = w_1$ as defined earlier, by an iterated application of the three sphere inequality, we can conclude that

$$\begin{aligned} \|f(\cdot, w)\|_{L^\infty(B_{\bar{r}}(y_1))} &\le c\, \|f(\cdot, w)\|^{\tau^s}_{L^\infty(B_{\bar{r}}(\bar{y}))} \|f(\cdot, w)\|^{1-\tau^s}_{L^\infty(\mathcal{G})} \\ &\le c(h^{2-2n})^{A''} \varepsilon^{\beta^2\, \tau^{k_h - 1}}, \end{aligned}$$

where $A' = 1 - \beta + A''\, \tau^S$, $\beta = \tau^S$.

By applying the three sphere inequality inside the cone of vertex O defined earlier over a chain of balls with shrinking radii, we obtain the inequality

$$\|f(\cdot, w)\|_{L^\infty(B_{\rho_{k_h}}(y_{k_h}))} \le c(h^{A'})^{1 - \tau^{k_h - 1}} (\varepsilon^{\beta^2\, \tau^{k_h - 1}})^{\tau^{k_h - 1}}.$$

Choosing $y = w = y_h$, we can conclude that

$$|f(y_h, y_h)| \le ch^{-A}(\varepsilon^{\beta^2\,\tau^{k_h-1}})^{\tau^{k_h-1}},$$

where $A = -(2-2n)\,A'\,(1-\tau^{k_h-1}) > 0$. Since $k_h \le c|\ln h| = -c\ln h$, we have

$$\tau^{k_h} = e^{-c\,\ln h\,\ln\tau} = h^{-c\,\ln\tau} = h^F, \qquad \text{where } F = c|\ln h|.$$

In conclusion,

$$|f(y_h, y_h)| \le c_1 h^{-A}\varepsilon^{\beta^2\,\tau^{2(k_h-1)}} = c_1 h^{-A} e^{\beta^2\tau^{2(k_h-1)}\ln\varepsilon} = c_1 h^{-A}\varepsilon^{B\,h^F},$$

where $B = \beta^2$. □

Lower bound

The proof of Theorem 4.0.1 requires finding a lower bound for the integral solution f in terms of certain geometric quantities. Proposition 4.3.4 provides such a lower bound under certain assumptions.

Proposition 4.3.4. *Under the same assumptions as in Proposition 4.3.1 and Lemma 4.1.2, there exist positive constants $c_2, c_3, \bar{h} \in (0, 1/2)$ that depend only on the a-priori data such that*

$$|f(y,y)| \ge c_2\,h^{2-n} - c_3(\operatorname{dist}(P, D_2) - h)^{2-2n} \quad \text{for } 0 < h < \bar{h}\bar{r}_2, \tag{4.56}$$

where $y = P + h\nu(P)$, where $\nu(P)$ is the outward unit normal of ∂D_1 at P, $\bar{r}_2 \in (0, h_2)$, and $h_2 := \min\left\{\tilde{C}r_0, \operatorname{dist}(P, D_2)\right\}$, for $\tilde{C}$ depending only on L_0.

The proof of Proposition 4.3.4 relies on the application of asymptotic estimates for the Green functions that are solutions of (4.23), in relation to two auxiliary families of Green functions. These auxiliary families will now be described.

First, we find it convenient to flatten the boundary $\partial\Omega_D$ near the point at which the Hausdorff distance is attained. Let $P \in \partial D_1 \cap \partial\Omega_D$ be the point from Lemma 4.1.2. Up to a certain rigid transformation, P can be assumed to coincide with the origin O and $D_1 \cap Q_{r_0}$ is the set $\{x \in Q_{r_0} : x_n > \varphi(x')\}$, where $\varphi \in C^2(B'_{r_0})$. Let $\tau \in C^\infty(\mathbb{R})$ be a smooth function such that $0 \le \tau(s) \le 1$, $\tau(s) = 1$ for $s \in (-1,1)$ and $\tau(s) = 0$ for $s \in \mathbb{R}\setminus(-2,2)$ and $|\tau'(s)| \le 2$ for every $s \in \mathbb{R}$. Set

$$r_1 = \frac{r_0}{2}\min\left\{(8L)^{-1}, \frac{1}{2}\right\}.$$

The change of coordinates given by

$$\xi = \phi(x) = \begin{cases} \xi' = x' \\ \xi_n = x_n - \varphi(x')\tau\left(\frac{|x'|}{r_1}\right)\tau\left(\frac{x_n}{r_1}\right) \end{cases}$$

is a $C^{1,1}$ diffeomorphism of $\mathbb{R}^n$ to itself. This change of coordinates allows us to locally flatten the boundary of the inclusion. Throughout the rest of the discussion, we will stick to using the notation with x, as the exponent in the asymptotic estimates does not depend on the change of coordinates.

For the proof of the asymptotic estimates of the Green function defined in Lemma 4.2.1, we consider a generic inclusion D and coefficients σ and q that satisfy the a priori assumptions with a jump at ∂D. For simplicity, we assume that the portion near point O is flat. We denote $a^- = a_b(0)$, $a^+ = a_D(0)$, and $A = A(0)$, and define $\sigma_0(x)$ and $q_0(x)$ as follows:

$$\begin{aligned} \sigma_0(x) &= \left(a^- + (a^+ - a^-)\chi^+\right) A, \\ q_0(x) &= q_b(0) + (q_D(0) - q_b(0))\,\chi^+(x), \end{aligned} \tag{4.57}$$

where $\chi^+ = \chi_{\mathbb{R}^n_+}$ is the characteristic function of the upper half space. Let $G_0(\cdot, y)$ be the weak solution of the following boundary value problem for $y \in \Omega_0$:

$$\begin{cases} \operatorname{div}(\sigma_0 \nabla G_0(\cdot, y)) + q_0\, G_0(\cdot, y) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ G_0(\cdot, y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_0 \nabla G_0(\cdot, y) \cdot \nu + i\, G_0(\cdot, y) = 0 & \text{on } \Sigma_0, \end{cases} \tag{4.58}$$

where δ denotes the Dirac delta function.

Let Γ denote the fundamental solution associated with the Laplacian operator defined on $\mathbb{R}^n$. Similarly, let H be the fundamental solution of $\operatorname{div}(\sigma_0 \nabla H(\cdot, y)) = -\delta(\cdot - y)$ in $\mathbb{R}^n$, which has the expression (A.15).

We now introduce the asymptotic estimates for the Green functions G with respect to H.

Proposition 4.3.5. *Under the same assumptions as in Proposition 4.3.3, there exist positive constants C_2, C_3, C_4, and $\theta_1, \theta_2 \in (0,1)$ that depend on the a-priori data such that for every $x \in D \cap B_r$ and $y = h\nu(O)$, where r and $h \in (0, \tilde{c} r_1)$, the following inequalities hold:*

$$|G(x, y) - H(x, y)| \le C_2\, |x - y|^{3-n}, \tag{4.59}$$
$$|\nabla_x G(x, y) - \nabla_x H(x, y)| \le C_3\, |x - y|^{1-n+\theta_1}, \tag{4.60}$$
$$|\nabla_y \nabla_x G(x, y) - \nabla_y \nabla_x H(x, y)| \le C_4\, |x - y|^{-n+\theta_2}, \tag{4.61}$$

where $\tilde{c} \in (0, 1/2)$.

Since for any x and y as in the assumptions, the following estimate holds:

$$|G(x, y) - H(x, y)| \le |G(x, y) - G_0(x, y)| + |G_0(x, y) - H(x, y)|.$$

We find it convenient to split the proof of Proposition 4.3.5 into two claims.

Claim 4.3.6. *There exist positive constants C_5, C_6, and $\theta_1 \in (0,1)$ that depend on the a-priori data such that for every $x \in D \cap B_r$ and $y = h\,\nu(O)$ where $h, r \in (0, r_1)$ the following inequalities hold:*

$$|G(x, y) - G_0(x, y)| \le C_5\, |x - y|^{3-n}, \tag{4.62}$$
$$|\nabla_x G(x, y) - \nabla_x G_0(x, y)| \le C_6\, |x - y|^{1-n+\theta_1}. \tag{4.63}$$

Proof of Claim 4.3.6. Let G denote the Green function associated with the elliptic operator $\mathrm{div}(\sigma\nabla\cdot) + q\,\cdot$ such that for every $y \in \Omega_0$, $G(\cdot, y)$ is a distributional solution of (4.23). For $O \in \partial D$, let σ_0 and q_0 be as defined in (4.57). For $y \in \Omega_0$, let $G_0(\cdot, y)$ be the Green function that is a distributional solution of (4.58). We define

$$R(x, y) = G(x, y) - G_0(x, y).$$

Subtracting the first equation of (4.58) to (4.23), it follows that $R(x, y)$ is a weak solution in Ω_0 to the equation

$$\mathrm{div}(\sigma\nabla R(\cdot, y)) + q\, R(\cdot, y) = -\,\mathrm{div}((\sigma - \sigma_0)\nabla G_0(\cdot, y)) - (q - q_0)\, G_0(\cdot, y), \quad \text{in } \Omega_0,$$

with boundary conditions

$$\begin{cases} R(\cdot, y) = 0 & \text{in } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma\, \nabla_x R(\cdot, y) \cdot \nu + i\, R(\cdot, y) = (\sigma_0 - \sigma)\nabla_x G_0(\cdot, y) \cdot \nu & \text{in } \Sigma_0. \end{cases}$$

Then the following representation formula holds

$$\begin{aligned} -R(x, y) = &\int_{\Omega_0} (\sigma(z) - \sigma_0(z))\nabla_z G(z, x) \cdot \nabla_z G_0(z, y)\, \mathrm{d}z \\ &+ \int_{\Omega_0} (q_0(z) - q(z))G(z, x)G_0(z, y)\, \mathrm{d}z \\ &+ \int_{\Sigma_0} [\sigma_0(z)\nabla_z G_0(z, y) \cdot \nu G(z, x) - \sigma(z)\nabla_z G(z, x) \cdot \nu G_0(z, y)]\, \mathrm{d}S(y). \end{aligned} \tag{4.64}$$

The boundary integral on the right-hand side is bounded, for instance, by the Schwarz inequality and standard trace estimates. The second volume integral in (4.64) is less singular than the first volume integral, so it is convenient to study the first volume integral only. Let us split the domain of integration into the union of the subdomains $\Omega \cap Q_{\bar r_0}$ and $\Omega \setminus Q_{\bar r_0}$ for $\bar r_0 = \frac{r_0}{4}\min\{(8L)^{-1}, 1\}$. For $x \in \Omega \cap Q_{\bar r_0}$, we have

$$|\sigma(z) - \sigma_0(z)| \le C|z|.$$

Hence, we can apply the same argument of [51, Proposition 4.1] and conclude that

$$|R(x, y)| \le C_5|x - y|^{3-n}, \tag{4.65}$$

where C_4 is a positive constant which depends only on the *a-priori* data.

Regarding the estimate for the gradient of R, recalling that the boundary of D is of class C^2 and hence $C^{1,1}$, for $x \in D \cap B_r$, we consider a cube $Q \subset B^+_{r/4}$ of side length $\frac{c\bar r_0}{4}$, where $c \in (0, 1)$ so that $y \notin Q$ and $x \in \partial Q$. By [7, Lemma 3.2], the following interpolation formula holds

$$\|\nabla R(\cdot, y)\|_{L^\infty(Q)} \le C\|R(\cdot, y)\|^{1/2}_{L^\infty(Q)}|\nabla_x R(\cdot, y)|^{1/2}_{1,Q}, \tag{4.66}$$

where C depends only on L. For $y = h\,\nu(O)$, from the piecewise Hölder continuity of $\nabla_x G(x,y)$ and $\nabla_x G_0(x,y)$ (see [85, Theorem 16.2]),we have

$$|\nabla_x G(\cdot,y)|_{1,Q},\ \ |\nabla_x G_0(\cdot,y)|_{1,Q} \leq C\, h^{-n}.$$

Therefore,

$$|\nabla_x R(\cdot,y)|_{1,Q} \leq C\, h^{-n}, \tag{4.67}$$

and combining (4.65), (4.66), and (4.67), it follows that

$$|\nabla_x R(x,y)| \leq C_6 |x-y|^{1-n+\theta_1}, \qquad \text{where } \theta_1 = \frac{1}{2}.$$

□

Claim 4.3.7. *There exist positive constants C_6, C_7 that depend only on the a-priori data such that for every $x \in D \cap B_r$ and $y = h\,\nu(O)$ where $r \in (0, r_1)$ and $h \in (0, r_1/2)$, the following inequalities hold:*

$$|G_0(x,y) - H(x,y)| \leq C_7 |x-y|^{4-n}, \tag{4.68}$$

$$|\nabla_x G_0(x,y) - \nabla_x H(x,y)| \leq C_8 |x-y|^{2-n}. \tag{4.69}$$

Proof of Claim 4.3.7. We will follow the argument presented in [51, Proposition 4.2]. Let y and z be elements of Ω_0, and let $G_0(\cdot,y)$ be the weak solution of (4.58), and let $H(\cdot,y)$ be the fundamental solution of the elliptic equation $\mathrm{div}_x(\sigma_0 \nabla_x H(\cdot,y)) = -\delta(\cdot - y)$ in $\mathbb{R}^n$.

We can define the distribution $R_0(x,y)$ as follows:

$$R_0(x,y) = G_0(x,y) - H(x,y).$$

It turns out that R_0 is a weak solution of the following boundary value problem:

$$\begin{cases} \mathrm{div}(\sigma_0 \nabla R_0(\cdot,y)) = -q_0\, G_0(\cdot,y) & \text{in } \Omega_0, \\ R_0(\cdot,y) = -H(\cdot,y) & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_0\, \nabla R_0(\cdot,y)\cdot\nu + i\, R_0(\cdot,y) = -\sigma_0\, \nabla H(\cdot,y)\cdot\nu - i\, H(\cdot,y) & \text{on } \Sigma_0. \end{cases}$$

Its representation formula is

$$\begin{aligned} -R_0(x,y) = &-\int_{\Omega_0} q_0(z) G_0(z,x) H(z,y)\, \mathrm{d}x + \\ &+ \int_{\partial\Omega_0} \sigma_0(z) \left[\nabla_z H(z,y)\cdot\nu G_0(z,x) - \nabla_z G_0(z,x)\cdot\nu H(z,y)\right]\, \mathrm{d}S(z). \end{aligned} \tag{4.70}$$

The surface integral can be easily bounded from above using Schwarz inequality by a constant that depends on the *a-priori* data. Regarding the volume integral, by (4.25) it follows that

$$\begin{aligned} \left|\int_{\Omega_0} q_0(z) G_0(z,x) H(z,y)\, \mathrm{d}x\right| &\leq \|q_0\|_{L^\infty(\Omega_0)} \int_{\Omega_0} |G_0(z,x)|\, |H(z,y)|\, \mathrm{d}z \\ &\leq C \int_{\Omega_0} |z-x|^{2-n}\, |z-y|^{2-n}\, \mathrm{d}z. \end{aligned}$$

Let $\tilde{r} = |x-y|$, and let $N \in \mathbb{N}$ be such that $B_{\tilde{r}/N}(x) \cap B_{\tilde{r}/N}(y) = \emptyset$. Define $\mathcal{O} = \Omega_0 \setminus (B_{\tilde{r}/N}(x) \cup B_{\tilde{r}/N}(y))$. It turns out that

$$\int_{\Omega_0} |z-x|^{2-n}\, |z-y|^{2-n}\, \mathrm{d}z = \int_{\Omega_0 \setminus (B_{\tilde{r}/N}(x) \cup B_{\tilde{r}/N}(y))} |z-x|^{2-n}\, |z-y|^{2-n}\, \mathrm{d}z$$
$$+ \int_{B_{\tilde{r}/N}(x)} |z-x|^{2-n}\, |z-y|^{2-n}\, \mathrm{d}z + \int_{B_{\tilde{r}/N}(y)} |z-x|^{2-n}\, |z-y|^{2-n}\, \mathrm{d}z.$$

For $z \in B_{\tilde{r}/N}(y)$, we can use the triangle inequality to obtain

$$|x-z| \geq |x-y| - |y-z| \geq \frac{|x-y|}{\tilde{c}}$$

for a suitable constant $\tilde{c}$. Hence,

$$\int_{B_{\tilde{r}/N}(y)} |z-x|^{2-n} |z-y|^{2-n}\, \mathrm{d}z \leq c|x-y|^{2-n} \int_{B_{\tilde{r}/N}(y)} |z-y|^{2-n}\, \mathrm{d}z \leq c\,|x-y|^{4-n}.$$

Similarly,

$$\int_{B_{\tilde{r}/N}(x)} |z-x|^{2-n} |z-y|^{2-n}\, \mathrm{d}z \leq c\,|x-y|^{4-n}.$$

For $z \in \mathcal{O}$, we have $|x-z| \geq \frac{|z-y|}{N}$. Therefore,

$$\int_{\mathcal{O}} |z-x|^{2-n} |z-y|^{2-n}\, \mathrm{d}z \leq c \int_{\mathcal{O}} |z-y|^{4-2n}\, \mathrm{d}z \leq c \int_{\Omega \setminus B_{\tilde{r}/N}(y)} |z-y|^{4-2n}\, \mathrm{d}z \leq c|x-y|^{4-n}.$$

The constants c appearing in the inequalities depend only on the *a-priori* data. Thus, we can conclude that

$$|R(x,y)| \leq C_7 |x-y|^{4-n}, \tag{4.71}$$

where C_7 depends only on the *a-priori* data.

The next quantity that we want to estimate is the gradient of R_0. Let $y = h\,\nu(O)$. By a similar argument as in Claim 4.3.6, for $x \in D \cap B_r$, we consider a cube $Q \subset B^+_{r/4}$ with side length $\frac{c\bar{r}_0}{4}$, where $c \in (0,1)$ is such that $y \notin Q$ and $x \in \partial Q$. By [7, Lemma 3.2], the following interpolation formula holds

$$\|\nabla R_0(\cdot,y)\|_{L^\infty(Q)} \leq C \|R_0(\cdot,y)\|^{1/2}_{L^\infty(Q)} |\nabla_x R_0(\cdot,y)|^{1/2}_{1,Q}, \tag{4.72}$$

where C only depends on L. Since G_0 and H are Hölder continuous, the following estimates hold

$$|\nabla_x G_0(\cdot,y)|_{1,Q} \leq c\,|x-y|^{-n} \quad \text{and} \quad |\nabla_x H(\cdot,y)|_{1,Q} \leq c\,|x-y|^{-n},$$

where c depends on L. By (4.66) and(4.71), we have

$$\|\nabla_x R_0(\cdot,y)\|_{L^\infty(Q)} \leq C_8\,|x-y|^{2-n},$$

where C_7 depends on the *a-priori* data. For the proof of (4.61), we can follow the lines of the proof of Proposition 3.2.2. □

The next step is to prove Proposition 4.3.4, which is based on the argument used in [5, Proposition 3.5] and [11, Theorem 6.5].

Proof of Proposition 4.3.4. Let $P \in \partial D_1$ be the point from Lemma 4.1.2, and assume that $P \equiv O$. Let $y = h\,\nu(O)$, where $\nu(O)$ is the exterior unit normal of ∂D_1 at O. Recall the definition of S_1:

$$\begin{aligned} S_1(y,y) = &\int_{D_1} (a_{D_1}(x) - a_0(x))\, A(x)\nabla_x G_1(x,y)\cdot \nabla_x G_2(x,y)\, \mathrm{d}x \\ &- \int_{D_1} (q_{D_1}(x) - q_0(x))\, G_1(x,y)\, G_2(x,y)\, \mathrm{d}x. \end{aligned} \tag{4.73}$$

Define $\bar r_2 := \min\{\mathrm{dist}(O, D_2), \tilde C r_0\}$, where $\tilde C$ depends on L_0, and fix $r \in (0, \bar r_2)$. Since for $y = h\,\nu(O)$, with $h \in (0, \bar h \bar r_2)$, where $\bar h$ will be defined later, the first term on the right-hand side of (4.73) is the dominant term as $h \to 0^+$, let us represent the domain of integration as $D_1 = (D_1 \cap B_r) \cup (D_1 \setminus B_r)$.

Then (4.73) can be written as follows:

$$S_1(y,y) = I_1 + R_1 + R_2 + R_3 + Q_1, \tag{4.74}$$

where

$$\begin{aligned} I_1 &= \int_{D_1\cap B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x H_1(x,y)\cdot\nabla_x H_2(x,y)\,\mathrm{d}x,\\ R_1 &= \int_{D_1\cap B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x H_1(x,y)\cdot\nabla_x (G_2(x,y) - H_2(x,y))\,\mathrm{d}x+\\ &\quad + \int_{D_1\cap B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x (G_1(x,y) - H_1(x,y))\cdot\nabla_x (G_2(x,y) - H_2(x,y))\,\mathrm{d}x,\\ R_2 &= \int_{D_1\cap B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x (G_1(x,y) - H_1(x,y))\cdot\nabla_x H_2(x,y)\,\mathrm{d}x,\\ R_3 &= \int_{D_1\setminus B_r(O)} (a_{D_1}(x) - a_b(x))A(x)\nabla_x G_1(x,y)\cdot\nabla_x G_2(x,y)\,\mathrm{d}x,\\ Q_1 &= \int_{D_1} (q_{D_1}(x) - q_b(x))G_1(x,y)\,G_2(x,y)\,\mathrm{d}x. \end{aligned}$$

Hence, applying the triangle inequality to (4.74), we obtain

$$|S_1(y,y)| \geq |I_1| - |R_1| - |R_2| - |R_3| - |Q_1|.$$

The term Q_1 exhibits less singular behaviour as $h \to 0^+$ compared to the other terms on the right-hand side of (4.73). Therefore, our focus is on finding a bound for the other terms.

For the term I_1, we can observe that

$$H_1(x,y) = \tilde c\,\Gamma(Jx, Jy) \quad \text{and} \quad H_2(x,y) = \tilde c\,\Gamma(Jx, Jy),$$

where $J = \sqrt{A(O)^{-1}}$ and $\tilde c$ is a constant depending only on a^+ and a^-. Hence, by the uniform ellipticity condition,

$$|I_1| \geq c\int_{D_1\cap B_r(O)} |x-y|^{2-2n}\,\mathrm{d}x \geq c\, r^{2-n} \geq c\, h^{2-n}.$$

Regarding the term R_2, Proposition 4.3.5 implies that

$$|\nabla_x G_1(x,y) - \nabla_x H_1(x,y)| \leq C\,|x-y|^{1-n+\theta_1}.$$

Thus,

$$|R_2| \leq \tilde{c} \int_{D_1 \cap B_r(O)} |x-y|^{2-2n+\theta_2}\,\mathrm{d}x \leq c\,h^{2-n+\theta_1}.$$

We can bound the term R_3 using a similar argument as we did for I_1 and R_2.

