Some decidability issues concerning C^n real functions

Gabriele Buriola[1,*], Domenico Cantone[2], Gianluca Cincotti[2], Eugenio G. Omodeo[3] and Gaetano T. Spartà[4]

[1] *Dept. of Computer Science, University of Verona, Italy*
[2] *Dept. of Mathematics and Computer Science, University of Catania, Italy*
[3] *Dept. of Mathematics, Informatics and Geosciences, University of Trieste, Italy*
[4] *Pontifical Gregorian University, Rome, Italy*

Abstract

This paper adapts preexisting decision algorithms to a family $\mathcal{RDF} = \{RDF^n \mid n \in \mathbb{N}\}$ of languages regarding one-argument real functions; each RDF^n is a quantifier-free theory about the differentiability class C^n, embodying a fragment of Tarskian elementary algebra. The limits of decidability are also highlighted, by pointing out that certain extensions of RDF^n are undecidable. The possibility of extending RDF^n into a language RDF^∞ regarding the class C^∞, without disrupting decidability, is briefly discussed.

Two sorts of individual variables, namely real variables and function variables, are available in each RDF^n. The former are used to construct terms and formulas that involve basic arithmetic operations and comparison relators between real terms, respectively. In contrast, terms designating functions involve function variables, constructs for addition of functions and scalar multiplication, and—outermost—i-th order differentiation $D^i[\bullet]$ with $i \leqslant n$. An array of predicate symbols designate various relationships between functions, as well as function properties, that may hold over intervals of the real line; those are: function comparisons, strict and non-strict monotonicity / convexity / concavity, comparisons between a function (or one of its derivatives) and a real term.

The decidability of RDF^n relies, on the one hand, on Tarski's celebrated decision algorithm for the algebra of real numbers, and, on the other hand, on reduction and interpolation techniques. An interpolation method, specifically designed for the case $n = 1$, has been previously presented; another method, due to Carla Manni, can be used when $n = 2$. For larger values of n, further research on interpolation is envisaged.

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Keywords

Decidable theories, Tarski's elementary algebra, Functions of a real variable

Introduction

This paper addresses the decision problem for a fragment of real analysis exploiting the renowned decidability result for elementary real algebra due to Tarski [22, 23]. The Tarskian algebra being referred to here is the first-order theory of the ordered field $(\mathbb{R}, 0, 1, +, \cdot, =, \leqslant)$ of real numbers: within its context, unlike with other first-order theories about numerical domains—most prominent, among those, the Dedekind–Peano integer arithmetic (see, e.g., [16, Chapter 3])—, an algorithm can establish whether or not any given sentence is true. This motivates one in seeking extensions of elementary real algebra where this decidability result is preserved: e.g., the decidability of Tarskian algebra enriched with the exponential function resists, since long, as an unsolved issue [14]. We undertook years ago a systematic study on enhancements of the Tarskian language, or fragments thereof, endowed with provisions regarding real functions.

The language, dubbed RDF^n, to be discussed in this paper is devoid of quantifiers but embodies, in addition to the algebraic operators and relators, predicate symbols expressing strict and nonstrict

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*Corresponding author.
gabriele.buriola@univr.it (G. Buriola); domenico.cantone@unict.it (D. Cantone); cincotti@dmi.unict.it (G. Cincotti); eomodeo@units.it (E. G. Omodeo); g.sparta@unigre.it (G. T. Spartà)
https://www.dmi.unict.it/cantone (D. Cantone); https://sites.units.it/eomodeo/index.html/ (E. G. Omodeo)
0000-0002-1612-0985 (G. Buriola); 0000-0002-1306-1166 (D. Cantone); 0000-0001-8460-1708 (G. Cincotti); 0000-0003-3917-1942 (E. G. Omodeo); 0000-0002-8993-5851 (G. T. Spartà)

monotonicity, concavity, and convexity of C^n functions of one real variable, as well as strict and non-strict comparisons '$>$' and '$\geqslant$' between functions, over bounded or unbounded intervals. Further primitive constructs available in the language are: operators designating pointwise addition of functions, multiplication of a function by a scalar, and differentiation operators (up to the n-th derivative).[1] We reduce the satisfiability problem regarding the formulas of *RDF*^{n} to the truth problem for purely *existential sentences* of the elementary algebra of reals; we can thus rely upon improved versions of Tarski's original method.[2]

This paper is a sequel of [2] and [1]—hence, indirectly, of their antecedents [3, 5, 8]. As for semantics, *RDF*^{n} deals with functions endowed with continuous derivatives (up to the n-th order): in consequence of this, a satisfiable *RDF*^{m} formula might cease to be satisfiable in *RDF*^{n} when $m < n$.[3] Our present language *RDF*^{n} differs from the language *RDF** studied in [1] in that its syntax is richer: now we have a batch of differentiation constructs only one of which was available in *RDF**; this calls for an enhancement of the decision algorithm, to wit, an enhanced reduction to Tarskian algebra.

Another novelty of the subject matter of this paper, with respect to its antecedents, is greater attention bestowed to assessing where the boundary between decidable and undecidable fragments of analysis precisely lies.

The ongoing is organized as follows. In Sec. 1, we introduce syntax and semantics of the language of interest, and illustrate its expressive power through a gallery of small examples. Before providing the detailed specification of our decision algorithm in Sec. 3, in Sec. 2 we exemplify its use by manually working out an emulation of how it would process a specific, valid formula. Then Sec. 4 provides clues on the correctness of the proposed decision algorithm, specifying the role of an *ad hoc* interpolation method. Sec. 5 explores the other side of the problem in asking which further enrichments lead to undecidability. To end, we outline possible connections with related work, and draw conclusions.[4]

1. The interpreted *RDF*^{n} language

The augmented version *RDF*^{n} of the theory *RDF** of Reals with Differentiable Functions [1] is an unquantified first-order theory dealing with reals and with real functions of class C^n of one real variable, namely functions with continuous n-th derivative. The function symbols of *RDF*^{n} designate the basic operations of real arithmetic and pointwise addition, scalar multiplication, and differentiation (up to the nth order) of functions. Its predicate symbols designate: comparisons between reals, pointwise comparisons of functions; strict and non-strict monotonicity, convexity, and concavity; comparisons between functions, and comparison between their derivatives (up to the n-th order), and real terms.

This section introduces the language underlying *RDF*^{n}, explains the intended meanings of its constructs, and briefly illustrates its use.

Syntax and semantics

The language *RDF*^{n} has two infinite supplies of individual variables, belonging to the respective sorts: *numerical variables* $x, y, z, \ldots$ and *function variables* $f, g, h, \ldots$ Numerical and function variables are supposed to range, respectively, over the set $\mathbb{R}$ of real numbers and over the collection of functions which interests us. Four constants are also available:

- the symbols 0 and 1;
- the distinguished symbols $+\infty$ and $-\infty$, occurring as ends of *interval specifications* (see below).

[1] Usage of the differentiation operators $D^i[\cdot]$ must be reasonably restrained, e.g., each of them can only appear as lead operator in a function term $\mathfrak{g}$ (which will then coincide with a term of the form $D[\mathfrak{f}]$).

[2] To make an example, polynomial methods for existential formulas with a fixed number of variables are available [19].

[3] Partial evidence of this emerges from an example provided in [1, p. 128].

[4] To ease the comparison between *RDF*^{n} and similar previous theories, mainly *RDF*$^+$ and *RDF**, the first three sections follow much the same structure as the companion papers [2] and [1].

We next specify the syntax of *terms*, *atoms*, and *formulas* for *RDF*^{n}.

Definition 1.1. FUNCTION TERMS, NUMERICAL TERMS, and INTERVAL SPECS are so defined:

a.1) Function variables are *function terms*;

a.2) if $\mathfrak{f}$ and $\mathfrak{g}$ are function terms, then $\mathfrak{f} + \mathfrak{g}$ is a *function term*;

a.3) if $\mathfrak{f}$ is a function term, then any "scalar multiple" $s\,\mathfrak{f}$, with s a numerical term, is a *function term.*

b.1) Numerical variables and the constants $0, 1$ are *numerical terms*;

b.2) if s and t are numerical terms, the following also are *numerical terms*:

$$s + t\,, \quad s - t\,, \quad \text{and} \quad s \cdot t\,;$$

b.3) if t is a numerical term, α a natural number between 1 and n, and $\mathfrak{f}$ is a function term, then

$$\mathfrak{f}(t) \quad \text{and} \quad D^{\alpha}[\mathfrak{f}](t)$$

are *numerical terms.*[5]

c.1) An *interval spec* A is an expression of any of the forms

$$[e_1, e_2]\,, \quad [e_1, e_2[\,, \quad]e_1, e_2]\,, \quad \text{and} \quad]e_1, e_2[\,,$$

where e_1 stands for either a numerical term or $-\infty$, and e_2 for either a numerical term or $+\infty$;

c.2) we dub the "extended" numerical terms e_1, e_2 of such an A the ENDS of A. ⊣

Throughout, $\mathfrak{f}$ and $\mathfrak{g}$ stand for function terms, s and t for numerical terms, and A stands for an interval spec.

Definition 1.2. An ATOM of *RDF*^{n} is an expression of one of the forms

$$\begin{array}{rrrr}
s = t, & s > t, & \mathfrak{f}(s) = t, & D^{\alpha}[\mathfrak{f}](s) = t, \\
(\mathfrak{f} = \mathfrak{g})_A, & (\mathfrak{f} > \mathfrak{g})_A, & (\mathfrak{f} \bowtie t)_A, & (D^{\alpha}[\mathfrak{f}] \bowtie t)_A, \\
\mathsf{Up}(\mathfrak{f})_A, & \mathsf{Down}(\mathfrak{f})_A, & \mathsf{Convex}(\mathfrak{f})_A, & \mathsf{Concave}(\mathfrak{f})_A, \\
\mathsf{S_Up}(\mathfrak{f})_A, & \mathsf{S_Down}(\mathfrak{f})_A, & \mathsf{S_Convex}(\mathfrak{f})_A, & \mathsf{S_Concave}(\mathfrak{f})_A,
\end{array}$$

where $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$ and α is a natural number between 1 and n. ⊣

Definition 1.3. A FORMULA of *RDF*^{n} is any truth-functional combination of *RDF*^{n} atoms. ⊣

For definiteness, we will construct the *RDF*^{n} formulas from *RDF*^{n} atoms by means of the usual propositional connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

The semantics of *RDF*^{n} revolves around the designation rules listed in our next definition, with which any truth-value *assignment* for the formulas of *RDF*^{n} must comply.

Definition 1.4. An ASSIGNMENT for *RDF*^{n} is a mapping M whose domain consists of all terms and formulas of *RDF*^{n}, satisfying the following conditions:

0. $M0$ and $M1$ are the real numbers 0 and 1.

[5] As for the syntax, the availability of $D^{\alpha}[\cdot]$ with $\alpha > 1$ is the only novelty with respect to [1]. We will signal in the body of the decision algorithm which changes this enrichment entails. The true challenge, with this enrichment, is the need to redesign the algorithm correctness proof.

1. For each numerical variable x, Mx is a real number.

2. For each function variable f, (Mf) is an everywhere defined real function of one real variable of class C^n, i.e., with all the first n-th derivatives continuous.

3. For each function term of the form $\mathfrak{f} + \mathfrak{g}$, the image $\big(M(\mathfrak{f} + \mathfrak{g})\big)(r)$ of any real number r is $(M\mathfrak{f})(r) + (M\mathfrak{g})(r)$.

4. For each function term of the form $s\,\mathfrak{f}$, the function $M(s\,\mathfrak{f})$ is defined to be $Ms\,(M\mathfrak{f})$.
Namely, the image $\big(M(s\,\mathfrak{f})\big)(r)$ of any real number r is $Ms\,\big((M\mathfrak{f})(r)\big)$.

5. For each numerical term of the form $t_1 \otimes t_2$ with $\otimes \in \{+, -, \cdot\}$, $M(t_1 \otimes t_2)$ is the real number $Mt_1 \otimes Mt_2$.

6. For each numerical term of the form $\mathfrak{f}(t)$, $M(\mathfrak{f}(t))$ is the real number $(M\mathfrak{f})(Mt)$; for each numerical term $D^\alpha[\mathfrak{f}](t)$, $M(D^\alpha[\mathfrak{f}](t))$ is the real number $D^\alpha[(M\mathfrak{f})](Mt)$, where $D^\alpha[(M\mathfrak{f})]$ denotes the α-th derivative of $(M\mathfrak{f})$.

7. For each interval specification A, MA is an interval of $\mathbb{R}$ of the appropriate kind, whose endpoints are the evaluations via M of the ends of A.[6]
For example, when $A = \,]t_1, t_2]$, then $MA = \,]Mt_1, Mt_2]$.

8. Truth values are assigned to formulas of *RDF*^{n} according to the following rules, where s and t stand for numerical terms and $\mathfrak{f}, \mathfrak{g}$ for function terms:
   a) $s = t$ (respectively $s > t$) is true iff $Ms = Mt$ (resp. $Ms > Mt$) holds;
   b) $(\mathfrak{f} = \mathfrak{g})_A$ is true iff $(M\mathfrak{f})(x) = (M\mathfrak{g})(x)$ holds for all x in MA;
   c) $(\mathfrak{f} > \mathfrak{g})_A$ is true iff $(M\mathfrak{f})(x) > (M\mathfrak{g})(x)$ holds for all x in MA;
   d) $(\mathfrak{f} \bowtie t)_A$, with $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$, is true iff $(M\mathfrak{f})(x) \bowtie Mt$ holds for all x in MA;
   e) $(D[\mathfrak{f}] \bowtie t)_A$, with $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$, is true iff $D[(M\mathfrak{f})](x) \bowtie Mt$ holds for all x in MA;
   f) $\mathsf{Up}(\mathfrak{f})_A$ (respectively $\mathsf{S\_Up}(\mathfrak{f})_A$) is true iff $(M\mathfrak{f})$ is a monotone nondecreasing (resp. strictly increasing) function in MA;
   g) $\mathsf{Convex}(\mathfrak{f})_A$ (respectively $\mathsf{S\_Convex}(\mathfrak{f})_A$) is true iff $(M\mathfrak{f})$ is a convex (resp. strictly convex) function in MA;
   h) the truth values of $\mathsf{Down}(\mathfrak{f})_A$, $\mathsf{Concave}(\mathfrak{f})_A$, $\mathsf{S\_Down}(\mathfrak{f})_A$, and $\mathsf{S\_Concave}(\mathfrak{f})_A$ are defined in close analogy with items **f**) and **g**);
   i) the truth value which M assigns to a formula whose lead symbol is any of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ complies with the usual semantics of the propositional connectives.

An assignment M is said to MODEL a set Φ of formulas when $M\varphi$ is true for every φ in Φ. $\dashv$

Note that *RDF*^{n} coincides with *RDF** (see [1]) when $n = 1$.

Definition 1.5 (Derived symbols)**.** In light of the above semantics, we tacitly enrich our language, much as in [2], with derived dyadic and triadic comparators involving numerical terms t_1, t_2, and t_3; namely $t_1 \triangleright t_2$ and $t_1 \bowtie t_2/t_3$, where $\triangleright \in \{\neq, <, \leqslant, \geqslant\}$ and $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$.
Additional relators intermixing function terms and numerical terms, e.g., the construct

$$\mathsf{Linear}(f)_A \quad \leftrightarrow_{\text{Def}} \quad \mathsf{Concave}(f)_A \wedge \mathsf{Convex}(f)_A\,,$$

can also be introduced by means of shortening definitions. $\dashv$

[6]It goes without saying what is meant when M is undefined at either end of A (actually, $M(-\infty)$ and $M(+\infty)$ are undefined).

Some examples

Basic facts of real analysis stateable by means of *RDF*^{n} formulas, with $n > 1$, are:

- $\mathsf{Linear}(f)_{]-\infty,+\infty[} \;\leftrightarrow\; \big(D^2[f] = 0\big)_{]-\infty,+\infty[}\,.$

  A function f of class C^2 is linear if and only if its second derivative is constantly null.

- $\big\{(a < x < b) \;\wedge\; \big[(\mathsf{S\_Convex}(f)_{[a,x]} \;\wedge\; \mathsf{S\_Concave}(f)_{[x,b]}) \;\vee$ $(\mathsf{S\_Concave}(f)_{[a,x]} \wedge \mathsf{S\_Convex}(f)_{[x,b]})\big]\big\} \;\rightarrow\; D^2[f](x) = 0\,.$

  Let x be an inflection point of a C^2 function f, then the second derivative of f in x is null.

- $$\Big[\big(D^{k-1}[f] = y\big)_{]-\infty,+\infty[} \;\rightarrow\; \big(D^k[f] = 0\big)_{]-\infty,+\infty[}\Big] \;\wedge$$
  $$\Big\{\big(D^k[f] = 0\big)_{]-\infty,+\infty[} \;\rightarrow\; \big[D^{k-1}[f](x) = y \;\rightarrow\; \big(D^{k-1}[f] = y\big)_{]-\infty,+\infty[}\big]\Big\}\,.$$

  Let f be a function of class C^n, and k an integer, $0 < k \leqslant n$. Then the $(k-1)$-st derivative of f is constant if and only if the k-th derivative of f is null everywhere. (Note: $D^{k-1}[f]$ stands for f when $k = 1$.)