Next, we estimate the term R_1. The problem here is that, due to our choice of the radius r, we do not have any asymptotic estimate for the term $\nabla_x(G_2(x,y)-H_2(x,y))$. However, we can solve this problem by using the following trick. Recall Lemma 4.2.1, which states that G_2 has the form

$$G_2(x,y) = \tilde{G}_2(x,y) + \sum_{j=1}^{J+1} R_j(x,y).$$

Here, $\tilde{G}_2$ is a weak solution of the following system

$$\begin{cases} \mathrm{div}(\sigma_2 \nabla \tilde{G}_2(\cdot,y)) = -\delta(\cdot - y) & \text{in} \quad \Omega_0, \\ \tilde{G}_2(\cdot,y) = 0 & \text{on} \quad \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_2\,\nabla\tilde{G}_2(\cdot,y)\cdot\nu + i\,\tilde{G}_2(\cdot,y) = 0 & \text{on} \quad \Sigma_0. \end{cases} \tag{4.75}$$

Hence,

$$|\nabla_x(G_2(x,y) - H_2(x,y))| \leq |\nabla_x(\tilde{G}_2(x,y) - H_2(x,y))| + \sum_{j=1}^{J+1} |\nabla_x R_j(x,y)|.$$

From the proof of Lemma 4.2.1, we can establish that for every $j = 1,\dots,J-1$,

$$|\nabla_x R_j(x,y)| \leq c|x-y|^{2j+1-n}.$$

Thus, it follows that

$$\sum_{j=1}^{J+1} |\nabla_x R_j(x,y)| \leq \sum_{j=1}^{J+1} (\mathrm{dist}(O,D_2) - h)^{2j+1-n} \leq c\,(\mathrm{dist}(O,D_2)-h)^{2-n}.$$

Next, let us consider $\tilde{G}_{2,0}(\cdot,y)$ as the Green function that is a weak solution of the following system

$$\begin{cases} \mathrm{div}(\sigma_{2,0} \nabla \tilde{G}_{2,0}(\cdot,y)) = -\delta(\cdot - y) & \text{in} \quad \Omega_0, \\ \tilde{G}_{2,0}(\cdot,y) = 0 & \text{on} \quad \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_{2,0}\,\nabla\tilde{G}_{2,0}(\cdot,y)\cdot\nu + i\,\tilde{G}_{2,0}(\cdot,y) = 0 & \text{on} \quad \Sigma_0, \end{cases} \tag{4.76}$$

where

$$\sigma_{2,0}(x) = (a^- + (a^+ - a^-)\chi_+(x))\,A(0).$$

Hence, we have

$$|\nabla_x(\tilde{G}_2(x,y) - H_2(x,y))| \leq |\nabla_x(\tilde{G}_2(x,y) - \tilde{G}_{2,0}(x,y))| + |\nabla_x(\tilde{G}_{2,0}(x,y) - H_2(x,y))|. \tag{4.77}$$

For the second term on the right-hand side of (4.77), notice that $\tilde{G}_{2,0}(\cdot,y) - H_2(\cdot,y)$ is a weak solution of

$$\begin{cases} \mathrm{div}(\sigma_{2,0}\nabla(\tilde{G}_{2,0}(\cdot,y) - H_2(\cdot,y))) = 0 & \text{in } B_r(O), \\ \left(\tilde{G}_{2,0}(\cdot,y) - H_2(\cdot,y)\right)|_{\partial B_r(O)} \leq c\, r^{2-n}. \end{cases} \tag{4.78}$$

We can apply the Maximum Principle [66, Theorem 8.1] to obtain

$$|\tilde{G}_{2,0}(x,y) - H_2(x,y)| \leq c\, r^{2-n}.$$

Using interior gradient estimates, it follows that

$$|\nabla_x(\tilde{G}_{2,0}(x,y) - H_2(x,y))| \leq c\, r^{1-n}. \tag{4.79}$$

Now, let us focus on the first term on the right-hand side of (4.77). Define

$$\tilde{R}_2(x,y) = \tilde{G}_2(x,y) - \tilde{G}_{2,0}(x,y).$$

We notice that $\tilde{R}_2(\cdot,y)$ is a weak solution of

$$\begin{cases} \mathrm{div}(\sigma_2\nabla\tilde{R}_2(\cdot,y)) = -\,\mathrm{div}\left((\sigma_2 - \sigma_{2,0})\nabla\tilde{G}_{2,0}(\cdot,y)\right) & \text{in } \Omega_0, \\ \tilde{R}_2(\cdot,y) = 0 & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma_2\nabla\tilde{R}_2(\cdot,y)\cdot\nu + i\tilde{R}_2(\cdot,y) = (\sigma_{2,0} - \sigma_2)\nabla\tilde{G}_{2,0}(\cdot,y)\cdot\nu & \text{on } \Sigma_0. \end{cases}$$

By the representation formula, the remainder has the form

$$\begin{aligned} -\tilde{R}_2(x,y) = & \int_{\Omega_0} (\sigma_2(z) - \sigma_{2,0}(z))\nabla_z\tilde{G}_2(z,x)\cdot\nabla_z\tilde{G}_{2,0}(z,y)\,\mathrm{d}z \\ & + \int_{\partial\Omega_0} \sigma_{2,0}(z)\nabla_z\tilde{G}_{2,0}(z,y)\cdot\nu\left[\tilde{G}_2(z,x) - \tilde{G}_{2,0}(z,x)\right]\,\mathrm{d}S(z) \\ & + \int_{\partial\Omega_0} \sigma_2(z)\nabla_z\left[\tilde{G}_2(z,x) - \tilde{G}_{2,0}(z,x)\right]\cdot\nu\,\tilde{G}_{2,0}(z,y)\,\mathrm{d}S(z). \end{aligned} \tag{4.80}$$

The integral over $\partial\Omega_0$ is bounded from above by a positive constant that depends on the *a-priori* data only. In order to estimate the volume integral, first notice that $|\sigma_2(z) - \sigma_{2,0}(z)| \leq C|z|$, where C is a positive constant depending only on *a-priori* data. Therefore, by Proposition 4.3.3, we have

$$\begin{aligned} & \left|\int_{\Omega_0} (\sigma_2(z) - \sigma_{2,0}(z))\nabla_z\tilde{G}_2(z,x)\cdot\nabla_z\tilde{G}_{2,0}(z,y)\,\mathrm{d}z\right| \\ & \leq c\int_{\Omega_0} |z|\,|z-x|^{1-n}\,|z-y|^{1-n}\,\mathrm{d}z, \end{aligned} \tag{4.81}$$

where c is a suitable positive constant. We define $\tilde{h} = |x-y|$ and set

$$I_1 = \int_{B_{4\tilde{h}}} |z|\,|z-x|^{1-n}\,|z-y|^{1-n}\,\mathrm{d}z, \tag{4.82}$$

$$I_2 = \int_{\mathbb{R}^n \setminus B_{4\tilde{h}}} |z|\,|z-x|^{1-n}\,|z-y|^{1-n}\,\mathrm{d}z. \tag{4.83}$$

We can rewrite the expression for the remainder as

$$|\tilde{R}_2(x,y)| \le c(I_1 + I_2). \tag{4.84}$$

Next, we can estimate I_1. Set $z = \tilde{h}w$, $t = \frac{x}{\tilde{h}}$ and $s = \frac{y}{\tilde{h}}$. Then

$$\begin{aligned} I_1 &= \int_{B_4} \tilde{h}|w|\,|\tilde{h}(w-t)|^{1-n}\,|\tilde{h}(w-s)|^{1-n}\tilde{h}\,\mathrm{d}w \\ &= 4\tilde{h}^{3-n} \int_{B_4} |w-t|^{1-n}\,|w-s|^{1-n}\ \mathrm{d}w \\ &\le c\tilde{h}^{3-n}, \end{aligned}$$

since $\int_{B_4} |w-t|^{1-n}\,|w-s|^{1-n}\ \mathrm{d}w \le c$ (see [99, Chapter 2, section 11]). Hence,

$$I_1 \le c(\mathrm{dist}(O, D_2) - h)^{3-n}. \tag{4.85}$$

Similarly, for I_2, we notice that since $y = h\,\nu(O) = -h\,e_n$ in a suitable coordinate system

$$|y| = -h \le |x-y| = \tilde{h},$$

and

$$|x| \le |x-y| + |y| \le 2\tilde{h}.$$

For any $z \in \mathbb{R}^n \setminus B_{4\tilde{h}}$, since $|z| > 4\tilde{h}$, we have

$$\frac{3}{4}|z| \le |z-y| \quad \text{and} \quad \frac{1}{2}|z| \le |z-x|.$$

Therefore, we can estimate I_2 as follows:

$$I_2 \le \left(\frac{8}{3}\right)^{1-n} \int_{\mathbb{R}^n \setminus B_{4\tilde{h}}} |z|^{3-2n}\,\mathrm{d}z \le c\,\tilde{h}^{3-n} \le c\,(\mathrm{dist}(O, D_2) - h)^{3-n}. \tag{4.86}$$

By using (4.85) and (4.86), we can conclude that

$$|\tilde{R}_2(x,y)| \le c\,|x-y|^{3-n}. \tag{4.87}$$

Now, let us determine an upper bound for $\nabla_x \tilde{R}_2$. Consider a cube $Q \subset D_1 \cap B_r(O)$. Since $\tilde{G}_2(\cdot, y)$ and $\tilde{G}_{2,0}(\cdot, y)$ are Hölder continuous, it follows that

$$|\nabla \tilde{R}_2(x,y)|_{1,Q} \le c\,|x-y|^{-n}.$$

By using the following inequality:

$$\|\nabla\tilde{R}_2(\cdot,y)\|_{L^\infty(Q)} \leq \|\tilde{R}_2(\cdot,y)\|^{1/2}_{L^\infty(Q)}|\nabla\tilde{R}_2(\cdot,y)|^{1/2}_{1,Q},$$

and by (4.87), it follows that

$$|\nabla\tilde{R}_2(x,y)| \leq c|x-y|^{1-n+\theta_3} \quad \text{where} \quad \theta_3 = \frac{1}{2}. \tag{4.88}$$

By collecting (4.77), (4.79) and (4.88), we obtain

$$|\nabla_x(\tilde{G}_2(x,y) - H_2(x,y))| \leq c\,h^{1-n+\theta_3}. \tag{4.89}$$

In conclusion, we have that the lower bound of S_1 is given by

$$|S_1(y,y)| \geq ch^{2-n}.$$

As for the estimate for S_2, from Proposition 4.3.3, it follows that

$$|S_2(y,y)| \leq C\int_{D_2}|x-y|^{1-n}|x-y|^{1-n}\,\mathrm{d}x \leq C(\mathrm{dist}(O,D_2)-h)^{2(1-n)}.$$

In conclusion, by the triangle inequality, we obtain

$$\begin{aligned} |f(y,y)| = |S_1(y,y) - S_2(y,y)| &\geq |S_1(y,y)| - |S_2(y,y)| \\ &\geq c_2\,h^{2-n} - c_3\,(\mathrm{dist}(O,D_2)-h)^{2(1-n)}, \end{aligned} \tag{4.90}$$

for suitable $c_2, c_3 > 0$ constants depending on the *a-priori* data. □

4.4 The stability estimate

In this Section, we provide the proof of Theorem 4.0.1 by applying the results proved in Sections 4.1 and 4.3.

Proof of Theorem 4.0.1. Let $P \in \partial D_1 \cap \partial\Omega_D$ be the point of Lemma 4.1.2 such that

$$\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2) \leq \tilde{c}_0\,\mathrm{dist}(P, D_2),$$

where $\tilde{c}_0$ is a positive constant depending only on the *a-priori* data. Assume we are in a coordinate system in which the point P coincides with the origin O and $\nu(O) = -e_n$, where $e_n = (0,\ldots,0,1)$. Define $y_h := O + h\,\nu(O)$ for $0 < h < \min\{h_1, h_2\}$, where h_1 and h_2 are the constants of Proposition 4.3.1 and Proposition 4.3.4. Combining the upper bound of Proposition 4.3.1 and the lower bound of Proposition 4.3.4 for the singular solution f evaluated at y_h, we obtain

$$c_2\,h^{2-n} - c_3\,(\mathrm{dist}(O,D_2)-h)^{2-2n} \leq c_1\frac{\varepsilon^{B\,h^F}}{h^A}. \tag{4.91}$$

Here, $c_1, c_2, c_3, A, B,$ and F are constants that depend on the *a-priori* data. From

(4.91), we derive

$$c_4(\mathrm{dist}(O, D_2) - h)^{2(1-n)} \geq c_5\,(1 - \varepsilon^{B\,h^F} h^{\tilde{A}}) h^{2-n}, \tag{4.92}$$

where $\tilde{A} = n - 2 - A$. Now, let $h = h(\varepsilon) = \min\{|\ln\varepsilon|^{-1/2F}, \mathrm{dist}(O, D_2)\}$. Let $\varepsilon_1 \in (0,1)$ be such that $\exp(-B|\ln\varepsilon_1|^{1/2}) = 1/2$. We can divide the proof into two cases.

a) Assume that $\varepsilon \in (0, \varepsilon_1)$. If $\mathrm{dist}(O, D_2) \leq |\ln\varepsilon|^{-1/2F}$ then by applying Lemma 4.1.1 and Lemma 4.1.2, the thesis follows straightforwardly. If $\mathrm{dist}(O, D_2) \geq |\ln\varepsilon|^{-1/2F}$, then $h = |\ln\varepsilon|^{-1/2F}$. Since $\varepsilon^{B\,h^F} h^{\tilde{A}} \leq \exp(-B|\ln\varepsilon|^{1/2})$, by (4.92) we obtain
$$(\mathrm{dist}(O, D_2) - h)^{2(1-n)} \geq c_6 h^{2-n}.$$
Therefore, we have
$$\mathrm{dist}(O, D_2) \leq c_7|\ln\varepsilon|^{-\eta} \qquad \text{for } \eta = \frac{n-2}{4F(n-1)}.$$

b) Assume that $\varepsilon \in [\varepsilon_1, 1)$. In this case, since $\mathrm{dist}(O, D_2) \leq diam(\Omega)$, it follows that
$$\mathrm{dist}(O, D_2) \leq diam(\Omega) \cdot \frac{|\ln\varepsilon|^{-1/2F}}{|\ln\varepsilon_1|^{-1/2F}} \leq C|\ln\varepsilon|^{-1/2F}.$$
This completes the proof.

□

4.5 The misfit functional

We conclude this chapter by establishing a stability result using a novel functional called misfit functional. In the previous chapter, we have introduced the misfit functional as a tool to measure discrepancies in boundary data for the Calderón problem (equation (3.15)). Minimising this functional allows for conductivity reconstruction. In our case, the coefficients σ and q are known, and the unknown quantity is the shape and location for the inclusion. We provide an optimal stability estimate in terms of the misfit functional. Before stating this estimate, we make a small variation in the definition of the coefficients.

Consider two inclusions D_1 and D_2 contained in Ω. Let $\sigma_i \in L^\infty(\Omega, Sym_n)$ for $i = 1, 2$ be positive definite matrix functions defined as

$$\sigma_i(x) := (a_b(x) + (a_D(x) - a_b(x))\chi_{D_i}(x))\,A(x), \tag{4.93}$$

where $a_b, a_D \in C^{0,1}(\Omega)$, and $A \in C^{0,1}(\Omega)$ are known. Assume that σ_i satisfies (4.8). Let $q_i \in L^\infty(\Omega)$ be defined as

$$q_i(x) := q_b(x) + (q_D(x) - q_b(x))\chi_{D_i}(x), \tag{4.94}$$

where $q_b, q_D \in L^\infty(\Omega)$ are known. Let G_j be the Green functions associated with the operator $\mathrm{div}(\sigma_j\nabla\cdot) + q_j\cdot$ for $j = 1, 2$, such that $G_j(\cdot, y)$ is a distributional solution of the boundary value problem (4.23). Choose suitable Lipschitz domains D_y and D_z that are compactly contained in D_0. For $(y,z) \in D_y \times D_z$, we have

$$f(y,z) = \int_\Sigma \Big[\sigma_1(x)\nabla G_1(x,y)\cdot\nu(x)\, G_2(x,z) - \sigma_2(x)\nabla G_2(x,z)\cdot\nu(x)\, G_1(x,y)\Big]\, dS(x). \tag{4.95}$$

Here, f plays a role similar to S_0 as defined in (3.13). The *misfit functional* $\mathcal{J}(D_1, D_2)$ is defined as

$$\mathcal{J}(D_1, D_2) = \int_{D_y \times D_z} |f(y,z)|^2 \, \mathrm{d}y \, \mathrm{d}z. \tag{4.96}$$

As the misfit functional already introduced in Chapter 3, (4.96) encodes the error that occurs when the boundary data induced by the inclusion D_1 is approximated by the boundary data induced by the inclusion D_2.

Now, we state the main result of this section, Theorem 4.5.1.

Theorem 4.5.1. *Let $\Omega \subset \mathbb{R}^n$, D_1, and D_2 be a bounded domain and be two inclusions of class C^2 contained in Ω satisfying the a-priori assumptions. Let σ_1 and σ_2 be the anisotropic coefficients as defined in (4.93), and let q_1 and q_2 be the coefficients of the zero-order term as defined in (4.94). If the misfit functional $\mathcal{J}(D_1, D_2)$ is less than a given small positive value $\varepsilon \in (0,1)$, then the following inequality holds:*

$$\mathrm{d}_{\mathcal{H}}(\partial D_1, \partial D_2) \le \omega(\varepsilon), \tag{4.97}$$

where $C > 0$ is a constant that depends only on the a-priori data, and ω satisfies (4.16).

The proof of Theorem 4.5.1 follows the same lines as the proof of Theorem 4.0.1, but instead of Proposition 4.3.1, a modified version, Proposition 4.5.2, is required. We note that using the Green's identity (4.40), for $y, z \in \Omega_0 \setminus \Omega$, the integral function $f(y,z)$ can be expressed as

$$\begin{aligned} f(y,z) = &\int_\Omega (\sigma_1(x) - \sigma_2(x))\nabla G_1(x,y)\cdot\nabla G_2(x,z) \\ &+ \int_\Omega (q_2(x) - q_1(x))\, G_1(x,y)\, G_2(x,z)\, \mathrm{d}x. \end{aligned} \tag{4.98}$$

Proposition 4.5.2. *Under the same assumptions of Theorem 4.5.1, if the misfit functional $\mathcal{J}(D_1, D_2)$ is smaller than a given constant $\varepsilon \in (0,1)$, then for $y = P + h\,\nu(P)$ there exists a positive constant C that depends on the a-priori data such that*

$$|f(y,y)| \le C\, \frac{\varepsilon^{B\, h^F}}{h^A},$$

where

$$0 < h \le h_1 := \bar{d}\left(1 - \frac{\sin\theta_0}{4}\right) \quad \text{and} \quad \theta_0 = \arctan\left(\frac{1}{L_0}\right), \tag{4.99}$$

P is the point of the Lemma 4.1.2, and $\nu(P)$ is the exterior unit normal of ∂D_1 at P.

Proof of Proposition 4.5.2. Let $\bar{y} \in D_0$ be fixed. Then for every $\bar{w} \in \mathbb{R}^n \setminus \overline{\Omega_D}$, the integral function $f(\bar{y},\bar{w})$ is a weak solution of

$$\mathrm{div}_w(a_b(\bar{w})A(\bar{w})\,\nabla_w f(\bar{y},\bar{w})) + q_b(\bar{w})\,f(\bar{y},\bar{w}) = 0.$$

By using the same argument presented as in the derivation of (3.108), it follows that

$$\max_{(y,w)\in(D_0)_r\times(D_0)_r} f(y,w) \le c\,(\mathcal{J}(D_1,D_2))^{1/2},$$

where $(D_0)_r = \{x \in D_0 \,:\, \mathrm{dist}(x,\partial\,D_0) > r\}$ for some $r > 0$, and the constant c depends on the *a-priori* data. Then, for any $(\bar{y},\bar{w}) \in D_y \times D_z$, we have

$$f(\bar{y},\bar{w}) \le c\,\varepsilon.$$

The remaining part of the proof is simply a straightforward adaptation of Proposition 4.5.2. □

CHAPTER 5

Lipschitz stability for a simultaneous coefficient identification problem

Contents

In this chapter, we consider the inverse problem of the simultaneous determination of the coefficients for the generalised Schrödinger equation on a bounded domain Ω from the given boundary data. In particular, we focus on the study of stability and derive a Lipschitz stability estimate. The main result, presented in Theorem 5.1.2, establishes the Lipschitz dependence of the coefficients σ and q on the distance between two sets of local Cauchy data. Corollary 5.1.3 provides a boundary stability result of Hölder type. This result is based on the application of the method of singular solution, which has proved to be effective since the pioneering work of Alessandrini and Vessella [21].

To obtain better stability estimates, we introduce additional assumptions. We assume that Ω can be divided into subdomains with regular boundaries of class C^2. Across these subdomains, the coefficients σ and q vary. Moreover, the coefficient σ exhibits an anisotropic behaviour, modelled by a $C^{1,1}$ matrix function A. To account for the boundary condition, we consider the local Cauchy data, which was defined in

Chapter 4.

This Chapter is divided into three sections. In Section 5.1, we introduce the a priori assumptions on the domain and the coefficients. We state the stability estimate, Theorem 5.1.2, and the boundary estimate, Corollary 5.1.3.

In Section 5.2, we define the Green function and describe its asymptotic behaviour near the discontinuity interfaces. The Green function is a weak solution of a boundary value problem defined on an enlarged domain Ω_0 with complex Robin boundary data prescribed on a small portion of $\partial\Omega_0$ that is not contained in $\partial\Omega$ and with homogeneous Dirichlet condition on the remaining portion. In Section 5.3, we introduce the singular integrals and the quantitative estimates of propagation of smallness.

Finally, Section 5.4 is devoted to the proof of Theorem 5.1.2. First, we derive a Hölder type estimate in the first subdomain labelled D_1, the one that shares a boundary portion with the portion Σ at which the measurements are taken. Then, we fix a chain of contiguous subdomains of the partition of Ω that joins D_1 to the domain D_K where the maximum between the L^∞ norm of σ and q is reached. On each domain D_k in the chain, an iterative procedure is applied to derive first a stability estimate for the scalar part of the coefficient σ and then an estimate for the coefficient q.

5.1 Notation and main result

A priori information about the domain

Consider $\Omega \subset \mathbb{R}^n$ a bounded, measurable domain with a C^2 boundary $\partial\Omega$ with positive constants r_0 and L. Assume that

$$|\Omega| \leq C r_0^n, \tag{5.1}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω and C is a positive constant. Let Σ be a flat portion of size r_0 on the boundary $\partial\Omega$. We assume that there exists a partition of bounded domains $\{D_m\}_{m=1}^N$, where N is a positive integer greater than 1, such that the following conditions are satisfied:

a) Each D_m for $m = 1, \dots, N$ is a connected domain with C^2 boundary with constants r_0 and L. These domains are pairwise non-overlapping.

b) The closure of Ω is the union of the closures of D_m for $m = 1, \dots, N$,

$$\overline{\Omega} = \bigcup_{m=1}^{N} \overline{D}_m.$$

c) There exists a domain, denoted by D_1, such that the intersection $\partial D_1 \cap \Sigma$ contains a flat portion Σ_1 of size $r_0/3$. For any index $m \in \{2, \dots, N\}$, the intersection $\partial D_m \cap \partial D_{m+1}$ contains a flat portion $\Sigma_{m+1} \subset \Omega$ of size $r_0/3$. Furthermore, we assume that there exists a point $P_{m+1} \in \Sigma_{m+1}$ and a rigid transformation under which P_{m+1} coincides with the origin O and

$$\Sigma_{m+1} \cap B_{r_0/3} = \Big\{x \in B_{r_0/3} \,:\, x_n = 0\Big\},$$
$$D_m \cap B_{r_0/3} = \Big\{x \in B_{r_0/3} \,:\, x_n < 0\Big\},$$
$$D_{m+1} \cap B_{r_0/3} = \Big\{x \in B_{r_0/3} \,:\, x_n > 0\Big\}.$$

Note that since the boundary is of class C^2, for each pair of contiguous subdomains, it is always possible to apply a local diffeomorphism that flattens the boundary. However, in order to prove the stability estimate, it is convenient to assume this condition.

A priori information on the coefficients

Consider the elliptic equation

$$\mathrm{div}(\sigma\,\nabla u) + q\,u = 0 \quad \text{in } \Omega. \tag{5.2}$$

The coefficient σ is a bounded, measurable real $n \times n$ matrix function of the form

$$\sigma(x) = \gamma(x)\,A(x), \qquad x \in \Omega, \tag{5.3}$$

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j(x)\,\chi_{D_j}(x), \qquad \gamma_j(x) = a_j + b_j \cdot x, \qquad x \in \Omega, \tag{5.4}$$

for $a_j \in \mathbb{R}$, $b_j \in \mathbb{R}^n$, and D_j for $j = 1, \dots, N$ are given subdomains of the given partition. Moreover, there exists a constant $\bar{\gamma} > 1$ such that for almost every $x \in \Omega$,

$$\bar{\gamma}^{-1} \le \gamma_j(x) \le \bar{\gamma}, \qquad \text{for any } j = 1, \dots, N. \tag{5.5}$$

The matrix function A belongs to the space $C^{1,1}(\Omega, Sym_n)$ and there is a constant $\bar{A} > 0$ such that

$$\|a_{ij}\|_{C^{1,1}(\Omega)} \le \bar{A}, \qquad \text{for } i, j = 1, \dots, n, \tag{5.6}$$

where

$$\|a_{ij}\|_{C^{1,1}(\Omega)} = \|a_{ij}\|_{C^1(\Omega)} + r_0 \sup_{x,y\in\Omega, x\neq y} \frac{|\nabla a_{ij}(x) - \nabla a_{ij}(y)|}{|x-y|}.$$

The matrix function σ satisfies the uniform ellipticity condition, namely there exists a constant $\bar{\lambda} > 1$ such that

$$\bar{\lambda}^{-1}|\xi|^2 \le \sigma(x)\xi \cdot \xi \le \bar{\lambda}|\xi|^2, \qquad \text{for any } \xi \in \mathbb{R}^n, \text{ for a.e. } x \in \Omega. \tag{5.7}$$

The coefficient $q \in L^\infty(\Omega)$ is a piecewise affine function of the form

$$q(x) = \sum_{j=1}^{N} q_j(x)\chi_{D_j}(x), \qquad q_j(x) = c_j + d_j \cdot x, \qquad x \in \Omega,$$

for $c_j \in \mathbb{R}$, $d_j \in \mathbb{R}^n$, and D_j for $j = 1, \dots, N$ are the given subdomains of the partition. Moreover, we assume that there are $\bar{\sigma}, \bar{q} > 0$ such that

$$\|\sigma\|_{L^\infty(\Omega)} \le \bar{\sigma}, \qquad \|q\|_{L^\infty(\Omega)} \le \bar{q}. \tag{5.8}$$

The collection of constants $\{r_0, L, N, \bar{\lambda}, \bar{A}, \bar{\gamma}, \bar{\sigma}, \bar{q}\}$ along with the dimension $n \ge 3$ are called the *a-priori* data. We will follow the so-called *constant variable convention*, where positive constants that depend only on the *a-priori* data and may vary from line to line in the inequalities will be denoted as the letter C.

Remark 5.1.1. *The class of functions $\gamma(x)$ and $q(x)$ form a finite dimensional linear subspace. The L^∞ norms of γ and q can be expressed in terms of the following norms:*

$$|||\gamma||| = \max_{j=1,\dots,N}\{|a_j| + |b_j|\}, \qquad |||q||| = \max_{j=1,\dots,N}\{|c_j| + |d_j|\},$$

modulo some constants that depend on the a-priori data.

Local Cauchy data set

For $f \in H^{1/2}_{00}(\Sigma)$, consider the boundary value problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u) + q\, u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{5.9}$$

The boundary value problem (5.9) may not have a unique solution. In this general framework, the Dirichlet-to-Neumann map may not be defined. To address this issue, we can introduce a set to model the pairs $(u|_{\partial\Omega}, \sigma\, \nabla u \cdot \nu|_{\partial\Omega})$, which we call the local Cauchy data.

Definition 5.1.1. *The* local Cauchy data $(u|_{\partial\Omega}, \sigma\nabla u \cdot \nu|_{\partial\Omega})$ *associated to σ, q having zero first component on $\partial\Omega \setminus \overline{\Sigma}$ is the set*

$$\begin{aligned} \mathcal{C}_{\sigma,q}(\Sigma) = \Big\{(f,g) \in H^{1/2}_{00}(\Sigma) \times H^{-1/2}(\partial\Omega)|_\Sigma \,:\, & \textit{there exists } u \in H^1(\Omega) \textit{ such that} \\ & \operatorname{div}(\sigma\nabla u) + qu = 0 \quad \textit{in } \Omega, \\ & u|_{\partial\Omega} = f, \\ & \langle \sigma\nabla u \cdot \nu|_{\partial\Omega}, \varphi\rangle = \langle g, \varphi\rangle \quad \textit{for any } \varphi \in H^{1/2}_{00}(\Sigma)\Big\}. \end{aligned}$$

The local Cauchy data is a subset of the Hilbert space $H^{1/2}_{00}(\Sigma) \times H^{-1/2}(\partial\Omega)|_\Sigma$ with norm as in (4.12).

The distance between two closed subsets $\mathcal{F}$ and $\mathcal{G}$ of a given Hilbert space is defined as

$$d(\mathcal{F}, \mathcal{G}) = \max\left\{ \sup_{h \in \mathcal{G}\setminus\{0\}} \inf_{k \in \mathcal{F}} \frac{\|h-k\|_{\mathcal{H}}}{\|h\|_{\mathcal{H}}}, \sup_{k \in \mathcal{F}\setminus\{0\}} \inf_{h \in \mathcal{G}} \frac{\|h-k\|_{\mathcal{H}}}{\|k\|_{\mathcal{H}}} \right\}.$$

Let $\{\sigma_k, q^{(k)}\}_{k=1,2}$ be two set of coefficients, we denote by $\mathcal{C}_k$, $k = 1, 2$, the corresponding local Cauchy data. Since we deal with sets that are quite close to each other, we can assume that the distance between two local Cauchy data is simply given by

$$d(\mathcal{C}_1, \mathcal{C}_2) = \sup_{(f_2,g_2)\in\mathcal{C}_2\setminus\{(0,0)\}} \inf_{(f_1,g_1)\in\mathcal{C}_1} \frac{\|(f_2,g_2)-(f_1,g_1)\|_{\mathcal{H}}}{\|(f_2,g_2)\|_{\mathcal{H}}}, \tag{5.10}$$

with $\mathcal{H}$ as in (4.11). It is important to notice that the Cauchy data $\mathcal{C}_1, \mathcal{C}_2$ are closed sets. Furthermore, if the direct problem is well-posed, then the local Cauchy data represents the graph of the local Dirichlet to Neumann map.

We state the stability estimate that will be proven in Section 5.4.

Theorem 5.1.2. *Let $\Omega \subset \mathbb{R}^n$ and $\Sigma \subset \partial\Omega$ be a bounded domain and a non-empty portion as stated above. Let $\{\sigma_k, q^{(k)}\}$ for $k = 1,2$ be two sets of parameters that satisfy the assumptions stated above. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the corresponding local Cauchy data and assume that $d(\mathcal{C}_1, \mathcal{C}_2) < 1$. Then, there exists a constant $C > 0$ depending on the a-priori data only such that*

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} \le C\, d(\mathcal{C}_1, \mathcal{C}_2). \tag{5.11}$$

The following corollary is a direct consequence of the proof of Theorem 5.1.2, hence we omit its proof.

Corollary 5.1.3. *Under the assumptions of Theorem 5.1.2, there exist constants $C > 0$, $0 < \eta < 1$ depending on the a-priori data only such that*

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Sigma)} + \|q^{(1)} - q^{(2)}\|_{L^\infty(\Sigma)} \le C(d(\mathcal{C}_1, \mathcal{C}_2) + E)^{1-\eta} d(\mathcal{C}_1, \mathcal{C}_2)^\eta, \tag{5.12}$$

with $E = \max\{\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Sigma)}, \|q^{(1)} - q^{(2)}\|_{L^\infty(\Sigma)}\}$.

5.2 Green functions and asymptotic estimates

We recall that by the a priori assumptions on the domain, there exists a point $P_1 \in \Sigma_1$ such that, up to a rigid transformation, we have that P_1 coincides with the origin. Without loss of generality, we can assume that $\Sigma = \Sigma_1$. We define

$$D_0 = \left\{ x \in (\mathbb{R}^n \setminus \Omega) \cap B_{r_0} \,:\, |x_i| < \frac{2}{3} r_0, \text{ for } i = 1,\dots,n-1, \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6} r_0 \right\}.$$

The enlarged domain is defined as

$$\Omega_0 = \text{Int}_{\mathbb{R}^n}(\overline{\Omega \cup D_0}).$$

The set Ω_0 is a bounded domain with boundary of Lipschitz class of constants $r_0/3$ and $\tilde{L}$, where $\tilde{L}$ depends on L. Moreover, we introduce the following sets

$$\Sigma_0 = \left\{ x \in \partial\Omega_0 \setminus \partial\Omega \,:\, |x_i| < \frac{2}{3} r_0, \text{ for } i = 1,\dots,n-1, \ x_n = -\frac{2}{3} r_0 \right\},$$

$$(\Omega_0)_r = \{ x \in \Omega_0 \,:\, \text{dist}(x, \partial\Omega_0) \ge r \}, \qquad \text{for some } r \in (0, r_0/6).$$

Let σ, q be a pair of coefficients of (5.2) as described above. We extend them on D_0 by setting $\sigma|_{D_0} = Id_n$, $\gamma|_{D_0} = 1$ and $q|_{D_0} = 1$, where Id_n denotes the $n \times n$

identity matrix. With an abuse of notation, we denote with the same letters the two extended coefficients when we deal with the enlarged domain Ω_0.

Let G be the Green function of Lemma 4.2.1 such that, for every $y \in \Omega_0$, $G(\cdot, y)$ is the unique distributional solution of the mixed boundary value problem

$$\begin{cases} \mathrm{div}(\sigma \nabla G(\cdot, y)) + q\, G(\cdot, y) = -\delta(\cdot - y) & \text{in } \Omega_0, \\ G(\cdot, y) = 0 & \text{on } \partial\,\Omega_0 \setminus \Sigma_0, \\ \sigma \nabla G(\cdot, y) \cdot \nu + i\, G(\cdot, y) = 0 & \text{on } \Sigma_0, \end{cases} \tag{5.13}$$

where $\delta(\cdot - y)$ is a Dirac distribution centred at y. Moreover, there exists a positive constant C that depends only on λ and n such that

$$0 < |G(x, y)| < C\, |x - y|^{2-n}, \qquad \text{for any } x, y \in \Omega_0,\ x \neq y. \tag{5.14}$$

Proposition 5.2.1. *For all $y \in \Omega_0$ and every $r > 0$, the following inequality holds:*

$$\int_{\Omega_0 \setminus B_r(y)} |\nabla G(\cdot, y)|^2 \leq C\, r^{2-n}, \tag{5.15}$$

where C is a positive constant depending on the a-priori data.

Proof. The proof can be derived by combining the Caccioppoli inequality with equation (5.14). □

Fix an index $m \in \{0, \ldots, N-1\}$, let $P_{m+1} \in \Sigma_{m+1}$ and assume that, up to a rigid transformation, P_{m+1} coincides with the origin O and Σ_{m+1} is a flat hyperplane of size $r_0/3$. Define the following quantities: $\gamma^+ = \gamma_{m+1}(0)$, $\gamma^- = \gamma_m(0)$, $A = A(0)$, $J = \sqrt{A(0)^{-1}}$, and $|J| = \det J$. We define

$$\sigma_0(x) := (\gamma^+ \chi_+(x) + \gamma^- \chi_-(x))A,$$

where $\chi_\pm = \chi_{\mathbb{R}^n_\pm}$.

The fundamental solution H associated to the elliptic operator $\mathrm{div}(\sigma_0(\cdot))A\nabla\cdot)$ in $\mathbb{R}^n$ is given by the formula (A.15).

Proposition 5.2.2. *Fix $m \in \{0, \ldots, N-1\}$. Let $Q_{m+1} \in B_{r_0/4}(P_{m+1}) \cap \Sigma_{m+1}$, where Σ_{m+1} is the flat portion as described in the a priori assumptions. For $r \in (0, r_0/8)$, set $y_{m+1} = Q_{m+1} - r\nu(Q_{m+1})$, where $\nu(Q_{m+1})$ is the outward unit normal of $\partial\, D_m$ at Q_{m+1} and let $x \in B_{r_0/4}(Q_{m+1}) \cap D_{m+1}$. Then there exist C_1, C_2, C_3, C_4 positive constants, $0 < \theta_1, \theta_2, \theta_3 < 1$ that depend on the a-priori data only such that*

$$|\nabla_x G(x, y_{m+1}) - \nabla_x H(x, y_{m+1})| \leq C_1\, |x - y_{m+1}|^{1-n+\theta_1}, \tag{5.16}$$
$$|\nabla_x \nabla_y G(x, y_{m+1}) - \nabla_x \nabla_y H(x, y_{m+1})| \leq C_2\, |x - y_{m+1}|^{-n+\theta_2}, \tag{5.17}$$
$$|\nabla_y G(x, y_{m+1}) - \nabla_y H(x, y_{m+1})| \leq C_3\, |x - y_{m+1}|^{1-n+\theta_3}, \tag{5.18}$$
$$|\nabla^2_y G(x, y_{m+1}) - \nabla^2_y H(x, y_{m+1})| \leq C_4\, |x - y_{m+1}|^{1-n}. \tag{5.19}$$

Proof of Proposition 5.2.2. For a proof of (5.16) and (5.17), see Proposition 4.3.5.

We prove (5.18) and (5.19). Fix $m \in \{0, \ldots, N-1\}$, and let $Q_{m+1} \in \Sigma_{m+1} \cap B_{r_0/4}(P_{m+1})$. Up to a rigid transformation, we can assume that Q_{m+1} coincides with the origin, so that $y_{m+1} = O - r\nu(O)$.

For simplicity, we write y in place of y_{m+1}. We define the residual $R(x, y)$ as

$$R(x, y) := G(x, y) - H(x, y).$$

It follows that for $y \in \Omega_0$, $R(\cdot, y)$ is a weak solution of

$$\begin{cases} \operatorname{div}(\sigma \nabla R(\cdot, y)) + q\, R(\cdot, y) = -\operatorname{div}((\sigma - \sigma_0)\nabla H(\cdot, y)) - q\, H(\cdot, y) & \text{in } \Omega_0, \\ R(\cdot, y) = -H(\cdot, y) & \text{on } \partial\Omega_0 \setminus \Sigma_0, \\ \sigma \nabla R(\cdot, y) \cdot \nu + i\, R(\cdot, y) = -\sigma \nabla H(\cdot, y) \cdot \nu - iH(\cdot, y) & \text{on } \Sigma_0. \end{cases}$$

By Green's identity, one derives

$$\begin{aligned} R(x, y) = &\int_{\Omega_0} (\sigma_0(z) - \sigma(z)) \nabla_z H(z, y) \cdot \nabla_z G(z, x)\, \mathrm{d}z + \int_{\Omega_0} q(z)\, H(z, y)\, G(z, x)\, \mathrm{d}z \\ &+ \int_{\partial\Omega_0} (\sigma(z) - \sigma_0(z)) \nabla_z H(z, y) \cdot \nu\, G(z, x)\, \mathrm{d}S(z). \end{aligned}$$

Let us define $B = B_{r_0/4}$ and introduce the term $\tilde{R}(x, y)$ as follows:

$$\tilde{R}(x, y) = \int_B (\sigma_0(z) - \sigma(z)) \nabla_z H(z, y) \cdot \nabla_z G(z, x)\, \mathrm{d}z + \int_B q(z)\, H(z, y)\, G(z, x)\, \mathrm{d}z.$$

We can observe that since $|\nabla_y(R(x, y) - \tilde{R}(x, y))| \leq C$ and $|\nabla_y^2(R(x, y) - \tilde{R}(x, y))| \leq C$, our analysis only needs to focus on the asymptotic behaviour of $\nabla_y \tilde{R}(x, y)$ and $\nabla_y^2 \tilde{R}(x, y)$. Let us establish an upper bound for $\nabla_y \tilde{R}(x, y)$.

Define $B' = B'_{r_0/4}$, and let us introduce the following quantities:

$$\begin{aligned} &B^+ = \{x \in B \,:\, x_n > 0\} && B^- = \{x \in B \,:\, x_n < 0\}, \\ &q^+ = q|_{B^+}, \quad q^- = q|_{B^-}, && [q] = (q^+ - q^-)|_{B'}, \\ &\gamma^+ = \gamma|_{B^+}, \quad \gamma^- = \gamma|_{B^-}, && [\sigma] = (\sigma^+ - \sigma^-)|_{B'} = (\gamma^+ - \gamma^-)|_{B'} A|_{B'}. \end{aligned}$$

For $i = 1, \ldots, n$, we have the expression

$$\begin{aligned} \partial_{y_i} \tilde{R}(x, y) = &- \int_B \partial_{y_i}((\sigma - \sigma_0)(z) \nabla_z H(z, y)) \cdot \nabla_z G(z, x)\, \mathrm{d}z + \int_B \partial_{y_i} H(z, y) q(z) G(z, x)\, \mathrm{d}z \\ = &\int_B \partial_{z_i} \left((\sigma(z) - \sigma_0(z)) \nabla_z H(z, y)\right) \cdot \nabla_z G(z, x)\, \mathrm{d}z - \int_B \partial_{z_i} H(z, y) q(z) G(z, x)\, \mathrm{d}z = \\ = &\int_{\partial B} (\sigma(z) - \sigma_0(z)) \nabla_z H(z, y) \cdot \nabla_z G(z, x) e_i \cdot \nu\, \mathrm{d}z - \int_{\partial B} H(z, y) q(z) G(z, x) e_i \cdot \nu\, \mathrm{d}z - \\ - &\int_{B'} [\sigma(z') - \sigma_0(z')] \nabla_z H(z', y) \cdot \nabla_z G(z', x) e_i \cdot e_n\, \mathrm{d}z' + \int_{B'} H(z', y) [q(z')] G(z', x) e_i \cdot e_n\, \mathrm{d}z' - \\ - &\int_B (\sigma(z) - \sigma_0(z)) \nabla_z H(z, y) \cdot \partial_{z_i} \nabla_z G(z, x)\, \mathrm{d}z + \int_B H(z, y)\, \partial_{z_i}(q(z) G(z, x))\, \mathrm{d}z. \end{aligned} \tag{5.20}$$

Notice that $\partial_{z_i}(\sigma - \sigma_0)(z)$ and $\partial_{z_i} q(z)$ are well-defined on $B \setminus B'$. The first and second integrals on the right-hand side of (5.20) can be easily bounded by a positive constant that depends only on the *a-priori* data. The fifth and sixth integrals are dominated by

$$\begin{aligned}&\int_B |(\sigma-\sigma_0)(z)|\,|\nabla_z H(z,y)|\,|\,\partial_{z_i}\,\nabla_z G(z,x)|\;\mathrm{d}z\\ &\le C\int_B |z||z-y|^{1-n}|z-x|^{-n} \le C|x-y|^{1-n+\theta_3},\end{aligned}$$

with $\theta_3 \in (0,1)$. We can further simplify this expression by using $|x-y|^2 = |x_n + r|^2 + |x'|^2 \ge r^2$, which gives us

$$\int_B |(\sigma-\sigma_0)(z)|\,|\nabla_z H(z,y)|\,|\,\partial_{z_i}\,\nabla_z G(z,x)|\;\mathrm{d}z \le Cr^{1-n+\theta_3}.$$

When $i \ne n$, the third and fourth integrals are equal to zero, hence we have

$$|\,\partial_{y_i}\,\tilde{R}(x,y)| \le C\,|x-y|^{1-n+\theta_3}.$$

For the case $i=j=n$, we have

$$\begin{aligned}&\Big|\int_{B'}[(\sigma-\sigma_0)(z')]\nabla_z H(z',y)\cdot\nabla_z G(z',x)\;\mathrm{d}z' + \int_{B'}\partial_{y_j}\,H(z',y)[q(z)]G(z',x)\;\mathrm{d}z'\Big|\\ &\le C\int_{B'}|z'|\cdot|z'-y|^{-n}\cdot|z'-x|^{1-n}\;\mathrm{d}z \le C|x-y|^{2-n-\alpha},\end{aligned}$$

with $0<\alpha<1$. Therefore, we conclude that

$$|\,\partial_{y_n}\,\tilde{R}(x,y)| \le C|x-y|^{1-n+\theta_3}, \qquad \text{with } \theta_3 \in (0,1).$$

To find the upper bound for $\nabla_y^2\tilde{R}(x,y)$, we perform similar calculations. Further differentiation gives us

$$\begin{aligned}\partial_{y_j}\,\partial_{y_i}\,\tilde{R}(x,y) &= \int_{\partial B}((\sigma-\sigma_0)(z)\,\partial_{y_j}\,\nabla_z H(z,y))\cdot\nabla_z G(z,x)e_i\cdot\nu\;\mathrm{d}z\\ &-\int_{\partial B}\partial_{y_j}\,H(z,y)q(z)G(z,x)e_i\cdot\nu\;\mathrm{d}z\\ &-\int_{B'}[(\sigma-\sigma_0)(z')]\,\partial_{y_j}\,\nabla_z H(z',y)\cdot\nabla_z G(z',x)e_i\cdot e_n\;\mathrm{d}z'\\ &+\int_{B'}\partial_{y_j}\,H(z',y)[q(z')]G(z',x)e_i\cdot e_n\;\mathrm{d}z'\\ &-\int_B \partial_{y_j}(\sigma-\sigma_0)(z)\nabla_z H(z,y))\cdot\partial_{z_i}\,\nabla_z G(z,x)\;\mathrm{d}z\\ &+\int_B \partial_{y_j}\,H(z,y)\,\partial_{z_i}(q(z)G(z,x))\;\mathrm{d}z.\end{aligned} \tag{5.21}$$

The first and second integrals on the right-hand side of (5.21) can be easily bounded. As for the fifth and sixth integrals, they are dominated by

$$\Big|\int_B \partial_{y_j}(\sigma-\sigma_0)(z)\,\nabla_z H(z,y)\;\partial_{z_i}\,\nabla_z G(z,x)\;\mathrm{d}z\Big| \le C\int_B |z-y|^{1-n}|z-x|^{-n} \le C|x-y|^{1-n}.$$

Since $|x-y|^2 = |x_n + r|^2 + |x'|^2 \geq r^2$, we can derive

$$\Big|\int_B \partial_{y_j}(\sigma-\sigma_0)(z)\,\nabla_z H(z,y)\;\partial_{z_i}\,\nabla_z G(z,x)\;\mathrm{d}z\Big| \leq C\,r^{1-n}.$$

Notice that when $(i,j)\neq(n,n)$, the third and fourth integrals are equal to zero. Therefore, we have

$$|\,\partial_{y_j}\,\partial_{y_i}\,\tilde{R}(x,y)| \leq C|x-y|^{1-n}.$$

When $i=j=n$, we have

$$\begin{aligned}\Big|\int_{B'}&[(\sigma-\sigma_0)(z')]\,\partial_{y_j}\,\nabla_z H(z',y)\cdot\nabla_z G(z,x)e_i\cdot e_n\;\mathrm{d}z'\\ &+\int_{B'}\partial_{y_j}\,H(z',y)[q(z)]G(z',x)e_i\cdot e_n\;\mathrm{d}z'\Big|\\ &\leq C\int_{B'}|z'|\cdot\frac{1}{|z'-y|^n}\cdot\frac{1}{|z'-x|^{n-1}}\;\mathrm{d}z \leq C|x-y|^{2-n}.\end{aligned}$$

Thus, we conclude that

$$|\,\partial^2_{y_n}\,\tilde{R}(x,y)| \leq C|x-y|^{1-n}.$$

□

5.3 Quantitative estimates of unique continuation

In this subsection, we define the singular solutions, and we establish suitable quantitative estimates of unique continuation. Define the following sets:

$$\begin{aligned}&\mathcal{W}_k = \bigcup_{m=0}^{k} D_m,\\ &\mathcal{U}_0=\Omega,\qquad \mathcal{U}_k=\Omega_0\setminus\overline{\mathcal{W}_k}\quad\text{for } k=1,\dots,N.\end{aligned}$$

For $y,z\in\mathcal{W}_k$, define the singular solution

$$\begin{aligned}S_k(y,z)=\int_{\mathcal{U}_k}&(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x G_1(x,y)\cdot\nabla_x G_2(x,z)\;\mathrm{d}x\\ &+\int_{\mathcal{U}_k}(q^{(2)}-q^{(1)})(x)G_1(x,y)G_2(x,z)\;\mathrm{d}x,\end{aligned}\tag{5.22}$$

where G_j are the weak solutions to (5.13). Moreover, for $i,j=1,\dots,n$, the following partial derivatives are well defined:

$$\begin{aligned}\partial_{y_i}\,\partial_{z_j}\,S_k(y,z)=\int_{\mathcal{U}_k}&(\sigma^{(1)}-\sigma^{(2)})(x)\,\partial_{y_i}\,\nabla_x G_1(x,y)\cdot\partial_{z_j}\,\nabla_x G_2(x,z)\;\mathrm{d}x\\ &+\int_{\mathcal{U}_k}(q^{(2)}-q^{(1)})(x)\,\partial_{y_i}\,G_1(x,y)\,\partial_{z_j}\,G_2(x,z)\;\mathrm{d}x,\end{aligned}\tag{5.23}$$

and

$$\partial^2_{y_iy_j}\,\partial^2_{z_iz_j}\,S_k(y,z) = \int_{\mathcal{U}_k} (\sigma^{(1)}-\sigma^{(2)})(x)\,\partial^2_{y_iy_j}\,\nabla_x G_1(x,y)\cdot\partial^2_{z_iz_j}\,\nabla_x G_2(x,z)\,\mathrm{d}x + \int_{\mathcal{U}_k} (q^{(2)}-q^{(1)})(x)\,\partial^2_{y_iy_j}\,G_1(x,y)\,\partial^2_{z_iz_j}\,G_2(x,z)\,\mathrm{d}x. \tag{5.24}$$

For any $y,z\in\mathcal{W}_k$, by adapting the argument of Proposition 3.2.4, it can be shown that $S_k(\cdot,z)$ and $S_k(y,\cdot)$ belong to $H^1_{loc}(\mathcal{W}_k)$ and are weak solutions, respectively, to the following equations:

$$\begin{aligned} \mathrm{div}_y(\sigma^{(1)}\nabla_y S_k(\cdot,z)) + q^{(1)}\,S_k(\cdot,z) = 0 \qquad &\text{in } \mathcal{W}_k,\\ \mathrm{div}_z(\sigma^{(2)}\nabla_z S_k(y,\cdot)) + q^{(2)}\,S_k(y,\cdot) = 0 \qquad &\text{in } \mathcal{W}_k. \end{aligned}$$

Recall the definition of E as

$$E := \max\{\|\gamma^{(1)}-\gamma^{(2)}\|_{L^\infty(\Omega)}, \|q^{(1)}-q^{(2)}\|_{L^\infty(\Omega)}\}. \tag{5.25}$$

Notice that by Proposition 5.2.1, for any $y,z\in\mathcal{W}_k$, the inequality

$$|S_k(y,z)| \le C\,E\,(\mathrm{dist}(y,\mathcal{U}_k)\,\mathrm{dist}(z,\mathcal{U}_k))^{1-n/2} \tag{5.26}$$

holds, where C is a positive constant that depends on $\bar{A}$ and the *a-priori* data.