- $$\Big\{(a < x < b) \;\wedge\; D[f](x) = 0 \;\wedge\; \big(D^2[f] \geqslant 0\big)_{[a,b]} \;\wedge\; f(x) = y\Big\} \;\rightarrow\; (f \geqslant y)_{[a,b]};$$
  $$\Big\{(a < x < b) \;\wedge\; D[f](x) = 0 \;\wedge\; \big(D^2[f] \leqslant 0\big)_{[a,b]} \;\wedge\; f(x) = y\Big\} \;\rightarrow\; (f \leqslant y)_{[a,b]}.$$

  Let f be a function of class C^2, whose first derivative vanishes at some point x and whose second derivative is non-negative (resp., non-positive) all over a neighborhood $[a, b]$ of that x. Then x is a relative minimum (resp., maximum) point for f. Note that an analogous conclusion can be drawn about any function of class C^{2k+2}, whose first, second, …, and $(2k+1)$-st derivative vanish at some point x and whose $(2k+2)$-nd derivative is non-negative (resp., non-positive) all over a neighborhood $[a, b]$ of that x.

- $$\left\{\begin{array}{c} (a < x < b) \;\wedge\; D[f](x) = 0 \;\wedge\; D^2[f](x) = 0 \;\wedge \\ \Big[\big(D^3[f] < 0\big)_{[a,b]} \;\vee\; \big(D^3[f] > 0\big)_{[a,b]}\Big] \end{array}\right\} \;\rightarrow$$
  $$\left\{\begin{array}{c} \big(\mathsf{S\_Convex}(f)_{[a,x]} \;\wedge\; \mathsf{S\_Concave}(f)_{[x,b]}\big) \;\vee \\ \big(\mathsf{S\_Concave}(f)_{[a,x]} \;\wedge\; \mathsf{S\_Convex}(f)_{[x,b]}\big) \end{array}\right\}.$$

  Let f be a function of class C^3, whose first and second derivative vanish at some point x, where the third derivative of f assumes a nonzero value. Then x is an inflection point of f. (A generalized variant of this statement is left to the insightful reader.)

2. The decision algorithm at work

Establishing that an *RDF*^{n} formula φ is valid is the same as establishing that its negation $\neg\varphi$ is unsatisfiable; moreover, once $\neg\varphi$ has been put in disjunctive normal form, satisfying it amounts to satisfying one of its clauses. Thus, the core task regarding the decidability of *RDF*^{n} is: how to determine whether or not a given conjunction of *RDF*^{n} *literals* (that is, *RDF*^{n} atoms and negations thereof) is satisfiable?

Main steps of the decision algorithm

Via routinary flattening techniques, and in view of some basic properties of C^n functions, the said task can be converted to the one of determining the satisfiability of an arbitrary conjunction φ_0 of atoms of the forms

$$\begin{array}{rcrcrcrc}
x = y & , & x > y & , & z = x \cdot y & , & z = x + y & , \\
y = f(x) & , & (f > g)_A & , & (f = z\,g)_A & , & (h = f + g)_A & , \\
y = D^{\alpha}[f](x) & , & (D^{\alpha}[f] \bowtie z)_A & , & \mathsf{S_Up}(f)_A & , & \mathsf{S_Down}(f)_A & , \\
\mathsf{Convex}(f)_A & , & \mathsf{Concave}(f)_A & , & \mathsf{S_Convex}(f)_A & , & \mathsf{S_Concave}(f)_A & ,
\end{array}$$

and of literals that are the complements of atoms of these forms involving an interval spec.[7] As always, x, y, z stand for numerical variables and f, g, h stand for function variables.

Through a possibly furcating process, φ_0 will undergo a series $\varphi_0 \rightsquigarrow \varphi_1 \rightsquigarrow \varphi_2 \rightsquigarrow \varphi_3 \rightsquigarrow \varphi_4 = \widehat{\varphi}$ of transformations, with no function variables occuring in the ending formula $\widehat{\varphi}$; thereby, the satisfiability of $\widehat{\varphi}$ can be tested by means of Tarski's renowned decision algorithm [23, 9]. With a slight, harmless ambiguity we dub "our algorithm" at times our rewriting technique alone, and at times the entire validity test consisting of it, preceded by various preparations (e.g., flattening), and supplemented with Tarski's decision method.

The transformations $\varphi_{i-1} \rightsquigarrow \varphi_i$ ($i = 1, 2, 3, 4$) aim to the following purposes:

1. Behavior at the ends: For a thorough comparison between relevant values (e.g., the values of a derivative at the endpoints of specific open or semi-open intervals), we must divide each literal containing a function- or derivative-comparison into subcases, thus of either the form $(f > g)_A$ or the form $(D^{\alpha}[f] \bowtie t)_A$, unless A is a closed interval. By relying on function continuity, we split each such literal into a finite disjunction covering all possible behaviors at ends. E.g., $(D^{\alpha}[f] < t)_{[v,w[}$ becomes $(D^{\alpha}[f] < t)_{[v,w]} \vee \big((D^{\alpha}[f] < t)_{[v,w[} \wedge D^{\alpha}[f](w) = t\big)$.

2. Negative clause removal: Each negative literal with an interval specification is replaced by an implicit existential assertion. E.g.,$\neg(f = g)_{[v,w]}$ is replaced by $v \leqslant x \leqslant w \wedge f(x) \neq g(x)$, where x is a new variable.

3. Explicit evaluation of function variables: With certain salient variables v_j, dubbed "domain variables" (e.g., the variable v in $f(v) = y$), associate new variables $y_j^f, t_j^{1,f}, \ldots, t_j^{n,f}$ (one for each function variable f) subject to the constraints $y_j^f = f(v_j), t_j^{1,f} = D[f](v_j), \ldots, t_j^{n,f} = D^n[f](v_j)$. E.g., if in *RDF*^{2} a formula involves one function variable f and three domain variables v_1, v_2, v_3 altogether, then this step brings 9 new numerical variables in, along with 9 equations:

$$\begin{array}{lll}
y_1^f = f(v_1), & t_1^{1,f} = D[f](v_1), & t_1^{2,f} = D^2[f](v_1), \\
y_2^f = f(v_2), & t_2^{1,f} = D[f](v_2), & t_2^{2,f} = D^2[f](v_2), \\
y_3^f = f(v_3), & t_3^{1,f} = D[f](v_3), & t_3^{2,f} = D^2[f](v_3).
\end{array}$$

4. Elimination of function variables: Get rid of all literals involving function variables, whose behaviours are already mimicked by the variables $y_j^f, t_j^{1,f}, \ldots, t_j^{n,f}$ introduced above. This elimination is obtained by introducing new number variables subject to suitable algebraic constraints. E.g., roughly speaking, $(D^2[f] < s)_{[v,w]}$ becomes $t_v^{2,f} < s \wedge t_w^{2,f} < s \wedge \frac{t_w^{1,f} - t_v^{1,f}}{w - v} < s$.

A worked example

Our decision algorithm for *RDF*^{n} is specified in full in Sec. 3; here, to convey a feel of how it works, we consider a paradigmatic formula ψ and carry out one by one the key transformations leading from ψ to a formula directly submittable to Tarski's algorithm for elementary real algebra.

Suppose that we want to establish whether the formula

$$\big\{(a < x < b) \;\wedge\; \mathsf{S\_Convex}(f)_{[a,x]} \;\wedge\; \mathsf{S\_Concave}(f)_{[x,b]}\big\} \quad\rightarrow\quad D^2[f](x) = 0,$$

[7] Here again $\alpha \in \{1, \ldots, n\}$.

dubbed ψ in the ongoing, is true under every value assignment; equivalently, we can check whether its negation $\neg\psi$ is unsatisfiable. Using classical properties of implication, the negation amounts to the following formula φ:

$$(a < x < b) \quad \wedge \quad \mathsf{S\_Convex}(f)_{[a,x]} \quad \wedge \quad \mathsf{S\_Concave}(f)_{[x,b]} \quad \wedge \quad D^2[f](x) \neq 0.$$

Then φ undergoes the following transformations:

1. Behavior at the ends: Generally speaking, function-comparison literals of the form $(f > g)_A$ must be bestowed special care, possibly leading to a subcase analysis. Since no such literal appears in our φ, this phase produces $\varphi_1 \coloneqq \varphi$.

2. Negative clause removal: This phase removes negative literals with interval specifications, such as $\neg\left(D^2[f] = y\right)_{[a,b]}$, substituting them with suitable witnesses; for example, $\neg\left(D^2[f] = y\right)_{[a,b]}$ would be replaced by the following conjunction:

$$a \leqslant x \leqslant b \wedge D^2[f](x) = s \wedge s \neq y.$$

   Since the only negated literal in φ, $D^2[f](x) \neq 0$, is pointwise, this phase produces $\varphi_2 \coloneqq \varphi_1$.