The following Proposition then introduces the quantitative estimates of unique continuation for the singular integrals.

Proposition 5.3.1. *Suppose that for some positive ε_0 we have*

$$|S_k(y,z)| \le \varepsilon_0 \qquad \text{for every } (y,z)\in D_0\times D_0. \tag{5.27}$$

Then there exist constants $\bar{r}>0$ and $C>0$ that depend only on the a-priori data such that the following inequalities hold true for every $r\in(0,\bar{r}/8)$:

$$|S_k\,(y_{k+1},y_{k+1}))| \le C_5 r^{-2\tilde{\gamma}}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)^{\tau_r^2\beta^{2N_1}}(\varepsilon_0+E), \tag{5.28}$$

$$\left|\partial_{y_j}\partial_{z_i}S_k\,(y_{k+1},y_{k+1})\right| \le C_6 r^{-2\tilde{\gamma}-2}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)^{\tau_r^2\beta^{2N_1}}(\varepsilon_0+E), \tag{5.29}$$

$$|\,\partial^2_{y_iy_j}\,\partial^2_{z_iz_j}\,S_k(y_{k+1},y_{k+1})| \le C_7 r^{-2\tilde{\gamma}-4}\left(\frac{\varepsilon_0}{\varepsilon_0+E}\right)^{\tau_r^2\beta^{2N_1}}(\varepsilon_0+E), \tag{5.30}$$

for any $i,j=1,\dots,n$, $y_{k+1}=P_{k+1}-r\nu(P_{k+1})$, where $\nu(P_{k+1})$ is the exterior unit normal to ∂D_k at the point P_{k+1}, $\tilde{\gamma}=\frac{n}{2}-1$, $0<\beta<1$, $N_1\in\mathbb{N}$ and, for $r_1=\bar{r}/8$,

$$\tau_r = \ln\left(\frac{12r_1-2r}{12r_1-3r}\right)/\ln\left(\frac{6r_1-r}{2r_1}\right). \tag{5.31}$$

Remark 5.3.2. *Notice that since*

$$\frac{\tau_r}{r} \ge \frac{1}{12r_1\ln 3}, \tag{5.32}$$

we can replace τ_r with r in Proposition 5.3.1.

To prove Proposition 5.3.1, we apply a result of propagation of smallness for elliptic PDEs with piecewise Lipschitz coefficients.

First, we recall the three sphere inequality in terms of L^∞ norms derived in Chapter 2.

Lemma 5.3.3. *Let $u \in H^1(B_{\bar{r}})$ be a weak solution of*

$$\operatorname{div}(\sigma \nabla u) + q\, u = 0 \qquad \text{in } B_{\bar{r}},$$

with $B_{\bar{r}} \subset (\Omega)_{r_0/3}$, $\bar{r} > 0$. We assume that σ, q satisfy the a priori assumptions. Then, for any $0 < r_1 < r_2 < r_3 \leq \bar{r}$, the following inequality holds:

$$\|u\|_{L^\infty(B_{r_2})} \leq C_\infty \|u\|^{\beta}_{L^\infty(B_{r_1})} \|u\|^{1-\beta}_{L^\infty(B_{r_3})}, \tag{5.33}$$

where $\beta = \ln\left(\dfrac{2r_3}{r_2 + r_3}\right) / \ln\left(\dfrac{r_3}{r_1}\right)$, $\beta \in (0,1)$ and $C_\infty > 1$ depends on $r_1, r_2, r_3, r_0, L, \lambda$ and the bounds on σ and q.

In the following Proposition we derive a result of propagation of smallness valid in our setting (see also [17, Lemma 4.1] and [39, Proposition 3.9]).

Proposition 5.3.4. *For $k = 0, \dots, N-1$ assume that there is a weak solution $v \in H^1(\mathcal{W}_k)$ to*

$$\operatorname{div}(\sigma\, \nabla v) + q\, v = 0 \qquad \text{in } \mathcal{W}_k. \tag{5.34}$$

Suppose that for any given positive number E_0 and ε_0, the function v satisfies

$$|v(x)| \leq \varepsilon_0 \qquad \text{for any } x \in D_0, \tag{5.35}$$

and

$$|v(x)| \leq C(E_0 + \varepsilon_0)\, \operatorname{dist}(x)^{-\tilde{\gamma}} \qquad \text{for any } x \in \mathcal{W}_k, \tag{5.36}$$

with $\tilde{\gamma} = n/2 - 1$. Let $\bar{r}$ be the constant of Lemma 5.3.3. Then, for any $r \in (0, \bar{r}/4)$, there exist constants $C > 1$ and $N_1 \in \mathbb{N}$ such that

$$|v(y_{k+1})| \leq C(E_0 + \varepsilon_0)\Big(\frac{\varepsilon_0}{\varepsilon_0 + E_0}\Big)^{\tau_r \beta^{N_1}} r^{-\tilde{\gamma}}, \tag{5.37}$$

where C, N_1 depend on r_0, L, λ, $\bar{\sigma}, \bar{q}$ only, $y_{k+1} = P_{k+1} - r\nu(P_{k+1})$ with $\nu(P_{k+1})$ the exterior unit normal of ∂D_k at P_{k+1}, $0 < \beta < 1$, $N_1 \in \mathbb{N}$ and τ_r as in (5.31).

Proof of Proposition 5.3.4. We begin the proof by following the lines of [60, Theorem 4.1] and [39, Proposition 3.9]. Let $P_0 \in (D_0)_{r_0/3}$ and let $r_{00} > 0$ be such that $B_{r_{00}}(P_0) \subset (D_0)_{r_0/3}$. By (5.35), we have

$$|v(x)| \leq \varepsilon_0 \qquad \text{for any } x \in B_{r_{00}}(P_0).$$

Next, let $P_{k+1} \in \Sigma_{k+1}$, and consider $\bar{y}_{k+1} = P_{k+1} - 3\, r_1\, \nu(P_{k+1})$, where $\nu(P_{k+1})$ is the exterior unit normal of ∂D_k at P_{k+1} and r_1 will be chosen later. For any point $y_0 \in B_{r_{00}}(P_0)$, we can find a Jordan curve contained in $\mathcal{W}_k$ that connects y_0 to $\bar{y}_{k+1}$. Let us call this curve $c(t) \in C([0,1], \mathcal{W}_k)$, with $c(0) = y_0$ and $c(1) = \bar{y}_{k+1}$.

Next, we define the radii $r_3 = \bar{r}/2$, $r_2 = 3\, r_3/4$, and $r_1 = r_3/4$. This allows us to have $B_{r_1}(y_0) \subset B_{r_3}(y_0) \subset (D_0)_{r_0/3}$. Now, let us consider a partition $0 = t_0 < t_1 < \cdots < t_{\bar{N}} = 1$ of the interval $[0,1]$ and let us define a sequence of points $c(t_k)$ on the Jordan curve as follows:

$$t_{k+1} = \max\{t : |c(t) - c(t_k)| = 2r_1\} \qquad \text{as long as} \quad |\bar{y}_{k+1} - c(t_k)| > 2r_1,$$
$$\text{otherwise } \bar{N} = k+1,\ t_{\bar{N}} = 1.$$

Notice that $B_{r_1}(c(t_k)) \cap B_{r_1}(c(t_{k+1})) = \emptyset$ and $B_{r_1}(c(t_{k+1})) \subset B_{r_2}(c(t_k))$ for $k = 1, \ldots, \bar{N}-1$. Using Lemma 5.3.3, we can propagate the estimate $|v(y_0)|$ along the Jordan curve up to a ball centered at $\bar{y}_{k+1}$ of radius r_1, passing through the flat interfaces Σ_m for $m \in \{1, \ldots, k\}$. This propagation leads to the following inequality:

$$|v(y_{k+1})| \le C\, \varepsilon_0^{\beta^{N_1}} (\varepsilon_0 + E_0)^{1-\beta^{N_1}},$$

where $0 < \beta < 1$, $N_1 \in \mathbb{N}$, and $C > 0$ depend only on the *a-priori* data.

Now, consider $r < r_1$ and let $y_{k+1} = P_{k+1} - r\, \nu(P_{k+1})$. We can apply Lemma 5.3.3 to spheres centred at $\bar{y}_{k+1}$ with radii $r_1, 3r_1 - r$, and $3r_1 - r/2$. This gives us the inequality

$$\|v\|_{L^\infty(B_{3r_1 - r}(\bar{y}_{k+1})} \le C r^{-(1-\tau_r)\tilde{\gamma}} \left(\frac{\varepsilon_0}{\varepsilon_0 + E_0}\right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E_0),$$

where

$$\tau_r = \log\left(\frac{12r_1 - 2r}{12r_1 - 3r}\right) / \log\left(\frac{6r_1 - r}{2r}\right).$$

Finally, we observe that

$$C_1 r^{-\tilde{\gamma}} \le r^{-(1-\tau_r)\tilde{\gamma}}. \le C_2 r^{-\tilde{\gamma}}.$$

This completes the proof. □

We are ready to prove the quantitative estimates of unique continuation for the singular solutions.

Proof of Proposition 5.3.1. First, fix $z \in (D_0)_{r_0/3}$ and define $v(y) = S_k(y,z)$. It can be observed that v is a weak solution of the equation

$$\mathrm{div}(\sigma^{(1)} \nabla v) + q^{(1)}\, v = 0 \qquad \text{in } \mathcal{W}_k.$$

Furthermore, for $y \in \mathcal{W}_k$, by (5.26) we have the estimate:

$$|v(y)| \le C\, E\, [\mathrm{dist}(y, \mathcal{U}_k)]^{1-n/2}.$$

By applying Proposition 5.3.4, with $r \in (0, \bar{r}/4)$, $z \in (D_0)_{r_0/3}$, and $y_{k+1} = P_{k+1} - r\, \nu(P_{k+1})$, we obtain

$$|S_k(y_{k+1}, z)| \le C\, r^{-\tilde{\gamma}} \left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)^{\tau_r \beta^{N_1}} (\varepsilon_0 + E),$$

where $\tilde{\gamma} = n/2 - 1$. We define $\tilde{v}(z) = S_k(y_{k+1}, z)$ for $z \in \mathcal{W}_k$. Then $\tilde{v}$ is a weak solution of the equation

$$\operatorname{div}(\sigma^{(2)} \nabla \tilde{v}) + q^{(2)}\, \tilde{v} = 0 \qquad \text{in } \mathcal{W}_k.$$

Since for any $z \in \mathcal{W}_k$ we have

$$|\tilde{v}(z)| \le C\, E\, (r \operatorname{dist}(z, \Sigma_{k+1}))^{1-n/2},$$

we obtain the bound

$$|S_k(y_{k+1}, y_{k+1})| \le C\, r^{-2\tilde{\gamma}} \left(\frac{\varepsilon_0}{\varepsilon_0 + E}\right)^{\tau_r^2 \beta^{2N_1}} (\varepsilon_0 + E).$$

Next, we derive the estimates for the partial derivatives of the integral solution. The function $S_k(y_1, \dots, y_n, z_1, \dots, z_n)$ is a weak solution of the equation

$$\operatorname{div}_y(\sigma^{(1)} \nabla_y S_k(y,z)) + \operatorname{div}_z(\sigma^{(2)} \nabla_z S_k(y,z)) + q^{(1)} S_k(y,z) + q^{(2)} S_k(y,z) = 0 \qquad \text{in } D_k \times D_k.$$

By applying the Schauder interior estimates (see [2] or [125]), we obtain the following inequalities at $y_{k+1} = P_{k+1} - 2\, r\, \nu(P_{k+1})$:

$$\|\, \partial_{y_j} \partial_{z_i} S_k(y,z)\|_{L^\infty(B_{r/2}(y_{k+1}) \times B_{r/2}(y_{k+1}))} \le \frac{C}{r^2} \|S_k(y,z)\|_{L^\infty(B_r(y_{k+1}) \times B_r(y_{k+1}))},$$

and

$$\|\, \partial^2_{y_j} \partial^2_{z_i} S_k(y,z)\|_{L^\infty(B_{r/4}(y_{k+1}) \times B_{r/4}(y_{k+1}))} \le \frac{C}{r^2} \|\, \partial_{y_j} \partial_{z_i} S_k(y,z)\|_{L^\infty(B_{r/2}(y_{k+1}) \times B_{r/2}(y_{k+1}))}.$$

Hence, the desired estimates for the partial derivatives of the integral solution can be obtained from the previous steps. □

5.4 The Lipschitz stability estimate

In this section we will provide the proof of the Lipschitz stability estimate. Before proving it, we will first introduce some notation and some additional observations.

Let $\eta > 0$. We define a non-decreasing function $\omega_\eta(t)$ on the interval $(0, +\infty)$ as follows:

$$\omega_\eta(t) = \begin{cases} 2^\eta\, e^{-2}\, |\ln t|^{-\eta} & \text{for } t \in (0, e^{-2}), \\ e^{-2} & \text{for } t \in [e^{-2}, +\infty). \end{cases} \tag{5.38}$$

We recall that the function $t\omega_\eta\left(\frac{1}{t}\right)$ is non-decreasing on $[0, +\infty)$, and for every $\beta \in (0,1)$, we have the following inequalities:

$$\omega_\eta\left(\frac{t}{\beta}\right) \le |\ln e\, \beta^{-1/2}|^\eta\, \omega_\eta(t) \quad \text{and} \quad \omega_\eta(t^\beta) \le \left(\frac{1}{\beta}\right)^\eta \omega_\eta(t).$$

We also introduce $\omega_\eta^{(0)} = t^\eta$ for $0 < \eta < 1$, and define the iterated composition of ω with itself as $\omega_\eta^{(1)} = \omega_\eta$ and $\omega_\eta^{(j)} = \omega_\eta \circ \omega_\eta^{(j-1)}$ for $j = 2, 3, \dots$.

Let $u_i \in H^1(\Omega)$ for $i = 1, 2$ be two weak solutions to

$$\operatorname{div}(\sigma_i \nabla u_i) + q_i\, u_i = 0 \quad \text{in } \Omega,$$

with $u_i|_{\partial\Omega} \in H^{1/2}_{00}(\Sigma)$. By using the weak formulation for $i = 1, 2$, we obtain the following equation:

$$\begin{aligned}\int_\Omega [(\sigma^{(1)} - \sigma^{(2)})(x)\nabla u_1 \cdot \nabla u_2 + (q^{(2)} - q^{(1)})(x) u_1(x) u_2(x)]\, \mathrm{d}x \\ = \langle \sigma^{(2)} \nabla \bar{u}_2 \cdot \nu, u_1\rangle - \langle \sigma^{(1)} \nabla u_1 \cdot \nu, u_2\rangle. \end{aligned} \tag{5.39}$$

By (4.49), we obtain the following inequality:

$$\begin{aligned}\Big| \int_\Omega [(\sigma^{(1)} - \sigma^{(2)})(x)\nabla u_1(x) \cdot \nabla u_2(x) + (q^{(2)} - q^{(1)})(x) u_1(x) u_2(x)]\, \mathrm{d}x \Big| \\ \le d(\mathcal{C}_1, \mathcal{C}_2)\, \|(u_1, \sigma^{(1)} \nabla u_1 \cdot \nu)\|_{\mathcal{H}}\, \|(\bar{u}_2, \sigma^{(2)} \nabla \bar{u}_2 \cdot \nu)\|_{\mathcal{H}}. \end{aligned} \tag{5.40}$$

Set

$$\begin{aligned} &\varepsilon = d(\mathcal{C}_1, \mathcal{C}_2), \\ &\delta_k = \|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\mathcal{W}_k)}, \qquad \tilde{\delta}_k = \|q^{(1)} - q^{(2)}\|_{L^\infty(\mathcal{W}_k)}, \\ &\delta^*_k = \max\{\delta_k, \tilde{\delta}_k\} \qquad \text{for } k = 1, \dots, \max\{K, \tilde{K}\}. \end{aligned}$$

Proof of Theorem 5.1.2. Let $\{\sigma_i, q_i\}$ for $i = 1, 2$ be two sets of coefficients and let $\mathcal{C}_1, \mathcal{C}_2$ be the corresponding local Cauchy data. By (5.6), the following inequality

$$\|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \le C d(\mathcal{C}_1, \mathcal{C}_2)$$

is equivalent to

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} \le C d(\mathcal{C}_1, \mathcal{C}_2),$$

where $C > 1$ is a constant that depends on the *a-priori* data.

For $K \in \{1, \dots, N\}$, let D_K be the subdomain of the known partition of Ω such that

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(\Omega)} = \|\gamma^{(1)}_K - \gamma^{(2)}_K\|_{L^\infty(D_K)}.$$

Similarly, for $\tilde{K} \in \{1, \dots, N\}$, let $D_{\tilde{K}}$ be such that

$$\|q^{(1)} - q^{(2)}\|_{L^\infty(\Omega)} = \|q^{(1)}_{\tilde{K}} - q^{(2)}_{\tilde{K}}\|_{L^\infty(D_{\tilde{K}})}.$$

Our goal is to prove that

$$\|q^{(1)}_{\tilde{K}} - q^{(2)}_{\tilde{K}}\|_{L^\infty(D_{\tilde{K}})} + \|\gamma^{(1)}_K - \gamma^{(2)}_K\|_{L^\infty(D_K)} \le C d(\mathcal{C}_1, \mathcal{C}_2).$$

Let Ω_0 be the augmented domain and let σ_i and q_i for $i = 1, 2$ be the extended coefficient on D_0, with $\sigma_i|_{D_0} = Id_n$ and $q_i = 1$. Let $D_0, D_1, \dots, D_K$ be the chain of contiguous domains such that $\Sigma_m = \partial D_m \cap \partial D_{m+1}$ and $\Sigma_1 = \partial D_0 \cap \partial D_1$.

Let $\{x_1, \dots, x_n\}$ be a coordinate system with origin at P_k. Let Σ_k be the flat interface contained in the tangential hyperplane of $\partial D_1 \cap B_{r_0/4}$ at P_k. For any scalar function f, we denote with $D_T f(x)$ the $(n-1)$ dimensional vector of the tangential partial derivatives of f at x on Σ_k, and with $\partial_\nu f(x)$ the normal partial derivative of f at x. The function $(\gamma_k^{(1)} - \gamma_k^{(2)})$ can be bounded from above in D_k in terms of the quantities

$$\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))} \quad \text{and} \quad |\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)|. \tag{5.41}$$

Indeed, set

$$A_k + B_k \cdot x = (\gamma_k^{(1)} - \gamma_k^{(2)})(x) \qquad \text{for } A_k \in \mathbb{R},\ B_k \in \mathbb{R}^n, \text{ and } x \in D_k.$$

Fix an orthonormal basis $\{e_j\}_{j=1}^{n-1}$ of Σ_k and let $\nu = e_n$ be the direction of the normal of ∂D_1. One can evaluate $(\gamma_k^{(1)} - \gamma_k^{(2)})$ at the points P_k and $P_k + \frac{r_0}{6} e_j$ for $j = 1, \ldots, n$ and derive

$$|A_k + B_k \cdot P_k| + \frac{r_0}{6} \sum_{j=1}^{n-1} |(B_k)_j| \leq C \|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))},$$

and

$$|B_k \cdot e_n| = |\,\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)|.$$

Hence, it turns out that

$$\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(D_k)} \leq C \left(\|\gamma_k^{(1)} - \gamma_k^{(2)}\|_{L^\infty(\Sigma_k \cap B_{r_0/4}(P_k))} + |\,\partial_\nu(\gamma_k^{(1)} - \gamma_k^{(2)})(P_k)| \right),$$

for $C > 0$ constant that depends on the *a-priori* data. A similar consideration holds for $q_k^{(i)}$, $i = 1, 2$ and $k \in \{1, \ldots, N\}$.

Our goal is to estimate δ_k^* for any $k = 1, \ldots, \max\{K, \tilde{K}\}$.

When $k = 1$, we obtain the following Hölder estimates at the boundary:

$$\delta_1 \leq C(E + \varepsilon)\Big(\frac{\varepsilon}{\varepsilon + E}\Big)^{\eta_1}, \tag{5.42}$$

$$\tilde{\delta}_1 \leq C(E + \varepsilon)\Big(\frac{\varepsilon}{\varepsilon + E}\Big)^{\tilde{\eta}_1}, \tag{5.43}$$

with $0 < \eta_1, \tilde{\eta}_1 < 1$ that depend on $\theta_1, \theta_2, \theta_3$ and C are positive constants that depend on the *a-priori* data only.

Our approach involves estimating the L^2 norm of $(\gamma^{(1)} - \gamma^{(2)})$ on Ω, denoted as δ_1, using δ_1^*. We then proceed to estimate the L^2 norm of $(q^{(2)} - q^{(1)})$ on Ω, denoted as $\tilde{\delta}_1$, in terms of δ_1.

Stability at the boundary for σ.

Consider the coordinate system with origin at P_1 and let $\{x_1, \ldots, x_n\}$ be the coordinates. For any $y, z \in D_0$, we have the following identities:

$$\begin{aligned} &\int_\Sigma [\sigma^{(2)}(x) \nabla_x G_2(x,z) \cdot \nu\, G_1(x,y) - \sigma^{(1)}(x) \nabla_x G_1(x,y) \cdot \nu\, G_2(x,z)]\, \mathrm{d}S(x) = \\ &= \int_\Omega [(\sigma^{(1)} - \sigma^{(2)})(x) \nabla_x G_1(x,y) \cdot \nabla_x G_2(x,z) + \\ &+ (q^{(2)} - q^{(1)})(x) G_1(x,y) G_2(x,z)]\, \mathrm{d}x, \end{aligned} \tag{5.44}$$

and

$$\int_{\Sigma}[\sigma^{(2)}(x)\nabla_x\,\partial_{z_n}\,G_2(x,z)\cdot\nu\;\partial_{y_n}\,G_1(x,y)-\sigma^{(1)}(x)\nabla_x\,\partial_{y_n}\,G_1(x,y)\cdot\nu\;\partial_{z_n}\,G_2(x,z)]\;\mathrm{d}S(x)=$$
$$=\int_{\Omega}[(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\,\partial_{y_n}\,G_1(x,y)\cdot\nabla_x\,\partial_{z_n}\,G_2(x,z)+$$
$$+(q^{(2)}-q^{(1)})(x)\,\partial_{y_n}\,G_1(x,y)\,\partial_{z_n}\,G_2(x,z)]\;\mathrm{d}x, \tag{5.45}$$

Using (5.40) and (5.44), we can derive the following inequality:

$$\Big|\int_{\Sigma}[\sigma^{(2)}\nabla_x G_2(x,z)\cdot\nu\,G_1(x,y)-\sigma^{(1)}\nabla_x G_1(x,y)\cdot\nu\,G_2(x,z)]\;\mathrm{d}S(x)\Big|$$
$$\leq C\varepsilon\,(\mathrm{dist}(y)\mathrm{dist}(z))^{1-n/2}, \tag{5.46}$$

where $\mathrm{dist}(y)$ denotes the distance between y and Ω. We can also express the norm $\|\gamma_1^{(1)}-\gamma_1^{(2)}\|_{L^\infty(D_1)}$ in terms of the quantities

$$\|\gamma_1^{(1)}-\gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))} \quad \text{and} \quad |\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1)|.$$

Let $\rho=r_0/4$, and let $r\in(0,\bar{r}/8)$. Set $w=P_1+r\,\nu(P_1)$, where $\nu(P_1)$ is the exterior unit normal of $\partial\,D_1$ at P_1. Consider

$$S_0(w,w)=I_1(w)+I_2(w), \tag{5.47}$$

where

$$I_1(w)=\int_{B_\rho(P_1)\cap D_1}(\gamma_1^{(1)}-\gamma_1^{(2)})(x)\;A(x)\,\nabla_x G_1(x,w)\cdot\nabla_x G_2(x,w)\;\mathrm{d}x$$
$$+\int_{B_\rho(P_1)\cap D_1}(q_1^{(2)}-q_1^{(1)})(x)\,G_1(x,w)\cdot G_2(x,w)\;\mathrm{d}x,$$

and

$$I_2(w)=\int_{\Omega\setminus(B_\rho(P_1)\cap D_1)}(\sigma^{(1)}-\sigma^{(2)})(x)\,\nabla_x G_1(x,w)\cdot\nabla_x G_2(x,w)\;\mathrm{d}x$$
$$+\int_{\Omega\setminus(B_\rho(P_1)\cap D_1)}(q^{(2)}-q^{(1)})(x)\,G_1(x,w)\cdot G_2(x,w)\;\mathrm{d}x.$$

The volume integrals of $I_2(w)$ can be bounded from above via Caccioppoli inequality A.2.2:

$$|I_2(w)|\leq CE\rho^{2-n}. \tag{5.48}$$

Regarding $I_1(w)$, it is important to note that there exists a point x^* in the closure of $\Sigma_1\cap B_{r_0/4}(P_1)$ such that

$$(\gamma_1^{(1)}-\gamma_1^{(2)})(x^*)=\|\gamma_1^{(1)}-\gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4}(P_1))}. \tag{5.49}$$

Using (5.49), we have

$$\begin{aligned} I_1(w) = &\int_{B_\rho(P_1)\cap D_1} (\gamma_1^{(1)} - \gamma_1^{(2)})(x^*)\, A(x)\, \nabla_x G_1(x,w) \cdot \nabla_x G_2(x,w)\, \mathrm{d}x + \\ &+ \int_{B_\rho(P_1)\cap D_1} B_1 \cdot (x - x^*)\, A(x)\, \nabla_x G_1(x,w) \cdot \nabla_x G_2(x,w)\, \mathrm{d}x + \\ &+ \int_{B_\rho(P_1)\cap D_1} (q_1^{(2)} - q_1^{(1)})(x)\, G_1(x,w) \cdot G_2(x,w)\, \mathrm{d}x. \end{aligned}$$

By the asymptotic estimate (5.16), we obtain

$$\begin{aligned} I_1(w) \geq &\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4})} \Big\{ \int_{B_\rho(P_1)\cap D_1} A(x) \nabla_x H_1(x,w) \cdot \nabla_x H_2(x,w)\, \mathrm{d}x - \\ &- \int_{B_\rho(P_1)\cap D_1} |x-w|^{2(1-n)+\theta_1}\, \mathrm{d}x - \int_{B_\rho(P_1)\cap D_1} |x-w|^{2(1-n+\theta_1)}\, \mathrm{d}x \Big\} - \\ &- CE \int_{B_\rho(P_1)\cap D_1} |x||x-w|^{2(1-n)}\, \mathrm{d}x - CE \int_{B_\rho(P_1)\cap D_1} |x-w|^{2(2-n)}\, \mathrm{d}x. \end{aligned}$$

This implies that

$$|I_1(w)| \geq C \|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1 \cap B_{\frac{r_0}{4}}(P_1))} r^{2-n} - CE r^{2-n+\theta_1} - CE r^{3-n}. \tag{5.50}$$

By rearranging the inequalities (5.50) and (5.48) together with (5.46), we obtain

$$\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))} r^{2-n} \leq CE r^{3-n} + CE r^{2-n+\theta_1} + C\varepsilon r^{2-n} + CE\rho^{2-n}. \tag{5.51}$$

Multiplying (5.51) by r^{n-2} and taking the limit as $r \to 0^+$, we obtain

$$\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))} \leq C\varepsilon. \tag{5.52}$$

Now, we proceed to derive an estimate for the normal partial derivative of $\gamma_1^{(1)} - \gamma_1^{(2)}$ at P_1. Applying Taylor's formula in a neighbourhood of the point P_1, we derive

$$\begin{aligned} (\gamma_1^{(1)} - \gamma_1^{(2)})(x) = &(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1) + (D_T(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)) \cdot (x - P_1)' + \\ &+ (\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)) \cdot (x - P_1)_n. \end{aligned}$$

Hence,

$$|\,\partial_{y_n} \partial_{z_n} S_0(w,w)| \geq I_{11} - I_{12} - I_{13} - I_{14} - I_{15} - I_{16},$$

where

$$I_{11} = \Big| \int_{B_\rho(P_1)\cap D_1} \partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)\cdot(x-P_1)_n A(x)\nabla_x\,\partial_{y_n} G_1(x,w)\cdot\nabla_x\,\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,$$
$$I_{12} = \Big| \int_{B_\rho(P_1)\cap D_1} D_T(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)\cdot(x-P_1)' A(x)\nabla_x\,\partial_{y_n} G_1(x,w)\cdot\nabla_x\,\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,$$
$$I_{13} = \Big| \int_{B_\rho(P_1)\cap D_1} (\gamma_1^{(1)} - \gamma_1^{(2)})(P_1) A(x)\nabla_x\,\partial_{y_n} G_1(x,w)\cdot\nabla_x\,\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,$$
$$I_{14} = \Big| \int_{B_\rho(P_1)\cap D_1} (q_1^{(2)} - q_1^{(1)})(x)\,\partial_{y_n} G_1(x,w)\cdot\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,$$
$$I_{15} = \Big| \int_{\Omega\setminus(B_\rho(P_1)\cap D_1)} (\sigma^{(1)} - \sigma^{(2)})(x)\,\partial_{y_n}\nabla_x G_1(x,w)\cdot\partial_{z_n}\nabla_x G_2(x,w)\,\mathrm{d}x\Big|,$$
$$I_{16} = \Big| \int_{\Omega\setminus(B_\rho(P_1)\cap D_1)} (q^{(1)} - q^{(2)})(x)\,\partial_{y_n} G_1(x,w)\cdot\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|.$$

To estimate I_{11} from below, we add and subtract the fundamental solution. Using (5.17), we derive

$$I_{11} \geq C\,|\,\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| r^{1-n} - C\,E\,r^{1-n+\theta_2}. \tag{5.53}$$

To estimate the terms I_{12} and I_{13}, we notice that

$$|(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| + C\,|D_T(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| \leq C\,\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(\Sigma_1\cap B_{r_0/4})} \leq C\,\varepsilon.$$

This implies that

$$I_{12},\ I_{13} \leq C\,\varepsilon\,r.$$

To estimate the integral I_{14}, we have

$$\begin{aligned} I_{14} &\leq \|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(D_1)} \int_{D_1\cap B_\rho} |\,\partial_{y_n} G_1(x,w)|\,|\,\partial_{z_n} G_2(x,w)|\,\mathrm{d}x \\ &\leq C \int_{D_1\cap B_\rho} |x-w|^{2(1-n)} \leq C\,r^{2-n}. \end{aligned}$$

Using [21, Proposition 3.1], we can bound the integrals I_{15} and I_{16} as follows:

$$I_{15},\ I_{16} \leq C\,E\,\rho^{-n}.$$

To summarise, we have

$$|\,\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| r^{1-n} \leq |\,\partial_{y_n}\,\partial_{z_n} S_0(w,w)| + C\{E r^{1-n+\theta_2} + \varepsilon r^{-n}\}. \tag{5.54}$$

Since

$$|\,\partial_{y_n}\,\partial_{z_n} S_0(w,w)| \leq C\varepsilon r^{-n},$$

we can derive

$$|\,\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| r^{1-n} \leq C\{E r^{1-n+\theta_2} + \varepsilon r^{-n}\}. \tag{5.55}$$

Multiplying (5.55) by r^{n-1}, we obtain

$$|\,\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| \le C\{Er^{\theta_2} + \varepsilon r^{-1}\}.$$

By optimising with respect to r, we find that

$$|\,\partial_\nu(\gamma_1^{(1)} - \gamma_1^{(2)})(P_1)| \le C(\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\frac{\theta_2}{\theta_2+1}}. \tag{5.56}$$

We can set $\eta_1 = \dfrac{\theta_2}{\theta_2 + 1}$. Hence, we conclude that

$$\|\gamma_1^{(1)} - \gamma_1^{(2)}\|_{L^\infty(D_1)} \le C(\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\eta_1}. \tag{5.57}$$

Stability at the boundary for q

Our goal is to derive a bound for $\|q_1^{(1)} - q_1^{(2)}\|_{L^\infty(D_1)}$ in terms of (5.57). A suitable bound for the norm $\|q_1^{(2)} - q^{(1)}\|_{L^\infty(D_1)}$ can be obtained by the following quantities:

$$\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))} \quad \text{and} \quad |\,\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1)|. \tag{5.58}$$

Let us consider $\rho = r_0/4$ and $r \in (0, \bar{r}/8)$, and set $w = P_1 + r\,\nu(P_1)$. Consider

$$\partial_{y_n}\partial_{z_n} S_0(w,w) = \partial_{y_n}\partial_{z_n} I_1(w) + \partial_{y_n}\partial_{z_n} I_2(w),$$

with $w = P_1 + r\nu(P_1)$, as above. The term $\partial_{y_n}\partial_{z_n} I_2(w)$ can be bounded from above as

$$\partial_{y_n}\partial_{z_n} I_2(w) \le C\,E\rho^{-n}.$$

To determine a lower bound for $\partial_{y_n}\partial_{z_n} I_1(w)$, first notice that there exists a point $\bar{x} \in \overline{\Sigma_1 \cap B_{r_0/4}(P_1)}$ such that

$$(q_1^{(2)} - q_1^{(1)})(\bar{x}) = \|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))}.$$

Using (5.16) and (5.57), it follows that

$$C\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{r_0/4}(P_1))} r^{2-n} \le |\,\partial_{y_n}\partial_{z_n} I_1(w)| + CEr^{2-n+\theta_1} + C(\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\eta_1} r^{-n}.$$

From (5.45), we have

$$|\,\partial_{y_n}\partial_{z_n} S_0(w,w)| \le C\,\varepsilon\, r^{-n}.$$

Hence, by combining the upper bound for $I_2(w)$ and the lower bound for $I_1(w)$, we obtain

$$\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{\frac{r_0}{4}}(P_1))} r^{2-n} \le C\Big\{\varepsilon\, r^{-n} + Er^{2-n+\theta_1} + (\varepsilon + E)\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\eta_1} r^{-n} + E\Big\}.$$

Multiplying by r^{n-2} leads to

$$\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{\frac{r_0}{4}}(P_1))} \le C(\varepsilon + E)\Big\{\left(\frac{\varepsilon}{\varepsilon + E}\right)^{\eta_1} r^{-2} + Er^{\theta_1}\Big\}.$$

By optimising with respect to r, we conclude that

$$\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1 \cap B_{\frac{r_0}{4}}(P_1))} \leq C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\eta_1\theta_1}{\theta_1+2}}. \tag{5.59}$$

To estimate $|\,\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1)|$, consider the singular solution $\partial^2_{y_iy_j}\,\partial^2_{z_iz_j}\,S_0(w,w)$ and split it as the sum of the terms

$$I_1^{ij}(w) = \int_{D_1\cap B_\rho(P_1)} (\sigma_1^{(1)} - \sigma_1^{(2)})(x)\nabla_x\,\partial^2_{y_iy_j}\,G_1(x,w)\cdot\partial^2_{z_iz_j}\,G_2(x,w)\,\mathrm{d}x + \\ + \int_{D_1\cap B_\rho(P_1)} (q_1^{(2)} - q_1^{(1)})(x)\,\partial^2_{y_iy_j}\,G_1(x,w)\cdot\partial^2_{z_iz_j}\,G_2(x,w)\,\mathrm{d}x,$$

and

$$I_2^{ij}(w) = \int_{\Omega\setminus(D_1\cap B_\rho(P_1))} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x\,\partial^2_{y_iy_j}\,G_1(x,w)\cdot\nabla_x\,\partial^2_{z_iz_j}\,G_2(x,w)\,\mathrm{d}x + \\ + \int_{\Omega\setminus(D_1\cap B_\rho(P_1))} (q^{(2)} - q^{(1)})(x)\,\partial^2_{y_iy_j}\,G_1(x,w)\cdot\partial^2_{z_iz_j}\,G_2(x,w)\,\mathrm{d}x.$$

Set $I_m(w) = \{I_m^{ij}(w)\}_{i,j=1,\dots,n}$. Denote by $|I_m(w)|$ the Euclidean norm of the matrix $I_m(w)$. The upper bound for $|I_2(w)|$ is given by

$$|I_2(w)| \leq CE\rho^{-(n+2)},$$

where C is a positive constant that depends on the *a-priori* data only. For the lower bound for $I_1(w)$,

$$|I_1(w)| \geq \frac{1}{n}\sum_{i,j=1}^{n}\Big\{\Big|\int_{D_1\cap B_\rho(P_1)} (\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1))\cdot(x-P_1)_n\,\partial^2_{y_iy_j}\,G_1(x,w)\cdot\partial^2_{z_iz_j}\,G_2(x,w)\,\mathrm{d}x\Big| - \\ - \Big|\int_{D_1\cap B_\rho(P_1)} (D_T(q_1^{(2)} - q_1^{(1)})(P_1))\cdot(x-P_1)'\,\partial^2_{y_iy_j}\,G_1(x,w)\cdot\partial^2_{z_iz_j}\,G_2(x,w)\,\mathrm{d}x\Big| - \\ - \Big|\int_{D_1\cap B_\rho(P_1)} (q_1^{(2)} - q_1^{(1)})(P_1)\,\partial^2_{y_iy_j}\,G_1(x,w)\cdot\partial^2_{z_iz_j}\,G_2(x,w)\,\mathrm{d}x\Big|\Big\} - \\ - \Big|\int_{D_1\cap B_\rho(P_1)} (\sigma_1^{(2)} - \sigma_1^{(1)})(x)\,\partial^2_{y_iy_j}\,\nabla_x G_1(x,w)\cdot\partial^2_{z_iz_j}\,\nabla_x G_2(x,w)\,\mathrm{d}x\Big|.$$

Since

$$|(q_1^{(2)} - q_1^{(1)})(P_1)| + C|(D_T(q_1^{(2)} - q_1^{(1)})(P_1))| \leq C\|q_1^{(2)} - q_1^{(1)}\|_{L^\infty(\Sigma_1\cap B_{\frac{r_0}{4}}(P_1))},$$

by (5.59) and (5.19), one derives

$$|I_1(w)| \geq C|(\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1))|r^{1-n} - C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\eta_1\theta_1}{\theta_1+2}} r^{-n} - \\ - CEr^{1+\theta_2-n} - C(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_1} r^{-2-n}. \tag{5.60}$$

Since for $y, z \in (D_0)_{r_0/3}$,

$$\int_\Sigma [\sigma^{(2)}(x)\nabla_x \partial^2_{z_n} G_2(x,z)\cdot\nu\, \partial^2_{y_n} G_1(x,y) - \sigma^{(1)}(x)\nabla_x \partial^2_{y_n} G_1(x,y)\cdot\nu\, \partial^2_{z_n} G_2(x,z)]\,\mathrm{d}S(x) =$$
$$= \int_\Omega [(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x \partial^2_{y_n} G_1(x,y)\cdot\nabla_x \partial^2_{z_n} G_2(x,z)\,\mathrm{d}x$$
$$+ \int_\Omega (q^{(2)}-q^{(1)})(x)\,\partial^2_{y_n} G_1(x,y)\,\partial^2_{z_n} G_2(x,z)]\,\mathrm{d}x,$$

it turns out that

$$|\,\partial^2_{y_n}\partial^2_{z_n} S_0(w,w)| \le C\,\varepsilon\, r^{-2-n}. \tag{5.61}$$

By (5.60) and (5.61), one derives

$$|(\partial_\nu(q_1^{(2)}-q_1^{(1)})(P_1))|r^{1-n} \le C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\frac{\eta_1\theta_1}{\theta_1+2}} r^{-n}+$$
$$+C(\varepsilon+E)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_1} r^{-2-n} + CEr^{1+\theta_2-n}C\varepsilon r^{-1-n}.$$

Multiply by r^{n-1} the last equation and optimise with respect to r leads to the estimate

$$|(\partial_\nu(q_1^{(2)}-q_1^{(1)})(P_1))| \le C(E+\varepsilon)\left(\frac{\varepsilon}{\varepsilon+E}\right)^{\eta_2},$$

with $\eta_2 \in (0,1)$.

We proceed by estimating δ_2^*. Our approach is to proceed similarly as for the first domain. We summarise the main steps. We claim that the following inequalities hold:

$$\delta_2 \le C(\varepsilon+E)\omega^{(2)}_{\eta_2}\Big(\frac{\varepsilon}{\varepsilon+E}\Big), \tag{5.62}$$
$$\tilde\delta_2 \le C(\varepsilon+E)\omega^{(3)}_{\tilde\eta_2}\Big(\frac{\varepsilon}{\varepsilon+E}\Big), \tag{5.63}$$

with $0<\eta_2,\tilde\eta_2<1$.

For any $y, z \in D_0$, the following identities hold:

$$\int_\Sigma [\sigma^{(2)}(x)\nabla_x G_2(x,z)\cdot\nu\, G_1(x,y) - \sigma^{(1)}(x)\nabla_x G_1(x,y)\cdot\nu\, G_2(x,z)]\,\mathrm{d}S(x) =$$
$$= S_1(y,z) + \int_{\mathcal{W}_1} [(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x G_1(x,y)\cdot\nabla_x G_2(x,z) + (q^{(2)}-q^{(1)})(x)G_1(x,y)G_2(x,z)]\,\mathrm{d}x, \tag{5.64}$$

and

$$\int_\Sigma [\sigma^{(2)}(x)\nabla_x \partial_{z_n} G_2(x,z)\cdot\nu\, \partial_{y_n} G_1(x,y) - \sigma^{(1)}(x)\nabla_x \partial_{y_n} G_1(x,y)\cdot\nu\, \partial_{z_n} G_2(x,z)]\,\mathrm{d}S(x) =$$
$$= \partial_{y_n}\partial_{z_n} S_1(y,z) + \int_{\mathcal{W}_1} (\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x \partial_{y_n} G_1(x,y)\cdot\nabla_x \partial_{z_n} G_2(x,z)\,\mathrm{d}x+$$
$$+ \int_{\mathcal{W}_1} (q^{(2)}-q^{(1)})(x)\,\partial_{y_n} G_1(x,y)\,\partial_{z_n} G_2(x,z)\,\mathrm{d}x. \tag{5.65}$$

Let $\rho = r_0/4$ and $r \in (0, \bar{r}/8)$, and set $w = P_2 + r\nu(P_2)$, where $\nu(P_2)$ is the exterior unit normal of ∂D_2 at P_2. Consider

$$S_1(w, w) = I_1(w) + I_2(w), \tag{5.66}$$

with

$$\begin{aligned} I_1(w) = &\int_{B_\rho(P_2)\cap D_2} (\gamma_2^{(1)} - \gamma_2^{(2)})(x)\, A(x)\, \nabla_x G_1(x, w) \cdot \nabla_x G_2(x, w)\, \mathrm{d}x + \\ &+ \int_{B_\rho(P_2)\cap D_2} (q_2^{(2)} - q_2^{(1)})(x)\, G_1(x, w) \cdot G_2(x, w)\, \mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} I_2(w) = &\int_{\mathcal{U}_1\setminus(B_\rho(P_2)\cap D_2)} (\sigma^{(1)} - \sigma^{(2)})(x)\, \nabla_x G_1(x, w) \cdot \nabla_x G_2(x, w)\, \mathrm{d}x + \\ &+ \int_{\mathcal{U}_1\setminus(B_\rho(P_2)\cap D_2)} (q^{(2)} - q^{(1)})(x)\, G_1(x, w) \cdot G_2(x, w)\, \mathrm{d}x. \end{aligned}$$

The volume integrals of $I_2(w)$ can be bounded from above via Caccioppoli inequality (see also [21, Proposition 3.1]):

$$|I_2(w)| \le CE\rho^{2-n}. \tag{5.67}$$

Regarding $I_1(w)$, by proceeding as for the boundary, we derive the following lower bound:

$$|I_1(w)| \ge C\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} r^{2-n} - CEr^{2-n+\theta_1} - CEr^{3-n}. \tag{5.68}$$

Since for every $y, z \in (D_0)_{r_0/3}$,

$$|S_1(y, z)| \le Cr_0^{2-n}(\varepsilon + \delta_1^*),$$

by (5.28), we have

$$|S_1(y, z)| \le C(\varepsilon + \delta_1^* + E)\Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)^{\beta^{2N_1}\tau_r^2} r^{2-n}. \tag{5.69}$$

Rearranging the inequalities (5.68) and (5.67) together with (5.69) and (5.32), we derive

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} r^{2-n} \le C(\varepsilon + \delta_1^* + E)\Big\{\Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)^{\beta^{2N_1}(12r_1 \ln 3)^{-2} r^2} r^{2-n} + r^{\theta_1}\Big\}. \tag{5.70}$$

To minimise the right-hand side of (5.70), we follow the lines of the procedure introduced in [19, Theorem 5.3]. The function that we want to minimise is

$$f(r) = \zeta^{r^2}\, r^{2-n} + r^{\theta_1} \qquad 0 < r \le r_1,$$

where

$$\zeta = \Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)^{\beta^{2N_1}(12r_1 \ln 3)^{-2}}.$$

Set

$$z = r^2, \quad \sigma = \frac{n-2}{2}, \quad D = \frac{\theta_1}{2}, \quad z_0 = r_1^2,$$

and consider the function

$$\phi(z) = z^D + z^{-\sigma}\,\zeta^z.$$

We want to determine an upper bound to $\inf_{0<z\le z_0}\phi(z)$. We introduce the parameter l as

$$z = \Big(\frac{1}{\log\frac{1}{\zeta}}\Big)^l.$$

We assume by now that $0 < l < 1$ and by the inequality $e^{-s} < 1/s$, we derive

$$\phi(z) \le \Big(\frac{1}{\log\frac{1}{\zeta}}\Big)^{l\,D} + \Big(\frac{1}{\log\frac{1}{\zeta}}\Big)^{1-l(\sigma+1)}.$$

Let us choose l such that $l\,D = 1 - l(\sigma+1)$, namely $l = 2/(\theta_1 + n)$. Set

$$\mu = \min\{lD, 1 - l(\sigma+1)\} = \frac{\theta_1}{\theta_1 + n},$$

then

$$\phi(z) \le 2\Big(\frac{1}{\log\frac{1}{\zeta}}\Big)^{\mu} = 2\Big(\frac{1}{\log\frac{1}{\zeta}}\Big)^{\frac{\theta_1}{\theta_1+n}}.$$

Let

$$r = \Big|\ln\Big(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\Big)^{\beta^{2N_1}(12r_1\ln 3)^{-2}}\Big|^{-\frac{1}{n+\theta_1}},$$

then it turns out that

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} \le C(\varepsilon + \delta_1^* + E)\Big|\ln\Big(\frac{\varepsilon+\delta_1^*}{\varepsilon+\delta_1^*+E}\Big)\Big|^{-\frac{\theta_1}{n+\theta_1}}. \tag{5.71}$$

By the properties of ω_η, one derives

$$\|\gamma_2^{(1)} - \gamma_2^{(2)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} \le C(\varepsilon + E)\omega_\eta\Big(\frac{\varepsilon}{\varepsilon+E}\Big), \tag{5.72}$$

with $0 < \eta < 1$ depending on θ_1.