3. Explicit evaluation of function variables: This phase introduces a new variable to designate each function-application term $\ell(v)$, where ℓ stands for a function variable of φ and v for one of its so-called 'domain' variables. To describe evaluation more transparently, let us do the renaming: $a \rightsquigarrow v_1, \;\; x \rightsquigarrow v_2, \;\; b \rightsquigarrow v_3$. From the previous formula φ_2 we get the following φ_3:

$$\begin{array}{ccccccc}
(v_1 < v_2 < v_3) & \wedge & \mathsf{S_Convex}(f)_{[v_1,v_2]} & \wedge & \mathsf{S_Concave}(f)_{[v_2,v_3]} & \wedge & D^2[f](v_2) \neq 0 \quad \wedge \\
f(v_1) = y_1^f & \wedge & f(v_2) = y_2^f & \wedge & f(v_3) = y_3^f & \wedge & \\
D^1[f](v_1) = t_1^f & \wedge & D^1[f](v_2) = t_2^f & \wedge & D^1[f](v_3) = t_3^f & \wedge & \\
D^2[f](v_1) = s_1^f & \wedge & D^2[f](v_2) = s_2^f & \wedge & D^2[f](v_3) = s_3^f & \wedge & s_2^f \neq 0.
\end{array}$$

4. Elimination of function variables: This final phase removes all literals still containing function variables. We get rid of them by suitable replacements involving algebraic conditions, such as the difference quotient for literals regarding derivatives. At the end we obtain an equisatisfiable formula that can be tested for satisfiability by Tarski's algorithm.
   From the previous formula φ_3 we get the following final formula φ_4:

$$\begin{array}{ccccccc}
(v_1 < v_2 < v_3) & \wedge & s_2^f \neq 0 & \wedge & & & \\
t_1^f < \frac{y_2^f - y_1^f}{v_2 - v_1} < t_2^f & \wedge & s_1^f \geqslant 0 & \wedge & s_2^f \geqslant 0 & \wedge \\
t_2^f > \frac{y_3^f - y_2^f}{v_3 - v_2} > t_3^f & \wedge & s_2^f \leqslant 0 & \wedge & s_3^f \leqslant 0. &
\end{array}$$

In particular, the unsatisfiability of this last formula is given by the conjunction:

$$s_2^f \neq 0 \wedge s_2^f \geqslant 0 \wedge s_2^f \leqslant 0.$$

3. The decision algorithm, in detail

When one deals with an unquantified language such as RDF^n, which is closed with respect to propositional connectives, being able to determine algorithmically whether or not a formula is valid amounts to establishing whether the negation thereof is *satisfiable* or *unsatisfiable*. (E.g., ascertaining the validity of the formula $\left\{(a < x < b) \;\wedge\; \mathsf{S\_Convex}(f)_{[a,x]} \;\wedge\; \mathsf{S\_Concave}(f)_{[x,b]}\right\} \rightarrow D^2[f](x) = 0$ amounts to checking $(a < x < b) \wedge \mathsf{S\_Convex}(f)_{[a,x]} \wedge \mathsf{S\_Concave}(f)_{[x,b]} \wedge D^2[f](x) \neq 0$ for unsatisfiability.)

We prefer to address the satisfiability problem for *RDF*^{n} here, so our algorithm is supposed to produce a yes/no answer, where 'yes' means that φ admits a model.

The idea is to transform, through a finite number of steps, the given *RDF*^{n} formula φ to be tested for satisfiability into a finite collection of formulas ψ_i, still devoid of quantifiers, each belonging to elementary algebra of real numbers; this will be done so that φ is satisfiable if and only if at least one of the resulting ψ_i's is satisfiable. Each resulting ψ_i can be tested via Tarski's decision algorithm.

First we discuss how to reduce our formula φ to a particular format, called *ordered form*.

Normalization

Let $\mathscr{T}$ be an unquantified, possibly multi-sorted, first-order theory, endowed with: equality $=$, a denumerable infinity of individual variables $x_1,\ x_2,\ldots$, function symbols $F_1,\ F_2,\ldots$, and predicate symbols $P_1,\ P_2,\ldots$.

Definition 3.1. A formula φ of $\mathscr{T}$ is said to be FLAT if it is a *conjunction* of literals of the forms:

$$x = y\,,\quad x = F(x_1,\ldots,x_n)\,,\quad x \neq y\,,\qquad P(x_1,\ldots,x_n)\,,\quad \neg P(x_1,\ldots,x_n)\,,$$

with x, y, and the x_i's numerical variables, F a function symbol and P a predicate symbol.

Let $\mathscr{S}$ be the class of all flat formulas of $\mathscr{T}$; the following holds:

Lemma 3.1. *The decision problem for $\mathscr{T}$, to wit, the problem of algorithmically determining whether or not any given formula φ in $\mathscr{T}$ is satisfiable, reduces to the analogous problem regarding $\mathscr{S}$.*

Proof. Each satisfiability algorithm for formulas in $\mathscr{T}$ clearly works also for formulas in the sublanguage $\mathscr{S}$ of $\mathscr{T}$. For the converse, suppose that an algorithmic satisfiability test for $\mathscr{S}$ is available, and let φ be any formula of $\mathscr{T}$. Via routinary techniques, which in our case include rewriting rules such as

$$\begin{array}{rclcll}
(h > f+g)_A & \rightsquigarrow & (h > k)_A & \wedge & (k = f+g)_{]-\infty,+\infty[} & , \\
(D^\alpha[f+g] = y)_A & \rightsquigarrow & (D^\alpha[k] = y)_A & \wedge & (k = f+g)_{]-\infty,+\infty[} & , \\
(\mathsf{S_Up}(f+g))_A & \rightsquigarrow & (\mathsf{S_Up}(k))_A & \wedge & (k = f+g)_{]-\infty,+\infty[} & , \\
(f+g = h+l)_A & \rightsquigarrow & (k = h+l)_A & \wedge & (k = f+g)_{]-\infty,+\infty[} & , \\
f(f(x)) = y & \rightsquigarrow & y = f(z) & \wedge & z = f(x) & , \\
(\mathsf{Up}(f))_A & \rightsquigarrow & (D[f] \geqslant 0)_A & & &
\end{array} \qquad (\ddagger)$$

(likewise, $D[f](D[f](x)) = y$ reduces to $y = D[f](z) \wedge z = D[f](x)$, etc.), one transforms φ into an equisatisfiable formula ψ such that

(1) every term occurring in ψ either is an individual variable or has the form $F(x_1,\ x_2,\ldots,\ x_n)$, where $x_1,\ x_2,\ldots,\ x_n$ are individual variables and F is a function symbol;

(2) every atom in ψ either has the form $x = t$, where x and t are an individual variable and a term, respectively, or has the form $P(x_1,\ x_2,\ldots,\ x_n)$, where $x_1,\ x_2,\ldots,\ x_n$ are individual variables and P is a predicate symbol.

Then one brings ψ to disjunctive normal form, thus obtaining a formula $\psi_1 \vee \cdots \vee \psi_\kappa$, where all ψ_i's are conjunctions. Additionally, we may assume that each ψ_i is flat, because any literal of type $\neg\, x = F(y_1,\ldots,y_n)$ within it can be replaced by the conjunction $x \neq z \wedge z = F(y_1,\ldots,y_n)$, where z is a brand new variable. Our claim follows, since

$$\varphi \text{ is satisfiable} \leftrightarrow \psi \text{ is satisfiable} \leftrightarrow \psi_i \text{ is satisfiable for some } i$$

and since all transformations used to obtain the conjunctions $\psi_1,\ldots,\psi_\kappa$ are effective. □

We can now proceed to define an *ordered form* for *RDF*^{n} formulas.

Definition 3.2. A DOMAIN VARIABLE in a formula φ of *RDF*^{n} is a numerical variable x that occurs in φ either as the argument of a term of one of the forms $f(x)$ and $D^\alpha[f](x)$, with f a function variable, or as an end of some interval mentioned in φ (as exemplified by $\mathsf{Convex}(f)_{[x,+\infty[}$).

Definition 3.3. An *RDF*^{n} formula is said to be in ORDERED FORM if it is flat and has the form $\varphi \ \wedge \bigwedge_{i=1}^{n-1}(x_i < x_{i+1})$, where $\{x_1,\ldots,x_n\}$ is the set of all distinct domain variables in φ.

The family *RDF*$^n_{ord}$ of all ordered formulas of *RDF*^{n} is a strict subset of *RDF*^{n}; notwithstanding:

Lemma 3.2. (Cf. [8, Lemma 1.4.3 on p.15]) *RDF*^{n} *is decidable if and only if RDF*$^n_{ord}$ *is decidable.* □

The algorithm

We describe next the decision algorithm for satisfiability of formulas of *RDF*^{n}. In view of Lemma 3.2, w.l.o.g. we assume that φ is given in ordered form. Moreover, using new function variables h subject to constraints of either the form $(h = f + g)_A$ or the form $(h = z\, g)_A$, all literals, except those defining these new h's have been superseded by literals where no compound function terms occur. For example, $(f + g = 3 \cdot k)_{[0,1]}$ gets rewritten as $(h_1 = f + g)_{]-\infty,+\infty[} \wedge (h_2 = 3 \cdot k)_{]-\infty,+\infty[} \wedge (h_1 = h_2)_{[0,1]}$ – further examples can be found in the rules in (‡). Hence, in view of some basic properties of C^n functions, the algorithm will only need to consider atomic formulas of the types

$$\begin{array}{rcrcrcrc}
x = y & , & x > y & , & z = x \cdot y & , & z = x + y & , \\
y = f(x) & , & (f > g)_A & , & (f = z\, g)_A & , & (h = f + g)_A & , \\
y = D^\alpha[f](x) & , & (D^\alpha[f] \bowtie z)_A & , & \mathsf{S_Up}(f)_A & , & \mathsf{S_Down}(f)_A & , \\
\mathsf{Convex}(f)_A & , & \mathsf{Concave}(f)_A & , & \mathsf{S_Convex}(f)_A & , & \mathsf{S_Concave}(f)_A & ,
\end{array}$$

where $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$. (These types form a streamlined subset of the ones seen in Def. 1.2, since they result from the flattening process mentioned earlier.) Moreover,

Remark 1. Leaving out of consideration literals of the types $(f > g)_A, (D^\alpha[f] > y)_A, (D^\alpha[f] < y)_A$, it suffices to take into account only closed intervals A; in fact, by continuity, the other properties are valid in an open or semi-open interval if and only if they are valid in its closure, e.g., $(f = g)_{]w_1,w_2[}$ holds iff $(f = g)_{[w_1,w_1]}$ holds. ⊣

We can now focus on the algorithm which takes a formula φ of *RDF*^{n} and reduces it, via a series $\varphi \rightsquigarrow \varphi_1 \rightsquigarrow \varphi_2 \rightsquigarrow \varphi_3 \rightsquigarrow \varphi_4 = \psi$ of transformations, to a formula ψ such that:

1. φ and ψ are equisatisfiable,
2. ψ is a Tarskian formula, i.e., one containing only numerical variables, the arithmetical operators $+, \cdot$ and the predicate symbols $=, <$.

As recalled in the introduction, there exists a decision algorithm for Tarskian formulas (cf. [23, 9]). A decision algorithm for *RDF*^{n} results from integrating Tarski's decision algorithm with the reduction $\varphi \rightsquigarrow \psi$ we are about to present.

In the following, w_i denotes a numerical variable, z_i an "extended" numerical variable and α a natural number between 1 and n.

The series of transformations we need goes as follows:

1. $\varphi \rightsquigarrow \varphi_1$: BEHAVIOR AT THE ENDPOINTS.

   a) We rewrite each atom of the form $(f > g)_{]-\infty,w_2[}$, where f, g are function variables and w_2 is a numerical variable, as the formula $(f > g)_{]-\infty,w_1]} \wedge (f > g)_{[w_1,w_2[} \wedge w_1 < w_2$, where w_1 is the first variable in the ordering of domain variables, if w_2 is preceded by at least one such variable; otherwise, w_1 is a brand new domain variable.

   We also perform the specular rewriting:
   $$(f > g)_{]w_1,+\infty[} \rightsquigarrow (f > g)_{]w_1,w_2]} \wedge (f > g)_{[w_2,+\infty[} \wedge w_1 < w_2\,.$$

   Thanks to the rewritings just made, every comparison between functions will refer either to a closed interval or to a bounded interval. (The rewritings to be made at step **c)** will serve a similar aim.)

   b) Let a, b be real numbers such that $a < b$, and f, g be real continuous functions in the closed interval $[a, b]$; then $f > g$ holds in the open interval $]a, b[$ if and only if either

   i. $f > g$ all over $[a, b]$; or
   ii. $f > g$ all over $[a, b[$, and $f(b) = g(b)$; or
   iii. $f > g$ all over $]a, b]$, and $f(a) = g(a)$; or
   iv. $f > g$ all over $]a, b[$, and $f(a) = g(a) \wedge f(b) = g(b)$

   holds. By virtue of the previous equivalences, we perform the following actions:

b$_1$**)** We rewrite a conjunct of this or of an alike form, namely an atom of one of the forms

$$(f > g)_{]w_1,w_2[}\,,\ (f > g)_{[w_1,w_2[}\,,\ (f > g)_{]w_1,w_2]}\,,$$

as an equivalent disjunction comprising 4 or just 2 alternatives; in particular:

$$\begin{array}{lcl}
(f > g)_{]w_1,w_2[} & \rightsquigarrow & (f > g)_{[w_1,w_2]} \vee \\
 & & ((f > g)_{[w_1,w_2[} \wedge f(w_2) = g(w_2)) \vee \\
 & & ((f > g)_{]w_1,w_2]} \wedge f(w_1) = g(w_1)) \vee \\
 & & ((f > g)_{]w_1,w_2[} \wedge f(w_1) = g(w_1) \wedge f(w_2) = g(w_2))\,, \\
(f > g)_{[w_1,w_2[} & \rightsquigarrow & (f > g)_{[w_1,w_2]} \vee ((f > g)_{[w_1,w_2[} \wedge f(w_2) = g(w_2))\,, \\
(f > g)_{]w_1,w_2]} & \rightsquigarrow & (f > g)_{[w_1,w_2]} \vee ((f > g)_{]w_1,w_2]} \wedge f(w_1) = g(w_1))\,,
\end{array}$$

b$_2$**)** Each such rewriting disrupts the structure of the overall formula, which we can readily restore by bringing it again to disjunctive normal form $\delta_1 \vee \delta_2 \vee \cdots \vee \delta_n$ (where $n \in \{2, 4\}$) by means of the distributive law $(\alpha \vee \beta) \wedge \gamma \ \leftrightarrow\ (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$, and then working on each δ_i separately in the sequel of this algorithm.

b$_3$**)** Let w_1, w_2 be numerical variables and f, g be function variables. In each δ_i where the literals $(f > g)_{]w_1,w_2[}$, $f(w_1) = g(w_1)$, and $f(w_2) = g(w_2)$ occur together, when $w_1 < w_2$ as ordered domain variables and there are no domain variables between w_1 and w_2, we add the literals $w_1 < w$, $w < w_2$ and $f(w) = z$, where w and z are new numerical variables. Plainly, the resulting formula and the original one are equisatisfiable.

c) We then rewrite each atom of the form $(D^\alpha[f] > y)_{]-\infty,w_2[}$, where f is a function variable and y, w_2 are numerical variables, as the formula $(D^\alpha[f] > y)_{]-\infty,w_1]} \ \wedge\ (D^\alpha[f] > y)_{[w_1,w_2[} \ \wedge\ w_1 < w_2$, where w_1 is the first variable in the ordering of domain variables if w_2 is preceded by at least one such variable; otherwise, w_1 is a brand new domain variable.

We also perform the specular rewriting:

$$(D^\alpha[f] > y)_{]w_1,+\infty[} \rightsquigarrow (D^\alpha[f] > y)_{]w_1,w_2]} \ \wedge\ (D^\alpha[f] > y)_{[w_2,+\infty[} \ \wedge\ w_1 < w_2\,.$$

We handle similarly also the two cases $(D^\alpha[f] < y)_{]-\infty,w_2[}$ and $(D^\alpha[f] < y)_{]w_1,+\infty[}$.

By these transformations we obtain an equisatisfiable formula.

d) Let a, b, and t be real numbers and f a function, with $f \in C^n([a, b])$. Then f^α, the α-th derivative of f, is greater than t in $]a, b[$, $f^\alpha > t$, if and only if one of the following holds:

i. $f^\alpha > t$ in $[a, b]$,
ii. $f^\alpha > t$ in $[a, b[$ and $f^\alpha(b) = t$,
iii. $f^\alpha > t$ in $]a, b]$ and $f^\alpha(a) = t$,
iv. $f^\alpha > t$ in $]a, b[$, $f^\alpha(a) = t$ and $f^\alpha(b) = t$.

The actions to be made are similar to the ones made under **b**):

d$_1$**)** We rewrite conjuncts of the forms $(D^\alpha[f] > y)_{]w_1,w_2[}$, $(D^\alpha[f] > y)_{[w_1,w_2[}$, and $(D^\alpha[f] > y)_{]w_1,w_2]}$, as equivalent disjunctions; for example:

$$(D^\alpha[f] > y)_{]w_1,w_2]} \rightsquigarrow (D^\alpha[f] > y)_{[w_1,w_2]} \vee \big((D^\alpha[f] > y)_{]w_1,w_2]} \ \wedge\ D^\alpha[f](w_1) = y\big)\,.$$

We proceed similarly for $(D^\alpha[f] < y)_{]w_1,w_2[}$, $(D^\alpha[f] < y)_{[w_1,w_2[}$, $(D^\alpha[f] < y)_{]w_1,w_2]}$.

d$_2$**)** We bring again the overall formula into disjunctive normal form, taking the distributive law into account.

d$_3$**)** If in a formula three literals $(D^\alpha[f] > y)_{]w_1,w_2[}$, $D^\alpha[f](w_1) = y$, and $D^\alpha[f](w_2) = y$ occur together, and they are such that $w_1 < w_2$ as ordered domain variables, and there are no domain variables between w_1 and w_2, we add the following literals: $w_1 < w$, $w < w_2$ and $f(w) = z$. We treat $(D^\alpha[f] < y)_{]w_1,w_2[}$ likewise.