A similar estimate can be derived for $\partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})$. From Taylor's formula, one derives

$$(\gamma_2^{(1)} - \gamma_2^{(2)})(x) = (\gamma_2^{(1)} - \gamma_2^{(2)})(P_2) + (D_T(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2))\cdot(x-P_2)' + \\ + (\partial_\nu(\gamma_2^{(1)} - \gamma_2^{(2)})(P_2))\cdot(x-P_2)_n.$$

Hence, it follows that

$$|\,\partial_{y_n}\partial_{z_n} S_1(w,w)| \ge I_{21} - I_{22} - I_{23} - I_{24} - I_{25} - I_{26}$$

$$
\begin{aligned}
I_{21} &= \Big|\int_{B_\rho(P_2)\cap D_2} \partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(P_2)\cdot(x-P_2)_n A(x)\nabla_x\,\partial_{y_n} G_1(x,w)\cdot\nabla_x\,\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,\\
I_{22} &= \Big|\int_{B_\rho(P_2)\cap D_2} D_T(\gamma_2^{(1)}-\gamma_2^{(2)})(P_2)\cdot(x-P_2)' A(x)\nabla_x\,\partial_{y_n} G_1(x,w)\cdot\nabla_x\,\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,\\
I_{23} &= \Big|\int_{B_\rho(P_2)\cap D_2} (\gamma_2^{(1)}-\gamma_2^{(2)})(P_2) A(x)\nabla_x\,\partial_{y_n} G_1(x,w)\cdot\nabla_x\,\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,\\
I_{24} &= \Big|\int_{B_\rho(P_2)\cap D_2} (q_2^{(2)}-q_2^{(1)})(x)\,\partial_{y_n} G_1(x,w)\cdot\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|,\\
I_{25} &= \Big|\int_{\mathcal{U}_1\setminus(B_\rho(P_2)\cap D_2)} (\sigma^{(1)}-\sigma^{(2)})(x)\,\partial_{y_n}\nabla_x G_1(x,w)\cdot\partial_{z_n}\nabla_x G_2(x,w)\,\mathrm{d}x\Big|,\\
I_{26} &= \Big|\int_{\mathcal{U}_1\setminus(B_\rho(P_2)\cap D_2)} (q^{(1)}-q^{(2)})(x)\,\partial_{y_n} G_1(x,w)\cdot\partial_{z_n} G_2(x,w)\,\mathrm{d}x\Big|.
\end{aligned}
$$

by (5.17), with minor calculations, we derive

$$
|\,\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(P_2)|r^{1-n} \le |\,\partial_{y_n}\partial_{z_n} S_1(w,w)|+C\{Er^{1-n+\theta_2}+C(\varepsilon+E)\omega_\eta\Big(\frac{\varepsilon}{\varepsilon+E}\Big)r^{-n}\}. \tag{5.73}
$$

Since for $y,z\in(D_0)_{r_0/3}$,

$$
|\,\partial_{y_n}\partial_{z_n} S_1(y,z)| \le C(\varepsilon+\delta_1^*)r^{-n},
$$

then by (5.29) and (5.32),

$$
\Big|\partial_{y_j}\partial_{z_i} S_1(w,w)\Big| \le C(\varepsilon+\delta_1^*+E)\left(\frac{\varepsilon+\delta_1^*}{\varepsilon+\delta_1^*+E}\right)^{\beta^{2N_1}(12r_1\ln 3)^{-2}r^2} r^{-n}. \tag{5.74}
$$

Hence, one derives

$$
\begin{aligned}
|\,\partial_\nu(\gamma_2^{(1)}-\gamma_2^{(2)})(P_1)|r^{1-n} \le C\Big\{&Er^{1-n+\theta_2}+(\varepsilon+\delta_1^*+E)\left(\frac{\varepsilon+\delta_1^*}{\varepsilon+\delta_1^*+E}\right)^{\beta^{2N_1}(12r_1\ln 3)^{-2}r^2} r^{-n}+\\
&+(\varepsilon+E)\omega_\eta\Big(\frac{\varepsilon}{\varepsilon+E}\Big)r^{-n}\Big\}.
\end{aligned} \tag{5.75}
$$

Multiplying (5.75) by r^{n-1} and optimising with respect to r leads to

$$
|\,\partial_\nu(\gamma_1^{(1)}-\gamma_1^{(2)})(P_1)| \le C(\varepsilon+E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon+E}\right), \tag{5.76}
$$

with $0<\eta_2<1$. Hence, we conclude that

$$
\delta_2 \le C(\varepsilon+E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon+E}\right). \tag{5.77}
$$

Now, let us derive a bound for δ_2^*. Notice that the norm $\|q_2^{(2)}-q_2^{(1)}\|_{L^\infty(D_2)}$ can be evaluated in terms of the following quantities:

$$
\|q_2^{(2)}-q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} \quad \text{and} \quad |\,\partial_\nu(q_2^{(2)}-q_2^{(1)})(P_2)|. \tag{5.78}
$$

Let ρ, r, w be as above. Consider

$$\partial_{y_n}\partial_{z_n} S_1(w,w) = \partial_{y_n}\partial_{z_n} I_1(w) + \partial_{y_n}\partial_{z_n} I_2(w).$$

We determine a lower bound for $\partial_{y_n}\partial_{z_n} I_1(w)$ in terms of $\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))}$. By the asymptotic estimate (5.17) and (5.77), one derives

$$\begin{aligned}|\,\partial_{y_n}\partial_{z_n} I_1(w)| \geq \|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))}\Big\{\int_{B_\rho(P_2)\cap D_2}\partial_{y_n} H_1(x,w)\,\partial_{z_n} H_2(x,w)\,\mathrm{d}x-\\ -\int_{B_\rho(P_2)\cap D_2}|x-w|^{2(1-n)+\theta_3}\,\mathrm{d}x - \int_{B_\rho(P_2)\cap D_2}|x-w|^{2(1-n+\theta_3)}\,\mathrm{d}x\Big\}-\\ -CE\int_{B_\rho(P_2)\cap D_2}|x||x-w|^{2(1-n)}\,\mathrm{d}x - C(\varepsilon+E)\omega^{(2)}\Big(\frac{\varepsilon}{\varepsilon+E}\Big)\int_{B_\rho(P_2)\cap D_2}|x-w|^{-n}\,\mathrm{d}x.\end{aligned}$$

It turns out that

$$C\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} r^{2-n} \leq |\,\partial_{y_n}\partial_{z_n} I_1(w)| + CEr^{2-n+\theta_3} + C(\varepsilon+E)\omega^{(2)}_{\eta_2}\left(\frac{\varepsilon}{\varepsilon+E}\right) r^{-n}.$$

By (5.74), due to the fact that

$$|\,\partial_{y_n}\partial_{z_n} I_1(w)| \leq |\,\partial_{y_n}\partial_{z_n} S_1(w,w)| + |\,\partial_{y_n}\partial_{z_n} I_2(w)|,$$

by the upper bound for $I_2(w)$, (5.29) and (5.32) we derive

$$\begin{aligned}\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))} r^{2-n} \leq C\Big\{(\varepsilon+\delta_1^*+E)\left(\frac{\varepsilon+\delta_1^*}{\varepsilon+\delta_1^*+E}\right)^{\beta^{2N_1}(12r_1\ln 3)^{-2}r^2} r^{-n}+\\ + Er^{2-n+\theta_1} + (\varepsilon+E)\omega^{(2)}_{\eta_2}\left(\frac{\varepsilon}{\varepsilon+E}\right) r^{-n}\Big\}.\end{aligned}$$

Multiply by r^{n-2} to obtain

$$\begin{aligned}\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{\frac{r_0}{4}}(P_2))} \leq C\Big\{(\varepsilon+\delta_1^*+E)\left(\frac{\varepsilon+\delta_1^*}{\varepsilon+\delta_1^*+E}\right)^{\beta^{2N_1}(12r_1\ln 3)^{-2}r^2} r^{-2}+\\ + Er^{\theta_1} + (\varepsilon+E)\omega^{(2)}_{\eta_2}\left(\frac{\varepsilon}{\varepsilon+E}\right) r^{-2}\Big\}.\end{aligned}$$

By optimising with respect to r, one concludes that

$$\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{\frac{r_0}{4}}(P_2))} \leq C(\varepsilon+E)\omega^{(3)}_{\bar{\eta}_2}\left(\frac{\varepsilon}{\varepsilon+E}\right), \tag{5.79}$$

with $0<\bar{\eta}_2<1$ that depends on $\theta_1,\theta_2,\theta_3$.

To estimate $|\,\partial_\nu(q_2^{(2)} - q_2^{(1)})(P_2)|$, consider the singular solution $\partial^2_{y_iy_j}\partial^2_{z_iz_j} S_1(w,w)$ and split it as the sum of the terms

$$\begin{aligned}I_1^{ij}(w) = \int_{D_2\cap B_\rho(P_2)}(\sigma_2^{(2)} - \sigma_2^{(1)})(x)\nabla_x\partial^2_{y_iy_j} G_1(x,w)\cdot\nabla_x\partial^2_{z_iz_j} G_2(x,w)\,\mathrm{d}x+\\ + \int_{D_2\cap B_\rho(P_2)}(q_2^{(2)} - q_2^{(1)})(x)\,\partial^2_{y_iy_j} G_1(x,w)\cdot\partial^2_{z_iz_j} G_2(x,w)\,\mathrm{d}x,\end{aligned}$$

and

$$I_2^{ij}(w) = \int_{\mathcal{U}_1\setminus(D_2\cap B_\rho(P_2))} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x\, \partial^2_{y_iy_j} G_1(x, y_r) \cdot \nabla_x\, \partial^2_{z_iz_j} G_2(x, y_r)\, \mathrm{d}x +$$
$$+ \int_{\mathcal{U}_1\setminus(D_2\cap B_\rho(P_2))} (q^{(2)} - q^{(1)})(x)\, \partial^2_{y_iy_j} G_1(x, w) \cdot \partial^2_{z_iz_j} G_2(x, w)\, \mathrm{d}x.$$

Set $I_m(w) = \{I_m^{ij}(w)\}_{i,j=1,\dots,n}$ for $m = 1, 2$. Denote by $|I_m(w)|$ the Euclidean norm of the matrix $I_m(w)$. The upper bound for $|I_2(w)|$ is given by

$$|I_2(w)| \leq CE\rho^{-(n+2)},$$

where C is a positive constant that depends on the *a-priori* data only. For the lower bound for $I_1(w)$,

$$|I_1(w)| \geq \frac{1}{n}\sum_{i,j=1}^{n} \Big\{ \Big| \int_{D_2\cap B_\rho(P_2)} (\partial_\nu(q_2^{(2)} - q_2^{(1)})(P_2)) \cdot (x - P_2)_n\, \partial^2_{y_iy_j} G_1(x, w) \cdot \partial^2_{z_iz_j} G_2(x, w)\, \mathrm{d}x \Big| -$$
$$- \Big| \int_{D_2\cap B_\rho(P_2)} (D_T(q_2^{(2)} - q_2^{(1)})(P_2)) \cdot (x - P_2)'\, \partial^2_{y_iy_j} G_1(x, w) \cdot \partial^2_{z_iz_j} G_2(x, w)\, \mathrm{d}x \Big| -$$
$$- \Big| \int_{D_2\cap B_\rho(P_2)} (q_2^{(2)} - q_2^{(1)})(P_2)\, \partial^2_{y_iy_j} G_1(x, w) \cdot \partial^2_{z_iz_j} G_2(x, w)\, \mathrm{d}x \Big| \Big\} -$$
$$- \Big| \int_{D_2\cap B_\rho(P_2)} (\sigma^{(1)} - \sigma^{(2)})(x)\, \partial^2_{y_iy_j} \nabla_x G_1(x, w) \cdot \partial^2_{z_iz_j} \nabla_x G_2(x, w)\, \mathrm{d}x \Big|.$$

Since

$$|(q_2^{(2)} - q_2^{(1)})(P_2)| + C|(D_T(q_2^{(2)} - q_2^{(1)})(P_2))| \leq C\|q_2^{(2)} - q_2^{(1)}\|_{L^\infty(\Sigma_2\cap B_{r_0/4}(P_2))},$$

by (5.79) and (5.19), one derives

$$|I_1(w)| \geq C|(\partial_\nu(q_2^{(2)} - q_2^{(1)})(P_2))|r^{1-n} - C(\varepsilon + E)\omega^{(3)}_{\bar\eta_2}\left(\frac{\varepsilon}{\varepsilon + E}\right) r^{-n} -$$
$$-CEr^{1+\theta_2-n} - C(E+\varepsilon)\omega^{(2)}_{\bar\eta_2}\left(\frac{\varepsilon}{\varepsilon + E}\right) r^{-2-n}. \tag{5.80}$$

Since for $y, z \in (D_0)_{r_0/3}$,

$$\int_\Sigma [\sigma^{(2)}(x)\nabla_x\, \partial^2_{z_n} G_2(x, z) \cdot \nu\, \partial^2_{y_n} G_1(x, y) - \sigma^{(1)}(x)\nabla_x\, \partial^2_{y_n} G_1(x, y) \cdot \nu\, \partial^2_{z_n} G_2(x, z)]\, \mathrm{d}S(x) =$$
$$= \partial^2_{y_n}\, \partial^2_{z_n} S_1(y, z) + \int_{\mathcal{W}_1} (\sigma^{(1)} - \sigma^{(2)})(x)\nabla_x\, \partial^2_{y_n} G_1(x, y) \cdot \nabla_x\, \partial^2_{z_n} G_2(x, z)\, \mathrm{d}x +$$
$$+ \int_{\mathcal{W}_1} (q^{(2)} - q^{(1)})(x)\, \partial^2_{y_n} G_1(x, y)\, \partial^2_{z_n} G_2(x, z)\, \mathrm{d}x,$$

by (5.30) and (5.32), it turns out that

$$|\,\partial^2_{y_n}\, \partial^2_{z_n} S_1(y_r, y_r)| \leq C\left(\frac{\varepsilon + \delta_1^*}{\varepsilon + \delta_1^* + E}\right)^{\beta^{2N_1}(12r_1\ln 3)^{-2}r^2} (\varepsilon + \delta_1^* + E)r^{-2-n}. \tag{5.81}$$

Collecting together (5.80) and (5.81), one derives

$$|(\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1))|r^{1-n} \leq C(\varepsilon+E)\omega_{\eta_2}^{(2)}\left(\frac{\varepsilon}{\varepsilon+E}\right)r^{-2-n} + CEr^{1+\theta_2-n} + \\ + C\left(\frac{\varepsilon+\delta_1^*}{\varepsilon+\delta_1^*+E}\right)^{\beta^{2N_1}(12r_1\ln 3)^{-2}r^2}(\varepsilon+\delta_1^*+E)r^{-2-n}.$$

Multiply by r^{n-1} the last equation and optimise with respect to r leads to the estimate

$$|(\partial_\nu(q_1^{(2)} - q_1^{(1)})(P_1))| \leq C(E+\varepsilon)\omega_{\tilde{\eta}_2}^{(3)}\left(\frac{\varepsilon}{\varepsilon+E}\right), \tag{5.82}$$

with $0 < \tilde{\eta}_2 < 1$ that depends on θ_1, θ_2.

For the general case, consider the following identities:

$$\begin{aligned} &\int_\Sigma [\sigma^{(2)}(x)\nabla_x G_2(x,z)\cdot\nu\, G_1(x,y) - \sigma^{(1)}(x)\nabla_x G_1(x,y)\cdot\nu\, G_2(x,z)]\, \mathrm{d}S(x) = \\ &= \int_{\mathcal{W}_{k-1}} (\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x G_1(x,y)\cdot\nabla_x G_2(x,z)\, \mathrm{d}x \\ &+ \int_{\mathcal{W}_{k-1}} (q^{(2)}-q^{(1)})(x) G_1(x,y) G_2(x,z)\, \mathrm{d}x + S_{k-1}(y,z), \end{aligned} \tag{5.83}$$

$$\begin{aligned} &\int_\Sigma \sigma^{(2)}(x)\nabla_x\, \partial_{z_n} G_2(x,z)\cdot\nu\ \partial_{y_n} G_1(x,y) - \sigma^{(1)}(x)\nabla_x\, \partial_{y_n} G_1(x,y)\cdot\nu\ \partial_{z_n} G_2(x,z)]\, \mathrm{d}S(x) = \\ &= \int_{\mathcal{W}_{k-1}} [(\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\, \partial_{y_n} G_1(x,y)\cdot\nabla_x\, \partial_{z_n} G_2(x,z)\, \mathrm{d}x \\ &+ \int_{\mathcal{W}_{k-1}} (q^{(2)}-q^{(1)})(x)\, \partial_{y_n} G_1(x,y)\, \partial_{z_n} G_2(x,z)\, \mathrm{d}x + \partial_{y_n}\, \partial_{z_n} S_{k-1}(y,z), \end{aligned} \tag{5.84}$$

and

$$\begin{aligned} &\int_\Sigma [\sigma^{(2)}(x)\nabla_x\, \partial^2_{z_n} G_2(x,z)\cdot\nu\ \partial^2_{y_n} G_1(x,y) - \sigma^{(1)}(x)\nabla_x\, \partial^2_{y_n} G_1(x,y)\cdot\nu\ \partial^2_{z_n} G_2(x,z)]\, \mathrm{d}S(x) = \\ &= \int_{\mathcal{W}_{k-1}} (\sigma^{(1)}-\sigma^{(2)})(x)\nabla_x\, \partial^2_{y_n} G_1(x,y)\cdot\nabla_x\, \partial^2_{z_n} G_2(x,z)\, \mathrm{d}x \\ &+ \int_{\mathcal{W}_{k-1}} (q^{(2)}-q^{(1)})(x)\, \partial^2_{y_n} G_1(x,y)\, \partial^2_{z_n} G_2(x,z)\, \mathrm{d}x + \partial^2_{y_n}\, \partial^2_{z_n} S_{k-1}(y,z). \end{aligned} \tag{5.85}$$

To estimate the norms

$$\|\gamma^{(1)}-\gamma^{(2)}\|_{L^\infty(D_k)} \quad \text{and} \quad \|q^{(1)}-q^{(2)}\|_{L^\infty(D_k)}$$

one can follow the procedure in Step $k=2$. Consider $\rho = r_0/4$, $r \in (0, \bar{r}/8)$ and set $w = P_k + r\nu(P_k)$, then we split the integral solutions $S_{k-1}(w,w)$, $\partial_{y_n}\partial_{z_n} S_{k-1}(w,w)$ and $\partial^2_{y_n}\partial^2_{z_n} S_{k-1}(w,w)$ into the sum of two integrals over the domains $B_\rho(P_k)\cap D_k$ and $\mathcal{U}_{k-1}\setminus(B_\rho(P_k)\cap D_k)$. We can determine a lower bound for the integral on the smallest domain and an upper bound for the integral on the largest domain using the estimates of Proposition 5.3.1. This leads to the inequality

$$\|\gamma^{(1)} - \gamma^{(2)}\|_{L^\infty(D_k)} \leq C(\varepsilon + E)\omega_{\eta_k}^{(3k-4)}\Big(\frac{\varepsilon}{\varepsilon + E}\Big), \tag{5.86}$$

and then, by applying (5.86), we have

$$\|q^{(1)} - q^{(2)}\|_{L^\infty(D_k)} \leq C(\varepsilon + E)\omega_{\tilde{\eta}_k}^{(3(k-1))}\Big(\frac{\varepsilon}{\varepsilon + E}\Big), \tag{5.87}$$

where $0 < \eta_k, \tilde{\eta}_k < 1$ are constants that depend only on the *a-priori* data.

Let $\bar{K} = \max\{K, \tilde{K}\}$. Since $E = \delta^*_{\bar{K}}$, we can derive the inequality

$$E \leq C(\varepsilon + E)\omega_{\tilde{\eta}_{\bar{K}}}^{(3(\bar{K}-1))}\left(\frac{\varepsilon}{\varepsilon + E}\right).$$

If $E \geq e^2\varepsilon$ (otherwise, the statement holds), it follows that

$$1 \leq C\omega_{\tilde{\eta}_{\bar{K}}}^{(3(\bar{K}-1))}\left(\frac{\varepsilon}{E}\right). \tag{5.88}$$

Taking the inverse of $\omega_{\tilde{\eta}_{\bar{K}}}^{(3(\bar{K}-1))}$ and applying it to (5.88), we can conclude that

$$E \leq C_1\,\varepsilon,$$

where C_1 is a positive constant that depends only on the *a-priori* data. This completes the proof of Theorem 5.1.2.

□

Appendices

APPENDIX A

Miscellanea

A.1 Spaces of functions

A.1.1 Hölder spaces

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We say that a function $u : \Omega \to \mathbb{R}$ is *α-Hölder continuous* with $\alpha \in (0,1]$ on Ω if

$$|u|_{\alpha,\Omega} = \sup\Big\{\frac{|u(x)-u(y)|}{|x-y|^\alpha} \,:\, x,y \in \Omega, x \neq y\Big\} < +\infty.$$

We say that u is a Lipschitz continuous function if $\alpha = 1$. The space of α-Hölder functions is defined as the set

$$C^{0,\alpha}(\Omega) = \{u : \Omega \to \mathbb{R} \,:\, |u|_{\alpha,\Omega} < +\infty\}.$$

The number α is called the *Hölder exponent* of the space $C^{0,\alpha}$. Equipped with the norm

$$\|u\|_{\alpha,\Omega} = \|u\|_{L^\infty(\Omega)} + |u|_{\alpha,\Omega},$$

the space $C^{0,\alpha}(\Omega)$ is a Banach space. For $k \in \mathbb{N}, \alpha \in (0,1]$, define the (k,α) Hölder space as the set

$$C^{k,\alpha}(\Omega) = \{u \in C^k(\Omega) \,:\, \sum_{|\alpha|=k} |D^\gamma u|_{\gamma,\Omega} < +\infty\}.$$

It is a Banach space with norm

$$\|u\|_{k,\alpha,\Omega} = \sum_{|\gamma|<k} \|D^\gamma u\|_{L^\infty(\Omega)} + \sum_{|\gamma|=k} |D^\gamma u|_{\alpha,\Omega}.$$

Theorem A.1.1. *i) The immersion $C^{k,\gamma}(\overline{\Omega}) \subset C^k(\overline{\Omega})$ is continuous and compact for $\gamma \in (0,1]$.*

ii) The immersion $C^{k,\gamma}(\overline{\Omega}) \subset C^{k,\beta}(\overline{\Omega})$ is continuous and compact for any $0 < \beta \leq \gamma \leq 1$.

A.1.2 Sobolev spaces

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

Definition A.1.1. *For $1 \leq p \leq \infty$, m a positive integer, we define the space $W^{m,p}(\Omega)$ as follows:*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \,:\, D^\alpha u \in L^p(\Omega) \text{ for all multi-indices } |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ is equipped with the norm:

$$\|u\|_{W^{m,p}(\Omega)} = \Big[\sum_{|\alpha|\leq m} \|D^\alpha u\|^p_{L^p(\Omega)}\Big]^{1/p}, \qquad \text{for } 1 \leq p < \infty,$$
$$\|u\|_{W^{m,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha u\|_{L^\infty(\Omega)}.$$

The Sobolev space $W^{m,p}(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$, it is separable for $\leq p < \infty$ and reflexive for $1 < p < \infty$. See [1, 42] for other properties.

In the case $p = 2$, the fractional Sobolev space $W^{s,2}(\Omega)$ turns out to be a Hilbert space and it is denoted by $H^s(\Omega)$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We define

$$H^1(\Omega) = \{u \in L^2(\Omega) \,:\, \nabla u \in L^2(\Omega)\}.$$

where ∇u is in the sense of distributions. The Sobolev space $H^1(\Omega)$ is a Hilbert space with scalar product

$$(u,v)_{H^1(\Omega)} = \int_\Omega u\, v + \int_\Omega \nabla u \cdot \nabla v,$$

and with norm

$$\|u\|_{H^1(\Omega)} = \sqrt{(u,u)_{H^1(\Omega)}} = \Big(\int_\Omega u^2 + \int_\Omega |\nabla u|^2\Big)^{1/2}.$$

The Sobolev space $H^1_0(\Omega)$ is the closure of $C^\infty_c(\Omega)$ under the $H^1(\Omega)$ norm.

Theorem A.1.2 (Trace Theorem)**.** *Let Ω in $\mathbb{R}^n$ be a bounded domain with Lipschitz boundary with constants r_0 and L. Let $p \in [1,+\infty)$, d_0 be the diameter of Ω. Then* *there exists a unique bounded linear operator*

$$\mathrm{T} : W^{1,p}(\Omega) \to L^p(\partial\,\Omega)$$

which satisfies the following conditions:

1. $\mathrm{T}(u) = u|_{\partial\,\Omega}$ *for every* $u \in C^0(\bar{\Omega}) \cap W^{1,p}(\Omega)$.
2. *There exists a positive constant* C *depending only on* r_0, L, d_0 *and* p *such that*

$$\|\,\mathrm{T}(u)\|_{L^p(\partial\,\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, \qquad \text{for any } u \in W^{1,p}(\Omega).$$

3. *For every* $u \in W^{1,p}(\Omega), \phi \in C^1(\bar{\Omega}, \mathbb{R}^n)$,

$$\int_\Omega u\,\mathrm{div}(\phi)\;\mathrm{d}x = -\int_\Omega \nabla u \cdot \phi\;\mathrm{d}x + \int_{\partial\,\Omega} (\phi \cdot \nu)\,\mathrm{T}(u)\;\mathrm{d}S.$$

The function $\mathrm{T}(u)$ *is called the* trace *of* u *on* $\partial\,\Omega$.

We give the following characterisation of trace spaces when $p = 2$.

Theorem A.1.3. *Let* Ω *in* $\mathbb{R}^n$ *be a bounded domain with Lipschitz boundary with constants* r_0 *and* L. *Let*

$$\mathrm{T} : H^1(\Omega) \to L^2(\partial\,\Omega)$$

be the trace operator defined in Theorem A.1.2. Then

$$\mathrm{T}(H^1(\Omega)) = H^{1/2}(\partial\,\Omega).$$

Moreover,

1. *There exists a positive constant* C *which depends only on* r_0, L *and* n *such that*

$$\|\,\mathrm{T}(u)\|_{H^{1/2}(\partial\,\Omega)} \leq C\|u\|_{H^1(\Omega)}, \qquad \text{for all } u \in H^1(\Omega).$$

2. *There exists a bounded, linear map*

$$R : H^{1/2}(\partial\,\Omega) \to H^1(\Omega)$$

such that

$$\mathrm{T}(R(h)) = h, \qquad \text{for all } h \in H^{1/2}(\partial\,\Omega).$$

In particular, by $u = R(h)$, *it turns out that*

$$\|u\|_{H^1(\Omega)} \leq C\|h\|_{H^{1/2}(\partial\,\Omega)} \qquad \text{for all } h \in H^{1/2}(\partial\,\Omega).$$

The trace space $H^{1/2}(\partial\,\Omega)$ is the space of traces of $H^1(\Omega)$ functions on $\partial\,\Omega$. It can be canonically endowed with a scalar product induced by that of $H^1(\Omega)$. The trace space $H^{1/2}(\partial\,\Omega)$ can be defined equivalently as the following quotient space

$$H^{1/2}(\partial\,\Omega) = H^1(\Omega)/H^1_0(\Omega).$$

The dual space of $H^{1/2}(\partial\Omega)$ is denoted as $H^{-1/2}(\partial\Omega)$ and it is the space of trace distributions acting on $H^{1/2}(\partial\Omega)$. The following chain of inclusions holds: $H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega) \subset H^{-1/2}(\partial\Omega)$. For any non-empty portion $\Sigma \subset \partial\Omega$ of $\partial\Omega$, we define the trace space $H^{1/2}_{co}(\Sigma)$ as the subset of $H^{1/2}(\partial\Omega)$ of trace functions whose support is compactly contained in Σ:

$$H^{1/2}_{co}(\Sigma) = \{f \in H^{1/2}(\partial\Omega) \,:\, \mathrm{supp}(f) \subset \Sigma\}.$$

Denote with $H^{1/2}_{00}(\Sigma)$ its closure under the norm $\|\cdot\|_{H^{1/2}(\partial\Omega)}$. Similarly, let

$$H^{1/2}_{co}(\partial\Omega \setminus \bar{\Sigma}) = \left\{f \in H^{1/2}(\partial\Omega) \,:\, \mathrm{supp}(f) \subset \partial\Omega \setminus \bar{\Sigma}\right\},$$

and denote with $H^{1/2}_{00}(\partial\Omega \setminus \bar{\Sigma})$ its closure under the norm $\|\cdot\|_{H^{1/2}(\partial\Omega)}$.