By applying rules **a**), **b**), **c**), and **d**) to a formula ψ in ordered form, we obtain a finite disjunction of ψ_i ordered formulas such that ψ is satisfiable if and only if at least one of the ψ_i is satisfiable. To each ψ_i we apply the rest of the algorithm.

2. $\varphi_1 \rightsquigarrow \varphi_2$: NEGATIVE-CLAUSE REMOVAL.

   From φ_1 we construct an equisatisfiable formula φ_2 within which literals refferring to intervals have no negative occurrences. The general idea applied in this step is to substitute every negative clause involving a function symbol along with an interval spec with an implicit existential assertion.

   For the sake of simplicity, in the following:

   - $x, x_1, x_2, x_3, y_1, y_2, y_3$ will be numerical variables, new w.r.t. the formula considered;
   - we use the notation $x \preccurlyeq y$ as a shorthand for $x \leqslant y$ when $x,\ y$ are both numerical variables; when either x is $-\infty$ or y is $+\infty$, $x \preccurlyeq y$ stands for a true literal (e.g., $0 = 0$).

   a) Replace each literal $\neg(f = z\, g)_{[z_1,z_2]}$ occurring in φ_1 by the formula (involving a new function variable h) $(z_1 \preccurlyeq x \preccurlyeq z_2) \wedge y_1 = f(x) \wedge y_2 = h(x) \wedge \neg(y_1 = y_2) \wedge (h = z\, g)_{[z_1,z_2]}$.

   b) Replace each literal $\neg(h = f + g)_{[z_1,z_2]}$ occurring in φ_1 by the formula (involving a new function variable l) $(z_1 \preccurlyeq x \preccurlyeq z_2) \wedge y_1 = h(x) \wedge y_2 = l(x) \wedge \neg(y_1 = y_2) \wedge (l = f + g)_{[z_1,z_2]}$.

   c) Replace each literal $\neg(f > g)_{[z_1,z_2]}$ occurring in φ_1 by the formula:
   $$(z_1 \preccurlyeq x \preccurlyeq z_2) \wedge y_1 = f(x) \wedge y_2 = g(x) \wedge (y_1 \leqslant y_2).$$

   d) Replace each literal $\neg(D^\alpha[f] \bowtie y)_{[z_1,z_2]}$ occurring in φ_1 by the formula:
   $$(z_1 \preccurlyeq x \preccurlyeq z_2) \wedge y_1 = D^\alpha[f](x) \wedge \neg(y_1 \bowtie y)\,, \quad \text{where} \quad \bowtie \in \{<, \leqslant, =, \geqslant, >\}\,.$$

   e) Replace each literal $\neg$S_Up $(f)_{[z_1,z_2]}$ (resp. $\neg$ S_Down $(f)_{[z_1,z_2]}$) occurring in φ_1 by the formula $\Gamma \wedge y_1 \geqslant y_2$ (resp. $\Gamma \wedge y_1 \leqslant y_2$), where $\Gamma \coloneqq (z_1 \preccurlyeq x_1 < x_2 \preccurlyeq z_2) \wedge y_1 = f(x_1) \wedge y_2 = f(x_2)$.

   f) Replace each literal $\neg$ Convex$(f)_{[z_1,z_2]}$ (resp. $\neg$ S_Convex $(f)_{[z_1,z_2]}$) occurring in φ_1 by $\Gamma \wedge (y_2 - y_1)(x_3 - x_1) > (x_2 - x_1)(y_3 - y_1)$ (resp. $\Gamma \wedge (y_2 - y_1)(x_3 - x_1) \geqslant (x_2 - x_1)(y_3 - y_1)$), where $\Gamma \coloneqq (z_1 \preccurlyeq x_1 < x_2 < x_3 \preccurlyeq z_2) \wedge y_1 = f(x_1) \wedge y_2 = f(x_2) \wedge y_3 = f(x_3)$. Literals of the forms $\neg$ Concave$(f)_{[z_1,z_2]}$, $\neg$ S_Concave $(f)_{[z_1,z_2]}$ are handled similarly.

   Analogously, with only slight changes, we can remove literals about open and semi-open intervals: e.g., $\neg(f > g)_{]w_1,w_2]}$ becomes $(w_1 < x \leqslant w_2) \wedge y_1 = f(x) \wedge y_2 = g(x) \wedge (y_1 \leqslant y_2)$.

   Equisatisfiability of the formulas φ_1 and φ_2 is straightforward to prove. According to Lemma 3.1 and Lemma 3.2, we can transform φ_2, to obtain an equivalent formula in ordered form with domain variables $v_1, v_2, \ldots, v_r$.

3. $\varphi_2 \rightsquigarrow \varphi_3$: EXPLICIT EVALUATION OF FUNCTION VARIABLES.

   This step is preparatory to the elimination of the functional clauses, by explicit evaluation of function variables over domain variables. For each such variable v_j and for every function variable f occurring in φ_2, introduce $n+1$ new numerical variables $y_j^f, t_j^{1,f}, \ldots, t_j^{n,f}$ and add the literals $y_j^f = f(v_j), t_j^{1,f} = D[f](v_j), \ldots, t_j^{n,f} = D^n[f](v_j)$ to φ_2. Moreover, for each literal $x = f(v_j)$ already occurring in φ_2, add the literal $x = y_j^f$ into φ_3; and similarly, for each literal of type $x = D^\alpha[f](v_j)$ already occurring in φ_2, insert the literal $x = t_j^{\alpha,f}$ into φ_3.

   The formula φ_3 resulting from these insertions and the original φ_2 are clearly equisatisfiable.

4. $\varphi_3 \rightsquigarrow \varphi_4$: ELIMINATION OF FUNCTION VARIABLES.

   As a final step, we get rid of all literals containing function variables.

Define the index function $ind\colon V \cup \{-\infty, +\infty\} \to \{1, 2, \ldots, r\}$ over the set $\{v_1, v_2, \ldots, v_r\}$ of distinct domain variables of φ_3 as follows:

$$ind(x) := \begin{cases} 1 & \text{if } x = -\infty, \\ l & \text{if } x = v_l \text{ for some } l \in \{1, 2, \ldots, r\}, \\ r & \text{if } x = +\infty. \end{cases}$$

For each function symbol f occurring in φ_3, introduce new numerical variables $k_0^f, k_r^f, \gamma_0^{1,f}, \ldots, \gamma_0^{n,f}$, $\gamma_r^{1,f}, \ldots, \gamma_r^{n,f}$, and proceed as follows:

a) For each literal $(f = g)_{[z_1,z_2]}$ occurring in φ_3, add all literals $y_i^f = y_i^g$, $t_i^{1,f} = t_i^{1,g}, \ldots$, $t_i^{n,f} = t_i^{n,g}$ whose subscript i satisfies $ind(z_1) \leqslant i \leqslant ind(z_2)$; moreover, if $z_1 = -\infty$ introduce the literals $\gamma_0^{1,f} = \gamma_0^{1,g}, \ldots, \gamma_0^{n,f} = \gamma_0^{n,g}$, and if $z_2 = +\infty$ introduce the literals $\gamma_r^{1,f} = \gamma_r^{1,g}, \ldots, \gamma_r^{n,f} = \gamma_r^{n,g}$.

b) For each literal $(f = z\, g)_{[z_1,z_2]}$ occurring in φ_3, add all literals $y_i^f = z\, y_i^g$, $t_i^{1,f} = z\, t_i^{1,g}, \ldots, t_i^{n,f} = z\, t_i^{n,g}$, whose subscript i satisfies $ind(z_1) \leqslant i \leqslant ind(z_2)$; moreover, if $z_1 = -\infty$ then introduce the literals $\gamma_0^{1,f} = z\, \gamma_0^{1,g}, \ldots, \gamma_0^{n,f} = z\, \gamma_0^{n,g}$, and if $z_2 = +\infty$ then introduce the literals $\gamma_r^{1,f} = z\, \gamma_r^{1,g}, \ldots, \gamma_r^{n,f} = z\, \gamma_r^{n,g}$.

c) For each literal $(h = f + g)_{[z_1,z_2]}$ occurring in φ_3, add all literals $y_i^h = y_i^f + y_i^g$, $t_i^{1,h} = t_i^{1,f} + t_i^{1,g}, \ldots, t_i^{n,h} = t_i^{n,f} + t_i^{n,g}$ whose subscript i satisfies $ind(z_1) \leqslant i \leqslant ind(z_2)$; moreover, if $z_1 = -\infty$ then introduce the literals $\gamma_0^{1,h} = \gamma_0^{1,f} + \gamma_0^{1,g}, \ldots, \gamma_0^{n,h} = \gamma_0^{n,f} + \gamma_0^{n,g}$, and if $z_2 = +\infty$ then introduce the literals $\gamma_r^{1,h} = \gamma_r^{1,f} + \gamma_r^{1,g}, \ldots, \gamma_r^{n,h} = \gamma_r^{n,f} + \gamma_r^{n,g}$.

d) For literals of type $(f > g)_A$, we consider separately bounded and unbounded intervals:

d$_1$) For each literal $(f > g)_{[w_1,w_2]}$ (resp. $(f > g)_{]w_1,w_2[}$, $(f > g)_{[w_1,w_2[}$, $(f > g)_{]w_1,w_2]}$) occurring in φ_3, add the literals $y_i^f > y_i^g$ with $ind(w_1) \leqslant i \leqslant ind(w_2)$ (resp. $ind(w_1) < i < ind(w_2)$, $ind(w_1) \leqslant i < ind(w_2)$, and $ind(w_1) < i \leqslant ind(w_2)$). Moreover, if $w_1 < w_2$ as domain variables, in the case $(f > g)_{]w_1,w_2[}$ (resp. $(f > g)_{[w_1,w_2[}$, $(f > g)_{]w_1,w_2]}$) also add the literals $t_{ind(w_1)}^{1,f} \geqslant t_{ind(w_1)}^{1,g}$, $t_{ind(w_2)}^{1,f} \leqslant t_{ind(w_2)}^{1,g}$ (resp. $t_{ind(w_2)}^{1,f} \leqslant t_{ind(w_2)}^{1,g}$ or $t_{ind(w_1)}^{1,f} \geqslant t_{ind(w_1)}^{1,g}$).

d$_2$) For each literal $(f > g)_{]-\infty,+\infty[}$ (resp. $(f > g)_{]-\infty,w_1]}$, $(f > g)_{[w_1,+\infty[}$) occurring in φ_3, add the literal $y_i^f > y_i^g$ with $1 \leqslant i \leqslant r$ (resp. $1 \leqslant i \leqslant ind(w_1)$, $ind(w_1) \leqslant i \leqslant r$), and the literals $k_0^f \geqslant k_0^g$, $k_r^f \geqslant k_r^g$ (resp. $k_0^f \geqslant k_0^g$ or $k_r^f \geqslant k_r^g$).

e) For literals of type $(D^\alpha[f] \bowtie y)_A$, we consider separately closed and unclosed interval specifications:[8]

e$_1$) For each literal $(D[f]^\alpha \bowtie y)_{[z_1,z_2]}$ occurring in φ_3, where $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$, add the following formulas:

$$t_i^{\alpha,f} \bowtie y, \quad \frac{t_{j+1}^{\alpha-1,f} - t_j^{\alpha-1,f}}{v_{j+1} - v_j} \bowtie y,$$

for $ind(z_1) \leqslant i \leqslant ind(z_2)$ and $ind(z_1) \leqslant j < ind(z_2)$, and if $\bowtie \in \{\leqslant, \geqslant\}$ add the implication:

$$\left(\frac{t_{j+1}^{\alpha-1,f} - t_j^{\alpha-1,f}}{v_{j+1} - v_j} = y \right) \to (t_j^{\alpha,f} = y \wedge t_{j+1}^{\alpha,f} = y);$$

moreover, if $z_1 = -\infty$, introduce the literal $\gamma_0^{\alpha,f} \bowtie y$, and if $z_2 = +\infty$, introduce the literal $\gamma_r^{\alpha,f} \bowtie y$.

e$_2$) For each literal $(D^\alpha[f] \bowtie y)_{]w_1,w_2[}$ (resp. $(D^\alpha[f] \bowtie y)_{]w_1,w_2]}$, $(D^\alpha[f] \bowtie y)_{[w_1,w_2[}$) occurring in φ_3, where $\bowtie \in \{=, <, >, \leqslant, \geqslant\}$, add the formulas:

[8] The treatment of these literals, novel with respect to [1], is a straightforward refinement of the corresponding algorithm step [1, step 4.e)].

$$t_i^{\alpha,f} \bowtie y, \quad \frac{t_{j+1}^{\alpha-1,f} - t_j^{\alpha-1,f}}{v_{j+1}-v_j} \bowtie y\,,$$

for $ind(w_1) \leqslant j < ind(w_2)$ and $ind(w_1) < i < ind(w_2)$ (resp. $ind(w_1) < i \leqslant ind(w_2)$ and $ind(w_1) \leqslant i < ind(w_2)$).

f) For each literal $\mathsf{S\_Up}(f)_{[z_1,z_2]}$ (resp. $\mathsf{S\_Down}(f)_{[z_1,z_2]}$) occurring in φ_3, add the literals

$$t_i^{1,f} \geqslant 0 \;\; (\text{resp. } \leqslant), \quad y_{j+1}^f > y_j^f \;\; (\text{resp. } <),$$

for $ind(z_1) \leqslant i \leqslant ind(z_2)$ and $ind(z_1) \leqslant j < ind(z_2)$; moreover, if $z_1 = -\infty$, introduce the literal $\gamma_0^{1,f} > 0$ (resp. $<$) and, if $z_2 = +\infty$, introduce the formula $\gamma_r^{1,f} > 0$ (resp. $<$).

g) For each literal $\mathsf{Convex}(f)_{[z_1,z_2]}$ (resp. $\mathsf{Concave}(f)_{[z_1,z_2]}$) occurring in φ_3, add:

$$t_i^{1,f} \leqslant \frac{y_{i+1}^f - y_i^f}{v_{i+1}-v_i} \leqslant t_{i+1}^{1,f} \;\; (\text{resp. } \geqslant), \quad t_i^{2,f} \geqslant 0 \;\; (\text{resp. } \leqslant)$$

$$\left(\frac{y_{i+1}^f - y_i^f}{v_{i+1}-v_i} = t_i^{1,f} \;\vee\; \frac{y_{i+1}^f - y_i^f}{v_{i+1}-v_i} = t_{i+1}^{1,f}\right) \rightarrow \left(t_i^{1,f} = t_{i+1}^{1,f}\right),$$

for each $ind(z_1) \leqslant i < ind(z_2)$; moreover, if $z_1 = -\infty$, introduce the literal $\gamma_0^{1,f} \leqslant t_1^{1,f}$ (resp. $\geqslant$), and, if $z_2 = +\infty$, introduce the literal $\gamma_r^{1,f} \geqslant t_r^{1,f}$ (resp. $\leqslant$).

h) For each literal $\mathsf{S\_Convex}(f)_{[z_1,z_2]}$ (resp. $\mathsf{S\_Concave}(f)_{[z_1,z_2]}$) occurring in φ_3, add:

$$t_i^{1,f} < \frac{y_{i+1}^f - y_i^f}{v_{i+1}-v_i} < t_{i+1}^{1,f} \;\; (\text{resp. } >), \quad t_i^{2,f} \geqslant 0 \;\; (\text{resp. } \leqslant)$$

for $ind(z_1) \leqslant i < ind(z_2)$; moreover, if $z_1 = -\infty$, introduce the literal $\gamma_0^{1,f} < t_1^{1,f}$ (resp. $>$), and, if $z_2 = +\infty$, introduce the literal $\gamma_r^{1,f} > t_r^{1,f}$ (resp. $<$).

i) If there are literals involving variables of type k, i.e., literals of the form $k_i^f \geqslant k_i^g$ with $i \in \{0, r\}$ and f, g function variables, perform the following steps:

i. for each variable k_i^f, add the formula $-1 \leqslant k_i^f \leqslant +1$, with $i \in \{0, r\}$;

ii. if both literals $k_i^f \geqslant k_i^g$ and $k_i^g \geqslant k_i^h$ occur in φ_4, add literals $k_i^f \geqslant k_i^h$ and $y_i^f > y_i^h$, with $i \in \{0, r\}$;

iii. if literals $k_0^f \geqslant k_0^g$, $\gamma_0^{1,f} \trianglerighteq m$ and $\gamma_0^{1,g} \trianglelefteq n$ occur together, with $\trianglerighteq \in \{\geqslant, >, =\}$ and $\trianglelefteq \in \{\leqslant, <, =\}$, add literal $m \leqslant n$; specularly, in the case $k_r^f \geqslant k_r^g$, $\gamma_r^{1,f} \trianglelefteq m$ and $\gamma_r^{1,g} \trianglerighteq n$, add the literal $m \geqslant n$.

j) Remove all literals that involve function variables.

The formula φ_4 resulting at the end involves only numerical variables, hence it can be decided by means of Tarski's method.

4. Remarks on the correctness of the algorithm

Proving the correctness of the algorithm amounts to showing that each one of the (*terminating*) transformations $\varphi \rightsquigarrow \varphi_1$, $\varphi_1 \rightsquigarrow \varphi_2$, $\varphi_2 \rightsquigarrow \varphi_3$, $\varphi_3 \rightsquigarrow \varphi_4$ is *satisfiability preserving*. As for the first three transformations (behavior at the endpoints, negative-clause removal, explicit evaluation of function variables), this emerges as a rather straightforward fact.

We must focus on the equisatisfiability of the formulas φ_3 and φ_4, because the transformation $\varphi_3 \rightsquigarrow \varphi_4$ is less transparent than the previous ones: we are, in fact, comparing a formula whose predicates regard the behavior of functions in real intervals with another one, which only involves relations between numerical variables. Let us sketch the idea behind the proof, which as usual consists of two parts: *soundness* and *completeness*. Recall that φ_4 is obtained from φ_3 by adding some formulas that involve only numerical variables and removing all predicates that refer to function variables.