The trace space of the form $H^{1/2}_{00}(\Sigma)$ was introduced by Jacques-Louis Lions and Enrico Magenes. We recall their definition as presented in the Lectures by Tataru [131]. We start with the definition of the interpolation spaces. Let E_0 and E_1 be two normed spaces that are continuously embedded in a topological vector space $\mathcal{E}$. We assume that $E_0 + E_1$ is a normed space equipped with the norm $\|a\|_{E_0+E_1} = \inf_{a=a_0+a_1}\{\|a_0\|_0 + \|a_1\|_1\}$.

For any $a \in E_0 + E_1$ and $t > 0$, we define

$$K(t;a) = \inf_{a=a_0+a_1}\{\|a_0\|_0 + t\|a_1\|_1\}.$$

For $0 < \theta < 1$ and $p \in [1, +\infty]$, we define the interpolation space

$$(E_0, E_1)_{\theta,p} = \left\{a \in E_0 + E_1 \,:\, t^{-\theta}K(t;a) \in L^p\left(\mathbb{R}_+; \frac{\mathrm{d}t}{t}\right)\right\}, \tag{A.1}$$

equipped with the norm

$$\|a\|_{(E_0,E_1)_{\theta,p}} = \|t^{-\theta}K(t;a)\|_{L^p(0,\infty;\,\mathrm{d}t/t)}.$$

These spaces satisfy nice embedding properties similar to Hölder spaces and interpolation properties [131, Lemma 22.2, Lemma 22.3].

When $E_0 = H^1_0(\Omega)$ and $E_1 = L^2(\Omega)$, the interpolation spaces $(H^1_0(\Omega), L^2(\Omega))_{\theta,2}$ for Ω a bounded domain with a Lipschitz boundary, Lions and Magenes discovered that for $\theta \neq 1/2$ we get the space $H^{1-\theta}_0(\Omega)$, whereas for $\theta = 1/2$ we get a new space, denoted by $H^{1/2}_{00}(\Omega)$.

Lemma A.1.4. *If $u \in H^{1/2}_{00}(\mathbb{R}_+) = (H^1_0(\mathbb{R}_+), L^2(\mathbb{R}_+))_{1/2,2}$, then $\frac{u}{\sqrt{x}} \in L^2(\mathbb{R}_+)$.*

Thanks to Lemma A.1.4, we can characterise the space $H^{1/2}_{00}(\Omega)$ as the space of functions of $H^{1/2}(\Omega)$ such that $\frac{u}{\sqrt{d(x)}} \in L^2(\Omega)$, where $d(x) = \mathrm{dist}(x, \partial\Omega)$.

A.2 Linear Elliptic Equations

Let Ω be a bounded domain of $\mathbb{R}^n$ with C^1 boundary and let ν be the normal derivative of $\partial\Omega$. Let **F** be a C^1 vector field on Ω, then the following relation holds:

$$\int_\Omega \operatorname{div}(\mathbf{F})dx = \int_{\partial\Omega} \mathbf{F}\cdot \nu dS, \tag{A.2}$$

where dS denotes the $(n-1)$-dimensional surface element of $\partial\Omega$. Equation (A.2) is known as the *divergence theorem*.

As a consequence of the divergence theorem, we are able to derive important identities known as *Green Identities*. Let $u, v \in C^2(\bar{\Omega})$, set $\mathbf{F} = v\nabla u$, then we obtain the first Green identity

$$\int_\Omega v\Delta u + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu} dS. \tag{A.3}$$

If we interchange the role of v with u in (A.3) and then subtract, we obtain the second Green identity,

$$\int_\Omega (v\Delta u - u\Delta v)dx = \int_{\partial\Omega}\left(v\frac{\partial u}{\partial\nu} - u\frac{\partial v}{\partial \nu}\right). \tag{A.4}$$

Theorem A.2.1 (Poincaré inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. There exists a constant $C = C(|\Omega|) > 0$ such that for any $u \in H_0^1(\Omega)$,*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}.$$

Theorem A.2.2 (The Caccioppoli inequality)**.** *Let Ω be a bounded domain of $\mathbb{R}^n$. Let $A \in L^\infty(\Omega, Sym_n)$ be a real symmetric $n\times n$ matrix function. Suppose that there are constants $\lambda, \tilde{\lambda} \in \mathbb{R}^+$ such that*

$$\lambda|\xi|^2 \leq A(x)\xi\cdot\xi \leq \tilde{\lambda}|\xi|^2, \qquad \textit{for a.e. } x\in\Omega, \textit{ for any } \xi \in \mathbb{R}^n. \tag{A.5}$$

Let $u \in H^1(\Omega)$ be a weak solution of the elliptic equation

$$\operatorname{div}(A(x)\nabla u(x)) = 0 \quad \textit{for } x \in \Omega.$$

Let $r \in (0, R)$ and $\bar{x}$ be such that $B_r(\bar{x}) \subset\subset B_R(\bar{x}) \subset\subset \Omega$. Then, there exists a positive constant $C = C(\lambda)$ such that

$$\int_{B_r(\bar{x})} |\nabla u|^2 \,\mathrm{d}x \leq \frac{C}{(R-r)^2}\int_{B_R(\bar{x})\setminus B_r(\bar{x})} |u|^2 \,\mathrm{d}x. \tag{A.6}$$

Proof. Let $\varphi \in C_c^1(\Omega)$ be a cut-off function such that

$$\begin{aligned} \varphi &\equiv 1 && \text{in } B_r(\bar{x}),\\ 0 < \varphi &< 1 && \text{in } B_R(\bar{x})\setminus B_r(\bar{x}),\\ \varphi &\equiv 0 && \text{in } \Omega\setminus B_R(\bar{x}),\\ |\nabla\varphi| &\leq \frac{2}{R-r} && \text{in } B_R(\bar{x})\setminus B_r(\bar{x}). \end{aligned} \tag{A.7}$$

We choose $u\varphi^2$ as a test function, then

$$\int_{B_R(\bar{x})} A(x)\nabla u \cdot \nabla(u\varphi^2)\, \mathrm{d}x = 0.$$

By the product rule,

$$\int_{B_R(\bar{x})} A(x)\nabla u(x) \cdot \nabla u(x)\varphi^2(x)\, \mathrm{d}x + 2\int_{B_R(\bar{x})} A(x)u(x)\nabla u(x) \cdot \nabla\varphi(x)\varphi(x)\, \mathrm{d}x = 0. \tag{A.8}$$

Let us estimate the two integrals in (A.8). By (A.5),

$$\int_{B_R(\bar{x})} A(x)\nabla u(x) \cdot \nabla u(x)\varphi^2(x)\, \mathrm{d}x \geq \lambda \int_{B_R(\bar{x})} |\nabla u(x)|^2\varphi^2(x)\, \mathrm{d}x.$$

By Hölder inequality,

$$\begin{aligned}\int_{B_R(\bar{x})} A(x)u(x)\nabla u(x) \cdot \nabla\varphi(x)\varphi(x)\, \mathrm{d}x &\leq \tilde{\lambda} \int_{B_R(\bar{x})} u(x)\nabla u(x) \cdot \nabla\varphi(x)\varphi(x)\, \mathrm{d}x \\ &\leq \tilde{\lambda}\Big(\int_{B_R(\bar{x})} |\varphi(x)\nabla u(x)|^2\, \mathrm{d}x\Big)^{1/2}\Big(\int_{B_R(\bar{x})} |u(x)\nabla\varphi(x)|^2\, \mathrm{d}x\Big)^{1/2}.\end{aligned}$$

Hence,

$$\lambda\int_{B_R(\bar{x})} |\varphi(x)\nabla u(x)|^2\, \mathrm{d}x \leq 2\tilde{\lambda}\Big(\int_{B_R(\bar{x})} |\varphi(x)\nabla u(x)|^2\, \mathrm{d}x\Big)^{1/2}\Big(\int_{B_R(\bar{x})} |u(x)\nabla\varphi(x)|^2\, \mathrm{d}x\Big)^{1/2}. \tag{A.9}$$

Dividing by $\lambda\Big(\int_{B_R(\bar{x})} |\varphi(x)\nabla u(x)|^2\, \mathrm{d}x\Big)^{1/2}$ and then take the square in (A.9):

$$\int_{B_R(\bar{x})} |\varphi(x)\nabla u(x)|^2\, \mathrm{d}x \leq \Big(\frac{2\tilde{\lambda}}{\lambda}\Big)^2 \int_{B_R(\bar{x})} |u(x)\nabla\varphi(x)|^2\, \mathrm{d}x. \tag{A.10}$$

From (A.7) and (A.10), it follows that

$$\int_{B_r(\bar{x})} |\nabla u|^2\, \mathrm{d}x \leq \frac{16\tilde{\lambda}^2}{\lambda^2(R-r)^2}\int_{B_R(\bar{x})\setminus B_r(\bar{x})} |u|^2\, \mathrm{d}x.$$

□

A.3 Fundamental Solutions

In this section, we will derive explicit formulas for the fundamental solutions of certain types of elliptic equations.

The fundamental solution for the Laplace operator, denoted as Γ, can be expressed as follows:

$$\Gamma(x,y) = \begin{cases} -\frac{1}{2\pi}\ln|x-y| & n=2, \\ \frac{1}{(n-2)\omega_n}|x-y|^{2-n} & n\geq 3. \end{cases}$$

The solution is defined for $x \neq y$, where $x, y \in \mathbb{R}^n$. The following estimates for Γ can be established:

$$|\nabla_x \Gamma(x,y)| \le C|x-y|^{1-n},$$
$$|\nabla_x \nabla_y \Gamma(x,y)| \le C|x-y|^{-n},$$
$$|\nabla_x \nabla_y^2 \Gamma(x,y)| \le C|x-y|^{-1-n},$$

where C is a positive constant depending on n (see [56, Section 2.2] or [66, Section 2.4]).

Consider a point $x = (x', x_n) \in \mathbb{R}^n$ and define the reflected point $x^* = (x', -x_n)$ with respect to the hyperplane $\{x_n = 0\}$. We will now consider the elliptic equation given by

$$\mathrm{L}(u) := \mathrm{div}((\gamma_- + (\gamma_+ - \gamma_-)\chi_+(x))\nabla u(x)) = -\delta(x-y), \tag{A.11}$$

where $\chi_+ = \chi_{\mathbb{R}^n_+}$ is the characteristic function of the upper half plane. The existence and uniqueness of a solution to (A.11) was proved in [95] under the assumption that γ_+ and γ_- are positive numbers in $\mathbb{R}$, and for any $\xi \in \mathbb{R}^n$, the following inequalities hold: for some $\gamma_1, \gamma_2 > 0$,

$$\gamma_1|\xi|^2 \le (\gamma_- + (\gamma_+ - \gamma_-)\chi_+(x))\xi \cdot \xi \le \gamma_2|\xi|^2, \qquad \text{for any } x, \xi \in \mathbb{R}^n.$$

By [108, Theorem 1.1], for $y \in \mathbb{R}^n$, the unique solution $\tilde{\Gamma}(\cdot, y)$ of

$$\mathrm{L}(\tilde{\Gamma}(x,y)) = -\delta(x-y) \qquad \text{in } \mathbb{R}^n$$

is:

$$\tilde{\Gamma}(x,y) = \begin{cases} \frac{1}{(n-2)\omega_n\gamma_+}\left[\frac{1}{r^{n-2}} + \frac{b}{R^{n-2}}\right] & y_n > 0, \\ \frac{1}{(n-2)\omega_n\gamma_-}\left[\frac{1}{r^{n-2}} - \frac{b}{R^{n-2}}\right] & y_n < 0, \end{cases} \tag{A.12}$$

for

$$b = \frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-}, \qquad r = |x-y|,$$
$$R = \sqrt{(x_1-y_1)^2 + \cdots + (x_{n-1}-y_{n-1})^2 + (|x_n| + |y_n|)^2}.$$

If we distinguish the cases where $x_n \gtrless 0$, the expression for $\tilde{\Gamma}(x,y)$ becomes:

$$\tilde{\Gamma}(x,y) = \begin{cases} \frac{1}{(n-2)\omega_n\gamma_+}\left[\frac{1}{|x-y|^{n-2}} + \frac{\gamma_+-\gamma_-}{\gamma_++\gamma_-}\frac{1}{|x-y^*|^{n-2}}\right] & x_n, y_n > 0, \\ \frac{2}{(n-2)\omega_n(\gamma_++\gamma_-)}\frac{1}{|x-y|^{n-2}} & x_n \cdot y_n < 0, \\ \frac{1}{(n-2)\omega_n\gamma_-}\left[\frac{1}{|x-y|^{n-2}} - \frac{\gamma_+-\gamma_-}{\gamma_++\gamma_-}\frac{1}{|x-y^*|^{n-2}}\right] & x_n, y_n < 0. \end{cases} \tag{A.13}$$

Next, we introduce the explicit formulation of the fundamental solution for the case where the conductivity coefficient is anisotropic. Consider the conductivity given by

$$\sigma(x) = (\gamma_- + (\gamma_+ - \gamma_-)\chi_+(x))A, \qquad \text{for } x \in \mathbb{R}^n,$$

where $A \in Sym_n$ is any real symmetric $n \times n$ positive definite matrix. We denote H as the fundamental solution associated with the elliptic operator $\mathrm{div}(\sigma\nabla\cdot)$. The weak formulation of H is defined as follows:

$$\int_{\mathbb{R}^n} \sigma(x)\nabla_x H(x,y)\cdot\nabla\varphi(x)\,\mathrm{d}x = \varphi(y) \qquad \text{for any } \varphi\in C_c^1(\mathbb{R}^n). \tag{A.14}$$

Let L be a non-singular $n\times n$ matrix, and we introduce the change of coordinates

$$x=\phi(\xi)=L\xi, \qquad y=L\eta, \qquad \frac{\partial x}{\partial \xi}=D\phi(x)=L.$$

By changing the variables in (A.14), we derive

$$\int_{\mathbb{R}^n} L^{-1}\sigma(L\xi)L^{-T}|\det L|\nabla_\xi H(L\xi,L\eta)\cdot\nabla\varphi(L\xi)\,\mathrm{d}\xi=\varphi(L\eta).$$

Thus, the linear map ϕ gives $\gamma_- + (\gamma_+ - \gamma_-)\chi_+(\xi) = L^{-1}\sigma(L\xi)L^{-T}$. If we set $\psi(\xi)=\varphi(L\xi)$, we further derive

$$\int_{\mathbb{R}^n}(\gamma_- + (\gamma_+ - \gamma_-)\chi_+(\xi))|\det L|\nabla H(L\xi,L\eta)\cdot\nabla\psi(\xi)\,\mathrm{d}\xi=\psi(\eta).$$

Now, we define $\gamma(x_n)$ as follows:

$$\gamma(x_n)=\begin{cases}\gamma_+ & \text{if } x_n>0,\\ \gamma_- & \text{if } x_n<0.\end{cases}$$

To satisfy the given conditions, the linear map ϕ or the corresponding matrix L must have the following properties:

i) $A=L^{-1}\cdot L^{-T}$,

ii) $(L\xi)\cdot e_n=\lambda\xi\cdot e_n$, where $\lambda>0$.

We can select the linear map ϕ as follows:

$$\begin{aligned}\phi :&\mathbb{R}^n\to\mathbb{R}^n\\ &\xi\ \mapsto \phi(\xi):=R\,J\xi.\end{aligned}$$

Here, $J=\sqrt{A^{-1}}$ and R is an orthogonal matrix representing the planar rotation in $\mathbb{R}^n$, which rotates the unit vector $\frac{v}{\|v\|}$, where $v=\sqrt{A}e_n$, to the nth standard unit vector e_n. Moreover,

$$R|_{(\pi)^\perp}=Id|_{(\pi)^\perp},$$

where π is the plane generated by e_n and v, and $(\pi)^\perp$ is the orthogonal complement of π. Moreover, $(L\xi)\cdot e_n=\frac{1}{\|v\|}\xi\cdot e_n$.

As a result, the explicit form of the fundamental solution for $\mathrm{div}_\xi((\gamma_- + (\gamma_+ - \gamma_-)\chi_+(\xi))A\nabla_\xi\cdot)$ is given by:

$$H(\xi,\eta)=|J|\begin{cases}\frac{1}{\gamma^+}\Gamma(L\xi,L\eta)+\frac{\gamma^+-\gamma^-}{\gamma^+(\gamma^++\gamma^-)}\Gamma(L\xi,(L\eta)^*) & \text{if } \xi_n,\eta_n>0,\\ \frac{2}{\gamma^++\gamma^-}\Gamma(L\xi,L\eta) & \text{if } \xi_n\cdot\eta_n<0,\\ \frac{1}{\gamma^-}\Gamma(L\xi,L\eta)+\frac{\gamma^--\gamma^+}{\gamma^-(\gamma^++\gamma^-)}\Gamma(L\xi,(L\eta)^*) & \text{if } \xi_n,\eta_n<0.\end{cases} \tag{A.15}$$

In this thesis, when dealing with the asymptotic estimates, we have considered the case

$$H(\xi,\eta) = \frac{2|J|}{\gamma^+ + \gamma^-}\Gamma(L\xi, L\eta), \quad \text{for } \xi_n \cdot \eta_n < 0,\ \xi,\eta \in \mathbb{R}^n.$$

APPENDIX **B**

List of publications

B.1 Articles

1. Sonia Foschiatti, Romina Gaburro, and Eva Sincich. *Stability for the Calderón's problem for a class of anisotropic conductivities via an ad hoc misfit functional*. In: Inverse Problems 37.12 (2021), Paper No. 125007, 34.
2. Sonia Foschiatti and Eva Sincich. *Stable determination of an anisotropic inclusion in the Schrödinger equation from local Cauchy data*. In: Inverse Probl. Imaging 17.3 (2023), pp. 584–613.
3. Sonia Foschiatti. *Lipschitz stability estimate for the simultaneous recovery of two coefficients in the anisotropic Schrödinger type equation via local Cauchy data*. In: Journal of Mathematical Analysis and Applications 531.1, Part 1 (2024), p. 127753.

Bibliography

[1] R. A. Adams. *Sobolev spaces*. Pure and Applied Mathematics, Vol. 65. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975, pp. xviii+268.

[2] S. Agmon, A. Douglis, and L. Nirenberg. "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I". In: *Comm. Pure Appl. Math.* 12 (1959), pp. 623–727. DOI: 10.1002/cpa.3160120405.

[3] G. S. Alberti, Á. Arroyo, and M. Santacesaria. "Inverse problems on low-dimensional manifolds". In: *Nonlinearity* 36.1 (2023), pp. 734–808. DOI: 10.1088/1361-6544/aca73d.

[4] G. Alessandrini. "Open issues of stability for the inverse conductivity problem". In: *Journal of Inverse and Ill-posed Problems* 15.5 (2007), pp. 451–460. DOI: doi:10.1515/jiip.2007.025.

[5] G. Alessandrini and M. Di Cristo. "Stable determination of an inclusion by boundary measurements". In: *SIAM J. Math. Anal.* 37.1 (2005), pp. 200–217. DOI: 10.1137/S003614100444191X.

[6] G. Alessandrini. "Determining Conductivity by Boundary Measurements, the Stability Issue". In: *Applied and Industrial Mathematics: Venice - 1, 1989*. Ed. by R. Spigler. Dordrecht: Springer Netherlands, 1991, pp. 317–324. DOI: 10.1007/978-94-009-1908-2_26.

[7] G. Alessandrini. "Singular solutions of elliptic equations and the determination of conductivity by boundary measurements". In: *J. Differential Equations* 84.2 (1990), pp. 252–272. DOI: 10.1016/0022-0396(90)90078-4.

[8] G. Alessandrini. "Stable determination of conductivity by boundary measurements". In: *Appl. Anal.* 27.1-3 (1988), pp. 153–172. DOI: 10.1080/00036818808839730.

[9] G. Alessandrini, E. Beretta, E. Rosset, and S. Vessella. "Optimal stability for inverse elliptic boundary value problems with unknown boundaries". In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* 29.4 (2000), pp. 755–806.

[10] G. Alessandrini, M. Di Cristo, E. Francini, and S. Vessella. "Stability for quantitative photoacoustic tomography with well-chosen illuminations". In: *Ann. Mat. Pura Appl. (4)* 196.2 (2017), pp. 395–406. DOI: 10.1007/s10231-016-0577-4.

[11] G. Alessandrini, M. Di Cristo, A. Morassi, and E. Rosset. "Stable determination of an inclusion in an elastic body by boundary measurements". In: *SIAM J. Math. Anal.* 46.4 (2014), pp. 2692–2729. DOI: 10.1137/130946307.

[12] G. Alessandrini and R. Gaburro. "Determining conductivity with special anisotropy by boundary measurements". In: *SIAM J. Math. Anal.* 33.1 (2001), pp. 153–171. DOI: 10.1137/S0036141000369563.

[13] G. Alessandrini and R. Gaburro. "The local Calderòn problem and the determination at the boundary of the conductivity". In: *Comm. Partial Differential Equations* 34.7-9 (2009), pp. 918–936. DOI: 10.1080/03605300903017397.

[14] G. Alessandrini, M. V. de Hoop, F. Faucher, R. Gaburro, and E. Sincich. "Inverse problem for the Helmholtz equation with Cauchy data: reconstruction with conditional well-posedness driven iterative regularization". In: *ESAIM Math. Model. Numer. Anal.* 53.3 (2019), pp. 1005–1030. DOI: 10.1051/m2an/2019009.

[15] G. Alessandrini, M. V. de Hoop, and R. Gaburro. "Uniqueness for the electrostatic inverse boundary value problem with piecewise constant anisotropic conductivities". In: *Inverse Problems* 33.12 (2017), pp. 125013, 24. DOI: 10.1088/1361-6420/aa982d.

[16] G. Alessandrini, M. V. de Hoop, R. Gaburro, and E. Sincich. "EIT in a layered anisotropic medium". In: *Inverse Probl. Imaging* 12.3 (2018), pp. 667–676. DOI: 10.3934/ipi.2018028.

[17] G. Alessandrini, M. V. de Hoop, R. Gaburro, and E. Sincich. "Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data". In: *Asymptot. Anal.* 108.3 (2018), pp. 115–149. DOI: 10.3233/asy-171457.

[18] G. Alessandrini, M. V. de Hoop, R. Gaburro, and E. Sincich. "Lipschitz stability for the electrostatic inverse boundary value problem with piecewise linear conductivities". In: *J. Math. Pures Appl. (9)* 107.5 (2017), pp. 638–664. DOI: 10.1016/j.matpur.2016.10.001.

[19] G. Alessandrini, L. Rondi, E. Rosset, and S. Vessella. "The stability for the Cauchy problem for elliptic equations". In: *Inverse Problems* 25.12 (2009), pp. 123004, 47. DOI: 10.1088/0266-5611/25/12/123004.

[20] G. Alessandrini and E. Sincich. "Cracks with impedance; stable determination from boundary data". In: *Indiana Univ. Math. J.* 62.3 (2013), pp. 947–989. DOI: 10.1512/iumj.2013.62.5124.

[21] G. Alessandrini and S. Vessella. “Lipschitz stability for the inverse conductivity problem”. In: *Adv. in Appl. Math.* 35.2 (2005), pp. 207–241. DOI: 10.1016/j.aam.2004.12.002.

[22] V. A. Ambartsumyan. “Abereine Frage der Eigenwerttheorie”. In: *Zeitschrift für Physik* 53 (1929), pp. 690–695.

[23] S. R. Arridge. “Optical tomography in medical imaging”. In: *Inverse Problems* 15.2 (1999), R41–R93. DOI: 10.1088/0266-5611/15/2/022.