Soundness: If a model exists for φ_3, it can be extended to a model that also verifies the numerical formulas added in φ_4, since these formulas reflect the properties of the functions in φ_3 at specific points in real intervals.

Completeness: Conversely, if there exists a model for φ_4, it should be possible to extend it to φ_3 by interpreting the function variables with suitable interpolating functions. Thus, showing the correctness of the fourth transformation calls for an *ad hoc* interpolation method, which we have produced explicitly for the case $n = 1$ in [1]; when $n = 2$, we could borrow an interpolation method due to Manni [15]; when $n > 2$, we hope for, and remain in debt with the reader of, a proof of existence of the suitable interpolating function.

Completeness for *RDF*^{2}, a bird's-eye view

The completeness for *RDF*^{2} relies on the interpolation method developed by Manni [15] which is a shape-preserving C^2 interpolation method. This means that, given a grid of real numbers $v_1 < v_2 < \cdots < v_r$ and real values y_i, t_i, s_i with $i \in \{1, \ldots, r\}$, Manni's method builds a C^2 real function f such that, for all $i \in \{1, \ldots, r\}$:

$$f(v_i) = y_i, \quad f'(v_i) = t_i \ \text{ and } \ f''(v_i) = s_i. \tag{1}$$

Moreover, Manni's method has three properties relevant for our aims:

1. It preserves the "geometric properties" [10] of the given data v_i, y_i, t_i, s_i; e.g, if the data are increasing, $y_i \leqslant y_{i+1}$ and $t_i \geqslant 0$ for all i, so will be the interpolating function.

2. The interpolation of a sum of data is the sum of the two interpolations; namely given two series of values y_i^f, t_i^f, s_i^f and y_i^g, t_i^g, s_i^g over the same grid $v_1 < v_2 < \cdots < v_r$ with, respectively, two interpolating functions f and g, then $f + g$ is the interpolating function for the values $y_i^f + y_i^g, t_i^f + t_i^g, s_i^f + s_i^g$.

3. Beside the given data v_i, y_i, t_i, s_i, the interpolating function f built by the method depends on a shrinking parameter k and we will use the notation f_k to emphasize the dependency from this parameter k. As k tends toward 0, the function f_k tends to the piece-wise linear interpolation passing through the points $(v_1, y_1), \ldots, (v_r, y_r)$; more precisely $\lim_{k\to 0} ||f_k - q||_\infty = 0$ where $||f_k - q||_\infty = \sup_{v_1 \leqslant x \leqslant v_r} |f_k(x) - q(x)|$ and q is the piece-wise linear interpolation passing through the points (v_i, y_i).[9]

Roughly speaking, having Manni's method at our disposal the completeness proof goes as follows.

Given a numerical model M for φ_4, i.e., a set of real values which satisfies all the algebraic constraints in φ_4, we use Manni's method to build from M a functional model $\mathcal{F}$ for φ_3, namely a set of real functions interpreting all the function variables of φ_3 and satisfying all the function requirements. More precisely, let $\bar{v}$ denote the interpretation Mv of the numerical variable v under the model M. Given a function variable f in φ_3, we apply Manni's method to the values $\bar{v}_i, \bar{y}_i^f, \bar{t}_i^f$ and $\bar{s}_i^f$ to obtain a C^2 function $\bar{f}$ which will be the interpretation of f. By (1), $\bar{f}$ satisfies all point-wise conditions such as $D[f](v_3) = t_3^f$. The remaining part consists in proving that, for k small enough, the interpolating function $\bar{f}_k$ satisfies also all the other possible atomic formulas, e.g. $(D[f] > s)_{[v_i, v_j[}$; this part heavily relies on the three interpolation properties previously exposed.

5. The threshold of undecidability

Tarski himself showed that decidability of his full elementary algebra of real numbers [23] would be disrupted if its language were enriched with a periodic real function, e.g., $\sin x$. Richardson proved in [20] the undecidability of the existential theory of reals extended with the numbers $\log 2$ and π, and with the functions e^x, $\sin x$; Richardson's results have been subsequently improved by Caviness [7], Wang [24] and Laczkovich [13]. More precisely: Caviness removed the use of e^x and $\ln 2$; Wang extended Caviness' result from undecidability for comparison with 0, namely of type $A(x) < 0$, to

[9]Over the interval $[v_i, v_{i+1}]$, q is defined as $q(x) := y_i + \frac{x - v_i}{v_{i+1} - v_i} \cdot (y_{i+1} - y_i)$.

undecidability for equality to 0, i.e., $A(x) = 0$; Laczkovich removed the need of π and reduced the use of function composition.

In consequence of Laczkovich's result and of our reduction of *RDF*^{n} to Tarskian algebra, any extension of *RDF*^{n} enabling us to express $\sin x$ turns out to be undecidable. For example, an atomic formula $\left(D^2[f] = g\right)_A$ for equality between a second derivative and a function would allow one to specify $f = \sin x$ through the differential characterization:

$$f(0) = 0 \ \wedge \ D^1[f](0) = 1 \ \wedge \ (D^2[f] = -f)_{]-\infty,+\infty[} \, .$$

Thus, for $n \geqslant 2$, the introduction of atomic formulas of type $\left(D^2[f] = g\right)_A$ would make *RDF*^{n} undecidable. Establishing whether or not an analogous extension of *RDF*^{1} is decidable is harder. As far as we now, having comparison between first derivatives and functions does not allow one to define $\sin x$; however, it enables the definition of e^x via

$$f(0) = 1 \ \wedge \ (D^1[f] = f)_{]-\infty,+\infty[} \, ,$$

but, since even the decidability of Tarskian algebra extended with the exponential function is still an open problem [14], we cannot judge whether such an enrichment, i.e., function-derivative comparison $\left(D^1[f] = g\right)_A$, would disrupt the decidability of *RDF*^{1}.

Conclusions

Applications

The decidability result presented in this paper is not merely of theoretical interest, but can be seen as a contribution to the automated reasoning field; as a matter of fact *RMCF*$^+$, one of the decision algorithms from which it originates, was discussed in the monograph [21], which is a companion of the proof-verification system ÆtnaNova: if it were implemented inside such a system, a decision algorithm akin to it could play the role of a sophisticated and specialized inference mechanism.[10] Regrettably, though, the worst-case algorithmic complexity of the known algorithms concerned with real functions, in their present forms, is not encouraging [3]; discouragement can be alleviated by considerations about the behaviour in typical cases, as discussed in the same context (cf. [3, pp. 775–776]).

Envisaged applications of a system such as ÆtnaNova regard proof checking as well as program-correctness verification; parts of it specialized on real algebra and analysis might also assist in formal hardware validation and in the study of hybrid systems.

Given the enduring popularity of resolution, research on decision algorithms—whether focusing on fragments of mathematical theories or logical calculi, or addressing entire theories—has only sporadically influenced the field of automated deduction. Notable exceptions include the influential papers by Nelson-Oppen [17, 18]. These works are significant not because they add to the inventory of decidable theories, but because they address the integration of decision algorithms. In this regard, their impact and long-term influence are comparable to that of the DPLL algorithm, which, besides being directly applicable as a test for propositional logic, often serves as a crucial and ubiquitous inference mechanism.

Related and future work

This article has presented a decision algorithm for a fragment *RDF*^{n} of real analysis, which extends the unquantified part of Tarski's elementary algebra EAR of real numbers with variables designating functions of a real variable endowed with continuous derivatives up the n-th order.

The decidability of the theory *RDF*^{n} is a follow-up of a series of previous results, regarding the theories *RMCF*, *RMCF*$^+$, *RDF*, *RDF*$^+$ and *RDF** [5, 3, 8, 4, 2, 1]. A general survey on those results, save the last two, can be found in [6], where other decidability results on real analysis are also treated, in particular the *FS* theory [11, 12].

It is hoped that decision methods regarding differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ are amenable to the approach discussed above: an encouraging indication in this direction comes from [5, Sec. 4] which,

[10] Actually, less specialized than it may seem at first glance: as shown in [5, Sec. 5], a decision algorithm conceived for real numbers and univariate real functions could be exploited to reason about a totally ordered set $(S, <)$ and monotone functions from S to S.

however, deals with continuous function (with no concern about differentiability).

A decidability problem that seems worth being investigated regards the theory RDF^{∞}, whose set of formulas is the union of all RDF^{n} formulas with $n \in \mathbb{N}$ (hence we have a differentiation operator $D^{i}[\cdot]$ for every natural number i); the intended semantics will refer to the real functions of class C^{∞}. The decision algorithm will proceed in full analogy with the one of each RDF^{n}; the correctness proof seems more challenging, though, because the interpolating method needs to accommodate an arbitrarily high number of derivative constraints. Thus far, we have no evidence in favor or against the existence of the sought C^{∞} interpolant.

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