[24] S. R. Arridge and W. R. B. Lionheart. “Nonuniqueness in diffusion-based optical tomography”. In: *Opt. Lett.* 23.11 (June 1998), pp. 882–884. DOI: 10.1364/OL.23.000882.

[25] S. R. Arridge and J. C. Schotland. “Optical tomography: forward and inverse problems”. In: *Inverse Problems* 25.12 (2009), pp. 123010, 59. DOI: 10.1088/0266-5611/25/12/123010.

[26] A. Aspri, E. Beretta, E. Francini, and S. Vessella. “Lipschitz stable determination of polyhedral conductivity inclusions from local boundary measurements”. In: *SIAM J. Math. Anal.* 54.5 (2022), pp. 5182–5222. DOI: 10.1137/22M1480550.

[27] K. Astala and L. Päivärinta. “Calderón’s inverse conductivity problem in the plane”. In: *Ann. of Math. (2)* 163.1 (2006), pp. 265–299. DOI: 10.4007/annals.2006.163.265.

[28] K. Astala, L. Päivärinta, and M. Lassas. “Calderón’s inverse problem for anisotropic conductivity in the plane”. In: *Comm. Partial Differential Equations* 30.1-3 (2005), pp. 207–224. DOI: 10.1081/PDE-200044485.

[29] V. Bacchelli and S. Vessella. “Lipschitz stability for a stationary 2D inverse problem with unknown polygonal boundary”. In: *Inverse Problems* 22.5 (Aug. 2006), p. 1627. DOI: 10.1088/0266-5611/22/5/007.

[30] G. Bal and G. Uhlmann. “Inverse diffusion theory of photoacoustics”. In: *Inverse Problems* 26.8 (2010), pp. 085010, 20. DOI: 10.1088/0266-5611/26/8/085010.

[31] A. Bamberger and T. Ha-Duong. “Diffraction d’une onde acoustique par une paroi absorbante: nouvelles équations intégrales”. In: *Math. Methods Appl. Sci.* 9.4 (1987), pp. 431–454. DOI: 10.1002/mma.1670090131.

[32] T. Barceló, D. Faraco, and A. Ruiz. “Stability of Calderón inverse conductivity problem in the plane”. In: *Journal de Mathématiques Pures et Appliquées* 88.6 (2007), pp. 522–556. DOI: https://doi.org/10.1016/j.matpur.2007.07.006.

[33] M. Bellassoued and M. Yamamoto. “Lipschitz stability in determining density and two Lamé coefficients”. In: *J. Math. Anal. Appl.* 329.2 (2007), pp. 1240–1259. DOI: 10.1016/j.jmaa.2006.06.094.

[34] E. Beretta and E. Francini. “Global Lipschitz stability estimates for polygonal conductivity inclusions from boundary measurements”. In: *Appl. Anal.* 101.10 (2022), pp. 3536–3549. DOI: 10.1080/00036811.2020.1775819.

[35] E. Beretta and E. Francini. “Lipschitz stability for the electrical impedance tomography problem: the complex case”. In: *Comm. Partial Differential Equations* 36.10 (2011), pp. 1723–1749. DOI: 10.1080/03605302.2011.552930.

[36] E. Beretta, E. Francini, and S. Vessella. “Lipschitz stable determination of polygonal conductivity inclusions in a two-dimensional layered medium from the Dirichlet-to-Neumann map”. In: *SIAM J. Math. Anal.* 53.4 (2021), pp. 4303–4327. DOI: 10.1137/20M1369609.

[37] E. Beretta, M. V. de Hoop, F. Faucher, and O. Scherzer. “Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates”. In: *SIAM J. Math. Anal.* 48.6 (2016), pp. 3962–3983. DOI: 10.1137/15M1043856.

[38] E. Beretta, M. V. de Hoop, E. Francini, S. Vessella, and J. Zhai. “Uniqueness and Lipschitz stability of an inverse boundary value problem for time-harmonic elastic waves”. In: *Inverse Problems* 33.3 (2017), pp. 035013, 27. DOI: 10.1088/1361-6420/aa5bef.

[39] E. Beretta, M. V. de Hoop, and L. Qiu. “Lipschitz stability of an inverse boundary value problem for a Schrödinger-type equation”. In: *SIAM J. Math. Anal.* 45.2 (2013), pp. 679–699. DOI: 10.1137/120869201.

[40] L. Borcea. “Addendum to: “Electrical impedance tomography” [Inverse Problems **18** (2002), no. 6, R99–R136; 1955896]”. In: *Inverse Problems* 19.4 (2003), pp. 997–998. DOI: 10.1088/0266-5611/19/4/501.

[41] L. Borcea. “Electrical impedance tomography”. In: *Inverse Problems* 18.6 (2002), R99–R136. DOI: 10.1088/0266-5611/18/6/201.

[42] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011, pp. xiv+599.

[43] R. M. Brown and R. H. Torres. “Uniqueness in the inverse conductivity problem for conductivities with 3/2 derivatives in L p, p> 2 n”. In: *Journal of Fourier Analysis and Applications* 9.6 (2003), pp. 563–574.

[44] R. Brummelhuis. “Three-spheres theorem for second order elliptic equations”. In: *J. Anal. Math.* 65 (1995), pp. 179–206. DOI: 10.1007/BF02788771.

[45] A.-P. Calderón. “On an inverse boundary value problem”. In: *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro, 1980)*. Soc. Brasil. Mat., Rio de Janeiro, 1980, pp. 65–73.

[46] P. Caro and K. M. Rogers. “Global Uniqueness for the Calderoń problem with Lipschitz conductivities”. In: *Forum of Mathematics, Pi* 4 (2016), e2. DOI: 10.1017/fmp.2015.9.

[47] C. I. Cârstea and J.-N. Wang. “Propagation of smallness for an elliptic PDE with piecewise Lipschitz coefficients”. In: *J. Differential Equations* 268.12 (2020), pp. 7609–7628. DOI: 10.1016/j.jde.2019.11.088.

[48] M. Cheney, D. Isaacson, and J. C. Newell. “Electrical impedance tomography”. In: *SIAM Rev.* 41.1 (1999), pp. 85–101. DOI: 10.1137/S0036144598333613.

[49] J. Curran, R. Gaburro, C. Nolan, and E. Somersalo. "Time-harmonic diffuse optical tomography: Hölder stability of the derivatives of the optical properties of a medium at the boundary". In: *Inverse Probl. Imaging* 17.2 (2023), pp. 338–361.

[50] M. Di Cristo. "Stability analysis of the inverse inclusion problem". In: *Rend. Istit. Mat. Univ. Trieste* 48 (2016), pp. 433–462. DOI: 10.13137/2464-8728/13166.

[51] M. Di Cristo. "Stability estimates in the inverse transmission scattering problem". In: *Inverse Probl. Imaging* 3.4 (2009), pp. 551–565. DOI: 10.3934/ipi.2009.3.551.

[52] M. Di Cristo and Y. Ren. "Stable determination of an inclusion for a class of anisotropic conductivities". In: *Inverse Problems* 33.9 (2017), pp. 095003, 15. DOI: 10.1088/1361-6420/aa7b90.

[53] M. Di Cristo and L. Rondi. "Examples of exponential instability for inverse inclusion and scattering problems". In: *Inverse Problems* 19.3 (2003), pp. 685–701. DOI: 10.1088/0266-5611/19/3/313.

[54] O. Doeva, R. Gaburro, W. R. B. Lionheart, and C. J. Nolan. "Lipschitz stability at the boundary for time-harmonic diffuse optical tomography". In: *Appl. Anal.* 101.10 (2022), pp. 3697–3715. DOI: 10.1080/00036811.2020.1758314.

[55] S. Eberle, B. Harrach, H. Meftahi, and T. Rezgui. "Lipschitz stability estimate and reconstruction of Lamé parameters in linear elasticity". In: *Inverse Probl. Sci. Eng.* 29.3 (2021), pp. 396–417. DOI: 10.1080/17415977.2020.1795151.

[56] L. C. Evans. *Partial differential equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749. DOI: 10.1090/gsm/019.

[57] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015, pp. xiv+299.

[58] J. Feldman, M. Salo, and G. Uhlmann. "The Calderón problem - An Introduction to Inverse Problems".

[59] S. Foschiatti, R. Gaburro, and E. Sincich. "Stability for the Calderón's problem for a class of anisotropic conductivities via an ad hoc misfit functional". In: *Inverse Problems* 37.12 (2021), Paper No. 125007, 34. DOI: 10.1088/1361-6420/ac349c.

[60] E. Francini, S. Vessella, and J.-N. Wang. "Propagation of smallness and size estimate in the second order elliptic equation with discontinuous complex Lipschitz conductivity". In: *J. Differential Equations* 343 (2023), pp. 687–717. DOI: 10.1016/j.jde.2022.10.028.

[61] R. Gaburro and W. R. B. Lionheart. "Recovering Riemannian metrics in monotone families from boundary data". In: *Inverse Problems* 25.4 (2009), pp. 045004, 14. DOI: 10.1088/0266-5611/25/4/045004.

[62] R. Gaburro and E. Sincich. "Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities". In: *Inverse Problems* 31.1 (2015), pp. 015008, 26. DOI: 10.1088/0266-5611/31/1/015008.

[63] H. Garde and N. Hyvönen. "Optimal depth-dependent distinguishability bounds for electrical impedance tomography in arbitrary dimension". In: *SIAM J. Appl. Math.* 80.1 (2020), pp. 20–43. DOI: 10.1137/19M1258761.

[64] N. Garofalo and F.-H. Lin. "Monotonicity properties of variational integrals, A_p weights and unique continuation". In: *Indiana Univ. Math. J.* 35.2 (1986), pp. 245–268. DOI: 10.1512/iumj.1986.35.35015.

[65] B. Gebauer. "Localized potentials in electrical impedance tomography". In: *Inverse Probl. Imaging* 2.2 (2008), pp. 251–269. DOI: 10.3934/ipi.2008.2.251.

[66] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Reprint of the 1998 edition. Springer-Verlag, Berlin, 2001, pp. xiv+517.

[67] A. Greenleaf, M. Lassas, and G. Uhlmann. "Anisotropic conductivities that cannot be detected by EIT". In: *Physiological Measurement* 24.2 (2003), p. 413. DOI: 10.1088/0967-3334/24/2/353.

[68] A. Greenleaf, M. Lassas, and G. Uhlmann. "On nonuniqueness for Calderón's inverse problem". In: *Math. Res. Lett.* 10.5-6 (2003), pp. 685–693. DOI: 10.4310/MRL.2003.v10.n5.a11.

[69] M. Grüter and K.-O. Widman. "The Green function for uniformly elliptic equations". In: *Manuscripta Math.* 37.3 (1982), pp. 303–342. DOI: 10.1007/BF01166225.

[70] J. Hadamard. "Sur les problèmes aux derivées partielles". In: *Bull. Univ. Princeton* 13 (1902).

[71] J. Hadamard. "Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques". In: *Paris* 11 (1932), pp. 243–264.

[72] B. Harrach. "On uniqueness in diffuse optical tomography". In: *Inverse Problems* 25.5 (2009), pp. 055010, 14. DOI: 10.1088/0266-5611/25/5/055010.

[73] B. Harrach. "Simultaneous determination of the diffusion and absorption coefficient from boundary data". In: *Inverse Probl. Imaging* 6.4 (2012), pp. 663–679. DOI: 10.3934/ipi.2012.6.663.

[74] B. Harrach and Y.-H. Lin. "Simultaneous recovery of piecewise analytic coefficients in a semilinear elliptic equation". In: *Nonlinear Anal.* 228 (2023), Paper No. 113188, 14. DOI: 10.1016/j.na.2022.113188.

[75] B. Harrach and J. K. Seo. "Detecting inclusions in electrical impedance tomography without reference measurements". In: *SIAM J. Appl. Math.* 69.6 (2009), pp. 1662–1681. DOI: 10.1137/08072142X.

[76] D. S. Holder. *Electrical Impedance Tomography: Methods, History and Applications*. Institute of Physics Publishing, 2005.

[77] V. Isakov. *Inverse problems for partial differential equations*. Third. Vol. 127. Applied Mathematical Sciences. Springer, Cham, 2017, pp. xv+406. DOI: 10.1007/978-3-319-51658-5.

[78] V. Isakov. "On uniqueness of recovery of a discontinuous conductivity coefficient". In: *Comm. Pure Appl. Math.* 41.7 (1988), pp. 865–877. DOI: 10.1002/cpa.3160410702.

[79] M. Kac. "Can one hear the shape of a drum?" In: *Amer. Math. Monthly* 73.4, part II (1966), pp. 1–23. DOI: 10.2307/2313748.

[80] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Reprint of the 1980 edition. Springer-Verlag, Berlin, 1995, pp. xxii+619.

[81] A. Knyazev, A. Jujunashvili, and M. Argentati. "Angles between infinite dimensional subspaces with applications to the Rayleigh-Ritz and alternating projectors methods". In: *J. Funct. Anal.* 259.6 (2010), pp. 1323–1345. DOI: 10.1016/j.jfa.2010.05.018.

[82] R. V. Kohn and M. Vogelius. "Determining conductivity by boundary measurements. II. Interior results". In: *Comm. Pure Appl. Math.* 38.5 (1985), pp. 643–667. DOI: 10.1002/cpa.3160380513.

[83] R. Kohn and M. Vogelius. "Determining conductivity by boundary measurements". In: *Comm. Pure Appl. Math.* 37.3 (1984), pp. 289–298. DOI: 10.1002/cpa.3160370302.

[84] R. V. Kohn and M. Vogelius. "Identification of an unknown conductivity by means of measurements at the boundary". In: *Inverse problems (New York, 1983)*. Vol. 14. SIAM-AMS Proc. Amer. Math. Soc., Providence, RI, 1984, pp. 113–123. DOI: 10.1002/cpa.3160370302.

[85] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and quasilinear elliptic equations*. Translated from the Russian by Scripta Technica, Inc, Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968, pp. xviii+495.

[86] R. Langer. "An inverse problem in differential equations". In: *Bulletin of the American Mathematical Society* 39 (1933), pp. 814–820.

[87] M. Lassas, M. Taylor, and G. Uhlmann. "The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary". In: *Comm. Anal. Geom.* 11.2 (2003), pp. 207–221. DOI: 10.4310/CAG.2003.v11.n2.a2.

[88] M. Lassas and G. Uhlmann. "I: Global Uniqueness and Reconstruction". In: *Communications on Pure and Applied Mathematics* 56 (2003), pp. 0001–0025.

[89] M. Lassas and G. Uhlmann. "On determining a Riemannian manifold from the Dirichlet-to-Neumann map". In: *Ann. Sci. École Norm. Sup. (4)* 34.5 (2001), pp. 771–787. DOI: 10.1016/S0012-9593(01)01076-X.

[90] M. M. Lavrent'ev. "On the problem of Cauchy for linear elliptic equations of the second order". In: *Doklady Akademii Nauk* 112 (1957), pp. 195–197.

[91] J. M. Lee and G. Uhlmann. "Determining anisotropic real-analytic conductivities by boundary measurements". In: *Comm. Pure Appl. Math.* 42.8 (1989), pp. 1097–1112. DOI: 10.1002/cpa.3160420804.

[92] Y. Y. Li and M. Vogelius. “Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients”. In: *Arch. Ration. Mech. Anal.* 153.2 (2000), pp. 91–151. DOI: 10.1007/s002050000082.

[93] W. R. B. Lionheart. “Conformal uniqueness results in anisotropic electrical impedance imaging”. In: *Inverse Problems* 13.1 (Feb. 1997), p. 125. DOI: 10.1088/0266-5611/13/1/010.

[94] W. R. B. Lionheart. “EIT reconstruction algorithms: pitfalls, challenges and recent developments”. In: *Physiological Measurement* 25.1 (2004), p. 125. DOI: 10.1088/0967-3334/25/1/021.

[95] W. Littman, G. Stampacchia, and H. F. Weinberger. “Regular points for elliptic equations with discontinuous coefficients”. In: *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* 17 (1963), pp. 43–77.

[96] D. Lynden-Bell and V. Gurzadyan. “Victor Amazaspovich Ambartsumian. 18 September 1908-12 August 1996”. In: *Biographical Memoirs of Fellows of the Royal Society* 44 (), pp. 22–34.

[97] N. Mandache. “Exponential instability in an inverse problem for the Schrödinger equation”. In: *Inverse Problems* 17.5 (Aug. 2001), p. 1435. DOI: 10.1088/0266-5611/17/5/313.

[98] P. Metherall, D. C. Barber, R. H. Smallwood, and B. H. Brown. “Three-dimensional electrical impedance tomography”. In: *Nature* 380 (1996), pp. 509–512.

[99] C. Miranda. *Partial differential equations of elliptic type*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 2. Second revised edition. Translated from the Italian by Zane C. Motteler. Springer-Verlag, New York-Berlin, 1970, pp. xii+370.

[100] A. Morassi and E. Rosset. “Stable determination of an inclusion in an inhomogeneous elastic body by boundary measurements”. In: *Rend. Istit. Mat. Univ. Trieste* 48 (2016), pp. 101–120. DOI: 10.13137/2464-8728/13153.

[101] A. I. Nachman. “Reconstructions from boundary measurements”. In: *Ann. of Math. (2)* 128.3 (1988), pp. 531–576. DOI: 10.2307/1971435.

[102] L. Päivärinta, A. Panchenko, and G. Uhlmann. “Complex geometrical optics solutions for Lipschitz conductivities”. In: *Revista Matematica Iberoamericana* 19.1 (2003), pp. 57–72.

[103] L. E. Payne. “Bounds in the Cauchy problem for the Laplace equation”. In: *Arch. Rational Mech. Anal.* 5 (1960), pp. 35–45. DOI: 10.1007/BF00252897.

[104] L. E. Payne. “On a priori bounds in the Cauchy problem for elliptic equations”. In: *SIAM J. Math. Anal.* 1 (1970), pp. 82–89. DOI: 10.1137/0501008.

[105] L. E. Payne and H. F. Weinberger. “New bounds for solutions of second order elliptic partial differential equations”. In: *Pacific J. Math.* 8 (1958), pp. 551–573.

[106] I. G. Petrowski. *Vorlesungen über partielle Differentialgleichungen.* B. G. Teubner Verlagsgesellschaft, Leipzig,, 1955, p. 296.

[107] A. Pliś. “On non-uniqueness in Cauchy problem for an elliptic second order differential equation”. In: *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 11 (1963), pp. 95–100.

[108] A. G. Ramm. “Fundamental solutions to some elliptic equations with discontinuous senior coefficients and an inequality for these solutions”. In: *Math. Inequal. Appl.* 1.1 (1998), pp. 99–104. DOI: 10.7153/mia-01-07.

[109] F. Rellich. “Darstellung der Eigenwerte von $\Delta u + \lambda u = 0$ durch ein Randintegral”. In: *Math. Z.* 46 (1940), pp. 635–636. DOI: 10.1007/BF01181459.

[110] T. Rivière. “Exploring the unknown: The work of Louis Nirenberg on partial differential equations”. In: *EMS Surv. Math. Sci.* 9.1 (2022), pp. 1–29. DOI: 10.4171/emss/52.

[111] L. Rondi. “A remark on a paper by Alessandrini and Vessella”. In: *Advances in Applied Mathematics* 36.1 (2006), pp. 67–69. DOI: https://doi.org/10.1016/j.aam.2004.12.003.

[112] L. Rondi. “A remark on a paper by G. Alessandrini and S. Vessella: “Lipschitz stability for the inverse conductivity problem” [Adv. in Appl. Math. **35** (2005), no. 2, 207–241; MR2152888]”. In: *Adv. in Appl. Math.* 36.1 (2006), pp. 67–69. DOI: 10.1016/j.aam.2004.12.003.

[113] L. Rondi, E. Sincich, and M. Sini. “Stable determination of a rigid scatterer in elastodynamics”. In: *SIAM J. Math. Anal.* 53.2 (2021), pp. 2660–2689. DOI: 10.1137/20M1352867.

[114] A. Rüland and E. Sincich. “Lipschitz stability for the finite dimensional fractional Calderón problem with finite Cauchy data”. In: *Inverse Probl. Imaging* 13.5 (2019), pp. 1023–1044. DOI: 10.3934/ipi.2019046.

[115] A. Rüland and E. Sincich. “On Runge approximation and Lipschitz stability for a finite-dimensional Schrödinger inverse problem”. In: *Appl. Anal.* 101.10 (2022), pp. 3655–3666. DOI: 10.1080/00036811.2020.1738403.

[116] P. C. Sabatier. “Inverse problems: anniversary and short review of generalized inverse scattering transforms”. In: *Inverse Problems* 25.12 (2009), pp. 123001, 20. DOI: 10.1088/0266-5611/25/12/123001.

[117] M. Salo. “Unique continuation for elliptic equations Notes, Fall 2014”.

[118] C. Schlumberger. *Étude sur la prospection electrique eu sous soil*. Paris: Gauthier Villars, 1920.

[119] K. Seidel and G. Lange. “Direct Current Resistivity Methods”. In: *Environmental Geology: Handbook of Field Methods and Case Studies*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007, pp. 205–237. DOI: 10.1007/978-3-540-74671-3_8.

[120] E. Sincich. “Lipschitz stability for the inverse Robin problem”. In: *Inverse Problems* 23.3 (2007), pp. 1311–1326. DOI: 10.1088/0266-5611/23/3/027.

[121] E. Sincich. “Stability and reconstruction for the determination of boundary terms by a single measurement”. PhD thesis. Sissa, 2005.

[122] G. Stampacchia. “Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus”. In: *Ann. Inst. Fourier (Grenoble)* 15.fasc. 1 (1965), pp. 189–258.

[123] P. Stefanov and G. Uhlmann. “Algebraic Analysis of Differential Equations: from Microlocal Analysis to Exponential Asymptotics Festschrift in Honor of Takahiro Kawai”. In: ed. by T. Aoki, H. Majima, Y. Takei, and N. Tose. Tokyo: Springer Japan, 2008. Chap. Boundary and lens rigidity, tensor tomography and analytic microlocal analysis, pp. 275–293.

[124] P. Stefanov, G. Uhlmann, A. Vasy, and H. Zhou. “Travel time tomography”. In: *Acta Math. Sin. (Engl. Ser.)* 35.6 (2019), pp. 1085–1114. DOI: 10.1007/s10114-019-8338-0.

[125] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970, pp. xiv+290.

[126] Z. Sun. “On a quasilinear inverse boundary value problem”. In: *Mathematische Zeitschrift* 221.1 (1996), pp. 293–305.

[127] J. Sylvester. “An anisotropic inverse boundary value problem”. In: *Comm. Pure Appl. Math.* 43.2 (1990), pp. 201–232. DOI: 10.1002/cpa.3160430203.

[128] J. Sylvester and G. Uhlmann. “A global uniqueness theorem for an inverse boundary value problem”. In: *Ann. of Math. (2)* 125.1 (1987), pp. 153–169. DOI: 10.2307/1971291.

[129] J. Sylvester and G. Uhlmann. “A uniqueness theorem for an inverse boundary value problem in electrical prospection”. In: *Communications on Pure and Applied Mathematics* 39.1 (1986), pp. 91–112.

[130] J. Sylvester and G. Uhlmann. “Inverse boundary value problems at the boundary—continuous dependence”. In: *Communications on pure and applied mathematics* 41.2 (1988), pp. 197–219.

[131] L. Tartar. *An introduction to Sobolev spaces and interpolation spaces*. Vol. 3. Lecture Notes of the Unione Matematica Italiana. Springer, Berlin; UMI, Bologna, 2007, pp. xxvi+218.

[132] D. Tataru. “Carleman estimates, unique continuation and applications”.

[133] A. N. Tikhonov. “On the stability of inverse problems”. In: *C. R. (Doklady) Acad. Sci. URSS (N.S.)* 39 (1943), pp. 176–179.

[134] G. N. Trytten. “Pointwise bounds for solutions of the Cauchy problem for elliptic equations”. In: *Arch. Rational Mech. Anal.* 13 (1963), pp. 222–244. DOI: 10.1007/BF01262694.

[135] G. Uhlmann. “Electrical impedance tomography and Calderón’s problem”. In: *Inverse Problems* 25.12 (Dec. 2009), p. 123011. DOI: 10.1088/0266-5611/25/12/123011.

[136] S. Vessella. *Notes on unique continuation properties for Partial Differential Equations – Introduction to the stability estimates for inverse problems*. 2023. arXiv: 2305.04765 [math.AP].

[137] H. Weyl. “Über die Asymptotische Verteilung der Eigenwerte”. In: *Nachr. Konigl. Ges. Wiss. G ̈ottingen* (1911), pp. 110–117.

[138] M. S. Zhdanov and G. V. Keller. *The geoelectrical methods in geophysical exploration*. Vol. 31. Elsevier, Amsterdam-London, 1994.